Voevodsky's conjecture for cubic fourfolds and Gushel-Mukai fourfolds via noncommutative K3 surfaces

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June 5, 2018

Abstract

In the first part of this paper we will prove the Voevodsky's nilpotence conjecture for smooth cubic fourfolds and ordinary generic Gushel-Mukai fourfolds. Then, making use of noncommutative motives, we will prove the Voevodsky's nilpotence conjecture for generic Gushel-Mukai fourfolds containing a τ-plane $\mathrm{Gr}(2,3)$ and for ordinary Gushel-Mukai fourfolds containing a quintic del Pezzo surface.

Introduction and statement of the results

In 1995 Voevodsky conjectured the following statement for the algebraic cycles of a smooth projective k-scheme X:

Conjecture (V). ([26], Conjecture 4.2) $\mathcal{Z}^*_{\otimes_{\mathrm{nil}}}(X)_F$ coincides with $\mathcal{Z}^*_{\otimes_{\mathrm{num}}}(X)_F$.

Here, $\mathcal{Z}^*(X)_F$ denotes the group of algebraic cycles of X, $\otimes_{\mathrm{nil}}$ denotes the smash-nilpotence equivalence relation on $\mathcal{Z}^*(X)_F$, introduced in [26], and $\otimes_{\mathrm{num}}$ denotes the classical numerical equivalence relation on $\mathcal{Z}^*(X)_F$ (see §1).
Voevodsky's nilpotence conjecture was classically known for curves, surfaces, abelian threefolds and uniruled threefolds (see [1], [10], [19], [21], [26]

2000 *Mathematics Subject Classification*: 14A22, 14C15, 14J35
Key words and phrases: Voevodsky's nilpotence conjecture, Non-commutative motives, Non-commutative algebraic geometry, Derived category, Cubic fourfolds, Gushel-Mukai fourfolds, Noncommutative K3 surfaces.

and [27]).

In 2014, motivated by the above conjecture, Bernardara, Marcolli and Tabuada stated the following conjecture for a smooth and proper dg category $\mathcal{A}$:

Conjecture ($\mathrm{V_{nc}}$). ([4], Introduction) $\mathrm{K}_0(\mathcal{A})/_{\sim \otimes_{\mathrm{nil}}}$ is equal to $\mathrm{K}_0(\mathcal{A})/_{\sim \otimes_{\mathrm{num}}}$.

Here, $K_0(\mathcal{A})$ denotes the Grothendieck group of the full subcategory $\mathcal{D}_{\mathrm{C}}(\mathcal{A})$ of compact objects of the derived category of $\mathcal{A}$, $\sim \otimes_{\mathrm{nil}}$ and $\sim \otimes_{\mathrm{num}}$ denote two equivalence relations on the Grothendieck group $K_0(\mathcal{A})$, as explained in §3.

They also reformulate Voevodsky's conjecture in the following way:

Theorem (BMT). ([4], Theorem 1.1) *Let X be a smooth projective k-scheme. The conjecture $\mathrm{V}(X)$ is equivalent to the conjecture $\mathrm{V_{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$.*

Here, $\mathrm{perf}_{\mathrm{dg}}(X)$ denotes the unique enhancement of the derived category of perfect complexes on X. Making use of noncommutative motives, Voevodsky's conjecture was proven for quadric fibrations, intersection of quadrics, linear sections of Grassmannians, linear sections of determinantal varieties, homological projective duals and a particular Moishenzon manifold (see [4] and [5] for details).

The main result of this paper is the proof of Voevodsky's conjecture for cubic fourfolds and ordinary generic Gushel-Mukai fourfolds. We recall that a cubic fourfold is a smooth complex hypersurface of degree 3 in $\mathbb{P}^5$, while a Gushel-Mukai fourfold is a smooth and transverse intersection of the form $\mathrm{Cone}(\mathrm{Gr}(2, V_5)) \cap \mathrm{Q}$, where Q is a quadric hypersurface in $\mathbb{P}^8 \subset \mathbb{P}(\bigwedge^2 V_5 \oplus \mathbb{C})$.

Theorem (A). *Let X be a cubic fourfold or an ordinary generic Gushel-Mukai fourfold; then the conjecture $\mathrm{V}(X)$ holds.*

In order to prove this conjecture, we use the decomposition in rational Chow motives of a flat morphism, computed by Vial in [25], Corollary 4.4. Indeed, we recall that a cubic fourfold X admits a flat conic fibration obtained by blowing up a line inside X and, then, projecting on a $\mathbb{P}^3$ which does not intersect the line (Lemma 2.1). On the other hand, in Proposition 2.3, we construct a flat conic fibration over a $\mathbb{P}^3$ by blowing up a smooth del Pezzo surface of degree four in an ordinary generic Gushel-Mukai fourfold. By Vial's result, we deduce a decomposition of the Chow motive of a cubic fourfold and of an ordinary generic Gushel-Mukai fourfold, whose summands correspond to varieties of smaller dimension and for which the

Voevodsky's conjecture is known.

As a direct consequence of Theorem (A), we prove the noncommutative version of Voevodsky's nilpotence conjecture for the Kuznetsov category of a cubic fourfold and the GM category of an ordinary generic Gushel-Mukai fourfold. Indeed, we recall from [13] that the derived category of a cubic fourfold X has a semiorthogonal decomposition of the form $\mathrm{perf}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle$. Here, the line bundles $\mathcal{O}_X$, $\mathcal{O}_X(H)$ and $\mathcal{O}_X(2H)$ are exceptional objects and $\mathcal{A}_X$ is a noncommutative K3 surface in the sense of Kontsevich. Analogously, in [15], Proposition 2.3, they proved that the derived category of a Gushel-Mukai fourfold X admits a semiorthogonal decomposition with four exceptional objects and a non-trivial part $\mathcal{A}_X$; again, the subcategory $\mathcal{A}_X$ is a noncommutative K3 surface in the sense of Kontsevich (see §4).

