Spectral Multipliers on 2-Step Stratified Groups, II

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Abstract

Given a graded group G and commuting, formally self-adjoint, left-invariant, homogeneous differential operators $\mathcal{L}_1, \dots, \mathcal{L}_n$ on G, one of which is Rockland, we study the convolution operators $m(\mathcal{L}_1, \dots, \mathcal{L}_n)$ and their convolution kernels, with particular reference to the case in which G is abelian and $n = 1$, and the case in which G is a 2-step stratified group which satisfies a slight strenghtening of the Moore-Wolf condition and $\mathcal{L}_1, \dots, \mathcal{L}_n$ are either sub-Laplacians or central elements of the Lie algebra of G. Under suitable conditions, we prove that: i) if the convolution kernel of the operator $m(\mathcal{L}_1, \dots, \mathcal{L}_n)$ belongs to L^1, then m equals almost everywhere a continuous function vanishing at ∞ ('Riemann-Lebesgue lemma'); ii) if the convolution kernel of the operator $m(\mathcal{L}_1, \dots, \mathcal{L}_n)$ is a Schwartz function, then m equals almost everywhere a Schwartz function.

1 Introduction

Given a Rockland family[1] $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ on a homogeneous group G, following [32, 39] (see also [15]) we define a 'kernel transform' $\mathcal{K}$ which to every measurable function $m\colon \mathbb{R}^n \to \mathbb{C}$ such that $m(\mathcal{L}_1, \dots, \mathcal{L}_n)$ is defined on $\mathcal{D}(G)$ associates a unique distribution $\mathcal{K}(m)$ such that

$$m(\mathcal{L}_1, \dots, \mathcal{L}_n)\,\varphi = \varphi * \mathcal{K}(m)$$

for every $\varphi \in \mathcal{D}(G)$. The so-defined kernel transform $\mathcal{K}$ enjoys some relevant properties, which we list below; see [32, 39] for their proofs and further information.

- there is a unique positive Radon measure β on $\mathbb{R}^n$ such that $\mathcal{K}(m) \in L^2(G)$ if and only if $m \in L^2(\beta)$, and $\mathcal{K}$ induces an isometry of $L^2(\beta)$ into $L^2(G)$;
- there is a unique $\chi \in L^\infty(\mathbb{R}^n \times G, \beta \otimes \nu)$, where ν denotes a Haar measure on G, such that for every $m \in L^1(\beta)$

  $$\mathcal{K}(m)(g) = \int_{\mathbb{R}^n} m(\lambda)\chi(\lambda, g)\,\mathrm{d}\beta(\lambda)$$

  for almost every $g \in G$;

[1] see Section 2 for precise definitions

- $\mathcal{K}$ maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(G)$.

We consider also some additional properties of particular interest, such as:

(RL) if $\mathcal{K}(m) \in L^1(G)$, then we can take m so as to belong to $C_0(\mathbb{R}^n)$;

(S) if $\mathcal{K}(m) \in \mathcal{S}(G)$, then we can take m so as to belong to $\mathcal{S}(\mathbb{R}^n)$.

In this paper, we shall investigate the validity of properties (RL) and (S) in two particular cases: that of a Rockland operator on an abelian group, and that of homogeneous sub-Laplacians and elements of the centre on an MW^+ group (cf. Definition 4.1).

Here is a plan of the following sections. In Section 2 we recall the basic definitions and notation, as well as some relevant results proved in [15]. In Section 3, we then consider abelian groups, and characterize the Rockland operators which satisfy property (S) thereon. In Section 4 we prepare the machinery for the study of homogeneous sub-Laplacians and elements of the centre on MW^+ groups, referring to [15] for the proof of analogous statements when necessary. In contrast with the situation considered in [15], the structure of MW^+ groups will allow us to treat more than one homogeneous sub-Laplacian at a time. In Sections 5 and 6, then, we prove some sufficient conditions for properties (RL) and (S) in this context.

In Section 7 we present a particularly elegant result where all the good properties we consider are proved to be equivalent for the families which are invariant (in some sense) under the action of suitable groups of isometries. In particular, this result covers the case of Heisenberg groups, thanks to the results of Section 3. Finally, in Section 8 we consider products of Heisenberg groups and 'decomposable' homogeneous sub-Laplacian thereon. In addition, we exhibit a Rockland family which is 'functionally complete' (cf. Definition 2.11) but does not satisfy property (S).

2 Preliminaries

In this section we recall some basic results and definitions from [15]. We shall then prove some useful results that were not considered therein.

2.1 General Definitions and Notation

We adopt Schwartz's notation for the spaces of smooth functions and distributions. So, if Ω is an open subset of some euclidean space, $r \in \mathbb{N} \cup \{\infty\}$, and F is a Fréchet space, then we denote by $\mathcal{E}^r(\Omega; F)$ the space of F-valued functions of class C^r on the open set Ω, endowed with the topology of locally uniform convergence of all derivative up to the order r; we simply write $\mathcal{E}(\Omega; F)$ instead of $\mathcal{E}^\infty(\Omega; F)$, and we omit to write F explicitly when it is $\mathbb{C}$. We denote by $\mathcal{E}'^r(\Omega)$ the dual of $\mathcal{E}^r(\Omega)$, endowed with the strong topology, and we denote by $\mathcal{E}'^r_c(\Omega)$ the dual of $\mathcal{E}^r(\Omega)$, endowed with the topology of uniform convergence on the compact subsets of $\mathcal{E}^r(\Omega)$. The spaces $\mathcal{D}^r(\Omega; F)$, $\mathcal{D}'^r(\Omega)$ and $\mathcal{D}'^r_c(\Omega)$ are defined analogously; for example, $\mathcal{D}^r(\Omega; F)$ denotes the space of compactly supported F-valued functions of class C^r on Ω, endowed with the usual inductive limit topology. See [38] for more details.

Given a metric space (X, d) and $k \in \mathbb{R}_+$, we denote by $\mathcal{H}^k$ the k-th dimensional Hausdorff measure on X, that is

$$\mathcal{H}^k(E) = \frac{\pi^{k/2}}{2^k \Gamma\left(\frac{k}{2}+1\right)} \sup_{\delta>0} \inf \left\{ \sum_{j\in\mathbb{N}} \operatorname{diam}(E_j)^k \colon E \subseteq \bigcup_{j\in\mathbb{N}} E_j,\ \forall j \in \mathbb{N} \quad \operatorname{diam}(E_j) < \delta \right\}$$

for every $E \subseteq X$ (cf., for instance, [2, Definition 2.46]).

As in [15], a Rockland family on a homogeneous group G (cf. [22]) is a jointly hypoelliptic,[2] commutative, finite family $\mathcal{L}_A = (\mathcal{L}_\alpha)_{\alpha\in A}$ of formally self-adjoint, homogeneous, left-invariant differential operators without constant terms. In this case, the $\mathcal{L}_\alpha$ are essentially self-adjoint on $\mathcal{D}(G)$ (as unbounded operators of $L^2(G)$), and their closures commute. In addition, $\mathcal{L}_A$ is a weighted subcoercive system of operators (cf. [32, Proposition 3.6.3]), so that the theory developed in [32] applies.

Definition 2.1. To every (Borel, say) measurable function $m\colon \mathbb{R}^A \to \mathbb{C}$ such that $m(\mathcal{L}_A)$ is defined (at least) on $\mathcal{D}(G)$, we associate a unique distribution $\mathcal{K}_{\mathcal{L}_A}(m)$ (its 'kernel') on G such that

$$m(\mathcal{L}_A)(\varphi) = \varphi * \mathcal{K}_{\mathcal{L}_A}(m)$$

for every $\varphi \in \mathcal{D}(G)$.

We denote by $E_{\mathcal{L}_A}$ the space $\mathbb{R}^A$ endowed with the dilations defined by

$$r \cdot (\lambda_\alpha) := (r^{\delta_\alpha} \lambda_\alpha)$$

for every $r > 0$ and for every $(\lambda_\alpha) \in \mathbb{R}^A$, where δ_α is the homogeneous degree of $\mathcal{L}_\alpha$. We shall often employ the following short-hand notation: $L^1_{\mathcal{L}_A}(G)$ and $\mathcal{S}_{\mathcal{L}_A}(G)$ will denote $\mathcal{K}_{\mathcal{L}_A}(L^\infty(\beta)) \cap L^1(G)$ and $\mathcal{K}_{\mathcal{L}_A}(L^\infty(\beta)) \cap \mathcal{S}(G)$, respectively, while $\mathcal{S}(G, \mathcal{L}_A)$ will denote $\mathcal{K}_{\mathcal{L}_A}(\mathcal{S}(E_{\mathcal{L}_A}))$.

Now, by [32, Theorem 3.2.7] there is a unique positive Radon measure $\beta_{\mathcal{L}_A}$ on $E_{\mathcal{L}_A}$ such that a Borel function $m\colon E_{\mathcal{L}_A} \to \mathbb{C}$ is square-integrable if and only if $\mathcal{K}_{\mathcal{L}_A}(m) \in L^2(G)$ and such that, in this case,

$$\|m\|_{L^2(\beta_{\mathcal{L}_A})} = \|\mathcal{K}_{\mathcal{L}_A}(m)\|_{L^2(G)}.$$

The measure $\beta_{\mathcal{L}_A}$ is then equivalent to the spectral measure associated with $\mathcal{L}_A$. Using the existence of $\beta_{\mathcal{L}_A}$ and the fact that $\mathcal{K}_{\mathcal{L}_A}$ maps $\mathcal{S}(E_{\mathcal{L}_A})$ in $\mathcal{S}(G)$ (cf. [32, Proposition 4.2.1]), it is not hard to prove

[2] That is, it T is a distribution on G and $\mathcal{L}_\alpha T$ has a density of class C^∞ on some open set Ω for every $\alpha \in A$, then T has a density of class C^∞ on Ω.

that a $\beta_{\mathcal{L}_A}$-measurable function admits a kernel in the sense of Definition 2.1 if and only if there is a positive polynomial P on $E_{\mathcal{L}_A}$ such that $\frac{m}{1+P} \in L^2(\beta_{\mathcal{L}_A})$.

Now, $\mathcal{K}_{\mathcal{L}_A}$ can be extended to a continuous linear mapping from $L^1(\beta_{\mathcal{L}_A})$ into $C_0(G)$ (cf. [32, Proposition 3.2.12]), and there is a unique $\chi_{\mathcal{L}_A} \in L^\infty(\beta_{\mathcal{L}_A} \otimes \nu_G)$, where ν_G denotes a fixed Haar measure on G, such that

$$\mathcal{K}_{\mathcal{L}_A}(m)(g) = \int_{E_{\mathcal{L}_A}} m(\lambda)\chi_{\mathcal{L}_A}(\lambda, g)\,\mathrm{d}\beta_{\mathcal{L}_A}(\lambda)$$

for every $m \in L^1(\beta_{\mathcal{L}_A})$ and for almost every $g \in G$.[3]

Further, we denote by $\mathcal{M}_{\mathcal{L}_A} \colon L^1(G) \to L^\infty(G)$ the transpose of the mapping $m \mapsto \mathcal{K}_{\mathcal{L}_A}(m)\check{}$, so that

$$\mathcal{M}_{\mathcal{L}_A}(f)(\lambda) = \int_G f(g)\overline{\chi_{\mathcal{L}_A}(\lambda, g)}\,\mathrm{d}g$$

for every $f \in L^1(G)$ and for $\beta_{\mathcal{L}_A}$-almost every $\lambda \in E_{\mathcal{L}_A}$. Observing that $\mathcal{M}_{\mathcal{L}_A}$ equals the adjoint of the isometry $\mathcal{K}_{\mathcal{L}_A} \colon L^2(\beta_{\mathcal{L}_A}) \to L^2(G)$ on $L^1(G) \cap L^2(G)$, one may then prove that $\mathcal{K}_{\mathcal{L}_A} \circ \mathcal{M}_{\mathcal{L}_A}$ is the identity on $L^1_{\mathcal{L}_A}(G)$.

Observe that, if $\mathrm{Card}(A) = 1$, then $\chi_{\mathcal{L}_A}$ has a bounded continuous representative (cf. [15, Proposition 3.14 and Theorem 3.17]), so that $\mathcal{L}_A$ satisfies property (RL).

2.2 Products

Assume that we are given two Rockland families $\mathcal{L}_A$ and $\mathcal{L}'_{A'}$ on two homogeneous groups G and G', respectively. Denote by $\mathcal{L}''_{A''}$ the family whose elements are the operators on $G \times G'$ induced by the elements of $\mathcal{L}_A$ and $\mathcal{L}'_{A'}$, and observe that $\mathcal{L}''_{A''}$ is a Rockland family.

Theorem 2.2 ([15], Theorems 4.5 and 4.10)**.** *The families $\mathcal{L}_A$ and $\mathcal{L}'_{A'}$ satisfy property (RL) (resp. (S)) if and only if $\mathcal{L}''_{A''}$ does.*

2.3 Composite Functions

Assume that we are given a Rockland family $\mathcal{L}_A$ and a polynomial mapping P on $E_{\mathcal{L}_A}$ such that $P(\mathcal{L}_A)$ is still a Rockland family (this is equivalent to saying that P is proper and that its components are homogeneous *with respect to the dilations of* $E_{\mathcal{L}_A}$). Then, for every bounded measurable function m we have $\mathcal{K}_{P(\mathcal{L}_A)}(m) = \mathcal{K}_{\mathcal{L}_A}(m \circ P)$. As a consequence, if we want to establish properties (RL) or (S) for $P(\mathcal{L}_A)$ on the base of our knowledge of $\mathcal{L}_A$, it is of importance to infer some properties of m from the properties of $m \circ P$. The results of this section address this problem.

We begin with a definition.

Definition 2.3. Let X be a locally compact space, Y a set, μ a positive Radon measure on X, and π a

[3]The existence of $\chi_{\mathcal{L}_A}$ is basically a consequence of the Dunford–Pettis theorem, cf. [39].

mapping from X into Y. We say that two points x, x' of $\mathrm{Supp}\,(\mu)$ are (μ,π)-connected if $\pi(x)=\pi(x')$ and there are $x=x_1,\dots,x_k=x'\in\pi^{-1}(\pi(x))\cap\mathrm{Supp}\,(\mu)$ such that, for every $j=1,\dots,k$, for every neighbourhood U_j of x_j in $\mathrm{Supp}\,(\mu)$, and for every neighbourhood U_{j+1} of x_{j+1} in $\mathrm{Supp}\,(\mu)$, the set $\pi^{-1}(\pi(U_j)\cap\pi(U_{j+1}))$ is not μ-negligible. We say that μ is π-connected if every pair of elements of $\mathrm{Supp}\,(\mu)$ having the same image under π are (μ,π)-connected.

Proposition 2.4. *Let E_1, E_2 be two finite-dimensional affine spaces, $L\colon E_1\to E_2$ an affine mapping and μ a positive Radon measure on E_1. Assume that the support of μ is either a convex set and that L is proper on it, or that the support of μ is the boundary of a convex polyhedron on which L is proper Then, μ is L-connected.*

Proof. **0.** The assertion is a consequence of [15, Proposition 6.3] when the support of μ is convex. Then, assume that the support of μ is the boundary of a convex polyhedron C, and that L is proper on C.

1. Consider first the case in which C is has non-empty interior, $E_1=\mathbb{R}^n$, $E_2=\mathbb{R}^{n-1}$ and $L(x_1,\dots,x_n)=(x_1,\dots,x_{n-1})$ for every $(x_1,\dots,x_n)\in\mathbb{R}^n$. Define $C':=L(C)$, so that C' is a convex polyhedron of E_2. Let $\mathcal{F}$ be the (finite) set of $(n-1)$-dimensional facets of C, and observe that $\partial C=\bigcup_{F\in\mathcal{F}}F$ since C is a convex polyhedron. Now, since L is proper on C, for every $x'\in L(C)$ the set $L^{-1}(L(x'))\cap C$ is convex and compact, hence a closed segment, whose end-points must belong to ∂C. By the arbitrariness of x', it follows that $L(C)=L(\partial C)$. In addition, let $\mathcal{F}_0$ be the set of $F\in\mathcal{F}$ such that L is one-to-one on F, and observe that $C':=L(C)=L(\partial C)$ is a closed convex set with non-empty interior, which differs from the closed set $\bigcup_{F\in\mathcal{F}_0}L(F)$ by an $\mathcal{H}^{n-1}$-negligible set (contained in $\bigcup_{F\in\mathcal{F}\setminus\mathcal{F}_0}L(F)$). Since the support of $\chi_{C'}\mathcal{H}^{n-1}$ is C', this implies that $C'=\bigcup_{F\in\mathcal{F}_0}L(F)$.

Now, observe that the functions

$$f_-\colon C'\ni x'\mapsto \min\{y\in\mathbb{R}\colon (x',y)\in C\}\qquad\text{and}\qquad f_+\colon C'\ni x'\mapsto \max\{y\in\mathbb{R}\colon (x',y)\in C\}$$

are well-defined by the preceding remarks; in addition, f_- is convex and f_+ is concave; let $\Gamma_\pm$ be the graph of $f_\pm$. By convexity, f_- and f_+ are continuous on $\mathring{C'}$ by [13, Corollary to Proposition 21 of Chapter II, § 2, No. 10]. Now, $f_-\leqslant f_+$ by definition; if $f_-(x')=f_+(x')$ for some $x'\in\mathring{C'}$, then $f_-=f_+$ on C' by convexity, and this contradicts the assumption that C has non-empty interior. Therefore, f_--f_+ is nowhere zero on $\mathring{C'}$ and $\{(x',y)\colon x'\in\mathring{C'}, f_-(x')<y<f_+(x')\}$ is the interior of C. Now, take $F\in\mathcal{F}_0$ and let F' be the interior of F in the affine space generated by F; observe that $L(F')$ is an open set contained in C'. Since $F\subseteq\partial C$, the preceding remarks imply that F' is either contained in Γ_- or in Γ_+; assume, for the sake of definiteness, that $F'\subseteq\Gamma_-$. Let A be the affine function on $\mathbb{R}^{n-1}$ such that F is the graph of the restriction of A to $L(F)$. Then, $A=f_-$ on $L(F')$, so that, by convexity, $f_-\geqslant A$ on the closure $L(F)$ of $L(F')$. On the other hand, $f_-\leqslant A$ on $L(F)$ since $F\subseteq C$. It then follows that $f_-=A$ on $L(F)$, so that $F\subseteq\Gamma_-$. Let $\mathcal{F}_\pm$ be the set of $F\in\mathcal{F}_0$ such that $F\subseteq\Gamma_\pm$. Then, the preceding remarks

show that $(\mathcal{F}_-, \mathcal{F}_+)$ form a partition of $\mathcal{F}_0$, so that f_- and f_+ are continuous and piecewise linear on C', and $\Gamma_\pm = \bigcup_{F \in \mathcal{F}_\pm} F$.[4] Since L induces a homeomorphism of Γ_- and Γ_+ onto C', it is easily seen that μ is L-connected.

2. Now, consider the general case. Observe first that we may assume that C has non-empty interior. Take $x_1, x_2 \in \partial C$ such that $x_1 \neq x_2$ and $L(x_1) = L(x_2)$. Let L' be an affine mapping defined on E_1 such that $L'(x_1) = L'(x_2)$ and such that the fibres of L' have dimension 1. Then, we may apply **1** above and deduce that x_1, x_2 are (μ, L')-connected. It is then easily seen that x_1, x_2 are also (μ, L)-connected, whence the result. □

Remark 2.5. Notice that Proposition 2.4 is false when the support of μ is the boundary of a more general convex set (on which L is proper). Indeed, choose $E_1 = \mathbb{R}^3$, $E_2 = \mathbb{R}^2$, $L = \mathrm{pr}_{1,2}$ and

$$C_1 \coloneqq \{(x, y, z) \in E_1 \colon 2yz \geqslant x^2, z \in [0, 1], y \geqslant 0\}.$$

Define C as the union of C_1 and $\pi(C_1)$, where π is the reflection along the plane $\mathrm{pr}_3^{-1}(1)$. Then, ∂C is the union of

$$C_1' \coloneqq \{(x, y, z) \in E_1 \colon 2yz = x^2, z \in [0, 1], y \geqslant 0\}$$

and $\pi(C_1')$. Choose any continuous function $m_1 \colon C_1' \to \mathbb{C}$, and define $m \colon \partial C \to \mathbb{C}$ so that it equals m_1 on C_1' and $m_1 \circ \pi$ on $\pi(C_1')$. Then, m is clearly continuous. In addition, it is clear that C_1' intersects the fibres of L at one point at most, except for $L^{-1}(0, 0)$. Since m can be chosen so that it is *not* constant on $\{(0, 0)\} \times [0, 2]$, Proposition 2.6 below shows that $\chi_{\partial C} \cdot \mathcal{H}^2$ cannot be L-connected.

Proposition 2.6 ([15], Proposition 6.2)**.** *Let X, Y, Z be three locally compact spaces, $\pi \colon X \to Y$ a μ-measurable mapping, and μ a π-connected positive Radon measure on X. Assume that $\pi_*(\mu)$ is a Radon measure and that there is a disintegration $(\lambda_y)_{y \in Y}$ of μ relative to π such that* $\mathrm{Supp}\,(\lambda_y) \supseteq \mathrm{Supp}\,(\mu) \cap \pi^{-1}(y)$ *for $\pi_*(\mu)$-almost every $y \in Y$.*

Take a continuous mapping $m_0 \colon X \to Z$ such that there is mapping $m_1 \colon Y \to Z$ such that $m_0(x) = (m_1 \circ \pi)(x)$ for μ-almost every $x \in X$. Then, there is a $\pi_*(\mu)$-measurable mapping $m_2 \colon Y \to Z$ such that $m_0 = m_2 \circ \pi$ pointwise *on* $\mathrm{Supp}\,(\mu)$*.*

If, in addition, π is proper on $\mathrm{Supp}\,(\mu)$*, Y is metrizable, and $Z = \mathbb{C}$, then m_2 can be chosen so as to be continuous.*

Notice that the last assertion is an almost immediate consequence of [12, Corollary to Theorem 2 of Chapter IX, § 4, No. 2].

Concerning the assumption on the disintegration, we shall often make use of a general result by Federer (cf. [20, Theorem 3.2.22]), which basically provides the disintegration of a wide family of measures. We

[4]Indeed, $\bigcup_{F \in \mathcal{F}_\pm} F$ is a closed set contained in $\Gamma_\pm$, and its image under L contains the interior of C'.

shall also derive Lemma 5.6 from it.

For what concerns the composition of Schwartz functions, the techniques employed to prove [5, Theorem 6.1] can be effectively used to derive from [7, Theorem 0.2] and [9, Theorem 0.2.1] the following result:

Theorem 2.7 ([15], Theorem 7.2)**.** *Let $P\colon \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial mapping, and assume that $\mathbb{R}^n$ and $\mathbb{R}^m$ are endowed with dilations such that $P(r\cdot x) = r\cdot P(x)$ for every $r>0$ and for every $x\in\mathbb{R}^n$. Let C be a dilation-invariant subanalytic closed subset of $\mathbb{R}^n$, and assume that P is proper on C and that $P(C)$ is Nash subanalytic. Then, the canonical mapping*

$$\Phi\colon \mathcal{S}(\mathbb{R}^m) \ni \varphi \mapsto \varphi\circ P \in \mathcal{S}_{\mathbb{R}^n}(C)$$

has a closed range and admits a continuous linear section defined on $\Phi(\mathcal{S}(\mathbb{R}^m))$. In addition, $\psi\in\mathcal{S}_{\mathbb{R}^n}(C)$ belongs to the image of Φ if and only if it is a 'formal composite' of P, that is, for every $y\in\mathbb{R}^m$ there is $\varphi_y\in\mathcal{E}(\mathbb{R}^m)$ such that, for every $x\in C\cap P^{-1}(y)$, the Taylor series of $\varphi_y\circ P$ and ψ at x differ by the Taylor series of a function of class C^∞ which vanishes on C.

In the statement, we denoted by $\mathcal{S}_{\mathbb{R}^n}(C)$ the quotient of $\mathcal{S}(\mathbb{R}^n)$ by the space of $f\in\mathcal{S}(\mathbb{R}^n)$ which vanish on the closed set C. We refer the reader to [7, 8, 9] for the notion of (Nash) subanalytic sets; as a matter of fact, in the applications we shall only need to know that any convex subanalytic set is automatically Nash subanalytic, since it is contained in an affine space of the same dimension, and that semianalytic sets are Nash subanalytic (cf. [7, Proposition 2.3]).

With similar techniques one may prove the following result.

Theorem 2.8. *Let $P\colon \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial mapping, and assume that $\mathbb{R}^n$ and $\mathbb{R}^m$ are endowed with dilations such that $P(r\cdot x) = r\cdot P(x)$ for every $r>0$ and for every $x\in\mathbb{R}^n$. Let C be a dilation-invariant closed subanalytic subset of $\mathbb{R}^n\setminus\{0\}$, and assume that P is proper on C and that $P(C)$ is Nash subanalytic in $\mathbb{R}^m\setminus\{0\}$. Then, the canonical mapping*

$$\Phi\colon \mathcal{S}_\infty(\mathbb{R}^m) \ni \varphi \mapsto \varphi\circ P \in \mathcal{S}_{\mathbb{R}^n,\infty}(C)$$

has a closed range and admits a continuous linear section defined on $\Phi(\mathcal{S}_\infty(\mathbb{R}^m))$. In addition, $\psi\in\mathcal{S}_{\mathbb{R}^n,\infty}(C)$ belongs to the image of Φ if and only if it is a 'formal composite' of P.

In the statement, we denoted by $\mathcal{S}_\infty(\mathbb{R}^m)$ the space of Schwartz functions on $\mathbb{R}^m$ which vanish of order ∞ at 0, endowed with the semi-norms $\varphi\mapsto \sup\limits_{y\in\mathbb{R}^m} \max(|y|^{-1},|y|)^k \sum_{|\alpha|\leqslant k}|\partial^\alpha\varphi(y)|$ for $k\in\mathbb{N}$, and by $\mathcal{S}_{\mathbb{R}^n,\infty}(C)$ the quotient of $\mathcal{S}_\infty(\mathbb{R}^n)$ by the space of $f\in\mathcal{S}(\mathbb{R}^n)$ which vanish on the closed set C.

Proof. Take $\tau\in C^\infty_c(\mathbb{R}^m\setminus\{0\})$ such that $\sum_{j\in\mathbb{Z}}\tau(2^j\cdot y)^2 = 1$ for every $y\in\mathbb{R}^m\setminus\{0\}$; define $\eta\in$

$C_c^\infty(\mathbb{R}^n\setminus\{0\})$ in such a way that $\tau\circ P=\eta$ on a dilation-invariant neighbourhood U of $C\setminus\{0\}$, so that $\tau_j\circ P=\eta_j$ on U for every $j\in\mathbb{Z}$, where $\tau_j=\tau(2^j\,\cdot)$ and $\eta_j=\eta(2^j\,\cdot)$; observe that $\sum_{j\in\mathbb{Z}}\eta_j^2$ is of class C^∞ on $\mathbb{R}^n\setminus\{0\}$, and equals 1 on U.

Now, by [7, Theorem 0.2] and [9, Theorem 0.2.1] there is a continuous linear section Ψ of the continuous linear mapping $\mathcal{E}(\mathbb{R}^m\setminus\{0\})\ni\varphi\mapsto\varphi\circ P\in\mathcal{E}_{\mathbb{R}^n\setminus\{0\}}(C\setminus\{0\})$. Then, for every $\psi\in\mathcal{S}_{\mathbb{R}^n,\infty}(C)$ define

$$\Psi'(\psi)\coloneqq\sum_{j\in\mathbb{Z}}[\tau\Psi(\psi(2^{-j}\,\cdot\,)\eta)](2^j\,\cdot\,),$$

so that clearly $\Psi'(\psi)\circ P=\sum_{j\in\mathbb{Z}}\eta_j^2\psi=\psi$ on C, since $\psi(0)=0$. It only remains to prove that Ψ' induces a continuous mapping $\mathcal{S}_\infty(\mathbb{R}^n)\to\mathcal{S}_\infty(\mathbb{R}^m)$.

Observe first that for every $k\in\mathbb{N}$ there are $h_k\in\mathbb{N}$ and a constant $C_k>0$ such that

$$\sum_{|\alpha|\leqslant k}\|\partial^\alpha(\tau\Psi(\psi(2^{-j}\,\cdot\,)\eta))\|_\infty\leqslant C_k\sup_{x\in K}\sum_{|\alpha|\leqslant h_k}2^{-j\mathrm{d}_\alpha}|\partial^\alpha\psi(2^{-j}\cdot x)|$$

for every $\psi\in\mathcal{S}_\infty(\mathbb{R}^n)$ and for every $j\in\mathbb{Z}$, where K is a compact neighbourhood of the support of η in $\mathbb{R}^n\setminus\{0\}$, while d_α is the homogeneous degree of ∂^α (assuming that homogeneous coordinates have been chosen on $\mathbb{R}^n$ and $\mathbb{R}^m$).

Notice that we may assume that the support of τ is contained in $\{y\in\mathbb{R}^m\colon\frac{1}{2}\leqslant|y|\leqslant2\}$. Therefore,

$$\max(|y|,|y|^{-1})^k\sum_{|\alpha'|\leqslant k}|\partial^{\alpha'}\Psi'(\psi)(y)|\leqslant C_k\max(|y|,|y|^{-1})^k\sum_{j=-\log_2(|y|)-1}^{-\log_2(|y|)+1}\sum_{|\alpha'|\leqslant k}\sum_{|\alpha|\leqslant h_k}2^{j(\mathrm{d}'_{\alpha'}-\mathrm{d}_\alpha)}\sup_{2^{-j}\cdot K}|\partial^\alpha\psi|.$$

Now, let right-hand side of the preceding expression is a continuous semi-norm of $\mathcal{S}_\infty(\mathbb{R}^n)$ computed at ψ, so that Ψ' is well defined and continuous.

The second assertion is clear when $\psi\in\mathcal{S}_{\mathbb{R}^n,\infty}(C)$ is supported in a compact subset of $C\setminus\{0\}$, thanks to [7, Theorem 0.2] and [9, Theorem 0.2.1], so that it follows by continuity in the general case, since $\sum_{j=-k}^k\eta_j^2\psi$ converges to ψ in $\mathcal{S}_{\mathbb{R}^n,\infty}(C)$ as $k\to\infty$. □

Corollary 2.9 ([15], Corollary 7.3)**.** *Let V and W be two finite-dimensional vector spaces, C a subanalytic closed convex cone in V, and L a linear mapping of V into W which is proper on C. Take $m_1\in\mathcal{S}(V)$ and assume that there is $m_2\colon W\to\mathbb{C}$ such that $m_1=m_2\circ L$ on C. Then, there is $m_3\in\mathcal{S}(W)$ such that $m_1=m_3\circ L$ on C.*

We now present a useful consequence of the preceding results with the purpose of showing how one may apply the preceding techniques to deduce properties (RL) and (S) for 'image families.' The notation is the same as in the beginning of this subsection.

