ON NONLOCAL CAHN-HILLIARD-NAVIER-STOKES SYSTEMS IN TWO DIMENSIONS

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Abstract

We consider a diffuse interface model which describes the motion of an incompressible isothermal mixture of two immiscible fluids. This model consists of the Navier-Stokes equations coupled with a convective nonlocal Cahn-Hilliard equation. Several results were already proven by two of the present authors. However, in the two-dimensional case, the uniqueness of weak solutions was still open. Here we establish such a result even in the case of degenerate mobility and singular potential. Moreover, we show the weak-strong uniqueness in the case of viscosity depending on the order parameter, provided that either the mobility is constant and the potential is regular or the mobility is degenerate and the potential is singular. In the case of constant viscosity, on account of the uniqueness results we can deduce the connectedness of the global attractor whose existence was obtained in a previous paper. The uniqueness technique can be adapted to show the validity of a smoothing property for the difference of two trajectories which is crucial to establish the existence of an exponential attractor. The latter is established even in the case of variable viscosity, constant mobility and regular potential.

Keywords: Incompressible binary fluids, Navier-Stokes equations, nonlocal Cahn-Hilliard equations, weak solutions, uniqueness, strong solutions, global attractors, exponential attractors.

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1 Introduction

In a series of recent papers (see [9, 14, 15, 16, 17]) the following nonlinear evolution system has been analyzed

$$u_t - 2\mathrm{div}(\nu(\varphi)Du) + (u \cdot \nabla)u + \nabla\pi = \mu\nabla\varphi + h(t), \tag{1.1}$$

$$\mathrm{div}(u) = 0, \tag{1.2}$$

$$\varphi_t + u \cdot \nabla\varphi = \mathrm{div}(m(\varphi)\nabla\mu), \tag{1.3}$$

$$\mu = a\varphi - J * \varphi + F'(\varphi), \tag{1.4}$$

on a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, for $t > 0$. This system describes the evolution of an isothermal mixture of two incompressible and immiscible fluids through the (relative) concentration φ of one species and the (averaged) velocity field u. Here m denotes the mobility, μ is the so-called chemical potential, J is a spatial-dependent interaction kernel and $J * \varphi$ stands for spatial convolution over Ω, a is defined as follows $a(x) = \int_\Omega J(x-y)dy$, F is a double well potential, ν is the viscosity and h is an external force acting on the mixture. The density is supposed to be constant and equal to one (i.e., matched densities).

Such a system is the nonlocal version of the well-known Cahn-Hilliard-Navier-Stokes system which has been the subject of a number of papers (cf., e.g., [1, 2, 7, 8, 18, 19, 20, 33, 35] and references therein, see also the review [26] for modelling and numerical simulation issues). We recall that the nonlocal term seems physically more appropriate than its approximation, i.e., when in place of $a\varphi - J * \varphi$ there is $-\Delta\varphi$. For this issue, we refer the reader to the basic papers [23, 24, 25] (see also [5, 21, 22, 28, 29]). However, from the mathematical viewpoint, the present system is more challenging since the regularity of φ is lower and so the Korteweg force $\mu\nabla\varphi$ acting on the fluid can be less regular than the convective term $(u \cdot \nabla)u$, even in dimension two (cf. [9, (3.7)]). Therefore, it is not straightforward to extend some of the results which holds for the Navier-Stokes equations as well as for the standard Cahn-Hilliard-Navier-Stokes system. This is particularly meaningful in dimension two. In fact, in dimension three, the only known results are comparable with the standard ones for the Navier-Stokes equations, namely, the existence of a global weak solution under various assumptions on m and F and a generalized notion of attractor (cf. [9, 14, 15, 17]).

In dimension two, under reasonable assumptions on F which ensure a suitable regularity of φ, it is possible to prove that there exists a weak solution which satisfies the energy identity. Therefore, such a solution is strongly continuous in time (see [9]). In addition, taking advantage of the energy identity, it is also possible to prove the existence of a the global attractor for the corresponding semiflow (cf. [14, 15, 17]). More recently, in [16], assuming that ν and m are constant and taking a regular potential F, it has been shown the existence of a (unique) strong solution and that any weak solution which satisfies the energy identity regularizes in finite time. This entails some smoothness for the global attractor. Also, the convergence of any weak to a single equilibrium was established through the Łojasiewicz-Simon inequality approach. However, uniqueness of weak solutions was still an open issue in [9, 14, 15, 17].

The main goal of this paper is to prove the uniqueness of weak solutions when ν is constant; while, when ν is non constant, we are able to show the existence of a strong solution and then the weak-strong uniqueness. It is interesting to note that in the case of the standard Cahn-Hilliard-Navier-Stokes system, uniqueness of solutions in two dimensions is known in the case of constant mobility and regular potential (see, e.g., [7, 18, 33]). However, if the potential is singular (e.g., logarithmic), to the best of our knowledge, the only (conditional) uniqueness result was proven in [1] for constant mobility and nonconstant viscosity.

Uniqueness entails the connectedness of the global attractor. In addition, modifying the uniqueness argument we can also show the validity of a suitable smoothing property of the difference of two trajectories (see [11, 12]). This is the basic step to establish the existence of an exponential attractor. The fractal dimension of the global attractor is thus finite.

As in the previous contributions we take the following boundary and initial conditions

$$\frac{\partial\mu}{\partial n} = 0, \quad u = 0 \quad \text{on } \partial\Omega \times (0,T) \tag{1.5}$$

$$u(0) = u_0, \quad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{1.6}$$

The plan of the paper is the following. In the next section we recall the basic assumptions and the related existence of a weak solution. Section 3 is devoted to the uniqueness of weak solutions for constant viscosity. The weak-strong uniqueness is shown in Section 4. The final Section 5 is concerned with the connectedness of the global attractor and the existence of an exponential attractor.

2 Functional setup and preliminary results

Let us introduce the classical Hilbert spaces for the Navier-Stokes equations with no-slip boundary condition (see, e.g., [34])

$$G_{div} := \overline{\{u \in C_0^\infty(\Omega)^d : \ \mathrm{div}(u) = 0\}}^{L^2(\Omega)^d},$$

and

$$V_{div} := \{u \in H_0^1(\Omega)^d : \ \mathrm{div}(u) = 0\}.$$

We set $H := L^2(\Omega)$, $V := H^1(\Omega)$, and denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the scalar product, respectively, on both H and G_{div}. The notation $\langle\cdot,\cdot\rangle$ will stand for the duality pairing between a Banach space X and its dual X'. V_{div} is endowed with the scalar product

$$(u,v)_{V_{div}} = (\nabla u, \nabla v) = 2(Du, Dv), \qquad \forall u, v \in V_{div},$$

where D is the symmetric gradient, defined by $Du := (\nabla u + (\nabla u)^{tr})/2$. The trilinear form b which appears in the weak formulation of the Navier-Stokes equations is defined as usual

$$b(u,v,w) = \int_\Omega (u\cdot\nabla)v\cdot w, \qquad \forall u,v,w \in V_{div},$$

and the associated bilinear operator $\mathcal{B}$ from $V_{div} \times V_{div}$ into V'_{div} is defined by $\langle \mathcal{B}(u,v), w\rangle := b(u,v,w)$, for all $u,v,w \in V_{div}$. We recall that we have $b(u,w,v) = -b(u,v,w)$, for all $u,v,w \in V_{div}$, and that the following estimate holds in dimension two

$$|b(u,v,w)| \le c\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|\|w\|^{1/2}\|\nabla w\|^{1/2}, \quad \forall u,v,w \in V_{div}.$$

In particular we have the following standard estimate in 2D which holds for all $u \in V_{div}$, $\|\mathcal{B}(u,u)\|_{V'_{div}} \le c\|u\|\|\nabla u\|$. For every $f \in V'$ we denote by $\overline{f}$ the average of f over Ω, i.e., $\overline{f} := |\Omega|^{-1}\langle f, 1\rangle$. Here $|\Omega|$ is the Lebesgue measure of Ω. We assume that $\partial\Omega$ is smooth enough (say of class $\mathcal{C}^2$).

We also need to introduce the Hilbert spaces

$$V_0 := \{v \in V : \overline{v} = 0\}, \qquad V_0' := \{f \in V' : \overline{f} = 0\},$$

and the operator $A_N : V \to V'$, $A_N \in \mathcal{L}(V, V')$, defined by

$$\langle A_N u, v\rangle := \int_\Omega \nabla u \cdot \nabla v \qquad \forall u, v \in V.$$

We recall that A_N maps V onto V_0' and the restriction B_N of A_N to V_0 maps V_0 onto V_0' isomorphically. Further, we denote by $B_N^{-1} : V_0' \to V_0$ the inverse map. As is well known, for every $f \in V_0'$, $B_N^{-1} f$ is the unique solution with zero mean value of the Neumann problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

In addition, we have

$$\langle A_N u, B_N^{-1} f \rangle = \langle f, u \rangle, \qquad \forall u \in V, \quad \forall f \in V_0',$$
$$\langle f, B_N^{-1} g \rangle = \langle g, B_N^{-1} f \rangle = \int_\Omega \nabla(B_N^{-1} f) \cdot \nabla(B_N^{-1} g), \qquad \forall f, g \in V_0'.$$

Furthermore, B_N can be also viewed as an unbounded linear operator on H with domain $D(B_N) = \{v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \partial\Omega\}$. If X is a Banach space and $\tau \in \mathbb{R}$, we shall denote by $L^p_{tb}(\tau, \infty; X)$, $1 \leq p < \infty$, the space of functions $f \in L^p_{loc}([\tau, \infty); X)$ that are translation bounded in $L^p_{loc}([\tau, \infty); X)$, that is,

$$\|f\|^p_{L^p_{tb}(\tau,\infty;X)} := \sup_{t \geq \tau} \int_t^{t+1} \|f(s)\|^p_X ds < \infty.$$

We now recall the result on existence of weak solutions and on the validity of the energy identity and of a dissipative estimate in dimension two for the nonlocal Cahn-Hilliard-Navier-Stokes system in the case of constant mobility, nonconstant viscosity and regular potential. This is the main case we shall deal with in this paper.

Let us list the assumptions (see [9]).

(H1) $J \in W^{1,1}(\mathbb{R}^d)$, $\quad J(x) = J(-x)$, $\quad a \geq 0$, a.e. in Ω.

(H2) The mobility $m(s) = 1$ for all $s \in \mathbb{R}$, the viscosity ν is locally Lipschitz on $\mathbb{R}$ and there exist $\nu_1, \nu_2 > 0$ such that $\nu_1 \leq \nu(s) \leq \nu_2$, for all $s \in \mathbb{R}$.

(H3) $F \in C^{2,1}_{loc}(\mathbb{R})$ and there exists $c_0 > 0$ such that $F''(s) + a(x) \geq c_0$, for all $s \in \mathbb{R}$, a.e. $x \in \Omega$.

(H4) $F \in C^2(\mathbb{R})$ and there exist $c_1 > 0$, $c_2 > 0$ and $q > 0$ such that $F''(s) + a(x) \geq c_1 |s|^{2q} - c_2$, for all $s \in \mathbb{R}$, a.e. $x \in \Omega$.

(H5) There exist $c_3 > 0$, $c_4 \geq 0$ and $r \in (1, 2]$ such that $|F'(s)|^r \leq c_3 |F(s)| + c_4$, for all $s \in \mathbb{R}$.

Remark 1. Assumption $J \in W^{1,1}(\mathbb{R}^d)$ can be weakened. Indeed, it can be replaced by $J \in W^{1,1}(B_\delta)$, where $B_\delta := \{z \in \mathbb{R}^d : |z| < \delta\}$ with $\delta := \mathrm{diam}(\Omega)$, or also by (see, e.g., [5])

$$\sup_{x \in \Omega} \int_\Omega \big(|J(x - y)| + |\nabla J(x - y)|\big) dy < \infty.$$

Remark 2. Since F is bounded from below, it is easy to see that (H5) implies that F has polynomial growth of order r', where $r' \in [2, \infty)$ is the conjugate index to r. Namely, there exist $c_5 > 0$ and $c_6 \geq 0$ such that

$$|F(s)| \leq c_5 |s|^{r'} + c_6, \qquad \forall s \in \mathbb{R}. \tag{2.1}$$

Observe that assumption (H5) is fulfilled by a potential of arbitrary polynomial growth. For example, (H3)–(H5) are satisfied for the case of the well-known double well potential $F(s) = (s^2 - 1)^2$.

The following result follows from [9, Theorem 1, Corollaries 1 and 2].

Theorem 1. *Assume that (H1)–(H5) are satisfied. Let $u_0 \in G_{div}$, $\varphi_0 \in H$ such that $F(\varphi_0) \in L^1(\Omega)$ and $h \in L^2_{loc}([0,\infty); V'_{div})$. Then, for every given $T > 0$, there exists a weak solution $[u, \varphi]$ to* (1.3)–(1.6) *such that*

$$u \in L^\infty(0,T;G_{div}) \cap L^2(0,T;V_{div}), \quad \varphi \in L^\infty(0,T;L^{2+2q}(\Omega)) \cap L^2(0,T;V), \tag{2.2}$$

$$u_t \in L^{4/3}(0,T;V'_{div}), \quad \varphi_t \in L^{4/3}(0,T;V'), \qquad d = 3, \tag{2.3}$$

$$u_t \in L^2(0,T;V'_{div}), \quad d = 2, \tag{2.4}$$

$$\varphi_t \in L^2(0,T;V'), \quad d = 2 \quad \text{or} \quad d = 3 \text{ and } q \geq 1/2, \tag{2.5}$$

and satisfying the energy inequality

$$\mathcal{E}(u(t),\varphi(t)) + \int_0^t \Big(2\|\sqrt{\nu(\varphi)}Du\|^2 + \|\nabla\mu\|^2\Big)d\tau \leq \mathcal{E}(u_0,\varphi_0) + \int_0^t \langle h(\tau), u\rangle d\tau, \tag{2.6}$$

for every $t > 0$, where we have set

$$\mathcal{E}(u(t),\varphi(t)) = \frac{1}{2}\|u(t)\|^2 + \frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi(x,t)-\varphi(y,t))^2 dxdy + \int_\Omega F(\varphi(t)).$$

If $d = 2$, then any weak solution satisfies the energy identity

$$\frac{d}{dt}\mathcal{E}(u,\varphi) + 2\|\sqrt{\nu(\varphi)}Du\|^2 + \|\nabla\mu\|^2 = \langle h(t), u\rangle, \tag{2.7}$$

In particular we have $u \in C([0,\infty); G_{div})$, $\varphi \in C([0,\infty); H)$ and $\int_\Omega F(\varphi) \in C([0,\infty))$. Furthermore, if $d = 2$ and $h \in L^2_{tb}(0,\infty; V'_{div})$, then any weak solution satisfies also the dissipative estimate

$$\mathcal{E}(u(t),\varphi(t)) \leq \mathcal{E}(u_0,\varphi_0)e^{-kt} + F(m_0)|\Omega| + K, \qquad \forall t \geq 0, \tag{2.8}$$

where $m_0 = (\varphi_0, 1)$ and k, K are two positive constants which are independent of the initial data, with K depending on Ω, ν, J, F and $\|h\|_{L^2_{tb}(0,\infty;V'_{div})}$.

Henceforth we shall denote by Q a continuous function monotone increasing with respect to each of its arguments. As a consequence of energy inequality (2.6) it is easy to deduce the following bound

$$\begin{aligned}&\|u\|_{L^\infty(0,T;G_{div})\cap L^2(0,T;V_{div})} + \|\varphi\|_{L^\infty(0,T;L^{2+2q}(\Omega))\cap L^2(0,T;V)} + \|F(\varphi)\|_{L^\infty(0,T;L^1(\Omega))} \\ &\leq Q\big(\mathcal{E}(u_0,\varphi_0), \|h\|_{L^2(0,T;V'_{div})}\big),\end{aligned} \tag{2.9}$$

where Q also depends on F, J, ν_1 and Ω. In all the following sections we take $d = 2$.

