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# On compact trees with the coarse wedge topology

by

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**Abstract.** We investigate the class of compact trees, endowed with the coarse wedge topology, in connection with the area of non-separable Banach spaces. We describe Valdivia compact trees in terms of inner structures and we characterize the space of continuous functions on them. Moreover we prove that the space of continuous functions on an arbitrary tree with height less than $\omega_1 \cdot \omega_0$ is a Plichko space.

**1. Introduction.** A *tree* is a partially ordered set $(T, <)$ such that the set $\{s \in T : s < t\}$ of predecessors of any $t \in T$ is well-ordered by $<$. There are several natural topologies that can be defined by using the order structures of trees [19]. Among them, the coarse wedge topology is a topology for which the tree $T$ is a compact Hausdorff space whenever $T$ satisfies certain structural properties.

In the present paper we investigate the relations between the coarse wedge topology and the classes of Valdivia compacta and of Plichko Banach spaces. Plichko spaces are a wide class of Banach spaces that extend the class of weakly Lindelöf determined (WLD) Banach spaces. They were introduced in [20] and were studied under equivalent definitions in [7], [24] and [25]; we refer to [14] for a detailed survey. Plichko spaces and the related class of compact spaces, called Valdivia compacta, appear in many different areas; see [12] for the details and [6], [2], [3] for some recent results in Banach spaces, $C^*$-algebras and topology.

W. Kubiś introduced the concept of projectional skeletons in [15], where he adapted the definition of retractional skeleton (see [16], [4]) from the topological setting to Banach spaces. Roughly speaking, a projectional skeleton decomposes the Banach space into smaller separable subspaces [5], [6], [13]. Banach spaces (resp. compact spaces) with a projectional skeleton (resp.

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a retractional skeleton) are the non-commutative counterpart of the aforementioned classes, in the sense that a Banach space (resp. a compact space) is Plichko (resp. Valdivia) if and only if it admits a commutative projectional skeleton (resp. a commutative retractional skeleton). Although Plichko spaces and Banach spaces with a projectional skeleton, as well as Valdivia compacta and compact spaces with a retractional skeleton, share many structural and topological properties, they do not coincide. A simple example of non-Valdivia compact space with a rectractional skeleton is the compact ordinal interval $[0, \omega_2]$ (see [16]). It is more difficult to prove that the Banach space $C([0, \omega_2])$, which has a projectional skeleton, is not Plichko [11].

In [21] we studied the class of trees endowed with the coarse wedge topology, providing new examples of non-Valdivia compact spaces with retractional skeletons. In the same paper it was proved that every tree with height less than or equal to $\omega_1 + 1$ is Valdivia, and no Valdivia tree has height greater than $\omega_2$. Moreover, an example of a non-Valdivia tree with height $\omega_1 + 2$ was given.

In the present paper we follow the same research line. In particular we investigate the space of continuous functions on compact trees. We prove that $C(T)$ is Plichko whenever the height of $T$ is less than $\omega_1 \cdot \omega_0$. Finally, we extend Theorem 4.1 of [21], characterizing Valdivia compact trees with height less than $\omega_2$. It turns out that this characterization depends only on the behavior of the tree on levels with uncountable cofinality.

We now outline how the paper is organized. In the remaining part of the introductory section, notation and basic notions addressed in this paper are given. Section 2 contains details of notation, basic definitions and some preliminary results on trees. Section 3 is devoted to characterizing Valdivia compact trees with height less than $\omega_2$. Section 4 deals with the class of continuous functions on a compact tree. It is shown that if $C(T)$ is 1-Plichko, then $T$ is Valdivia. We also prove that if $T$ is an arbitrary tree with height less than $\omega_1 \cdot \omega_0$, then $C(T)$ is a Plichko space.

We denote by $\omega_0$ the set of natural numbers (including 0) with the usual order. Given a set $X$ we denote by $|X|$ the cardinality of $X$, and by $[X]^{\leq\omega_0}$ the family of all countable subsets of $X$.

All the topological spaces considered are assumed to be Hausdorff and completely regular. Given a topological space $X$ we denote by $\overline{A}$ the closure of $A \subset X$. We say that $A \subset X$ is *countably closed* if $\overline{C} \subset A$ for every $C \in [A]^{\leq\omega_0}$.

Given a topological compact space (resp. a locally compact space) $K$ we use $C(K)$ (resp. $C_0(K)$) to denote the space of all real-valued or complex-valued continuous functions on $K$ (resp. all real-valued or complex-valued continuous functions on $K$ vanishing at infinity) with the usual norm. By the Riesz representation theorem the elements of $C(K)^*$ are considered as measures. If $\mu \in C(K)^*$, we denote by $\|\mu\|$ its norm. If $\mu$ is a non-negative measure, we denote by $\operatorname{supp}(\mu)$ the support of $\mu$, i.e. the set of $x \in K$ such that each neighborhood of $x$ has positive $\mu$-measure. The support of a measure $\mu \in C(K)^*$ coincides with the support of its total variation $|\mu|$.

Given a Banach space $X$ and a subset $A \subset X$ we denote by $\operatorname{span}(A)$ the linear hull of $A$, by $B_X$ the norm-closed unit ball of $X$, and by $X^*$ the (topological) dual space of $X$. A set $D \subset X^*$ is said to be $\lambda$-*norming* if

$$\|x\| \le \lambda \sup\{|x^*(x)| : x^* \in D \cap B_{X^*}\}$$

for every $x \in X$, and *norming* if it is $\lambda$-*norming* for some $\lambda \ge 1$. A subspace $S \subset X^*$ is called a *Σ-subspace* if there is a set $M \subset X$ such that $\overline{\operatorname{span}}(M) = X$ and

$$S = \bigl\{f \in X^* : \{m \in M : f(m) \neq 0\} \text{ is countable}\bigr\}.$$

A Banach space $X$ is called a *Plichko* (resp. a $\lambda$-*Plichko*) *space* if $X^*$ has a norming (resp. $\lambda$-norming) $\Sigma$-subspace.

Let $\Gamma$ be an arbitrary set. We put

$$\Sigma(\Gamma) = \bigl\{x \in \mathbb{R}^\Gamma : |\{\gamma \in \Gamma : x(\gamma) \neq 0\}| \le \omega_0\bigr\}.$$

Let $K$ be a compact space. We say that $A \subset K$ is a *Σ-subset* of $K$ if there exists a homeomorphic injection $h$ of $K$ into some $\mathbb{R}^\Gamma$ such that $h(A) = h(K) \cap \Sigma(\Gamma)$. Finally, $K$ is a *Valdivia compact space* if $K$ has a dense $\Sigma$-subset.

**2. Basic notions on trees.** We recall that a *tree* is a partially ordered set $(T, <)$ such that the set $\{s \in T : s < t\}$ of predecessors of any $t \in T$ is well-ordered by $<$. A tree $T$ is said to be *rooted* if it has only one minimal element, called the *root.* A totally ordered subset of $T$ is called a *chain.* A maximal chain is called a *branch.* The tree is called *chain complete* if every chain has a supremum. For any $t \in T$, $\operatorname{ht}(t,T)$ denotes the order type of $\{s \in T : s < t\}$. For any ordinal $\alpha$, the set $\operatorname{Lev}_\alpha(T) = \{t \in T : \operatorname{ht}(t,T) = \alpha\}$ is called the *αth level* of $T$. The *height* of $T$, denoted by $\operatorname{ht}(T)$, is the least $\alpha$ such that $\operatorname{Lev}_\alpha(T) = \emptyset$. For an element $t \in T$, $\operatorname{cf}(t)$ denotes the cofinality of $\operatorname{ht}(t,T)$, where $\operatorname{cf}(t) = 0$ when $\operatorname{ht}(t,T)$ is a successor ordinal; and $\operatorname{ims}(t) = \{s \in T : t < s,\ \operatorname{ht}(s,T) = \operatorname{ht}(t,T)+1\}$ denotes the set of *immediate successors* of $t$. Given a subset $S$ of a tree $T$, and an element $t \in S$, we denote by $\operatorname{ims}_S(t)$ the set of immediate successors of $t$ in $S$ with the inherited order. In a tree $T$ of height $\alpha$, for each $\beta < \alpha$ we denote $T_\beta = \bigcup_{\gamma \le \beta} \operatorname{Lev}_\gamma(T)$ and $T_{<\beta} = \bigcup_{\gamma<\beta} \operatorname{Lev}_\gamma(T)$.

For $t \in T$ we put $V_t = \{s \in T : s \ge t\}$ and $\hat{t} = \{s \in T : s \le t\}$. In the present work we consider $T$ endowed with the *coarse wedge topology* (defined below). This topology coincides with the path topology on the set of all initial chains of $T$, which is a tree itself when ordered by inclusion; we refer to [23], [9] for the details.

The coarse wedge topology on a tree $T$ is the one whose subbase is the set of all $V_t$ and their complements, where $t$ is either minimal or on a successor level. If $\mathrm{ht}(t,T)$ is a successor or $t$ is the minimal element, a local base at $t$ is formed by all sets of the form

$$W_t^F = V_t \setminus \bigcup\{V_s : s \in F\},$$

where $F$ is a finite set of immediate successors of $t$. If $\mathrm{ht}(t,T)$ is limit, a local base at $t$ is formed by all sets of the form

$$W_s^F = V_s \setminus \bigcup\{V_r : r \in F\},$$

where $s < t$, $\mathrm{ht}(s,T)$ is a successor and $F$ is a finite set of immediate successors of $t$. We refer to [22] and [19] for further information.

Since we are interested in compact spaces, we recall that, by [19, Corollary 3.5], a tree $T$ is compact Hausdorff in the coarse wedge topology if and only if $T$ is chain complete and has finitely many minimal elements. For this reason, from now on we will consider only chain complete trees with a unique minimal element. The operation $t \wedge s = \max(\hat{t} \cap \hat{s})$ is then well-defined for all $s, t \in T$.

Given a subset $S$ of a tree $T$, there are two natural topologies on $S$: the subspace topology and the coarse wedge topology generated by the inherited order. We shall prove that these topologies sometimes coincide.

LEMMA 2.1. *Let $S$ be a closed subset of a tree $T$. Suppose that $S$ is closed under $\wedge$ (i.e. if $s, t \in S$, then $s \wedge t \in S$). Then the subspace topology coincides with the coarse wedge topology on $S$.*

*Proof.* We first observe that if $S$ is a branch of $T$, then the two topologies coincide with the interval topology. We shall prove that if $S$ is endowed with the coarse wedge topology, then it is compact. We observe that, since $T$ is chain complete, any chain in $S$ has a supremum in $T$. By the closedness of $S$, the supremum belongs to $S$. Moreover, since $S$ is closed under $\wedge$ and $T$ is rooted, we deduce that $S$ is rooted too. Therefore, by [19, Corollary 3.5] the set $S$, endowed with the coarse wedge topology, is a compact Hausdorff space.

We shall prove that the coarse wedge topology on $S$ is coarser than the subspace topology.

Let $x \in S$ be on a successor level in $T$. Since $S$ is closed, $x$ is also on a successor level in $S$. Let $W_x^F \subset S$ be an open basic neighborhood of $x$, where $F = \{t_i\}_{i=1}^n \subset \mathrm{ims}_S(x)$. For each $t_i \in T$ there exists a unique $u(t_i) \in \mathrm{ims}_T(x)$ such that $t_i \geq u(t_i)$. Let $F_1 = \{u(t_i)\}_{i=1}^n$. Then $W_x^F \supset W_x^{F_1} \cap S$.