Let X be a cubic fourfold or a Gushel-Mukai fourfold; we denote by $\mathcal{A}_X^{\mathrm{dg}}$ the dg enhancement of the category $\mathcal{A}_X$ induced from $\mathrm{perf}_{\mathrm{dg}}(X)$. By Theorem (A), it is immediate to deduce the proof of conjecture V_{nc} for $\mathcal{A}_X^{\mathrm{dg}}$ as explained below.

Theorem (B). *Let X be a cubic fourfold or an ordinary generic Gushel-Mukai fourfold; then $\mathrm{V}_{\mathrm{nc}}(\mathcal{A}_X^{\mathrm{dg}})$ holds.*

An application of the part of Theorem (A) concerning cubic fourfolds is the following result, which states the Voevodsky's conjecture for generic Gushel-Mukai fourfolds containing a τ-plane.

Theorem (C). *Let X be a generic Gushel-Mukai fourfold containing a plane P of type $\mathrm{Gr}(2,3)$; then $\mathrm{V}(X)$ holds.*

We point out that the proof of Theorem (C) is based on the fact that the semiorthogonal decomposition of $\mathrm{perf}(X)$ contains the Kuznetsov category associated to a cubic fourfold, as showed in [15], Theorem 1.3.

To the best of the authors knowledge, Theorem (A) and Theorem (C) prove Voevodsky's nilpotence conjecture in new cases.
We believe that Theorem (B) provides a new tool for the proof of Voevodsky's conjecture of a smooth projective k-scheme whose derived category of perfect complexes contains the noncommutative K3 surface $\mathcal{A}_X$.

The plan of the paper is the following. In Section 1 we briefly survey some constructions and basic properties of pure motives. In Section 2.1 we

recall the construction of a conic fibration associated to a cubic fourfold, obtained by blowing up a line and then projecting on a disjoint $\mathbb{P}^3$. The main result of Section 2.2 is the construction of a conic fibration associated to an ordinary generic Gushel-Mukai fourfold (Proposition 2.3). Section 2.3 is devoted to prove Theorem (A). In Section 3 we recall the formulation of the noncommutative Voevodsky's conjecture and its connection with the classical version. In Section 4 we give a brief review of the notion of Kuznetsov category (resp. GM category) associated to a cubic fourfold (resp. a Gushel-Mukai variety). Then, we prove Theorem (B). In Section 5 we show some consequences of the results proved in the previous sections. In particular, we prove the Voevodsky's conjecture for Gushel-Mukai fourfolds containing some particular surfaces (Theorem (C) and Theorem (D)).

Notations and conventions: The letter k will stand for a field. The letter F will denote a commutative ring. Throughout the article we will assume that all cubic fourfolds and Gushel-Mukai varieties are smooth. We will denote by $\mathrm{perf}(X)$ the derived category of perfect complexes of $\mathcal{O}_X$-modules and by $\mathrm{perf}_{\mathrm{dg}}(X)$ the corresponding (unique) dg enhancement. We point out that if X is a smooth and projective scheme, then $\mathrm{perf}(X)$ coincides with the derived category $\mathcal{D}^b(X)$ of bounded complexes of coherent sheaves on X. Moreover, if there exists a semiorthogonal decomposition of the form $\mathrm{perf}(X) = \langle \mathcal{T}_1, ..., \mathcal{T}_l \rangle$, then $\mathcal{T}_i^{\mathrm{dg}}$ is the dg enhancement induced from $\mathrm{perf}_{\mathrm{dg}}(X)$.

1 Background in pure motives

In this first section we give some information about the theory of pure motives. In particular, we define the group of algebraic cycles and some adequate equivalence relations on it. Then, we give an idea of the construction of the category of Chow motives. Our refence is [1], Chapitre 3 and Chapitre 4. Finally, we recall some basic properties of rational motivic decomposition we will use in the next.

Let k be a field.

Definition 1.1 (Group of algebraic cycles). Let X be a smooth projective k-scheme. We define the group of algebraic cycles $\mathcal{Z}^*(X)$ to be the direct sum $\bigoplus_{d \in \mathbb{N}} \mathcal{Z}^d(X)$, where $\mathcal{Z}^d(X)$ denotes the group

$$\mathcal{Z}^d(X) := \left\{ V = \sum_i n_i V_i, \quad \begin{array}{l} \text{s.t. } n_i \in \mathbb{Z} \text{ and } V_i \text{ is an irreducible reduced} \\ \text{closed subscheme with } \operatorname{codim}_X(V_i) = d \end{array} \right\}.$$

Remark 1. If F is a commutative ring, we set $\mathcal{Z}^*(X)_F = \mathcal{Z}^*(X) \otimes F$.

For any pair α, $\beta \in \mathcal{Z}^*(X)_F$, we denote by $\alpha \cdot \beta$ their intersection product. In order to define a ring structure on the group of algebraic cycles induced by the intersection product, it is necessary to quotient the group by an adequate equivalence relation. We give some examples of adequate relations.

Example 1.1 (Rational equivalence). We say that two algebraic cycles α and β in $\mathcal{Z}^d(X)_F$ are rationally equivalent ($\alpha \sim_{\mathrm{rat}} \beta$) if there exists an algebraic cycle $\gamma \in \mathcal{Z}^d(X \times \mathbb{P}^1)_F$, flat over $\mathbb{P}^1$, such that $i_0^{-1}\gamma - i_\infty^{-1}\gamma = \alpha - \beta$. The maps $i_0 : X \times \{0\} \to X \times \mathbb{P}^1$ and $i_\infty : X \times \{\infty\} \to X \times \mathbb{P}^1$ are the respective inclusions. In the case of divisors, the condition above is equivalent to say that there exists a rational function f on X such that $\alpha - \beta = Z(f)$. We call *Chow ring* the ring $\mathcal{Z}^*(X)_F/_{\sim_{\mathrm{rat}}}$ and we denote it by $\mathrm{Chow}(X)$. In the sequel, we will also use the notation $\mathrm{CH}_i(X)$ to denote the Chow group of i-dimensional cycles modulo rational equivalence.