Proposition 2.10. *Assume that P is linear and that $\sigma(\mathcal{L}_A)$ is convex. Then, the following hold:*

1. *if $\mathcal{L}_A$ satisfies property (RL) and $\beta_{\mathcal{L}_A}$ is equivalent to $\chi_{\sigma(\mathcal{L}_A)} \cdot \mathcal{H}^k$, where k is the dimension of the convex set $\sigma(\mathcal{L}_A)$, then $P(\mathcal{L}_A)$ satisfies property (RL);*

2. *if $\mathcal{L}_A$ satisfies property (S), $P(\mathcal{L}_A)$ satisfies property (RL), and $\sigma(\mathcal{L}_A)$ is subanalytic, then $P(\mathcal{L}_A)$ satisfies property (S).*

Proof. **1.** Take $\varphi \in L^1_{P(\mathcal{L}_A)}(G)$ and let m be a representative of $\mathcal{M}_{P(\mathcal{L}_A)}(\varphi)$. Since $\mathcal{L}_A$ satisfies property (RL), there is $m_0 \in C_0(\sigma(\mathcal{L}_A))$ such that $\varphi = \mathcal{K}_{\mathcal{L}_A}(m_0)$, so that $m_0 = m \circ P$ $\mathcal{H}^k$-almost everywhere thanks to the assumptions. Write $\Sigma \coloneqq \sigma(\mathcal{L}_A)$ to simplify the notation, and let V be the vector space generated by Σ. Define $\Sigma' \coloneqq P(\Sigma)$ and $V' \coloneqq P(V)$. Now, observe that $\chi_\Sigma \cdot \mathcal{H}^k$ is P-connected thanks to Proposition 2.4. In addition, by means of Tonelli's theorem we see that $P_*(\chi_\Sigma \cdot \mathcal{H}^k)$ is equivalent to $\chi_{\Sigma'} \cdot \mathcal{H}^{k'}$, where k' is the dimension of V', and that $\chi_\Sigma \cdot \mathcal{H}^k$ has a disintegration $(\beta'_\lambda)_{\lambda \in \Sigma'}$ relative to P, with β'_λ equivalent to $\chi_{L^{-1}(\lambda)\cap\Sigma} \cdot \mathcal{H}^{k'-k}$ for $\mathcal{H}^{k'}$-almost every $\lambda \in \Sigma'$. Now, let U be the interior of Σ in V, and let U' be the interior of Σ' in V'; observe that $P(U) = U'$ since P is linear and Σ convex.[5] Now, for every $\lambda \in U'$, $L^{-1}(\lambda) \cap \Sigma$ is then the closure of the convex set $L^{-1}(\lambda) \cap U$ (cf. Lemma 5.10 below), which has non-empty interior in $L^{-1}(\lambda)$. Since the boundary of $L^{-1}(\lambda) \cap U$ in $L^{-1}(\lambda)$ is $\mathcal{H}^{k'-k}$-negligible, the support of β_λ is $L^{-1}(\lambda) \cap \Sigma$. In addition, $\Sigma' \setminus U'$ is $\mathcal{H}^{k'}$-negligible, so that Proposition 2.6 can be applied. Hence, there is $m_1 \in \mathcal{E}^0(E_{P(\mathcal{L}_A)})$ such that $m_1 \circ P = m_0$ on $\sigma(\mathcal{L}_A)$, so that $m_1 = m$ $\beta_{P(\mathcal{L}_A)}$-almost everywhere. The assertion follows.

2. Take $\varphi \in \mathcal{S}_{P(\mathcal{L}_A)}(G)$ and let m be a representative of $\mathcal{M}_{P(\mathcal{L}_A)}(\varphi)$. Notice that we may assume that m is continuous, since $P(\mathcal{L}_A)$ satisfies property (RL). Since $\mathcal{L}_A$ satisfies property (S), there is $m_0 \in \mathcal{S}(E_{\mathcal{L}_A})$ such that $\varphi = \mathcal{K}_{\mathcal{L}_A}(m_0)$, so that $m_0 = m \circ P$ on $\sigma(\mathcal{L}_A)$. Notice that $\sigma(\mathcal{L}_A)$ is a subanalytic convex cone since it is dilation-invariant and P is linear and homogeneous. Then, Corollary 2.9 implies that there is $m_1 \in \mathcal{S}(E_{P(\mathcal{L}_A)})$ such that $m_1 \circ P = m_0$ on $\sigma(\mathcal{L}_A)$, so that $m_1 = m$ on $\sigma(P(\mathcal{L}_A))$. The assertion follows. □

2.4 Equivalence and Completeness

Let us now add some definitions to those presented in [15].

Definition 2.11. We say that two Rockland families $\mathcal{L}_{A_1}$ and $\mathcal{L}_{A_2}$ are functionally equivalent if there are two Borel functions $m_1 \colon E_{\mathcal{L}_{A_1}} \to E_{\mathcal{L}_{A_2}}$ and $m_2 \colon E_{\mathcal{L}_{A_2}} \to E_{\mathcal{L}_{A_1}}$ such that $m_1(\mathcal{L}_{A_1}) = \mathcal{L}_{A_2}$ and $m_2(\mathcal{L}_{A_2}) = \mathcal{L}_{A_1}$.

We shall say that a Rockland family $\mathcal{L}_A$ is functionally complete if every $\beta_{\mathcal{L}_A}$-measurable function $m \colon E_{\mathcal{L}_A} \to \mathbb{C}$ such that $m(\mathcal{L}_A)$ is a differential operator equals a polynomial $\beta_{\mathcal{L}_A}$-almost everywhere.

[5] Indeed, U' is clearly an open convex subset of Σ'; in addition, Σ is the closure of U, so that $\Sigma' = P(\Sigma) \subseteq \overline{U'} \subseteq \Sigma'$; since U' is open and convex, it equals the interior of its closure by [13, Corollary 1 to Proposition 16 of Chapter II, § 2, No. 6].

Notice that there exist Rockland families which are not functionally complete; for example, if $\mathcal{L}$ is a positive Rockland operator, then $(\mathcal{L}^2)$ is a Rockland family which is not functionally complete. Further, observe that we cannot talk of a 'completion' of $\mathcal{L}_A$ unless we know that the algebra of differential operators arising as functions of $\mathcal{L}_A$ is (algebraically) finitely generated.

The main point for considering functional completeness is the following result, which shows that property (S) implies functional completeness; nevertheless, the converse fails in general (cf. Proposition 8.5).

Proposition 2.12. *Let $\mathcal{L}_A$ be a Rockland family on a homogeneous group G. If $\mathcal{L}_A$ satisfies property (S), then it is functionally complete.*

Proof. Take a function of $\mathcal{L}_A$ which is a homogeneous left-invariant differential operator of homogeneous degree δ, and let T be its convolution kernel; assume that $\mathcal{L}_A$ satisfies property (S). Take $\tau \in \mathcal{S}(E_{\mathcal{L}_A})$ such that $\tau(\lambda) \neq 0$ for every $\lambda \in E_{\mathcal{L}_A}$; then $\mathcal{K}_{\mathcal{L}_A}(\tau) * T \in \mathcal{S}(G)$, so that there is $m_1 \in \mathcal{S}(E_{\mathcal{L}_A})$ such that $\mathcal{K}_{\mathcal{L}_A}(m_1) = \mathcal{K}_{\mathcal{L}_A}(\tau) * T$. If we define $m := \frac{m_1}{\tau}$, then $m \in \mathcal{E}(E_{\mathcal{L}_A})$ and $\mathcal{K}_{\mathcal{L}_A}(m) = T$. By means of [22, Theorem 1.37], we see that there are a family with finite support $(P_{\delta'})_{0\leqslant\delta'\leqslant\delta}$ of homogeneous polynomials, where $P_{\delta'}$ has homogeneous degree δ' for every $\delta' \in [0,\delta]$, and a function ω, such that

$$m(\lambda) = \sum_{0\leqslant\delta'\leqslant\delta} P_{\delta'}(\lambda) + \omega(\lambda)$$

for every $\lambda \in E_{\mathcal{L}_A}$, and such that

$$\lim_{\lambda\to 0} \frac{\omega(\lambda)}{|\lambda|^\delta} = 0.$$

Now, $m(r\cdot\lambda) = r^\delta m(\lambda)$ for every $r > 0$ and for every $\lambda \in \sigma(\mathcal{L}_A)$ since T is homogeneous of homogeneous degree δ; fix a non-zero $\lambda \in \sigma(\mathcal{L}_A)$. Then,

$$r^\delta m(\lambda) = m(r\cdot\lambda) = \sum_{0\leqslant\delta'\leqslant\delta} P_{\delta'}(r\cdot\lambda) + \omega(r\cdot\lambda) = \sum_{0\leqslant\delta'\leqslant\delta} r^{\delta'} P_{\delta'}(\lambda) + o\left(r^\delta\right)$$

for $r \to 0^+$, so that $P_{\delta'}(\lambda) = 0$ for every $\delta' \in [0,\delta[$ and $P_\delta(\lambda) = m(\lambda)$. Hence, $m = P_\delta$ on $\sigma(\mathcal{L}_A)$, so that $m = P_\delta$ $\beta_{\mathcal{L}_A}$-almost everywhere. By the arbitrariness of T, the assertion follows (cf. [36]). ☐

3 Abelian Groups

In this section, G denotes a homogeneous *abelian* group. In other words, G is the euclidean space $\mathbb{R}^n$ endowed with dilations of the form $r\cdot x = (r^{\mathrm{d}_1}x_1, \dots, r^{\mathrm{d}_n}x_n)$ for $r > 0$, $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, and some fixed $\mathrm{d}_1,\dots,\mathrm{d}_n > 0$. Then, $\partial = (\partial_1,\dots,\partial_n)$ is a homogeneous basis of the Lie algebra of G. We shall consequently put a scalar product and the associated Hausdorff measures on G, and identify the Fourier transform $\mathcal{F}$ with a mapping from $\mathcal{S}'(G)$ onto $\mathcal{S}'(E_{-i\partial})$.

Proposition 3.1. *Let P be a polynomial mapping with homogeneous components from $E_{-i\partial}$ into $\mathbb{R}^A$ for some finite set A. Then, $\mathcal{L}_A = P(-i\partial)$ is a Rockland family if and only if P is proper. In this case, the following hold:*

1. *$\sigma(\mathcal{L}_A) = P(E_{-i\partial})$;*

2. *a $\beta_{\mathcal{L}_A}$-measurable function m admits a kernel in the sense of Definition 2.1 if and only if $m \circ P$ is a polynomial times an element of $L^2(E_{-i\partial})$; in this case,*

$$\mathcal{K}_{\mathcal{L}_A}(m) = \mathcal{F}^{-1}(m \circ P).$$

Proof. Since $\sigma(-i\partial) = E_{-i\partial}$ and $-i\partial$ is Rockland, the assertions follow easily from the properties of the Fourier transform. □

By means of [20, Theorem 3.2.22], one may obtain some relatively explicit formulae for $\beta_{\mathcal{L}_A}$ and $\chi_{\mathcal{L}_A}$.

In the following result, we give complete answers to our main questions in the case of *one* operator.

Theorem 3.2. *Let $\mathcal{L}$ be a positive Rockland operator on G.*[6] *Then, $\chi_{\mathcal{L}}$ has a continuous representative which is of class C^∞ on $\mathbb{R}_+^* \times G$; in particular, property (RL) holds.*

In addition, take $m \in C_b(\beta_{\mathcal{L}})$, and let k be the greatest $k' \in \mathbb{N}^*$ such that $P^{\frac{1}{k'}}$ is a polynomial. Then, the following conditions are equivalent:

1. *$\mathcal{K}_{\mathcal{L}}(m) \in \mathcal{S}(G)$;*

2. *there are $m_0, \dots, m_{k-1} \in \mathcal{S}(\mathbb{R})$ such that $m(\lambda) = \sum_{h=0}^{k-1} \lambda^{\frac{h}{k}} m_h(\lambda)$ for every $\lambda \geqslant 0$.*

In particular, property (S) holds if and only if $k = 1$.

Before we prove the preceding result, we need to establish a lemma.

Lemma 3.3. *Let A be a non-empty finite set and endow $\mathbb{R}^A$ with a family of (not-necessarily isotropic) dilations. Take a positive, non-constant, homogeneous polynomial P in $\mathbb{R}[A]$ and assume that there is a homogeneous element x of $\mathbb{R}^A$ such that $P(x) \neq 0$. Then, the following statements are equivalent:*

1. *there are no positive homogeneous polynomials $Q \in \mathbb{R}[A]$ and no $k \in \mathbb{N}$ such that $k \geqslant 2$ and $P = Q^k$;*

2. *if m is a complex-valued function defined on $\mathbb{R}_+$ such that $m \circ P$ is of class C^∞ on $\mathbb{R}^A$, then m may be extended to an element of $\mathcal{E}(\mathbb{R})$.*

[6] Notice that $\mathcal{L} = P(-i\partial)$ where P is a proper polynomial; unless $G = \mathbb{R}$, in which case our analysis is trivial, P must have a constant sign, so that we may assume that $\mathcal{L}$ is positive without loss of generality.

Proof. **1** $\implies$ **2.** Take $m\colon \mathbb{R}_+ \to \mathbb{C}$ and assume that $m \circ P$ is of class C^∞ on $\mathbb{R}^A$. Notice that there is a homogeneous polynomial $P_x \in \mathbb{R}[X]$ such that $P(\lambda x) = P_x(\lambda)$ for every $\lambda \in \mathbb{R}$.[7] In particular, $m \circ P_x$ is of class C^∞. In addition, $P_x(X) = a_x X^{d_x}$ for some $a_x \neq 0$ and $d_x \in \mathbb{N}^*$; we may assume that $a_x = 1$. It is then clear that m is of class C^∞ on $\mathbb{R}_+^*$; further, $m \circ P_x$ admits a Taylor series $\sum_{j\in\mathbb{N}} a_{x,j} X^j$ at 0. Therefore, m admits the asymptotic development $\sum_{j\in\mathbb{N}} a_{x,j} \lambda^{\frac{j}{d_x}}$ for $\lambda \to 0^+$. Suppose that there are some $j \in \mathbb{N} \setminus (d_x \mathbb{N})$ such that $a_{x,j} \neq 0$, and let j_x be the least of them. Let q_x, r_x be the quotient and the remainder, respectively, of the division of j_x by d_x.

Define $\widetilde{m} \coloneqq m - \sum_{j=0}^{j_x d_x} a_{x,j} (\,\cdot\,)^{\frac{j}{d_x}}$. Then, $\widetilde{m} \circ P_x$ is of class C^∞ and $(\widetilde{m} \circ P_x)(\lambda) = o\left(|\lambda|^{j_x d_x}\right)$.[8] Hence, it is not hard to see that $\widetilde{m}$ may be extended to an element of $\mathcal{E}^{j_x}(\mathbb{R})$. Let us then prove that

$$a_{x,j_x} \partial_x^{j_x} P^{\frac{j_x}{d_x}} = \partial_x^{j_x}(m \circ P) - \sum_{j=0}^{q_x} a_{x,d_x j} \partial_x^{j_x} P^j - \sum_{j=j_x+1}^{j_x d_x} a_{x,j} \partial_x^{j_x} P^{\frac{j}{d_x}} - \partial_x^{j_x}(\widetilde{m} \circ P)$$

extends to a continuous function on $E \coloneqq \{x' \in \mathbb{R}^A \colon P(x') \neq 0\} \cup \{0\}$. Indeed, this is clear for the first two terms, and follows from the above remarks for the fourth one. Let us then consider the third term. Notice that both ∂_x and P are homogeneous, and that $\partial_x^{j_x} P^{\frac{j_x}{d_x}}$ is homogeneous of homogeneous degree 0 on the x axis, hence on $\mathbb{R}^A$. Hence, $\partial_x^{j_x}$ must be homogeneous of homogeneous degree $d\frac{j_x}{d_x}$, where d is the homogeneous degree of P. Then, $\partial_x^{j_x} P^{\frac{j}{d_x}}$ is homogeneous of homogeneous degree $d\frac{j-j_x}{d_x} > 0$, so that it may be extended by continuity at 0.

Therefore, $\partial_x^{j_x} P^{\frac{j_x}{d_x}}$ is a continuous function on E which is homogeneous of degree 0; hence, it is constant, and its constant value must be $j_x! \neq 0$. Now, Faà di Bruno's formula shows that

$$P^{1-\frac{r_x}{d_x}} = \frac{1}{j_x!} P^{1-\frac{r_x}{d_x}} \partial_x^{j_x} P^{\frac{j_x}{d_x}} = \sum_{\sum_{\ell=1}^{j_x} \ell\beta_\ell = j_x} \frac{1}{\beta!} \left(\frac{j_x}{d_x}\right)_{|\beta|} P^{1+q_x-|\beta|} \prod_{\ell=1}^{j_x} \left(\frac{\partial_x^\ell P}{\ell!}\right)^{\beta_\ell},$$

where $\left(\frac{j_x}{d_x}\right)_{|\beta|} \coloneqq \frac{j_x}{d_x}\left(\frac{j_x}{d_x} - 1\right) \cdots \left(\frac{j_x}{d_x} - |\beta| + 1\right)$ is the Pochhammer symbol. Then, $P^{1-\frac{r_x}{d_x}}$ is a rational function, so that there are $N, D \in \mathbb{R}[A]$, with $D \neq 0$, such that $P^{1-\frac{r_x}{d_x}} = \frac{N}{D}$. Hence, $P^{d_x - r_x} = \frac{N^{d_x}}{D^{d_x}}$, so that D^{d_x} divides N^{d_x} in $\mathbb{R}[A]$. Since $\mathbb{R}[A]$ is factorial, it follows that D divides N, so that $P^{1-\frac{r_x}{d_x}}$ is a (positive) polynomial. Next, let g be the greatest common divisor of d_x and $d_x - r_x$, and take $d', r' \in \mathbb{N}^*$ so that $d_x = g\, d'$ and $d_x - r_x = g\, r'$. Then,

$$\left(P^{1-\frac{r_x}{d_x}}\right)^{d'} = P^{r'}.$$

Since $\mathbb{R}[A]$ is factorial, this proves that there is a polynomial $Q \in \mathbb{R}[A]$ such that $Q^{r'} = P^{1-\frac{r_x}{d_x}}$ and $Q^{d'} = P$. Now, $d' \geqslant 2$ since d_x does not divide $d_x - r_x$; in addition, Q is positive since both $P^{1-\frac{r_x}{d_x}}$ and

[7] Notice that λx denotes the scalar multiplication of x by λ, *not* the dilate $\lambda \cdot x$ of x by λ, which is meaningful only for $\lambda > 0$.

[8] Here, $|\lambda|$ denotes the usual absolute value of $\lambda \in \mathbb{R}$.

P are positive and d', r' are coprime: contradiction. Therefore, $a_{x,j} = 0$ for every $j \notin d_x \mathbb{N}$, so that the conclusion follows easily.

2 $\implies$ 1. Suppose by contradiction that there are a positive homogeneous polynomial $Q \in \mathbb{R}[A]$ and $k \geqslant 2$ such that $P = Q^k$. Define $m \colon \lambda \mapsto \lambda^{\frac{1}{k}}$ on $\mathbb{R}_+$. Then, m is not right-differentiable at 0; nevertheless, $m \circ P = Q$ since Q is positive, so that $m \circ P$ is of class C^∞: contradiction. □

Proof of Theorem 3.2. Notice that $\chi_{\mathcal{L}}(\lambda, \cdot\,)$ is an eigenfunction of positive type and of class C^∞ of $\mathcal{L}$, with eigenvalue λ, and that $\chi_{\mathcal{L}}(r \cdot \lambda, g) = \chi_{\mathcal{L}}(\lambda, r \cdot g)$ for every $r > 0$ and for $(\beta_{\mathcal{L}} \otimes \nu_G)$-almost every (λ, g) (cf. [39]). It is then easily seen that $\chi_{\mathcal{L}}$ has a continuous representative which is of class C^∞ on $\mathbb{R}_+^* \times G$.

Now, take $m \in C_b(\beta_{\mathcal{L}})$ such that $\mathcal{K}_{\mathcal{L}}(m) \in \mathcal{S}(G)$. Then, Proposition 3.1 implies that $m \circ P \in \mathcal{S}(E_{-i\partial})$. Take a positive polynomial Q on $E_{-i\partial}$ such that $P = Q^k$. Since $[m \circ (\,\cdot\,)^k] \circ Q = m \circ P$, Lemma 3.3 implies that we may take $\widetilde{m} \in \mathcal{E}(\mathbb{R})$ so that $m \circ (\,\cdot\,)^k = \widetilde{m}$ on $\mathbb{R}_+$. In addition, it is clear that we may assume that $\widetilde{m} \in \mathcal{S}(\mathbb{R})$. Now, let $\sum_{\ell \in \mathbb{N}} a_\ell \lambda^\ell$ be the Taylor development of $\widetilde{m}$ at 0. Take, for $h = 1, \dots, k-1$, $m_h \in \mathcal{D}(\mathbb{R})$ so that its Taylor development at 0 is $\sum_{\ell \in \mathbb{N}} a_{h+k\ell} \lambda^\ell$ (cf. [26, Theorem 1.2.6]), and define $m_0 \coloneqq m - \sum_{h=1}^{k-1} (\,\cdot\,)^{\frac{h}{k}} m_h$ on $\mathbb{R}_+$. Since clearly m_0 has the asymptotic development $\sum_{\ell \in \mathbb{N}} a_{k\ell} \lambda^\ell$ for $\lambda \to 0$, and since $m_0 \circ (\,\cdot\,)^k$ equals a Schwartz function on $\mathbb{R}_+$, it is easily seen that m_0 may be extended to an element of $\mathcal{S}(\mathbb{R})$. Therefore, $m(\lambda) = \sum_{h=0}^{k-1} \lambda^{\frac{h}{k}} m_h(\lambda)$ for every $\lambda \geqslant 0$.

Conversely, suppose that there are $m_0, \dots, m_{k-1} \in \mathcal{S}(\mathbb{R})$ such that $m(\lambda) = \sum_{h=0}^{k-1} \lambda^{\frac{h}{k}} m_h(\lambda)$ for every $\lambda \geqslant 0$. Then, $m \circ P \in \mathcal{S}(E_{-i\partial})$, so that $\mathcal{K}_{\mathcal{L}}(m) \in \mathcal{S}(G)$ by Proposition 3.1. □

Corollary 3.4. *Let $L \colon \mathbb{R}^n \to \mathbb{R}^{n'}$ be a linear mapping which is proper on $\mathbb{R}^n_+$. Then, $L(-\partial_1^2, \dots, -\partial_n^2)$ satisfies properties (RL) and (S).*

Proof. This is a consequence of Theorems 2.2 and 3.2 when L is the identity. The general case then follows by means of Proposition 2.10. □

4 MW^+ Groups

Definition 4.1. Let G be a 2-step stratified group, that is, a simply connected Lie group whose Lie algebra is decomposed as $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}]$ and $[\mathfrak{g}, \mathfrak{g}_2] = 0$. For every $\omega \in \mathfrak{g}_2^*$, define

$$B_\omega \colon \mathfrak{g}_1 \times \mathfrak{g}_1 \ni (X, Y) \mapsto \langle \omega, [X, Y] \rangle\,.$$

We say that G is an MW^+ group if B_ω is non-degenerate for some $\omega \neq 0$. A Heisenberg group is an MW^+ group with one-dimensional centre.

Notice that a 2-step stratified group satisfies property MW^+ if and only if it satisfies the Moore-Wolf condition (cf. [35]) and $[\mathfrak{g}, \mathfrak{g}]$ is the centre of $\mathfrak{g}$.

We shall endow a 2-step stratified group with the canonical dilations, so that

$$r \cdot (X + Y) = rX + r^2 Y$$

for every $r > 0$, for every $X \in \mathfrak{g}_1$ and for every $Y \in \mathfrak{g}_2$. Since $\exp_G \colon \mathfrak{g} \to G$ is a diffeomorphism, these dilations transfer to G.

Now, to every symmetric bilinear form Q on $\mathfrak{g}_1^*$ we associate a differential operator on G as follows:

$$\mathcal{L} \coloneqq -\sum_{\ell,\ell'} Q(X_\ell^*, X_{\ell'}^*) X_\ell X_{\ell'},$$

where (X_ℓ) is a basis of $\mathfrak{g}_1$ with dual basis (X_ℓ^*). As the reader may verify, $\mathcal{L}$ does not depend on the choice of (X_ℓ); actually, one may prove that $-\mathcal{L}$ is the symmetrization of the quadratic form induced by Q on $\mathfrak{g}^*$ (cf. [24, Theorem 4.3]).

By a 'sum of squares' we mean a differential operator of the form $\mathcal{L} = -\sum_{j=1}^k Y_j^2$, where $Y_1, \dots, Y_k$ are elements of $\mathfrak{g}$. If, in addition, $Y_1, \dots, Y_k$ generate $\mathfrak{g}$ as a Lie algebra, then we say that $\mathcal{L}$ is a sub-Laplacian. Thanks to [25], this is equivalent to saying that $\mathcal{L}$ is hypoelliptic.

Lemma 4.2. *Let Q be a symmetric bilinear form on $\mathfrak{g}_1^*$, and let $\mathcal{L}$ be the associated operator. Then, $\mathcal{L}$ is formally self-adjoint if and only if Q is real. In addition, $\mathcal{L}$ is formally self-adjoint and hypoelliptic if and only if Q is non-degenerate and either positive or negative.*

Definition 4.3. Let V be a finite-dimensional vector space and Φ a bilinear form on V. Then, define

$$\mathrm{s}_\Phi \colon V \ni v \mapsto \Phi(v, \cdot\,) \in V^* \qquad \text{and} \qquad \mathrm{d}_\Phi \colon V \ni v \mapsto \Phi(\,\cdot\,, v) \in V^*.$$

Recall that, if Φ is non-degenerate, then its inverse $\widehat{\Phi}$ is the bilinear form $\Phi \circ (\mathrm{s}_\Phi^{-1}, \mathrm{d}_\Phi^{-1})$ on V^* (cf. [10, Definition 8 of Chapter IX, §1, No. 7]).

Definition 4.4. G is a group of Heisenberg type, or an H-type group, if $\mathfrak{g}_2 \neq 0$, $\mathfrak{g}$ is endowed with a scalar product for which $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are orthogonal, and, denoting by Q the scalar product induced on $\mathfrak{g}_1^*$, we have $(\mathrm{d}_Q \circ \mathrm{d}_{B_\omega})^2 = -|\omega|^2 \mathrm{id}_{\mathfrak{g}_1}$ for every $\omega \in \mathfrak{g}_2^*$.

Proposition 4.5. *Let Q_1 and Q_2 be two symmetric bilinear forms on $\mathfrak{g}_1^*$, and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the associated operators. Then, $\mathcal{L}_1$ and $\mathcal{L}_2$ commute if and only if*

$$\mathrm{d}_{Q_1} \circ \mathrm{d}_{B_\omega} \circ \mathrm{d}_{Q_2} = \mathrm{d}_{Q_2} \circ \mathrm{d}_{B_\omega} \circ \mathrm{d}_{Q_1}$$

for every $\omega \in \mathfrak{g}_2^*$.

Proof. Choose a basis $(X_j)_{j \in J}$ of $\mathfrak{g}_1$ and a basis $(T_k)_{k \in K}$ of $\mathfrak{g}_2$. Let $(X_j^*)_{j \in J}$ and $(T_k^*)_{k \in K}$ be the corresponding dual bases. Define $a_{h,j_1,j_2} \coloneqq Q_h(X^*_{j_1}, X^*_{j_2})$ for $h = 1,2$ and for every $j_1, j_2 \in J$, so that d_{Q_h} is identified with the matrix $A_h \coloneqq (a_{h,j_1,j_2})_{j_1,j_2\in J}$ for $h = 1,2$. Analogously, define $b_{k,j_1,j_2} \coloneqq B_{T_k^*}(X_{j_1}, X_{j_2})$ for every $k \in K$ and for every $j_1, j_2 \in J$, so that $\mathrm{d}_{B_{T_k^*}}$ is identified with the matrix $B_k \coloneqq (b_{k,j_1,j_2})_{j_1,j_2\in J}$ for every $k \in K$. Now, define $Y_{j_1,j_2} \coloneqq \frac{1}{2}(X_{j_1}X_{j_2} + X_{j_2}X_{j_1})$ for every $j_1, j_2 \in J$. Then,

$$\mathcal{L}_h = \sum_{j_1,j_2\in J} a_{h,j_1,j_2} Y_{j_1,j_2}$$

since Q_h is symmetric. In addition, for every $j_1, j_2, j_3, j_4 \in J$,

$$[Y_{j_1,j_2}, Y_{j_3,j_4}] = Y_{j_2,j_4}[X_{j_1}, X_{j_3}] + Y_{j_2,j_3}[X_{j_1}, X_{j_4}] + Y_{j_1,j_4}[X_{j_2}, X_{j_3}] + Y_{j_1,j_3}[X_{j_2}, X_{j_4}]$$

since the elements of $\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1]$ lie in the centre of $\mathfrak{U}(\mathfrak{g})$. Next, observe that, for every $j_1, j_2 \in J$,

$$[X_{j_1}, X_{j_2}] = \sum_{k\in K} b_{k,j_1,j_2} T_k.$$

Therefore,

$$\begin{aligned}[\mathcal{L}_1, \mathcal{L}_2] &= \sum_{j_1,j_2,j_3,j_4\in J} \sum_{k\in K} a_{1,j_1,j_2} a_{2,j_3,j_4} [b_{k,j_1,j_3} Y_{j_2,j_4} + b_{k,j_1,j_4} Y_{j_2,j_3} + b_{k,j_2,j_3} Y_{j_1,j_4} + b_{k,j_2,j_4} Y_{j_1,j_3}] T_k \\ &= 2 \sum_{j_1,j_2\in J} \sum_{k\in K} c_{k,j_1,j_2} Y_{j_1,j_2} T_k,\end{aligned}$$

where

$$c_{k,j_1,j_2} = \sum_{j_3,j_4\in J} (a_{1,j_1,j_3} a_{2,j_2,j_4} + a_{1,j_2,j_3} a_{2,j_1,j_4}) b_{k,j_3,j_4}$$

for every $k \in K$ and for every $j_1, j_2 \in J$. Now, the distinct monomials in the family of the $Y_{j_1,j_2}T_k$, as $j_1, j_2 \in J$ and $k \in K$, are linearly independent (cf., for example, [14, Corollary 4 to Theorem 1 of Chapter I, § 2, No. 7]). In addition, denote by C_k the matrix $(c_{k,j_1,j_2})_{j_1,j_2\in J}$ for every $k \in K$. Since A_1 and A_2 are symmetric and since B_k is skew-symmetric, we have

$$C_k = A_1 B_k A_2 + {A_2}^t B_k A_1 = A_1 B_k A_2 - A_2 B_k A_1$$

for every $k \in K$. The assertion follows easily. □

Now we shall present some results which will enable us to put our homogeneous sub-Laplacians in a particularly convenient form. We state them in terms of the associated quadratic forms.

Proposition 4.6. *Let (V, σ) be a finite-dimensional symplectic vector space over $\mathbb{R}$. Let $(Q_\iota)_{\iota\in I}$ be a family of positive, non-degenerate bilinear forms on V such that the $\mathrm{d}_{Q_\iota}^{-1} \circ \mathrm{d}_\sigma$, as ι runs through I, commute.*

Then, there is a finite family $(P_\gamma)_{\gamma\in\Gamma}$ of projectors of V such that the following hold:

- *P_γ is σ- and Q_ι-self-adjoint for every $\iota \in I$ and for every $\gamma \in \Gamma$;*
- *$\mathrm{id}_V = \sum_{\gamma\in\Gamma} P_\gamma$ and $P_\gamma P_{\gamma'} = 0$ for $\gamma, \gamma' \in \Gamma$, $\gamma \neq \gamma'$;*
- *the bilinear forms $Q_\iota(P_\gamma \cdot\,, P_\gamma \cdot\,)$, as $\iota \in I$, are all multiples of one another for every $\gamma \in \Gamma$.*

For the proof, basically follow that of [28, Theorem 3.1 (c)] using commutativity in order to get simultaneous diagonalizations. Applying [28, Theorem 3.1 (c)] (or simply [1, Corollary 5.6.3]) to the range of each P_γ, we may find a symplectic basis of V which is Q_ι-orthogonal for every $\iota \in I$.