3 Uniqueness of weak solutions (constant viscosity)

Here we prove that the weak solution of the nonlocal Cahn-Hilliard-Navier-Stokes system with constant viscosity ν is unique and we provide a continuous dependence estimate. In Subsection 3.1 we shall first address the case of constant mobility ($m = 1$) and regular potential F. Nevertheless, we shall see in Subsection 3.2 and Subsection 3.3 that the arguments used for this case can also be applied to the cases of singular potential and constant or degenerate mobility (see [15] or [17] for the existence).

3.1 Regular potential and constant mobility

The main result is the following.

Theorem 2. *Let $d = 2$ and suppose that assumptions (H1)–(H5) are satisfied with ν constant. Let $u_0 \in G_{div}$, $\varphi_0 \in H$ with $F(\varphi_0) \in L^1(\Omega)$ and $h \in L^2_{loc}([0,\infty); V'_{div})$. Then, the weak solution $[u,\varphi]$ corresponding to $[u_0,\varphi_0]$ and given by Theorem 1 is unique. Furthermore, let $z_i := [u_i,\varphi_i]$ be two weak solutions corresponding to two initial data $z_{0i} := [u_{0i},\varphi_{0i}]$ and external forces h_i, with $u_{0i} \in G_{div}$, $\varphi_{0i} \in H$ such that $F(\varphi_{0i}) \in L^1(\Omega)$ and $h_i \in L^2_{loc}([0,\infty); V'_{div})$. Then the following continuous dependence estimate holds*

$$\begin{aligned}
&\|u_2(t) - u_1(t)\|^2 + \|\varphi_2(t) - \varphi_1(t)\|^2_{V'} \\
&+ \int_0^t \Big(\frac{c_0}{2}\|\varphi_2(\tau) - \varphi_1(\tau)\|^2 + \frac{\nu}{4}\|\nabla\big(u_2(\tau) - u_1(\tau)\big)\|^2\Big) d\tau \\
&\le \big(\|u_2(0) - u_1(0)\|^2 + \|\varphi_2(0) - \varphi_1(0)\|^2_{V'}\big)\Lambda_0(t) \\
&+ \big|\overline{\varphi_2(0)} - \overline{\varphi_1(0)}\big| Q\big(\mathcal{E}(z_{01}), \mathcal{E}(z_{02}), \|h_1\|_{L^2(0,t;V'_{div})}, \|h_2\|_{L^2(0,t;V'_{div})}\big)\Lambda_1(t) \\
&+ \|h_2 - h_1\|^2_{L^2(0,T;V'_{div})}\Lambda_2(t), \qquad (3.1)
\end{aligned}$$

for all $t \in [0,T]$, where Λ_0, Λ_1 and Λ_2 are continuous functions which depend on the norms of the two solutions. The functions Q and Λ_i also depend on F, J, ν and Ω.

Proof. Let us start by rewriting the Korteweg force by making explicit the dependence on φ. Indeed, we have

$$\mu\nabla\varphi = \big(a\varphi - J*\varphi + F'(\varphi)\big)\nabla\varphi = \nabla\Big(F(\varphi) + a\frac{\varphi^2}{2}\Big) - \nabla a\frac{\varphi^2}{2} - (J*\varphi)\nabla\varphi.$$

Hence we can write the Navier-Stokes equation with an extra-pressure $\widetilde{\pi} := \pi - F(\varphi) + a\frac{\varphi^2}{2}$ as follows

$$u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla\widetilde{\pi} - h = -\nabla a\frac{\varphi^2}{2} - (J*\varphi)\nabla\varphi =: K(\varphi).$$

Let us now consider two weak solutions $[u_i,\varphi_i]$ corresponding to two initial data $[u_{0i},\varphi_{0i}]$ and two external forces h_i, with $u_{0i} \in G_{div}$, $\varphi_{i0} \in H$, $F(\varphi_{0i}) \in L^1(\Omega)$ and $h_i \in L^2_{loc}([0,\infty); V'_{div})$, $i = 1, 2$. Set $u := u_2 - u_1$ and $\varphi := \varphi_2 - \varphi_1$. Then, the difference $[u,\varphi]$ satisfies the system

$$\varphi_t = \Delta\widetilde{\mu} - u\cdot\nabla\varphi_1 - u_2\cdot\nabla\varphi, \qquad (3.2)$$

$$\widetilde{\mu} = a\varphi - J*\varphi + F'(\varphi_2) - F'(\varphi_1), \qquad (3.3)$$

$$\begin{aligned}
&u_t - \nu\Delta u + (u_2\cdot\nabla)u_2 - (u_1\cdot\nabla)u_1 + \nabla\widetilde{\pi} \\
&= -\varphi(\varphi_1 + \varphi_2)\frac{\nabla a}{2} - (J*\varphi)\nabla\varphi_2 - (J*\varphi_1)\nabla\varphi + h, \qquad (3.4)
\end{aligned}$$

where $\widetilde{\pi} := \widetilde{\pi}_2 - \widetilde{\pi}_1$ and $h := h_2 - h_1$. We multiply (3.4) by u in G_{div}. After standard calculations, the following terms (cf. (3.4))

$$I_1 = -\frac{1}{2}\left(\varphi\left(\varphi_1 + \varphi_2\right)\nabla a, u\right), \; I_2 = -\left(\left(J*\varphi\right)\nabla\varphi_2, u\right), \; I_3 = -\left(\left(J*\varphi_1\right)\nabla\varphi, u\right),$$

can be estimated in this way

$$I_1 \le \big|\big(\varphi(\varphi_1 + \varphi_2)\nabla a, u\big)\big| \le \|\varphi\|\|\varphi_1 + \varphi_2\|_{L^4}\|\nabla a\|_{L^\infty}\|u\|_{L^4}$$

$$
\begin{aligned}
&\leq c\|\varphi\|\|\varphi_1+\varphi_2\|_{L^4}\|\nabla a\|_{L^\infty}\|u\|^{1/2}\|\nabla u\|^{1/2}\\
&\leq \frac{c_0}{10}\|\varphi\|^2 + c\|\varphi_1+\varphi_2\|_{L^4}^2\|\nabla a\|_{L^\infty}^2\|u\|\|\nabla u\|\\
&\leq \frac{c_0}{10}\|\varphi\|^2 + \frac{\nu}{6}\|\nabla u\|^2 + c\|\varphi_1+\varphi_2\|_{L^4}^4\|\nabla a\|_{L^\infty}^4\|u\|^2, \qquad (3.5)\\
I_2 &\leq \big|\big(\varphi_2, (\nabla J*\varphi)u\big)\big| \leq \|\varphi_2\|_{L^4}\|\nabla J*\varphi\|\|u\|_{L^4}\\
&\leq c\|\varphi_2\|_{L^4}\|\nabla J\|_{L^1}\|\varphi\|\|u\|^{1/2}\|\nabla u\|^{1/2}\\
&\leq \frac{c_0}{10}\|\varphi\|^2 + c\|\nabla J\|_{L^1}^2\|\varphi_2\|_{L^4}^2\|u\|\|\nabla u\|\\
&\leq \frac{c_0}{10}\|\varphi\|^2 + \frac{\nu}{6}\|\nabla u\|^2 + c\|\nabla J\|_{L^1}^4\|\varphi_2\|_{L^4}^4\|u\|^2, \qquad (3.6)\\
I_3 &\leq \big|\big((\nabla J*\varphi_1)\varphi, u\big)\big| \leq \|\nabla J*\varphi_1\|_{L^4}\|\varphi\|\|u\|_{L^4}\\
&\leq c\|\nabla J\|_{L^1}\|\varphi_1\|_{L^4}\|\varphi\|\|u\|^{1/2}\|\nabla u\|^{1/2}\\
&\leq \frac{c_0}{10}\|\varphi\|^2 + c\|\nabla J\|_{L^1}^2\|\varphi_1\|_{L^4}^2\|u\|\|\nabla u\|\\
&\leq \frac{c_0}{10}\|\varphi\|^2 + \frac{\nu}{6}\|\nabla u\|^2 + c\|\nabla J\|_{L^1}^4\|\varphi_1\|_{L^4}^4\|u\|^2. \qquad (3.7)
\end{aligned}
$$

Taking estimates (3.5)–(3.7) into account, it is easy see that from (3.4) we are led to the following differential inequality

$$
\frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{\nu}{4}\|\nabla u\|^2 \leq \frac{3}{10}c_0\|\varphi\|^2 + \alpha\|u\|^2 + \frac{1}{\nu}\|h\|_{V_{div}'}^2, \qquad (3.8)
$$

where the function α is given by

$$
\alpha := c\|\nabla J\|_{L^1}^4\big(\|\varphi_1\|_{L^4}^4 + \|\varphi_2\|_{L^4}^4\big) + c\|\nabla u_2\|^2.
$$

Since $\varphi_1, \varphi_2 \in L^\infty(0,T;H)\cap L^2(0,T,V)$ and $L^\infty(0,T;H)\cap L^2(0,T,V) \hookrightarrow L^4(0,T;L^4(\Omega))$, thanks to the Gagliardo-Nirenberg inequality, we have $\alpha \in L^1(0,T)$.

Let us now multiply (3.2) by $B_N^{-1}(\varphi-\overline{\varphi})$ (notice that we have $\overline{\varphi} = \overline{\varphi}_{01} - \overline{\varphi}_{02}$). We get

$$
\frac{1}{2}\frac{d}{dt}\|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2 + (a\varphi + F'(\varphi_1) - F'(\varphi_2), \varphi) = (J*\varphi, \varphi) + |\Omega|\overline{\varphi}\overline{\widetilde{\mu}} + I_4 + I_5, \qquad (3.9)
$$

where

$$
I_4 = -\left(u\cdot\nabla\varphi_1, B_N^{-1}(\varphi-\overline{\varphi})\right), \quad I_5 = -\left(u_2\cdot\nabla\varphi, B_N^{-1}(\varphi-\overline{\varphi})\right).
$$

By using assumption (H3), we find

$$
\frac{1}{2}\frac{d}{dt}\|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2 + c_0\|\varphi\|^2 \leq |(J*\varphi,\varphi)| + |\Omega|\overline{\varphi}\overline{\widetilde{\mu}} + I_4 + I_5. \qquad (3.10)
$$

The first term on the right-hand side of (3.10) can be controlled as follows

$$
\begin{aligned}
&\big|(J*\varphi, \varphi-\overline{\varphi})\big| + |(J*\varphi,\overline{\varphi})| = \big|\big(B_N^{1/2}(J*\varphi - \overline{J*\varphi}), B_N^{-1/2}(\varphi-\overline{\varphi})\big)\big| + |(J*\varphi,\overline{\varphi})|\\
&\leq \frac{c_0}{10}\|\varphi\|^2 + c\|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2 + \frac{c_0}{4}\|\varphi\|^2 + c\overline{\varphi}^2, \qquad (3.11)
\end{aligned}
$$

where we have used the fact that $\|B_N^{1/2}u\|^2 = (B_N u, u) = \|\nabla u\|^2$, for all $u \in D(B_N)$ and hence $\|B_N^{1/2}u\| = \|\nabla u\|$, which also holds, by density, for all $u \in D(B_N^{1/2}) = V_0$. The terms I_4 and I_5 can be estimated in this way

$$
I_4 \leq \big|\big(u\cdot\nabla B_N^{-1}(\varphi-\overline{\varphi}), \varphi_1\big)\big| \leq \|u\|_{L^4}\|\nabla B_N^{-1}(\varphi-\overline{\varphi})\|\|\varphi_1\|_{L^4}
$$

$$\leq \frac{\nu}{8}\|\nabla u\|^2 + c\|\varphi_1\|_{L^4}^2\|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2, \tag{3.12}$$

$$I_5 \leq \big|(u_2\cdot\nabla B_N^{-1}(\varphi-\overline{\varphi}),\varphi)\big| \leq \|\varphi\|\|u_2\|_{L^4}\|\nabla B_N^{-1}(\varphi-\overline{\varphi})\|_{L^4}$$
$$\leq \frac{c_0}{20}\|\varphi\|^2 + c\|u_2\|_{L^4}^2\|\nabla B_N^{-1}(\varphi-\overline{\varphi})\|_{L^4}^2$$
$$\leq \frac{c_0}{20}\|\varphi\|^2 + c\|u_2\|_{L^4}^2\|\nabla B_N^{-1}(\varphi-\overline{\varphi})\|\|\nabla B_N^{-1}(\varphi-\overline{\varphi})\|_{H^1}. \tag{3.13}$$

Observe that the H^2-norm of ϕ on $D(B_N)$ is equivalent to the L^2-norm of $B_N\phi+\phi$ (recall that $\phi := B_N^{-1}(\varphi-\overline{\varphi}) \in D(B_N)$). Thus we have

$$\|\nabla B_N^{-1}(\varphi-\overline{\varphi})\|_{H^1} \leq \|B_N^{-1}(\varphi-\overline{\varphi})\|_{H^2} \leq c\|(B_N+I)B_N^{-1}(\varphi-\overline{\varphi})\| \leq c\|\varphi-\overline{\varphi}\|.$$

Therefore, from (3.13) we get

$$I_5 \leq \frac{c_0}{10}\|\varphi\|^2 + c\|u_2\|_{L^4}^4\|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2 + |\Omega|\overline{\varphi}^2. \tag{3.14}$$

Recalling estimate (3.8) and plugging estimates (3.11)–(3.14) into (3.10), we deduce the differential inequality

$$\frac{1}{2}\frac{d}{dt}\Big(\|u\|^2 + \|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2\Big) + \frac{c_0}{4}\|\varphi\|^2 + \frac{\nu}{8}\|\nabla u\|^2$$
$$\leq \beta\Big(\|u\|^2 + \|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2\Big) + c\overline{\varphi}^2 + |\Omega|\overline{\varphi}\overline{\widetilde{\mu}} + \frac{1}{\nu}\|h\|_{V_{div}'}^2, \tag{3.15}$$

where β is given by

$$\beta := \alpha + c(1 + \|\varphi_1\|_{L^4}^2 + \|u_2\|_{L^4}^4) \in L^1(0,T).$$

If we consider two weak solutions corresponding to the same initial data and to the same external force, then we have $\overline{\varphi}=0$ and $h=0$. Therefore, by using Gronwall's lemma, from (3.15) we get $u=0$ and $\varphi=0$ on $[0,T]$ and this proves uniqueness. If the two weak solutions correspond to different initial data and to different external forces, we have

$$|\Omega||\overline{\widetilde{\mu}}| \leq \int_\Omega \big(|F'(\varphi_2)| + |F'(\varphi_1)|\big) \leq c\int_\Omega \big(|F(\varphi_2)| + |F(\varphi_1)|\big) + c$$
$$\leq Q\big(\mathcal{E}(z_{01}),\mathcal{E}(z_{02}),\|h_1\|_{L^2(0,T;V_{div}')},\|h_2\|_{L^2(0,T;V_{div}')}\big), \qquad \forall t\geq 0, \tag{3.16}$$

where we have used (H5) (which implies that $|F'(s)| \leq cF(s)+c$, for all $s\in\mathbb{R}$) and (2.9). Therefore (3.15) can be rewritten as

$$\frac{d}{dt}\Big(\|u\|^2 + \|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2\Big) + \frac{c_0}{2}\|\varphi\|^2 + \frac{\nu}{4}\|\nabla u\|^2$$
$$\leq \beta\Big(\|u\|^2 + \|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2\Big) + |\overline{\varphi}|Q\big(\mathcal{E}(z_{01}),\mathcal{E}(z_{02}),\|h_1\|_{L^2(0,T;V_{div}')},\|h_2\|_{L^2(0,T;V_{div}')}\big)$$
$$+ \frac{2}{\nu}\|h\|_{V_{div}'}^2. \tag{3.17}$$