Let $x \in S$ belong to a limit level in $T$. Now two cases are possible:

- $x$ is on a limit level in $S$. Let $W_s^F \subset S$ be an open basic neighborhood of $x$, where $s < x$ is on a successor level in $S$ and $F = \{t_i\}_{i=1}^n \subset \mathrm{ims}_S(x)$. Let $F_1 = \{u(t_i)\}_{i=1}^n$ and $\{s+1\} = \hat{x} \cap \mathrm{Lev}_{\mathrm{ht}(s,T)+1}(T)$. Then $W_{s+1}^{F_1}$ is an open basic neighborhood of $x$ in $T$ and $W_{s+1}^{F_1} \cap S \subset W_s^F$.
- $x$ is on a successor level in $S$. Let $W_x^F \subset S$ be an open basic neighborhood of $x$, where $F = \{t_i\}_{i=1}^n \subset \mathrm{ims}_S(x)$. Since $x$ is on a successor level in $S$, it has a unique immediate predecessor, say $x-1$. Since $x$ is on a limit level in $T$ and $S$ is closed in $T$, there exists $s \in T$ on a successor level such that $x-1 < s < x$ and $W_s^x \cap S = \emptyset$. Indeed, let $x-1 < s < x$ and suppose that $y \in W_s^x \cap S$. Then $x-1 < s \le x \wedge y < x$. Since $x \wedge y \in S$, this contradicts the maximality of $x-1$. Hence $W_s^{F_1} \cap S \subset W_x^F$, where $F_1 = \{u(t_i)\}_{i=1}^n$.

Since $S$ is a compact Hausdorff space in both topologies, we obtain the assertion. ■

As a consequence we obtain the following result.

Corollary 2.2. *Let $C$ be a countable subset of a tree $T$. Then $\overline{C}$ is a metrizable subspace of $T$.*

*Proof.* Let $C_\wedge$ be the smallest subset of $T$ containing $C$ and closed under $\wedge$. It is clear that $C_\wedge$ is countable. We shall prove that $\overline{C_\wedge}$ is closed under $\wedge$. Let $s, t \in \overline{C_\wedge}$. Suppose that $s, t$ are incomparable; otherwise $s \wedge t \in \overline{C_\wedge}$ follows immediately. Let $\{u(t)\} = \mathrm{ims}(s \wedge t) \cap \hat{t}$ and $\{u(s)\} = \mathrm{ims}(s \wedge t) \cap \hat{s}$. Since $s, t \in \overline{C_\wedge}$, there are $s_1, t_1 \in C_\wedge$ such that $s_1 \in V_{u(s)}$ and $t_1 \in V_{u(t)}$. Then $s \wedge t = s_1 \wedge t_1 \in C_\wedge$.

We observe that if $t \in \overline{C_\wedge} \setminus C_\wedge$, then $t$ belongs to a limit level. Indeed, suppose that $t \in \overline{C_\wedge} \setminus C_\wedge$ and $t$ belongs to a successor level. Then there exists an infinite set $A$ such that for each $\alpha \in A$ there exists $t_\alpha \in \mathrm{ims}(t)$ satisfying $V_{t_\alpha} \cap C_\wedge \neq \emptyset$. Pick $s_{\alpha_i} \in V_{t_{\alpha_i}} \cap C_\wedge$ such that $\alpha_i \in A$ with $i = 1, 2$; then $s_{\alpha_1} \wedge s_{\alpha_2} = t \in C_\wedge$, a contradiction.

Thus, any $t \in \overline{C_\wedge} \setminus C_\wedge$ belongs to a limit level and so any chain is at most countable. Combining Lemma 2.1 with [18, Theorem 2.8] we find that $\overline{C_\wedge}$ is a separable Corson compact space, hence metrizable. Since $\overline{C} \subset \overline{C_\wedge}$, we obtain the assertion. ■

Let $X$ be a topological space. A family $\mathcal{U}$ of subsets of $X$ is $T_0$-*separating* in $X$ if for any distinct $x, y \in X$ there is $U \in \mathcal{U}$ satisfying $|\{x, y\} \cap U| = 1$. A family $\mathcal{U}$ is *point countable on* $D \subset X$ if

$$|\{U \in \mathcal{U} : x \in U\}| \le \omega_0 \quad \text{for every } x \in D.$$

Since we are interested in compact trees, we are going to state [14, Proposition 1.9] in these terms.

Theorem 2.3. *Let $T$ be a tree and $D = \{t \in T : \mathrm{cf}(t) \le \omega_0\}$. Then the following are equivalent:*

(i) *$T$ is Valdivia.*
(ii) *$D$ is a $\Sigma$-subset of $T$.*
(iii) *There is a $T_0$-separating family of basic clopen sets point-countable exactly on $D$.*

The equivalence of (i) and (ii) follows from [21, Proposition 3.2], while (ii)$\Leftrightarrow$(iii) follows from [10, Theorem 19.11], by observing that a tree $T$ endowed with the coarse wedge topology is a zero-dimensional space.

We conclude this section with a description of Radon measures on trees, useful to investigate the spaces of continuous functions on trees. Observe that a chain complete and rooted tree $T$ endowed with the coarse wedge topology is the Stone space of the Boolean algebra $\mathrm{Clop}(T)$ of clopen subsets of $T$. For a tree $T$ endowed with the coarse wedge topology, $\mathrm{Clop}(T)$ is generated by the family $\{V_t : t \text{ is on a successor level}\}$.

Combining this observation with [8, Lemma 3.2], we will be able to prove that every Radon measure on a tree $T$ has metrizable support. We recall that a partial order is *$\sigma$-centered* if it is a countable union of centered subsets.

Lemma 2.4 ([8, Lemma 3.2]). *Suppose that a Boolean algebra $\mathcal{U}$ is generated by a subfamily $\mathcal{G}$ such that if $a, b \in \mathcal{G}$, then $a \le b$, $b \le a$ or $a \cdot b = 0$, and that $\mathcal{U}$ is not $\sigma$-centered. Then $\mathcal{U}$ carries no strictly positive measure.*

Proposition 2.5. *Let $T$ be a tree and $\mu$ a Radon measure on $T$. Then the support of $\mu$ is metrizable.*

*Proof.* Let $S = \mathrm{supp}(\mu)$. Then $S$ is a compact subspace of $T$, and hence a Stone space. Since the Boolean algebra $\mathrm{Clop}(S)$ is generated by the family $\mathcal{G} = \{V_t \cap S : t \text{ is on a successor level}\}$, the previous lemma implies that $\mathrm{Clop}(S)$ and in particular $\mathcal{G}$ are $\sigma$-centered.

Thus $\mathcal{G}$ may be written as a union of countably many chains. We claim that all these chains are countable. If not, let $\{V_{t_\alpha} \cap S : \alpha < \omega_1\}$ be a chain of size $\omega_1$ such that $t_\alpha \le t_\beta$ if $\alpha \le \beta$. Let $U_\alpha = (V_{t_{\alpha+1}} \setminus V_{t_\alpha}) \cap S$ for each $\alpha < \omega_1$. Then $\{U_\alpha\}_{\alpha<\omega_1}$ is an uncountable family of disjoint open subsets of $S$ with positive measure, a contradiction. Hence $\mathcal{G}$ is countable, as also is $\mathrm{Clop}(S)$. Therefore $S$ is metrizable. ∎

As an immediate consequence, by [14, Theorem 5.3], $C(T)$ is a WLD Banach space if and only if $T$ is a Corson compact space. Moreover, from the previous proposition we easily obtain the following result.

Corollary 2.6. *Let $T$ be a tree with height $\eta + 1$ where $\mathrm{cf}(\eta) \ge \omega_1$, and $\mu$ a continuous Radon measure on $T$. Then there exists $\beta < \eta$ such that $\mathrm{supp}(\mu) \subset T_\beta$.*

**3. Characterization of Valdivia compact trees.** The purpose of this section is to describe relations between trees and Valdivia compacta. We will characterize trees of height less than $\omega_2$. We recall the following definition.

Definition 3.1. Let $X$ be a topological space. We say that a subset $A \subset X$ is *$\omega_1$-relatively discrete* if it can be written as a union of $\omega_1$-many relatively discrete subsets of $X$.

The main results of this section are contained in the following theorem.

Theorem 3.2. *Let $T$ be a tree. Let $R = \{t \in T : \mathrm{cf}(t) = \omega_1 \;\&\; \mathrm{ims}(t) \neq \emptyset\}$. Consider the following conditions:*

(i) $|\mathrm{ims}(t)| < \omega_0$ *for every* $t \in R$*;*
(ii) $R \cap \mathrm{Lev}_\alpha(T)$ *is $\omega_1$-relatively discrete for each* $\alpha < \omega_2$ *with* $\mathrm{cf}(\alpha) = \omega_1$.

*Then the following two statements hold.*

(1) *If $T$ is Valdivia, then* $\mathrm{ht}(T) \leq \omega_2$ *and* (i) *and* (ii) *hold.*
(2) *If* $\mathrm{ht}(T) < \omega_2$ *and* (i) *and* (ii) *hold, then $T$ is Valdivia.*

The proof is split into two parts. The first statement does not require any extra result; on the other hand, we postpone the second part to the end of the section because two lemmata are needed.

*Proof of Theorem 3.2(1).* Let $T$ be a Valdivia compact tree. Since $T$ is Valdivia, by [21, Theorem 4.1] we have $\mathrm{ht}(T) \leq \omega_2$. Moreover, $T$ has a retractional skeleton, hence by [21, Theorem 3.1] we have $|\mathrm{ims}(t)| < \omega_0$ for every $t \in \mathrm{Lev}_\alpha(T)$ with $\mathrm{cf}(\alpha) = \omega_1$, in particular for every $t \in R$, and (i) is fulfilled.

To prove (ii), let $\alpha < \omega_2$ with $\mathrm{cf}(\alpha) = \omega_1$. Since $T$ is a Valdivia compact, by Theorem 2.3 the subtree $T_{\alpha+1}$ is Valdivia as well. Hence, by (iii) of that theorem, there exists a family $\mathcal{U}_\alpha$ of clopen subsets of $T_{\alpha+1}$ that is $T_0$-separating and point countable on $D_\alpha = \{t \in T_{\alpha+1} : \mathrm{cf}(t) \leq \omega_0\}$. Each $U \in \mathcal{U}_\alpha$ is of the form $W_s^F$ for some $s \in T_{\alpha+1}$ and some finite subset $F \subset T_{\alpha+1}$, whose elements are larger than $s$ and on a successor level. For every $t \in R \cap \mathrm{Lev}_\alpha(T)$ there is $\eta(t) < \alpha$ such that if $U \in \mathcal{U}_\alpha$, $t \in U$, and $\mathrm{ht}(\min U, T_{\alpha+1}) > \eta(t)$, then $U \cap \mathrm{ims}(t) = \emptyset$. Indeed, since for every $s \in \mathrm{ims}(t)$ we have $s \in D_\alpha$, it follows that $s$ is contained in countably many elements of $\mathcal{U}_\alpha$. For this reason there are only countably many elements of $\mathcal{U}_\alpha$ containing both $t$ and $s$. It is enough to take

$$\eta(t) = \sup\{\mathrm{ht}(p, T_{\alpha+1}) : p < t,\ (\exists W_p^F \in \mathcal{U}_\alpha)(t \in W_p^F,\ \mathrm{ims}(t) \cap W_p^F \neq \emptyset)\}.$$

Let $R_\eta = \{t \in R \cap \mathrm{Lev}_\alpha(T) : \eta(t) = \eta\}$.