Notation 4.7. From now on, G will denote an MW^+ group, $(Q_\iota)_{\iota\in I}$ a family of positive symmetric bilinear forms on $\mathfrak{g}_1^*$, and $(T_1, \dots, T_{n_2})$ a basis of $\mathfrak{g}_2$. Notice that, since G is an MW^+ group, there is $n_1 \in \mathbb{N}^*$ such that $\dim \mathfrak{g}_1 = 2n_1$. We shall denote by $\mathcal{L}_\iota$ the sum of squares induced by Q_ι, and we shall assume that $\mathcal{L}_A \coloneqq ((\mathcal{L}_\iota)_{\iota\in I}, (-iT_k)_{k=1,\dots,n_2})$ is a Rockland family. Observe that this condition is equivalent to the fact that the sum of the $\mathcal{L}_\iota$ is hypoelliptic.[9] We may therefore assume that Q_ι is non-degenerate for every $\iota \in I$, in which case each $\mathcal{L}_\iota$ is a (homogeneous) sub-Laplacian.

We shall also endow $\mathfrak{g}$ with a scalar product which turns $\mathfrak{g}_1$ and $\mathfrak{g}_2$ into orthogonal subspaces, and which induces $\widehat{Q}_{\iota_0}$ on $\mathfrak{g}_1$ for some *fixed* $\iota_0 \in I$. Up to a normalization, *we may then assume that $(\exp_G)_*(\mathcal{H}^n)$ is the chosen Haar measure on G*, where n is the dimension of G. We endow $\mathfrak{g}_2^*$ with the scalar product induced by that of $\mathfrak{g}_2$, and then with the corresponding Lebesgue measure, that is, $\mathcal{H}^{n_2}$.

Proposition 4.8. *There is a finite family $(P_\gamma)_{\gamma\in\Gamma}$ of non-zero projectors of $\mathfrak{g}_1$ such that the following hold:*

- *$\mathrm{id}_{\mathfrak{g}_1} = \sum_{\gamma\in\Gamma} P_\gamma$ and $P_{\gamma_1} P_{\gamma_2} = 0$ for every $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1 \neq \gamma_2$;*
- *P_γ is B_ω- and $\widehat{Q}_\iota$-self-adjoint for every $\gamma \in \Gamma$, for every $\omega \in \mathfrak{g}_2^*$, and for every $\iota \in I$;*
- *for every $\gamma \in \Gamma$, the bilinear forms $Q_\iota\left({}^tP_\gamma \cdot\,, {}^tP_\gamma \cdot\right)$, as ι runs through I, are mutually proportional.*

Proof. Fix $\omega_0 \in \mathfrak{g}_2^*$ such that B_{ω_0} is non-degenerate. Then, Proposition 4.6 and the remarks which follow its statement imply that there is a basis $X_1, \dots, X_{2n_1}$ of $\mathfrak{g}_1$ such that $\mathrm{d}_{B_{\omega_0}}$ and d_{Q_ι} are represented by the matrices

$$\begin{pmatrix} 0 & I_{n_1} \\ -I_{n_1} & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} D_\iota & 0 \\ 0 & D_\iota \end{pmatrix},$$

respectively, for some diagonal matrix D_ι ($\iota \in I$). Here, I_{n_1} denotes the identity matrix of order n_1. Denote by $d_{\iota,1}, \dots, d_{\iota,n_1}$ the diagonal elements of D_ι, and denote by $(a_{\omega,j,k})$ the matrix associated with

[9] Indeed, if π_0 is the projection of G onto its abelianization, then $\mathrm{d}\pi_0(\mathcal{L}_A)$ is a Rockland family, so that $\mathcal{F}(\mathrm{d}\pi_0(\mathcal{L}_A))$ vanishes only at 0. Since $\mathcal{F}(\mathrm{d}\pi_0(\mathcal{L}_\iota)) \geqslant 0$ and $\mathrm{d}\pi_0(T_k) = 0$ for every $\iota \in I$ and for every $k = 1, \dots, n_2$, this implies that $\sum_{\iota\in I} \mathcal{F}(\mathrm{d}\pi_0(\mathcal{L}_\iota))$ vanishes only at 0, so that $\sum_{\iota\in I} Q_\iota$ is positive and non-degenerate and $\sum_{\iota\in I} \mathcal{L}_\iota$ is hypoelliptic.

d_{B_ω}, for every non-zero $\omega \in \mathfrak{g}_2^*$. Assume that I has exactly two elements ι_1, ι_2, and define

$$\Gamma \coloneqq \left\{ \frac{d_{\iota_1,j}}{d_{\iota_2,j}} : j \in \{1, \dots, n_1\} \right\};$$

for every $\gamma \in \Gamma$, let V_γ be the vector subspace of $\mathfrak{g}_1$ generated by the set

$$\left\{ X_j, X_{n_1+j} : \frac{d_{\iota_1,j}}{d_{\iota_2,j}} = \gamma \right\}.$$

Next, take $j, k \in \{1, \dots, n_1\}$ such that $\frac{d_{\iota_1,j}}{d_{\iota_2,j}} \neq \frac{d_{\iota_1,k}}{d_{\iota_2,k}}$. Apply Proposition 4.5, and observe that the (j,k)-th components of (the matrices representing) the equality

$$\mathrm{d}_{Q_{\iota_1}} \circ \mathrm{d}_{B_\omega} \circ \mathrm{d}_{Q_{\iota_2}} = \mathrm{d}_{Q_{\iota_2}} \circ \mathrm{d}_{B_\omega} \circ \mathrm{d}_{Q_{\iota_1}}$$

give the equality

$$d_{\iota_1,j} a_{\omega,j,k} d_{\iota_2,k} = d_{\iota_2,j} a_{\omega,j,k} d_{\iota_1,k},$$

whence $a_{\omega,j,k} = 0$. Considering the components $(n_1+j, k), (j, n_1+k)$, and (n_1+j, n_1+k), we see that $a_{\omega,n_1+j,k} = a_{\omega,j,n_1+k} = a_{\omega,n_1+j,n_1+k} = 0$. Therefore, $B_\omega(V_{\gamma_1}, V_{\gamma_2}) = \{0\}$ for every non-zero $\omega \in \mathfrak{g}_2^*$ and for every $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1 \neq \gamma_2$. Then, it suffices to define P_γ as the projector of $\mathfrak{g}_1$ onto V_γ with kernel $\bigoplus_{\gamma' \neq \gamma} V_{\gamma'}$. The general case follows easily. ☐

Definition 4.9. Define

$$J_{Q_\iota,\omega} \coloneqq \mathrm{d}_{Q_\iota} \circ \mathrm{d}_{B_\omega} : \mathfrak{g}_1 \to \mathfrak{g}_1$$

for every $\iota \in I$ and for every $\omega \in \mathfrak{g}_2^*$.

We shall denote by W the set of $\omega \in \mathfrak{g}_2^*$ such that B_ω is degenerate, that is, the set where the polynomial mapping $\omega \mapsto \det J_{\iota_0,\omega}$ vanishes. As a consequence, W is an algebraic variety.

Definition 4.10. Given two non-empty finite sets H_1, H_2, we shall often identify $(\mathbb{R}^{H_1})^{H_2}$ with $\mathbb{R}^{H_2 \times H_1}$, so that, for $S \in (\mathbb{R}^{H_1})^{H_2}$, $h_1 \in H_1$ and $h_2 \in H_2$, we shall denote by S_{h_2,h_1} the h_1-th component of the h_2-component S_{h_2} of S; thus, S_{h_2,h_1} is an abuse of notation for $(S_{h_2})_{h_1}$. Hence, we shall identify the elements S of $(\mathbb{R}^{H_1})^{H_2}$ with the corresponding matrices (S_{h_2,h_1}) of type $H_2 \times H_1$; we shall then let them act on elements of $\mathbb{R}^{H_1}$ in the usual way, so that $S(v) = \left(\sum_{h_1 \in H_1} S_{h_2,h_1} v_{h_1}\right)_{h_2 \in H_2}$ for every $S \in (\mathbb{R}^{H_1})^{H_2}$ and for every $v \in \mathbb{R}^{H_1}$. Notice that this notation is self-consistent: if $S \in (\mathbb{R}^{H_1})^{H_2}$, then S is identified with the linear mapping $\mathbb{R}^{H_1} \to \mathbb{R}^{H_2}$ whose h_2-th component is S_{h_2} for every $h_2 \in H_2$.

If H is a finite set, then we denote by $\mathbf{1}_H$ the element of $\mathbb{R}^H$ whose components are all 1.

Proposition 4.11. *There are a non-empty finite set H, a non-empty Zariski open subset Ω of $\mathfrak{g}_2^* \setminus W$,* *and two mappings*

$$\mu\colon \mathfrak{g}_2^* \to \left(\mathbb{R}_+^H\right)^I \qquad \text{and} \qquad P\colon \Omega \to \mathcal{L}(\mathfrak{g}_1)^H$$

such that the following hold:

- *Ω is dilation-invariant, μ is homogeneous of degree 1, and P is homogeneous of degree 0;*
- *μ is continuous on $\mathfrak{g}_2^*$ and analytic on Ω; in addition, $\mu(\omega) \in \left((\mathbb{R}_+^*)^H\right)^I$ for every $\omega \in \Omega$;*
- *P_h is analytic and $\operatorname{Tr} P_h$ is non-zero and constant on Ω for every $h \in H$;*
- *for every $h \in H$ and for every $\omega \in \Omega$, $P_h(\omega)$ is a B_ω- and $\widehat{Q}_I$-self-adjoint projector of $\mathfrak{g}_1$;*
- *$P_{h_1} P_{h_2} = 0$ for every $h_1, h_2 \in H$ such that $h_1 \neq h_2$;*
- *$\sum_{h\in H} P_h(\omega) = \mathrm{id}_{\mathfrak{g}_1}$ and $\sum_{h\in H} \mu_{\iota,h}(\omega) P_h(\omega) = |J_{Q_\iota,\omega}|$ for every $\omega \in \Omega$ and for every $\iota \in I$.*

Notice that, for the sake of simplicity, we do *not* assume that $(\mu_{\iota,h})_{\iota\in I} \neq (\mu_{\iota,h'})_{\iota\in I}$ for $h, h' \in H$ such that $h \neq h'$; besides that, this condition is completely irrelevant for our purposes.

Proof. The assertion follows from [33, Lemmas 4 and 5] when $\operatorname{Card}(I) = 1$. Assume that $\operatorname{Card}(I) > 1$ and apply Proposition 4.8. Then, there is a finite family $(P'_\gamma)_{\gamma\in\Gamma}$ of commuting projectors of $\mathfrak{g}_1$ which are B_ω- and $\widehat{Q}_\iota$-self-adjoint for every $\omega \in \mathfrak{g}_2^*$ and for ever $\iota \in I$; in addition, $\sum_{\gamma\in\Gamma} P'_\gamma = \mathrm{id}_{\mathfrak{g}_1}$, and, for every $\gamma \in \Gamma$ and for every $\iota \in I$, there is $c_{\gamma,\iota} > 0$ such that $Q_\iota = c_{\gamma,\iota} Q_{\iota_0}$ on ${}^t P'_\gamma(\mathfrak{g}_1^*)^2$.

Then, apply the case $\operatorname{Card}(I) = 1$ to the MW^+ group $\exp_G(P'_\gamma(\mathfrak{g}_1) \oplus \mathfrak{g}_2)$ and to the operator $\mathcal{L}_{\iota_0}^{(\gamma)}$ associated with the restriction of Q_{ι_0} to ${}^t P'_\gamma(\mathfrak{g}_1^*)^2$, for every $\gamma \in \Gamma$. We then find two mappings $\mu_{\iota_0}^{(\gamma)}\colon \mathfrak{g}_2^* \to \mathbb{R}_+^{H^{(\gamma)}}$ and $P^{(\gamma)}\colon \Omega^{(\gamma)} \to \mathcal{L}(P'_\gamma(\mathfrak{g}_1))^{H^{(\gamma)}}$ with the properties of the statement. Observe that, if we define $\mu_\iota^{(\gamma)} \coloneqq c_{\gamma,\iota}\mu_{\iota_0}^{(\gamma)}$, then $\mu_\iota^{(\gamma)}$ and $P^{(\gamma)}$ satisfy the properties of the statement for the operator $\mathcal{L}_\iota^{(\gamma)}$ associated with the restriction of Q_ι to ${}^t P'_\gamma(\mathfrak{g}_1^*)^2$. Now, define $\Omega \coloneqq \bigcap_{\gamma\in\Gamma} \Omega^{(\gamma)}$ and $H \coloneqq \bigcup_{\gamma\in\Gamma} \left(\{\gamma\} \times H^{(\gamma)}\right)$. Then, we may define $(\mu_\iota)_{(\gamma,h)} \coloneqq \left(\mu_\iota^{(\gamma)}\right)_h$ for every $\iota \in I$ and for every $(\gamma, h) \in H$. In addition, we may define $P_{(\gamma,h)}(\omega)$ as the composition of $P_{h'}^{(\gamma)}(\omega) P'_\gamma$ with the canonical inclusion of $P'_\gamma(\mathfrak{g}_1)$ in $\mathfrak{g}_1$, for every $\omega \in \Omega$ and for every $(\gamma, h) \in H$. The assertion follows. □

Definition 4.12. We define Ω, H, μ and P as in Proposition 4.11. In addition, we define $\mathbf{n_1} = (n_{1,h})_{h\in H} \in (\mathbb{N}^*)^H$ in such a way that $n_{1,h}$ is the constant value of $\operatorname{Tr} P_h$ on Ω, for every $h \in H$. Furthermore, denote by $\widetilde{\mu}\colon \mathfrak{g}_2^* \to (\mathbb{R}_+^{n_1})^I$ a continuous mapping which is analytic on Ω and such that, for every $\omega \in \mathfrak{g}_2^*$, $\pm i(\widetilde{\mu}_{\iota,1})_\iota, \dots, \pm i(\widetilde{\mu}_{\iota,n_1})_\iota$ are the joint eigenvalues of $(J_{Q_\iota,\omega})_\iota$, for every $\omega \in \mathfrak{g}_2^*$.[10]

[10] The existence of a mapping $\widetilde{\mu}$ with the required properties follows from the existence of μ and the fact that $\operatorname{Tr} P_h$ is constant on Ω for every $h \in H$. Even though $\widetilde{\mu}$ and μ are essentially the same thing, in some situations it will be convenient to work with $\widetilde{\mu}$ instead of μ.

By an abuse of notation, we shall denote by (x,t) the elements of G, where $x \in \mathfrak{g}_1$ and $t \in \mathfrak{g}_2$, thus identifying (x,t) with $\exp_G(x,t)$. For every $x \in \mathfrak{g}_1$, for every $\omega \in \Omega$, and for every $h \in H$, define

$$x_h(\omega) \coloneqq \sqrt{\mu_{\iota_0,h}(\omega)} P_h(\omega)(x).$$

We shall then define $x(\omega) \coloneqq \sum_{h\in H} x_h(\omega)$, so that $|x(\omega)|^2 = \sum_{h\in H} |x_h(\omega)|^2 = \langle |J_{Q_{\iota_0},\omega}|(x)|x\rangle$.

The following two results are easy and their proof is omitted.

Proposition 4.13. *The function $\omega \mapsto \mu_\iota(\omega)(\mathbf{n_1}) = \widetilde{\mu}_\iota(\omega)(\mathbf{1}_{n_1}) = \frac{1}{2}\|J_{Q_\iota,\omega}\|_1$ is a norm on $\mathfrak{g}_2^*$ which is analytic on $\mathfrak{g}_2^* \setminus W$ for every $\iota \in I$.*[11]

Proposition 4.14. *The mapping*

$$\mathfrak{g}_1 \times \Omega \ni (x,\omega) \mapsto x(\omega) = \sqrt[4]{-J^2_{Q_{\iota_0},\omega}}(x)$$

extends uniquely to a continuous function on $\mathfrak{g}_1 \times \mathfrak{g}_2^*$ which is analytic on $\mathfrak{g}_1 \times (\mathfrak{g}_2^* \setminus W)$.

Definition 4.15. Define G_ω, for every $\omega \in \mathfrak{g}_2^*$, as the quotient of G by its normal subgroup $\exp_G(\ker\omega)$; we denote by π_ω the canonical projection of G onto G_ω.

Then, G_0 is the abelianization of G, and we identify it with $\mathfrak{g}_1$. If $\omega \neq 0$, then we shall identify G_ω with $\mathfrak{g}_1 \oplus \mathbb{R}$, endowed with the product

$$(x_1,t_1)(x_2,t_2) \coloneqq \left(x_1 + x_2, t_1 + t_2 + \frac{1}{2} B_\omega(x_1,x_2)\right)$$

for every $x_1, x_2 \in \mathfrak{g}_1$ and for every $t_1, t_2 \in \mathbb{R}$. Hence,

$$\pi_\omega(x,t) = (x, \omega(t))$$

for every $(x,t) \in G$.

Proposition 4.16. *Define*

$$\widetilde{\pi} \colon \bigcup_{\omega\in\Omega} (\{\omega\} \times G_\omega) \ni (\omega,(x,t)) \mapsto \omega \in \Omega,$$

and identify the domain of $\widetilde{\pi}$ with $\Omega \times (\mathfrak{g}_1 \oplus \mathbb{R})$ as an analytic manifold, so that $\widetilde{\pi}$ becomes an analytic submersion.

Then, $\widetilde{\pi}$ defines a fibre bundle with base Ω and fibres isomorphic to $\mathbb{H}^{n_1}$.[12] *More precisely, for every $\omega_0 \in \Omega$, there is an analytic trivialization (U,ψ) of $\widetilde{\pi}$ such that the following hold:*

[11] Here, $\|J_{Q_\iota,\omega}\|_1$ denotes the trace-norm of the endomorphism $J_{Q_\iota,\omega}$, that is, $\mathrm{Tr}|J_{Q_\iota,\omega}|$.

[12] We denote by $\mathbb{H}^{n_1}$ the $(2n_1+1)$-dimensional Heisenberg group, identified with $\mathbb{R}^{2n_1} \times \mathbb{R}$ with product $(x,t)(x',t') \coloneqq \left(x + x', t + t' + \frac{1}{2}\sum_{j=1}^{n_1}(x_j x'_{n_1+j} - x_{n_1+j} x'_j)\right)$.

- *U is an open neighbourhood of ω_0 in Ω;*

- *$\psi\colon \widetilde{\pi}^{-1}(U) \to U \times \mathbb{H}^{n_1}$ is an analytic diffeomorphism such that $\mathrm{pr}_1 \circ \psi = \widetilde{\pi}$ and such that $\mathrm{pr}_2 \circ \psi$ induces a group isomorphism $\psi_\omega\colon \widetilde{\pi}^{-1}(\omega) \to \mathbb{H}^{n_1}$ for every $\omega \in U$;*

- *if $(X_1, \dots, X_{2n_1}, T)$ is a basis of left-invariant vector fields on $\mathbb{H}^{n_1}$ which induce the partial derivatives along the coordinate axes at the origin, then*

$$\mathrm{d}(\psi_\omega \circ \pi_\omega)(\mathcal{L}_\iota) = -\sum_{k=1}^{n_1} \widetilde{\mu}_{\iota,k}(\omega)(X_k^2 + X_{n_1+k}^2)$$

and

$$\mathrm{d}(\psi_\omega \circ \pi_\omega)(T_j) = \omega(T_j)T$$

for every $\iota \in I$, for every $j = 1, \dots, n_2$, and for every $\omega \in U$.

The proof is omitted. It basically consists in using the projectors P_h to propagate locally a given basis of eigenvectors and then in 'symplectifying' the new basis in order to meet the requirements.

Definition 4.17. For every $\omega \in \mathfrak{g}_2^* \setminus W$, define the Pfaffian of ω as follows (cf. [3]):

$$|\mathrm{Pf}(\omega)| \coloneqq \prod_{h\in H} \mu_{\iota_0,h}(\omega)^{n_{1,h}} = \prod_{k=1}^{n_1} \widetilde{\mu}_{\iota_0,k}(\omega)$$

Furthermore, take $m, \gamma \in \mathbb{N}$. Then, denote by Λ_γ^m the γ-th Laguerre polynomial of order m. In other words, $\Lambda_\gamma^m(X) = \sum_{j=0}^{\gamma} \binom{\gamma+m}{\gamma-j} \frac{(-X)^j}{j!}$.

Proposition 4.18. *Define $\Sigma_\omega \coloneqq \mu(\omega)\left(\mathbf{n_1} + 2\mathbb{N}^H\right)$ for every $\omega \in \Omega$; then, for every $\varphi \in \mathcal{D}^0(E_{\mathcal{L}_A})$,*

$$\int_{E_{\mathcal{L}_A}} \varphi \,\mathrm{d}\beta_{\mathcal{L}_A} = \frac{1}{(2\pi)^{n_1+n_2}} \int_\Omega \sum_{\gamma\in\Sigma_\omega} c_{\gamma,\omega} \varphi(\gamma, \omega(\mathbf{T})) |\mathrm{Pf}(\omega)| \,\mathrm{d}\omega,$$

where

$$c_{\gamma,\omega} \coloneqq \sum_{\substack{\gamma'\in\mathbb{N}^H \\ \mu(\omega)(\mathbf{n_1}+2\gamma')=\gamma}} \binom{\mathbf{n_1} + \gamma' - \mathbf{1}_H}{\gamma'}$$

for every $\omega \in \Omega$ and for every $\gamma \in \Sigma_\omega$.

In addition, for $(\beta_{\mathcal{L}_A} \otimes \nu_G)$-almost every $((\gamma, \omega(\mathbf{T})), (x, t))$,

$$\chi_{\mathcal{L}_A}((\gamma, \omega(\mathbf{T})), (x, t)) = \frac{1}{c_{\gamma,\omega}} \sum_{\mu(\omega)(\mathbf{n_1}+2\gamma')=\gamma} e^{-\frac{1}{4}|x(\omega)|^2 - i\omega(t)} \prod_{h\in H} \Lambda_{\gamma'_h}^{n_{1,h}-1}\left(\frac{1}{2}|x_h(\omega)|^2\right).$$

Proof. We follow the construction of the Plancherel measure of [3] as in [32, 4.4.1]. Observe first that, for every $\omega \in \mathfrak{g}_2^* \setminus W$ there is (up to unitary equivalence) a unique irreducible unitary representation ϖ_ω of G in a hilbertian space H_ω such that $\pi_\omega(0, t) = e^{i\omega(t)}\mathrm{id}_{H_\omega}$: indeed, any such representation must be of the form $\widetilde{\varpi}_\omega \circ \pi_\omega$ for some irreducible unitary representation of the Heisenberg group G_ω (cf. Definition 4.15), so that the assertion follows from the Stone–Von Neumann theorem (cf. [21, (6.49)]). Then, [3, Section 2] implies that, for every $f \in L^1(G) \cap L^2(G)$,[13]

$$\|f\|_2^2 = \frac{1}{(2\pi)^{n_1+n_2}} \int_\Omega \|\varpi_\omega(f)\|_2^2 |\mathrm{Pf}(\omega)| \,\mathrm{d}\omega.$$

In particular, the Plancherel measure of $\widehat{G}$ is concentrated on the set of (equivalence classes of) the representations ϖ_ω, as ω runs through Ω.

Now, fix $\omega \in \Omega$, and let us describe a little further a ϖ_ω as above. Observe first that we may find a B_ω-symplectic basis $(X_{\omega,1}, \dots, X_{\omega,n_1}, Y_{\omega,1}, \dots, Y_{\omega,n_1})$ of $\mathfrak{g}_1$ such that $\mathcal{L}_\iota = -\sum_{k=1}^{n_1} \widetilde{\mu}_{\iota,k}(\omega)(X_{\omega,k}^2 + Y_{\omega,k}^2)$ (cf. Proposition 4.16). Then, we may choose H_ω as the space of holomorphic functions in $L^2(\mathbb{C}^{n_1}, \nu)$, where $\nu = e^{-2|\,\cdot\,|^2} \cdot \mathcal{H}^{2n_1}$, and define ϖ_ω so that

$$\mathrm{d}\varpi_\omega(X_{\omega,k} + iY_{\omega,k}) f(z) = 2 z_k f(z) \qquad \text{and} \qquad \mathrm{d}\varpi_\omega(X_{\omega,k} - iY_{\omega,k}) f(z) = -\partial_{z_k} f(z)$$

for every $f \in C^\infty(\varpi_\omega)$ and for every $z \in \mathbb{C}^{n_1}$ (this is a version of the 'Bargmann(–Fock)' representation, cf. [29]). Then,

$$\mathrm{d}\varpi_\omega(\mathcal{L}_\iota) f(z) = \sum_{k=1}^{n_1} \widetilde{\mu}_{\iota,k}(\omega)\left(2 z_k \partial_{z_k} f(z) + f(z)\right)$$

for every $f \in C^\infty(\varpi_\omega)$, for every $z \in \mathbb{C}^{n_1}$, and for every $\iota \in I$. Now, define $w_\gamma(z) := \sqrt{\frac{2^{n_1+2|\gamma|}}{\pi^{n_1}\gamma!}} z^\gamma$ for every $z \in \mathbb{C}^{n_1}$ and for every $\gamma \in \mathbb{N}^{n_1}$, and observe that $(w_\gamma)_{\gamma \in \mathbb{N}^{n_1}}$ is an orthonormal basis of H_ω (cf. [29]), and that

$$\mathrm{d}\pi(\mathcal{L}_\iota) w_\gamma = \widetilde{\mu}_\iota(\omega)(\mathbf{1}_{n_1} + 2\gamma) w_\gamma$$

for every $\gamma \in \mathbb{N}^{n_1}$. Therefore, for every $\varphi \in \mathcal{D}(E_{\mathcal{L}_A})$,

$$\|\mathcal{K}_{\mathcal{L}_A}(\varphi)\|_2^2 = \frac{1}{(2\pi)^{n_1+n_2}} \int_{\mathfrak{g}_2^*} \sum_{\gamma \in \mathbb{N}^H} \binom{\mathbf{n_1} + \gamma - \mathbf{1}_H}{\gamma} |\varphi\left(\mu(\omega)(\mathbf{n_1} + 2\gamma), \omega(\mathbf{T})\right)|^2 |\mathrm{Pf}(\omega)| \,\mathrm{d}\omega,$$

whence the stated formula for $\beta_{\mathcal{L}_A}$.

Next, observe that, from the stated Plancherel formula for G, we deduce the following inversion formula:

$$f(x,t) = \frac{1}{(2\pi)^{n_1+n_2}} \int_{\mathfrak{g}_2^*} \mathrm{Tr}(\varpi_\omega(x,t)^* \varpi_\omega(f)) |\mathrm{Pf}(\omega)| \,\mathrm{d}\omega$$

for every $f \in \mathcal{S}(G)$ and for almost every $(x,t) \in G$. If $\varphi \in \mathcal{D}(E_{\mathcal{L}_A})$, then

$$\mathcal{K}_{\mathcal{L}_A}(\varphi)(x,t) = \frac{1}{(2\pi)^{n_1+n_2}} \int_{\mathfrak{g}_2^*} \sum_{\gamma \in \mathbb{N}^{n_1}} \varphi\left(\widetilde{\mu}_\omega(\mathbf{n_1} + 2\gamma), \omega(\mathbf{T})\right) \langle \varpi_\omega(x,t)^* w_\gamma | w_\gamma \rangle \, |\mathrm{Pf}(\omega)| \,\mathrm{d}\omega$$

[13] Here, $\|T\|_2$ denotes the Hilbert–Schmidt norm of the endomorphism T of H_ω.

for almost every $(x,t)\in G$. In addition, for every $(x,t)\in G$, for every $\omega\in\Omega$, and for every $\gamma\in\mathbb{N}^{n_1}$, [29, Proposition 2] implies that

$$\langle\varpi_\omega(x,t)^* w_\gamma|w_\gamma\rangle = e^{-\frac{1}{4}\sum_{k=1}^{2n_1}|x_k|^2 - i\omega(t)}\prod_{k=1}^{n_1}\Lambda^0_{\gamma_k}\left(\frac{1}{2}|(x_k,x_{n_1+k})|^2\right)$$

for every $x=\sum_{k=1}^{n_1}(x_k X_{\omega,k}+x_{n_1+k}Y_{\omega,k})\in\mathfrak{g}_1$ and for every $t\in\mathfrak{g}_2$. Now, observe that $|X_{\omega,k}|^2 = |Y_{\omega,k}|^2=\widehat{Q}_{\iota_0}(X_{\omega,k},X_{\omega,k})=\frac{1}{\widetilde{\mu}_{\iota_0,k}(\omega)}$ for every $k=1,\dots,n_1$, so that, for every $x\in\mathfrak{g}_1$,

$$|x(\omega)|^2=\sum_{k=1}^{2n_1}|x_k|^2 \qquad \text{and} \qquad |x_h(\omega)|^2=\sum_{k\in K_h}|x_k|^2,$$

where $(K_h)_{h\in H}$ is a suitable partition of $\{1,\dots,n_1\}$ such that $\widetilde{\mu}_{\iota,k}=\mu_{\iota,h}$ and $\mathrm{Card}(K_h)=n_{1,h}$ for every $\iota\in I$, for every $k\in K_h$, and for every $h\in H$. In addition, observe that [19, Formula (41) of p. 192] implies that

$$\Lambda^m_\gamma(z_1+\cdots+z_{m+1})=\sum_{|\gamma'|=\gamma}\prod_{j=1}^{m+1}\Lambda^0_{\gamma'_j}(z_j)$$

for every $m,\gamma\in\mathbb{N}$, and for every $z_1,\dots,z_{m+1}\in\mathbb{C}$. Hence, the asserted formula for $\chi_{\mathcal{L}_A}$ follows. □

Corollary 4.19. *Every $m\in L^\infty(\beta_{\mathcal{L}_A})$ such that $\mathcal{K}_{\mathcal{L}_A}(m)\in L^1(G)$ has a representative which is continuous on $\{(\mu(\omega)(\mathbf{n_1}),\omega(\mathbf{T}))\colon\omega\in\mathfrak{g}_2^*\}$.*

Proof. Simply define

$$\widetilde{m}(\mu(\omega)(\mathbf{n_1}),\omega(\mathbf{T})) := \int_G \mathcal{K}_{\mathcal{L}_A}(m)(x,t)e^{-\frac{1}{4}|x(\omega)|^2-i\omega(t)}\,\mathrm{d}(x,t)$$

for every $\omega\in\mathfrak{g}_2^*$, and observe that $\widetilde{m}$ is continuous on $C:=\{(\mu(\omega)(\mathbf{n_1}),\omega(\mathbf{T}))\colon\omega\in\mathfrak{g}_2^*\}$ thanks to Propositions 4.13 and 4.14; in addition, Proposition 4.18 implies that $\widetilde{m}=m$ $\beta_{\mathcal{L}_A}$-almost everywhere on C, whence the result. □

Corollary 4.20. *Take a bounded Borel function $m\colon E_{\mathcal{L}_A}\to\mathbb{C}$ such that $\mathcal{K}_{\mathcal{L}_A}(m)\in L^1(G)$. Then, there is a dilation-invariant negligible subset N of $\mathfrak{g}_2^*$ such that, for every $\omega\in\mathfrak{g}_2^*\setminus N$ (cf. Definition 4.15),*

$$(\pi_\omega)_*(\mathcal{K}_{\mathcal{L}_A}(m))=\mathcal{K}_{\mathrm{d}\pi_\omega(\mathcal{L}_A)}(m).$$

Proof. Keep the notation of the proof of Proposition 4.18. Notice first that, up to replace m with $\tau(2^{-j}\,\cdot)m$ for some $\tau\in\mathcal{D}(E_{\mathcal{L}_A})$ which equals 1 at 0, we may assume that m is compactly supported. Therefore, for every $\omega\in\mathfrak{g}_2^*$ we have (at least) $\mathcal{K}_{\mathrm{d}\pi_\omega(\mathcal{L}_A)}(m)\in L^2(G_\omega)$.