By using Gronwall's lemma once more, we deduce from (3.17) that

$$\|u(t)\|^2 + \|B_N^{-1/2}(\varphi(t)-\overline{\varphi})\|^2 \leq \big(\|u(0)\|^2 + \|B_N^{-1/2}(\varphi(0)-\overline{\varphi})\|^2\big)\Gamma_0(t)$$
$$+ |\overline{\varphi}|Q\big(\mathcal{E}(z_{01}),\mathcal{E}(z_{02}),\|h_1\|_{L^2(0,T;V_{div}')},\|h_2\|_{L^2(0,T;V_{div}')}\big)\Gamma_1(t) + \frac{2}{\nu}\Gamma_0(t)\|h\|_{L^2(0,T;V_{div}')}^2, \tag{3.18}$$

where $\Gamma_0(t) := e^{\int_0^t \beta(s)ds}$ and $\Gamma_1(t) := \int_0^t e^{\int_s^t \beta(\tau)d\tau} ds$. By integrating (3.17) between 0 and t and using (3.18), we find

$$\begin{aligned}
&\|u(t)\|^2 + \|B_N^{-1/2}(\varphi(t)-\overline{\varphi})\|^2 + \int_0^t \Big(\frac{c_0}{2}\|\varphi\|^2 + \frac{\nu}{4}\|\nabla u\|^2\Big)d\tau \\
&\leq \big(\|u(0)\|^2 + \|B_N^{-1/2}(\varphi(0)-\overline{\varphi})\|^2\big)\Gamma_2(t) \\
&+ |\overline{\varphi}| Q\big(\mathcal{E}(z_{01}), \mathcal{E}(z_{02}), \|h_1\|_{L^2(0,T;V_{div}')}, \|h_2\|_{L^2(0,T;V_{div}')}\big)\Gamma_3(t) \\
&+ \frac{2}{\nu}\Gamma_0(t)\|h\|^2_{L^2(0,T;V_{div}')},
\end{aligned} \tag{3.19}$$

for all $t \in [0,T]$, where $\Gamma_2(t) := 1 + \int_0^t \beta(s)\Gamma_0(s)ds$ and $\Gamma_3(t) := \int_0^t \beta(s)\Gamma_1(s)ds + T$. Finally, by suitably defining the functions Λ_0, Λ_1 in terms of Γ_0, Γ_2 and Γ_3, we deduce (3.1) from (3.19). ∎

3.2 Singular potential and constant mobility

The proof of existence of a weak solution with initial data $u_0 \in G_{div}$ and $\varphi_0 \in L^\infty(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ is given in [15], where also a nonconstant viscosity is considered. We recall that in this case the assumption $|\overline{\varphi}_0| < 1$ is needed in order to control the average of the chemical potential. For the assumptions on the singular potential F we refer the reader to [15]. We recall, in particular, the physically relevant case of the so-called logarithmic potential, that is,

$$F(s) = -\frac{\theta_c}{2}s^2 + \frac{\theta}{2}\big((1+s)\log(1+s) + (1-s)\log(1-s)\big), \tag{3.20}$$

where $0 < \theta < \theta_c$, θ being the absolute temperature and θ_c a given critical temperature below which the phase separation takes place.

It is easy to see that, assuming the viscosity ν constant and $d = 2$, the uniqueness argument can also be applied to the present case. Indeed, estimates (3.5)-(3.8) obviously still hold. Moreover, considering (3.9) we immediately see that (3.10) still follows from (3.9), since in the case of singular potential we have

$$F''(s) + a(x) \geq c_0, \qquad \forall s \in (-1,1), \qquad c_0 > 0.$$

In particular, this assumption is ensured by [15, (A6)]. Therefore, uniqueness follows from (3.15) on account of the fact that in this inequality we have $\overline{\varphi} = 0$ (and $h = 0$).

Concerning the proof of the continuous dependence estimate (3.1), we have to be a bit more careful since estimate (3.16) cannot be applied in the present situation. On the other hand, recalling [15, Proof of Theorem 1], we have

$$\|F'(\varphi_i)\|_{L^2(0,T;L^1(\Omega))} \leq Q\big(\overline{\varphi}_{0i}, \mathcal{E}(z_{0i}), \|h_i\|_{L^2(0,T;V_{div}')}\big), \qquad i = 1,2.$$

By applying these last estimates we see that the term $|\Omega|\overline{\varphi}\overline{\widetilde{\mu}}$ on the right-hand side of (3.15) can be written in the form $\overline{\varphi}\Gamma_4$ with a function Γ_4 such that

$$\|\Gamma_4\|_{L^2(0,T)} \leq Q\big(\eta, \mathcal{E}(z_{01}), \mathcal{E}(z_{02}), \|h_1\|_{L^2(0,T;V_{div}')}, \|h_2\|_{L^2(0,T;V_{div}')}\big),$$

where $\eta \in [0,1)$ is such that $|\overline{\varphi}_{0i}| \leq \eta$, $i = 1,2$. Starting now from (3.15) and using Gronwall's lemma like in the proof of Theorem 2, we find a continuous dependence estimate of the same form as (3.1) where now the function Q depends also on η. We can therefore state the following

Theorem 3. *Let $d = 2$ and suppose that assumptions (A1)–(A8) of [15] are satisfied with ν constant. Let $u_0 \in G_{div}$, $\varphi_0 \in L^\infty(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$, $|\overline{\varphi}_0| < 1$ and $h \in L^2_{loc}([0,\infty); V'_{div})$. Then, the weak solution $[u, \varphi]$, corresponding to $[u_0, \varphi_0]$ and given by [15, Theorem 1], is unique. Furthermore, let $z_i := [u_i, \varphi_i]$ be two weak solutions corresponding to two initial data $z_{0i} := [u_{0i}, \varphi_{0i}]$ and two external forces h_i, with $u_{0i} \in G_{div}$, $\varphi_{0i} \in L^\infty(\Omega)$ such that $F(\varphi_{0i}) \in L^1(\Omega)$, $|\overline{\varphi}_{0i}| \leq \eta$ for some constant $\eta \in [0, 1)$ and $h_i \in L^2_{loc}([0,\infty); V'_{div})$, $i = 1, 2$. Then estimate (3.1) holds with Q also depending on η.*

3.3 Singular potential and degenerate mobility

This physically relevant case was addressed in [17] from which we recall all the assumptions on the degenerate mobility m and on the singular potential F as well as the weak formulation. We assume that the mobility m is degenerate at ± 1 and that the double well potential F is singular (e.g. logarithmic like) and defined in $(-1, 1)$. More precisely, we assume that $m \in C^1([-1, 1])$, $m \geq 0$, that $m(s) = 0$ if and only if $s = -1$ or $s = 1$, and that there exists $\epsilon_0 > 0$ such that m is non-increasing in $[1 - \epsilon_0, 1]$ and non-decreasing in $[-1, -1 + \epsilon_0]$. Furthermore, we suppose that m and F fulfill the conditions

(A1) $F \in C^2(-1, 1)$ and $mF'' \in C([-1, 1])$.

As far as F is concerned, we assume that it can be written in the following form $F = F_1 + F_2$, where the singular component F_1 and the regular component $F_2 \in C^2([-1, 1])$ satisfy the following assumptions.

(A2) There exist $\kappa > 4(a^* - a_* - b_*)$, where $b_* := \min_{[-1,1]} F_2''$, and $\epsilon_0 > 0$ such that $F_1''(s) \geq \kappa$, for all $s \in (-1, -1 + \epsilon_0] \cup [1 - \epsilon_0, 1)$.

(A3) There exists $\epsilon_0 > 0$ such that F_1'' is non-decreasing in $[1 - \epsilon_0, 1)$ and non-increasing in $(-1, -1 + \epsilon_0]$.

(A4) There exists $c_0 > 0$ such that $F''(s) + a(x) \geq c_0$, for all $s \in (-1, 1)$, a.e. $x \in \Omega$.

The constants a^* and a_* in (A2) are given by

$$a^* := \sup_{x\in\Omega} \int_\Omega |J(x - y)| dy < \infty, \qquad a_* := \inf_{x\in\Omega} \int_\Omega J(x - y) dy.$$

Moreover, we denote by ϵ_0 a positive constant the value of which may possibly vary from line to line.

As far as the weak formulation in dimension two is concerned, we point out that, if the mobility degenerates then the gradient of the chemical potential μ is not controlled in some L^p space. For this reason, and also in order to pass to the limit to prove existence of a weak solution, a suitable reformulation of the definition of weak solution should be introduced in such a way that μ does not appear explicitly (cf. [13], see also [17]).

Definition 1. *Let $u_0 \in G_{div}$, $\varphi_0 \in H$ with $F(\varphi_0) \in L^1(\Omega)$, $h \in L^2(0, T; V'_{div})$ and $0 < T < +\infty$ be given. A couple $[u, \varphi]$ is a weak solution to* (1.3)-(1.6) *on* $[0, T]$ *corresponding to* $[u_0, \varphi_0]$ *if*

- *u, φ satisfy*

$$u \in L^\infty(0, T; G_{div}) \cap L^2(0, T; V_{div}),$$

$$u_t \in L^2(0,T;V_{div}'),$$
$$\varphi \in L^\infty(0,T;H) \cap L^2(0,T;V),$$
$$\varphi_t \in L^2(0,T;V'),$$

and

$$\varphi \in L^\infty(Q_T), \qquad |\varphi(x,t)| \leq 1 \quad a.e.\ (x,t) \in Q_T := \Omega \times (0,T);$$

- *for every* $\psi \in V$*, every* $v \in V_{div}$ *and for almost any* $t \in (0,T)$ *we have*

$$\langle \varphi_t, \psi \rangle + \int_\Omega m(\varphi) F''(\varphi) \nabla\varphi \cdot \nabla\psi + \int_\Omega m(\varphi) a \nabla\varphi \cdot \nabla\psi$$
$$+ \int_\Omega m(\varphi)(\varphi\nabla a - \nabla J * \varphi) \cdot \nabla\psi = (u\varphi, \nabla\psi),$$
$$\langle u_t, v\rangle + \nu(\nabla u, \nabla v) + b(u,u,v) = \big((a\varphi - J*\varphi)\nabla\varphi, v\big) + \langle h, v\rangle;$$

- *the initial conditions* $u(0) = u_0$*,* $\varphi(0) = \varphi_0$ *hold.*

Recall also that from the regularity properties of the weak solution we have $u \in C_w([0,T];G_{div})$ and $\varphi \in C_w([0,T];H)$. Therefore, the initial conditions $u(0) = u_0$, $\varphi(0) = \varphi_0$ make sense. In [17, Theorem 2] the existence of a weak solution was established with initial data $u_0 \in G_{div}$ and $\varphi_0 \in L^\infty(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ and $M(\varphi_0) \in L^1(\Omega)$, where $M \in C^2(-1,1)$ is defined by $m(s)M''(s) = 1$ for all $s \in (-1,1)$ and $M(0) = M'(0) = 0$. Furthermore, in [17, Proposition 4] uniqueness of the weak solution was proven for the convective nonlocal Cahn-Hilliard equation with degenerate mobility for a given velocity $u \in L^2_{loc}([0,\infty);V_{div} \cap L^\infty(\Omega)^2)$. To this purpose, the following additional conditions were assumed.

(A5) There exists $\rho \in [0,1)$ such that $\rho F_1''(s) + F_2''(s) + a(x) \geq 0$, for all $s \in (-1,1)$, a.e. Ω.

(A6) There exists $\alpha_0 > 0$ such that $m(s)F_1''(s) \geq \alpha_0$, for all $s \in [-1,1]$.

By combining the proof of [17, Proposition 4] with the arguments of Theorem 2 we can now prove uniqueness of weak solutions for the nonlocal Cahn-Hilliard-Navier-Stokes system with singular potential and degenerate mobility. Indeed we have

Theorem 4. *Let* $d = 2$ *and suppose that assumptions (A1)–(A6) are satisfied with* ν *constant. Let* $u_0 \in G_{div}$*,* $\varphi_0 \in L^\infty(\Omega)$ *with* $F(\varphi_0) \in L^1(\Omega)$*,* $M(\varphi_0) \in L^1(\Omega)$ *and* $h \in L^2_{loc}([0,\infty);V_{div}')$*. Then, the weak solution to system* (1.3)*-*(1.6) *is unique. Moreover, let* $z_i := [u_i, \varphi_i]$ *be two weak solutions corresponding to two initial data* $z_{0i} := [u_{0i}, \varphi_{0i}]$ *and external forces* h_i*, with* $u_{0i} \in G_{div}$*,* $\varphi_{0i} \in L^\infty(\Omega)$ *such that* $F(\varphi_{0i}) \in L^1(\Omega)$*,* $M(\varphi_{0i}) \in L^1(\Omega)$ *and* $h_i \in L^2_{loc}([0,\infty);V_{div}')$*. Then the following continuous dependence estimate holds*

$$\begin{aligned}
&\|u_2(t) - u_1(t)\|^2 + \|\varphi_2(t) - \varphi_1(t)\|^2_{V'} \\
&+ \int_0^t \Big((1-\rho)\alpha_0 \|\varphi_2(\tau) - \varphi_1(\tau)\|^2 + \frac{\nu}{2}\|\nabla\big(u_2(\tau) - u_1(\tau)\big)\|^2 \Big) d\tau \\
&\leq \big(\|u_2(0) - u_1(0)\|^2 + \|\varphi_2(0) - \varphi_1(0)\|^2_{V'}\big)\Lambda_0(t) + \big|\overline{\varphi_2(0)} - \overline{\varphi_1(0)}\big|^2 \Lambda_1(t) \\
&+ \|h_2 - h_1\|^2_{L^2(0,T;V_{div}')}\Lambda_2(t),
\end{aligned} \tag{3.21}$$

for all $t \in [0,T]$*, where* Λ_0*,* Λ_1 *and* Λ_2 *are continuous functions which depend on the norms of the two solutions. The functions* Q *and* Λ_i *also depend on* F, J, ν *and* Ω*.*

Proof. Arguing as in the first part of the proof of Theorem 2 we can obtain (3.8) that we now write in the following form

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{\nu}{2}\|\nabla u\|^2 \leq \frac{1}{4}(1-\rho)\alpha_0\|\varphi\|^2 + \alpha\|u\|^2 + \frac{1}{\nu}\|h\|^2_{V'_{div}}, \tag{3.22}$$

where the function α is still given by (3.8) and we have set $\varphi := \varphi_2 - \varphi_1$, $u := u_2 - u_1$, $h := h_2 - h_1$. Regarding the estimates for the difference of the nonlocal Cahn-Hilliard, let us first recall the approach used in the proof of [17, Proposition 4]. Following [25], we introduce

$$\widetilde{\Lambda}_1(s) := \int_0^s m(\sigma)F_1''(\sigma)d\sigma, \qquad \widetilde{\Lambda}_2(s) := \int_0^s m(\sigma)F_2''(\sigma)d\sigma, \qquad \Gamma(s) := \int_0^s m(\sigma)d\sigma,$$

for all $s \in [-1,1]$, and see that the assumptions on m and on F imply that $\widetilde{\Lambda}_1 \in C^1([-1,1])$ and $0 < \alpha_0 \leq \widetilde{\Lambda}_1'(s) \leq \alpha_1$ for some positive constant α_1. The weak formulation of the convective nonlocal Cahn-Hilliard equation with degenerate mobility can then be rewritten as follows

$$\langle \varphi_t, \psi\rangle + \big(\nabla\Lambda(\cdot,\varphi), \nabla\psi\big) - \big(\Gamma(\varphi)\nabla a, \nabla\psi\big) + \big(m(\varphi)(\varphi\nabla a - \nabla J * \varphi), \nabla\psi\big) = \big(u\varphi, \nabla\psi\big), \tag{3.23}$$

for all $\psi \in V$, where $\Lambda(x,s) := \widetilde{\Lambda}_1(s) + \widetilde{\Lambda}_2(s) + a(x)\Gamma(s)$ for all $s \in [0,T]$ and almost any $x \in \Omega$.