Let $t \in R_\eta$. Since $t \notin D_\alpha$, there exists an unbounded subset $S_t$ of $\hat{t}$ such that for each $s \in S_t$ there exists $W_s^F \in \mathcal{U}_\alpha$ with $t \in W_s^F$. In particular, since $S_t$ is unbounded, there exists $s_0 \in S_t$ and an open basic subset $W_{s_0}^F \in \mathcal{U}_\alpha$

with $\mathrm{ht}(s_0, T_{\alpha+1}) > \eta$. Since $F$ is finite and $\mathrm{ims}(p) \cap W^F_{s_0} = \emptyset$ if $p \in R_\eta$, we have $|W^F_{s_0} \cap R_\eta| < \omega_0$. Therefore there exists $r \in T_{\alpha+1}$ on a successor level such that $s_0 \le r < t$ and $V_r \cap R_\eta = \{t\}$. Hence $R_\eta$ is relatively discrete for each $\eta < \omega_1$, which gives the assertion. ∎

We observe that the second statement of Theorem 3.2 cannot be reversed. Indeed, there are several examples of Valdivia trees with height $\omega_2$. Here we provide an easy example. Let $X$ be the topological sum of the ordinal intervals $X_\alpha = [0, \alpha]$ where $\alpha < \omega_2$. Let $X_0 = X \cup \{\infty\}$ be the one-point compactification of $X$. By [14, Theorem 3.35], $X_0$ is a Valdivia compact space. Consider the following relation on $X_0$:

- $\infty$ is the least element,
- $x < y$ in $X$ if and only if there exists $\alpha < \omega_2$ such that $x, y \in X_\alpha$ and $x < y$ in $X_\alpha$.

It is clear that $(X_0, <)$ is a tree and, if endowed with the coarse wedge topology, it is homeomorphic to $X_0$ with the topology given by the compactification. Therefore we have obtained the desired tree.

Much more interesting is the following problem, which seems to be open.

Problem 3.3. *Can the first statement of Theorem* 3.2 *be reversed?*

In order to prove Theorem 3.2(2), we need to describe a natural way to extend relatively open subsets to the whole tree. Let $T$ be a tree of height $\alpha$ and let $\beta < \alpha$ be on a successor level. Let $U \subset T_\beta$ be a relatively open set in $T_\beta$. We extend $U$ to the whole tree as follows:

$$\widetilde{U} = U \cup \bigcup_{x \in \mathrm{Lev}_\beta(T) \cap U} V_x.$$

It is clear that $\widetilde{U}$ is open in $T$. Given a family $\mathcal{U}_\beta$ of open subsets of $T_\beta$ we denote by $\widetilde{\mathcal{U}}_\beta$ the family of the extended elements of $\mathcal{U}_\beta$.

Given a family $\mathcal{U}$ of clopen subsets of $T$ we put $\mathcal{U}(t) = \{U \in \mathcal{U} : t \in U\}$ for every $t \in T$. If $A, B \subset T$ and $A \cap B = \{t\}$, then by abuse of notation, $\mathcal{U}(A \cap B)$ means $\mathcal{U}(t)$. We need three technical lemmata.

Let $T$ be a tree with height $\le \alpha + 1$, where $\alpha$ has uncountable cofinality. Let $\{\alpha_\gamma\}_{\gamma < \mathrm{cf}(\alpha)}$ be a continuous increasing transfinite sequence converging to $\alpha$. Denote by $I(\mathrm{cf}(\alpha))$ the set of all successor ordinals less than $\mathrm{cf}(\alpha)$.

Suppose that for each $\gamma \in I(\mathrm{cf}(\alpha))$, there exists a $T_0$-separating family $\mathcal{U}_\gamma$ in $T_{\alpha_\gamma+1}$, and that each $t \in T_{\alpha_\gamma+1}$ belongs to at least one element of $\mathcal{U}_\gamma$. For each $\gamma \in I(\mathrm{cf}(\alpha))$, $U \in \mathcal{U}_\gamma$ and $t \in \mathrm{Lev}_{\alpha_{(\gamma-1)}+1}(T)$ define $U_t = V_t \cap U$. Finally, we define

$$\mathcal{U} = \bigcup_{\gamma \in I(\mathrm{cf}(\alpha))} \bigcup_{U \in \mathcal{U}_\gamma} \{\widetilde{U}_t : t \in \mathrm{Lev}_{\alpha_{(\gamma-1)}+1}(T)\}.$$

Now we can state the first lemma.

LEMMA 3.4. *Let $T$ be a tree with height $\leq \alpha+1$, where $\alpha$ has uncountable cofinality. Let $\mathcal{U}$ be the family of clopen subsets of $T$ defined as above. Then $\mathcal{U}$ is $T_0$-separating in $T$.*

*Proof.* Let $s, t \in T$ with $\mathrm{ht}(s,T) \leq \mathrm{ht}(t,T)$.

- If $t \in T_{\alpha_\gamma+1} \setminus T_{\alpha_{(\gamma-1)}}$ for some $\gamma \in I(\mathrm{cf}(\alpha))$, then the assertion follows from the fact that the family $\mathcal{U}_\gamma$ is $T_0$-separating in $T_{\alpha_\gamma+1}$.
- If $t \in \mathrm{Lev}_{\alpha_\gamma}(T)$ with $\gamma$ limit, then $\mathrm{ht}(s \wedge t, T) < \mathrm{ht}(t,T)$. Since $\alpha_\gamma$ is limit too, there is a successor ordinal $\xi < \gamma$ such that $\mathrm{ht}(s \wedge t, T) < \alpha_{\xi-1}$. We define $\{u\} = \hat{t} \cap \mathrm{Lev}_{\alpha_\xi+1}(T)$. Let us consider two cases. If $\{v\} = \hat{s} \cap \mathrm{Lev}_{\alpha_\xi+1}(T)$, then since $u, v \in T_{\alpha_\xi+1}$ and $\mathcal{U}_\xi$ is $T_0$-separating in $T_{\alpha_\xi+1}$, there is $U \in \mathcal{U}_\xi$ such that $|U \cap \{u,v\}| = 1$. It follows that $|\widetilde{U} \cap \{s,t\}| = 1$. Otherwise suppose that $\hat{s} \cap \mathrm{Lev}_{\alpha_\xi+1}(T) = \emptyset$. Similarly there exists $U \in \mathcal{U}_\xi$ such that $|U \cap \{u,s\}| = 1$. Thus $|\widetilde{U} \cap \{s,t\}| = 1$.
- If $t \in \mathrm{Lev}_\alpha(T)$ we use the same argument as in the previous item.

Therefore $\mathcal{U}$ is a $T_0$-separating family in $T$. ∎

LEMMA 3.5. *Let $T$ be a tree with height greater than $\eta$ where $\mathrm{cf}(\eta) \geq \omega_1$. Let $N$ be a countable subset of $\mathrm{Lev}_\eta(T)$. Then there exists $\delta < \eta$ such that if $t_1, t_2 \in N$, then $\mathrm{ht}(t_1 \wedge t_2, T) < \delta$.*

*Proof.* Let $N = \{t_n\}_{n\in\omega_0}$ where $\{t_n\}_{n\in\omega_0}$ is a one-to-one sequence. Define $\delta_n^m = \mathrm{ht}(t_n \wedge t_m, T)$. The assertion follows by taking

$$\delta = \sup_{n,m\in\omega_0,\, n\neq m} \delta_n^m + 1.$$

If $N$ is finite, we use the same argument as in the infinite case. ∎

LEMMA 3.6. *Let $T$ be a tree of height $\leq \eta+2$ where $\eta<\omega_2$ and $\mathrm{cf}(\eta)=\omega_1$. Suppose that:*

(1) *$R = \{t \in \mathrm{Lev}_\eta(T) : \mathrm{ims}(t) \neq \emptyset\}$ has cardinality at most $\omega_1$;*
(2) *$|\mathrm{ims}(t)| < \omega_0$ for every $t \in R$;*
(3) *$T_{\gamma+1}$ is a Valdivia compactum for every $\gamma < \eta$ .*

*Then $T$ is a Valdivia compact space.*

*Proof.* We split the proof into two parts. In the first part we define a function $\theta : [0,\omega_1) \to [0,\eta)$ satisfying certain properties; it will be defined separately in three different cases: when $R$ is uncountable, infinite countable, and finite. In the second part we use $\theta$ to define a family $\mathcal{U}$ of clopen subsets of $T$ that is $T_0$-separating and point-countable on $D = \{t \in T : \mathrm{cf}(t) \leq \omega_0\}$.

Suppose that $|R| = \omega_1$, and enumerate $R$ as $\{t_\alpha\}_{\alpha<\omega_1}$. Let $\{\eta_\gamma\}_{\gamma<\omega_1}$ be a continuous increasing transfinite sequence converging to $\eta$. We may suppose that $\eta_0 = 0$. We define the mapping $\theta$ by transfinite induction. Let $\theta(0) = 0$

and for each $\zeta < \omega_1$ set

$$\theta(\zeta) = \max\big(\eta_\zeta, \sup\{\mathrm{ht}(t_\beta \wedge t_\gamma, T) + 1 : \beta, \gamma < \zeta, t_\beta \neq t_\gamma\}, \sup\{\theta(\xi) + 1 : \xi < \zeta\}\big).$$

Then $\theta$ satisfies the following conditions:

- for every $\alpha < \omega_1$ and $\beta, \gamma < \alpha$ ($t_\beta \neq t_\gamma$), $\mathrm{ht}(t_\beta \wedge t_\gamma, T) < \theta(\alpha)$,
- $\theta$ is increasing, continuous and $\sup_{\zeta<\omega_1} \theta(\zeta) = \eta$.

Let us prove that $\theta$ is continuous; the other properties are clear. Let $\zeta < \omega_1$ be a limit ordinal; we need to show that $\sup_{\xi<\zeta} \theta(\xi) = \theta(\zeta)$. We observe that $\eta_\zeta = \sup_{\xi<\zeta} \eta_\xi \leq \sup_{\xi<\zeta}(\theta(\xi) + 1)$, and furthermore

$$\begin{aligned}\sup\{\mathrm{ht}(t_\beta \wedge t_\gamma, T) + 1 : \beta, \gamma < \zeta,\ t_\beta \neq t_\gamma\} &= \sup_{\xi<\zeta} \sup\{\mathrm{ht}(t_\beta \wedge t_\gamma, T) + 1 : \beta, \gamma < \xi,\ t_\beta \neq t_\gamma\} \\ &\leq \sup_{\xi<\zeta} \theta(\xi) \leq \sup_{\xi<\zeta}(\theta(\xi) + 1).\end{aligned}$$

Hence, by definition of $\theta(\zeta)$, we obtain $\sup_{\xi<\zeta}(\theta(\xi) + 1) = \theta(\zeta)$. This proves the continuity.

Suppose that $|R| = \omega_0$ and $R = \{t_n\}_{n<\omega_0}$. Let $\{\theta(\alpha)\}_{\alpha<\omega_1}$ be any continuous increasing sequence with $\theta(0) = 0$ and $\theta(n) = \sup\{\mathrm{ht}(t_{m_1} \wedge t_{m_2}, T) : m_1 < m_2 < n\}$ for each $n \leq \omega_0$. When $R$ is finite one can define $\theta$ similarly.

We observe that for every $t \in T$ on a successor level and with $\mathrm{ht}(t, T) < \eta$, there exists a unique $\alpha < \omega_1$ such that $\mathrm{ht}(t, T) \in [\theta(\alpha), \theta(\alpha + 1))$, and at most one $\beta < \alpha$ such that $t < t_\beta$.