Now, by [34, Proposition 5.4] there is a negligible subset N' of $\mathfrak{g}_2^*$ such that $W\subseteq N'$ and such that, for every $\omega \in \mathfrak{g}_2^* \setminus N'$,

$$\int_G \mathcal{K}_{\mathcal{L}_A}(m)(x,t)\varpi_\omega(x,t)^* \,\mathrm{d}(x,t) = \varpi_\omega^*(\mathcal{K}_{\mathcal{L}_A}(m)) = m(\mathrm{d}\varpi_\omega(\mathcal{L}_A)).$$

Now, there is a negligible subset N'' of the unit sphere S of $\mathfrak{g}_2^*$ such that $N' \cap \mathbb{R}\omega$ is a negligible subset of $\mathbb{R}\omega$ for every $\omega \in S \setminus N''$; fix $\omega \in \mathfrak{g}_2^* \setminus \mathbb{R}_+ N''$. Then, for every $\rho \in \mathbb{R}^*$ there is a unique representation $\varpi_{\omega,\rho}$ of G_ω into $H_{\rho\omega}$ such that[14]

$$\varpi_{\rho\omega} = \varpi_{\omega,\rho} \circ \pi_\omega.$$

Therefore, the preceding remarks imply that

$$\varpi_{\omega,\rho}^*((\pi_\omega)_*(\mathcal{K}_{\mathcal{L}_A}(m))) = m(\mathrm{d}\varpi_{\omega,\rho}(\mathrm{d}\pi_\omega(\mathcal{L}_A)))$$

for every $\rho \in \mathbb{R}^*$ such that $\rho\omega \notin N'$. Now, arguing as in the proof of [34, Proposition 5.4], we see that there is a negligible subset N'_ω of $\mathbb{R}^*$ such that, for every $\rho \in \mathbb{R}^* \setminus N'_\omega$, we have $\rho\omega \notin N'$ and

$$\mathcal{F}(\mathcal{K}_{\mathrm{d}\pi_\omega(\mathcal{L}_A)}(m))(\varpi_{\omega,\rho}) = m(\mathrm{d}\varpi_{\omega,\rho}(\mathrm{d}\pi_\omega(\mathcal{L}_A))),$$

where $\mathcal{F}(\mathcal{K}_{\mathrm{d}\pi_\omega(\mathcal{L}_A)}(m))$ is (a fixed representative of) the Fourier transform of $\mathcal{K}_{\mathrm{d}\pi_\omega(\mathcal{L}_A)}(m)$.[15] Since the (equivalence classes of the) representations $\varpi_{\omega,\rho}$, as ρ runs through $\mathbb{R}^* \setminus N'_\omega$, form a co-negligible subset of the dual of G_ω (cf., for instance, [3, Section 2]), it follows that $(\pi_\omega)_*(\mathcal{K}_{\mathcal{L}_A}(m)) = \mathcal{K}_{\mathrm{d}\pi_\omega(\mathcal{L}_A)}(m)$. It then suffices to define $N \coloneqq \mathbb{R}_+ N''$. □

5 Property (RL)

In this section we keep the setting of Section 4; we shall present several sufficient conditions for the validity of property (RL). Unlike in the cases considered in [15], we are able to prove continuity results for $\chi_{\mathcal{L}_A}$, even though under rather strong assumptions (cf. Theorem 5.3); we then deduce property (RL) under slightly weaker assumptions (cf. Theorem 5.4). Let us comment a little more on the assumptions of Theorem 5.4. Besides the condition that μ is constant where the norm $\omega \mapsto \mu_{\iota_0}(\omega)(\mathbf{n_1})$ is constant, we need to add the condition that $\dim_\mathbb{R} \mu(\omega)(\mathbb{R}^H) = \dim_\mathbb{Q} \mu(\omega)(\mathbb{Q}^H)$ for every $\omega \in \Omega$. Even though this condition may appear peculiar, we cannot get rid of it without running into counterexamples, as Theorem 7.4 shows. Furthermore, observe that, even though Theorem 7.4 is the main application of Theorems 5.3 and 5.4, the latter result can be applied to more general homogeneous sub-Laplacians on MW^+ groups. For example, consider the complexified Heisenberg group $\mathbb{H}^1_\mathbb{C}$, whose Lie algebra

[14] With the notation of the proof of Proposition 4.18, $\varpi_{\omega,1} = \widetilde{\varpi}_\omega$.

[15] Define $(\mathcal{F}f)(\varpi_{\omega,\rho}) = \varpi^*(f)$ for every $f \in L^1(G_\omega) \cap L^2(G_\omega)$ and for every $\rho \neq 0$; then, $\mathcal{F}$ extends to an isometry of $L^2(G_\omega)$ onto $\int_{\mathbb{R}^*}^\oplus H_{\rho\omega}\, c|\rho|^{n_1} \,\mathrm{d}\rho$ for some $c > 0$; cf., for instance, [3, Section 2].

is endowed with an orthonormal basis $X_1, X_2, X_3, X_4, T_1, T_2$ such that $[X_1, X_3] = [X_4, X_2] = T_1$ and $[X_2, X_3] = [X_1, X_4] = T_2$, while the other commutators vanish. If $\mathrm{Card}(I) = 1$ and $\mathcal{L} = -(aX_1^2 + bX_2^2 + cX_3^2 + dX_4^2)$ with $a, b, c, d > 0$, $\sqrt{\frac{a}{b}}, \sqrt{\frac{c}{d}} \in \mathbb{Q}$, and either $a = b$ or $c = d$, then Theorem 5.4 applies, but Theorem 7.4 does not unless $a = b$ and $c = d$. *We are not aware of any applications of Theorem 5.3 besides Theorem 7.4.*

The next results concern families of the form $(\mathcal{L}, (-iT_1, \dots, -iT_{n_2'}))$ for $n_2' < n_2$. Notice that, in this case, we do not only reduce the number of elements of $\mathfrak{g}_2$, but we restrict to the case in which $\mathrm{Card}(I) = 1$. In this case, indeed, the spectrum of $(\mathcal{L}, (-iT_1, \dots, -iT_{n_2'}))$ is no longer a countable union of closures of analytic submanifolds, but a convex cone, so that things are somewhat easier and we can prove more general results than for the 'full family' $\mathcal{L}_A$. In Theorem 5.8, we show that property (RL) holds if $W = \{0\}$. With reference to the above example in the complexified Heisenberg group, this is the case when $ac \neq bd$ and $ad \neq bc$.

Our last result concerns the case of general MW^+ groups (cf. Theorem 5.9); even though its hypotheses are more restrictive than in the preceding one, it nonetheless applies when G is a product of Heisenberg groups and $\mathcal{L}$ is a sum of homogeneous sub-Laplacians on each factor (cf. Proposition 8.3).

5.1 The Case $n_2' = n_2$

We begin with some technical lemmas.

Lemma 5.1. *Let V and $\widetilde{V}$ be two finite-dimensional vector spaces over $\mathbb{R}$, L a discrete subgroup of V, and $\mu\colon V \to \widetilde{V}$ an $\mathbb{R}$-linear mapping. Then, the following conditions are equivalent:*

1. *$\mu(L)$ is a discrete subgroup of $\widetilde{V}$;*
2. *$L \cap \ker \mu$ generates $\langle L \rangle_{\mathbb{R}} \cap \ker \mu$ as a vector space over $\mathbb{R}$, where $\langle L \rangle_{\mathbb{R}}$ denotes the vector subspace of V over $\mathbb{R}$ generated by L;*
3. *$\dim_{\mathbb{R}} \mu(\langle L \rangle_{\mathbb{R}}) = \dim_{\mathbb{Q}} \mu(\langle L \rangle_{\mathbb{Q}})$, where $\langle L \rangle_{\mathbb{Q}}$ denotes the vector subspace of V over $\mathbb{Q}$ generated by L.*

Proof. The equivalence between **1** and **2** follows from [31, Theorem 1.1.2 and Proposition 1.1.4]. The equivalence between **2** and **3** follows from the fact that $\dim_{\mathbb{Q}} \mu(\langle L \rangle_{\mathbb{Q}}) = \dim_{\mathbb{Q}} \langle L \rangle_{\mathbb{Q}} - \dim_{\mathbb{Q}} \langle L \rangle_{\mathbb{Q}} \cap \ker \mu$, which equals $\dim_{\mathbb{R}} \langle L \rangle_{\mathbb{R}} - \dim_{\mathbb{R}} \langle L \cap \ker \mu \rangle_{\mathbb{R}}$ since $\langle L \rangle_{\mathbb{Q}} \cap \ker \mu = (\mathbb{Q}L) \cap \ker \mu = \mathbb{Q}(L \cap \ker \mu) = \langle L \cap \ker \mu \rangle_{\mathbb{Q}}$ and since $L \cap \ker \mu$ is a discrete subgroup of $\ker \mu$ by [31, Proposition 1.1.3]. □

Lemma 5.2. *Let V and $\widetilde{V}$ be two finite-dimensional vector spaces over $\mathbb{R}$, L a discrete subgroup of V, C the convex cone (with vertex 0) generated by some finite subset of L which generates V, and $\mu\colon V \to \widetilde{V}$ a* *linear mapping which is proper on C. Assume that $L \cap \ker \mu$ generates $\ker \mu$, and take $\xi \in \mu(C)$. Define*

$$V_\xi \coloneqq \mu^{-1}(\xi) \qquad\qquad S_\xi \coloneqq V_\xi \cap C$$
$$n_\xi \coloneqq \dim_{\mathbb{R}} S_\xi \qquad\qquad \nu_\xi \coloneqq \frac{1}{\mathcal{H}^{n_\xi}(S_\xi)} \chi_{S_\xi} \cdot \mathcal{H}^{n_\xi}.$$

Take $x_0 \in C$ and define, for every $\lambda \in \mathbb{R}_+^*$ and for every $\gamma \in \mu(x_0 + L \cap C)$,

$$\nu_{\lambda,\gamma} = \frac{1}{c_\gamma} \sum_{\substack{\gamma' \in L \cap C \\ \gamma = \mu(x_0+\gamma')}} \delta_{\lambda(x_0+\gamma')},$$

where $c_\gamma = \operatorname{Card}\left(\mu^{-1}(\gamma) \cap (x_0 + L \cap C)\right)$. Then,

$$\lim_{\substack{(\lambda\gamma,\lambda) \to (\xi,0) \\ \gamma \in \mu(x_0+L\cap C)}} \nu_{\lambda,\gamma} = \nu_\xi$$

in $\mathcal{E}'^0_c(V)$.

Proof. **1.** Define $\Sigma \coloneqq \mu(x_0 + L \cap C)$, and define $\mathfrak{F}_\xi$ as the filter '$(\lambda,\gamma) \in \mathbb{R}_+^* \times \Sigma$, $(\lambda\gamma,\lambda) \to (\xi,0)$.' Observe that it will suffice to prove that $\nu_{\lambda,\gamma}$ converges vaguely to ν_ξ along $\mathfrak{F}_\xi$. Indeed, the $\nu_{\lambda,\gamma}$ are probability measures supported in

$$S_{\lambda\gamma} \subseteq C \cap \mu^{-1}(K) \tag{1}$$

eventually along $\mathfrak{F}_\xi$, where K is any compact neighbourhood of ξ in $\widetilde{V}$. Since μ is proper on C, the assertion follows.

Now, let us prove that we may reduce to the case in which $x_0 = 0$. Indeed, define

$$\nu^0_{\lambda,\gamma} \coloneqq \frac{1}{c_\gamma} \sum_{\substack{\gamma' \in L \cap C \\ \gamma = \mu(x_0+\gamma')}} \delta_{\lambda\gamma'}.$$

It will then suffice to prove that $\nu_{\lambda,\gamma} - \nu^0_{\lambda,\gamma}$ converges vaguely to 0 along $\mathfrak{F}_\xi$. However, take $\varphi \in \mathcal{D}^0(V)$ and $\varepsilon > 0$. Then, there is a neighbourhood U of 0 in V such that $|\varphi(x_1) - \varphi(x_2)| < \varepsilon$ for every $x_1, x_2 \in V$ such that $x_1 - x_2 \in U$. Therefore, $\left|\left\langle \nu_{\lambda,\gamma} - \nu^0_{\lambda,\gamma}, \varphi \right\rangle\right| < \varepsilon$ as long as $\lambda x_0 \in U$, hence eventually along $\mathfrak{F}_\xi$. The assertion follows.

2. Observe that C is a polyhedral convex cone. In addition, let n be the dimension of V, and let $(F_\zeta)_{\zeta \in Z}$ be the (finite) family of $(n-1)$-dimensional facets of C; observe that F_ζ is a convex cone for every $\zeta \in Z$, so that $0 \in F_\zeta$. Take, for every $\zeta \in Z$, some $p_\zeta \in V^*$ such that $F_\zeta = \ker p_\zeta \cap C$ and $p_\zeta(C) \subseteq \mathbb{R}_+$. Then, C is the set of $x \in V$ such that $p_\zeta(x) \geqslant 0$ for every $\zeta \in Z$, and $L \cap \ker p_\zeta$ generates $\ker p_\zeta$ for every $\zeta \in Z$.

In addition, let Z_ξ be the set of $\zeta \in Z$ such that $p_\zeta(S_\xi) = \{0\}$, and let Z'_ξ be its complement in Z. We shall write p_{Z_ξ} and $p_{Z'_\xi}$ instead of $(p_\zeta)_{\zeta\in Z_\xi}$ and $(p_\zeta)_{\zeta\in Z'_\xi}$, respectively. Define $V'_\xi := V_\xi \cap \ker p_{Z_\xi}$. Then, $V'_\xi \cap p_{Z'_\xi}^{-1}\left((\mathbb{R}_+^*)^{Z'_\xi}\right)$ is the interior of S_ξ in V'_ξ; since, by convexity, $V'_\xi \cap p_{Z'_\xi}^{-1}\left((\mathbb{R}_+^*)^{Z'_\xi}\right)$ is not empty, V'_ξ is the affine space generated by S_ξ.

3. Define $W_\xi := V'_\xi - V'_\xi$, and observe that $L \cap W_\xi$ generates W_ξ. Indeed, the linear mapping $(\mu, p_{Z_\xi})\colon V \to \widetilde{V} \times \mathbb{R}^{Z_\xi}$ maps L into the discrete subgroup $\mu(L) \times \prod_{\zeta\in Z_\xi} p_\zeta(L)$ of $\widetilde{V} \times \mathbb{R}^{Z_\xi}$ (cf. Lemma 5.1), and W_ξ is the kernel of (μ, p_{Z_ξ}), whence the assertion by Lemma 5.1.

Therefore, there are two subspaces W'_ξ and W''_ξ of V such that the following hold (cf. [12, Exercises 2 and 3 of Chapter VII, § 1] or [31, Proposition 1.1.3]):

- $W_\xi \oplus W'_\xi = V_0 = \ker \mu$ and $V_0 \oplus W''_\xi = V$;
- $L \cap W'_\xi$ and $L \cap W''_\xi$ generate W'_ξ and W''_ξ, respectively, over $\mathbb{R}$;
- $(L \cap W_\xi) \oplus (L \cap W'_\xi) \oplus (L \cap W''_\xi) = L$ as abelian groups.

Therefore, we may endow V and $\widetilde{V}$ with two scalar products such that W_ξ, W'_ξ, and W''_ξ are orthogonal, and μ induces an isometry of W''_ξ into $\widetilde{V}$. We may further assume that $\|p_\zeta\| \leqslant 1$ for every $\zeta \in Z$.

4. Define, for $\lambda > 0$ and $\gamma \in \Sigma$,

$$r_{\xi,\lambda,\gamma} := \inf\{r > 0\colon S_{\lambda\gamma} \subseteq B(S_\xi, r)\} + \lambda,$$

so that $S_{\lambda\gamma} \subseteq B(S_\xi, r_{\xi,\lambda,\gamma})$. Here (and later in the proof), we denote by $B(K, r)$ the r-dilate of the set K, that is, $\bigcup_{x\in K} B(x, r)$. Let us prove that $r_{\xi,\lambda,\gamma}$ converges to 0 along $\mathfrak{F}_\xi$.

Indeed, let $\mathfrak{U}$ be an ultrafilter finer than $\mathfrak{F}_\xi$. Denote by $\mathcal{K}$ the space of non-empty compact subsets of V, endowed with the Hausdorff distance d_H, defined by $d_H(K_1, K_2) := \inf\{r > 0\colon K_1 \subseteq B(K_2, r), K_2 \subseteq B(K_1, r)\}$ for every $K_1, K_2 \in \mathcal{K}$. By (1), [11, Proposition 10 of Chapter I, § 6, No. 6], and [2, Theorem 6.1], it follows that $S_{\lambda\gamma}$ has a (unique) limit S in $\mathcal{K}$ along $\mathfrak{U}$. Now, for every closed neighbourhood K of ξ in $\widetilde{V}$,

$$S_{\lambda\gamma} \subseteq C \cap \mu^{-1}(K)$$

as long as $\lambda\gamma \in K$, so that, by passing to the limit along $\mathfrak{U}$,

$$S \subseteq C \cap \mu^{-1}(K).$$

By the arbitrariness of K, it follows that $S \subseteq S_\xi$. Therefore,

$$r_{\xi,\lambda,\gamma} \leqslant d_H(S, S_{\lambda\gamma}) + \lambda,$$

so that $r_{\xi,\lambda,\gamma}$ converges to 0 along $\mathfrak{U}$. Thanks to [11, Proposition 2 of Chapter I, § 7, No. 1], the arbitrariness of $\mathfrak{U}$ implies that $r_{\xi,\lambda,\gamma}$ converges to 0 along $\mathfrak{F}_\xi$.

Now, if $n_\xi = 0$, then **1** and the preceding arguments show that $\nu_{\lambda,\gamma}$ has a limit in $\mathcal{E}'^0_c(V)$ along $\mathfrak{F}_\xi$, and this limit is necessarily a probability measure supported on S_ξ. Since, in this case, $\mathrm{Card}(S_\xi) = 1$, this measure must be ν_ξ, so that the arbitrariness of $\mathfrak{U}$ and [11, Proposition 2 of Chapter I, § 7, No. 1] show that $\nu_{\lambda,\gamma}$ converges to ν_ξ in $\mathcal{E}'^0_c(V)$ along $\mathfrak{F}_\xi$, whence the result when $n_\xi = 0$.

5. Now, let π_ξ be the affine projection of V onto V'_ξ with fibres parallel to $W'_\xi \oplus W''_\xi$. Arguing as in **1** and taking **4** into account, we see that $\nu_{\lambda,\gamma} - (\pi_\xi)_*(\nu_{\lambda,\gamma})$ converges vaguely to 0 along $\mathfrak{F}_\xi$, so that it will suffice to prove that $(\pi_\xi)_*(\nu_{\lambda,\gamma})$ converges vaguely to ν_ξ along $\mathfrak{F}_\xi$. Observe, in addition, that from now on we may assume that $n_\xi > 0$, thanks to **4**.

Take $\varepsilon > 0$, $x \in \pi_\xi(\lambda L)$, and $y \in S_{\xi,\lambda,\gamma} \coloneqq \mathrm{Supp}\left((\pi_\xi)_*(\nu_{\lambda,\gamma})\right)$. Assume that $B(x,\varepsilon) \cap p^{-1}_{Z_\xi}\left(\mathbb{R}_+^{Z_\xi}\right) \subseteq C$, and that $r_{\xi,\lambda,\gamma} < \varepsilon$. Take $y' \in \mathrm{Supp}\left(\nu_{\lambda,\gamma}\right)$ such that $\pi_\xi(y') = y$, and let us prove that $y' + x - y \in \mathrm{Supp}\left(\nu_{\lambda,\gamma}\right)$. Indeed, it is clear that $x - y \in \lambda L \cap W_\xi$, so that $y' + x - y \in \lambda L$. Hence, it will suffice to prove that $y' + x - y \in C$. Now, since $y' \in S_{\lambda\gamma} \subseteq B(S_\xi, \varepsilon)$ by **4**, there is $x' \in S_\xi$ such that $|y' - x'| < \varepsilon$, so that

$$\varepsilon^2 > |y' - x'|^2 = |y - x'|^2 + |y' - y|^2$$

since $y - x' \in W_\xi$ and $y' - y \in W'_\xi \oplus W''_\xi$. Therefore, $|y' - y| < \varepsilon$; since, in addition, $p_\zeta(y' + x - y) = p_\zeta(y') \geqslant 0$ for every $\zeta \in Z_\xi$, it follows that $y' + x - y \in B(x,\varepsilon) \cap p^{-1}_{Z_\xi}\left(\mathbb{R}_+^{Z_\xi}\right) \subseteq C$. In particular, it follows that $x = \pi_\xi(y' + x - y) \in S_{\xi,\lambda,\gamma}$.

6. By the arguments of **5** above, we see that there is a function $c_{\xi,\lambda,\gamma}$ on $S_{\xi,\lambda,\gamma}$ such that

$$(\pi_\xi)_*(\nu_{\lambda,\gamma}) = \sum_{x \in S_{\xi,\lambda,\gamma}} c_{\xi,\lambda,\gamma}(x)\delta_x,$$

and such that $c_{\xi,\lambda,\gamma}(x) \geqslant c_{\xi,\lambda,\gamma}(y) \geqslant 1$ whenever $x, y \in S_{\xi,\lambda,\gamma}$, $B(x,\varepsilon) \cap p^{-1}_{Z_\xi}\left(\mathbb{R}_+^{Z_\xi}\right) \subseteq C$, and $r_{\xi,\lambda,\gamma} < \varepsilon$ for some fixed $\varepsilon > 0$. In particular, the function $c_{\xi,\lambda,\gamma}$ is constant on the set of $x \in S_{\xi,\lambda,\gamma}$ such that $B(x,\varepsilon) \cap p^{-1}_{Z_\xi}\left(\mathbb{R}_+^{Z_\xi}\right) \subseteq C$, as long as $r_{\xi,\lambda,\gamma} < \varepsilon$.

Now, take $\varepsilon > 0$ and $x \in V'_\xi$; let us prove that, if $p_\zeta(x) \geqslant \varepsilon$ for every $\zeta \in Z'_\xi$, then $B(x,\varepsilon) \cap p^{-1}_{Z_\xi}\left(\mathbb{R}_+^{Z_\xi}\right) \subseteq C$. Indeed, take $y \in B(x,\varepsilon)$, and assume that $p_\zeta(y) \geqslant 0$ for every $\zeta \in Z_\xi$. Take $\zeta \in Z'_\xi$, and observe that $|p_\zeta(y - x)| \leqslant |y - x| < \varepsilon$, so that $p_\zeta(y) = p_\zeta(x) + p_\zeta(y - x) \geqslant p_\zeta(x) - \varepsilon \geqslant 0$ by our choice of x. By the arbitrariness of ζ, it follows that $y \in C$.

7. Now, take a fundamental parallelotope P_ξ of $L \cap W_\xi$, and extend $c_{\xi,\lambda,\gamma}$ to a function on V which is constant on $x + \lambda P_\xi$ for every $x \in \pi_\xi(\lambda L)$, and vanishes outside $S_{\xi,\lambda,\gamma} + \lambda P_\xi$. Then, $\nu_{\xi,\lambda,\gamma} \coloneqq \frac{1}{\mathcal{H}^{n_\xi}(\lambda P_\xi)} c_{\xi,\lambda,\gamma} \cdot \mathcal{H}^{n_\xi}$ is a probability measure; in addition, as in **1** we see that $(\pi_\xi)_*(\nu_{\lambda,\gamma}) - \nu_{\xi,\lambda,\gamma}$ converges vaguely to 0 along $\mathfrak{F}_\xi$, so that it will suffice to show that $\nu_{\xi,\lambda,\gamma}$ converges vaguely to ν_ξ along $\mathfrak{F}_\xi$. Let us prove that, if S'_ξ denotes the boundary of S_ξ in V'_ξ, then $\frac{1}{\mathcal{H}^{n_\xi}(\lambda P_\xi)} c_{\xi,\lambda,\gamma}$ is uniformly bounded eventually along $\mathfrak{F}_\xi$, and converges on $V \setminus S'_\xi$ to the function $\frac{1}{\mathcal{H}^{n_\xi}(S_\xi)} \chi_{S_\xi}$: this will complete the proof.

Indeed, for every $\varepsilon > 0$ define $V'_{\xi,\varepsilon}$ as the set of $x \in V'_\xi$ such that $p_\zeta(x) \geqslant \varepsilon$ for every $\zeta \in Z'_\xi$. Observe that the union of the decreasing family $(V'_{\xi,\varepsilon})_\varepsilon$ is clearly $V'_\xi \cap p^{-1}_{Z'_\xi}\left((\mathbb{R}^*_+)^{Z'_\xi}\right)$, which in turn equals the interior of S_ξ in V'_ξ by **2** above. In addition, **5** and **6** above imply that $V'_{\xi,\varepsilon} \cap \pi_\xi(\lambda L) \subseteq S_{\xi,\lambda,\gamma}$ as long as $r_{\xi,\lambda,\gamma} < \varepsilon$. In particular, take $\varepsilon_0 > 0$ so that V'_{ξ,ε_0} has non-empty interior in V'_ξ; then, for every $\varepsilon \in]0, \varepsilon_0]$, $V'_{\xi,\varepsilon}$ contains at least an element of $S_{\xi,\lambda,\gamma}$, eventually along $\mathfrak{F}_\xi$. Next, for every $\lambda > 0$, define $V'_{\xi,\varepsilon,\lambda}$ as the union of the sets $x + \lambda P_\xi$, as $x \in \pi_\xi(\lambda L)$ and $(x + \lambda P_\xi) \cap V'_{\xi,\varepsilon} \neq \emptyset$. Then, $V'_{\xi,\varepsilon,\lambda}$ is contained in $V'_{\xi,\varepsilon/2}$ for λ sufficiently small, so that **5**, **6** and the above remarks imply that the function $c_{\xi,\lambda,\gamma}$ is constantly equal to $\max c_{\xi,\lambda,\gamma}$ on $V'_{\xi,\varepsilon}$, eventually along $\mathfrak{F}_\xi$ ($\varepsilon \in]0, \varepsilon_0]$).

Now, **4** above shows that $S_{\xi,\lambda,\gamma} \subseteq B(S_\xi, r_{\xi,\lambda,\gamma}) \cap V'_\xi$, since π_ξ is an orthogonal projection. Therefore, $\mathrm{Supp}\,(\nu_{\xi,\lambda,\gamma}) = S_{\xi,\lambda,\gamma} + \lambda \overline{P_\xi} \subseteq B(S_\xi, \varepsilon) \cap V'_\xi$, eventually along $\mathfrak{F}_\xi$, for every fixed $\varepsilon > 0$. Now, $\nu_{\xi,\lambda,\gamma}$ is a probability measure, so that

$$\mathcal{H}^{n_\xi}(V'_{\xi,\varepsilon}) \max c_{\xi,\lambda,\gamma} \leqslant \int_V \mathrm{d}\nu_{\xi,\lambda,\gamma} = 1 \leqslant \mathcal{H}^{n_\xi}(B(S_\xi, \varepsilon) \cap V'_\xi) \max c_{\xi,\lambda,\gamma}$$

eventually along $\mathfrak{F}_\xi$ ($\varepsilon \in]0, \varepsilon_0]$). As a consequence,

$$\frac{1}{\mathcal{H}^{n_\xi}(B(S_\xi, \varepsilon) \cap V'_\xi)} \leqslant \max c_{\xi,\lambda,\gamma} \leqslant \frac{1}{\mathcal{H}^{n_\xi}(V'_{\xi,\varepsilon})}$$

eventually along $\mathfrak{F}_\xi$ ($\varepsilon \in]0, \varepsilon_0]$); in particular, $c_{\xi,\lambda,\gamma}$ is uniformly bounded eventually along $\mathfrak{F}_\xi$. Now, the intersection of the increasing family $(B(S_\xi, \varepsilon) \cap V'_\xi)_{\varepsilon > 0}$ of $\mathcal{H}^{n_\xi}$-integrable sets is the closed set S_ξ, so that $\lim_{\varepsilon \to 0^+} \mathcal{H}^{n_\xi}(B(S_\xi, \varepsilon) \cap V'_\xi) = \mathcal{H}^{n_\xi}(S_\xi)$; analogously, since S'_ξ is $\mathcal{H}^{n_\xi}$-negligible, we have $\lim_{\varepsilon \to 0^+} \mathcal{H}^{n_\xi}(V'_{\xi,\varepsilon}) = \mathcal{H}^{n_\xi}(S_\xi)$. It is then clear that

$$\lim_{(\lambda,\gamma), \mathfrak{F}_\xi} c_{\xi,\lambda,\gamma}(x) = \frac{1}{\mathcal{H}^{n_\xi}(S_\xi)}$$

for every $x \in V'_{\xi,\varepsilon}$ and for every $\varepsilon \in]0, \varepsilon_0]$, hence for every $x \in S_\xi \setminus S'_\xi$. On the other hand,

$$\lim_{(\lambda,\gamma), \mathfrak{F}_\xi} c_{\xi,\lambda,\gamma}(x) = 0$$

for every $x \in V \setminus (B(S_\xi, \varepsilon) \cap V'_\xi)$ and for every $\varepsilon > 0$, hence for every $x \in V \setminus S_\xi$. Then, by means of the dominated convergence theorem we see that $\nu_{\xi,\lambda,\gamma}$ converges vaguely to ν_ξ along $\mathfrak{F}_\xi$, whence the result. □

Theorem 5.3. *Assume that* $\dim_{\mathbb{Q}} \mu(\omega)(\mathbb{Q}^H) = \dim_{\mathbb{R}} \mu(\omega)(\mathbb{R}^H)$ *for some non-zero* $\omega \in \mathfrak{g}_2^*$, *that* μ *is constant where the mapping* $\omega \mapsto \mu_{\iota_0}(\omega)(\mathbf{n_1})$ *is constant, and that* P *is constant. Then,* $\chi_{\mathcal{L}_A}$ *has a continuous representative.*

Proof. **1.** By an abuse of notation, we shall confuse P_h with its constant value for every $h \in H$. In addition, we denote by N the norm $\omega \mapsto \widetilde{\mu}_{\iota_0}(\omega)(\mathbf{1}_{n_1})$, and by S' the corresponding unit sphere; fix $\omega_0 \in S'$. Then, $\widetilde{\mu}$ is constant on S', so that $\widetilde{\mu}(\omega) = N(\omega)\widetilde{\mu}(\omega_0)$ by homogeneity for every $\omega \in \mathfrak{g}_2^*$. Define $\Sigma \coloneqq \widetilde{\mu}(\omega_0)(\mathbf{1}_{n_1} + 2\mathbb{N}^{n_1})$.

For every $\xi \in \widetilde{\mu}(\omega_0)(\mathbb{R}_+^{n_1})$, let $\mathfrak{F}_\xi$ denote the filter '$(\lambda, \gamma) \in \mathbb{R}_+^* \times \Sigma$, $(\lambda\gamma, \lambda) \to (\xi, 0)$.' In addition, define, for every $\lambda \in \mathbb{R}_+^*$ and for every $\gamma \in \Sigma$,

$$\nu'_{\lambda,\gamma} = \sum_{\gamma = \widetilde{\mu}(\omega_0)(\mathbf{1}_{n_1} + 2\gamma')} \delta_{\lambda(\mathbf{1}_{n_1} + 2\gamma')},$$

and $\nu_{\lambda,\gamma} \coloneqq \frac{1}{\nu'_{\lambda,\gamma}(\mathbb{R}^{n_1})} \cdot \nu'_{\lambda,\gamma}$, so that $\nu_{\lambda,\gamma}$ is a probability measure. Then, Lemma 5.2 implies that $\nu_{\lambda,\gamma}$ converges to some probability measure ν_ξ in $\mathcal{E}'^0_c(\mathbb{R}^{n_1})$ along $\mathfrak{F}_\xi$.