Consider now two weak solutions $[u_1,\varphi_1]$, $[u_2,\varphi_2]$ and take the difference between the two identities (3.23) corresponding to each solution. Then, choose $\psi = B_N^{-1}(\varphi - \overline{\varphi})$ as test function in the resulting identity. This yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2 + \big(\Lambda(\cdot,\varphi_2) - \Lambda(\cdot,\varphi_1), \varphi\big) - \big((\Gamma(\varphi_2) - \Gamma(\varphi_1))\nabla a, \nabla B_N^{-1}(\varphi-\overline{\varphi})\big)\\
&+ \big((m(\varphi_2) - m(\varphi_1))(\varphi_2\nabla a - \nabla J * \varphi_2), \nabla B_N^{-1}(\varphi-\overline{\varphi})\big)\\
&+ \big(m(\varphi_1)(\varphi\nabla a - \nabla J * \varphi), \nabla B_N^{-1}(\varphi-\overline{\varphi})\big)\\
&= \big(\Lambda(\cdot,\varphi_2) - \Lambda(\cdot,\varphi_1), \overline{\varphi}\big) + \big(u\varphi_1, \nabla B_N^{-1}(\varphi-\overline{\varphi})\big) + \big(u_2\varphi, \nabla B_N^{-1}(\varphi-\overline{\varphi})\big).
\end{aligned} \tag{3.24}$$

Observe first that, thanks to (A5) and(A6), we have

$$\partial_s\Lambda(x,s) = m(s)(F''(s) + a(x)) \geq (1-\rho)\alpha_0, \qquad \forall s \in [-1,1], \quad \text{a.e. } x \in \Omega,$$

and also

$$|\Lambda(x,s_2) - \Lambda(x,s_1)| \leq k|s_2 - s_1|, \qquad \forall s_1, s_2 \in [-1,1], \quad \text{a.e. } x \in \Omega,$$

where $k = \|mF''\|_{C([-1,1])} + \|m\|_{C([-1,1])}\|a\|_{L^\infty(\Omega)}$. Hence we have

$$\big(\Lambda(\cdot,\varphi_2) - \Lambda(\cdot,\varphi_1), \varphi\big) \geq (1-\rho)\alpha_0\|\varphi\|^2,$$

and also

$$\big(\Lambda(\cdot,\varphi_2) - \Lambda(\cdot,\varphi_1), \overline{\varphi}\big) \leq k|\Omega|^{1/2}\|\varphi\|\overline{\varphi} \leq \frac{1}{8}(1-\rho)\alpha_0\|\varphi\|^2 + c\overline{\varphi}^2.$$

Concerning the third, fourth and fifth term on the left-hand side of (3.24), it is easy to see that they can be estimated by

$$\frac{1}{8}(1-\rho)\alpha_0\|\varphi\|^2 + c\|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2.$$

Finally, the last two terms on the right-hand side of (3.24) can be controlled in this way

$$|\big(u\varphi_1, \nabla B_N^{-1}(\varphi-\overline{\varphi})\big)| \leq \|u\|_{L^4}\|\varphi_1\|_{L^4}\|\nabla B_N^{-1}(\varphi-\overline{\varphi})\|$$

$$
\begin{aligned}
&\leq \frac{\nu}{4}\|\nabla u\|^2 + c\|\varphi_1\|_{L^4}^2 \|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2,\\
|\big(u_2\varphi, \nabla B_N^{-1}(\varphi-\overline{\varphi})\big)| &\leq \|u_2\|_{L^4}\|\varphi\|\|\nabla B_N^{-1}(\varphi-\overline{\varphi})\|_{L^4}\\
&\leq \frac{1}{8}(1-\rho)\alpha_0\|\varphi\|^2 + c\|u_2\|_{L^4}^2\|\nabla B_N^{-1}(\varphi-\overline{\varphi})\|_{L^4}^2\\
&\leq \frac{1}{8}(1-\rho)\alpha_0\|\varphi\|^2 + c\|u_2\|_{L^4}^2\|\nabla B_N^{-1}(\varphi-\overline{\varphi})\|\|\nabla B_N^{-1}(\varphi-\overline{\varphi})\|_{H^1}\\
&\leq \frac{1}{8}(1-\rho)\alpha_0\|\varphi\|^2 + c\|u_2\|_{L^4}^2\|B_N^{-1/2}(\varphi-\overline{\varphi})\|\|\varphi-\overline{\varphi}\|\\
&\leq \frac{1}{4}(1-\rho)\alpha_0\|\varphi\|^2 + c\|u_2\|_{L^4}^4\|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2 + c\overline{\varphi}^2.
\end{aligned}
$$

Therefore, using the above estimates, we deduce from (3.24) the following differential inequality

$$
\frac{1}{2}\frac{d}{dt}\|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2 + \frac{3}{4}(1-\rho)\alpha_0\|\varphi\|^2 \leq \frac{\nu}{4}\|\nabla u\|^2 + \zeta\|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2 + c\overline{\varphi}^2, \tag{3.25}
$$

where $\zeta \in L^1(0,T)$ is given by $\zeta := c(1+\|\varphi_1\|_{L^4}^2 + \|u_2\|_{L^4}^4)$. Inequalities (3.22) and (3.25) finally give

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|u\|^2 + \|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2\Big) + (1-\rho)\alpha_0\|\varphi\|^2 + \frac{\nu}{2}\|\nabla u\|^2\\
&\leq \theta\Big(\|u\|^2 + \|B_N^{-1/2}(\varphi-\overline{\varphi})\|^2\Big) + c\overline{\varphi}^2 + \frac{2}{\nu}\|h\|_{V_{div}'}^2,
\end{aligned} \tag{3.26}
$$

where $\theta = 2(\alpha+\zeta) \in L^1(0,T)$. Inequality (3.26) has the same form as (3.15) without the term containing $\widetilde{\mu}$. Therefore, arguing as in the proof of Theorem 2 and using the standard Gronwall's lemma, we find (3.21). ∎

4 Weak-strong uniqueness (nonconstant viscosity)

Here we consider system (1.3)-(1.5) in dimension two with constant mobility, regular potential and nonconstant viscosity $\nu = \nu(\varphi)$. In this case we are not able to prove the uniqueness of weak solutions, due to the poor regularity of φ which makes difficult to estimate the difference of the dissipation term in the Navier-Stokes equations. However, we can prove a weak-strong uniqueness result. This means that, given a weak solution $[u_1, \varphi_1]$ and a strong solution $[u_2, \varphi_2]$ both corresponding to the same initial datum $[u_0, \varphi_0] \in G_{div} \times L^\infty(\Omega)$, then these two solutions coincide.

Before proving this result, let us first show that a global strong solution exists. Indeed, we observe that, while the existence of a weak solution with nonconstant viscosity easily follows easily from the same result for the constant viscosity case (see [9]), this does not occur as far as strong solutions are concerned. The difficulty essentially lies in the fact that the classical results for the Navier-Stokes equations in two dimensions with constant viscosity (see, e.g., [34]) cannot be used as in [16] to exploit the improved regularity for the convective term in the nonlocal Cahn-Hilliard equation.

The regularity result requires a slightly stronger assumption on the interaction kernel J. Thus, before stating the main results of this section we recall the definition of admissible kernel (see [6, Definition 1]).

Definition 2. *A kernel $J \in W^{1,1}_{loc}(\mathbb{R}^2)$ is admissible if the following conditions are satisfied:*

(A1) $J \in C^3(\mathbb{R}^2 \setminus \{0\})$*;*

(A2) *J is radially symmetric, $J(x) = \tilde{J}(|x|)$ and $\tilde{J}$ is non-increasing;*

(A3) *$\tilde{J}''(r)$ and $\tilde{J}'(r)/r$ are monotone on $(0, r_0)$ for some $r_0 > 0$;*

(A4) *$|D^3 J(x)| \leq C_\sharp |x|^{-3}$ for some $C_\sharp > 0$.*

We recall that the Newtonian and Bessel potentials are admissible. Moreover, we report the following (cf. [6, Lemma 2]).

Lemma 1. *Let J be admissible and $v = \nabla J * \psi$. Then, for all $p \in (1, \infty)$, there exists $C_p > 0$ such that*

$$\|\nabla v\|_p \leq C_{L^p} \|\psi\|_{L^p}.$$

We also recall the following proposition for an inhomogeneous Stokes system in non-divergence form:

$$\begin{cases} -\varpi(x)\,\Delta u + \nabla \pi = f(x)\,, & \text{in } \Omega, \\ \operatorname{div}(u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Proposition 1. *[36, Proposition 2.1] Let $f \in L^2(\Omega)^2$ and $\varpi \in C^\delta\left(\overline{\Omega}\right)$, for some $\delta \in (0,1)$, such that $0 < \lambda_0 \leq \varpi(x) \leq \lambda_1 < \infty$ for all $x \in \overline{\Omega}$. Then any solution $[u, \pi] \in H^2(\Omega)^2 \times H^1(\Omega)$ of (4.1) satisfies the estimate*

$$\|u\|_{H^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C\left(\|f\|_{L^2} + \|\pi\|_{L^2}\right),$$

for some constant $C = C(\lambda_0, \lambda_1, \Omega, \|\varpi\|_{C^\delta(\overline{\Omega})}) > 0$.

We first show a result which generalizes [21, Lemma 2.11] for the nonlocal Cahn-Hilliard equation with convection in two space dimensions.

Lemma 2. *Let $d = 2$ and assume (H1) and (H3). Let $u \in L^\infty(T', T; G_{div}) \cap L^2(T', T; V_{div})$, for some $T > T' \geq 0$ and let $\varphi \in L^\infty(T', T; L^\infty(\Omega))$ be a bounded generalized (weak) solution of*

$$\begin{cases} \partial_t \varphi = div(c(x, \varphi, \nabla\varphi)) - div(u\varphi)\,, & in\ \Omega \times (T', T), \\ c(x, \varphi, \nabla\varphi) \cdot n = 0, & on\ \Gamma \times (T', T), \end{cases} \tag{4.2}$$

where $c(x, \varphi, \nabla\varphi) := (a(x) + F''(\varphi))\nabla\varphi + \nabla a \varphi - \nabla J * \varphi$. There exist constants $C > 0$, $\alpha \in (0,1)$, depending on the $L^\infty(T', T; L^\infty(\Omega))$-norm of φ and $L^4(T', T; L^4(\Omega)^2)$-norm of u, respectively, such that

$$|\varphi(x,t) - \varphi(y,s)| \leq C(|x-y|^\alpha + |t-s|^{\alpha/2}), \tag{4.3}$$

for every $(x,t), (y,s) \in Q_{T',T} := [T', T] \times \overline{\Omega}$.

Proof. The proof is inspired by [31, Theorem 3.7] (cf. also [36, Lemma 3.2]) where it was observed that a Hölder continuous estimate holds for a similar parabolic equation with drift term $u \cdot \nabla\varphi$ whenever the vector field u is divergent free and belongs to the critical space $L^4\left(0, T; L^4(\Omega)\right)$. We begin by assuming that $\|\varphi\|_{L^\infty(T',T;L^\infty(\Omega))} \leq R$, for some $R > 0$ and observe that

$$L^\infty(T', T; G_{div}) \cap L^2(T', T; V_{div}) \hookrightarrow L^4(T', T; L^4(\Omega)^2).$$

Following [30], we let $k \in [0, R]$ and $\eta = \eta(x,t) \in [0,1]$ be a continuous piecewise-smooth function which is supported on the space-time cylinders $Q_{t_0, t_0+\tau}(\rho) := B_\rho(x_0) \times (t_0, t_0 + \tau)$, where $B_\rho(x_0)$ denotes the ball centered at x_0 of radius $\rho > 0$. As usual for the interior Hölder regularity in (4.3) one takes $x_0 \in \Omega$, while $x_0 \in \partial\Omega$ for the corresponding boundary estimate in (4.3) and then exploit a standard compactness argument in which $\overline{\Omega}$ may be covered by a finite number of such balls. We thus multiply the first equation of (4.2) by $\eta^2\varphi_k^+$, where $\varphi_k^+ := \max\{0, \varphi - k\}$, integrate the resulting identity over $Q_{t_0,t} := (t_0, t) \times \Omega$, where $T' \leq t_0 < t < t_0 + \tau \leq T$, to deduce

$$\begin{aligned}
&\int_{Q_{t_0,t}} \partial_t\varphi\eta^2\varphi_k^+ dxdt + \int_{Q_{t_0,t}} (a(x) + F''(\varphi))\nabla\varphi_k^+ \cdot \nabla\left(\eta^2\varphi_k^+\right) dxdt \\
&= \int_{Q_{t_0,t}} u\varphi \cdot \nabla\left(\eta^2\varphi_k^+\right) dxdt + \int_{Q_{t_0,t}} l(x,t) \cdot \nabla\left(\eta^2\varphi_k^+\right) dxdt,
\end{aligned} \tag{4.4}$$

owing to the boundary condition of (4.2) and the fact that $u \in L^2(T', T; V_{div})$. Here, we have set $l = -\varphi\nabla a + \nabla J * \varphi$ for the sake of simplicity. Also we notice that $\nabla\varphi_k^+ \equiv \nabla\varphi$ only on the sets where $\{\varphi(x,t) > k\}$ while $\nabla\varphi_k^+ \equiv 0$ elsewhere. In addition, if $J \in W^{1,1}(\mathbb{R}^2)$ then $l \in L^\infty(T', T; L^\infty(\Omega)^2)$, since φ is bounded and $a \in W^{1,\infty}(\Omega) \hookrightarrow C(\overline{\Omega})$. From (4.4) and assumption (H3), we obtain

$$\begin{aligned}
&\frac{1}{2}\sup_{t\in(t_0,t)} \int_\Omega \left(\eta\varphi_k^+\right)^2(t)\, dx + c_0 \int_{Q_{t_0,t}} \left|\nabla\left(\eta\varphi_k^+\right)\right|^2 dxdt \\
&\quad \leq \frac{1}{2}\int_\Omega \left(\eta\varphi_k^+\right)^2(t_0)\, dx + \int_{Q_{t_0,t}} \left(\varphi_k^+\right)^2 |\eta\partial_t\eta|\, dxdt \\
&\quad + L(R)\int_{Q_{t_0,t}} \left(\varphi_k^+\right)^2 |\nabla\eta|^2 dxdt + \int_{Q_{t_0,t}} u\varphi \cdot \nabla\left(\eta^2\varphi_k^+\right) dxdt \\
&\quad + \int_{Q_{t_0,t}} l(x,t) \cdot \nabla\left(\eta^2\varphi_k^+\right) dxdt,
\end{aligned} \tag{4.5}$$

for some function $L > 0$ such that $|a(x) + F''(\varphi)| \leq L(R)$. Indeed, we have

$$\nabla\varphi_k^+ \cdot \nabla\left(\eta^2\varphi_k^+\right) = \left|\nabla\left(\eta\varphi_k^+\right)\right|^2 - |\nabla\eta|^2\left(\varphi_k^+\right)^2, \quad \text{a.e. in } Q_{t_0,t}.$$