Since $T_{\theta(\alpha)+1}$ is Valdivia, by Theorem 2.3, there exists a family $\mathcal{U}_\alpha$ of clopen subsets of $T_{\theta(\alpha)+1}$ which is $T_0$-separating and point-countable on $D_\alpha = \{t \in T_{\theta(\alpha)+1} : \mathrm{cf}(t) \leq \omega_0\}$ for every $\alpha < \omega_1$. Moreover, the elements of $\mathcal{U}_\alpha$ are of the form $W_s^F$ for every $\alpha < \omega_1$. Finally, we may suppose that each element of $T_{\theta(\alpha)+1}$ is contained in some element of $\mathcal{U}_\alpha$ (if necessary, add $T_{\theta(\alpha)+1}$ to $\mathcal{U}_\alpha$).

In order to define a family $\mathcal{U}$ of clopen subsets of $T$ which is $T_0$-separating and point-countable on $D$, we are going to select and appropriately modify a suitable subfamily of $\bigcup_{\alpha<\omega_1} \mathcal{U}_\alpha$.

Let $\alpha < \omega_1$ be a successor ordinal and $U \in \mathcal{U}_\alpha$. For every $t \in \mathrm{Lev}_{\theta(\alpha-1)+1}(T)$ let $U_t = U \cap V_t$. Recall that if $t \in \mathrm{Lev}_{\theta(\alpha-1)+1}(T)$, then $|V_t \cap \{t_\beta\}_{\beta<\alpha-1}| \leq 1$. Therefore if $V_t \cap \{t_\beta\}_{\beta<\alpha-1} = \emptyset$, then we extend $U_t$ to $\widetilde{U}_t$, while if $V_t \cap \{t_\beta\}_{\beta<\alpha-1} = \{t_\gamma\}$ for some $\gamma < \alpha - 1$, then we extend $U_t$ to $\widetilde{U}_t \setminus \mathrm{ims}(t_\gamma)$, obtaining a clopen subset of $T$ that avoids $\bigcup_{\beta<\alpha-1} \mathrm{ims}(t_\beta)$.

Define $I(\omega_1)$ as the set of successor ordinals less than $\omega_1$. We define the following family of clopen subsets of $T$:

$$\mathcal{U} = \big\{\{t\} : t \in \mathrm{Lev}_{\eta+1}(T)\big\} \cup \bigcup_{\alpha\in I(\omega_1)} \bigcup_{U\in\mathcal{U}_\alpha} \big\{\widetilde{U}_t : t \in \mathrm{Lev}_{\theta(\alpha-1)+1}(T),\, V_t \cap \{t_\beta\}_{\beta<\alpha-1} = \emptyset\big\} \cup \bigcup_{\alpha\in I(\omega_1)} \bigcup_{U\in\mathcal{U}_\alpha} \big\{\widetilde{U}_t \setminus \mathrm{ims}(t_\xi) : t \in \mathrm{Lev}_{\theta(\alpha-1)+1}(T),\, \{t_\xi\} = V_t \cap \{t_\beta\}_{\beta<\alpha-1}\big\}.$$

To prove that $T$ is Valdivia, we first observe that $\mathcal{U}$ restricted to $T_\eta$ satisfies the hypothesis of Lemma 3.4. Therefore combining Lemma 3.4 with the fact that $\{\{t\} : t \in \mathrm{Lev}_{\eta+1}(T)\}$ is contained in $\mathcal{U}$ we find that $\mathcal{U}$ is $T_0$-separating in $T$.

It remains to prove that $\mathcal{U}$ is point-countable on $D$. Suppose that $t \in D$, and consider two cases.

First, suppose that $\mathrm{ht}(t,T) < \eta$. Noting that $|\{\alpha < \omega_1 : \mathrm{ht}(t,T) > \theta(\alpha)\}| \le \omega_0$, define $\alpha_0 = \sup\{\alpha < \omega_1 : \mathrm{ht}(t,T) > \theta(\alpha)\}$. Hence if $t \in U$ and $U \in \mathcal{U}$ then $U$ is extended from an element of a family $\mathcal{U}_\xi$ where $\xi \le \alpha_0$. Since $\mathcal{U}_\xi$ is point-countable on $D_\xi \subset T_{\theta(\xi)+1}$ we have

$$|\mathcal{U}(t)| \le \Big|\bigcup_{\xi\le\alpha_0} \mathcal{U}_\xi(\hat{t} \cap \mathrm{Lev}_{\theta(\xi)+1}(T))\Big| \le \omega_0.$$

Now suppose that $\mathrm{ht}(t,T) = \eta + 1$. Then there exists $t_\beta \in R$ such that $t \in \mathrm{ims}(t_\beta)$ for some $\beta < \omega_1$. Let $X \in \mathcal{U}$ be such that $t \in X$. Then there are the following possibilities:

- $X = \{t\}$; exactly one element of $\mathcal{U}$ has this form.
- There exist $\xi \in I(\omega_1)$ and $s \in \mathrm{Lev}_{\theta(\xi-1)+1}(T)$ such that $X = \widetilde{U}_s$ for some $U \in \mathcal{U}_\xi$. Since $V_s \cap \{t_\gamma\}_{\gamma<\xi-1} = \emptyset$ we obtain $\xi \le \beta + 1$, and moreover $\hat{t} \cap \mathrm{Lev}_{\theta(\xi)+1}(T) \subset U_s$ and $|\mathcal{U}_\xi(\hat{t} \cap \mathrm{Lev}_{\theta(\xi)+1}(T))| \le \omega_0$. Hence there are at most countably many elements of this form.
- There are $\xi \in I(\omega_1)$, $s \in \mathrm{Lev}_{\theta(\xi-1)+1}(T)$ and $p \in R$ such that $X = \widetilde{U}_s \setminus \mathrm{ims}(p)$. Since $V_s \cap \bigcup_{\gamma<\xi-1} \mathrm{ims}(t_\gamma) = \emptyset$, we have $\xi \le \beta + 1$ and since $(\hat{t} \cap \mathrm{Lev}_{\theta(\xi)+1}(T)) \subset U_s$ and $|\mathcal{U}_\xi(\hat{t} \cap \mathrm{Lev}_{\theta(\xi)+1}(T))| \le \omega_0$, there are at most countably many sets of this form.

Therefore $\mathcal{U}$ is point-countable on $D$, hence $T$ is Valdivia. ■

*Proof of Theorem 3.2(2).* We use transfinite induction on the height of the tree. Let $T$ be a tree as in the hypothesis; by [21, Theorem 4.1], if $\mathrm{ht}(T) \le \omega_1 + 1$, then $T$ is Valdivia.

Suppose that the assertion is true for each tree $T$ with $\mathrm{ht}(T) \le \alpha + 2$; we will prove it for trees with height $\le \alpha + 3$. Let $T$ satisfy $\mathrm{ht}(T) = \alpha + 3$. Then, by induction hypothesis, $T_{\alpha+1}$ is a Valdivia compact space. Hence, by Theorem 2.3, there exists a family $\mathcal{U}_\alpha$ of clopen subsets of $T_\alpha$ which is $T_0$-separating and point-countable on $D_\alpha = \{t \in T_{\alpha+1} : \mathrm{cf}(t) \le \omega_0\}$. Then

$\mathcal{U} = \widetilde{\mathcal{U}}_\alpha \cup \{\{t\} : t \in \mathrm{Lev}_{\alpha+2}(T)\}$ is a family of clopen subsets of $T$ which is clearly $T_0$-separating and point-countable on $D = \{t \in T : \mathrm{cf}(t) \leq \omega_0\}$. Therefore $T$ is Valdivia.

Suppose that the assertion is true for each tree $T$ with $\mathrm{ht}(T) < \alpha$ for some limit ordinal $\alpha$; we will prove it for trees with height $\leq \alpha + 2$. Suppose $T$ is such a tree. We consider two cases.

Suppose that $\alpha$ is a limit ordinal with countable cofinality. Then there exists an increasing sequence $\{\alpha_n\}_{n\in\omega_0}$ of ordinals converging to $\alpha$. By induction hypothesis the subtrees $T_{\alpha_n+1}$ are Valdivia compact spaces. Hence, by Theorem 2.3, for each $n \in \omega_0$ there exists a $T_0$-separating family $\mathcal{U}_n$ of clopen subsets that is point countable on $D_n = \{t \in T_{\alpha_n+1} : \mathrm{cf}(t) \leq \omega_0\}$. Now we are going to prove that $\mathcal{U} = \bigcup_{n\in\omega_0} \widetilde{\mathcal{U}}_n \cup \{\{t\} : t \in \mathrm{Lev}_{\alpha+1}(T)\}$ is a $T_0$-separating family of clopen subsets of $T$ that is point countable on $D = \{t \in T : \mathrm{cf}(t) \leq \omega_0\}$. Let $t \in T$, and consider three possibilities:

- if $t \in \mathrm{Lev}_\alpha(T)$, then $\mathcal{U}(t) = \bigcup_{n\in\omega_0}\{\widetilde{U} : U \in \mathcal{U}_n(\hat{t} \cap \mathrm{Lev}_{\alpha_n+1}(T))\}$;
- if $t \in \mathrm{Lev}_{\alpha+1}(T)$, then $\mathcal{U}(t) = \{\{t\}\} \cup \bigcup_{n\in\omega_0}\{\widetilde{U} : U \in \mathcal{U}_n(\hat{t} \cap \mathrm{Lev}_{\alpha_n+1}(T))\}$;
- if $t \in D \cap \bigcup_{n\in\omega_0} T_{\alpha_n+1}$, then $\mathcal{U}(t) = \bigcup_{n\in\omega_0}\{\widetilde{U} : U \in \mathcal{U}_n(t)\}$;

in all cases $\mathcal{U}(t)$ is countable, hence $\mathcal{U}$ is point-countable on $D$. To prove that $\mathcal{U}$ is $T_0$-separating on $T$, let $s, t \in T$, satisfy $\mathrm{ht}(s,T) \leq \mathrm{ht}(t,T)$. Three cases are possible:

- If $\mathrm{ht}(t,T) < \alpha$, then $s, t \in T_{\alpha_n+1}$ for some $n \in \omega_0$. The assertion follows from the fact that $\mathcal{U}_n$ is $T_0$-separating on $T_{\alpha_n+1}$.
- If $t \in \mathrm{Lev}_\alpha(T)$, then $\mathrm{ht}(s \wedge t, T) < \alpha$, hence $s \wedge t \in T_{\alpha_n+1}$ for some $n \in \omega_0$. Let $\{t_1\} = \hat{t} \cap \mathrm{Lev}_{\alpha_{n+1}+1}(T)$ and $\{s_1\} = \hat{s} \cap \mathrm{Lev}_{\alpha_{n+1}+1}(T)$. Then there exists $U \in \mathcal{U}_{n+1}$ such that $|U \cap \{s_1, t_1\}| = 1$, and the assertion follows by observing that $\widetilde{U} \in \mathcal{U}$ and $|\widetilde{U} \cap \{s,t\}| = 1$.
- If $t \in \mathrm{Lev}_{\alpha+1}(T)$, then the assertion follows since $\{t\} \in \mathcal{U}$.