2. Recall that Λ^m_γ denotes the γ-th Laguerre polynomial of order m, and denote by J_0 the Bessel function (of the first kind) of order 0. Define, for every $(x,t) \in \mathbb{R}^{n_1} \times \mathbb{R}$,

$$\chi_0(\lambda(\mathbf{1}_{n_1} + 2\gamma'), \lambda, x, t) = e^{-\frac{1}{4}\lambda|x|^2 - i\lambda t} \prod_{k=1}^{n_1} \Lambda^0_{\gamma'_k}\left(\frac{1}{2}\lambda|x_k|^2\right)$$

for every $\lambda \in \mathbb{R}_+^*$ and for every $\gamma' \in \mathbb{N}^{n_1}$, and

$$\chi_0(\xi', 0, x, t) \coloneqq \prod_{k=1}^{n_1} J_0\left(\sqrt{\xi'_k}|x_k|\right)$$

for every $\xi' \in \mathbb{R}_+^{n_1}$. We claim that χ_0 extends to a continuous function on $\mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}$. To prove the continuity of χ_0, one may argue directly, making use of the well-known series expansion of J_0; nonetheless, our assertion follows from the fact that χ_0 is closely related with the spherical functions of a suitable Gelfand pair; see, for example, [4, Sections 1 and 2], and also [6, Lemma 3.1] for a quite explicit *analytic* extension of (a function closely related to) χ_0.

Now, define $c_\gamma \coloneqq \sum_{\gamma = \mu(\omega_0)(\mathbf{n_1} + 2\gamma')} \binom{\mathbf{n_1} + \gamma' - \mathbf{1}_H}{\gamma'}$, and

$$\begin{aligned}\chi_1((N(\omega)\gamma, \omega(\mathbf{T})), (x,t)) &\coloneqq \frac{1}{c_\gamma} \sum_{\gamma = \mu(\omega_0)(\mathbf{n_1} + 2\gamma)} e^{-\frac{1}{4}|x(\omega)|^2 - i\omega(t)} \prod_{h \in H} \Lambda^{n_{1,h} - 1}_{\gamma'_h}\left(\frac{1}{2}\mu_{\iota_0,h}(\omega)|P_h(x)|^2\right) \\ &= \frac{1}{c_\gamma} \sum_{\gamma = \mu(\omega_0)(\mathbf{n_1} + 2\gamma)} e^{-\frac{N(\omega)}{4}|x(\omega_0)|^2 - i\omega(t)} \prod_{h \in H} \Lambda^{n_{1,h} - 1}_{\gamma'_h}\left(\frac{N(\omega)}{2}\mu_{\iota_0,h}(\omega_0)|P_h(x)|^2\right)\end{aligned}$$

for every non-zero $\omega \in \mathfrak{g}_2^*$, for every $\gamma \in \Sigma$, and for every $(x,t) \in G$, so that χ_1 induces a representative of $\chi_{\mathcal{L}_A}$ thanks to Proposition 4.18. In addition, χ_1 is clearly continuous on its domain. Let us prove that χ_1 extends by continuity to $\sigma(\mathcal{L}_A) \times G$.

Indeed, fix a B_{ω_0}-symplectic basis $(X_{\omega_0,1}, \dots, X_{\omega_0,n_1}, Y_{\omega_0,1}, \dots, Y_{\omega_0,n_1})$ of $\mathfrak{g}_1$ such that $X_{\omega_0,k}$ and $Y_{\omega_0,k}$ are joint eigenvectors of $(|J_{Q_\iota,\omega_0}|)_\iota$ with joint eigenvalue $(\widetilde{\mu}_{\iota,k}(\omega_0))_\iota$, for every $k = 1, \dots, n_1$. Then, for every $x \in \mathfrak{g}_1$, we denote by $\widetilde{x}_1, \dots, \widetilde{x}_{2n_1}$ the coordinates of x with respect to such basis. As in the proof of Proposition 4.18, we then see that

$$\chi_1((N(\omega)\gamma, \omega(\mathbf{T})), (x,t)) = \left\langle \nu_{N(\omega),\gamma}, \chi_0\left(\,\cdot\,, N(\omega), (|(\widetilde{x}_k, \widetilde{x}_{n_1+k})|)_{k=1}^{n_1}, \tfrac{\omega(t)}{N(\omega)}\right)\right\rangle,$$

for every non-zero $\omega \in \mathfrak{g}_2^*$, for every $\gamma \in \Sigma$, and for every $(x,t) \in G$. Now, fix $\xi \in \widetilde{\mu}(\omega_0)(\mathbb{R}_+^{n_1})$, and observe that

$$\lim_{(\lambda,\gamma),\mathfrak{F}_\xi} \chi_1((\lambda\gamma, \lambda\omega(\mathbf{T})), (x,t)) = \langle \nu_\xi, \chi_0(\,\cdot\,, 0, (|(\widetilde{x}_k, \widetilde{x}_{n_1+k})|)_{k=1}^{n_1}, \omega(t))\rangle$$

uniformly as ω runs through S', and as (x,t) runs through a compact subset of G. Since the function $\langle \nu_\xi, \chi_0(\,\cdot\,, 0, (|(\widetilde{x}_k, \widetilde{x}_{n_1+k})|)_{k=1}^{n_1}, \omega(t))\rangle$ does *not* depend on $\omega \in S'$, it follows that χ_1 is continuous at ξ. The assertion follows. □

Theorem 5.4. *Assume that* $\dim_{\mathbb{Q}} \mu(\omega)(\mathbb{Q}^H) = \dim_{\mathbb{R}} \mu(\omega)(\mathbb{R}^H)$ *for some non-zero* $\omega \in \mathfrak{g}_2^*$*, and that* μ *is constant where the mapping* $\omega \mapsto \mu_{\iota_0}(\omega)(\mathbf{n_1})$ *is constant. Then,* $\mathcal{L}_A$ *satisfies property* (RL)*.*

Observe that, in this situation, $W = \{0\}$.

Proof. Take $\varphi \in L^1_{\mathcal{L}_A}(G)$, and let S' be the unit sphere associated with the norm $N\colon \omega \mapsto \widetilde{\mu}_{\iota_0}(\omega)(\mathbf{1}_{n_1})$. Corollary 4.20 implies that there is a negligible subset N_1 of S' such that $(\pi_\omega)_*(\varphi) \in L^1_{\mathrm{d}\pi_\omega(\mathcal{L}_A)}(G_\omega)$ for every $\omega \in S' \setminus N_1$ (cf. Definition 4.15). Observe, in addition, that the mapping

$$\omega \mapsto (\pi_\omega)_*(\varphi) \in L^1(\mathfrak{g}_1 \oplus \mathbb{R})$$

is continuous on $\mathfrak{g}_2^* \setminus \{0\}$, hence on S'. Now, fix $\omega_0 \in S'$, and take (U, ψ) as in Proposition 4.16. Then, it is easily seen that the mapping

$$U \cap S' \ni \omega \mapsto (\psi_\omega \circ \pi_\omega)_*(\varphi) \in L^1(\mathbb{H}^{n_1})$$

is continuous. Furthermore, observe that, with the notation of Proposition 4.16,

$$\mathcal{L}'_I \coloneqq \mathrm{d}(\psi_\omega \circ \pi_\omega)(\mathcal{L}_I) = \left(-\sum_{k=1}^{n_1} \widetilde{\mu}_{\iota,k}(\omega)(X_k^2 + X_{n_1+k}^2)\right)_{\iota \in I}$$

does *not* depend on $\omega \in U \cap S'$ since $\widetilde{\mu}$ is constant on S', while

$$\mathrm{d}(\psi_\omega \circ \pi_\omega)(\mathbf{T}) = \omega(\mathbf{T})T.$$

Observe that $(\mathcal{L}'_I, -iT)$ satisfies property (RL) by Theorem 5.3, and that $(\psi_\omega \circ \pi_\omega)_*(\varphi) \in L^1_{(\mathcal{L}'_I, -iT)}(\mathbb{H}^{n_1})$

for every $\omega\in S'\setminus N_1$. Therefore, the mapping

$$U\cap S'\ni\omega\mapsto \mathcal{M}_{(\mathcal{L}'_I,-iT)}((\psi_\omega\circ\pi_\omega)_*(\varphi))\in C_0(\sigma(\mathcal{L}'_I,-iT))$$

is continuous. In addition, if $\omega\in U\cap S'\setminus N_1$, then [32, Proposition 3.2.4], applied to the right quasi-regular representation of $\mathbb{H}^{n_1}$ in $L^2(G_0)$, implies that

$$\mathcal{M}_{(\mathcal{L}'_I,-iT)}((\psi_\omega\circ\pi_\omega)_*(\varphi))(\lambda,0)=\mathcal{M}_{\mathrm{d}\pi_0(\mathcal{L}_I)}((\pi_0)_*(\varphi))(\lambda)$$

for every $\lambda\in\mathbb{R}^I$ such that $(\lambda,0)\in\sigma(\mathcal{L}'_I,-iT)$, that is, for every $\lambda\in\sigma(\mathrm{d}\pi_0(\mathcal{L}_I))$. By continuity, this proves that the mapping $U\cap S'\ni\omega\mapsto\mathcal{M}_{(\mathcal{L}'_I,-iT)}((\psi_\omega\circ\pi_\omega)_*(\varphi))(\lambda,0)$ is constant for every $\lambda\in\sigma(\mathrm{d}\pi_0(\mathcal{L}_I))$. Taking into account the arbitrariness of U, we infer that there is a unique $m\in C_0(\sigma(\mathcal{L}_A))$ such that

$$m(\lambda,\omega(\mathbf{T}))=\mathcal{M}_{(\mathcal{L}'_I,-iT)}((\psi_{U,\omega/N(\omega)}\circ\pi_{\omega/N(\omega)})_*(\varphi))(\lambda,N(\omega))$$

for every $(\lambda,\omega(\mathbf{T}))\in\sigma(\mathcal{L}_A)$ such that $\omega\neq 0$ and $\frac{\omega}{N(\omega)}\in U\cap S'$, where U runs through a finite open covering of S' and ψ_U is the associated local trivialization as above. Finally, observe that either Proposition 4.18, applied to G and the G_ω, or Corollary 4.20 implies that

$$m(\lambda,\omega(\mathbf{T}))=\mathcal{M}_{\mathcal{L}_A}(\varphi)(\lambda,\omega(\mathbf{T}))$$

for $\beta_{\mathcal{L}_A}$-almost every $(\lambda,\omega(\mathbf{T}))$, so that $\varphi=\mathcal{K}_{\mathcal{L}_A}(m)$ and the assertion follows. □

Here we prove a negative result.

Proposition 5.5. *Assume that G is the product of $k\geqslant 2$ MW^+ groups $G'_1,\dots,G'_k$, and assume that each G'_j is endowed with a homogeneous sub-Laplacian $\mathcal{L}'_j$. Assume that* $\mathrm{Card}(I)=1$ *and that $\mathcal{L}=\mathcal{L}'_1+\dots+\mathcal{L}'_k$. Then, $\mathcal{L}_A$ does not satisfy properties (RL) and (S).*

Proof. Take, for every $j=1,\dots,k$, a basis $\mathbf{T}'_j$ of the centre $\mathfrak{g}'_{j,2}$ of the Lie algebra of G'_j. By an abuse of notation, we may assume that $\mathcal{L}_A=(\mathcal{L},-i\mathbf{T}'_1,\dots,-i\mathbf{T}'_k)$; define $\mathcal{L}'_{A'}\coloneqq(\mathcal{L}'_1,\dots,\mathcal{L}'_k,-i\mathbf{T}'_1,\dots,-i\mathbf{T}'_k)$. Then, there is a unique linear mapping $L\colon E_{\mathcal{L}'_{A'}}\to E_{\mathcal{L}_A}$ such that $\mathcal{L}_A=L(\mathcal{L}'_{A'})$. For every $j\in\{1,\dots,k\}$, define Ω_j, H_j, μ_j, $\mathbf{n}_{\mathbf{1},j}$ as the objects defined in Definition 4.12 starting with the family $(\mathcal{L}'_j,-i\mathbf{T}'_j)$ on G'_j. Now, take $j\in\{1,\dots,k\}$ and $\gamma\in\mathbb{N}^{H_j}$, and define

$$C_{j,\gamma}\coloneqq\{(\mu_j(\omega)(\mathbf{n}_{\mathbf{1},j}+2\gamma),\omega(\mathbf{T}'_j))\colon\omega\in\Omega_j\}.$$

Define

$$C := \bigcup_{\gamma \in \prod_{j=1}^k \mathbb{N}^{H_j}} \prod_{j=1}^k C_{j,\gamma_j},$$

and observe that $\beta_{\mathcal{L}'_{A'}}$ is equivalent to $\chi_C \cdot \mathcal{H}^{n_2}$ thanks to Proposition 4.18 (up to a re-ordering of the coordinates). Next, define Z as the set of non-zero $\gamma \in \prod_{j=1}^k \mathbb{Z}^{H_j}$ such that $\gamma_{j,h_1}\gamma_{j,h_2} \geqslant 0$ for every $j = 1, \dots, k$ and for every $h_1, h_2 \in H_j$.[16] Then, define

$$N := \mathbb{R}^k \times \bigcup_{\gamma \in Z} \left\{ \omega(\mathbf{T}) \colon \omega \in \prod_{j=1}^k \Omega_j, \quad \sum_{j=1}^k \mu_j(\omega_j)(\gamma_j) = 0 \right\},$$

and observe that L is one-to-one on $C \setminus N$. Let us prove that N is $\beta_{\mathcal{L}'_{A'}}$-negligible. Indeed, for every $j = 1, \dots, k$, let U_j be a component of Ω_j and fix $\gamma \in Z$. Observe that U_j is dilation-invariant. In addition, observe that either

$$N_{(U_j),\gamma} := \left\{ \omega(\mathbf{T}) \colon \omega \in \prod_{j=1}^k U_j, \quad \sum_{j=1}^k \mu_j(\omega_j)(\gamma_j) = 0 \right\}$$

is an analytic set of dimension at most $n_2 - 1$, or the mapping

$$(\omega_j) \mapsto \sum_{j=1}^k \mu_j(\omega_j)(\gamma_j) = 0$$

vanishes identically on $\prod_{j=1}^k U_j$. Now, there is $j_0 \in \{1, \dots, k\}$ such that $\gamma_{j_0} \neq 0$. By the definition of Z, it follows that either γ_{j_0} or $-\gamma_{j_0}$ is an element of $\mathbb{N}^{H_{j_0}}$, so that we may choose $(\omega_j) \in \prod_{j=1}^k U_j$ in such a way that $\mu_{j_0}(\omega_{j_0})(\gamma_{j_0}) \neq 0$. Now, for every $r > 0$ define $\left(\omega_j^{(r)}\right)_j$ so that $\omega_j^{(r)} = \omega_j$ for $j \neq j_0$, while $\omega_{j_0}^{(r)} = r\omega_{j_0}$; observe that $\left(\omega_j^{(r)}\right) \in \prod_{j=1}^k U_j$. In addition, the mapping

$$r \mapsto \sum_{j=1}^k \mu_j\left(\omega_j^{(r)}\right)(\gamma_j) = r\mu_{j_0}(\omega_{j_0})(\gamma_{j_0}) + \sum_{j \neq j_0} \mu_j(\omega_j)(\gamma_j)$$

is strictly monotone, so that it cannot be identically zero. As a consequence, the preceding remarks imply that $\mathbb{R}^k \times N_{(U_j),\gamma}$ is $\beta_{\mathcal{L}'_{A'}}$-negligible. By the arbitrariness of the U_j and γ, it follows that N is $\beta_{\mathcal{L}'_{A'}}$-negligible.

Therefore, there is a unique $m \colon E_{\mathcal{L}_A} \to E_{\mathcal{L}'_{A'}}$ such that $m \circ L$ is the identity on $C \setminus N$, while m equals 0 on the complement of $L(C \setminus N)$. Then, m is $\beta_{\mathcal{L}_A}$-measurable, and $\mathcal{K}_{\mathcal{L}_A}(m) = \mathcal{L}'_{A'}\delta_e$. Now, let us prove that m is not equal $\beta_{\mathcal{L}_A}$-almost everywhere to any continuous functions. Assume by contradiction that $\mathcal{L}'_{A'}\delta_e = \mathcal{K}_{\mathcal{L}_A}(m')$ for some continuous function m', and let π_0 be the projection of G onto its abelianization G_0. Then, [32, Proposition 3.2.4], applied (arguing by approximation) to the right quasi-

[16]Here, $\gamma_{j,h}$ denotes the h-th component of the j-th component γ_j of γ, for every $j = 1, \dots, k$ and for every $h \in H_j$.

regular representation of G in $L^2(G_0)$, implies that the operators $\mathrm{d}\pi_0(\mathcal{L}'_1), \dots, \mathrm{d}\pi_0(\mathcal{L}'_k)$ belong to the functional calculus of $\mathrm{d}\pi_0(\mathcal{L})$, which is absurd.

To conclude, simply take $\tau \in \mathcal{S}(E_{\mathcal{L}_A})$ such that $\tau(\lambda) \neq 0$ for every $\lambda \in E_{\mathcal{L}_A}$, and observe that $\mathcal{K}_{\mathcal{L}_A}(m\tau) = \mathcal{L}'_{A'}\mathcal{K}_{\mathcal{L}_A}(\tau)$ is a family of elements of $\mathcal{S}(G)$, while $m\tau$ is not equal $\beta_{\mathcal{L}_A}$-almost everywhere to any continuous functions. $\square$

5.2 The Case $n'_2 < n_2$

Before we state our main results, let us consider some technical lemmas.

Lemma 5.6. *Let M be a separable analytic manifold of dimension n endowed with a positive Radon measure μ which is equivalent to Lebesgue measure on every local chart. In addition, take $k, h \in \mathbb{N}$ and a analytic mapping $P \colon M \to \mathbb{R}^k$ with generic rank h such that $P_*(\mu)$ is a Radon measure. Then, the following hold:*

1. *$P(M)$ is $\mathcal{H}^h$-measurable and countably $\mathcal{H}^h$-rectifiable;*
2. *$P_*(\mu)$ is equivalent to $\chi_{P(M)} \cdot \mathcal{H}^h$;*
3. *$\mathrm{Supp}\,(P_*(\mu)) = \overline{P(M)}$;*
4. *if $(\beta_y)_{y \in \mathbb{R}^k}$ is a disintegration of μ relative to P, then $\mathrm{Supp}\,(\beta_y) = P^{-1}(y)$ for $\mathcal{H}^h$-almost every $y \in P(M)$.*

Notice that it is worthwhile for our analysis to consider the case in which M is possibly disconnected.

Proof. Observe first that M may be embedded as a closed submanifold of class C^∞ of $\mathbb{R}^{2n+1}$ by Whitney embedding theorem (cf. [18, Theorem 5 of Chapter 1]). We may therefore assume that $\mu = f \cdot \mathcal{H}^n$ for some $f \in L^1_{\mathrm{loc}}(\chi_M \cdot \mathcal{H}^n)$. Now, [37] implies that the set where P has rank $< h$, which is $\mathcal{H}^n$-negligible by analyticity, has $\mathcal{H}^h$-negligible image under P. Since the image under P of the set where P has rank h is a countable union of analytic submanifolds of $\mathbb{R}^k$ of dimension h, we see that $P(M)$ is $\mathcal{H}^h$-measurable and countably $\mathcal{H}^h$-rectifiable. Therefore, we may make use of [20, Theorem 3.2.22], and infer that $P_*(\mu)$ is equivalent to the restriction of $\mathcal{H}^h$ to the set of y such that $\mathcal{H}^{n-h}(P^{-1}(y)) > 0$, and that we may find a disintegration (β_y) of μ relative to P such that β_y is equivalent to $\chi_{P^{-1}(y)} \cdot \mathcal{H}^{n-h}$ for $P_*(\mu)$-almost every $y \in \mathbb{R}^k$. Now, the preceding arguments show that $P^{-1}(y)$ is an analytic submanifold of dimension $n-h$ of M for $\mathcal{H}^h$-almost every $y \in P(M)$. As a consequence, $\mathrm{Supp}\,(\beta_y) = \mathrm{Supp}\,\big(\chi_{P^{-1}(y)} \cdot \mathcal{H}^{n-h}\big) = P^{-1}(y)$ for $\mathcal{H}^h$-almost every $y \in P(M)$; for the same reason, we also see that $P_*(\mu)$ is equivalent to $\chi_{P(M)} \cdot \mathcal{H}^h$. Finally, $\mathrm{Supp}\,(P_*(\mu)) = \overline{P(M)}$ since P is continuous and $\mathrm{Supp}\,(\mu) = M$. $\square$

Lemma 5.7. *Let E_1, E_2 be two finite-dimensional vector spaces, C a convex subset of E_1 with non-empty interior, and $L \colon E_1 \to E_2$ a linear mapping which is proper on ∂C. Assume that for every $x \in \partial C$*

either $L^{-1}(L(x))\cap\partial C=\{x\}$ *or* ∂C *is an analytic hypersurface of* E_1 *in a neighbourhood of* x*. Then,* L *induces an* open *mapping* $L'\colon \partial C\to L(\partial C)$*.*

Proof. Take $x\in\partial C$, and assume that $L^{-1}(L(x))\cap\partial C=\{x\}$. Define $U_{x,k}\coloneqq L^{-1}(\overline{B}(L(x),2^{-k}))\cap\partial C$ for every $k\in\mathbb{N}$. Since L is proper on ∂C, $U_{x,k}$ is a compact neighbourhood of x for every $k\in\mathbb{N}$. In addition, $\bigcap_{k\in\mathbb{N}}U_{x,k}=\{x\}$; hence, [11, Proposition 1 of Chapter 1, § 9, No. 2] implies that $(U_{x,k})$ is a fundamental system of neighbourhoods of x in ∂C, so that L' is open at x.

Now, assume that $L^{-1}(L(x))\cap\partial C\neq\{x\}$. Then, the hypotheses imply that there is an open neighbourhood U of $L^{-1}(L(x))\cap\partial C$ such that $\partial C\cap U$ is an analytic hypersurface of E_1. Observe that, if (U_j) is a decreasing fundamental system of relatively compact open convex neighbourhoods of $L(x)$ in E_2, then $(L^{-1}(\overline{U_j})\cap\partial C)$ is a decreasing sequence of compact neighbourhoods of $L^{-1}(L(x))\cap\partial C$ whose intersection is $L^{-1}(L(x))\cap\partial C$. By compactness, we then see that there is $j\in\mathbb{N}$ such that $L^{-1}(\overline{U_j})\cap\partial C\subseteq U$, so that $L^{-1}(U_j)$ is an open *convex* neighbourhood of $L^{-1}(L(x))\cap\partial C$ such that $\partial C\cap L^{-1}(U_j)$ is an analytic hypersurface of E_1. Hence, we may assume that U is convex. Assume by contradiction that $\ker L\subseteq T_x(\partial C\cap U)$, and take $x'\in\partial C$ such that $L(x')=L(x)$ but $x'\neq x$. Since C is convex, we have $[x,x']\subseteq\partial C$; let ℓ be the line passing through x and x'. Since $\partial C\cap U$ is an analytic hypersurface, and since $\ell\cap U$ is convex, it follows that $\ell\cap U\subseteq\partial C$. Then, $\ell\cap U=\ell\cap\partial C\cap U=\ell\cap\partial C$, so that $\ell\cap U$ is non-empty, compact, and open in ℓ: contradiction. Therefore, $\ker L\not\subseteq T_x(\partial C\cap U)$. Now, this implies that $L\colon\partial C\cap U\to L(E_1)$ is a submersion at x, since $T_x(\partial C\cap U)$ is a hyperplane. Hence, $L\colon\partial C\cap U\to L(E_1)$ is open at x; *a fortiori*, L' is open at x. The assertion follows. □

Theorem 5.8. *Assume that* $\mathrm{Card}(I)=1$ *and that* $W=\{0\}$*; take a positive integer* $n_2'<n_2$*. Then, the family* $(\mathcal{L},(-iT_j)_{j=1,\dots,n_2'})$ *satisfies property* (RL)*.*

Proof. **1.** Define $L\colon E_{\mathcal{L}_A}\to E_{(\mathcal{L},(-iT_j)_{j=1,\dots,n_2'})}$ as the unique linear mapping such that $L(\mathcal{L}_A)=(\mathcal{L},(-iT_j)_{j=1,\dots,n_2'})$. Define, in addition, L' in such a way that $L=\mathrm{id}_{\mathbb{R}}\times L'$, and identify $\mathfrak{g}_2^*$ with $\mathbb{R}^{n_2}$ by means of the mapping $\omega\mapsto\omega(\mathbf{T})$. Define, for every $\gamma\in\mathbb{N}^{n_1}$,

$$\beta_\gamma\colon \mathcal{D}^0(E_{\mathcal{L}_A})\ni\varphi\mapsto\int_{\mathfrak{g}_2^*}\varphi(\widetilde{\mu}(\omega)(\mathbf{1}_{n_1}+2\gamma),\omega)|\mathrm{Pf}(\omega)|\,\mathrm{d}\omega,$$

so that $\beta_{\mathcal{L}_A}=\frac{1}{(2\pi)^{n_1+n_2}}\sum_{\gamma\in\mathbb{N}^{n_1}}\beta_\gamma$ by Proposition 4.18. Now, arguing as in the proof of Proposition 2.4, we see that $C_0\coloneqq L(\mathrm{Supp}\,(\beta_0))=L(\sigma(\mathcal{L}_A))$ is a closed convex cone.

2. Observe that Proposition 4.13 implies that $\mathrm{Supp}\,(\beta_0)\setminus\{0\}$ is an analytic submanifold of $E_{\mathcal{L}_A}$. In addition, Lemma 5.6 implies that $L_*(\beta_0)$ is equivalent to $\chi_{C_0}\cdot\mathcal{H}^{n_2'+1}$ and that, if $(\beta_{0,\lambda})$ is a disintegration of β_0 relative to L, then $\mathrm{Supp}\,(\beta_{0,\lambda})=L^{-1}(\lambda)\cap\mathrm{Supp}\,(\beta_0)$ for $L_*(\beta_0)$-almost every $\lambda\in C_0$. In addition, Lemma 5.7 implies that the mapping $L\colon\mathrm{Supp}\,(\beta_0)\to C_0$ is open, so that, in particular, β_0 is L-connected. If we prove that $L_*(\beta_\gamma)$ is absolutely continuous with respect to $\mathcal{H}^{n_2'+1}$ for every $\gamma\in\mathbb{N}^{n_1}$, the assertion will then follow from Corollary 4.19 and Proposition 2.6.

Then, let us prove that $L_*(\beta_\gamma)$ is absolutely continuous with respect to $\mathcal{H}^{n'_2+1}$ for every $\gamma \in \mathbb{N}^{n_1}$. Notice that this will be the case if we prove that the analytic mapping $\Omega \ni \omega \mapsto (\widetilde{\mu}(\omega)(\mathbf{1}_{n_1}+2\gamma), L'(\omega))$ is generically a submersion for every $\gamma \in \mathbb{N}^{n_1}$ (cf. Lemma 5.6). Assume by contradiction that this is not the case, so that there are $\gamma \in \mathbb{N}^{n_1}$ and a component U of Ω such that $\frac{\mathrm{d}}{\mathrm{d}\omega}\widetilde{\mu}(\omega)(\mathbf{1}_{n_1}+2\gamma)$ vanishes on $\ker L'$ for every $\omega \in U$. As a consequence, there are $(r,\omega') \in \mathbb{R}\times\mathbb{R}^{n'_2}$ such that $L^{-1}(r,\omega')\cap \operatorname{Supp}(\beta_\gamma)$ contains an open segment. Then, there is a line ℓ in $L'^{-1}(\omega')$ such that $(\{r\}\times \ell)\cap \operatorname{Supp}(\beta_\gamma)$ contains an open segment; observe that $0 \notin \ell$ since the mapping $\omega \mapsto \widetilde{\mu}(\omega)(\mathbf{1}_{n_1}+2\gamma)$ is homogeneous and proper. Then, [30, Theorem 6.1 of Chapter II] implies that there is an analytic function $f\colon \ell \to \mathbb{R}^{n_1}$ such that $f(\omega)$ is a reordering of $\widetilde{\mu}(\omega)$ for every $\omega \in \ell$. As a consequence, for every $\gamma' \in \mathbb{N}^{n_1}$ the set of $\omega \in \ell$ such that $f(\omega)(\mathbf{1}_{n_1}+2\gamma') = r$ is compact, hence discrete by analyticity. Therefore, $(\{r\}\times\ell)\cap\sigma(\mathcal{L}_A)$ is countable, so that it cannot contain any open segments: contradiction. The proof is therefore complete. □

Theorem 5.9. *Assume that* $\operatorname{Card}(I) = 1$ *and define* $C_\gamma \coloneqq \{(\mu(\omega)(\mathbf{n_1}+2\gamma), \omega(\mathbf{T}))\colon \omega \in \mathfrak{g}_2^*\}$ *for every* $\gamma \in \mathbb{N}^H$*. In addition, take* $n'_2 < n_2$ *and define* $L \coloneqq \mathrm{id}_\mathbb{R}\times \mathrm{pr}_{1,\dots,n'_2}$ *on* $E_{\mathcal{L}_A}$*. Assume that the following hold:*

1. $\chi_{C_0}\cdot\beta_{\mathcal{L}_A}$ *is L-connected*
2. *for every* $f \in L^1_{\mathcal{L}_A}(G)$ *and for every* $\gamma \in \mathbb{N}^H$, $\mathcal{M}_{\mathcal{L}_A}(f)$ *equals* $\beta_{\mathcal{L}_A}$*-almost everywhere a continuous function on* C_γ*.*

Then, $L(\mathcal{L}_A)$ *satisfies property (RL).*

Observe that condition **1** holds if C_0 is the boundary of a polyhedron (cf. Proposition 2.4) and if $W = \{0\}$ (cf. Lemma 5.7). With a little effort, one may prove that condition **1** holds if $n'_2 = 1$.

We shall prepare the proof of Theorem 5.9 through several lemmas.

Lemma 5.10. *Let* V *be a topological vector space,* C *a convex subset of* V *with non-empty interior, and* W *an affine subspace of* V *such that* $W\cap \mathring{C} \neq \emptyset$*. Then,* $W \cap \partial C$ *is the boundary of* $W\cap C$ *in* W*.*

Proof. Indeed, take $x_0 \in W\cap\mathring{C}$, and take x in the interior of $W\cap C$ in W. Then, there is $y\in W\cap C$ such that $x \in [x_0, y[$, so that [13, Proposition 16 of Chapter II, § 2, No. 6] implies that $x\in\mathring{C}$. By the arbitrariness of x, this proves that $W\cap\mathring{C}$ is the interior of $W\cap C$ in W. Analogously, one proves that $W\cap\overline{C}$ is the closure of $W\cap C$ in W, whence the result. □

Lemma 5.11. *Let* $f\colon \mathbb{R}^n\to\mathbb{R}$ *be a convex function which is differentiable on an open subset* U *of* $\mathbb{R}^n$*. Let* L *be a linear mapping of* $\mathbb{R}^n$ *onto* $\mathbb{R}^k$ *for some* $k \leqslant n$*, and assume that* (f,L) *has rank* k *on* U*. Then, for every* $y\in(f,L)(U)$*, the fibre* $(f,L)^{-1}(y)$ *is a closed convex set which contains* $L^{-1}(y_2)\cap U$*.*

Proof. We may assume that U is not empty. Define $\pi \coloneqq (f,L)\colon \mathbb{R}^n\to\mathbb{R}\times\mathbb{R}^k$, and observe that $\ker L \subseteq \ker f'(x)$ for every $x\in U$ since π has rank k on U. Consequently, if $y\in\pi(U)$, then f is locally constant on $L^{-1}(y_2) \cap U$. Now, take two components C_1 and C_2 of $L^{-1}(y_2) \cap U$, and observe that they are open in $L^{-1}(y_2)$. Take $x_1 \in C_1$ and $x_2 \in C_2$. Then $[x_1, x_2] \subseteq L^{-1}(y_2)$, so that there are $x_1', x_2' \in]x_1, x_2[$ such that f is constant on $[x_1, x_1']$ and on $[x_2', x_2]$. By convexity, f must be constant on $[x_1, x_2]$, hence on $C_1 \cup C_2$. By the arbitrariness of C_1 and C_2, we infer that $\pi^{-1}(y) \supseteq L^{-1}(y_2) \cap U$.

