

CORRIGENDUM

Rank monotonicity in centrality measures—Corrigendum

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We correct the statement of Theorem 5 of our paper Boldi et al. (2017): the original statement is not true in general, as a simple counterexample shows. We, however, show that the statement holds under a mild positivity condition.

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1. Corrected statement and counterexample

Given a nonnegative square matrix M with spectral radius¹ $\rho(M)$, its *damped spectral ranking* (Vigna, 2016) is given by²

$$\mathbf{r} = \mathbf{v} \sum_{n \geq 0} (\alpha M)^n = \mathbf{v}(1 - \alpha M)^{-1} \quad (1)$$

where $0 \leq \alpha < 1/\rho(M)$ is a *damping factor*, and \mathbf{v} is a nonnegative *preference vector*.

In Boldi et al. (2017), we discussed under which conditions damped spectral rankings on graphs, such as Katz's index or PageRank, enjoy *strict rank monotonicity*, that is, if it happens that when the score of $z \neq y$ is smaller than or equal to the score of y , after adding an arc from x to y the score of z becomes smaller than the score of y . In this note we correct the statement of Theorem 5, which is not true without the condition $\mathbf{v} > 0$.

Adding the latter condition to Theorem 5 of Boldi et al. (2017), the statement becomes³:

Theorem 1. *Let M and M' be two nonnegative matrices, such that $M' - M = \mathbf{x}_x^T \boldsymbol{\delta}$ (i.e., the matrices differ only on the x -th row, and $\boldsymbol{\delta}$ is the corresponding row difference). Let also $\mathbf{v} > 0$ be a positive preference vector and $0 \leq \alpha < \min(1/\rho(M), 1/\rho(M'))$; let $\mathbf{r} = \mathbf{v} \sum_{n \geq 0} (\alpha M)^n$ and $\mathbf{r}' = \mathbf{v} \sum_{n \geq 0} (\alpha M')^n$ be the damped spectral rankings associated with M and M' , respectively. Assume further that:*

1. *there is exactly one y such that $\delta_y > 0$;*
2. *$r_y < r'_y$.*

Then, if $r_z \leq r_y$ we have $r'_z - r_z < r'_y - r_y$, and in particular $r'_z < r'_y$.

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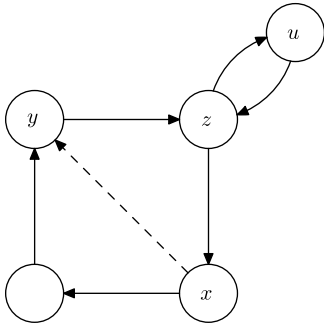


Figure 1. A counterexample.

The purpose of this note is to provide a proof of this corrected version of the theorem, highlighting the reasons behind the necessity of the condition $\mathbf{v} > 0$.

Let us start showing that the original statement was indeed false. Consider the graph in Figure 1 (without the dashed arrow): let M be its adjacency matrix and M' be the matrix of the graph obtained after the addition of the arc $x \rightarrow y$. M has dominant eigenvalue $\sqrt{\varphi}$, where $\varphi = (\sqrt{5} + 1)/2$ is the golden ratio. As usual, let $C = (1 - \alpha M)^{-1}$.

Let us first prove that $c_{yz} = c_{yy}$ iff $\alpha = 1/\varphi$. Note that every path from y to z can be seen as a (possibly empty) path from y to y , followed by a path from y to z which does not traverse again y . Hence the condition $c_{yz} = c_{yy}$ is equivalent to the fact that the sum s of the weights of the paths from y to z which do not traverse again y is exactly one.⁴ Since we are considering the bare adjacency matrix M , a path of length n has weight α^n , and thus looking at the graph

$$s = \alpha(1 + \alpha^2 + \alpha^4 + \dots) = \frac{\alpha}{1 - \alpha^2}$$

which is exactly 1 when $\alpha = 1/\varphi$. It is then easy to check that indeed, with this choice of α , we obtain⁵

$$c_{yy} = c_{yz} = \frac{1}{3 - \sqrt{5}}$$

Let us compute the damped spectral ranking of M (i.e., Katz's index of the graph in Figure 1). We observe that if we choose \mathbf{v} so that $v_u = v_z = 0$, the condition $s = 1$ implies $r_y = r_z$, since all paths giving score to z pass through y . Thus, in particular, the condition $v_u > 0$ of Lemma 4 is necessary: we have $r_y \leq r_z$, there is a path from u to z not passing through y , and nonetheless $c_{yy} = c_{yz}$.

The same graph shows that in general it is not possible to prove strict rank monotonicity for a generic spectral ranking without additional conditions (strong connectivity or $\mathbf{r} > 0$ being not sufficient). The key observation is that the property $r_y = r_z$ remains true even after adding an arc toward y , because s does not change. Thus, in the case of Katz's index, when we add an arc from x to y in the graph of Figure 1, the score of y will increase (i.e., $r'_y > r_y$), as Katz's index is score monotone (Boldi and Vigna, 2014). However, also the score of z will increase exactly by the same amount (i.e., $r'_z - r_z = r'_y - r_y$), violating strict rank monotonicity.⁶

The case of PageRank is analogous, but since we have to normalize the rows of the adjacency matrix the sum of the weights of the paths from y to z which do not traverse again y is

$$\alpha \left(1 + \frac{1}{2}\alpha^2 + \frac{1}{4}\alpha^4 + \dots \right) = \frac{\alpha}{1 - \frac{1}{2}\alpha^2}$$

which is exactly 1 when $\alpha = \sqrt{3} - 1$.

The counterexample of Figure 1 thus shows that Theorem 5 from Boldi et al. (2017) is not true in general, not even for strongly connected graphs; as a consequence, also Corollaries 1 and 2 of Boldi et al. (2017) are false in general.