Suppose that $\alpha$ is a limit ordinal with uncountable cofinality. Then there exists an increasing continuous transfinite sequence $\{\alpha_\gamma\}_{\gamma<\omega_1}$ of ordinals converging to $\alpha$. Since $T$ satisfies (ii) we have $\mathrm{Lev}_\alpha(T) \cap R = \bigcup_{\xi<\omega_1} A_\xi$ where $A_\xi$ is relatively discrete in $T$ for each $\xi < \omega_1$; we may suppose that the family $\{A_\xi\}_{\xi<\omega_1}$ is disjoint. We observe that any relatively discrete subset $B \subset \mathrm{Lev}_\alpha(T)$ can be decomposed as $B = \bigcup_{\beta<\omega_1} B_\beta$ in such a way that if $s \in T$ and $\mathrm{ht}(s,T) > \alpha_{\beta+1}$, then $V_s \cap B_\beta$ contains at most one point. Indeed, since $B$ is relatively discrete, for each $t \in B$ there exists $s_t < t$ on a successor level such that $V_{s_t} \cap B = \{t\}$. Define $B_\beta = \{t \in B : \alpha_\beta < \mathrm{ht}(s_t, T) \leq \alpha_{\beta+1}\}$. Hence $B = \bigcup_{\beta<\omega_1} B_\beta$ and if $s \in T$ is such that $\mathrm{ht}(s,T) > \alpha_{\beta+1}$, then $|V_s \cap B_\beta| \leq 1$. Therefore, since each $A_\xi$ is a relatively discrete subset of $\mathrm{Lev}_\alpha(T)$, it can be decomposed into $\omega_1$-many pieces as above. Hence we may suppose that for each $\xi < \omega_1$ there is $\beta(\xi) < \alpha$ such that for any $s \in T$

with $\mathrm{ht}(s,T) > \beta(\xi)$ we have $|V_s \cap A_\xi| \le 1$. Moreover we may suppose that $\beta$ is non-decreasing (replace $\beta(\xi)$ by $\sup\{\beta(\gamma) : \gamma \le \xi\}$).

First suppose that $\beta$ is bounded by an ordinal $\beta_0 < \alpha$. Let $p \in \mathrm{Lev}_{\beta_0+1}(T)$. Since the height of $p$ is greater than $\beta_0$, we have $|V_p \cap A_\xi| \le 1$ for every $\xi < \omega_1$. Hence $|V_p \cap \bigcup_{\xi<\omega_1} A_\xi| \le \omega_1$. By induction hypothesis $T_{\beta_0+1}$ is Valdivia, hence there exists a family $\mathcal{U}_0$ of clopen subsets of $T_{\beta_0+1}$ which is $T_0$-separating and point-countable on $D_0 = \{t \in T_{\beta_0+1} : \mathrm{cf}(t) \le \omega_0\}$. Further, for any $p \in \mathrm{Lev}_{\beta_0+1}(T)$, the subset $V_p \subset T$ is isomorphic to a tree satisfying the assumptions of Lemma 3.6. Hence $V_p$ is a Valdivia compact space. Therefore there is a family $\mathcal{U}_p$ of clopen sets that is $T_0$-separating and point countable on $D_p = \{t \in V_p : \mathrm{cf}(t) \le \omega_0\}$. We may assume that $V_p \in \mathcal{U}_p$ for every $p \in \mathrm{Lev}_{\beta_0+1}(T)$. Defining $\mathcal{U} = \widetilde{\mathcal{U}_0} \cup \bigcup_{p\in\mathrm{Lev}_{\beta_0+1}(T)} \mathcal{U}_p$ we obtain a family of clopen subsets of $T$. Let $t \in D = \{t \in T : \mathrm{cf}(t) \le \omega_0\}$. We consider two cases:

- If $\mathrm{ht}(t,T) \ge \beta_0 + 1$, take $\{p_t\} = \hat{t} \cap \mathrm{Lev}_{\beta_0+1}(T)$. Then
$$|\mathcal{U}(t)| \le |\mathcal{U}_0(p_t)| + |\mathcal{U}_{p_t}(t)| \le \omega_0.$$
- If $\mathrm{ht}(t,T) < \beta_0 + 1$, we have $|\mathcal{U}(t)| \le |\mathcal{U}_0(t)| \le \omega_0$.

Hence $\mathcal{U}$ is point-countable on $D$. To prove that $\mathcal{U}$ is $T_0$-separating, let $s,t \in T$ and suppose that $\mathrm{ht}(s,T) \le \mathrm{ht}(t,T)$.

- If either $s,t \in T_{\beta_0+1}$, or $s,t \in V_p$ for some $p \in \mathrm{Lev}_{\beta_0+1}(T)$, then we use the fact that $\mathcal{U}_0$ (resp. $\mathcal{U}_p$) is $T_0$-separating on $T_{\beta_0+1}$ (resp. $V_p$).
- If there exists $p \in \mathrm{Lev}_{\beta_0+1}(T)$ such that $t \in V_p$ and $s \notin V_p$, then $V_p \in \mathcal{U}$ and $s \notin V_p$.

Hence $\mathcal{U}$ is $T_0$-separating, and therefore $T$ is Valdivia.

Now suppose that the mapping $\beta$ is unbounded. Recall that if $t \in T$ and $\mathrm{ht}(t,T) > \beta(\xi)$ for some $\xi < \omega_1$, then $|V_t \cap \bigcup_{\eta\le\xi} A_\eta| \le \omega_0$. We are going to define a family $\{S_\xi\}_{\xi<\omega_1}$ of subsets of $T$ with the following properties:

(a) $S_\xi \subset T_{<\alpha}$;
(b) $S_\xi \cap S_\eta = \emptyset$ for $\xi \ne \eta$;
(c) if $t \in \bigcup_{\gamma<\xi} S_\gamma$ for some $\xi < \omega_1$, then $\{s \in T : s < t\} \subset \bigcup_{\gamma<\xi} S_\gamma$;
(d) if $t \in T_{<\alpha} \setminus \bigcup_{\gamma\le\xi} S_\gamma$, then $\mathrm{ht}(t,T) > \beta(\xi+1)$;
(e) if $t \in S_\xi$ for some $\xi < \omega_1$, then $V_t \cap \bigcup_{\gamma<\xi} A_\gamma$ is at most a singleton;
(f) if $t \in T_{<\alpha} \setminus \bigcup_{\gamma\le\xi} S_\gamma$, then $V_t \cap \bigcup_{\gamma\le\xi} A_\gamma$ is at most a singleton.

We use transfinite induction. First we define
$$S_0 = \{t \in T : \mathrm{ht}(t,T) \le \beta(1)\}.$$
We observe that $S_0 = T_{\beta(1)}$ and $\beta(1) < \alpha$, hence $S_0$ satisfies (a)–(f). Suppose that for every $\gamma < \eta$, $S_\gamma$ has been already defined so that (a)–(f) are fulfilled. To define $S_\eta$, we consider two cases:

Case 1: $\eta = \gamma + 1$. Let $M_\eta = \{t \in T : t \text{ is minimal in } T_{<\alpha} \setminus \bigcup_{\zeta\le\gamma} S_\zeta\}$. Fix $t \in M_\eta$. Then, by induction hypothesis, $\mathrm{ht}(t,T) > \beta(\eta)$. Consequently, $|V_t \cap \bigcup_{\zeta\le\eta} A_\zeta| \le \omega_0$, so by Lemma 3.5 there exists $\delta(t) < \alpha$ such that $\delta(t) > \beta(\eta) + 1$ and if $t_0, t_1 \in V_t \cap \bigcup_{\zeta\le\eta} A_\zeta$ then $\mathrm{ht}(t_0 \wedge t_1, T) < \delta(t)$. Let $z(t) = \max\{\delta(t), \beta(\eta+1)\}$ and

$$S_\eta = \bigcup_{t\in M_\eta} (V_t \cap T_{z(t)}).$$

We now prove that $S_\eta$ satisfies (a)–(f). For every $t \in M_\eta$ we have $\beta(\eta+1) \le z(t) < \alpha$, hence (a) and (d) are satisfied. By construction $S_\eta \subset T_{<\alpha} \setminus \bigcup_{\zeta\le\gamma} S_\zeta$, so (b) is satisfied.

By definition of $S_\eta$ and the induction hypothesis, if $t \in \bigcup_{\gamma<\eta+1} S_\gamma$ then $\{s \in T : s < t\} \subset \bigcup_{\gamma<\eta+1} S_\gamma$, hence $S_\eta$ satisfies (c).

Let $t \in S_\eta$. Then $t \in T_{<\alpha} \setminus \bigcup_{\zeta\le\gamma} S_\zeta$, hence, by induction hypothesis, $|V_t \cap \bigcup_{\zeta<\eta} A_\zeta| = |V_t \cap \bigcup_{\zeta\le\gamma} A_\zeta| \le 1$. Thus $S_\eta$ satisfies (e).

Finally, we prove that $S_\eta$ satisfies (f). Suppose that $t \in T_{<\alpha} \setminus \bigcup_{\zeta\le\eta} S_\zeta$ and $\{p\} = \hat{t} \cap M_\eta$. Then $\mathrm{ht}(t,T) > z(p) \ge \delta(p) > \mathrm{ht}(t_0 \wedge t_1, T)$ for all $t_0, t_1 \in V_p \cap \bigcup_{\zeta\le\eta} A_\zeta$. Hence $|V_t \cap \bigcup_{\zeta\le\eta} A_\zeta| \le 1$.

Case 2: $\eta$ is limit. The function $\beta$ is not necessarily continuous, so we define the set $S_\eta$ in two steps.

Let $M^0_\eta = \{t \in T : t \text{ is minimal in } T_{<\alpha} \setminus \bigcup_{\gamma<\eta} S_\gamma\}$. Suppose that $t \in M^0_\eta$. Then by induction hypothesis, $\mathrm{ht}(t,T) > \beta(\gamma)$ for every $\gamma < \eta$. Therefore $|V_t \cap \bigcup_{\gamma<\eta} A_\gamma| \le \omega_0$, so by Lemma 3.5 there exists $\delta^0(t) < \alpha$ such that $\delta^0(t) > \sup_{\gamma<\eta}(\beta(\gamma))+1$ and if $t_0, t_1 \in V_t \cap \bigcup_{\zeta<\eta} A_\zeta$ then $\mathrm{ht}(t_0\wedge t_1, T) < \delta(t)$. Set $z^0(t) = \max\{\delta^0(t), \beta(\eta)\}$ and

$$S^0_\eta = \bigcup_{t\in M^0_\eta} (V_t \cap T_{z^0(t)}).$$

Let $M_\eta = \{t \in T : t \text{ is minimal in } T_{<\alpha} \setminus (S^0_\eta \cup \bigcup_{\gamma<\eta} S_\gamma)\}$ and let $t \in M_\eta$. Then $\mathrm{ht}(t,T) > \beta(\eta)$. Hence $|V_t \cap \bigcup_{\zeta\le\eta} A_\zeta| \le \omega_0$, therefore by Lemma 3.5 there exists $\delta(t) < \alpha$ such that $\delta(t) > \beta(\eta) + 1$ and if $t_0, t_1 \in V_t \cap \bigcup_{\zeta\le\eta} A_\zeta$ then $\mathrm{ht}(t_0 \wedge t_1, T) < \delta(t)$. Set $z(t) = \max\{\delta(t), \beta(\eta+1)\}$ and

$$S_\eta = S^0_\eta \cup \bigcup_{t\in M_\eta} (V_t \cap T_{z(t)}).$$

Since the definition of $S_\eta$ is similar to the one given in the previous case, conditions (a)–(d) are verified analogously.

Let $t \in S_\eta$. Then $t \in T_{<\alpha} \setminus \bigcup_{\zeta<\gamma} S_\zeta$ for each $\gamma < \eta$. Hence, by induction hypothesis, $|V_t \cap \bigcup_{\zeta\le\gamma} A_\zeta| \le 1$ for each $\gamma < \eta$. It follows that $|V_t \cap \bigcup_{\zeta<\eta} A_\zeta| \le 1$. Therefore $S_\eta$ satisfies (e).

Finally, we prove that $S_\eta$ satisfies (f). Let $t \in T_{<\alpha} \setminus \bigcup_{\gamma\le\eta} S_\gamma$. If $\{p\} = \hat{t} \cap M_\eta$, then $\mathrm{ht}(t,T) > z(p) \ge \delta(p) > \mathrm{ht}(t_0 \wedge t_1, T)$ for all $t_0, t_1 \in V_p \cap \bigcup_{\zeta\le\eta} A_\zeta$.