Now, consider the convex set $C \coloneqq \{(\lambda, x) : x \in \mathbb{R}^n, \lambda \geqslant f(x)\}$, and observe that C is closed and that

$$\mathring{C} = \{(\lambda, x) : x \in \mathbb{R}^n, \lambda > f(x)\}$$

since f is continuous, so that ∂C is the graph of f. Next, define $W \coloneqq (\mathrm{id}_{\mathbb{R}} \times L)^{-1}(y) = \{y_1\} \times L^{-1}(y_2)$, and observe that $W \cap \partial C = \{y_1\} \times \pi^{-1}(y)$. Assume by contradiction that $W \cap \mathring{C} \neq \emptyset$. Then, Lemma 5.10 implies that $W \cap \partial C$ is the boundary of $W \cap C$ in W, so that $\pi^{-1}(y)$ has empty interior in $L^{-1}(y_2)$. However, $\pi^{-1}(y)$ contains $L^{-1}(y_2) \cap U$, which is non-empty and open in $L^{-1}(y_2)$: contradiction. Therefore, $\{y_1\} \times \pi^{-1}(y) = W \cap C$ is a closed convex set, whence the result. □

Lemma 5.12. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function which is analytic on some open subset Ω of $\mathbb{R}^n$ whose complement is $\mathcal{H}^n$-negligible. Let L be a linear mapping of $\mathbb{R}^n$ onto $\mathbb{R}^k$ for some $k \leqslant n$, and let U be the union of the components of Ω where (f, L) has rank k. Then,*

$$(f, L)^{-1}(y) = \overline{L^{-1}(y_2) \cap U}$$

for $\mathcal{H}^k$-almost every $y \in (f, L)(U)$.

Proof. We may assume that U is not empty; define $\pi \coloneqq (f, L)$. Since the complement of Ω is $\mathcal{H}^n$-negligible, there is an $\mathcal{H}^k$-negligible subset N_1 of $\mathbb{R}^k$ such that $L^{-1}(y) \setminus \Omega$ is $\mathcal{H}^{n-k}$-negligible for every $y \in \mathbb{R}^k \setminus N_1$ (cf. [20, Theorem 3.2.22]). In addition, observe that the set R_k of $x \in \Omega \setminus U$ such that $\ker L \subseteq \ker f'(x)$, that is, such that $\pi'(x)$ has rank k, is $\mathcal{H}^n$-negligible by the analyticity of f. Then, there is an $\mathcal{H}^k$-negligible subset N_2 of $\mathbb{R}^k$ such that $L^{-1}(y) \cap R_k$ is $\mathcal{H}^{n-k}$-negligible for every $y \in \mathbb{R}^k \setminus N_2$ (*loc. cit.*). Now, observe that there is a continuous nowhere vanishing function φ on $\mathbb{R}^n$ such that $(\chi_U \varphi) \cdot \mathcal{H}^n$ is a bounded measure (for example, take a Gaussian). Then, Lemma 5.6 implies that the measure $\chi_{\pi(U)} \cdot \mathcal{H}^k$ is equivalent to the measure $\pi_*((\chi_U \varphi) \cdot \mathcal{H}^n)$, which is in turn equivalent to the (not necessarily Radon) measure $\pi_*(\chi_U \cdot \mathcal{H}^n)$. Next, define $N \coloneqq \mathbb{R} \times (N_1 \cup N_2)$; since $U \cap \pi^{-1}(N) = U \cap L^{-1}(N_1 \cup N_2)$ is $\mathcal{H}^n$-negligible, it follows that $\pi(U) \cap N$ is $\mathcal{H}^k$-negligible.

Now, take $y \in \pi(U) \setminus N$. Then, Lemma 5.11 implies that $\pi^{-1}(y)$ is a closed convex set which contains $L^{-1}(y_2) \cap U$, so that its interior in $L^{-1}(y_2)$ is not empty. Let U' be a component of Ω which is not contained in U, and assume that $\pi^{-1}(y) \cap U' \neq \emptyset$. Since f is analytic on U', and since $\pi^{-1}(y)$ is a convex set with non-empty interior in $L^{-1}(y_2)$, we see that a component C of $L^{-1}(y_2) \cap U'$ is contained in R_k. By the choice of N_2, this implies that C is $\mathcal{H}^{n-k}$-negligible; since C is non-empty and open in $L^{-1}(y_2)$, this leads to a contradiction. Therefore,

$$L^{-1}(y_2) \cap U \subseteq \pi^{-1}(y) \subseteq L^{-1}(y_2) \cap [U \cup (\mathbb{R}^n \setminus \Omega)].$$

By our choice of N_1, the set $L^{-1}(y_2) \setminus \Omega$ is $\mathcal{H}^{n-k}$-negligible; on the other hand, the support of $\chi_{\pi^{-1}(y)} \cdot \mathcal{H}^{n-k}$ is $\pi^{-1}(y)$ by convexity. Hence, $L^{-1}(y_2) \cap U$ is dense in $\pi^{-1}(y)$, whence the result. □

Lemma 5.13. *Keep hypotheses and notation of Lemma 5.12. Assume, in addition, that* $\lim_{x \to \infty} f(x) = +\infty$ *and that* $\mathcal{H}^n$ *is* (f, L)*-connected. Then, for every* $m \in \mathcal{E}^0(\mathbb{R}^n)$ *such that* $m = m' \circ (f, L)$ $\mathcal{H}^n$*-almost everywhere for some* $m' \colon \mathbb{R} \times \mathbb{R}^k \to \mathbb{C}$*, there is* $m'' \in \mathcal{E}^0(\mathbb{R} \times \mathbb{R}^k)$ *such that* $m = m'' \circ (f, L)$ *pointwise.*

Proof. Define $\pi \coloneqq (f, L)$, and observe that π is proper. Let $(\beta_{1,y})_{y \in \mathbb{R} \times \mathbb{R}^k}$ be a disintegration of $\chi_U \cdot \mathcal{H}^n$ relative to π and let $(\beta_{2,y})_{y \in \mathbb{R} \times \mathbb{R}^k}$ be a disintegration of $\chi_{\Omega \setminus U} \cdot \mathcal{H}^n$ relative to π. Then, Lemma 5.6 implies that:

- $\pi_*(\chi_U \cdot \mathcal{H}^n)$ is equivalent to $\chi_{\pi(U)} \cdot \mathcal{H}^k$;
- $\pi_*(\chi_{\Omega \setminus U} \cdot \mathcal{H}^n)$ is equivalent to $\chi_{\pi(\Omega \setminus U)} \cdot \mathcal{H}^{k+1}$;
- $\operatorname{Supp}(\beta_{1,y}) = \overline{\pi^{-1}(y) \cap U}$ for $\mathcal{H}^k$-almost every $y \in \pi(U)$;
- $\operatorname{Supp}(\beta_{2,y}) = \overline{\pi^{-1}(y) \cap \Omega \setminus U}$ for $\mathcal{H}^{k+1}$-almost every $y \in \pi(\Omega \setminus U)$.

In addition, $\pi(U)$ (if non-empty) has Hausdorff dimension k, so that $\mathcal{H}^{k+1}(\pi(U)) = 0$; in particular, $\pi_*(\chi_U \cdot \mathcal{H}^n)$ and $\pi_*(\chi_{\Omega \setminus U} \cdot \mathcal{H}^n)$ are alien measures. If we define $\beta_y \coloneqq \beta_{1,y}$ for every $y \in \pi(U)$ and $\beta_y \coloneqq \beta_{2,y}$ for every $y \in (\mathbb{R} \times \mathbb{R}^k) \setminus \pi(U)$, then (β_y) is a disintegration of $\mathcal{H}^n$ relative to π.

Now, Lemma 5.12 implies that $\overline{\pi^{-1}(y) \cap U} = \pi^{-1}(y)$ for $\mathcal{H}^k$-almost every $y \in \pi(U)$. Next, let us prove that $\pi^{-1}(y) = \overline{\pi^{-1}(y) \cap \Omega \setminus U}$ for $\mathcal{H}^{k+1}$-almost every $y \notin \pi(U)$. To this end, we may assume that $\Omega \neq U$, so that $k + 1 \leqslant n$. Let us first prove that $\pi^{-1}(y)$ is the boundary of a compact convex set with non-empty interior in $L^{-1}(y_2)$ for $\mathcal{H}^{k+1}$-almost every $y \in \pi(\Omega \setminus U)$.

Indeed, by [37] there is an $\mathcal{H}^{k+1}$-negligible subset N of $\pi(\Omega \setminus U)$ such that $\pi'(x)$ has rank $k + 1$ for every $x \in \pi^{-1}(y) \cap \Omega \setminus U$ and for every $y \in \pi(\Omega \setminus U) \setminus N$. Now, define $C \coloneqq \{(\lambda, x) \colon x \in \mathbb{R}^n, \lambda \geqslant f(x)\}$, and observe that $\mathrm{id}_{\mathbb{R}} \times L$ is proper on C since $\lim_{x \to \infty} f(x) = +\infty$. Therefore, $\{y_1\} \times \pi^{-1}(y) = (\mathrm{id}_{\mathbb{R}} \times L)^{-1}(y) \cap \partial C$ is compact for every $y \in \mathbb{R} \times \mathbb{R}^k$. In addition, if $y \in \pi(\Omega \setminus U) \setminus N$, then $(\mathrm{id}_{\mathbb{R}} \times L)^{-1}(y) \cap \mathring{C} \neq \emptyset$, so that Lemma 5.10 implies that $\pi^{-1}(y)$ is the boundary of a compact convex set with non-empty interior in $L^{-1}(y_2)$.

Therefore, $\pi^{-1}(y)$ is bi-Lipschitz homeomorphic to $\mathbb{S}^{n-k-1}$, so that the support of $\chi_{\pi^{-1}(y)} \cdot \mathcal{H}^{n-k-1}$ is $\pi^{-1}(y)$ for such y. In addition, since $\mathbb{R}^n \setminus \Omega$ is $\mathcal{H}^n$-negligible, [20, Theorem 3.2.22] implies that $\pi^{-1}(y) \setminus \Omega$ is $\mathcal{H}^{n-k-1}$-negligible for $\mathcal{H}^{k+1}$-almost every $y \in \mathbb{R} \times \mathbb{R}^k$. Hence, $\overline{\pi^{-1}(y) \cap \Omega \setminus U} = \pi^{-1}(y)$ for $\mathcal{H}^{k+1}$-almost every $y \notin \pi(U)$.

Then, Proposition 2.6 implies that there is $m''' \colon \pi(\mathbb{R}^n) \to \mathbb{C}$ such that $m = m''' \circ \pi$; since π is proper, this implies that m''' is continuous on $\pi(\mathbb{R}^n)$. Finally, since π is proper, $\pi(\mathbb{R}^n)$ is closed, so that the assertion follows from [12, Corollary to Theorem 2 of Chapter IX, § 4, No. 2]. □

Proof of Theorem 5.9. Until the end of this proof, we shall identify $\mathbb{R}^{n_2}$ and $\mathfrak{g}_2^*$ by means of the bijection $\omega \mapsto \omega(\mathbf{T})$; L' will denote $\mathrm{pr}_{1,\dots,n_2'}$, so that $L = \mathrm{id}_{\mathbb{R}} \times L'$. In addition, for every $\gamma \in \mathbb{N}^H$, define $\pi_\gamma \colon \mathbb{R}^{n_2} \ni \omega \mapsto (\mu(\omega)(\mathbf{n_1} + 2\gamma), \omega)$, so that π_γ is continuous and C_γ is the graph of π_γ.

Take $f \in L^1_{L(\mathcal{L}_A)}(G)$ and let m be a representative of $\mathcal{M}_{L(\mathcal{L}_A)}(f)$. Take, for every $\gamma \in \mathbb{N}^H$, a continuous function m_γ on C_γ such that $m_\gamma = \mathcal{M}_{\mathcal{L}_A}(f)\, \chi_{C_\gamma} \cdot \beta_{\mathcal{L}_A}$-almost everywhere. Then, Lemma 5.13 implies that there is a continuous function $m_0' \colon E_{L(\mathcal{L}_A)} \to \mathbb{C}$ such that $m_0 = m_0' \circ L$ *on* C_0. Since $\beta_{L(\mathcal{L}_A)}$ need *not* be equivalent to $L_*(\chi_{C_0} \cdot \beta_{\mathcal{L}_A})$, though, this is not sufficient to conclude.

For every $\gamma \in \mathbb{N}^H$, define $\beta_\gamma \coloneqq \chi_{C_\gamma} \cdot \beta_{\mathcal{L}_A}$, and let $U_{\gamma,1}$ be the union of the components C of Ω such that $\frac{\mathrm{d}}{\mathrm{d}\omega}\mu(\omega)(\mathbf{n_1} + 2\gamma)$ does not vanish on $\ker L'$ for some $\omega \in C$. Let $U_{\gamma,2}$ be the complement of $U_{\gamma,1}$ in Ω. Notice that β_γ is equivalent to $(\pi_\gamma)_*(\mathcal{H}^{n_2})$ by Proposition 4.18. In addition, Lemma 5.6 implies that the following hold:

- $L_*(\chi_{\mathbb{R}\times U_{\gamma,1}} \cdot \beta_\gamma)$ is equivalent to $\chi_{L(\pi_\gamma(U_{\gamma,1}))} \cdot \mathcal{H}^{n_2'+1}$;
- $L_*(\chi_{\mathbb{R}\times U_{\gamma,2}} \cdot \beta_\gamma)$ is equivalent to $\chi_{L(\pi_\gamma(U_{\gamma,2}))} \cdot \mathcal{H}^{n_2'}$;
- $\chi_{\mathbb{R}\times U_{\gamma,2}} \cdot \beta_\gamma$ has a disintegration $(\beta_{\gamma,2,\lambda})_{\lambda \in E_{L(\mathcal{L}_A)}}$ relative to L such that $L^{-1}(\lambda) \cap \pi_\gamma(U_{\gamma,2}) \subseteq \mathrm{Supp}\,(\beta_{\gamma,2,\lambda})$ and $\beta_{\gamma,2,\lambda}$ is equivalent to the measure $\chi_{L^{-1}(\lambda)\cap\pi_\gamma(U_{\gamma,2})} \cdot \mathcal{H}^{n_2 - n_2'}$ for $\mathcal{H}^{n_2'}$-almost every $\lambda \in L(\pi_\gamma(U_{\gamma,2}))$.

In particular, $\beta_{L(\mathcal{L}_A)}$ is equivalent to $\chi_{\sigma(L(\mathcal{L}_A))} \cdot \mathcal{H}^{n_2'+1} + \mu$, where μ is a measure alien to $\mathcal{H}^{n_2'+1}$ and absolutely continuous with respect to $\mathcal{H}^{n_2'}$. Now, observe that $L_*(\chi_{\mathbb{R}\times U_{\gamma,1}} \cdot \beta_\gamma)$ is absolutely continuous with respect to $L_*(\beta_0)$; since $(m - m_0') \circ L$ is β_0-negligible, there is an $L_*(\beta_0)$-negligible subset N of $E_{L(\mathcal{L}_A)}$ such that $m = m_0'$ on $E_{L(\mathcal{L}_A)} \setminus N$. Since N is then $L_*(\chi_{\mathbb{R}\times U_{\gamma,1}} \cdot \beta_\gamma)$-negligible, this implies that $(m - m_0') \circ L$ vanishes $\chi_{\mathbb{R}\times U_{\gamma,1}} \cdot \beta_\gamma$-almost everywhere. Since $m \circ L = m_\gamma$ β_γ-almost everywhere, it follows that $m_0' \circ L = m_\gamma$ $\chi_{\mathbb{R}\times U_{\gamma,1}} \cdot \beta_\gamma$-almost everywhere, hence on

$$\mathrm{Supp}\left(\chi_{\mathbb{R}\times U_{\gamma,1}} \cdot \beta_\gamma\right) = \mathrm{Supp}\left((\pi_\gamma)_*(\chi_{U_{\gamma,1}} \cdot \mathcal{H}^{n_2})\right) = \pi_\gamma(\overline{U_{\gamma,1}}),$$

since $m_0' \circ L$ and m_γ are continuous, while π_γ is proper.

Next, consider $\chi_{\mathbb{R}\times U_{\gamma,2}} \cdot \beta_\gamma$. Tonelli's theorem implies that $L'^{-1}(\lambda_2) \setminus \Omega$ is $\mathcal{H}^{n_2 - n_2'}$-negligible for $\mathcal{H}^{n_2'}$-almost every $\lambda_2 \in \mathbb{R}^{n_2'}$. Now, if $\widetilde{N}$ is an $\mathcal{H}^{n_2'}$-negligible subset of $\mathbb{R} \times \mathbb{R}^{n_2'}$, then $\mathrm{pr}_2(\widetilde{N})$ is $\mathcal{H}^{n_2'}$-negligible since $\mathrm{pr}_2 \colon \mathbb{R} \times \mathbb{R}^{n_2'} \to \mathbb{R}^{n_2'}$ is Lipschitz. Therefore, there is an $\mathcal{H}^{n_2'}$-negligible subset N' of $\mathbb{R}^{n_2'}$ such that, for every $\lambda \in L(\pi_\gamma(U_{\gamma,2})) \setminus (\mathbb{R} \times N')$,

- $m \circ L = m_\gamma$ $\beta_{\gamma,2,\lambda}$-almost everywhere;

- $L^{-1}(\lambda) \cap \pi_\gamma(U_{\gamma,2}) \subseteq \operatorname{Supp}(\beta_{\gamma,2,\lambda})$;
- $L'^{-1}(\lambda_2) \setminus \Omega$ is $\mathcal{H}^{n_2 - n_2'}$-negligible.

Consequently, if $\lambda \in L(\pi_\gamma(U_{\gamma,2})) \setminus (\mathbb{R} \times N')$, then m_γ is constant on $L^{-1}(\lambda) \cap \pi_\gamma(U_{\gamma,2})$. In addition, fix $\lambda \in L(\pi_\gamma(U_{\gamma,2})) \setminus (\mathbb{R} \times N')$; then,

$$L'^{-1}(\lambda_2) = \overline{L'^{-1}(\lambda_2) \cap U_{\gamma,1}} \cup \overline{L'^{-1}(\lambda_2) \cap U_{\gamma,2}},$$

so that either $\overline{L'^{-1}(\lambda_2) \cap U_{\gamma,1}} \cap \overline{L'^{-1}(\lambda_2) \cap U_{\gamma,2}} \neq \emptyset$ or $L'^{-1}(\lambda_2) \cap U_{\gamma,1} = \emptyset$ by connectedness.

Now, let $\mathcal{C}$ be the set of components of $L'^{-1}(\lambda_2) \cap U_{\gamma,2}$; observe that $\mathcal{C}$ is finite since $L'^{-1}(\lambda_2) \cap \Omega$ is semi-algebraic (cf. [16, Proposition 4.13]) and since $L'^{-1}(\lambda_2) \cap U_{\gamma,2}$ is open and closed in $L'^{-1}(\lambda_2) \cap \Omega$. In addition, observe that $\mathrm{pr}_1 \circ \pi_\gamma$ is constant on each $C \in \mathcal{C}$; let $\lambda_{1,C}$ be its constant value. In particular, since $\mathrm{pr}_1 \circ \pi_\gamma$ is proper and since $\mathcal{C}$ is finite, this implies that $L'^{-1}(\lambda_2) \cap U_{\gamma,1} \neq \emptyset$. Further, m_γ is constant on $\pi_\gamma(C) \subseteq L^{-1}(\lambda_{1,C}, \lambda_2) \cap \pi_\gamma(U_{\gamma,2})$ for every $C \in \mathcal{C}$. Now, there is $C_1 \in \mathcal{C}$ such that $L'^{-1}(\lambda_2) \cap \overline{U_{\gamma,1}} \cap \overline{C_1} \neq \emptyset$; since $m_\gamma \circ \pi_\gamma = m_0' \circ L \circ \pi_\gamma$ on $\overline{U_{\gamma,1}}$, and since m_γ is continuous, it follows that $m_\gamma \circ \pi_\gamma = m_0' \circ L \circ \pi_\gamma$ on $\overline{C}_1$. Iterating this procedure, we eventually see that $m_\gamma \circ \pi_\gamma = m_0' \circ L \circ \pi_\gamma$ on $L'^{-1}(\lambda_2)$. Therefore, $m_\gamma = m_0' \circ L$ on $L^{-1}(\lambda) \cap C_\gamma$ for every $\lambda \in L(\pi_\gamma(U_{\gamma,2})) \setminus (\mathbb{R} \times N')$.

Now, observe that $L^{-1}(\mathbb{R} \times N') \cap \pi_\gamma(U_{\gamma,2})$ is $\mathcal{H}^{n_2}$-negligible since $\mathrm{pr}_2 \circ L \circ \pi_\gamma = L'$ and since $\mathcal{H}^{n_2'}$ is equivalent to the (non-Radon) measure $L'_*(\mathcal{H}^{n_2})$. Therefore, $m_\gamma = m_0' \circ L$ β_γ-almost everywhere, hence on C_γ by continuity. By the arbitrariness of γ, this implies that $m_0' \circ L$ is a representative of $\mathcal{M}_{\mathcal{L}_A}(f)$, so that m_0' is a continuous representative of $\mathcal{M}_{L(\mathcal{L}_A)}(f)$. The assertion follows. □

6 Property (S)

In this section we keep the setting of Section 4; our techniques are basically a generalization of those employed in [4, 5]. The first result has very restrictive hypotheses, for the same reasons explained while discussing property (RL), but holds for the 'full family' $\mathcal{L}_A$ (cf. Theorem 6.2); on the contrary, the second one holds under more general assumptions, but only for families of the form $(\mathcal{L}, (-iT_1, \dots, -iT_{n_2'}))$ with $n_2' < n_2$ (cf. Theorem 6.5).

Notice that, even though Theorem 7.4 is the main application of Theorem 6.2, there are other families to which it applies as well. This happens for the family we considered while discussing property (RL) in the case of Theorem 5.4 (notice that Theorems 5.4 and 6.2 have the same assumptions).

Observe that in all the results of this section we impose the condition $W = \{0\}$; this is unavoidable (with our methods), since on W we cannot infer any kind of regularity from the 'inversion formulae' employed. Indeed, our auxiliary function $|x(\omega)|^2$ is not differentiable on W, in general. Nevertheless, this does not mean that property (S) cannot hold when $W \neq \{0\}$, as Theorem 8.3 shows.

Before stating our first result, let us recall a lemma based on some techniques developed in [23] and then in [4].

Lemma 6.1 ([15], Lemma 11.1)**.** *Let $\mathcal{L}'_{A'}$ be a Rockland family on a homogeneous group G', and let $\mathbf{T}' = (T'_1, \dots, T'_{n'})$ be a free family of elements of the centre of the Lie algebra $\mathfrak{g}'$ of G'. Let π_1 be the canonical projection of G' onto its quotient by the normal subgroup $\exp_{G'}(\mathbb{R}T'_1)$, and assume that the following hold:*

- *$(\mathcal{L}'_{A'}, iT'_1, \dots, iT'_{n'})$ satisfies property (RL);*
- *$\mathrm{d}\pi_1(\mathcal{L}'_{A'}, iT'_2, \dots, iT'_{n'})$ satisfies property (S).*

Take $\varphi \in \mathcal{S}_{(\mathcal{L}'_{A'}, iT'_1, \dots, iT'_{n'})}(G')$. Then, there are two families $(\widetilde{\varphi}_\gamma)_{\gamma \in \mathbb{N}^{n'}}$ and $(\varphi_\gamma)_{\gamma \in \mathbb{N}^{n'}}$ of elements of $\mathcal{S}(G', \mathcal{L}'_{A'})$ and $\mathcal{S}_{(\mathcal{L}'_{A'}, iT'_1, \dots, iT'_{n'})}(G')$ (cf. Section 2), respectively, such that

$$\varphi = \sum_{|\gamma| < h} \mathbf{T}'^\gamma \widetilde{\varphi}_\gamma + \sum_{|\gamma| = h} \mathbf{T}'^\gamma \varphi_\gamma$$

for every $h \in \mathbb{N}$.

Theorem 6.2. *Assume that $\dim_{\mathbb{Q}} \mu(\omega)(\mathbb{Q}^H) = \dim_{\mathbb{R}} \mu(\omega)(\mathbb{R}^H)$ for some non-zero $\omega \in \mathfrak{g}_2^*$, and that μ is constant where the mapping $\omega \mapsto \mu_{\iota_0}(\omega)(\mathbf{n_1})$ is constant. Then, $\mathcal{L}_A$ satisfies property (S).*

Proof. We proceed by induction on $n_2 \geqslant 1$.

1. Observe that $\mathcal{L}_A$ satisfies property (RL) by Theorem 5.4, and that the hypotheses imply that $W = \{0\}$; fix $\varphi \in \mathcal{S}_{\mathcal{L}_A}(G)$. Notice that the inductive hypothesis, Corollary 3.4, and Lemma 6.1 imply that we may find a family $(\widetilde{\varphi}_\gamma)$ of elements of $\mathcal{S}(G, \mathcal{L}_I)$, and a family (φ_γ) of elements of $\mathcal{S}_{\mathcal{L}_A}(G)$ such that

$$\varphi = \sum_{|\gamma| < h} (-i\mathbf{T})^\gamma \widetilde{\varphi}_\gamma + \sum_{|\gamma| = h} (-i\mathbf{T})^\gamma \varphi_\gamma$$

for every $h \in \mathbb{N}$.

Define $\widetilde{m}_\gamma \coloneqq \mathcal{M}_{\mathcal{L}_I}(\widetilde{\varphi}_\gamma) \in \mathcal{S}(\sigma(\mathcal{L}_I))$ and $m_\gamma \coloneqq \mathcal{M}_{\mathcal{L}_A}(\varphi_\gamma) \in C_0(\sigma(\mathcal{L}_A))$ for every γ. Then,

$$m_0(\lambda, \omega) = \sum_{|\gamma| < h} \omega^\gamma \widetilde{m}_\gamma(\lambda) + \sum_{|\gamma| = h} \omega^\gamma m_\gamma(\lambda, \omega)$$

for every $h \in \mathbb{N}$ and for every $(\lambda, \omega) \in \sigma(\mathcal{L}_A)$.

2. Assume that $\widetilde{m}_\gamma = 0$ for every $\gamma \in \mathbb{N}^{n_2}$. Define $N(\omega) \coloneqq \mu_{\iota_0}(\omega)(\mathbf{n_1})$ for every $\omega \in \mathfrak{g}_2^*$, so that N is a norm on $\mathfrak{g}_2^*$ which is analytic on $\mathfrak{g}_2^* \setminus \{0\}$ thanks to Proposition 4.13. Define, in addition, $\Sigma \coloneqq \mu(\omega_0)(\mathbf{n_1} + 2\mathbb{N}^H)$ for some (hence every) $\omega_0 \in \mathfrak{g}_2^*$ such that $N(\omega_0) = 1$. Then, set $d \coloneqq \inf_{\gamma \in \Sigma} d(\gamma, \Sigma \setminus \{\gamma\})$, and observe that $d > 0$ since $\dim_{\mathbb{Q}} \mu(\omega_0)(\mathbb{Q}^H) = \dim_{\mathbb{R}} \mu(\omega_0)(\mathbb{R}^H)$ (cf. Lemma 5.1). Finally, identify $\mathfrak{g}_2^*$

with $\mathbb{R}^{n_2}$ by means of the mapping $\omega \mapsto \omega(\mathbf{T})$, take $r \in \left]0, \frac{\min_{\gamma\in\Sigma}|\gamma|}{4d}\right[$, and choose $\tau \in \mathcal{D}(\mathbb{R}^I)$ so that $\chi_{B(0,r)} \leqslant \tau \leqslant \chi_{B(0,2r)}$. Define

$$m(\lambda, \omega) := \begin{cases} \sum_{\gamma\in\Sigma} m_0(N(\omega)\gamma, \omega)\tau\left(\frac{1}{d}\left(\frac{\lambda}{N(\omega)} - \gamma\right)\right) & \text{if } \omega \neq 0 \\ 0 & \text{if } \omega = 0 \end{cases}$$

for every $(\lambda, \omega) \in E_{\mathcal{L}_A}$. Proceeding as in the proof of [4, Lemma 3.1], one sees that $m \in \mathcal{S}(E_{\mathcal{L}_A})$, so that $\varphi \in \mathcal{S}(G, \mathcal{L}_A)$.

3. Now, consider the general case. By a vector-valued version of Borel's lemma (cf. [26, Theorem 1.2.6] for the scalar, one-dimensional case), there is $\widehat{m} \in \mathcal{D}(\mathfrak{g}_2^*; \mathcal{S}(\mathbb{R}^I))$ such that $\partial^\gamma \widehat{m}(0) = \widetilde{m}_\gamma$ for every $\gamma \in \mathbb{N}^{n_2}$. Interpret $\widehat{m}$ as an element of $\mathcal{S}(E_{\mathcal{L}_A})$. Then, **2** implies that $m_0 - \widehat{m}$ induces an element of $\mathcal{S}(\sigma(\mathcal{L}_A))$. The assertion follows. □

Now we consider the case in which $\operatorname{Card}(I) = 1$, and $n_2' < n_2$. We begin with a suitable version of Morse lemma, which is an easy consequence of [27, Lemma C.6.1].

Lemma 6.3. *Let U be an open subset of $\mathbb{R}^k \times \mathbb{R}^n$, and φ a mapping of class C^∞ of U into $\mathbb{R}$. Assume that $\partial_1\varphi(x_0) = 0$ and that $\partial_1^2\varphi(x_0)$ is positive and non-degenerate for some $x_0 \in U$.*[17]

Then, there are an open neighbourhood V_1 of 0 in $\mathbb{R}^k$, an open neighbourhood V_2 of $x_{0,2}$ in $\mathbb{R}^n$, and a C^∞-diffeomorphism ψ from $V_1 \times V_2$ onto an open subset of U such that $\psi(0, x_{0,2}) = x_0$, $\psi_2 = \mathrm{pr}_2$, and

$$\varphi(\psi(y)) = \varphi(\psi(0, y_2)) + \|y_1\|^2$$

for every $y \in V_1 \times V_2$.

Corollary 6.4. *Keep the hypotheses and the notation of Lemma 6.3. Take a function $f \in \mathcal{E}(\psi(V_1 \times V_2) \times \mathbb{R})$ and a function $g\colon V_2 \times \mathbb{R} \to \mathbb{C}$ so that*

$$f(x, \varphi(x)) = g(x_2, \varphi(x))$$

for every $x \in \psi(V_1 \times V_2)$. Then, g can be modified so as to be of class C^∞ in a neighbourhood of $(x_{0,2}, \varphi(x_0))$.