Example 1.2 (Smash-nilpotence equivalence). We say that an algebraic cycle $\alpha \in \mathcal{Z}^*(X)_F$ is smash-nilpotent equivalent to zero if there exists a positive integer n such that $\alpha^{\otimes n}$ is equal to 0 in $\mathrm{Chow}(X^n)$. Two algebraic cycles α, $\beta \in \mathcal{Z}^*(X)_F$ are smash-nilpotent equivalent ($\alpha \sim_{\otimes_{\mathrm{nil}}} \beta$) if the algebraic cycle $\alpha - \beta$ is smash-nilpotent equivalent to zero.

Example 1.3 (Numerical equivalence). Let n be the dimension of X. We say that an algebraic cycle $\alpha \in \mathcal{Z}^d(X)_F$ is numerically trivial if for all

$\gamma \in \mathcal{Z}^{n-d}(X)_F$, we have $\gamma \cdot \alpha \sim_{\otimes_{\mathrm{rat}}} 0$. Two cycles α and β are numerically equivalent ($\alpha \sim_{\mathrm{num}} \beta$) if the algebraic cycle $\alpha - \beta$ is numerically trivial.

Roughly speaking, we can define the category of Chow motives whose objects are triples (X, p, r), where X is a smooth projective k-scheme, p is an idempotent endomorphism and r is an integer, and whose morphisms are obtained by the algebraic cycles. We denote by $\mathrm{Chow}(k)$ the category of Chow motives. For further details about the construction of $\mathrm{Chow}(k)$ and $\mathrm{Chow}(k)_F$ consult [1], Chapitre 4.
We have a contravariant symmetric monoidal functor

$$\mathfrak{h} : \mathrm{SmProj}(k) \to \mathrm{Chow}(k)$$
$$X \mapsto \mathfrak{h}(X),$$

where $\mathrm{SmProj}(k)$ denotes the category of smooth and projective k-schemes. It is well known that $\mathrm{Chow}(k)$ is an additive, idempotent complete and rigid symmetric monoidal category. We list some properties of the functor $\mathfrak{h}$.

1.1 Projective space

Let us denote by $\mathbb{1}$ the $\otimes$-unit of the category $\mathrm{Chow}(k)$; we recall that $\mathfrak{h}(\mathbb{P}^1) = \mathbb{1} \oplus \mathbb{L}$, where $\mathbb{L}$ denotes the Lefschetz motive.
In more general terms, for every positive integer n, we have the decomposition $\mathfrak{h}(\mathbb{P}^n) = \bigoplus_{i=0}^{n} \mathbb{1}(-i)$, where $\mathbb{1}(1)$ denotes the Tate motive (i.e. the inverse of $\mathbb{L}$, formally $\mathbb{1}(-1) = \mathbb{L}$) and $-(i)$ denotes $- \otimes \mathbb{1}(1)^{\otimes i}$.

1.2 Blowups

The functor $\mathfrak{h}$ is "well behaved" with respect to blowups. In detail, let X be a smooth projective variety over a field k and let $j : Y \hookrightarrow X$ be a smooth closed subvariety of codimension r. Then the blowup $\pi_Y : \mathrm{Bl}_Y(X) \to X$ of X in Y induces an isomorphism of Chow motives $\mathfrak{h}(X) \oplus \bigoplus_{i=1}^{r-1} \mathfrak{h}(Y)(i) \to \mathfrak{h}(\mathrm{Bl}_Y(X))$ (see [18], Section 7). As a consequence, if $\dim Y \leq 2$, then $\mathrm{V}(\mathrm{Bl}_Y(X))$ holds if and only if $\mathrm{V}(X)$ holds.

1.3 Flat morphisms

Consider a flat morphism $f : X \to B$ in $\mathrm{SmProj}(k)$, with X and B of dimension d_X and d_B, respectively. We denote by X_b the fiber of f over a point b in B and let Ω be a universal domain containing k. Assume

that $\mathrm{CH}_l(X_b) = \mathbb{Q}$, for all $0 \leq l < \frac{d_X - d_B}{2}$ and for all points $b \in B(\Omega)$. Then we have a direct sum decomposition of the Chow motive of X as $\mathfrak{h}(X) \simeq \bigoplus_{i=0}^{d_X - d_B} \mathfrak{h}(B)(i) \oplus (Z, r, \lfloor \frac{d_X - d_B + 1}{2} \rfloor)$, where Z is a smooth and projective variety of dimension

$$d_Z = \begin{cases} d_B - 1, \text{ if } d_X - d_B \text{ is odd,} \\ d_B, \text{ if } d_X - d_B \text{ is even.} \end{cases}$$

For a complete proof of this result, we refer to [25], Theorem 4.2, Corollary 4.4.

Remark 2. We point out that the same results hold for the category $\mathrm{Chow}(k)_F$ for any field F.

2 Cubic fourfolds and Gushel-Mukai varieties

The aim of this section is to prove Theorem (A). To this end, we discuss the construction of quadric fibrations over the projective space $\mathbb{P}^3$, obtained form the blow-up of a cubic fourfold (resp. a GM fourfold) over a line (resp. a surface). Then, we show how to use this geometric construction to prove Voevodsky's nilpotence conjecture for cubic fourfolds and ordinary generic GM fourfolds. From now on, the field k is the complex field $\mathbb{C}$.

2.1 Cubic fourfolds

Definition 2.1 (Cubic fourfold). A *cubic fourfold* is a smooth complex hypersurface of degree 3 in $\mathbb{P}^5$.