Hence $|V_t \cap \bigcup_{\zeta \leq \eta} A_\zeta| \leq 1$. The same follows in an analogous way if $\{p\} = \hat{t} \cap M^0_\eta$.

By the transfinite induction hypothesis the tree $T_{\alpha_\gamma+1}$ is a Valdivia compact space, hence, by Theorem 2.3, there is a family $\mathcal{U}_\gamma$ of clopen subsets of $T_{\alpha_\gamma+1}$ which is $T_0$-separating and point-countable on $D_\gamma = \{t \in T_{\alpha_\gamma+1} : \mathrm{cf}(t) \leq \omega_0\}$ for every $\gamma < \omega_1$. Moreover the elements of each $\mathcal{U}_\gamma$ are of the form $W^F_s$.

By construction, $\{S_\xi\}_{\xi<\omega_1}$ is a pairwise disjoint family of subsets of $T_{<\alpha}$. Indeed, since $\beta$ is unbounded, for every $t \in T_{<\alpha}$ there is $\xi < \omega_1$ with $\mathrm{ht}(t, T) < \beta(\xi)$ and so by (d), $t \in \bigcup_{\gamma<\xi} S_\xi$. Thus, taking into account (b), for every $t \in T_{<\alpha}$ there exists a unique $\xi < \omega_1$ with $t \in S_\xi$.

Let $\phi : T_{<\alpha} \to [0, \omega_1)$ be such that $t \in S_{\phi(t)}$ for every $t \in T_{<\alpha}$. Let $I(\omega_1)$ be the set of successor ordinals less than $\omega_1$. Let $\gamma \in I(\omega_1)$. Define $U_p = V_p \cap U$ for any $U \in \mathcal{U}_\gamma$ and $p \in \mathrm{Lev}_{\alpha_{(\gamma-1)}+1}(T)$. We observe that $\min(U_p) \in S_{\phi(\min(U_p))}$, hence (e) implies that $|\widetilde{U}_p \cap \bigcup_{\gamma<\phi(\min(U_p))} A_\gamma| \leq 1$. If $\widetilde{U}_p \cap \bigcup_{\gamma<\phi(\min(U_p))} A_\gamma = \emptyset$, then $\widetilde{U}_p$ is a clopen subset of $T$ that does not intersect $\bigcup_{\gamma<\phi(\min(U_p))} A_\gamma$. Similarly, if $\widetilde{U}_p \cap \bigcup_{\gamma<\phi(\min(U_p))} A_\gamma = \{s\}$, then $\widetilde{U}_p \setminus \mathrm{ims}(s)$ is a clopen subset of $T$ that avoids $\bigcup_{\gamma<\phi(\min(U_p))} \bigcup_{s\in A_\gamma} \mathrm{ims}(s)$. Therefore we define a family $\mathcal{U}$ of clopen subsets of $T$ as follows:

$$\begin{aligned}\mathcal{U} &= \big\{\{t\} : t \in \mathrm{Lev}_{\alpha+1}(T)\big\}\\ &\cup \bigcup_{\gamma\in I(\omega_1)} \bigcup_{U\in\mathcal{U}_\gamma} \Big\{\widetilde{U}_p : p \in \mathrm{Lev}_{\alpha_{(\gamma-1)}+1}(T),\ \widetilde{U}_p \cap \bigcup_{\eta<\phi(\min(U_p))} A_\eta = \emptyset\Big\}\\ &\cup \bigcup_{\gamma\in I(\omega_1)} \bigcup_{U\in\mathcal{U}_\gamma} \Big\{\widetilde{U}_p \setminus \mathrm{ims}(s) : p \in \mathrm{Lev}_{\alpha_{(\gamma-1)}+1}(T),\ \widetilde{U}_p \cap \bigcup_{\eta<\phi(\min(U_p))} A_\eta = \{s\}\Big\}.\end{aligned}$$

It remains to prove that $\mathcal{U}$ is a family of clopen subsets which is $T_0$-separating and point-countable on $D = \{t \in T : \mathrm{cf}(t) \leq \omega_0\}$. We observe that $\mathcal{U}$ restricted to $T_\alpha$ satisfies the hypothesis of Lemma 3.4. Therefore combining Lemma 3.4 with the fact that $\{\{t\} : t \in \mathrm{Lev}_{\eta+1}(T)\}$ is contained in $\mathcal{U}$ we find that $\mathcal{U}$ is $T_0$-separating in $T$.

To prove that $\mathcal{U}$ is point-countable on $D$, let $t \in D$ and consider two cases.

Case 1: $\mathrm{ht}(t, T) < \alpha$. We define $\gamma_0 = \min\{\gamma < \omega_1 : \mathrm{ht}(t, T) < \alpha_\gamma\}$; such a $\gamma_0$ exists since $\mathrm{ht}(t, T) < \alpha$ and $\{\alpha_\gamma\}_{\gamma<\omega_1}$ is a continuous increasing transfinite sequence converging to $\alpha$. Since $\{\alpha_\gamma\}_{\gamma<\omega_1}$ is continuous and $t$ belongs to a successor level, we have $\gamma_0 \in I(\omega_1)$.

Suppose $U_p \subset T_{\alpha_\xi+1} \setminus T_{\alpha_{\xi-1}}$ for some $\xi \in I(\omega_1)$. We consider three cases. If $\xi \geq \gamma_0 + 1$, then $t \notin \widetilde{U}_p$. If $\xi < \gamma_0$, then since $\widetilde{U}_p = U_p \cup \bigcup_{x\in\mathrm{Lev}_{\alpha_\xi+1}(T)\cap U} V_x$, it follows that $t \in \widetilde{U}_p$ if and only if $\hat{t} \cap \mathrm{Lev}_{\alpha_\xi+1}(T) \subset U_p$. Finally, if $\xi = \gamma_0$, we have $t \in \widetilde{U}_p$ if and only if $t \in U_p$.

Hence, since $\mathcal{U}_\xi$ is point-countable on $D_\xi \subset T_{\alpha_\xi+1}$ for each $\xi < \gamma_0 + 1$, we see that $\mathcal{U}(t)$ is countable as well.

CASE 2: $t \in \mathrm{Lev}_{\alpha+1}(T)$. Then there exists $s \in R \cap \mathrm{Lev}_\alpha(T)$ such that $t \in \mathrm{ims}(s)$. There exists $\gamma < \omega_1$ such that $s \in A_\gamma$. Take any element $X$ of $\mathcal{U}$ containing $t$. Then there are two possibilities:

If $X = \{t\}$ then exactly one element of $\mathcal{U}$ has this form.

If $X \neq \{t\}$, then it follows from (c) and (d) that the restriction of $\phi$ to $\{u \in T : u < s\}$ is non-decreasing and unbounded. Let $u_0 \in T$ be the minimal element $u < s$ such that $\phi(u) \geq \gamma$. Let $\xi_0 < \omega_1$ be the minimal ordinal $\xi < \omega_1$ such that $u_0 \in T_{\alpha_\xi+1}$. If $\eta \in I(\omega_1)$ and $p \in \mathrm{Lev}_{\alpha_{\eta-1}+1}(T)$ with $X = \widetilde{U}_p$ for some $U_p \subset T_{\alpha_\eta+1} \setminus T_{\alpha_{\eta-1}}$, then since $\widetilde{U}_p \cap \bigcup_{\zeta<\phi(\min(U_p))} A_\zeta = \emptyset$ and $s \in A_\gamma$, we obtain $\phi(\min(U_p)) \leq \gamma$ and therefore $\eta \leq \xi_0+1$. Similarly, if $X = \widetilde{U}_p \setminus \mathrm{ims}(u)$ for some $u \in R$, then since $(\widetilde{U}_p \setminus \mathrm{ims}(u)) \cap \bigcup_{\zeta<\phi(\min(U_p))} \bigcup_{r \in A_\zeta} \mathrm{ims}(r) = \emptyset$ and $t \in X$, we obtain $\eta \leq \xi_0 + 1$.

Since $\widetilde{U}_p = U_p \cup \bigcup_{x \in \mathrm{Lev}_{\alpha_\xi+1}(T) \cap U} V_x$, it follows that $t \in \widetilde{U}_p$ if and only if $\hat{t} \cap \mathrm{Lev}_{\alpha_\xi+1}(T) \subset U_p$ when $X = \widetilde{U}_p$, and $t \in \widetilde{U}_p \setminus \mathrm{ims}(u)$ if and only if $\hat{t} \cap \mathrm{Lev}_{\alpha_\xi+1}(T) \subset U_p$ when $X = \widetilde{U}_p \setminus \mathrm{ims}(u)$. Hence, since $\mathcal{U}_\xi$ is point-countable on $D_\xi \subset T_{\alpha_\xi+1}$ for each $\xi \leq \xi_0+1$, we conclude that $\mathcal{U}(t)$ is countable as well.

Therefore $T$ is Valdivia. This concludes the proof. ∎

**4. Banach spaces of continuous functions on trees.** In this section we deal with the space of continuous functions on a tree $T$. We will prove that Valdivia compact trees can be characterized by their space of continuous functions: $T$ is Valdivia if and only if $C(T)$ is 1-Plichko ($T$ has a retractional skeleton if and only if $C(T)$ has a 1-projectional skeleton). Notice that for general sets this is not true: there are examples of non-Valdivia compacta $K$ such that $C(K)$ is 1-Plichko (see [1], [12], [17]).

In the final part of this section, we will prove that each $C(T)$ space, where $T$ is a tree with height less than $\omega_1 \cdot \omega_0$, is a Plichko space. Using this result, we observe that the tree $T$ defined as in [21, Example 4.3] is an example of a compact space with retractional skeletons none of which is commutative, but $C(T)$ is a Plichko space, so it has a commutative projectional skeleton (see [15, Theorem 27]).

THEOREM 4.1. *Let $T$ be a tree. Then $T$ is a Valdivia compact space if and only if $C(T)$ is a 1-Plichko space.*

*Proof.* The "only if" part is a particular case of [14, Theorem 5.2]. Suppose that $C(T)$ is a 1-Plichko space and let $S \subset C(T)^*$ be a 1-norming

$\Sigma$-subspace. The compact space $T$ embeds canonically into $B_{C(T)^*}$ by identifying each $t \in T$ with the Dirac measure concentrated on $\{t\}$. This embedding will be denoted by $\delta$. We are going to prove that $\delta(t) = \delta_t \in S$ whenever $t \in T$ is on a successor level.

Pick $t \in T$ on a successor level. If $\mathrm{ims}(t)$ is finite, then $t$ is isolated, and since $S$ is 1-norming, we obtain $\delta_t \in S$.

Now suppose that $\mathrm{ims}(t)$ is infinite. Let $\{t_n\}_{n\in\omega_0}$ be an infinite subset of $\mathrm{ims}(t)$. Since $V_{t_n}$ is a clopen subset of $T$, the function $f_n = 1_{V_{t_n}}$ is continuous for every $n \in \omega_0$. Since $S$ is a 1-norming subset of $C(T)^*$, for each $k \in \omega_0$ there exists $\mu_n^k \in C(T)^*$ with $\|\mu_n^k\| = 1$ and $\mu_n^k(f_n) > 1 - 1/k$. Since $S$ is a $\Sigma$-subspace, by [14, Lemma 1.6] the closure of $\{\mu_n^k\}_{k\in\omega_0}$ is contained in $S$ and moreover there exist a measure $\mu_n$ contained in that closure and a subsequence $\{\mu_n^{k_j}\}_{j\in\omega_0}$ that $\mu_n^{k_j}$ converges to $\mu_n$. Hence $\mu_n(f_n) = \mu_n(V_{t_n}) = 1$. Observing that $\|\mu_n\| = |\mu_n|(T) \le 1$ and $1 = |\mu_n(V_{t_n})| \le |\mu_n|(V_{t_n})$, we easily deduce that $\mathrm{supp}(\mu_n) \subset V_{t_n}$.