Proof. Indeed, the assumption means that

$$f\left(\psi(y_1, y_2), \varphi(\psi(0, y_2)) + \|y_1\|^2\right) = g\left(y_2, \varphi(\psi(0, y_2)) + \|y_1\|^2\right)$$

[17] Here, ∂_1 denotes the differential along the factor $\mathbb{R}^k$ of $\mathbb{R}^k \times \mathbb{R}^n$.

for every $(y_1, y_2) \in V_1 \times V_2$. Define, for every $y_2 \in V_2$,

$$\widetilde{f}_{y_2} \colon V_1 \ni y_1 \mapsto f\left(\psi(y_1, y_2), \varphi(\psi(0, y_2)) + \|y_1\|^2\right) \qquad \text{and} \qquad \widetilde{g}_{y_2} \colon \mathbb{R} \ni t \mapsto g(y_2, \varphi(\psi(0, y_2)) + t).$$

Then, the mapping $V_2 \ni y_2 \mapsto \widetilde{f}_{y_2}$ belongs to $\mathcal{E}(V_2; \mathcal{E}(V_1))$, and

$$\widetilde{f}_{y_2}(y_1) = \widetilde{g}_{y_2}\left(\|y_1\|^2\right)$$

for every $y_1 \in V_1$ and for every $y_2 \in V_2$.

Now, [40] and the open mapping theorem easily imply that the mapping[18]

$$\Phi_1 \colon \mathcal{E}_{\mathbb{R}}(\mathbb{R}_+) \ni h \mapsto h \circ \|\cdot\|^2 \in \mathcal{E}(\mathbb{R}^k)$$

is an isomorphism onto the set of radial functions of class C^∞ on $\mathbb{R}^k$. Since there is a continuous linear extension operator $\mathcal{E}_{\mathbb{R}}(\mathbb{R}_+) \to \mathcal{E}(\mathbb{R})$ (cf., for instance, [7, Corollary 0.3]), we find a continuous linear mapping $\Phi_2 \colon \Phi_1(\mathcal{E}_{\mathbb{R}}(\mathbb{R}_+)) \to \mathcal{E}(\mathbb{R})$ such that

$$\Phi_2(h) \circ \|\cdot\|^2 = h$$

for every radial function $h \in \mathcal{E}(\mathbb{R}^k)$. Then, take $\tau \in \mathcal{D}(V_1)$ so that τ is radial and equals 1 on a neighbourhood V_1' of 0 in V_1, and define $\widetilde{G}_{y_2} := \Phi_2(\tau \widetilde{f}_{y_2})$. Then, $\widetilde{G}_{y_2}\left(\|y_1\|^2\right) = \widetilde{g}_{y_2}\left(\|y_1\|^2\right)$ for every $y_1 \in V_1'$ and for every $y_2 \in V_2$. In addition, the mapping $y_2 \mapsto \widetilde{G}_{y_2}$ belongs to $\mathcal{E}(V_2; \mathcal{E}(\mathbb{R}))$, so that there is $\widetilde{G} \in \mathcal{E}(V_2 \times \mathbb{R})$ such that $\widetilde{G}(y_2, t) = \widetilde{G}_{y_2}(t)$ for every $y_2 \in V_2$ and for every $t \in \mathbb{R}$. Then,

$$g\left(y_2, \varphi(\psi(0, y_2)) + \|y_1\|^2\right) = \widetilde{G}\left(y_2, \|y_1\|^2\right)$$

for every $y_2 \in V_2$ and for every $y_1 \in V_1'$. Define

$$G \colon V_2 \times \mathbb{R} \ni (y_2, t) \mapsto \widetilde{G}(y_2, t - \varphi(\psi(0, y_2))),$$

so that $G \in \mathcal{E}(V_2 \times \mathbb{R})$ and

$$f(x, \varphi(x)) = G(x_2, \varphi(x))$$

for every $x \in \psi(V_1' \times V_2)$, whence the result. □

Theorem 6.5. *Assume that* $\mathrm{Card}(I) = 1$ *and that* $W = \{0\}$*, and let* S' *be the analytic hypersurface* $\{\omega \in \mathfrak{g}_2^* \colon \mu(\omega)(\mathbf{n_1}) = 1\}$*. Take* $n_2' \in \{0, \dots, n_2 - 1\}$ *and assume that, for every* $\omega \in S'$ *such that*

[18] We denote by $\mathcal{E}_{\mathbb{R}}(\mathbb{R}_+)$ the quotient of $\mathcal{E}(\mathbb{R})$ by the set of $\varphi \in \mathcal{E}(\mathbb{R})$ which vanish on $\mathbb{R}_+$.

$\langle T_1, \dots, T_{n'_2} \rangle^\circ \subseteq T_\omega(S')$, *the normal curvatures of S' at ω along the one-dimensional vector subspaces of $\langle T_1, \dots, T_{n'_2} \rangle$ are all non-zero. Then, the family $\mathcal{L}_{A'} \coloneqq (\mathcal{L}, (-iT_1, \dots, -iT_{n'_2}))$ satisfies property (S).*

The condition on S' is satisfied, for example, if $\omega \mapsto \mu(\omega)(\mathbf{n_1})$ is a hilbertian norm. In addition, since S' is the boundary of a convex set, the condition on S' is satisfied also if, for every $\omega \in S'$ such that $\langle T_1, \dots, T_{n'_2} \rangle^\circ \subseteq T_\omega(S')$, the Gaussian curvature of S' at ω is not zero.

Proof. **1.** We proceed by induction on n'_2. The assertion follows from [34, Corollary 1.3 and Theorem 1.4] when $n'_2 = 0$. Then, assume that $n'_2 > 0$, so that the inductive assumption and Corollary 3.4 imply that $\mathrm{d}\pi_1(\mathcal{L}_{A'})$ satisfies property (S), where π_1 is the canonical projection of G onto $G/_{\exp_G(\mathbb{R}T_1)}$. In addition, Theorem 5.8 implies that $\mathcal{L}_{A'}$ satisfies property (RL). To simplify the notation, we shall identify $\mathfrak{g}_2^*$ with $\mathbb{R}^{n_2}$ by means of the mapping $\omega \mapsto \omega(\mathbf{T})$.

Take $\varphi \in \mathcal{S}_{\mathcal{L}_{A'}}(G)$. Then, Lemma 6.1 implies that we may find a family $(\widetilde{\varphi}_\gamma)_{\gamma \in \mathbb{N}^{n'_2}}$ of elements of $\mathcal{S}(G, \mathcal{L})$, and a family $(\varphi_\gamma)_{\gamma \in \mathbb{N}^{n'_2}}$ of elements of $\mathcal{S}_{\mathcal{L}_{A'}}(G)$ such that[19]

$$\varphi = \sum_{|\gamma| < h} (-i\mathbf{T})^\gamma \widetilde{\varphi}_\gamma + \sum_{|\gamma| = h} (-i\mathbf{T})^\gamma \varphi_\gamma$$

for every $h \in \mathbb{N}$.

Define $\widetilde{m}_\gamma \coloneqq \mathcal{M}_{\mathcal{L}}(\widetilde{\varphi}_\gamma) \in \mathcal{S}(\sigma(\mathcal{L}))$ and $m_\gamma \coloneqq \mathcal{M}_{\mathcal{L}_{A'}}(\varphi_\gamma) \in C_0(\sigma(\mathcal{L}_{A'}))$ for every γ. Then,

$$m_0(\lambda, \omega') = \sum_{|\gamma| < h} \omega'^\gamma \widetilde{m}_\gamma(\lambda) + \sum_{|\gamma| = h} \omega'^\gamma m_\gamma(\lambda, \omega')$$

for every $h \in \mathbb{N}$ and for every $(\lambda, \omega') \in \sigma(\mathcal{L}_{A'})$.

2. As in the proof of Theorem 6.2, we may reduce to the case in which $\widetilde{m}_\gamma = 0$ for every γ. Let $L\colon E_{\mathcal{L}_A} \to E_{\mathcal{L}_{A'}}$ be the unique linear mapping such that $L(\mathcal{L}_A) = \mathcal{L}_{A'}$; then, $\langle T_1, \dots, T_{n'_2} \rangle^\circ$ is identified with $\ker L$. In addition, if $L'\colon \mathbb{R}^{n_2} \to \mathbb{R}^{n'_2}$ is the projection onto the first n'_2 components of $\mathbb{R}^{n_2}$, then $L = \mathrm{id}_{\mathbb{R}} \times L'$. Now, define

$$\widetilde{M}(\omega) \coloneqq \int_G \varphi(x,t) e^{-\frac{1}{4}|x(\omega)|^2 - i\omega(t)} \,\mathrm{d}(x,t)$$

for every $\omega \in \mathbb{R}^{n_2}$. Let us prove that $\widetilde{M} \in \mathcal{S}(\mathbb{R}^{n_2})$ and that $\widetilde{M}$ vanishes of order ∞ at 0.

Indeed, define $N(\omega) \coloneqq \mu(\omega)(\mathbf{n_1})$, for every $\omega \in \mathbb{R}^{n_2}$, and choose $p_1, p_2, p_3 \in \mathbb{N}$; observe that, for

[19] By an abuse of notation, identify $\mathbb{N}^{n'_2}$ with $\mathbb{N}^{n'_2} \times \{0\}^{n_2 - n'_2}$, and define $\mathbf{T}^\gamma$ for $\gamma \in \mathbb{N}^{n'_2}$ accordingly.

every $\omega\in\mathbb{R}^{n_2}$,

$$
\begin{aligned}
N(\omega)^{p_3}\widetilde{M}(\omega)&=\int_G(\mathcal{L}^{p_3}\varphi)(x,t)e^{-\frac{1}{4}|x(\omega)|^2-i\omega(t)}\,\mathrm{d}(x,t)\\
&=\sum_{|\gamma|=p_2}\int_G((-i\mathbf{T})^{\gamma}\mathcal{L}^{p_3}\varphi_\gamma)(x,t)e^{-\frac{1}{4}|x(\omega)|^2-i\omega(t)}\,\mathrm{d}(x,t)\\
&=\sum_{|\gamma|=p_2}\omega^{\gamma}\int_G(\mathcal{L}^{p_3}\varphi_\gamma)(x,t)e^{-\frac{1}{4}|x(\omega)|^2-i\omega(t)}\,\mathrm{d}(x,t).
\end{aligned}
$$

Therefore, by means of Leibniz's rule and Faà di Bruno's formula, we see that

$$
\begin{aligned}
\widetilde{M}^{(p_1)}(\omega)=\sum_{|\gamma|=p_2}\sum_{q_1+q_2=p_1}\sum_{\sum_{\ell=1}^{q_2}\ell s_\ell=q_2}\frac{p_1!}{q_1!\ell!}\frac{\mathrm{d}^{q_1}}{\mathrm{d}\omega^{q_1}}\left(\frac{\omega^\gamma}{N(\omega)^{p_3}}\right)\int_G(\mathcal{L}^{p_3}\varphi_\gamma)(x,t)e^{-\frac{1}{4}|x(\omega)|^2-i\omega(t)}\times\\
\times\prod_{\ell=1}^{q_2}\left(\frac{1}{\ell!}\frac{\mathrm{d}^{\ell}}{\mathrm{d}\omega^{\ell}}\left(-\frac{1}{4}|x(\omega)|^2-i\omega(t)\right)\right)^{s_\ell}\,\mathrm{d}(x,t)
\end{aligned}
$$

for every non-zero $\omega\in\mathbb{R}^{n_2}$, where the last product must be interpreted as the symmetrized product of symmetric multilinear forms. Now, taking into account the fact that $|x(\omega)|^2=\langle|J_{Q,\omega}|x|x\rangle$ for every $\omega\in\mathbb{R}^{n_2}$ and for every $x\in\mathfrak{g}_1$, and arguing by homogeneity, we see that there is a constant $C_{p_1,p_2,p_3}>0$ such that

$$
\left|\frac{\mathrm{d}^{q_1}}{\mathrm{d}\omega^{q_1}}\left(\frac{\omega^\gamma}{N(\omega)^{p_3}}\right)\right|\leqslant C_{p_1,p_2,p_3}N(\omega)^{p_2-p_3-q_1}
$$

and such that

$$
\left|\prod_{\ell=1}^{q_2}\left(\frac{1}{\ell!}\frac{\mathrm{d}^{\ell}}{\mathrm{d}\omega^{\ell}}\left(-\frac{1}{4}|x(\omega)|^2-i\omega(t)\right)\right)^{s_\ell}\right|\leqslant C_{p_1,p_2,p_3}\left(|x|^2+|t|\right)^{|s|}N(\omega)^{|s|-q_2}
$$

for every non-zero $\omega\in\mathbb{R}^{n_2}$, for every $(x,t)\in G$, for every $\gamma\in\mathbb{N}^{n_2}$ such that $|\gamma|=p_2$, for every $q_1,q_2\in\mathbb{N}$ such that $q_1+q_2=p_1$, and for every $s\in\mathbb{N}^{q_2}$ such that $\sum_{\ell=1}^{q_2}\ell s_\ell=q_2$.

Therefore, there is a constant $C'_{p_1,p_2,p_3}>0$, such that

$$
\left|\widetilde{M}^{(p_1)}(\omega)\right|\leqslant C'_{p_1,p_2,p_3}(1+N(\omega))^{p_1}N(\omega)^{p_2-p_1-p_3}
$$

for every non-zero $\omega\in\mathbb{R}^{n_2}$.

Now, choosing $p_3=0$ and $p_2>p_1$, we see that $\widetilde{M}^{(p_1)}$ can be extended by continuity at 0, and that $\widetilde{M}^{(p_1)}(0)=0$. By the arbitrariness of p_1, we then see that $\widetilde{M}\in\mathcal{E}(\mathbb{R}^{n_2})$ and that $\widetilde{M}$ vanishes of order ∞ at 0. In addition, taking $p_2=0$ and p_3 arbitrarily large, we see that $\widetilde{M}^{(p_1)}$ decays at ∞ faster than any polynomial. Hence, $\widetilde{M}\in\mathcal{S}(\mathbb{R}^{n_2})$.

3. Now, let Σ be the graph of N, so that Σ equals $\mathbb{R}_+(\{1\}\times S')$ and $\Sigma\setminus\{0\}$ is a closed analytic subset of $E_{\mathcal{L}_A} \setminus \{0\}$. Take $\tau \in \mathcal{D}(\mathbb{R}_+^*)$ such that $\tau(1) = 1$, and define

$$\widetilde{m} \colon E_{\mathcal{L}_A} \ni (\lambda, \omega) \mapsto \widetilde{M}(\omega)\tau\left(\frac{\lambda}{N(\omega)}\right) \in \mathbb{C},$$

with the convention that $\widetilde{m}(\lambda, 0) = 0$ for every $\lambda \in \mathbb{R}$. Then, by means of **2** we see that $\widetilde{m}$ is an element of $\mathcal{S}(E_{\mathcal{L}_A})$ and vanishes of order infinity at 0; in addition, $\widetilde{m}(N(\omega), \omega) = \widetilde{M}(\omega)$ for every $\omega \in \mathbb{R}^{n_2}$, so that $\widetilde{m}$ equals a representative of $\mathcal{M}_{\mathcal{L}_A}(\varphi)$ *on* Σ thanks to Proposition 4.18.

Now, observe that

$$m_0 \circ L = \widetilde{m}$$

on Σ. In addition, L is proper on the convex envelope of Σ, so that $L(\Sigma \setminus \{0\}) = \sigma(\mathcal{L}_{A'}) \setminus \{0\}$ is a subanalytic closed convex cone in $E_{\mathcal{L}_{A'}} \setminus \{0\}$, hence Nash subanalytic in $E_{\mathcal{L}_{A'}} \setminus \{0\}$. Using the fact that $\widetilde{m}$ vanishes of order ∞ at 0 and applying Theorem 2.8, in order to prove that $m_0 \in \mathcal{S}_{E_{\mathcal{L}_{A'}}}(\sigma(\mathcal{L}_{A'}))$ it suffices to show that $\widetilde{m}$ is a formal composite of L (on Σ). Then, take $\omega \in S'$, and observe that $\ker L' = \ker L = \langle T_1, \dots, T_{n_2'} \rangle^\circ$. If $\ker L' \not\subseteq T_\omega(S')$, then the restriction of L' to S' is a submersion at ω, so that the assertion follows in this case. Otherwise, as in the proof of Lemma 5.7 we see that $L'^{-1}(L'(\omega)) = \{\omega\}$. Then, observe that, since N is convex and $S' = N^{-1}(\{1\})$, S' is the boundary of a convex set; thus, in the vicinity of ω S' equals the 'graph' $\{\omega + x + \psi(x)v \colon x \in U\}$, for some neighbourhood U of 0 in $T_\omega(S')$, and for some convex analytic function $\psi \colon U \to \mathbb{R}$; here, v denotes the inward-pointing normal unit vector of S' (oriented as the boundary of $N^{-1}([0,1])$) at ω. Then, the hypotheses and the convexity of ψ imply that $\partial^2_{\ker L'}\psi(\omega)$ is positive and non-degenerate. Consequently, we are able to apply Corollary 6.4, where φ corresponds to ψ, $\mathbb{R}^k$ corresponds to $\ker L'$, $\mathbb{R}^n$ corresponds to $T_\omega(S') \cap \ker L'^\perp$, x_0 corresponds to 0, f corresponds $\widetilde{m}$, and g corresponds to m_0 (under suitable identifications). Then, our assertion holds also in this case. By homogeneity, the assertion follows for every $\omega \neq 0$. Then, $m_0 \in \mathcal{S}_{E_{\mathcal{L}_{A'}}}(\sigma(\mathcal{L}_{A'}))$, whence the result. $\square$

7 Examples: H-Type Groups

In this section we shall deal with the following situation: G is an H-type group and there is a finite family $(\mathfrak{v}_\iota)_{\iota \in I}$ of subspaces of $\mathfrak{g}_1$ such that $\mathfrak{g}_1 = \bigoplus_{\iota \in I} \mathfrak{v}_\iota$, such that $\mathfrak{v}_\iota \oplus \mathfrak{g}_2$, with the induced structure, is an H-type Lie algebra for every $\iota \in I$, and such that $\mathfrak{v}_{\iota_1}$ and $\mathfrak{v}_{\iota_2}$ commute and are orthogonal for every $\iota_1, \iota_2 \in I$ such that $\iota_1 \neq \iota_2$. We shall define $\mathbf{n_1} \coloneqq \left(\frac{1}{2} \dim \mathfrak{v}_\iota\right)_{\iota \in I}$.

We shall then consider, for every $\iota \in I$, the group of linear isometries $O(\mathfrak{v}_\iota)$ of $\mathfrak{v}_\iota$, and define a canonical action of $O \coloneqq \prod_{\iota \in I} O(\mathfrak{v}_\iota)$ on the *vector space* subjacent to $\mathfrak{g}$ as follows: $(L_\iota)((v_\iota), t) \coloneqq ((L_\iota(v_\iota)), t)$ for every $(L_\iota) \in O$ and for every $((v_\iota), t) \in \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

A projector of $\mathcal{D}'(G)$ is then canonically defined as follows:

$$\pi_*(T) \coloneqq \int_O (L\,\cdot\,)_*(T)\,\mathrm{d}\nu_O(L)$$

for every $T \in \mathcal{D}'(G)$; here, ν_O denotes the *normalized* Haar measure on O.

Proposition 7.1. *The following hold:*

1. *π induces a continuous projection on $\mathcal{D}'^r(G)$, $\mathcal{S}'(G)$, $\mathcal{E}'^r(G)$, $\mathcal{E}^r(G)$, $\mathcal{S}(G)$, $\mathcal{D}^r(G)$ and $L^p(G)$ for every $r \in \mathbb{N} \cup \{\infty\}$ and for every $p \in [1,\infty]$;*

2. *if $\varphi_1, \varphi_2 \in \mathcal{D}(G)$, then*

$$\langle \pi_*(\varphi_1), \varphi_2 \rangle = \langle \varphi_1, \pi_*(\varphi_2) \rangle \qquad \textit{and} \qquad \langle \pi_*(\varphi_1) | \varphi_2 \rangle = \langle \varphi_1 | \pi_*(\varphi_2) \rangle\,;$$

3. *if μ is a positive measure on G, then also $\pi_*(\mu)$ is a positive measure; in addition, $\pi_*(\nu_G) = \nu_G$;*

4. *if $T \in \mathcal{D}'(G)$ is O-invariant, then also $\check{T}$ is O-invariant;*

5. *if T is supported at e, then $\pi_*(T)$ is supported at e;*

6. *if $\varphi_1, \varphi_2 \in \mathcal{D}(G)$ and either φ_1 or φ_2 is O-invariant, then*

$$\pi_*(\varphi_1 * \varphi_2) = \pi_*(\varphi_1) * \pi_*(\varphi_2) = \pi_*(\varphi_2) * \pi_*(\varphi_1) = \pi_*(\varphi_2 * \varphi_1).$$

The proof is based on [17] and is omitted.

Now, let $\mathcal{L}_\iota$ be the differential operator corresponding to the restriction of the scalar product to $\mathfrak{v}_\iota^*$; in other words, $\mathcal{L}_\iota$ is minus the sum of the squares of the elements of any orthonormal basis of $\mathfrak{v}_\iota$. Let $T_1, \dots, T_{n_2}$ be an orthonormal basis of $\mathfrak{g}_2$, and define $\mathcal{L}_A \coloneqq ((\mathcal{L}_\iota)_{\iota \in I}, (-iT_1, \dots, -iT_{n_2}))$.

Recall (cf. [17]) that a left-invariant differential operator X is π-radial if and only if $\pi_*(X_e) = X_e$, that is, if and only if X_e is O-invariant. Nevertheless, this does *not* imply that X is O-invariant.

Proposition 7.2. *$\mathcal{L}_A$ is a Rockland family and generates (algebraically) the unital algebra of left-invariant differential operators which are π-radial.*

Proof. Observe first that $T_1, \dots, T_{n_2}$ are clearly π-radial. On the other hand, a direct computation shows that, for every $\iota \in I$, $(\mathcal{L}_\iota)_e = -\sum_{v \in B_\iota} \partial_v^2$, where B_ι is any orthonormal basis of $\mathfrak{v}_\iota$. Hence, $(\mathcal{L}_\iota)_e$ is O-invariant. Consequently, Proposition 7.1 implies that the family $\mathcal{L}_A$ is commutative; since $\sum_{\iota \in I} \mathcal{L}_\iota$ is the operator associated with the scalar product of $\mathfrak{g}_1^*$, it is then clear that $\mathcal{L}_A$ is a Rockland family.

Now, take an O-invariant distribution S on G which is supported at e. Let $p\colon G \to G/_{[G,G]}$ be the canonical projection. Then, $p_*(S)$ is O-invariant and supported at $p(e)$. By means of the Fourier transform, we then see that there is a unique polynomial $P_0 \in \mathbb{R}[I]$ such that $p_*(S) = P_0(p_*(\mathcal{L}_I))_e$. Therefore, there are $S_1, \dots, S_{n_2} \in \mathcal{D}'(G)$ such that $\operatorname{Supp}(S_k) \subseteq \{e\}$ for every $k = 1, \dots, n_2$, and such that

$$S = P_0(\mathcal{L}_I)_e + \sum_{k=1}^{n_2} (T_k)_e * S_k.$$

Then, Proposition 7.1 implies that

$$S = \pi_*(S) = P_0(\mathcal{L}_I)_e + \sum_{k=1}^{n_2} (T_k)_e * \pi_*(S_k),$$

so that we may assume that $S_1, \dots, S_k$ are O-invariant. Arguing by induction, it then follows that S belongs to the unital algebra (algebraically) generated by $(\mathcal{L}_A)_e$. $\square$

Now, we shall consider some image families of $\mathcal{L}_A$. More precisely, we shall fix a non-empty finite set I' and $\mu \in (\mathbb{R}^I)^{I'}$ so that the induced linear mapping from $\mathbb{R}^I$ into $\mathbb{R}^{I'}$ is proper on $\mathbb{R}^I_+$. Then, we shall define $L\colon E_{\mathcal{L}_A} \ni (\lambda, \omega) \mapsto (\mu(\lambda), \omega) \in \mathbb{R}^{I'} \times \mathbb{R}^{n_2}$ and consider the family $L(\mathcal{L}_A)$. Then, $L(\mathcal{L}_A)$ is a Rockland family since L is proper on $\sigma(\mathcal{L}_A)$ by construction.

Proposition 7.3. *Set $d \coloneqq \dim_{\mathbb{Q}} \mu(\mathbb{Q}^I)$. Then, there are a $\beta_{L(\mathcal{L}_A)}$-measurable function $m\colon E_{L(\mathcal{L}_A)} \to \mathbb{C}^d$ and a linear mapping $L'\colon \mathbb{R}^d \to \mathbb{R}^{I'}$ such that the following hold:*

- *there is $\mu' \in (\mathbb{Q}^I)^d$ such that the associated linear mapping $\mu'\colon \mathbb{R}^I \to \mathbb{R}^d$ is proper on $\mathbb{R}^I_+$, and such that $m(L(\mathcal{L}_A)) = \mu'(\mathcal{L}_I)$;*
- *$(L'(m(L(\mathcal{L}_A)), (-iT_j)_{j=1}^{n_2}) = L(\mathcal{L}_A)$;*
- *m equals $\beta_{L(\mathcal{L}_A)}$-almost everywhere a continuous function if and only if $d = \dim_{\mathbb{R}} \mu(\mathbb{R}^I)$.*

Proof. Notice first that there are d linearly independent $\mathbb{Q}$-linear functionals $p_1, \dots, p_d$ on $\mu(\mathbb{Q}^I)$; define $\mu' \in (\mathbb{Q}^I)^d$ so that $\mu'_{h,\iota} \coloneqq p_h((\mu_{\iota',\iota})_{\iota' \in I'})$ for every $h = 1, \dots, d$ and for every $\iota \in I$. In addition, define $\mathcal{L}''_h \coloneqq \mu'_h(\mathcal{L}_I) = \sum_{\iota \in I} \mu'_{h,\iota} \mathcal{L}_\iota$ for every $h = 1, \dots, d$. Next, take $h \in \{1, \dots, d\}$, and observe that, if $\omega \in \mathbb{R}^{n_2} \setminus \{0\}$ and $\gamma_1, \gamma_2 \in \mathbb{N}^I$ are such that

$$(|\omega| \mu(\mathbf{n_1} + 2\gamma_1), \omega) = (|\omega| \mu(\mathbf{n_1} + 2\gamma_2), \omega),$$

then $\mu(\gamma_1 - \gamma_2) = 0$, so that $\mu'(\gamma_1 - \gamma_2) = 0$, and then

$$(|\omega| \mu'(\mathbf{n_1} + 2\gamma_1), \omega) = (|\omega| \mu'(\mathbf{n_1} + 2\gamma_2), \omega).$$

Hence, there is a $\beta_{\mathcal{L}_A}$-measurable function $m\colon E_{L(\mathcal{L}_A)} \to \mathbb{R}^d$ such that

$$m_h(L(\lambda, \omega)) = \mu'_h(\lambda)$$

for every $(\lambda, \omega) \in \sigma(\mathcal{L}_A) \cap (\mathbb{R}^I \times (\mathbb{R}^{n_2} \setminus \{0\}))$ (cf. Proposition 4.18). Then, $\mathcal{L}''_h \delta_e = \mathcal{K}_{L(\mathcal{L}_A)}(m_h)$ for every $h = 1, \dots, d$ (cf. Proposition 4.18). Next, observe that the mapping $p = (p_1, \dots, p_d) \colon \mu(\mathbb{Q}^I) \to \mathbb{Q}^d$ is an isomorphism; let $L' \colon \mathbb{Q}^d \to \mathbb{R}^{I'}$ be the composite of the inverse of p with the canonical inclusion $\mu(\mathbb{Q}^I) \subseteq \mathbb{R}^{I'}$. If $(L'_{\iota'})_{\iota' \in I'}$ are the components of L', then $\sum_{h=1}^d L'_{\iota',h} \mu'_h = \mu_{\iota'}$, whence $(L'(m(L(\mathcal{L}_A))), (-iT_j)_{j=1}^{n_2}) = L(\mathcal{L}_A)$.

If $d = \dim_\mathbb{R} \mu(\mathbb{R}^I)$, then the mapping $(\lambda, \omega) \mapsto (m(\lambda, \omega), \omega)$ induces a homeomorphism of $\sigma(L(\mathcal{L}_A))$ onto $\sigma(\mathcal{L}''_1, \dots, \mathcal{L}''_d, (-iT_h)_{h=1}^{n_2})$. Conversely, assume that m can be taken so as to be continuous, and denote by M the continuous mapping $(\lambda, \omega) \mapsto (m(\lambda, \omega), \omega)$. Then, the preceding arguments show that $M \circ (L' \times \mathrm{id}_{\mathbb{R}^{n_2}}) = \mathrm{id}_{\mathbb{R}^d \times \mathbb{R}^{n_2}}$ and $(L' \times \mathrm{id}_{\mathbb{R}^{n_2}}) \circ M = \mathrm{id}_{\mathbb{R}^{I'} \times \mathbb{R}^{n_2}}$ $\beta_{(\mathcal{L}''_1, \dots, \mathcal{L}''_d, (-iT_h)_{h=1}^{n_2})}$-almost everywhere and $\beta_{L(\mathcal{L}_A)}$-almost everywhere, respectively; by continuity, the same equalities hold on $\sigma(\mathcal{L}''_1, \dots, \mathcal{L}''_d, (-iT_h)_{h=1}^{n_2})$ and $\sigma(L(\mathcal{L}_A))$, respectively. Consequently, $L' \times \mathrm{id}_{\mathbb{R}^{n_2}}$ induces a homeomorphism of $\mu'(\mathbb{R}^I_+) \times \{0\}^{n_2}$ onto $\mu(\mathbb{R}^I_+) \times \{0\}^{n_2}$, so that these two convex cones must have the same dimension. Hence, $d = \dim_\mathbb{R}(\mu(\mathbb{R}^I))$. ◻

Theorem 7.4. *The following conditions are equivalent:*

(i) $\chi_{L(\mathcal{L}_A)}$ *has a continuous representative;*

(ii) $L(\mathcal{L}_A)$ *satisfies property (RL);*

(iii) for every $\varphi \in \mathcal{S}_{L(\mathcal{L}_A)}(G)$ *there is* $m \in C_0(\sigma(L(\mathcal{L}_A)))$ *such that* $\varphi = \mathcal{K}_{L(\mathcal{L}_A)}(m)$*;*

(iv) $L(\mathcal{L}_A)$ *satisfies property (S);*

(v) $L(\mathcal{L}_A)$ *is functionally complete;*

(vi) $\dim_\mathbb{Q} \mu(\mathbb{Q}^I) = \dim_\mathbb{R} \mu(\mathbb{R}^I)$.

Proof. **(i) $\implies$ (ii).** Obvious.

(ii) $\implies$ (iii). Obvious.

(iii) $\implies$ (vi). Assume, on the contrary, that $\dim_\mathbb{Q} \mu(\mathbb{Q}^I) > \dim_\mathbb{R} \mu(\mathbb{R}^I)$, and keep the notation of Proposition 7.3. Then, m_h cannot be taken so as to be continuous for some $h \in \{1, \dots, d\}$. Take $\varphi \in \mathcal{S}(E_{L(\mathcal{L}_A)})$ so that $\varphi(\lambda) \neq 0$ for every $\lambda \in E_{\mathcal{L}_A}$. Then,

$$\mathcal{K}_{L(\mathcal{L}_A)}(m_h \varphi) = \mu'_h(\mathcal{L}_I) \mathcal{K}_{L(\mathcal{L}_A)}(\varphi) \in \mathcal{S}(G),$$

but $m_h \varphi$ is not equal $\beta_{L(\mathcal{L}_A)}$-almost everywhere to any continuous functions, whence the result.