We observe that a cubic fourfold X contains (at least) a line l. Actually, by [3], the Fano variety parametrizing lines on X is an irreducible holomorphic symplectic fourfold, deformation equivalent to the Hilbert scheme of points of length two on a K3 surface.

We denote by l a line in X and let $\mathrm{Bl}_l(X)$ be the blow-up of X in l. The aim of this paragraph is to prove that the projection from the line l induces a flat quadric fibration from $\mathrm{Bl}_l(X)$.

Lemma 2.1. *Let X be a smooth cubic fourfold and let l be a line in X. Then the linear projection from the line l induces a flat quadric fibration from the blow-up $\mathrm{Bl}_l(X)$ to $\mathbb{P}^3$.*

Proof. Let V_6 be a six-dimensional vector space such that $X \subset \mathbb{P}(V_6) \cong \mathbb{P}^5$. Let V_2 be a two-dimensional subvector space of V_6 such that $l = \mathbb{P}(V_2) \cong \mathbb{P}^1$ and we set $V_4 := V_6/V_2$. We denote by $\mathrm{Bl}_l(\mathbb{P}(V_6))$ the blow-up of $\mathbb{P}(V_6)$ in l.

Then the projection from the line l defines a regular map $\pi : \mathrm{Bl}_l(\mathbb{P}(V_6)) \to \mathbb{P}(V_4) \cong \mathbb{P}^3$, which is a $\mathbb{P}^2$-bundle over $\mathbb{P}^3$. Let $\pi_l : \mathrm{Bl}_l(X) \to X$ be the blow-up of X along l. Then the restriction of π to $\mathrm{Bl}_l(X)$ induces a smooth flat conic fibration $f : \mathrm{Bl}_l(X) \to \mathbb{P}(V_4) \cong \mathbb{P}^3$. In other words, we have the following commutative diagram

$$
\begin{array}{ccc}
\mathrm{Bl}_l(X) & \hookrightarrow & \mathrm{Bl}_l(\mathbb{P}(V_6)) \\
\pi_l \downarrow & \searrow f & \downarrow \pi \\
X \subset \mathbb{P}(V_6) & & \mathbb{P}(V_4)
\end{array}
$$

where f is the map claimed in the statement.

$\square$

2.2 Gushel-Mukai varieties

Let V_5 be a k-vector space of dimension 5; considering the Plücker embedding, we have that $\mathrm{Cone}(\mathrm{Gr}(2, V_5)) \subset \mathbb{P}(k \oplus \wedge^2 V_5)$. We denote by W a linear subspace of dimension $n + 5$ of $\wedge^2 V_5 \oplus k$ (with $2 \leq n \leq 6$).

Definition 2.2 (Gushel-Mukai n-fold)**.** We define a *Gushel-Mukai n-fold* X to be a smooth and transverse intersection of the form

$$
X = \mathrm{Cone}(\mathrm{Gr}(2, V_5)) \cap \mathrm{Q},
$$

where Q is a quadric hypersurface in $\mathbb{P}(W)$.
We say that X is:

- *Ordinary* if X is isomorphic to a linear section of $\mathrm{Gr}(2, V_5) \subset \mathbb{P}^9$,

- *Special* if X is isomorphic to a double cover of a linear section of $\mathrm{Gr}(2, V_5)$ branched along a quadric section.

From now on, we will write GM instead of Gushel-Mukai.

Let X be a GM variety. Notice that X does not contain the vertex of the cone over $\mathrm{Gr}(2, V_5)$, because X is smooth. Thus, we have a regular map defined by the projection from the vertex:

$$
\gamma_X : X \to \mathrm{Gr}(2, V_5).
$$

Definition 2.3 (Gushel bundle)**.** Let $\mathscr{U}$ be the tautological bundle of rank 2 over $\mathrm{Gr}(2, V_5)$. We define the *Gushel bundle* to be the pullback $\mathscr{U}_X := \gamma_X^* \mathscr{U}$.

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We denote by $\pi : \mathbb{P}_X(\mathscr{U}_X) \to X$ the projectivization of the bundle $\mathscr{U}_X$. We can consider the map

$$\rho : \mathbb{P}_X(\mathscr{U}_X) \to \mathbb{P}(V_5)$$

induced by the embedding $\mathscr{U}_X \hookrightarrow V_5 \otimes \mathcal{O}_X$. By [8], Proposition 4.5, we have that ρ is a fibration in quadrics.

Now, let us suppose that X is an ordinary GM fourfold. By [8], Remark 3.15 and Remark B.4, the fibers of ρ are all conics in $\mathbb{P}^2$ except for the fiber over a point v_0 in $\mathbb{P}(V_5)$, which is a 2-dimensional quadric in $\mathbb{P}^3$. Let us fix a four-dimensional subvector space V_4 of V_5 such that the point v_0 is not contained in $\mathbb{P}(V_4)$. We set

$$\tilde{X} := \mathbb{P}_X(\mathscr{U}_X) \times_{\mathbb{P}(V_5)} \mathbb{P}(V_4)$$

and we denote by $\tilde{\rho}$ the restriction of ρ to $\tilde{X}$. Thus, we have the following commutative diagram

$$
\begin{array}{ccc}
 & \overset{\pi}{\overbrace{}} & \\
 & \tilde{X} \longrightarrow \mathbb{P}_X(\mathscr{U}_X) & \\
\sigma \swarrow \quad \tilde{\rho} \downarrow & & \downarrow \rho \\
X \longleftarrow \mathbb{P}(V_4) \longrightarrow & \mathbb{P}(V_5) &
\end{array}
\qquad (1)
$$

By the previous observations, we have that the restriction $\tilde{\rho}$ defines a flat conic fibration over $\mathbb{P}(V_4) \cong \mathbb{P}^3$. In the rest of this section, we prove that $\tilde{X}$ is smooth when X is generic.