Now, for $f \in C(T)$ and $\varepsilon > 0$ we define

$$Z_\varepsilon(f,t) = \Big\{ s \in \mathrm{ims}(t) : \sup_{p\in V_s} |f(p) - f(t)| > \varepsilon \Big\}.$$

By the continuity of $f$, the set $Z_\varepsilon(f,t)$ is finite. Hence there exists $n_0 \in \omega_0$ such that

$$\sup_{p\in V_{t_n}} |f(p) - f(t)| < \varepsilon \quad \text{for every } n \ge n_0.$$

If $n \ge n_0$, then

$$\begin{aligned} |\mu_n(f) - f(t)| &= \Big| \int_{V_{t_n}} f(x)\, d\mu_n(x) - f(t) \Big| = \Big| \int_{V_{t_n}} (f(x) - f(t))\, d\mu_n(x) \Big| \\ &\le \int_{V_{t_n}} |f(x) - f(t)|\, d\mu_n(x) < \varepsilon. \end{aligned}$$

Hence $\mu_n(f)$ converges to $\delta_t(f)$ for every $f \in C(T)$. By the weak$^*$ countable closedness of $S$ it follows that $\delta_t \in S$.

Therefore, since by [14, Theorem 5.2], $B_{C(T)^*}$ is a Valdivia compact space with $B_{C(T)^*} \cap S$ being a $\Sigma$-subset, and $S \cap \delta(T)$ is dense in $\delta(T)$, we conclude that $\delta(T)$ is a Valdivia compact space. ∎

We observe that the same result can be proved in the non-commutative setting. Using [4, Proposition 3.15] instead of [14, Theorem 5.2] we obtain the following result.

THEOREM 4.2. *Let $T$ be a tree. Then $T$ has a retractional skeleton if and only if $C(T)$ has a 1-projectional skeleton.*

Now we are going to investigate the space of continuous functions on trees with height less than $\omega_1 \cdot \omega_0$. It turns out that all such spaces are Plichko.

THEOREM 4.3. *Let $T$ be a tree such that* $\mathrm{ht}(T) < \omega_1 \cdot \omega_0$. *Then $C(T)$ is a Plichko space.*

This follows immediately from the next technical proposition, where, for every tree $T$ of height less than $\omega_1 \cdot \omega_0$, a norming $\Sigma$-subspace of $C(T)^*$ is explicitly described.

PROPOSITION 4.4. *Let $T$ be a tree and suppose that* $\mathrm{ht}(T) \leq \omega_1 \cdot n + 1$ *for some $n \geq 1$. Then*

$$\Lambda = \{\mu \in C(T)^* : (\forall j \in \{1, \ldots, n\})(\forall t \in \mathrm{Lev}_{\omega_1 \cdot j}(T))(\mu(V_t) = 0)\}$$

*is a $(2n-1)$-norming $\Sigma$-subspace of $C(T)^*$. If* $\mathrm{ht}(T) > \omega_1 \cdot (n-1) + 1$*, then the norming constant is exactly $2n - 1$.*

LEMMA 4.5. *Let $T$ be a tree such that* $\mathrm{ht}(T) \leq \omega_1 + 1$ *and $D = \{t \in T :$* $\mathrm{cf}(t) \leq \omega_0\}$*. Then the set*

$$S = \{\mu \in C(T)^* : \mathrm{supp}(\mu) \subset D\}$$

*is a 1-norming $\Sigma$-subspace of $C(T)^*$.*

*Proof.* Since $\mathrm{ht}(T) \leq \omega_1 + 1$, by [21, Theorem 4.1], $T$ is a Valdivia compact space and $D$ is a dense $\Sigma$-subspace. Hence, by [14, Proposition 5.1] the set

$$S = \{\mu \in C(T)^* : \mathrm{supp}(\mu) \text{ is a separable subset of } D\}$$

is a 1-norming $\Sigma$-subspace of $C(T)^*$. Finally, the assertion follows by Proposition 2.5. ∎

*Proof of Proposition 4.4.* If $n = 1$ the assertion follows from Lemma 4.5. Hence we assume that $n \geq 2$ and $T$ is a tree with $\omega_1 \cdot (n-1) + 1 < \mathrm{ht}(T) \leq \omega_1 \cdot n + 1$. As in Lemma 4.5 we define $D = \{t \in T : \mathrm{cf}(t) \leq \omega_0\}$. Let $S_0 = \emptyset$, $S_i = T_{\omega_1 \cdot i}$ for each $i \leq n-1$, and $S_n = T$. Then we obtain the following:

- $S_i$ is a closed subset of $T$ for every $i \in \{1, \ldots, n\}$.
- $S_1$ is isomorphic to a tree of height $\omega_1 + 1$, hence, by Lemma 4.5, $C(S_1)$ is a 1-Plichko space with $\Sigma_1 = \{\mu \in C(S_1)^* : \mathrm{supp}(\mu) \subset D \cap S_1\}$ being a $\Sigma$-subspace.
- For every $i \in \{1, \ldots, n-1\}$, the subset $S_{i+1} \setminus S_i$ is a locally compact space and $C_0(S_{i+1} \setminus S_i)$ is a 1-Plichko space. Indeed, let $t \in \mathrm{Lev}_{(\omega_1 \cdot i)+1}(T)$ and $U_t = V_t \cap S_{i+1}$. It is clear that $U_t$ is a closed subset of $T$ and it is isomorphic to a tree of height $\leq \omega_1 + 1$. Hence, by Lemma 4.5, $U_t$ is a Valdivia compact space and $C(U_t)$ is a 1-Plichko space with $\Sigma_{i,t} = \{\mu \in C(U_t)^* : \mathrm{supp}(\mu) \subset D \cap U_t\}$ being a $\Sigma$-subspace. Moreover $S_{i+1} \setminus S_i$ is the topological sum of all $U_t$, so $C_0(S_{i+1} \setminus S_i)$ is the $c_0$-sum of the $C(U_t)$ and its dual is the $\ell_1$-sum of the $C(U_t)^*$. Hence, by [14, Theorem 4.31 and Lemma 4.34], $C_0(S_{i+1} \setminus S_i)$ is a 1-Plichko space and

$$\begin{aligned}\Sigma_i = \big\{(\mu_t)_{t \in \mathrm{Lev}_{(\omega_1 \cdot i)+1}(T)} \in C_0(S_{i+1} \setminus S_i)^* : (\forall t \in \mathrm{Lev}_{(\omega_1 \cdot i)+1}(T))(\mu_t \in \Sigma_{i,t}) \\ \& \, \{t \in \mathrm{Lev}_{(\omega_1 \cdot i)+1}(T) : \mu_t \neq 0\} \text{ is countable}\big\}\end{aligned}$$

  is a $\Sigma$-subspace.

Suppose that $i \leq n-1$ and let $r_i : T \to T$ be the continuous retraction defined by

$$r_i(t) = \begin{cases} t & \text{if } t \in S_i, \\ s & \text{if } s \leq t \text{ and } s \in \mathrm{Lev}_{\omega_1 \cdot i}(T). \end{cases}$$

For simplicity we define $r_n : T \to T$ to be the identity map. These continuous retractions induce continuous linear projections on $C(T)$ defined by $P_i(f) = f \circ r_i$. Then for every $f \in C(T)$ and every $i \leq n-1$ the following conditions hold:

- $f{\restriction}_{S_i} = P_i f{\restriction}_{S_i}$;
- $(f - P_i f){\restriction}_{S_{i+1}} = (P_{i+1} f - P_i f){\restriction}_{S_{i+1}}$;
- $(f - P_i f){\restriction}_{S_{i+1} \setminus S_i} \in C_0(S_{i+1} \setminus S_i)$.

To get an isomorphism between $C(T)$ and a 1-Plichko space we define the following map:

$$\begin{aligned} G : C(T) &\to C(S_1) \oplus_\infty C_0(S_2 \setminus S_1) \oplus_\infty \cdots \oplus_\infty C_0(S_n \setminus S_{n-1}), \\ f &\mapsto (f{\restriction}_{S_1}, (f - P_1 f){\restriction}_{S_2 \setminus S_1}, \ldots, (f - P_{n-1} f){\restriction}_{S_n \setminus S_{n-1}}). \end{aligned}$$

The norm of $G$ is clearly at most 2. Now we define the inverse of $G$. For simplicity we denote

$$W = C(S_1) \oplus_\infty C_0(S_2 \setminus S_1) \oplus_\infty \cdots \oplus_\infty C_0(S_n \setminus S_{n-1}).$$

Let $(f_1, \ldots, f_n) \in W$. We define its preimage $f \in C(T)$ as follows:

$$f(t) = \begin{cases} f_1(t) & \text{if } t \in S_1, \\ f_{i+1}(t) + \sum_{j=1}^{i} f_j(r_j(t)) & \text{if } t \in S_{i+1} \setminus S_i. \end{cases}$$

It follows that the norm of the inverse of $G$ is at most $n$. Therefore $G$ is an isomorphism. Since each component of $W$ is a 1-Plichko space, it follows that $W$ is 1-Plichko, so $C(T)$ is a Plichko space. Moreover, $\Sigma = \{(\mu_i)_{i=1}^n \in W^* : \mu_i \in \Sigma_i\} \subset W^*$ is a 1-norming $\Sigma$-subspace. In order to compute the exact value of the norming constant of the $\Sigma$-subspace $G^*(\Sigma)$, we compute the adjoint map of $G$:

$$\begin{aligned} G^*(\mu_1, \ldots, \mu_n)(f) &= (\mu_1, \ldots, \mu_n)(Gf) \\ &= \mu_1(f{\restriction}_{S_1}) + \sum_{j=1}^{n-1} \mu_{j+1}((f - P_j f){\restriction}_{S_{j+1} \setminus S_j}) \\ &= \int_{S_1} f d\mu_1 + \sum_{j=1}^{n-1} \int_{S_{j+1} \setminus S_j} (f - P_j f)\, d\mu_{j+1} \\ &= \int_T f d\Big(\sum_{i=1}^{n} \mu_i\Big) - \sum_{j=1}^{n-1} \int_T f\, dr_j(\mu_{j+1}), \end{aligned}$$

hence

$$G^*(\mu_1,\dots,\mu_n)=\sum_{i=1}^{n}\mu_i-\sum_{j=1}^{n-1}r_j(\mu_{j+1})$$

where $r_i(\mu_j)(A)=\mu_j(r_i^{-1}(A))$ for every measurable $A\subset T$. Now we give a representation of the inverse of $G^*$. Let $\mu=G^*(\mu_1,\dots,\mu_n)=\sum_{i=1}^{n}\mu_i-\sum_{j=1}^{n-1}r_j(\mu_{j+1})$ and $k\le n-1$. Then

$$r_k(\mu)=\sum_{i=1}^{n}r_k(\mu_i)-\sum_{j=1}^{n-1}r_k(r_j(\mu_{j+1})).$$

Further we observe that $r_k(\mu_i)=\mu_i$ for $i\le k$ and

$$r_k(r_j(\mu_{j+1}))=\begin{cases}r_j(\mu_{j+1}) & \text{if } j<k,\\ r_k(\mu_{j+1}) & \text{if } j\ge k.\end{cases}$$