The two following corollaries are correct restatements of the same corollaries in Boldi et al. (2017), under the further assumption that the preference vector and damping factor be positive⁷:

Corollary 1. *PageRank satisfies the strict rank-monotonicity axiom, for every graph, positive damping factor and positive preference vector.*

Corollary 2. *Katz’s index satisfies the strict rank-monotonicity axiom, for every graph, positive damping factor and positive preference vector.*

2. Proof of Theorem 1

Let us consider the graph induced by a nonnegative matrix M as the graph whose arcs correspond to the nonzero entries of M ; one can also consider M as providing the weights of such arcs. Let us write $x \rightsquigarrow_{\neg y} z$ to mean that there exists, in the graph induced by M , a path from x to z not passing through y (note that, in particular, this implies $x, z \neq y$).

First of all, we need to extend a result from McDonald et al. (1995):

Theorem 2. *Let M be a nonnegative matrix, $0 < \alpha < 1/\rho(M)$ and $C = (1 - \alpha M)^{-1}$. Given indices x, y, z we have that*

- $c_{yz} = c_{yx}c_{xz}/c_{xx}$ if $y \not\rightsquigarrow_{-x} z$;
- $c_{yz} > c_{yx}c_{xz}/c_{xx}$ if $y \rightsquigarrow_{-x} z$.

The above statement extends Theorem 3.9 from McDonald et al. (1995) by removing the condition that x, y , and z be distinct.

Proof. We notice that if $x = y$ or $x = z$, the statement trivializes, as in the case $x = y$ we have $x \not\rightsquigarrow_{-x} z$, so the thesis reduces to the identity $c_{xz} = c_{xx}c_{xz}/c_{xx} = c_{xz}$ analogously if $x = z$.

The remaining case is $y = z$. In this case, the property

$$c_{xx}c_{zz} - c_{xz}c_{zx} > 0$$

is part of the statement of Theorem 1 of Willoughby (1977). □

Let us now recall Lemma 3 of Boldi et al. (2017):

Lemma 3. *Let M be a nonnegative matrix, $0 \leq \alpha < 1/\rho(M)$ a damping factor and \mathbf{v} a nonnegative preference vector. Let*

$$\mathbf{r} = \mathbf{v} \sum_{n \geq 0} (\alpha M)^n$$

be the associated damped spectral ranking and let $C = (1 - \alpha M)^{-1}$. Then, given y and z such that $c_{yz} > 0$ and letting $q = c_{yy}/c_{yz}$, we have $c_{wy} \leq q \cdot c_{wz}$ for all w . In particular, if $r_y > 0$

- *if $r_z \leq r_y$, then $c_{yz} \leq c_{yy}$;*
- *if $r_z < r_y$, then $c_{yz} < c_{yy}$.*

Using Theorem 2, we can strengthen part of its statement under some additional conditions:

Lemma 4. *With the notation of Lemma 3, if $r_z \leq r_y$ and there is some u with $v_u > 0$ and $u \rightsquigarrow_{\neg y} z$, then $c_{yz} < c_{yy}$.*

Proof. We recall from Theorem 2 that if there is no path from w to z not passing through y we have

$$q \cdot c_{wz} = c_{wy}$$

otherwise

$$q \cdot c_{wz} > c_{wy}$$

Now, if $v_u > 0$, $u \rightsquigarrow_{\neg y} z$, and $c_{yy} \leq c_{yz}$, then as above $q \leq 1$, but

$$r_y = \sum_w v_w c_{wy} = \sum_{w \rightsquigarrow_{\neg y} z} v_w c_{wy} + \sum_{w \not\rightsquigarrow_{\neg y} z} v_w c_{wy} < \sum_{w \rightsquigarrow_{\neg y} z} v_w c_{wz} + \sum_{w \not\rightsquigarrow_{\neg y} z} v_w c_{wz} = r_z$$

as $0 \neq v_u c_{uy} < v_u q c_{uz} \leq v_u c_{uz}$ appears in the first summation. □

Based on these results, we can finally provide a proof of Theorem 1:

Proof of Theorem 1. The proof is the same as that of Theorem 3 of Boldi et al. (2017). However, we use first the score-monotonicity condition to prove the case in which $r'_z \leq r_z$. Then, when analyzing the case $c_{yz} > 0$ at the end of the proof, since $\mathbf{v} > 0$ the empty path from z to z makes it possible to apply Lemma 4, obtaining the stricter bound $q = c_{yy}/c_{yz} > 1$, which turns the last sequence of inequalities of the proof into a strict one. □

Notes

- 1 The *spectral radius* of a matrix is the largest absolute value of an eigenvalue.
- 2 In this paper we use row vectors.
- 3 The original statement of Theorem 5 of Boldi et al. (2017) also assumed that $r_x, r_y > 0$; this is not needed anymore, because it is implied by $\mathbf{v} > 0$.
- 4 Recall that the summation $\sum_{n \geq 0} (\alpha M)^n$ can be interpreted as providing in row i and column j the sum of the weights of the paths from i to j , where the weight of a path is the product of the weights of its arcs.
- 5 Of course, one can get to the same conclusion by inverting symbolically $1 - \alpha M$ and then solving the equation $c_{yy}(\alpha) = c_{yz}(\alpha)$, but we believe that reasoning geometrically on the paths makes much clearer how the counterexample was concocted.
- 6 We remark that this counterexample is conceptually similar to the one for Seeley's index (Boldi et al., 2017): the choice of α transfers exactly the score of y to z .
- 7 Note that also Theorem 4 from Boldi et al. (2017) needs the condition $\alpha > 0$ to work in the strict case.

Conflict of interest. The authors have nothing to disclose.

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