Notice that for every x in X, the fiber of σ over x is equal to $\mathbb{P}(\mathscr{U}_{X,x} \cap V_4)$. In particular, we have that $\sigma^{-1}(x)$ is a point (resp. a line) if the dimension of $\mathscr{U}_{X,x} \cap V_4$ is equal to 1 (resp. if $\mathscr{U}_{X,x} \subset V_4$). It follows that the locus of non trivial fibers of σ is the intersection

$$E := \mathrm{Gr}(2, V_4) \cap X = \mathrm{Gr}(2, V_4) \cap \mathbb{P}(W) \cap \mathrm{Q} \subset \mathbb{P}(\overset{2}{\bigwedge} V_5) \cong \mathbb{P}^9 . \qquad (2)$$

Since the Grassmannian $\mathrm{Gr}(2, V_4)$ has degree 2, we have that the degree of E is at most 4. Moreover, the expected dimension of E is 2. On the other hand, by Lefschetz Theorem the fourfold X cannot contain a divisor with degree less than 10, because its class has to be cohomologous to the class of a hyperplane in X. Thus, we conclude that $\dim(E) \leq 2$. In the next lemma, we show that E is smooth under generality assumptions on $\mathbb{P}(W)$ and Q; in this case, E is a del Pezzo surface of degree 4.

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Lemma 2.2. *If W is a generic vector space of dimension 9 in $\bigwedge^2 V_5$ and Q is a generic quadric hypersurface in the linear system $|\mathcal{O}_{\mathbb{P}(W)}(2)|$, then E defined in (2) is a smooth and irreducible surface.*

Proof. We consider the intersection $Y := \mathbb{P}(W) \cap \mathrm{Gr}(2, V_4) \subset \mathbb{P}(\bigwedge^2 V_5) \cong \mathbb{P}^9$. By Bertini's Theorem on hyperplane sections (see [9], Chapter 1), we have that Y is smooth and irreducible, because $\mathbb{P}(W)$ is a generic hyperplane in $\mathbb{P}^9$.

Let $i : Y \hookrightarrow \mathbb{P}^8$ be the embedding of Y in $\mathbb{P}(W) \cong \mathbb{P}^8$. Notice that if Y is contained in the quadric Q, then $Y = E$ would be a smooth divisor in X with degree less than 10, in contradiction with the previous observation. Hence, we have that the quadric Q does not contain Y. Again by Bertini's Theorem, the intersection $Y \cap Q = E$ is smooth and irreducible. Indeed, we can consider the embedding of $\mathbb{P}^8$ in $\mathbb{P}(H^0(\mathbb{P}^8, \mathcal{O}(2))) \cong \mathbb{P}^N$ defined by $\mathcal{O}(2)$. The quadric hypersurfaces in $\mathbb{P}^8$ correspond to hyperplanes in $\mathbb{P}^N$ via this embedding. Thus, by Bertini's Theorem for hyperplane sections, we conclude that the intersection of the image of Y with the generic hyperplane in $\mathbb{P}^N$, corresponding to the generic quadric Q, is smooth and irreducible. Hence, we conclude that E is smooth and irreducible of dimension 2, as we wanted. $\square$

As a consequence, we obtain the smoothness of the restriction to a hyperplane of the conic fibration ρ.

Proposition 2.3. *Let X be an ordinary generic GM fourfold. Then $\tilde{X}$ is the blow-up of X in E (so it is smooth) and the map $\tilde{\rho} : \tilde{X} \to \mathbb{P}(V_4)$ defined in (1) is a flat conic fibration.*

Proof. We observe that the quadric Q which defines X is generic in the linear system $|\mathcal{O}_{\mathbb{P}(W)}(2)|$, because X is a generic quadric section of the intersection $\mathbb{P}(W) \cap \mathrm{Gr}(2, V_5)$. On the other hand, we recall that, by [8], Lemma 2.7, there exists a functor between the groupoid of polarized GM varieties to the groupoid of GM data, which is an equivalence by [8], Theorem 2.9. In particular, a generic X corresponds to a generic GM data $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$. Thus, the vector spaces and the linear maps which define this GM data are generic and, then, W is a generic subvector space in $\bigwedge^2 V_5$. By Lemma 2.2, we have that the locus E defined by (2) is smooth and irreducible.

Notice that $\sigma^{-1}(E)$ is by definition the projective bundle $\mathbb{P}_E(\mathcal{U}_X) \to E$. On the other hand, the exceptional divisor of the blow-up of X in E is isomorphic to the projectivized conormal bundle $\mathbb{P}_E(\mathcal{N}_{E|X}^*)$. Since E can be represented as the zero locus of a regular section of $\mathscr{U}_X^*$, the conormal

bundle of E in X is isomorphic to $\mathscr{U}_X$. Hence, we deduce that $\tilde{X}$ is the blow-up of X in E. It follows that $\tilde{X}$ is smooth and $\tilde{\rho} : \tilde{X} \to \mathbb{P}(V_4)$ is a flat conic fibration, as we claimed. $\qquad\square$

2.3 Proof of Voevodsky's nilpotence conjecture for cubic fourfolds and generic GM fourfolds

In [26] Voevodsky conjectured the following statement for the algebraic cycles:

Conjecture (V). Let X be a smooth projective k-scheme; let $\mathcal{Z}^*_{\otimes_{\mathrm{nil}}}(X)_F$ and $\mathcal{Z}^*_{\otimes_{\mathrm{num}}}(X)_F$ be the ring of algebraic cycles modulo the relation of Example 1.2 and of Example 1.3, respectively. Then $\mathcal{Z}^*_{\otimes_{\mathrm{nil}}}(X)_F$ coincides with $\mathcal{Z}^*_{\otimes_{\mathrm{num}}}(X)_F$.