To estimate the fourth term on the right-hand side of (4.5) we use the fact that $u \in L^4(T', T; L^4(\Omega)^2)$ is also divergent free, we argue by elementary Hölder's and Young's inequalities as in the proof of [36, Lemma 3.2] to find

$$\begin{aligned}
&\left|\int_{Q_{t_0,t}} u\varphi \cdot \nabla\left(\eta^2\varphi_k^+\right) dxdt\right| \\
&\leq \frac{1}{4}\left\|\eta\varphi_k^+\right\|^2_{L^2(Q_{t_0,t})} + \frac{c_0}{4}\left\|\nabla\left(\eta\varphi_k^+\right)\right\|^2_{L^2(Q_{t_0,t})} + C_0\left\|\nabla\eta\varphi_k^+\right\|^2_{L^2(Q_{t_0,t})},
\end{aligned} \tag{4.6}$$

where $C_0 > 0$ depends on $c_0 > 0$ and the $L^4(T', T; L^4(\Omega)^2)$-norm of u only. For the final term on the right-hand side of (4.5), we employ Hölder's and Young's inequalities again to deduce

$$\begin{aligned}
\left|\int_{Q_{t_0,t}} l(x,t) \cdot \nabla\left(\eta^2\varphi_k^+\right) dxdt\right| &= \left|\int_{Q_{t_0,t}} \left(l(x,t) \cdot \nabla\eta\varphi_k^+\eta + \eta l(x,t) \cdot \nabla\left(\eta\varphi_k^+\right)\right) dxdt\right| \\
&\leq C_1\int_{Q_{t_0,t}} |\eta|^2 dxdt + \frac{1}{2}\int_{Q_{t_0,t}} \left(\varphi_k^+\right)^2 |\nabla\eta|^2 dxdt
\end{aligned} \tag{4.7}$$

$$+\frac{c_0}{4}\int_{Q_{t_0,t}}\left|\nabla\left(\eta\varphi_k^+\right)\right|^2 dxdt,$$

where $C_1 > 0$ depends only on $c_0 > 0$ and the $L^\infty(T',T;L^\infty(\Omega)^2)$-norm of l, and hence on $R > 0$. Inserting estimates (4.6)-(4.7) into the right-hand side of (4.5), we infer the existence of a constant $C_2 = C_2(C_0, C_1) > 0$ such that

$$\begin{aligned}
&\frac{1}{2}\sup_{t\in(t_0,t)}\int_\Omega\left(\eta\varphi_k^+\right)^2(t)\,dx + c_0\int_{Q_{t_0,t}}\left|\nabla\left(\eta\varphi_k^+\right)\right|^2 dxdt \\
&\qquad\le \frac{1}{2}\int_\Omega\left(\eta\varphi_k^+\right)^2(t_0)\,dx \\
&\qquad + C_2\left(\int_{Q_{t_0,t}}\left(\varphi_k^+\right)^2|\eta\partial_t\eta|\,dxdt + \int_{Q_{t_0,t}}\left(\varphi_k^+\right)^2|\nabla\eta|^2 dxdt + \int_{Q_{t_0,t}}|\eta|^2 dxdt\right).
\end{aligned}\tag{4.8}$$

Arguing in a similar fashion, inequality (4.8) also holds with φ replaced by $-\varphi$. In particular, such inequalities imply that the generalized solution φ of (4.2) is an element of $\mathcal{B}_2(Q_{T',T}, R, \gamma, \omega, 0, \varkappa)$ in the sense of [30, Chapter II, Section 7], for some $\gamma = \gamma(c_0, R)$ and $\omega, \varkappa > 0$ (cf., in particular, the inequalities in [30, Section V, (1.12)-(1.13)]). Therefore, on account of [30, Chapter V, Theorem 1.1], the Hölder continuity (4.3) of the solution of (4.2) follows in a standard way. This ends the proof. ∎

Corollary 1. *Let $d = 2$. If $[u, \varphi]$ is any weak solution to problem* (1.3)–(1.6) *in the sense of Theorem 1 then, for every $\tau > 0$, we have*

$$\|\varphi\|_{C^{\delta/2,\delta}([\tau,\infty)\times\overline{\Omega})} \le C_\tau,$$

for some $C_\tau \sim \tau^{-\gamma}$, $\gamma > 0$, depending only on $\mathcal{E}(u_0, \varphi_0)$ and on the other parameters of the problem.

Proof. The claim follows from the statement of Theorem 1 and the application of Lemma 2 and [21, Lemma 2.10]. ∎

The following result on the existence of a strong solution generalizes [16, Theorem 2] to the case of nonconstant viscosity.

Theorem 5. *Let $d = 2$ and suppose that (H1)–(H5) are satisfied with either $J \in W^{2,1}(B_\delta)$ or J admissible. Assume that $u_0 \in V_{div}$, $\varphi_0 \in V \cap C^\beta(\overline{\Omega})$, for some $\beta > 0$, and $h \in L^2_{loc}(\mathbb{R}^+; G_{div})$. Then, for every $T > 0$, there exists a solution $[u, \varphi]$ to* (1.3)–(1.6) *such that*

$$u \in L^\infty(0,T;V_{div}) \cap L^2(0,T;H^2(\Omega)^2),\ \ u_t \in L^2(0,T;G_{div}) \tag{4.9}$$
$$\varphi, \mu \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(\Omega\times(0,T)), \tag{4.10}$$
$$\varphi_t, \mu_t \in L^2(0,T;H). \tag{4.11}$$

Furthermore, suppose in addition that $F \in C^3(\mathbb{R})$ and $\varphi_0 \in H^2(\Omega)$. Then, system (1.3)-(1.6) *admits a strong solution on $[0,T]$ satisfying* (4.9) *and*

$$\varphi, \mu \in L^\infty(0,T;H^2(\Omega)), \tag{4.12}$$
$$\varphi_t, \mu_t \in L^\infty(0,T;H) \cap L^2(0,T;V). \tag{4.13}$$

Proof. Step 1. We first need to establish the $L^\infty(0,T;V)$-regularity for μ and φ. The argument used here differs from the one devised in [16]. Indeed, we cannot easily exploit the regularity $u \in L^2(0,T;H^2)$ as it happens for the constant viscosity case. Let us consider equation (1.3) whose generalized (weak) solution also satisfies (4.2). First we recall that φ is bounded (see [21, Lemma 2.10], cf. also [16, Theorem 2]) and thus, by Lemma 2, we infer that $\varphi \in C^{\delta/2,\delta}\left([0,T]\times\overline{\Omega}\right)$ for some $0<\delta\leq \min\{\alpha,\beta\}$. By assumption (H2), $\nu(\varphi)\in C^{\delta/2,\delta}\left([0,T]\times\overline{\Omega}\right)$ since ν is a (locally) Lipschitz function on $\mathbb{R}$; moreover, there exists a positive constant $\nu_2=\nu_2(R)>0$ such that $\nu_2\geq\nu(\varphi)\geq\nu_1$, almost everywhere in $(0,T)\times\Omega$, owing once again to the boundedness of φ. In the same fashion, we define $b(x,t,\varphi)=a(x)+F''(\varphi)$ and observe that it is measurable and bounded (i.e., $c_0\leq b\leq b_0=b_0(R,\|a\|_{L^\infty})$) for all (x,t,φ), in light of $a\in W^{1,\infty}(\Omega)\hookrightarrow C(\overline{\Omega})$ and the fact that $F''(\varphi)\in C\left([0,T]\times\overline{\Omega}\right)$. In fact, as a function of $(x,t)\in Q_{0,T}$, $b(\cdot,\cdot,\varphi(\cdot,\cdot))$ is also continuous due to the Hölder continuity of φ. Henceforth we shall denote by R a constant such that $\|\varphi\|_{L^\infty(\Omega\times(0,T))}\leq R$.

We now test the nonlocal Cahn-Hilliard equation by $\mu_t=\big(a+F''(\varphi)\big)\varphi_t-J*\varphi_t$ in H to deduce

$$\begin{aligned}&\int_\Omega \varphi_t\mu_t+\int_\Omega (u\cdot\nabla\varphi)\mu_t+\frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2\\ &=\int_\Omega (a+F''(\varphi))\varphi_t^2-(\varphi_t,J*\varphi_t)+\int_\Omega(u\cdot\nabla\varphi)\mu_t+\frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2=0.\end{aligned}\tag{4.14}$$

This identity was considered in [16], but now we cannot use the H^2-norm of u to estimate the convective term (i.e., the third term in the second line of (4.14)). Here we exploit the identity

$$u\cdot\nabla\varphi=b^{-1}u\cdot\nabla\mu+b^{-1}u\cdot(\nabla J*\varphi-\nabla a\varphi)\tag{4.15}$$

and we find

$$\begin{aligned}\Big|\int_\Omega(u\cdot\nabla\varphi)\mu_t\Big|&=\Big|\int_\Omega\left(b^{-1}u\cdot\nabla\mu\right)\mu_t+\int_\Omega b^{-1}\left[u\cdot(\nabla J*\varphi-\nabla a\varphi)\right]\mu_t\Big| \qquad (4.16)\\ &\leq c_0^{-1}\left(\|u\cdot\nabla\mu\|\|\mu_t\|+\|u\cdot(\nabla J*\varphi-\nabla a\varphi)\|\|\mu_t\|\right)\\ &\leq Q_{J,c_0}(R)\|\varphi_t\|\left(\|u\cdot\nabla\mu\|+\|u\|\right)\\ &\leq\frac{c_0}{4}\|\varphi_t\|^2+Q_{c_0,J}(R)\left(\|u\|_{L^4}^2\|\nabla\mu\|_{L^4}^2+\|u\|^2\right)\\ &\leq\frac{c_0}{4}\|\varphi_t\|^2+Q_{c_0,J}(R)\|u\|\|\nabla u\|\|\nabla\mu\|\|\mu\|_{H^2}+Q_{c_0,J}(R)\,\|u\|^2\\ &\leq\frac{c_0}{4}\|\varphi_t\|^2+Q_{c_0,J,\epsilon}(R)\big(\|u\|^2\|\nabla u\|^2\big)\|\nabla\mu\|^2\\ &+Q_{c_0,J}(R)\,\|u\|^2+\epsilon\left(\|B_N\mu\|^2+\|\mu\|^2\right),\end{aligned}$$

for any $\epsilon>0$. Furthermore, we have

$$\begin{aligned}|(\varphi_t,J*\varphi_t)|&\leq\|\varphi_t\|_{V'}\|J*\varphi_t\|_V\leq\|\varphi_t\|_{V'}\|J\|_{W^{1,1}}\|\varphi_t\|\\ &\leq\frac{c_0}{4}\|\varphi_t\|^2+c\|J\|_{W^{1,1}}^2\|\varphi_t\|_{V'}^2.\end{aligned}\tag{4.17}$$

Inserting (4.16), (4.17) into (4.14), and keeping $\epsilon>0$ arbitrary, we get the following differential inequality

$$\frac{d}{dt}\|\nabla\mu\|^2+c_0\|\varphi_t\|^2\tag{4.18}$$

$$\leq Q_{c_0,J,\epsilon}(R)\big(\|u\|^2\|\nabla u\|^2\big)\|\nabla\mu\|^2 + c\|J\|^2_{W^{1,1}}\|\varphi_t\|^2_{V'}$$
$$+ Q_{c_0,J}(R)\,\|u\|^2 + \epsilon\left(\|B_N\mu\|^2 + \|\mu\|^2\right).$$

Moreover, observing that $\varphi_t = -B_N\mu - u\cdot\nabla\varphi$, we have

$$\|\varphi_t\|^2 \geq \frac{1}{2}\,\|B_N\mu\|^2 - \|u\cdot\nabla\varphi\|^2\,, \tag{4.19}$$

owing to the basic inequality $(a-b)^2 \geq (1/2)\,a^2 - b^2$. We can estimate the last term using (4.15). Thus, recalling (4.16), we obtain

$$\begin{aligned}\|u\cdot\nabla\varphi\|^2 &\leq 2c_0^{-2}\left(\|u\cdot\nabla\mu\|^2 + \|u\cdot(\nabla J * \varphi - \nabla a\varphi)\,\|^2\right)\\ &\leq Q_{c_0,J,\epsilon}(R)\big(\|u\|^2\|\nabla u\|^2\big)\|\nabla\mu\|^2 + Q_{c_0,J}(R)\,\|u\|^2\\ &+ \epsilon\left(\|B_N\mu\|^2 + \|\mu\|^2\right).\end{aligned}$$

Thus, from (4.18) by virtue of (4.19) we further derive

$$\begin{aligned}&\frac{d}{dt}\|\nabla\mu\|^2 + \frac{c_0}{2}\left(\|\varphi_t\|^2 + \frac{1}{2}\,\|B_N\mu\|^2\right)\\ &\leq Q_{c_0,J,\epsilon}(R)\big(\|u\|^2\|\nabla u\|^2\big)\|\nabla\mu\|^2 + c\|J\|^2_{W^{1,1}}\|\varphi_t\|^2_{V'}\\ &+ Q_{c_0,J}(R)\,\|u\|^2 + 2\epsilon\left(\|B_N\mu\|^2_{H^2} + \|\mu\|^2\right),\end{aligned} \tag{4.20}$$

for any $\epsilon > 0$. Let us now choose a sufficiently small $\epsilon \leq c_0/8$ in order to absorb the L^2-norm of $B_N\mu$ into the left-hand side and observe that $\mu \in L^\infty(\Omega\times(0,T))$ since φ is bounded. Thus, we find

$$\begin{aligned}&\varphi \in L^\infty(0,T;V),\ \ \varphi_t \in L^2(0,T;H),\\ &\mu \in L^\infty(0,T;V)\cap L^2\left(0,T;H^2(\Omega)\right),\end{aligned} \tag{4.21}$$

by means of Gronwall's inequality (cf. also Lemma 1), using the initial condition $\varphi_0 \in V\cap L^\infty(\Omega)$ (which implies $\mu_0 \in V$), the regularity properties of the weak solution given by the first of (2.2) and by (2.5), and the fact that

$$c_0\|\nabla\varphi\|^2 - Q(R) \leq \|\nabla\mu\|^2 \leq Q(R)\big(\|\nabla\varphi\|^2 + 1\big).$$

We now control $\nabla\varphi$ in terms of $\nabla\mu$ in L^p. In order to do that we take the gradient of $\mu = a\varphi - J*\varphi + F'(\varphi)$, multiply it by $\nabla\varphi|\nabla\varphi|^{p-2}$ and integrate the resulting identity on Ω. This gives

$$\int_\Omega \nabla\varphi|\nabla\varphi|^{p-2}\cdot\nabla\mu = \int_\Omega (a + F''(\varphi))|\nabla\varphi|^p + \int_\Omega (\varphi\nabla a - \nabla J*\varphi)\cdot\nabla\varphi|\nabla\varphi|^{p-2}.$$

So that, by (H3), we find

$$\begin{aligned}c_0\|\nabla\varphi\|^p_{L^p} &\leq \|\nabla\varphi\|^{p-1}_{L^p}\|\nabla\mu\|_{L^p} + (\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1})\|\varphi\|_{L^p}\|\nabla\varphi\|^{p-1}_{L^p}\\ &\leq \frac{c_0}{2}\|\nabla\varphi\|^p_{L^p} + c\|\nabla\mu\|^p_{L^p} + Q(R)(\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1})^p,\end{aligned}$$

which yields

$$\|\nabla\varphi\|_{L^p} \leq c\|\nabla\mu\|_{L^p} + Q(R). \tag{4.22}$$

This estimate implies in particular

$$\varphi \in L^4\left(0,T;W^{1,4}(\Omega)\right), \tag{4.23}$$

owing to the second of (4.21). We now control the H^2-norm of φ (or at least the L^2-norm of the second derivatives $\partial^2_{ij}\varphi := \frac{\partial^2\varphi}{\partial x_i \partial x_j}$) in terms of the H^2-norm of μ and (4.23). To this aim apply the second derivative operator ∂^2_{ij} to (1.4), multiply the resulting identity by $\partial^2_{ij}\varphi$ and integrate on Ω. This entails

$$\begin{aligned}\int_\Omega \partial^2_{ij}\mu\partial^2_{ij}\varphi &= \int_\Omega (a+F''(\varphi))(\partial^2_{ij}\varphi)^2 + \int_\Omega (\partial_i a\partial_j\varphi + \partial_j a\partial_i\varphi)\partial^2_{ij}\varphi \\ &+ \int_\Omega (\varphi\partial^2_{ij}a - \partial_i(\partial_j J * \varphi)\partial^2_{ij}\varphi + \int_\Omega F'''(\varphi)\partial_i\varphi\partial_j\varphi\partial^2_{ij}\varphi, \qquad i,j=1,2.\end{aligned}$$