(vi) $\implies$ (iv). This follows from Theorem 6.2.

(iv) $\implies$ (v). This follows from Proposition 2.12.

(v) $\implies$ (vi). This follows from Proposition 7.3.

(vi) $\implies$ (i). This follows from Theorem 5.3. ◻

8 Examples: Products of Heisenberg Groups

In this section, $(G_\alpha)_{\alpha\in A}$ will be a family of Heisenberg groups each of which is endowed with a homogeneous sub-Laplacian $\mathcal{L}_\alpha$. Define $\mathcal{L} \coloneqq \sum_{\alpha\in A} \mathcal{L}_\alpha$, and denote by $\mathcal{T}$ a finite family of elements of $\mathfrak{g}_2$, which is the centre of the Lie algebra of $G \coloneqq \prod_{\alpha\in A} G_\alpha$.

Before we proceed to the main results of these section, let us introduce some more notation. For every $\alpha \in A$, we shall denote by T_α a non-zero element of the centre of the Lie algebra of G_α, so that we may identify $\mathfrak{g}_2$ with $\bigoplus_{\alpha\in A} \mathbb{R}T_\alpha$. Then, Proposition 4.6 and the remarks following its statement imply that there is a basis $(X_{\alpha,1}, \dots, X_{\alpha,2n_{1,\alpha}}, T_\alpha)$ of the Lie algebra of G_α such that $[X_{\alpha,k}, X_{\alpha,n_{1,\alpha}+k}] = T_\alpha$ for every $k = 1, \dots, n_{1,\alpha}$, while the other commutators vanish, and such that there is $\mu_\alpha \in (\mathbb{R}_+^*)^{n_{1,\alpha}}$ such that

$$\mathcal{L}_\alpha = -\sum_{k=1}^{n_{1,\alpha}} \mu_{\alpha,k}(X_{\alpha,k}^2 + X_{\alpha,n_{1,\alpha}+k}^2).$$

We shall denote by $\mathfrak{g}_{1,\alpha}$ the vector space generated by $X_{\alpha,1}, \dots, X_{\alpha,2n_{1,\alpha}}$, and we shall define $\mathbf{n_1} \coloneqq (n_{1,\alpha})_{\alpha\in A}$.

Proposition 8.1. *Assume that* $\mathrm{Card}(A) \geqslant 2$. *If $\mathcal{T}$ generates $\mathfrak{g}_2$, then the families $(\mathcal{L}, -i\mathcal{T})$ and $(\mathcal{L}_A, -i\mathcal{T})$ are functionally equivalent. In addition, $(\mathcal{L}, -i\mathcal{T})$ does not satisfy properties (RL) and (S).*

Proof. See Theorem 5.5 and its proof. □

Lemma 8.2. *Let μ' be a linear mapping of* $\mathbb{R}^n$ onto $\mathbb{R}^m$ *which is proper on $\mathbb{R}^n_+$. Define $\Sigma_0 \coloneqq \mu'(\mathbb{R}^n_+) \times \{0\}$, and*

$$\Sigma \coloneqq \{(\lambda\mu'(\mathbf{1}_n + 2\gamma), \lambda) \colon \lambda > 0, \gamma \in \mathbb{N}^n\}.$$

If $\varphi \in \mathcal{E}(\mathbb{R}^m \times \mathbb{R})$ vanishes on Σ, then φ vanishes of order ∞ on Σ_0. In particular, the closure of Σ in the Zariski topology is $\mathbb{R}^m \times \mathbb{R}$.

Proof. Take $x = (\lambda\mu'(\mathbf{1}_n + 2\gamma), 0)$ for some $\lambda > 0$ and some $\gamma \in \mathbb{N}^n$. Then, for every $k \in \mathbb{N}$,

$$\left(x_1, \frac{\lambda}{2k+1}\right) = \left(\frac{\lambda}{2k+1}\mu'(\mathbf{1}_n + 2((2k+1)\gamma + k\mathbf{1}_n)), \frac{\lambda}{2k+1}\right) \in \Sigma.$$

Therefore, it is easily seen that $\partial_2^h \varphi(x) = 0$ for every $h \in \mathbb{N}$. Since the set

$$\{(\lambda\mu'(\mathbf{1}_n + 2\gamma), 0) \colon \lambda > 0, \gamma \in \mathbb{N}^n\}$$

is dense in Σ_0, it follows that $\partial_2^h \varphi$ vanishes on Σ_0 for every $h \in \mathbb{N}$. Now, since $\mu'(\mathbb{R}^n) = \mathbb{R}^m$, the closed convex cone Σ_0 generates $\mathbb{R}^m \times \{0\}$, so that Σ_0 is the closure of its interior in $\mathbb{R}^m \times \{0\}$. The assertion follows easily. □

Theorem 8.3. *Assume that* $\mathrm{Card}(A) \geqslant 2$*. If* $\mathcal{T}$ *does not generate* $\mathfrak{g}_2$*, then the family* $(\mathcal{L}, -i\mathcal{T})$ *satisfies properties* (RL) *and* (S)*.*

Proof. **1.** Let us prove that $(\mathcal{L}, -i\mathcal{T})$ satisfies property (RL). Consider the Rockland family $(\mathcal{L}, -iT_A)$, where $T_A = (T_\alpha)_{\alpha \in A}$, and define

$$C_\gamma \coloneqq \left\{ \left(\sum_{\alpha \in A} |\omega_\alpha| \mu_\alpha (\mathbf{1}_{n_{1,\alpha}} + 2\gamma_\alpha), \omega \right) : \omega \in \mathbb{R}^A \right\}$$

for every $\gamma \in \mathbb{N}^{\mathbf{n_1}}$, so that C_0 is the boundary of a convex polyhedron which contains C_γ for every $\gamma \in \mathbb{N}^{\mathbf{n_1}}$. If $L \colon E_{(\mathcal{L}, -iT_A)} \to E_{(\mathcal{L}, -i\mathcal{T})}$ is the unique linear mapping such that $L(\mathcal{L}, -iT_A) = (\mathcal{L}, -i\mathcal{T})$, then L is proper on the convex envelope of C_0, so that $\chi_{C_0} \cdot \beta_{(\mathcal{L}, -iT_A)}$ is L-connected by Proposition 2.4. In addition, $\sigma(\mathcal{L}, -i\mathcal{T}) = L(\sigma(\mathcal{L}, -iT_A)) = L(C_0)$ is a convex polyhedron. Now, define $\mathcal{L}'_{A'} \coloneqq ((-X_{\alpha,k}^2 - X_{\alpha, n_{1,\alpha}+k}^2)_{k=1,\dots,n_{1,\alpha}}, -iT_\alpha)_{\alpha \in A}$, so that $\mathcal{L}'_{A'}$ satisfies properties (RL) and (S) by Theorems 2.2 and 7.4. Take $f \in L^1_{(\mathcal{L}, -i\mathcal{T})}(G)$, and define $\widetilde{m} \coloneqq \mathcal{M}_{\mathcal{L}'_{A'}}(f) \in C_0(\sigma(\mathcal{L}'_{A'}))$. Then,

$$m_\gamma \colon C_\gamma \ni \left(\sum_{\alpha \in A} |\omega_\alpha| \mu_\alpha (\mathbf{1}_{n_{1,\alpha}} + 2\gamma_\alpha), \omega \right) \mapsto \widetilde{m}((|\omega_\alpha| (\mathbf{1}_{n_{1,\alpha}} + 2\gamma_\alpha), \omega_\alpha)_{\alpha \in A})$$

is a continuous function on C_γ which equals $\mathcal{M}_{(\mathcal{L}, -iT_A)}(f)\, \chi_{C_\gamma} \cdot \beta_{(\mathcal{L}, -iT_A)}$-almost everywhere.[20] Therefore, the assertion follows from Theorem 5.9.

2. Assume that $\mathcal{T}$ generates a hyperplane of $\mathfrak{g}_2$, and let us prove that $(\mathcal{L}, -i\mathcal{T})$ satisfies property (S). Take $m \in C_0(E_{(\mathcal{L}, -i\mathcal{T})})$ such that $\mathcal{K}_{(\mathcal{L}, -i\mathcal{T})}(m) \in \mathcal{S}(G)$, and consider the (unique) linear mapping

$$L' \colon E_{\mathcal{L}'_{A'}} \to E_{(\mathcal{L}, -i\mathcal{T})}$$

such that $L'(\mathcal{L}'_{A'}) = (\mathcal{L}, -i\mathcal{T})$. Since $\mathcal{L}'_{A'}$ satisfies property (S), there is $m_0 \in \mathcal{S}(E_{\mathcal{L}'_{A'}})$ such that $m \circ L' = m_0$ on $\sigma(\mathcal{L}'_{A'})$. Next, define, for every $\varepsilon \in \{-1, 1\}^A$ and for every $\gamma \in \mathbb{N}^{\mathbf{n_1}}$,

$$S_{\varepsilon, \gamma} \coloneqq \left\{ \left(\omega_\alpha (\mathbf{1}_{n_{1,\alpha}} + 2\gamma_\alpha), \varepsilon_\alpha \omega_\alpha \right)_{\alpha \in A} : \omega \in \mathbb{R}_+^A \right\},$$

so that $S_{\varepsilon,\gamma}$ is a closed convex semi-algebraic set of dimension $\mathrm{Card}(A)$.

Now, let L'' be the unique linear mapping such that $L''(\mathcal{L}'_{A'}) = (\mathcal{L}, -iT_A)$, so that $L \circ L'' = L'$. If we define $S_0 \coloneqq \bigcup_{\varepsilon \in \{-1,1\}^A} S_{\varepsilon, 0}$, then L'' induces a homeomorphism of S_0 onto C_0, which is the boundary of a convex polyhedron on which L is proper (cf. **1**). Therefore, arguing as in the proof of Proposition 2.4, we see that there is a finite subset E_+ of $\{-1, 1\}^A$ such that L induces a homeomorphism of $\bigcup_{\varepsilon \in E_+} L''(S_{\varepsilon, 0})$ onto $\sigma(\mathcal{L}, -i\mathcal{T})$. Hence, L' induces a homeomorphism of $\bigcup_{\varepsilon \in E_+} S_{\varepsilon, 0}$ onto $\sigma(\mathcal{L}, -i\mathcal{T})$.

[20] Indeed, $m_\gamma \circ L'' = \widetilde{m}$ on $C'_\gamma \coloneqq \{((|\omega_\alpha| (\mathbf{1}_{n_{1,\alpha}} + 2\gamma_\alpha), \omega_\alpha)_{\alpha \in A}) \colon \omega \in \mathbb{R}^A\}$, where $L''(\lambda, \omega) = \left(\sum_{\alpha \in A} \lambda_\alpha, \omega\right)$ for every $(\lambda, \omega) \in E_{\mathcal{L}'_{A'}}$. Since $L''_* \left(\chi_{C'_\gamma} \cdot \beta_{\mathcal{L}'_{A'}} \right)$ is equivalent to $\chi_{C_\gamma} \cdot \beta_{(\mathcal{L}, -iT_A)}$ by Proposition 4.18, the assertion follows.

Now, Corollary 2.9 implies that for every $\varepsilon \in E_+$ there is $m'_\varepsilon \in \mathcal{S}(E_{(\mathcal{L},-i\mathcal{T})})$ such that $m'_\varepsilon \circ L' = m_0$ on $S_{\varepsilon,0}$. Nevertheless, we must prove that these functions m'_ε can be patched together to form a Schwartz function which equals m on $\sigma(\mathcal{L},-i\mathcal{T})$. Then, take $\lambda' \in \bigcup_{\varepsilon\in E_+} S_{\varepsilon,0}$ and define $\lambda \coloneqq L'(\lambda')$. Let $E'_{\lambda'}$ be the set of $\varepsilon \in E_+$ such that $\lambda' \in S_{\varepsilon,0}$, so that $\bigcup_{\varepsilon \in E'_{\lambda'}} L'(S_{\varepsilon,0})$ is a neighbourhood of λ in $\sigma(\mathcal{L},-i\mathcal{T})$. Define $\Gamma_{\lambda'}$ as the set of $\gamma \in \mathbb{N}^{\mathbf{n_1}}$ such that $\gamma_\alpha = 0$ if for every $\varepsilon_1, \varepsilon_2 \in E'_{\lambda'}$ we have $\varepsilon_{1,\alpha} = \varepsilon_{2,\alpha}$, and observe that $\lambda' \in S_{\varepsilon,\gamma}$ for every $\varepsilon \in E'_{\lambda'}$ and for every $\gamma \in \Gamma_{\lambda'}$. Assume that $E'_{\lambda'}$ has at least two elements, so that $\Gamma_{\lambda'} \neq \{0\}$. Now, fix $\varepsilon \in E'_{\lambda'}$, and observe that Lemma 8.2 implies that the closure of $\bigcup_{\gamma\in\Gamma_{\lambda'}} S_{\varepsilon,\gamma}$ is $\prod_{\alpha\in A} V_\alpha$, where $V_\alpha = \mathbb{R}(\mathbf{1}_{n_{1,\alpha}}, \varepsilon_\alpha)$ if $\varepsilon_\alpha = \varepsilon'_\alpha$ for every $\varepsilon' \in E'_\lambda$, while $V_\alpha = \mathbb{R}^{n_{1,\alpha}+1}$ otherwise. In particular, $\prod_{\alpha\in A} V_\alpha$ does not depend on the choice of $\varepsilon \in E'_{\lambda'}$. Now, the preceding remarks show that, for every $\gamma \in \Gamma_{\lambda'}$, the union of the convex sets $L'(S_{\varepsilon',0}) \cap L'(S_{\varepsilon,\gamma})$, as ε' runs through $E'_{\lambda'}$, is a neighbourhood of λ in $L'(S_{\varepsilon,\gamma})$; as a consequence, there is $\varepsilon'_\gamma \in E'_{\lambda'}$ such that $L'(S_{\varepsilon'_\gamma,0}) \cap L'(S_{\varepsilon,\gamma})$ has non-empty interior in $L'(S_{\varepsilon,\gamma})$, so that λ is adherent to the interior of $L'(S_{\varepsilon'_\gamma,0}) \cap L'(S_{\varepsilon,\gamma})$ in $L'(S_{\varepsilon,\gamma})$. Hence, m_0, which is invariant on the fibres of L' *in* $\sigma(\mathcal{L}'_{A'})$, equals $m'_{\varepsilon'_\gamma} \circ L'$ on $S'_{\varepsilon,\gamma} \coloneqq S_{\varepsilon,\gamma} \cap (S_{\varepsilon'_\gamma,0} + \ker L')$, which is a convex subset of $S_{\varepsilon,\gamma}$ with non-empty interior in $S_{\varepsilon,\gamma}$.

Take $k \in \mathbb{N}$ and let $P_{\lambda',k}$ be the Taylor polynomial of order k of m_0 at λ'; in addition, let $P'_{\lambda,k,\varepsilon''}$ be the Taylor polynomial of $m'_{\varepsilon''}$ of order k at λ, for every $\varepsilon'' \in E'_{\lambda'}$. Then, the preceding remarks imply that $P_{\lambda',k} = P'_{\lambda,k,\varepsilon'_\gamma} \circ L'$ on the closed convex cone $C'_{\varepsilon,\gamma}$ with vertex λ' generated by the convex set $S'_{\varepsilon,\gamma}$. Hence, the same happens on the closure of $C'_{\varepsilon,\gamma}$ in the Zariski topology, which is the affine space $V'_{\varepsilon,\gamma}$ generated by $S'_{\varepsilon,\gamma}$. The preceding remarks then imply that $V'_{\varepsilon,\gamma}$ is the affine space generated by $S_{\varepsilon,\gamma}$, which, in turn, is the vector space $\{(\omega_\alpha(\mathbf{1}_{n_{1,\alpha}} + 2\gamma_\alpha), \varepsilon_\alpha \omega_\alpha) \colon \omega \in \mathbb{R}^A\}$.

The preceding remarks then imply that, for every $\varepsilon'' \in E'_\lambda$, we have $P_{\lambda',k} = P'_{\lambda,k,\varepsilon''} \circ L'$ on the closure $Z_{\varepsilon''}$, in the Zariski topology, of the union of the $V'_{\varepsilon,\gamma}$ as $\gamma \in \Gamma_{\lambda'}$ and $\varepsilon'_\gamma = \varepsilon''$. Now, $Z = \bigcup_{\varepsilon''\in E'_{\lambda'}} Z_{\varepsilon''}$ is the closure in the Zariski topology of $\bigcup_{\gamma\in\Gamma_{\lambda'}} S_{\varepsilon,\gamma}$, which is $\prod_{\alpha\in A} V_\alpha$ by the preceding remarks. Since Z is then an irreducible algebraic variety, it follows that $Z = Z_{\widetilde{\varepsilon}}$ for some $\widetilde{\varepsilon} \in E'_{\lambda'}$, so that $P_{\lambda',k} = P'_{\lambda,k,\widetilde{\varepsilon}} \circ L'$ on Z for every $k \in \mathbb{N}$. Now, $Z \supseteq S_{\varepsilon'',0}$ for every $\varepsilon'' \in E'_{\lambda'}$, so that $P'_{\lambda,k,\varepsilon''} \circ L' = P_{\lambda',k} = P'_{\lambda,k,\widetilde{\varepsilon}} \circ L'$ on $S_{\varepsilon'',0}$ for every $k \in \mathbb{N}$ and for every $\varepsilon'' \in E'_{\lambda'}$; in addition, $L'(S_{\varepsilon'',0})$ has non-empty interior, so that the arbitrariness of k implies that $m'_{\varepsilon''} - m'_{\widetilde{\varepsilon}}$ vanishes of order ∞ at λ for every $\varepsilon'' \in E'_{\lambda'}$, whence our assertion.

Hence, by means of Theorem 2.7 we see that there is $m' \in \mathcal{S}(E_{(\mathcal{L},-i\mathcal{T})})$ such that $m' \circ L = m_0$ on $\sigma(\mathcal{L}'_{A'})$, so that $m' = m$ on $\sigma(\mathcal{L},-i\mathcal{T})$, whence the result in this case.

3. Now, consider the general case. Take a finite subset $\mathcal{T}'$ of $\mathfrak{g}_2$ which contains $\mathcal{T}$ and generates a hyperplane of $\mathfrak{g}_2$, so that **2** implies that $(\mathcal{L},-i\mathcal{T}')$ satisfies property (S). Observe that $\sigma(\mathcal{L},-i\mathcal{T}')$ is a convex semi-algebraic set. Therefore, the assertion follows easily from Proposition 2.10. ☐

Theorem 8.4. *Let G' be a homogeneous group endowed with a positive Rockland operator $\mathcal{L}'$ which is homogeneous of degree 2. Then, the following hold:*

1. $(\mathcal{L}+\mathcal{L}', -i\mathcal{T})$ *satisfies property* (RL)*;*

2. if $\mathcal{T}$ *does* not *generate* $\mathfrak{g}_2$*, then* $(\mathcal{L}+\mathcal{L}', -i\mathcal{T})$ *satisfies property* (S)*;*

3. if $\mathcal{L}'$ *satisfies property* (S)*, then also* $(\mathcal{L}+\mathcal{L}', -i\mathcal{T})$ *satisfies property* (S)*.*

Notice that we do *not* require that G' is graded, so that the requirement that $\mathcal{L}'$ has homogeneous degree 2 can be met up to rescaling the dilations of G'. In addition, if $A \neq \emptyset$ and $\mathcal{L}'$ is not positive, then $(\mathcal{L}+\mathcal{L}', -i\mathcal{T})$ is *not* a Rockland family, since the mapping $\sigma(\mathcal{L}, -i\mathcal{T}, \mathcal{L}') \ni (\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1+\lambda_3, \lambda_2)$ is not proper.

Proof. **1.** Let us prove that $(\mathcal{L}+\mathcal{L}', -i\mathcal{T})$ satisfies property (RL). Assume first that $\mathcal{T}$ is a basis of $\mathfrak{g}_2$, define $\mathcal{L}'_{A'} \coloneqq (((-X_1^2 - X_{1+n_{1,\alpha}}^2, \dots, -X_{n_{1,\alpha}}^2 - X_{2n_{1,\alpha}}^2), -iT_\alpha)_{\alpha\in A}, \mathcal{L}')$, and observe that $\mathcal{L}'_{A'}$ satisfies property (RL) by Theorems 2.2 and 7.4. Let $L\colon E_{\mathcal{L}'_{A'}} \to E_{(\mathcal{L}+\mathcal{L}', -i\mathcal{T})}$ be the unique linear mapping such that $L(\mathcal{L}'_{A'}) = (\mathcal{L}+\mathcal{L}', -i\mathcal{T})$, and define

$$S_0 \coloneqq \left\{ \left(|\omega_\alpha| \mathbf{1}_{n_{1,\alpha}}, \omega_\alpha \right)_{\alpha\in A} : \omega \in \mathbb{R}^A \right\} \times \mathbb{R}_+,$$

so that S_0 is a closed semi-algebraic set of dimension $\mathrm{Card}(A)+1$. Then, apply Proposition 2.6 with $\beta = \chi_{S_0}\beta_{\mathcal{L}'_{A'}}$ (cf. Proposition 4.18), observing that L induces a proper bijective mapping from S_0 onto $\sigma(\mathcal{L}+\mathcal{L}', -i\mathcal{T})$, hence a homeomorphism. By means of Proposition 4.18 and [20, Theorem 3.2.22], we see that $\beta_{(\mathcal{L}+\mathcal{L}', -i\mathcal{T})}$ and $L_*(\beta)$ are both equivalent to $\chi_{\sigma(\mathcal{L}+\mathcal{L}', -i\mathcal{T})} \cdot \mathcal{H}^{\mathrm{Card}(A)+1}$, so that the assertion follows.

The general case follows by means of Proposition 2.10, thanks to the preceding remarks.

2. Now, assume that $\mathcal{T}$ does not generate $\mathfrak{g}_2$, so that A is not empty. Then, $(\mathcal{L}, -i\mathcal{T})$ satisfies property (S) by [34, Corollary 1.3 and Theorem 1.4] and Theorem 8.3. In addition, $\sigma(\mathcal{L}, -i\mathcal{T})$ is a closed convex cone containing $\mathbb{R}_+ \times \{0\}$, so that $\sigma(\mathcal{L}, -i\mathcal{T}) = \sigma(\mathcal{L}+\mathcal{L}', -i\mathcal{T})$. Then, take $\varphi \in \mathcal{S}_{(\mathcal{L}+\mathcal{L}', -i\mathcal{T})}(G\times G')$ and define $m \coloneqq \mathcal{M}_{(\mathcal{L}+\mathcal{L}', -i\mathcal{T})}(\varphi) \in C_0(\sigma(\mathcal{L}+\mathcal{L}', -i\mathcal{T}))$ (cf. **1**). Let $\pi\colon G\times G' \to G$ be the canonical projection. Then, [32, Theorem 3.2.4], applied to the right quasi-regular representation of $G\times G'$ in $L^2(G)$, implies that $\pi_*(\varphi) = \mathcal{K}_{(\mathcal{L}, -i\mathcal{T})}(m)$. Then, the preceding remarks show that $m \in \mathcal{S}(\sigma(\mathcal{L}+\mathcal{L}', -i\mathcal{T}))$, whence the result.

3. Finally, assume that $\mathcal{L}'$ satisfies property (S), and let us prove that $(\mathcal{L}+\mathcal{L}', -i\mathcal{T})$ satisfies property (S). Observe that, by **2**, we may assume that $\mathcal{T} = (T_\alpha)_{\alpha\in A}$.

Observe first that, with the notation of **1**, $\mathcal{L}'_{A'}$ satisfies property (S) by Theorems 2.2 and 7.4. Then, take $m \in C_0(\sigma(\mathcal{L}+\mathcal{L}', -i\mathcal{T}))$ such that $\mathcal{K}_{(\mathcal{L}+\mathcal{L}', -i\mathcal{T})}(m) \in \mathcal{S}(G\times G')$. It follows that there is $m_0 \in \mathcal{S}(E_{\mathcal{L}'_{A'}})$ such that

$$m \circ L = m_0$$

on $\sigma(\mathcal{L}'_{A'})$. The proof then proceeds along the lines of that of Theorem 8.3. More precisely, define, for every $\varepsilon \in \{-1,1\}^A$ and for every $\gamma \in \mathbb{N}^{\mathbf{n_1}}$,

$$S_{\varepsilon,\gamma} \coloneqq \{(\omega_\alpha(\mathbf{1}_{n_{1,\alpha}} + 2\gamma_\alpha), \varepsilon_\alpha\omega_\alpha) \colon \omega \in \mathbb{R}^A_+\} \times \mathbb{R}_+,$$

so that $S_{\varepsilon,\gamma}$ is a closed semi-algebraic subset of $\sigma(\mathcal{L}'_{A'})$. In particular, for every $\varepsilon \in \{-1,1\}^A$ there is $m'_\varepsilon \in \mathcal{S}(E_{(\mathcal{L}+\mathcal{L}',-i\mathcal{T})})$ such that $m'_\varepsilon \circ L = m_0$ on $S_{\varepsilon,0}$, thanks to Corollary 2.9. Now, observe that $L(S_{\varepsilon,\gamma}) \subseteq L(S_{\varepsilon,0})$ for every $\gamma \in \mathbb{N}^{\mathbf{n_1}}$, so that, since m_0 is constant on the fibres of L in $\sigma(\mathcal{L}'_{A'})$, we have $m'_\varepsilon \circ L = m_0$ on $S_{\varepsilon,\gamma}$ for every $\gamma \in \mathbb{N}^{\mathbf{n_1}}$. Then, fix $\lambda \in \sigma(\mathcal{L}+\mathcal{L}', -i\mathcal{T})$ and let λ' be the unique element of S_0 such that $\lambda = L(\lambda')$. Let $E'_{\lambda'}$ be the set of $\varepsilon \in \{-1,1\}^A$ such that $\lambda' \in S_{\varepsilon,0}$, so that $\bigcup_{\varepsilon\in E'_{\lambda'}} L(S_{\varepsilon,0})$ is a neighbourhood of λ in $\sigma(\mathcal{L}+\mathcal{L}',-i\mathcal{T})$. Then, define $\Gamma_{\lambda'}$ as the set of $\gamma \in \mathbb{N}^{\mathbf{n_1}}$ such that $\gamma_\alpha = 0$ for every $\alpha \in A$ such that $\varepsilon_{1,\alpha} = \varepsilon_{2,\alpha}$ for every $\varepsilon_1, \varepsilon_2 \in E'_{\lambda'}$. Observe that, for every fixed $\varepsilon \in E'_{\lambda'}$, the closure of $\bigcup_{\gamma\in\Gamma_{\lambda'}} S_{\varepsilon,\gamma}$ in the Zariski topology is $\left(\prod_{\alpha\in A} V_\alpha\right) \times \mathbb{R}$, where $V_\alpha = \mathbb{R}(\mathbf{1}_{n_{1,\alpha}}, \varepsilon_\alpha)$ for every $\alpha \in A$ such that $\varepsilon_\alpha = \varepsilon'_\alpha$ for every $\varepsilon' \in E'_{\lambda'}$, while $V_\alpha = \mathbb{R}^{n_{1,\alpha}} \times \mathbb{R}$ otherwise (use Lemma 8.2). Therefore, arguing as in the proof of Theorem 8.3, we see that $m'_\varepsilon - m'_{\varepsilon'}$ vanishes of order infinity at λ for every $\varepsilon' \in E'_{\lambda'}$, whence the result thanks to Theorem 2.7. □

As a complement to Theorem 8.4, we present the following pathological case.

Proposition 8.5. *Let (X,Y,T) be a standard basis of $\mathbb{H}^1$, and let $\mathcal{L}'$ be a positive Rockland operator on a homogeneous group G'. Assume that $(\mathcal{L}')$ satisfies property (S) and that $\mathcal{L}'^h$ is homogeneous of degree 2 for some $h \geqslant 2$. Then, the Rockland family $(-X^2 - Y^2 + \mathcal{L}'^h, -iT)$ is functionally complete and satisfies property (RL), but does* not *satisfy property (S).*

Proof. **1.** Define $\mathcal{L} \coloneqq -X^2 - Y^2$. Then, Theorem 8.4 implies that $(\mathcal{L} + \mathcal{L}'^h, -iT)$ is a Rockland family which satisfies the property (RL). Next, take some $\widetilde{m} \in \mathcal{D}(E_{(\mathcal{L},-iT,\mathcal{L}')})$ supported in $\{(\lambda'_1,\lambda'_2,\lambda'_3) \colon \lambda'_1 < 3|\lambda'_2| - \lambda'^h_3\}$ and equal to pr_3 on a neighbourhood of $(1,1,0)$. Then,

$$m \colon (\lambda_1,\lambda_2) \mapsto \widetilde{m}\left(|\lambda_2|, \lambda_2, \sqrt[h]{\lambda_1 - |\lambda_2|}\right)$$

is not equal to any elements of $\mathcal{S}\left(E_{(\mathcal{L}+\mathcal{L}'^h,-iT)}\right)$ on $\sigma(\mathcal{L}+\mathcal{L}'^h,-iT)$ (cf. Proposition 4.18). On the other hand, $\mathcal{K}_{(\mathcal{L}+\mathcal{L}'^h,-iT)}(m) = \mathcal{K}_{(\mathcal{L},-iT,\mathcal{L}')}(\widetilde{m}) \in \mathcal{S}(\mathbb{H}^1 \times \mathbb{R})$. Hence, $(\mathcal{L}+\mathcal{L}'^h,-iT)$ does not satisfy property (S).

2. Now, let us prove that $(\mathcal{L}+\mathcal{L}'^h,-iT)$ is functionally complete. Take $m \in \mathcal{E}^0(\mathbb{R}^2)$ such that $\mathcal{K}_{(\mathcal{L}+\mathcal{L}'^h,-iT)}(m)$ is supported in $\{e\}$. Notice that we may assume that m is continuous since $(\mathcal{L}+\mathcal{L}'^h,-iT)$ satisfies property (RL). Project onto the quotient by the normal subgroup $\{e_{\mathbb{H}^1}\}\times G'$ and make use of [32, Theorem 3.2.4], applied (arguing by approximation) to the right quasi-regular representation of $\mathbb{H}^1 \times G'$ in $\mathbb{H}^1$; since $(\mathcal{L},-iT)$, considered as a family on $\mathbb{H}^1$, is functionally complete (cf. Proposition 2.12), we see that there is a unique polynomial P on $\mathbb{R}^2$ which coincides with m on $\sigma(\mathcal{L}, -iT)$. On the other hand, the family $(\mathcal{L}, -iT, \mathcal{L}')$ is functionally complete since it satisfies property (S) (cf. Theorem 2.2 and Proposition 2.12). Hence, there is a unique polynomial Q on $\mathbb{R}^3$ such that

$$m(\lambda_1 + \lambda_3^h, \lambda_2) = Q(\lambda_1, \lambda_2, \lambda_3)$$

for every $(\lambda_1, \lambda_2, \lambda_3) \in \sigma(\mathcal{L}, -iT, \mathcal{L}')$. Hence,

$$P(\lambda_1 + \lambda_3^h, \lambda_2) = Q(\lambda_1, \lambda_2, \lambda_3)$$

for every $(\lambda_1, \lambda_2, \lambda_3) \in \left\{ \left(k_1 |r|, r, \sqrt[h]{k_2 |r|} \right) : r \in \mathbb{R}, k_1 \in 2\mathbb{N} + 1, k_2 \in 2\mathbb{N} \right\}$. Now, the closure of this latter set in the Zariski topology is $\mathbb{R}^3$, so that $m = P$ on $\sigma(\mathcal{L} + \mathcal{L}'^h, -iT)$. The assertion follows. □

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