State of art. Conjecture V was proven for curves, surfaces, abelian three-folds, uniruled threefolds (see [1], [10], [19], [21], [26], [27]), and, making use of noncommutative motives, for quadric fibrations, intersection of quadrics, linear sections of Grassmannians, linear sections of determinantal varieties and some homological projective duals (see [4]).

Theorem (A). *Let X be a cubic fourfold or an ordinary generic GM four-fold. Then the conjecture $\mathrm{V}(X)$ holds.*

Proof. Let X be a cubic fourfold and we consider the blow-up of X along a line l. By Subsection 1.3 and Lemma 2.1, the Chow motive of the blow-up decomposes as

$$\mathfrak{h}(\mathrm{Bl}_l(X)) \simeq \bigoplus_{k=0}^{1} \mathfrak{h}(\mathbb{P}^3)(k) \oplus (Z, r, 1) \simeq \bigoplus_{k=0}^{1}(\bigoplus_{i=0}^{3} \mathbb{1}(-i))(k) \oplus (Z, r, 1),$$

where $r \in \mathrm{End}(\mathfrak{h}(Z))$ and $\dim Z = \dim \mathbb{P}^3 - 1 = 2$. It means that conjecture V holds for $\mathrm{Bl}_l(X)$: by Subsection 1.2 we conclude that conjecture V holds for X, as we wanted.

If X is an ordinary generic GM fourfold, the same strategy applied to the conic fibration of Proposition 2.3 gives the required statement. $\qquad\square$

3 The noncommutative setting

The aim of this section is to recall the formulation of noncommutative Voevodsky's conjecture and its relation with the classical version. Our main references are [4] and [22].

3.1 Dg categories

Let $\mathcal{C}(k)$ be the category of differential graded k-modules. A differential graded category (shortly a dg-category) is a category enriched over $\mathcal{C}(k)$ (i.e. morphism sets are complexes), whose compositions fulfill the Leibniz rule:

$$d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g).$$

A dg functor is a functor between dg categories enriched over $\mathcal{C}(k)$. We denote by $\mathrm{dgcat}(k)$ the category whose objects are small dg categories and whose morphisms are dg functors. Consult [11] for a complete survey.

Let $\mathcal{A}$ be a dg category. A right dg $\mathcal{A}$-module is a dg functor $M : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{dg}}(k)$, where $\mathcal{C}^{\mathrm{dg}}(k)$ is the dg category of dg k-modules and $\mathcal{A}^{\mathrm{op}}$ is the opposite category. We denote by Mod-$\mathcal{A}$ the category of dg $\mathcal{A}$-modules. The derived category of $\mathcal{A}$, denoted by $\mathcal{D}(\mathcal{A})$, is the localization of Mod-$\mathcal{A}$ with respect to the class of objectwise quasi-isomorphisms.

A dg functor $F : \mathcal{A} \to \mathcal{B}$ is a Morita equivalence if the induced functor $\mathbb{L}F_! : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ on derived categories is an equivalence of triangulated categories.

We note that the tensor product of k-algebras gives rise to a symmetric monoidal structure - $\otimes$ - on dgcat. The $\otimes$-unit is the dg category with one object k.

Moreover we say that a dg category $\mathcal{A}$ is smooth if it is perfect as a bimodule over itself. We say that $\mathcal{A}$ is proper if for every couple of objects $x, y \in \mathcal{A}$ the complex of k-modules $\mathcal{A}(x, y)$ is perfect. The definitions of smooth and proper dg category are due to Kontsevich (see also [22], Definition 1.14).

3.2 Dg enhancements

Let X be a smooth projective k-scheme. We know that the category of perfect complexes $\mathrm{perf}(X)$ has a unique dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$ (cf. [16], Theorem 7.9, or [6], Proposition 6.10), which is smooth and proper as a dg category.

Moreover, suppose that the derived category of perfect complexes on X has a semiorthogonal decomposition of the form $\mathrm{perf}(X) = \langle \mathcal{A}_1, ..., \mathcal{A}_n \rangle$. Then, by [4], Lemma 2.1, we have that every dg category $\mathcal{A}_i^{\mathrm{dg}}$ is smooth and proper (where $\mathcal{A}_i^{\mathrm{dg}}$ denotes the dg enhancement of the subcategory $\mathcal{A}_i$ induced from $\mathrm{perf}_{\mathrm{dg}}(X)$).

3.3 Voevodsky conjecture in the noncommutative case

Let $\mathcal{A}$ be a smooth and proper dg category. We denote by $K_0(\mathcal{A})$ the Grothendieck group $K_0(\mathcal{D}_{\mathrm{C}}(\mathcal{A}))$, where $\mathcal{D}_{\mathrm{C}}(\mathcal{A})$ denotes the subcategory of compact objects in $\mathcal{D}(\mathcal{A})$. In analogy to algebraic cycles, we can define some equivalence relations on $K_0(\mathcal{A})$. We discuss the two examples we deal with. Our reference is [4], Sections 2.3 and 2.4.

Example 3.1 ($\otimes$-nilpotence equivalence relation)**.** We say that an element $[M]$ in $K_0(\mathcal{A})$ is $\otimes$-nilpotent if there exists a positive integer n such that $[M \times n] = 0$ in the Grothendieck group $K_0(\mathcal{A}^{\otimes n})$. Given $[M]$ and $[N]$ in $K_0(\mathcal{A})$, we say that $[M]$ and $[N]$ are $\otimes$-nilpotent equivalent (shortly $[M]\sim_{\otimes_{\mathrm{nil}}}[N]$) if $[M]$-$[N]$ is $\otimes$-nilpotent.

We have a bilinear form $\chi(-,-)$ on $K_0(\mathcal{A})$ defined as

$$(M, N) \to \sum_i (-1)^i \dim \mathrm{Hom}_{\mathcal{D}_{\mathrm{C}}(\mathcal{A})}(M, N[i]).$$

The left and right kernels of $\chi(-,-)$ are the same.