From this identity, thanks to (H3), we obtain

$$\begin{aligned}& c_0\|\partial^2_{ij}\varphi\|^2 \le c\|\partial^2_{ij}\mu\|^2 \\ &+ c\big(\|\nabla a\|^2_{L^\infty} + Q(R)\big)\|\nabla\varphi\|^2 + Q(R)\|\partial^2_{ij}a\|^2 \\ &+ \|\partial_i(\partial_j J * \varphi)\|^2 + Q(R)\|\nabla\varphi\|^4_{L^4},\end{aligned} \tag{4.24}$$

and an estimate like this still holds if $\|\partial^2_{ij}\varphi\|$ and $\|\partial^2_{ij}\mu\|$ are replaced by $\|\varphi\|_{H^2}$ and $\|\mu\|_{H^2}$, respectively. Thus, recalling (4.21), (4.23), and using the fact that $J \in W^{2,1}(B_\delta)$ or J is admissible, from (4.24) we easily get

$$\varphi \in L^2\left(0,T;H^2(\Omega)\right). \tag{4.25}$$

Step 2. We now establish the $L^\infty(0,T;V_{div}) \cap L^2\left(0,T;H^2(\Omega)^2\right)$-regularity for u. To this end, let us test the Navier-Stokes equations by u_t in G_{div} to deduce the identity

$$\|u_t\|^2 + 2\int_\Omega \nu(\varphi)\,(Du : Du_t)\,dx + b(u,u,u_t) = (l,u_t), \tag{4.26}$$

where the function l is given by

$$l := -\frac{\varphi^2}{2}\nabla a - (J*\varphi)\nabla\varphi + h.$$

Notice that, due to the assumption on the external force h and to the regularity of φ, we have $l \in L^2(0,T;L^2(\Omega)^2)$. From (4.26) we obtain

$$\frac{1}{2}\|u_t\|^2 + \frac{d}{dt}\int_\Omega \nu(\varphi)|Du|^2 + b(u,u,u_t) \le \frac{1}{2}\|l\|^2 + \int_\Omega |Du|^2\nu'(\varphi)\varphi_t. \tag{4.27}$$

Observe that

$$\begin{aligned}&\Big|\int_\Omega |Du|^2\nu'(\varphi)\varphi_t\Big| \le \|\nu'(\varphi)\|_{L^\infty}\|\varphi_t\|\|Du\|^2_{L^4} \\ &\le Q(R)\|\varphi_t\|\|Du\|\|u\|_{H^2} \\ &\le \delta\|u\|^2_{H^2} + Q_\delta(R)\|Du\|^2\|\varphi_t\|^2.\end{aligned} \tag{4.28}$$

Furthermore, we have

$$|b(u,u,u_t)| \le \frac{1}{4}\|u_t\|^2 + \|u\cdot\nabla u\|^2$$

$$
\begin{aligned}
&\leq \frac{1}{4}\|u_t\|^2 + 2\|u\|_{L^4}^2\|\nabla u\|_{L^4}^2 \\
&\leq \frac{1}{4}\|u_t\|^2 + c\|u\|\|\nabla u\|\|\nabla u\|\|u\|_{H^2} \\
&\leq \frac{1}{4}\|u_t\|^2 + \delta\|u\|_{H^2}^2 + c_\delta\big(\|u\|^2\|\nabla u\|^2\big)\|\nabla u\|^2. \qquad (4.29)
\end{aligned}
$$

Plugging (4.28) and (4.29) into (4.27), we get

$$
\begin{aligned}
&\frac{1}{4}\|u_t\|^2 + \frac{d}{dt}\int_\Omega \nu(\varphi)|Du|^2 \\
&\leq \frac{1}{2}\|l\|^2 + 2\delta\|u\|_{H^2}^2 + c_\delta\big(\|u\|^2\|\nabla u\|^2\big)\|Du\|^2 \\
&+ Q_\delta(R)\|Du\|^2\|\varphi_t\|^2, \qquad (4.30)
\end{aligned}
$$

for any $\delta > 0$ that will be fixed later.

It remains to absorb the term $2\delta\|u\|_{H^2}^2$ into the left-hand side of inequality (4.30). This can be done essentially by controlling it with $2\delta\,\|u_t\|^2$ plus some lower-order (bounded) perturbation. To achieve this we first rewrite the Navier-Stokes equations as an inhomogeneous elliptic system in divergence form, namely,

$$
\begin{cases}
-\mathrm{div}(2\nu(\varphi)Du) + \nabla\pi = \widetilde{h}, & \text{in } \Omega\times(0,T)\,, \\
\mathrm{div}\,(u) = 0, & \text{in } \Omega\times(0,T)\,, \\
u = 0, & \text{on } \partial\Omega\times(0,T)\,,
\end{cases} \qquad (4.31)
$$

where

$$
\widetilde{h} := \mu\nabla\varphi + h(t) - (u\cdot\nabla)u - u_t. \qquad (4.32)
$$

Since φ is bounded on $\Omega\times(0,T)$ (and therefore, $\nu(\varphi)$ is bounded by (H3)), by the application of Lax-Milgram lemma, we can infer that every solution $[u,\pi] \in V_{div}\times L^2(\Omega)$ to (4.31) such that $\overline{\pi} = 0$ satisfies the bound

$$
\|Du\| + \|\pi\| \leq C||\widetilde{h}||_{V'}, \qquad (4.33)
$$

for some $C > 0$ which depends on Ω and $R > 0$ only. On the other hand, we can also rewrite (4.31) as an inhomogeneous elliptic system in non-divergence form, that is,

$$
\begin{cases}
-\nu(\varphi)\Delta u + \nabla\pi = \widehat{h}, & \text{in } \Omega\times(0,T)\,, \\
\mathrm{div}\,(u) = 0, & \text{in } \Omega\times(0,T)\,, \\
u = 0, & \text{on } \partial\Omega\times(0,T)\,,
\end{cases} \qquad (4.34)
$$

where

$$
\widehat{h} := \widetilde{h} + 2\nu'(\varphi)\,\nabla\varphi\cdot Du.
$$

We can then apply Proposition 1 to (4.34) since $\nu(\varphi) \in C^{\delta/2,\delta}\big([0,T]\times\overline{\Omega}\big)$. Thus we obtain the bound (cf. also (4.33))

$$
\begin{aligned}
\|u\|_{H^2} + \|\pi\|_{H^1} &\leq C\left(||\widehat{h}|| + \|\pi\|\right) \leq C\left(||\widehat{h}|| + ||\widetilde{h}||_{V'}\right) \qquad (4.35)\\
&\leq C\left(||\widetilde{h}|| + \|\nabla\varphi\cdot Du\|\right),
\end{aligned}
$$

where $C = C(\nu_1,\nu_2,R,T,\Omega) > 0$. Recalling (4.32), we deduce

$$
\|u\|_{H^2} \leq C\,\|u_t\| + C\left(\|h\| + \|u\cdot\nabla u\| + \|\mu\nabla\varphi\| + \|\nabla\varphi\cdot Du\|\right) \qquad (4.36)
$$

$$
\begin{aligned}
&\leq C\|u_t\| + C\left(\|h\| + \|u\|_{L^4}\|\nabla u\|_{L^4} + \|\mu\|_{L^\infty}\|\nabla\varphi\| + \|\nabla\varphi\|_{L^4}\|Du\|_{L^4}\right)\\
&\leq C\|u_t\| + C\left(\|h\| + \|u\|^{1/2}\|Du\|\,\|u\|_{H^2}^{1/2}\right)\\
&+ C\left(\|\mu\|_{L^\infty}\|\nabla\varphi\| + \|\nabla\varphi\|_{L^4}\|Du\|^{1/2}\|u\|_{H^2}^{1/2}\right)\\
&\leq C\|u_t\| + C_\epsilon\left(\|h\| + (\|u\|\,\|\nabla u\|)\,\|Du\|\right)\\
&+ C_\epsilon\left(\|\mu\|_{L^\infty}\|\nabla\varphi\| + \|\nabla\varphi\|_{L^4}^2\right)\|Du\| + 2\epsilon\|u\|_{H^2},
\end{aligned}
$$

for any $\epsilon > 0$. Thus, for $\epsilon \in (0, \frac{1}{2})$ we can absorb the small term on the left-hand side and infer

$$
\begin{aligned}
\|u\|_{H^2}^2 \leq C\|u_t\|^2 &+ C\left(\|h\|^2 + \|\mu\|_{L^\infty}^2\|\nabla\varphi\|^2\right)\\
&+ C\left(\|u\|^2\|\nabla u\|^2 + \|\nabla\varphi\|_{L^4}^4\right)\|Du\|^2.
\end{aligned}
\tag{4.37}
$$

We can now insert the bound (4.37) into (4.30), take $\delta > 0$ small enough and obtain the differential inequality

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \nu(\varphi)|Du|^2 + \frac{1}{8}\|u_t\|^2\\
&\leq C\left(\|l\|^2 + \|h\|^2 + \|\mu\|_{L^\infty}^2\|\nabla\varphi\|^2\right)\\
&+ C(R)\,(\|u\|^2\|\nabla u\|^2 + \|\nabla\varphi\|_{L^4}^4 + \|\varphi_t\|^2)\|Du\|^2.
\end{aligned}
\tag{4.38}
$$

From (4.38), on account of (H2) and of the improved regularity for $[\varphi, \mu]$ given by (4.21) and (4.23), by means of Gronwall's inequality, we obtain

$$
u \in L^\infty(0,T;V_{div}) \cap L^2(0,T;H^2(\Omega)^2), \;\; u_t \in L^2(0,T;G_{div}). \tag{4.39}
$$

Moreover, owing to (4.35), we have $\pi \in L^2\left(0,T;H^1(\Omega)\right)$. With these regularity properties for u at disposal we can now argue exactly as in the second step of the proof of [16, Theorem 2] by differentiating (1.3) with respect to time, multiplying the resulting identity by μ_t in H and using the assumptions $F \in C^3(\mathbb{R})$ and $\varphi_0 \in H^2(\Omega)$ (this last assumption ensures that $\varphi_t(0) \in H$, see Lemma 1) to deduce

$$
\varphi_t \in L^\infty(0,T;H) \cap L^2(0,T,V). \tag{4.40}
$$

Furthermore, using (1.3), we find

$$
\begin{aligned}
\|\nabla\mu\|_{L^p} &\leq c\|\nabla\mu\|^{2/p}\|\nabla\mu\|_{H^1}^{1-2/p}\\
&\leq c\|\nabla\mu\|^{2/p}\|\mu\|_{H^2}^{1-2/p}\\
&\leq c\|\nabla\mu\|^{2/p}(\|\Delta\mu\|^{1-2/p} + \|\mu\|^{1-2/p})\\
&\leq Q(R, \|\varphi_0\|_V, \|u_0\|)\big(\|\varphi_t\|^{1-2/p} + \|u\cdot\nabla\varphi\|^{1-2/p} + 1\big)\\
&\leq Q(R, \|\varphi_0\|_V, \|u_0\|)\big(\|\varphi_t\|^{1-2/p} + \|u\|_{L^q}^{1-2/p}\|\nabla\varphi\|_{L^p}^{1-2/p} + 1\big).
\end{aligned}
\tag{4.41}
$$

Here we have used the fact that the H^2-norm of μ is equivalent to the L^2- norm of $(B_N + I)\,\mu$ (cf. (1.5)) and we have taken into account the improved regularity for μ given by the third of (4.21). By combining (4.21) with (4.41) we therefore get

$$
\|\nabla\varphi\|_{L^p} \leq Q(R, \|\varphi_0\|_V, \|u_0\|)\big(\|\varphi_t\|^{1-2/p} + \|u\|_{L^{2p/(p-2)}}^{(p-2)/2} + 1\big) \tag{4.42}
$$

$$\leq Q(R,\|\varphi_0\|_V,\|u_0\|)\big(\|\varphi_t\|^{1-2/p}+\|u\|^{(p-2)^2/2p}\|\nabla u\|^{1-2/p}+1\big).$$

Thanks to this property, on account of $(4.39)_1$, (4.40) and (4.41)-(4.42), we have

$$\varphi\in L^\infty(0,T;W^{1,p}(\Omega)). \tag{4.43}$$

Finally, by comparison in (1.3) (cf. [16]) we also get $\mu\in L^\infty(0,T;H^2(\Omega))$. This fact, thanks to (4.24) and using once more the regularity assumption on J, implies

$$\varphi\in L^\infty\left(0,T;H^2(\Omega)\right). \tag{4.44}$$

Step 3. We shall briefly explain the details of the approximation schemes which can be used to derive the estimates in Steps 1 and 2. Regarding estimates (4.21), (4.23), (4.25), it suffices to employ the usual Faedo-Galerkin truncation method as in [9, Theorem 1] since $u\in L^\infty(0,T;G_{div})\cap L^2(0,T;V_{div})$ is a weak solution to (1.1)-(1.2). Weak solutions are also enough to deduce (4.3). To deduce the higher-order estimate for $u\in L^\infty(0,T;V_{div})$ in Step 2, we can no longer exploit the usual Galerkin scheme in a standard fashion but we need to rely on a different scheme. We first mollify the Navier-Stokes equation in the following fashion: recall that $\varphi\in L^\infty(0,T;V)\cap L^2\left(0,T;H^2(\Omega)\right)$ is such that $\varphi\in C^{\delta/2}\left([0,T];C^\delta\left(\overline{\Omega}\right)\right)$ as provided by the Step 1 and that $\partial\Omega$ is of class $\mathcal{C}^2$. Let $\widetilde{\varphi}=E\varphi$, where $E:W^{2,p}(\Omega)\to W^{2,p}\left(\mathbb{R}^2\right)$ is an extension operator for any $p\in[1,\infty)$. Then set $\widetilde{\varphi}_\varepsilon=\eta_\varepsilon*\widetilde{\varphi}$ where $\eta_\varepsilon\in C^\infty\left(\mathbb{R}^2\right)$ is the usual Friedrich mollifier such that $\eta_\varepsilon\geq 0$ and $\int_{\mathbb{R}^2}\eta_\varepsilon dx=1$. Defining $\varphi_\epsilon=R\widetilde{\varphi}_\varepsilon$, where $R:W^{2,p}\left(\mathbb{R}^2\right)\to W^{2,p}(\Omega)$ is the restriction operator, it is clear that $\widetilde{\varphi}_\varepsilon(x,\cdot)$ is of class C^∞ in a neighborhood of $\overline{\Omega}$. Moreover φ_ϵ satisfies, for any $k\in\{0,1,2\}$ and $p\in[1,\infty)$, the bounds

$$\|\varphi_\varepsilon(t)\|_{W^{k,p}}\leq C\,\|\varphi(t)\|_{W^{k,p}}\,,\ \|\varphi_\varepsilon(t)\|_{W^{k+1,p}}\leq C_{k,p,\varepsilon}\,\|\varphi(t)\|_{W^{k,p}}$$

and $\varphi_\varepsilon(t)\to\varphi(t)$ strongly in $W^{k,p}(\Omega)$ for almost any $t\in(0,T)$ (see, e.g., [10, Chapter V]). We also have

$$\varphi_\varepsilon\in L^\infty\left(0,T;H^2(\Omega)\right)\cap C^{\delta/2}\left([0,T];C^\delta\left(\overline{\Omega}\right)\right). \tag{4.45}$$

We now consider the following mollified version of the original Navier-Stokes equations

$$u_t-2\mathrm{div}(\nu(\varphi_\varepsilon)Du)+(u\cdot\nabla)u+\nabla\pi=\mu\nabla\varphi+h(t), \tag{4.46}$$

$$\mathrm{div}\,(u_\varepsilon)=0 \tag{4.47}$$

in $\Omega\times(0,T)$ with initial condition $u_{\varepsilon|t=0}=u_0$ and no-slip boundary condition. Here μ and φ are as regular as specified in Step 1. Let us observe that (4.45) together with standard interpolation results in Sobolev spaces imply that $\varphi_\varepsilon\in BUC\left([0,T];W^{1,q}(\Omega)\right)$ for any $q>2$ (i.e., φ_ε is bounded and uniformly continuous with values in $W^{1,q}(\Omega)$ with $\|\varphi_\varepsilon\|_{BUC(0,T;W^{1,q})}\leq C_\varepsilon$, for some $C_\varepsilon\to\infty$ as $\varepsilon\to 0^+$). Thus, thanks to a result contained in the proof of [1, Theorem 8], we can find a sufficiently small time $T_\varepsilon\leq T$, a function u_ε such that

$$u_\varepsilon\in H^1(0,T_\varepsilon;G_{div})\cap L^2(0,T_\varepsilon;H^2(\Omega)^2)\cap L^\infty(0,T_\varepsilon;V_{div}) \tag{4.48}$$

and the associated pressure $\pi_\varepsilon\in L^2\left(0,T_\varepsilon;H^1(\Omega)/\mathbb{R}\right)$ such that u_ε is a strong solution to (4.46)-(4.47), provided that $u_0\in V_{div}$ and $h\in L^2_{\mathrm{loc}}(\mathbb{R}_+;G_{div})$ and $\mu\nabla\varphi\in L^2_{\mathrm{loc}}(\mathbb{R}_+;L^2(\Omega)^2)$ (for the latter see Step 1). The regularity (4.48) is enough to perform all the estimates of Step 2 on the fluid velocity rigorously. In particular, estimates (4.37)-(4.38) entail that u_ε can be extended to any interval $(0,T)$, for any given $T>0$. Moreover, u_ε is bounded in the spaces (4.48) uniformly with respect to ε (and π_ε is bounded in $L^2\left(0,T;H^1(\Omega)\right)$ uniformly with respect to ε). Thus, usual compactness arguments allows to pass to the limit as $\varepsilon \to 0$ in (4.46)-(4.47), owing to the strong convergence $\varphi_\varepsilon(t) \to \varphi(t)$ in V for almost any $t \in (0,T)$. This gives a strong solution $\tilde{u}$ to the same problem solved by the weak solution found in Step 1. Then uniqueness applied to the NS equations with given viscosity implies that $u = \tilde{u}$. We can now perform estimates (4.40)-(4.42) to show that φ satisfies (4.43) and (4.44). This ends the proof. ∎

We can now state the weak-strong uniqueness result for the nonconstant viscosity case.