Hence

$$\begin{aligned}r_k(\mu)&=\sum_{i=1}^{k}\mu_i+\sum_{i=k+1}^{n}r_k(\mu_i)-\sum_{j=1}^{k-1}r_j(\mu_{j+1})-\sum_{j=k}^{n-1}r_k(\mu_{j+1})\\&=\sum_{i=1}^{k}\mu_i-\sum_{j=1}^{k-1}r_j(\mu_{j+1}).\end{aligned}$$

Now we take the restriction of $\mu$ to $S_i\setminus S_{i-1}$:

$$\begin{aligned}&\mu\restriction_{S_1}=\mu_1-r_1(\mu_2),\\&\mu\restriction_{S_i\setminus S_{i-1}}=\mu_i-r_i(\mu_{i+1})\quad\text{ for } i\in\{2,\dots,n-1\},\\&\mu\restriction_{T\setminus S_{n-1}}=\mu_n.\end{aligned}$$

Hence, combining these formulae with $r_k(\mu)=\sum_{i=1}^{k}\mu_i-\sum_{j=1}^{k-1}r_j(\mu_{j+1})$, we obtain

$$\begin{aligned}&\mu_1=r_1(\mu),\\&\mu_i=r_i(\mu)-\mu\restriction_{S_{i-1}}\quad\text{ for } i\in\{2,\dots,n-1\},\\&\mu_n=\mu\restriction_{T\setminus S_{n-1}}.\end{aligned}$$

Therefore the inverse of $G^*$ can be represented as

$$\mu\mapsto(r_1(\mu),r_2(\mu)-\mu\restriction_{S_1},\dots,r_{n-1}(\mu)-\mu\restriction_{S_{n-2}},\mu\restriction_{T\setminus S_{n-1}}).$$

Hence

$$\begin{aligned}&G^*(\Sigma)\\&\ =\{\mu\in C(T)^*:(r_1(\mu),r_2(\mu)-\mu\restriction_{S_1},\dots,r_{n-1}(\mu)-\mu\restriction_{S_{n-2}},\mu\restriction_{T\setminus S_{n-1}})\in\Sigma\}\\&\ =\Big\{\mu\in C(T)^*:(\forall j\in\{1,\dots,n\})(\forall B\subset \mathrm{Lev}_{\omega_1\cdot j}(T))\Big(\mu\Big(\bigcup_{t\in B}V_t\Big)=0\Big)\Big\}\\&\ =\{\mu\in C(T)^*:(\forall j\in\{1,\dots,n\})(\forall t\in\mathrm{Lev}_{\omega_1\cdot j}(T))(\mu(V_t)=0)\}.\end{aligned}$$

Indeed, the first equality is obvious. Let us prove the second one:

$\subset$: Let $\mu \in G^*(\Sigma)$ and $B \subset \mathrm{Lev}_{\omega_1 \cdot j}(T)$ for some $j \in \{1, \dots, n-1\}$. Since $(r_j(\mu) - \mu\restriction_{S_{j-1}}) \in \Sigma_j$ we have $(r_j(\mu) - \mu\restriction_{S_{j-1}})(B) = 0$. Hence $0 = (r_j(\mu) - \mu\restriction_{S_{j-1}})(B) = \mu(\bigcup_{t\in B} V_t) - \mu(B \cap S_{j-1}) = \mu(\bigcup_{t\in B} V_t)$. If $j = n$ we have $\mu\restriction_{T\setminus S_{n-1}} \in \Sigma_n$, hence $0 = \mu\restriction_{T\setminus S_{n-1}}(B) = \mu(B)$.

$\supset$: Let $\mu \in C(T)^*$ be such that $\mu(\bigcup_{t\in B} V_t) = 0$ for each $j \in \{1, \dots, n\}$ and $B \subset \mathrm{Lev}_{\omega_1 \cdot j}(T)$. By Proposition 2.5, the support of the measure $r_j(\mu) - \mu\restriction_{S_{j-1}}$ is a metrizable subset of $T$ for each $j \in \{2, \dots, n-1\}$. Hence $(r_j(\mu) - \mu\restriction_{S_{j-1}})(V_t) = 0$ for all but countably many $t \in \mathrm{Lev}_{(\omega_1 \cdot j)+1}(T)$. Let $j \in \{2, \dots, n-1\}$ and let $s \in T$ be on a successor level and such that $\omega_1 \cdot (j-1) < \mathrm{ht}(s, T) \le \omega_1 \cdot (j+1)$. Then

$$\begin{aligned}(r_j(\mu) - \mu\restriction_{S_{j-1}})(V_s) &= r_j(\mu)(V_s) = \mu(V_s) \\ &= \mu(V_s \cap T_{<\omega_1 \cdot j}) + \mu((V_s \cap (T \setminus T_{<\omega_1 \cdot j})) \\ &= \mu(V_s \cap T_{<\omega_1 \cdot j}).\end{aligned}$$

In particular, for each $s \in \mathrm{Lev}_{(\omega_1\cdot(j-1))+1}(T)$ we have $(r_j(\mu) - \mu\restriction_{S_{j-1}})(V_s) = \mu(V_s \cap T_{<\omega_1 \cdot j})$, hence $(r_j(\mu) - \mu\restriction_{S_{j-1}})\restriction_{V_s} \subset V_s \cap D$. Therefore we obtain $(r_j(\mu) - \mu\restriction_{S_{j-1}}) \in \Sigma_j$ for each $j \in \{2, \dots, n-1\}$. Using a similar argument we obtain $r_1(\mu) \in \Sigma_1$ and $\mu\restriction_{T\setminus S_{n-1}} \in \Sigma_n$.

Let us prove the last equality:

$\subset$: This is trivial.

$\supset$: Let $\mu \in C(T)^*$ be such that $\mu(V_t) = 0$ for any $j \in \{1, \dots, n\}$ and $t \in \mathrm{Lev}_{\omega_1 \cdot j}(T)$. Fix $j \in \{1, \dots, n\}$. Denote by $\mu_c$ the continuous part of $\mu$. By Corollary 2.6, $\mathrm{supp}(\mu_c \restriction S_j) \subset T_\alpha$, where $\alpha < \omega_1 \cdot j$. Hence we may suppose that $\mu(V_t \cap S_j) = 0$ whenever $\mathrm{ht}(t, T) > \alpha$. Consequently, $\mu(A) = 0$ for each relatively open subset $A$ of $\mathrm{Lev}_{\omega_1 \cdot j}(T)$, and thus for any subset by the regularity of $\mu$.

Hence $\Lambda = G^*(\Sigma)$ is a $\Sigma$-subspace of $C(T)^*$. Now we show that the norming constant of $\Lambda$ is $2n - 1$. First we observe that if $f \in C(T)$ and $t \in \mathrm{Lev}_\alpha(T)$ with $\mathrm{cf}(t) = \omega_1$, then there exists $s \le t$ on a successor level such that $f$ is constant on $V_s \cap T_\alpha$. Indeed, since $f$ is a continuous function on $T_\alpha$ for each $n \in \omega_0$ there exists $t_n < t$ on a successor level such that $|f(t) - f(s)| < 1/n$ for each $s \in V_{t_n} \cap T_\alpha$. Then $f(s) = f(t)$ for each $s \in V_{t_0} \cap T_\alpha$, where $t_0 = \sup_{n\in\omega_0} t_n + 1$.

Now, let $f \in C(T)$; without loss of generality, suppose $\|f\| = 1$. Let $t \in T$ be such that $|f(t)| = 1$. Then there exists $i \in \{0, \dots, n-1\}$ such that $\omega_1 \cdot i \le \mathrm{ht}(t, T) < \omega_1 \cdot (i+1)$. We will show that there exists a measure $\mu \in \Lambda$ satisfying $\|\mu\| \le 2n - 1$ and $\mu(f) = 1$. Set

$$\mu = \Big(\sum_{k=1}^{i} \delta_{t_k} - \delta_{s_k}\Big) + \delta_{t_0},$$

where $t_i = t$, $\{s_k\} = \hat{t}_i \cap \mathrm{Lev}_{\omega_1 \cdot k}(T)$ and $t_k$ is such that $s_k < t_k < s_{k+1}$ (resp. $t_0 < s_1$) and $f(s_{k+1}) = f(t_k)$ for $k \geq 1$ (resp. for $k = 0$). Such elements exist since $f$ is constant near points of uncountable cofinality. Therefore $2n - 1 \geq 2i + 1 = \|\mu\|$ and we easily get $|\mu(f)| = 1$. Hence the norming constant of $\Lambda$ is at most $2n - 1$. On the other hand, suppose that $\mathrm{ht}(T) > \omega_1 \cdot (n-1) + 1$ and let $t_{n-1} \in T$ such that $\mathrm{ht}(t_{n-1}, T) > \omega_1 \cdot (n-1) + 1$ and

$$\{s_i\} = \hat{t}_{n-1} \cap \mathrm{Lev}_{\omega_1 \cdot i}(T) \qquad \text{for } i = 1, \ldots, n-1,$$
$$\{t_i\} = \hat{t}_{n-1} \cap \mathrm{Lev}_{(\omega_1 \cdot i)+1}(T) \qquad \text{for } i = 1, \ldots, n-2.$$

Let us consider the following continuous map:

$$f(t) = 1_{V_{t_{n-1}}}(t) + \delta_1 1_{T \setminus V_{t_1}}(t) + \sum_{i=1}^{n-2} \delta_{i+1} 1_{V_{t_i} \setminus V_{t_{i+1}}}(t),$$

where $\delta_1 = 1/(2n-1)$ and $\delta_i = (2i-1)\delta_1$ for $i = 1, \ldots, n-1$. Let $\mu \in \Lambda$ be such that $\mu(f) = 1$. We put

$$\begin{aligned} &\mu(T \setminus V_{s_1}) = a_0, \\ &\mu(V_{s_i} \setminus V_{t_i}) = b_i \qquad \text{for } i = 1, \ldots, n-1, \\ &\mu(V_{t_i} \setminus V_{s_{i+1}}) = a_i \qquad \text{for } i = 1, \ldots, n-2, \\ &\mu(V_{t_{n-1}}) = a_{n-1}. \end{aligned}$$

Since $\mu \in \Lambda$ we have $b_i = -a_i$ for every $i = 1, \ldots, n-1$. Therefore

$$\begin{aligned} 1 = \mu(f) &= a_{n-1} + \sum_{i=1}^{n-1} \delta_i (a_{i-1} - a_i) \\ &= \delta_1 a_0 + (1 - \delta_{n-1}) a_{n-1} + \sum_{i=1}^{n-2} a_i (\delta_{i+1} - \delta_i) \\ &\leq \Big( |a_0| + \sum_{i=1}^{n-1} 2|a_i| \Big) \cdot \max\Big\{ \delta_1, \frac{1 - \delta_{n-1}}{2} \Big\} \\ &= \Big( |a_0| + \sum_{i=1}^{n-1} 2|a_i| \Big) \cdot \frac{1}{2n-1}. \end{aligned}$$

Hence $\|\mu\| \geq 2n - 1$. This concludes the proof. ∎

Combining Theorems 3.2 and 4.1 we obtain several examples of trees $T$, also with height greater than $\omega_1 \cdot \omega_0$, such that $C(T)$ is a 1-Plichko space. However, in the final part of the proof of Theorem 4.3, the norming constant of the $\Sigma$-subspace grows as $2n - 1$. This means that, in general, this is not the optimal choice, and the following question is natural:

Problem 4.6. *Let $T$ be a tree with height $\omega_1 \cdot \omega_0$. Is $C(T)$ necessarily Plichko?*

Remark 4.7. We have assumed that every tree was rooted, but the above results can also be proved if the tree has finitely many minimal elements. Indeed, it can then be viewed as the topological direct sum of rooted trees.

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