Example 3.2 (Numerical equivalence relation)**.** We say that an element $[M]$ in $K_0(\mathcal{A})$ is numerically trivial if $\chi([M],[N]) = 0$ for all $[N] \in K_0(\mathcal{A})$. We say that $[M]$ and $[N]$ are numerically trivial equivalent (shortly $[M]\sim_{\otimes_{\mathrm{num}}}[N]$) if $[M]$-$[N]$ is numerically trivial.

Remark 3. The equivalence relations defined above give rise to well defined equivalence relations on $K_0(\mathcal{A})_F$.

In [4] Bernardara, Marcolli and Tabuada conjectured the following statement:

Conjecture ($\mathrm{V_{nc}}$)**.** Let $\mathcal{A}$ be a smooth proper dg category. Then $\mathrm{K_0}(\mathcal{A})/_{\sim \otimes_{\mathrm{nil}}}$ is equal to $\mathrm{K_0}(\mathcal{A})/_{\sim \otimes_{\mathrm{num}}}$.

Moreover, we have the following result which relates the Voevodsky's nilpotence conjecture and its noncommutative version:

Theorem (BMT)**.** ([4], Theorem 1.1) *Let X be a smooth projective k-scheme. The conjecture $V(X)$ holds if and only if $\mathrm{V_{nc}}(\mathrm{perf_{dg}}(X))$ holds.*

3.4 Noncommutative Chow motives

We denote by $\mathrm{Hmo}(k)$ the localization of $\mathrm{dgcat}(k)$ with respect to the class of Morita equivalences. We observe that the tensor product of dg categories gives rise to a symmetric monoidal structure on $\mathrm{Hmo}(k)$.

Definition 3.1 ([22], Definition 4.1)**.** The category $\mathrm{NChow}(k)$ of noncommutative Chow motives is the pseudoabelian envelope of the full subcategory of smooth and proper dg categories in $\mathrm{Hmo}(k)$.

We recall that the symmetric monoidal functor

$$U : \mathrm{dgcat} \to \mathrm{Hmo}(k)$$

extends to $\mathrm{NChow}(k)$ (see [22], Section 1.6 and Theorem 2.9).

Remark 4. Let X be an object in $\mathrm{SmProj}(k)$. If $\mathrm{perf}(X) = \langle \mathcal{T}_1, ..., \mathcal{T}_l \rangle$, then $U(\mathrm{perf}_{\mathrm{dg}}(X)) = U(\mathcal{T}_1^{\mathrm{dg}}) \oplus ... \oplus U(\mathcal{T}_l^{\mathrm{dg}})$, where $\mathcal{T}_i^{\mathrm{dg}}$ is the dg enhancement induced from $\mathrm{perf}_{\mathrm{dg}}(X)$ (see [5], Proposition 3.1).

4 Kuznetsov category and GM category

In this section we recall some facts about the decomposition of the derived category of a cubic fourfold X (resp. of a GM variety). In particular, we remark some properties about the Kuznetsov category (resp. the GM category) $\mathcal{A}_X$ associated to X. Then we prove Voevodsky's nilpotence conjecture for the Kuznetsov category of a cubic fourfold and for the GM category of an ordinary generic GM fourfold.

4.1 Kuznetsov category

Let X be a cubic fourfold. The derived category of perfect complexes $\mathrm{perf}(X)$ admits a semiorthogonal decomposition given by

$$\mathrm{perf}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle, \qquad (\star)$$

where H is a hyperplane section and $\mathcal{A}_X$ is defined as:

$$\mathcal{A}_X = \langle \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle^{\perp}$$
$$= \{ E \in \mathrm{perf}(X) \text{ s.t. } \mathbb{R}\mathrm{Hom}_{\mathrm{perf}(X)}(\mathcal{O}_X(i), E) = 0 \text{ for } i = 0, 1, 2 \}$$

We call $\mathcal{A}_X$ the Kuznetsov category.

We recall that the triangulated subcategory $\mathcal{A}_X$ is a Calabi-Yau category of dimension 2; indeed, the Serre functor is equal to the shift -[2], i.e. for every pair of objects F, E we have

$$\mathbb{R}\mathrm{Hom}_{\mathcal{A}_X}(E, F)^* \simeq \mathbb{R}\mathrm{Hom}_{\mathcal{A}_X}(F, E)[2].$$

Moreover, $\mathcal{A}_X$ has the same sized Hochschild (co)homology of the derived category of a K3 surface. Thus, the Kuznetsov category is a noncommutative K3 surface in the sense of Kontsevich (see [13], [12], Corollary 4.3 and [14], Proposition 4.1).

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Remark 5. We recall that if X is a cubic fourfold containing a plane, we can prove V-conjecture via noncommutative motives. In fact, if X contains a plane, we have that $\mathcal{A}_X$ is equivalent to $\mathcal{D}^b(S, \mathcal{B})$, where S is a K3 surface, $\mathcal{B}$ is a sheaf of Azumaya algebras on S and $\mathcal{D}^b(S, \mathcal{B})$ is the derived category of coherent $\mathcal{B}$-modules on S (see [13], Theorem 4.3). Then by [23], Theorem 2.1 and Remark 4, we have the following decomposition in NChow(k):

$$U(\mathrm{perf}_{\mathrm{dg}}(X)) \simeq U(\mathrm{perf}_{\mathrm{dg}}(S)) \oplus U(\mathbb{C}) \oplus U(\mathbb{C}) \oplus U(\mathbb{C}).$$

Using Theorem (BMT), since V(S) holds, we conclude that also V(X) holds, as we claimed.