Theorem 6. *Let $d = 2$ and assume that (H1)–(H5) are satisfied. Let $u_0 \in G_{div}$, $\varphi_0 \in L^\infty(\Omega)$ and let $[u_1, \varphi_1]$ be a weak solution and $[u_2, \varphi_2]$ a strong solution satisfying* (4.9) *and* (4.10) *both corresponding to $[u_0, \varphi_0]$ and to the same external force $h \in L^2(0,T;V'_{div})$. Then $u_1 = u_2$ and $\varphi_1 = \varphi_2$.*

Proof. Taking the difference between the variational formulation of (1.1) and (1.3) written for each solution and setting $u := u_2 - u_1$, $\varphi := \varphi_2 - \varphi_1$, we get

$$\begin{aligned}
&\langle u_t, v\rangle + 2\big((\nu(\varphi_2) - \nu(\varphi_1))Du_2, Dv\big) + 2\big(\nu(\varphi_1)Du, Dv\big) + b(u_2, u_2, v) - b(u_1, u_1, v) \\
&= -\frac{1}{2}\big(\varphi(\varphi_1 + \varphi_2)\nabla a, v\big) - \big((J * \varphi)\nabla\varphi_2, v\big) - \big((J * \varphi_1)\nabla\varphi, v\big), \qquad (4.49)\\
&\langle \varphi_t, \psi\rangle + (\nabla\mu, \nabla\psi) = -(u \cdot \nabla\varphi_2, \psi) - (u_1 \cdot \nabla\varphi, \psi), \qquad (4.50)
\end{aligned}$$

for all $v \in V_{div}$ and $\psi \in V$, where $\mu = \mu_2 - \mu_1 = a\varphi - J * \varphi + F'(\varphi_2) - F'(\varphi_1)$. Let us choose $v = u$ and $\psi = \varphi$ as test functions in (4.49) and (4.50), respectively, and add the resulting identities. Notice that the contribution from the second term on the right-hand side of (4.50) vanishes due to the incompressibility condition. Hence, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|u\|^2 + \|\varphi\|^2\big) + 2\big((\nu(\varphi_2) - \nu(\varphi_1))Du_2, Du\big) + 2\big(\nu(\varphi_1)Du, Du\big) + b(u, u_1, u) \\
&+ (\nabla\mu, \nabla\varphi) = I_1 + I_2 + I_3 + I_4, \qquad (4.51)
\end{aligned}$$

where I_1, I_2, I_3 are given again by

$$I_1 = -\frac{1}{2}\big(\varphi(\varphi_1 + \varphi_2)\nabla a, u\big), \quad I_2 = -\big((J * \varphi)\nabla\varphi_2, u\big), \quad I_3 = -\big((J * \varphi_1)\nabla\varphi, u\big),$$

while I_4 is given by

$$I_4 = -(u \cdot \nabla\varphi_2, \varphi).$$

Let us first estimate the terms in (4.51) coming from the Navier-Stokes equations. Due to assumption (H2) we have

$$\begin{aligned}
&2\big|\big((\nu(\varphi_2) - \nu(\varphi_1))Du_2, Du\big)\big| \le C\|\varphi\|_{L^4}\|Du_2\|_{L^4}\|\nabla u\| \\
&\le C\|\varphi\|^{1/2}\|\varphi\|_V^{1/2}\|Du_2\|^{1/2}\|Du_2\|_{H^1}^{1/2}\|\nabla u\| \\
&\le \frac{\nu_1}{12}\|\nabla u\|^2 + C\|\nabla u_2\|\|u_2\|_{H^2}\|\varphi\|^2 + C\|\nabla u_2\|\|u_2\|_{H^2}\|\varphi\|\|\nabla\varphi\| \\
&\le \frac{\nu_1}{12}\|\nabla u\|^2 + \frac{c_0}{4}\|\nabla\varphi\|^2 + C(1 + \|\nabla u_2\|^2\|u_2\|_{H^2}^2)\|\varphi\|^2, \qquad (4.52)
\end{aligned}$$

$$2\big(\nu(\varphi_1)Du, Du\big) \ge \nu_1\|\nabla u\|^2,$$

where henceforth in this proof C will denote a constant which depends on $\|\varphi_0\|_{L^\infty}$, and on $\|u_0\|$. Indeed, recall that, since $\varphi_0 \in L^\infty(\Omega)$, then we have $\|\varphi_i\|_{L^\infty(\Omega\times(0,T))} \le C_i = C_i\big(\|\varphi_0\|_{L^\infty}, \|u_0\|\big)$, for $i = 1, 2$.

The term in the trilinear form is standard

$$|b(u,u_1,u)| \leq c\|u\|\|\nabla u\|\|\nabla u_1\| \leq \frac{\nu_1}{12}\|\nabla u\|^2 + c\|\nabla u_1\|^2\|u\|^2,$$

while the terms I_1, I_2, I_3 can now be estimated more easily in this way

$$\begin{aligned}
I_1 &\leq \|\varphi\|\|\varphi_1+\varphi_2\|_{L^4}\|\nabla a\|_{L^\infty}\|u\|_{L^4} \\
&\leq \frac{\nu_1}{12}\|\nabla u\|^2 + c\big(\|\varphi_1\|_{L^4}^2 + \|\varphi_2\|_{L^4}^2\big)\|\varphi\|^2, \\
I_2 &\leq \|\varphi_2\|_{L^4}\|\nabla J\|_{L^1}\|\varphi\|\|u\|_{L^4} \\
&\leq \frac{\nu_1}{12}\|\nabla u\|^2 + c\|\varphi_2\|_{L^4}^2\|\varphi\|^2, \\
I_3 &\leq \frac{\nu_1}{12}\|\nabla u\|^2 + c\|\varphi_1\|_{L^4}^2\|\varphi\|^2.
\end{aligned}$$

Regarding the terms coming from the nonlocal Cahn-Hilliard equation we have

$$\begin{aligned}
(\nabla\mu,\nabla\varphi) &= \big((a+F''(\varphi_1))\nabla\varphi,\nabla\varphi\big) + \big(\varphi\nabla a - \nabla J * \varphi,\nabla\varphi\big) \\
&+ \big((F''(\varphi_2)-F''(\varphi_1))\nabla\varphi_2,\nabla\varphi\big),
\end{aligned} \tag{4.53}$$

and the last term on the right-hand side of this identity can be estimated as

$$\begin{aligned}
&\big|\big((F''(\varphi_2)-F''(\varphi_1))\nabla\varphi_2,\nabla\varphi\big)\big| \leq \|F''(\varphi_2)-F''(\varphi_1)\|_{L^4}\|\nabla\varphi_2\|_{L^4}\|\nabla\varphi\| \\
&\leq C\|\varphi\|_{L^4}\|\nabla\varphi_2\|_{L^4}\|\nabla\varphi\| \leq C(\|\varphi\| + \|\varphi\|^{1/2}\|\nabla\varphi\|^{1/2})\|\nabla\varphi_2\|_{L^4}\|\nabla\varphi\| \\
&\leq \frac{c_0}{4}\|\nabla\varphi\|^2 + C(1+\|\nabla\varphi_2\|_{L^4}^4)\|\varphi\|^2.
\end{aligned}$$

Hence, by means of assumption (H3), we get

$$\begin{aligned}
&(\nabla\mu,\nabla\varphi) \geq c_0\|\nabla\varphi\|^2 - 2\|\nabla J\|_{L^1}\|\varphi\|\|\nabla\varphi\| - \frac{c_0}{4}\|\nabla\varphi\|^2 - C(1+\|\nabla\varphi_2\|_{L^4}^4)\|\varphi\|^2 \\
&\geq \frac{c_0}{2}\|\nabla\varphi\|^2 - C(1+\|\nabla\varphi_2\|_{L^4}^4)\|\varphi\|^2.
\end{aligned}$$

Finally, the last term in (4.51) coming from the nonlocal Cahn-Hilliard equation can be controlled as follows

$$I_4 \leq \|u\|_{L^4}\|\nabla\varphi_2\|_{L^4}\|\varphi\| \leq \frac{\nu_1}{12}\|\nabla u\|^2 + c\|\nabla\varphi_2\|_{L^4}^2\|\varphi\|^2. \tag{4.54}$$

By plugging estimates (4.52)–(4.54) into (4.51) we are led to the following differential inequality

$$\frac{1}{2}\frac{d}{dt}\big(\|u\|^2 + \|\varphi\|^2\big) + \frac{\nu_1}{2}\|\nabla u\|^2 + \frac{c_0}{4}\|\nabla\varphi\|^2 \leq \Pi\big(\|u\|^2 + \|\varphi\|^2\big), \tag{4.55}$$

where the function Π is given by

$$\Pi = c\big(1 + \|\nabla u_2\|^2\|u_2\|_{H^2}^2 + \|\nabla u_1\|^2 + \|\varphi_1\|_{L^4}^2 + \|\varphi_2\|_{L^4}^2 + \|\nabla\varphi_2\|_{L^4}^2 + \|\nabla\varphi_2\|_{L^4}^4\big),$$

and due to the regularity properties of the weak solution $[u_1,\varphi_1]$ and of the strong solution $[u_2,\varphi_2]$ we have $\Pi \in L^1(0,T)$. Weak-strong uniqueness follows by applying Gronwall's lemma to (4.55). In addition, a continuous dependence estimate in $L^2(\Omega)^2$ can also be deduced by considering two solutions with different initial data and external forces. ∎

If the potential is singular and the mobility is constant, the weak-strong uniqueness does not seem to be easy to prove. However, if the mobility is degenerate, thanks to the particular weak formulation of the convective nonlocal Cahn-Hilliard (cf. (3.23)), the weak-strong uniqueness can be proven as stated in the next theorem. In order to do that, we just need to strengthen (A1) slightly, namely,

(A7) $mF'' \in C^1([-1,1])$.

We point out that in the case of singular potential, degenerate mobility and constant (or nonconstant) viscosity, existence of strong solutions in 2D for the nonlocal Cahn-Hilliard-Navier-Stokes system has not been proven yet. This result, which actually can be established, will be presented in a forthcoming paper.

Theorem 7. *Let $d = 2$ and suppose that assumptions (A1)–(A7) and (H2) are satisfied. Let $u_0 \in G_{div}$, $\varphi_0 \in L^\infty(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$, $M(\varphi_0) \in L^1(\Omega)$ and let $[u_1, \varphi_1]$ be a weak solution and $[u_2, \varphi_2]$ a strong solution to (1.1)–(1.6) satisfying (4.9) and (4.11) both corresponding to $[u_0, \varphi_0]$ and to the same external force $h \in L^2(0, T; V'_{div})$. Then $u_1 = u_2$ and $\varphi_1 = \varphi_2$.*

Proof. Let us write the variational formulation of (1.1)–(1.2) and (3.23) for each solution and take the difference, setting $u := u_2 - u_1$, $\varphi := \varphi_2 - \varphi_1$. Then we choose $v = u$ as test function in the first identity (4.49) and $\psi\zeta = \varphi$ as test function in the second. Concerning the first identity, we can argue exactly as in the proof of Theorem 6 and get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u\|^2 + 2\big((\nu(\varphi_2) - \nu(\varphi_1))Du_2, Du\big) + 2\big(\nu(\varphi_1)Du, Du\big) + b(u, u_1, u),\\ &= I_1 + I_2 + I_3.\end{aligned} \tag{4.56}$$

Then, by similarly estimating the terms in (4.56), we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{\nu_1}{2}\|\nabla u\|^2 \le \frac{1}{4}(1-\rho)\alpha_0\|\nabla\varphi\|^2\\ &+ C(1 + \|\nabla u_2\|^2\|u_2\|^2_{H^2} + \|\varphi_1\|^2_{L^4} + \|\varphi_2\|^2_{L^4})\|\varphi\|^2 + C\|\nabla u_1\|^2\|u\|^2.\end{aligned} \tag{4.57}$$

As far as the identity resulting from the difference in the Cahn-Hilliard is concerned, if we set

$$b(x,s) := \partial_s\Lambda(x,s) = m(s)(F''(s) + a(x)), \qquad \forall s \in [-1,1], \quad \text{a.e. } x \in \Omega,$$

this identity reads as follows

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 + \big(b(\cdot,\varphi_1)\nabla\varphi, \nabla\varphi\big) + \big((b(\cdot,\varphi_2) - b(\cdot,\varphi_1))\nabla\varphi_2, \nabla\varphi\big)\\ &+ \big((m(\varphi_2) - m(\varphi_1))(\varphi_2\nabla a - \nabla J * \varphi_2), \nabla\varphi\big)\\ &+ \big(m(\varphi_1)(\varphi\nabla a - \nabla J * \varphi), \nabla\varphi\big)\\ &= \big(u\varphi_2, \nabla\varphi\big).\end{aligned} \tag{4.58}$$

Observe now that, thanks to assumptions (A5), (A6) and (A7), we have $b(x,s) \ge (1-\rho)\alpha_0$ and $|b(x,s_2) - b(x,s_1)| \le k'|s_2 - s_1|$, for all $s, s_1, s_2 \in [-1,1]$ and for almost every $x \in \Omega$. Here $k' = \|(mF'')'\|_{C([-1,1])} + \|m'\|_{C([-1,1])}\|a\|_{L^\infty(\Omega)}$. Let us now estimate the terms in (4.58), taking the bounds $|\varphi_i| \le 1$, $i = 1, 2$, into account. The second and third term on the left-hand side can be estimated in the following way

$$\big(b(\cdot,\varphi_1)\nabla\varphi, \nabla\varphi\big) \ge (1-\rho)\alpha_0\|\nabla\varphi\|^2,$$

$$
\begin{aligned}
\big((b(\cdot,\varphi_2)-b(\cdot,\varphi_1))\nabla\varphi_2,\nabla\varphi\big) &\le k'\|\varphi\|_{L^4}\|\nabla\varphi_2\|_{L^4}\|\nabla\varphi\| \\
&\le c\|\varphi\|^{1/2}\|\varphi\|_V^{1/2}\|\nabla\varphi_2\|_{L^4}\|\nabla\varphi\| \\
&\le \frac{1}{32}(1-\rho)\alpha_0\|\nabla\varphi\|^2 + c\|\varphi\|\|\varphi\|_V\|\nabla\varphi_2\|_{L^4}^2 \\
&\le \frac{1}{16}(1-\rho)\alpha_0\|\nabla\varphi\|^2 + c\big(1+\|\nabla\varphi_2\|_{L^4}^4\big)\|\varphi\|^2.
\end{aligned}
$$

Furthermore, it is immediate to see that the last two terms on the left-hand side of (4.58) can be controlled in this way

$$
c\|\varphi\|\|\nabla\varphi\| \le \frac{1}{16}(1-\rho)\alpha_0\|\nabla\varphi\|^2 + c\|\varphi\|^2,
$$

and, finally, the term on the right-hand side can be controlled by

$$
c\|u\|\|\nabla\varphi\| \le \frac{1}{16}(1-\rho)\alpha_0\|\nabla\varphi\|^2 + c\|u\|^2.
$$

From (4.58), using the estimates above, we are therefore led to the following differential inequality

$$
\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 + \frac{3}{4}(1-\rho)\alpha_0\|\nabla\varphi\|^2 \le c\big(1+\|\nabla\varphi_2\|_{L^4}^4\big)\|\varphi\|^2 + c\|u\|^2. \tag{4.59}
$$

Thus, from (4.57) and (4.59) we deduce

$$
\frac{1}{2}\frac{d}{dt}\big(\|u\|^2+\|\varphi\|^2\big) + \frac{\nu_1}{2}\|\nabla u\|^2 + \frac{1}{2}(1-\rho)\alpha_0\|\nabla\varphi\|^2 \le \gamma\big(\|u\|^2+\|\varphi\|^2\big), \tag{4.60}
$$

where $\gamma \in L^1(0,T)$ has the same form as given at the end of the proof of Theorem 6. We conclude again by applying Gronwall's lemma to (4.60). Moreover, a continuous dependence estimate in $L^2(\Omega)^2$ can be deduced in the present situation as well by considering two solutions with different data. ∎

5 Global and exponential attractors

In this section we prove two results concerning the asymptotic behavior of the dynamical system generated by (1.3)–(1.5) in dimension two. The first result is related to the property of connectedness of the global attractor whose existence was established in [14] for nonconstant viscosity, constant mobility and regular potential. The second result is the existence of an exponential attractor. This will be proven in details when mobility and viscosity are constant and the potential is regular. This kind of result relies on a regularization argument devised in [16] and on an abstract theorem (see [12]) which generalizes a well known result on the existence of exponential attractors in Banach spaces (cf. [11]). A similar argument will be carried out in the nonconstant viscosity case albeit we will work with strong solutions.