4.2   GM category

Let X be a GM n-fold; in [15], Proposition 4.2, they proved that its derived category of perfect complexes has a semiorthogonal decomposition of the form

$$\mathrm{perf}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathscr{U}_X^*, \mathcal{O}_X(H), \mathscr{U}_X^*(H), ..., \mathcal{O}_X((n-3)H), \mathscr{U}_X^*((n-3)H) \rangle,$$
$$(*)$$

where $\mathscr{U}_X^*$ is the dual of the Gushel bundle previously defined and $\mathcal{A}_X$ is defined as:

$$\mathcal{A}_X = \langle \mathcal{O}_X, \mathscr{U}_X^*, \mathcal{O}_X(H), \mathscr{U}_X^*(H), ..., \mathcal{O}_X((n-3)H), \mathscr{U}_X^*((n-3)H) \rangle^{\perp}.$$

We call $\mathcal{A}_X$ the GM category of X.

Assume that X is a GM fourfold. Then, the Serre functor on $\mathcal{A}_X$ is the shift by two and the Hochschild cohomology of $\mathcal{A}_X$ is isomorphic to that of a K3 surface. In other words, the GM category of a GM fourfold is a noncommutative K3 surface in the sense of Kontsevich (see [15], Proposition 5.18).

4.3   Proof of conjecture $\mathbf{V_{nc}}$ for the Kuznetsov category and the GM category of a generic GM fourfold

Using the deep results in [4], [13] and [15] recalled in the previous sections, it is now clear that Theorem (B) follows easily from Theorem (A).

Theorem (B). *Let X be a cubic fourfold or an ordinary generic GM fourfold. Then $V_{\mathrm{nc}}(U(\mathcal{A}_X^{dg}))$ holds, where $\mathcal{A}_X^{dg}$ is the dg enhancement of $\mathcal{A}_X$ induced from $\mathrm{perf}_{\mathrm{dg}}(X)$.*

Proof. Let X be a cubic fourfold. Using the decomposition $(\star)$, we have that the dg enhancement of the triangulated category $\mathrm{perf}(X)$ admits the following decomposition in $\mathrm{NChow}(k)$:

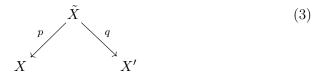$$U(\mathrm{perf}_{\mathrm{dg}}(X)) = U(\mathcal{A}_X^{\mathrm{dg}}) \oplus U(\mathbb{C}) \oplus U(\mathbb{C}) \oplus U(\mathbb{C}).$$

Hence, the result is a straightforward consequence of Theorem (A).

The proof in the case of an ordinary generic GM fourfold X is analogous, applying the decomposition $(*)$ and Theorem (A). $\square$

5 Voevodsky's nilpotence conjecture for GM fourfolds containing surfaces

In this section we will prove Voevodsky's nilpotence conjecture for generic GM fourfolds containing a τ-plane and for ordinary GM fourfolds containing a quintic del Pezzo surface. Let X be a GM fourfold containing a τ-plane P, i.e. a plane P of the form $\mathrm{Gr}(2, V_3)$ for some 3-dimensional subvector space V_3 of V_5. In [15], Lemma 7.8, they proved that there exists a cubic fourfold X' containing a smooth cubic surface scroll T such that the blow-up of X in P is identified to the blow-up of X' in T. More precisely, if $p : \tilde{X} \to X$ is the blow-up of X along P and q is the regular map induced by the linear projection from P, then the diagram

(3)

commutes and q is identified with the blow-up of X' along T. Moreover, they showed that if the GM fourfold X does not contain a plane of the form $\mathbb{P}(V_1 \wedge V_4)$ for some subvectorspaces satisfying $V_1 \subset V_3 \subset V_4 \subset V_5$, then the cubic fourfold X' is smooth. We point out that this construction had already been described in [7], Section 7.2.

They also observed that a generic GM fourfold containing a τ-plane does not contain a plane of the form $\mathbb{P}(V_1 \wedge V_4)$ as above; hence, the associated cubic fourfold X' obtained with this geometric construction is smooth. In this case, they proved that there exists an equivalence of Fourier-Mukai type

$$\phi : \mathcal{A}_X \simeq \mathcal{A}_{X'} \tag{4}$$

between the GM category of X and the Kuznetsov category of X' (see [15], Theorem 1.3).

Using this construction and Theorem (A) we can prove the Voevodsky's nilpotence conjecture for this class of GM fourfolds.

Theorem (C)**.** *Let X be a generic GM fourfold containing a plane P of type* $\mathrm{Gr}(2,3)$. *Then* $\mathrm{V}(X)$ *holds.*

Proof. The derived category of perfect complexes of X has the following decomposition:

$$\mathrm{perf}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathscr{U}_X^*, \mathcal{O}_X(H), \mathscr{U}_X^*(H) \rangle.$$

Since the functor ϕ defined in (4) is of Fourier-Mukai type, we know that ϕ has a dg lift, thanks to the works of [17], Theorem 1.1 and [24], Theorem 8.9. Then the proof is a consequence of Theorem (B) for cubic fourfolds and Theorem (BMT).

Alternatively, we can prove Theorem (C) by observing that the isomorphism of triangulated categories $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ is induced by diagram (3). Then, conjecture $\mathrm{V}(X)$ follows from Subsection 1.2 and Theorem (A). $\square$

In a similar fashion, we can prove conjecture V for the category of perfect complexes of ordinary GM fourfolds containing a quintic del Pezzo surface.

Theorem (D)**.** *Let X be an ordinary GM fourfold containing a quintic del Pezzo surface. Then* $\mathrm{V}(X)$ *holds.*

Proof. By [15], Theorem 1.2 we have that there exist a K3 surface Y and an equivalence $\psi : \mathcal{A}_X \simeq \mathcal{D}^b(Y)$ of Fourier-Mukai type. Since ψ has a dg lift and conjecture V holds for Y, the proof follows from Theorem (BMT). $\square$

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