Let us define the dynamical system in the autonomous case. Take $d=2$ and $h \in V_{div}'$. Then, as a consequence of Theorem 2, we have that for every fixed $\eta \ge 0$ system (1.3)–(1.5) generates a semigroup $\{S_\eta(t)\}_{t\ge 0}$ of *closed* operators (see [32]) on the metric space $\mathcal{X}_\eta$ given by

$$
\mathcal{X}_\eta := G_{div} \times \mathcal{Y}_\eta, \tag{5.1}
$$

where

$$
\mathcal{Y}_\eta := \{\varphi \in H : F(\varphi) \in L^1(\Omega),\ |\overline{\varphi}| \le \eta\}.
$$

It is convenient to endow the space $\mathcal{X}_\eta$ with the following metric

$$\rho_{\mathcal{X}_\eta}(z_2,z_1) = \|u_2-u_1\| + \|\varphi_2-\varphi_1\| + \Big|\int_\Omega F(\varphi_2) - \int_\Omega F(\varphi_1)\Big|, \quad \forall z_i := [u_i,\varphi_i] \in \mathcal{X}_\eta,\ i=1,2.$$

Notice that this metric is slightly different from the one which is naturally associated to the energy $\mathcal{E}$ (the difference is in the exponent in the third term, see [14]).

A first noteworthy consequence of the uniqueness result for weak solutions is the following

Theorem 8. *Let $d=2$ and let (H1)–(H5) be satisfied with ν constant. Assume also that that $h \in V_{div}'$. Then, the global attractor in $\mathcal{X}_\eta$ for the semigroup $S_\eta(t)$ is connected.*

Proof. The conclusion follows immediately by applying [4, Corollary 4.3]. Indeed, the space $\mathcal{X}_\eta$ is (arcwise) connected, thanks to the fact that F is a quadratic perturbation of a convex function. Moreover, we have the strong time continuity of each trajectory $z=[u,\varphi]$ from $[0,\infty)$ to the metric space $\mathcal{X}_\eta$ (see Theorem 1). Thus Kneser's property is satisfied thanks to uniqueness. ∎

Remark 3. Theorem 8 also holds in the case of constant (or degenerate) mobility and singular potential on account of Theorem 3 and [15, Proposition 4] (or Theorem 4 and [17, Proposition 3]). The argument is similar.

The second result is the existence of an exponential attractor. We first recall its definition.

Definition 3. *A compact set $\mathcal{M}_\eta \subset \mathcal{X}_\eta$ is an exponential attractor for the dynamical system $(\mathcal{X}_\eta, S_\eta(t))$ if the following properties are satisfied*

(i) positive invariance: $S_\eta(t)\mathcal{M}_\eta \subseteq \mathcal{M}_\eta$ for all $t \geq 0$;

(ii) finite dimensionality: $dim_F(\mathcal{M}_\eta, \mathcal{X}_\eta) < \infty$;

(iii) exponential attraction: $\exists\ Q : \mathbb{R}^+ \to \mathbb{R}^+$ increasing and $\kappa > 0$ such that, for all $R>0$ and for all $\mathcal{B} \subset \mathcal{X}_\eta$ with $\sup_{z\in\mathcal{B}} \rho_{\mathcal{X}_\eta}(z,0) \leq R$ there holds

$$dist_{\mathcal{X}_\eta}(S_\eta(t)\mathcal{B}, \mathcal{M}_\eta) \leq Q(R)e^{-\kappa t}, \qquad \forall t \geq 0.$$

Theorem 9. *Let $d=2$. Assume that (H1)–(H5) are satisfied with ν constant. Then the dynamical system $(\mathcal{X}_\eta, S_\eta(t))$ possesses an exponential attractor $\mathcal{M}_\eta$ which is bounded in $V_{div} \times W^{1,p}(\Omega)$, $2 < p < \infty$.*

The proof of Theorem 9 is based on four lemmas. These lemmas allow us to apply the abstract result in [12]. For their proof we shall need the following regularization result which is an easy consequence of [16, Theorem 2 and Proposition 1] and has an independent interest. In the statement and proof of this result we shall denote by $\Gamma_\tau = \Gamma_\tau\big(\mathcal{E}(z_0),\eta\big)$ a positive constant depending on a positive time τ, on the energy $\mathcal{E}(z_0)$ of the initial datum $z_0 := [u_0,\varphi_0]$ of a weak solution, and on η, where $\eta \geq 0$ is such that $|\overline{\varphi}_0| \leq \eta$ (Γ_τ may of course depend also on h, F, J, ν and Ω). The value of Γ_τ may change even on the same line.

Proposition 2. *Let $d=2$ and $h \in L^2_{tb}(0,\infty;G_{div})$. Assume that (H1)–(H5) are satisfied with ν constant, and suppose $F \in C^3(\mathbb{R})$. Let $u_0 \in G_{div}$, $\varphi_0 \in H$ with $F(\varphi_0) \in L^1(\Omega)$ and let $[u,\varphi]$ be the weak solution on $(0,\infty)$ to system (1.3)–(1.6) corresponding to $[u_0,\varphi_0]$. Then, for every $\tau > 0$ there exists $\Gamma_\tau > 0$ such that we have*

$$u \in L^\infty(\tau,\infty;V_{div}) \cap L^2_{tb}\big(\tau,\infty;H^2(\Omega)^2\big), \qquad u_t \in L^2_{tb}\big(\tau,\infty;G_{div}\big), \tag{5.2}$$

$$\varphi \in L^\infty\big(\tau,\infty;W^{1,p}(\Omega)\big), \qquad 2<p<\infty, \qquad \varphi_t \in L^\infty(\tau,\infty;H)\cap L^2_{tb}(\tau,\infty;V), \tag{5.3}$$

with norms controlled by Γ_τ. In addition, for every initial data $z_0 := [u_0,\varphi_0] \in G_{div}\times H$ with $F(\varphi_0)\in L^1(\Omega)$ and $|\overline{\varphi}_0|\le\eta$, there exists a constant $\Lambda=\Lambda(\eta)>0$ depending only on η (and on F, J, ν and Ω) and a time $t^*=t^*\big(\mathcal{E}(z_0)\big)\ge 0$ starting from which the weak solution corresponding to z_0 regularizes, that is,

$$\|\nabla u(t)\| + \|\varphi(t)\|_{W^{1,p}(\Omega)} + \int_t^{t+1}\|u(s)\|^2_{H^2(\Omega)^2}ds \le \Lambda(\eta), \qquad \forall t\ge t^*. \tag{5.4}$$

Remark 4. Notice that, differently from [16, Theorem 2], in Proposition 2 we do not require any further regularity assumption on J in addition to (H1).

Proof. Recalling the proof of [21, Lemma 2.10] and the dissipative estimate (2.8), observe first that, if $z_0\in\mathcal{X}_\eta$, then for every $\tau>0$ there exists $\Gamma_\tau=\Gamma_\tau\big(\mathcal{E}(z_0),\eta\big)$ such that

$$\|\varphi(t)\|_{L^\infty(\Omega)}\le\Gamma_\tau, \qquad \forall t\ge\tau. \tag{5.5}$$

This implies that $\|\mu(t)\|_{L^\infty(\Omega)}\le\Gamma_\tau$ for all $t\ge\tau$, and hence that the Korteweg term $\mu\nabla\varphi\in L^2(\tau,T;L^2(\Omega)^2)$. By Lemma 2, there also holds

$$\sup_{t\ge\tau}\|\varphi\|_{C^{\delta/2,\delta}\left([t,t+1]\times\overline{\Omega}\right)}\le\Gamma_\tau,\ \forall t\ge\tau. \tag{5.6}$$

We can now repeat exactly the same argument in the proof of [16, Theorem 2], by writing the same estimates which now hold starting from a positive time, say for $t\ge\tau/2>0$. We recall that these estimates are obtained by multiplying the nonlocal Cahn-Hilliard by μ_t in H and then by differentiating the nonlocal Cahn-Hilliard with respect to time and multiplying the resulting identity bu μ_t. By doing so we are led to a differential inequality of the following form

$$\frac{d}{ds}\log\Big(1+\int_\Omega\big(a+F''(\varphi)\big)\varphi_t^2\Big)\le\Gamma_\tau\big(\sigma(s)+\|\varphi_t\|^2\big), \qquad \forall s\ge\tau/2, \tag{5.7}$$

where $\sigma=\Gamma_\tau\big(1+\|u\|^2_{H^2}+\|u_t\|^2\big)$and we have $\sigma\in L^1(\tau/2,T)$, for all $T>\tau/2$. At this point we argue a bit differently from the proof of [16, Theorem 2]. Indeed, here we want to avoid the L^2-norm of φ_t in $\tau/2$ which would require the initial condition $\varphi(\tau/2)\in H^2(\Omega)$ and in addition would force us to make some further regularity assumptions on the kernel J (like, e.g., $J\in W^{2,1}(\mathbb{R}^2)$ or J admissible) in order to have $\varphi_t(\tau/2)\in H$. Therefore, we multiply (5.7) by $(s-\tau/2)$ and integrate with respect to s between $\tau/2$ and $t\in(\tau/2,T)$. We get

$$\begin{aligned}&\Big(t-\frac{\tau}{2}\Big)\log\Big(1+\int_\Omega\big(a+F''(\varphi)\big)\varphi_t^2\Big)\le\int_{\tau/2}^T\log\Big(1+\int_\Omega\big(a+F''(\varphi)\big)\varphi_t^2\Big)ds\\&+\Gamma_\tau\Big(T-\frac{\tau}{2}\Big)\big(\|\sigma\|_{L^1(\tau/2,T)}+\|\varphi_t\|^2_{L^2(\tau/2,T;H)}\big)\\&\le\Gamma_\tau\|\varphi_t\|^2_{L^2(\tau/2,T;H)}+\Gamma_\tau\Big(T-\frac{\tau}{2}\Big)\big(\|\sigma\|_{L^1(\tau/2,T)}+\|\varphi_t\|^2_{L^2(\tau/2,T;H)}\big), \qquad \forall t\in(\tau/2,T).\end{aligned}$$

From this inequality, on account of the fact that we have $\|\varphi_t\|_{L^2(\tau/2,T;H)}\le\Gamma_\tau$ (this was shown in the first step of the proof of [16, Theorem 2], before (5.7)) we deduce that

$$\varphi_t\in L^\infty(\tau,T;H). \tag{5.8}$$

This bound, together with the following estimate (cf. proof of [16, Theorem 2])

$$\|\nabla\mu\|_{L^p} \le \Gamma_\tau\big(1+\|\varphi_t\|^{1-2/p}\big), \qquad 2<p<\infty,$$

yield

$$\varphi \in L^\infty\big(\tau,T;W^{1,p}(\Omega)\big). \tag{5.9}$$

Finally, arguing as in the proof of [16, Proposition 1] by applying the uniform Gronwall's lemma, and taking (5.8), (5.9) (together with the bounds for u on (τ,T)) into account, we get (5.2), (5.3) and (5.4), respectively. ∎

For the statements and proofs of the following lemmas we shall denote by C_τ a positive constant depending on a positive time τ, on the energies $\mathcal{E}(z_{01})$, $\mathcal{E}(z_{02})$ of the initial data $z_{01}, z_{02} \in \mathcal{X}_\eta$ of two weak solutions, and on η, where $\eta>0$ is such that $|\overline{\varphi}_{01}|, |\overline{\varphi}_{02}| \le \eta$ (of course, C_τ will generally depend also on h, F, J, ν and Ω). The value of C_τ may change even within the same line. Furthermore, we shall always set $u := u_2 - u_1$, $\varphi := \varphi_2 - \varphi_1$.

Lemma 3. *Let $d=2$. Assume that (H1)–(H5) are satisfied with ν constant and that $F\in C^3(\mathbb{R})$. Let $u_{0i}\in G_{div}$, $\varphi_{0i}\in H$ with $F(\varphi_{0i})\in L^1(\Omega)$ and $[u_i,\varphi_i]$ be the corresponding weak solutions, $i=1,2$. Then, for every $\tau>0$ there exists $C_\tau>0$ such that we have*

$$\begin{aligned}&\|u_2(t)-u_1(t)\|^2+\|\varphi_2(t)-\varphi_1(t)\|^2+\int_\tau^t\Big(\frac{\nu}{4}\|\nabla(u_2(s)-u_1(s))\|^2+\frac{c_0}{4}\|\nabla(\varphi_2(s)-\varphi_1(s))\|^2\Big)ds\\&\le e^{C_\tau t}\big(\|u_2(\tau)-u_1(\tau)\|^2+\|\varphi_2(\tau)-\varphi_1(\tau)\|^2\big), \qquad \forall t\ge\tau.\end{aligned} \tag{5.10}$$

Proof. Let us multiply (3.2) by φ in $L^2(\Omega)$. We get

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 = -(u\cdot\nabla\varphi_2,\varphi) - (\nabla\widetilde{\mu},\nabla\varphi) \tag{5.11}$$

Taking the gradient of $\widetilde{\mu}$, on account of (3.3) we have

$$\begin{aligned}&(\nabla\widetilde{\mu},\nabla\varphi) = \int_\Omega\big(a+F''(\varphi_1)\big)|\nabla\varphi|^2 + (\varphi\nabla a - \nabla J*\varphi,\nabla\varphi)\\&+\big((F''(\varphi_2)-F''(\varphi_1))\nabla\varphi_2,\nabla\varphi\big) \ge c_0\|\nabla\varphi\|^2 - c\|\varphi\|\|\nabla\varphi\|\\&-\|F''(\varphi_2)-F''(\varphi_1)\|_{L^4}\|\nabla\varphi_2\|_{L^4}\|\nabla\varphi\| \ge \frac{c_0}{2}\|\nabla\varphi\|^2 - c\|\varphi\|^2 - C_\tau\|\varphi\|_{L^4}\|\nabla\varphi_2\|_{L^4}\|\nabla\varphi\|\\&\ge \frac{c_0}{2}\|\nabla\varphi\|^2 - c\|\varphi\|^2 - C_\tau\big(\|\varphi\|+\|\varphi\|^{1/2}\|\nabla\varphi\|^{1/2}\big)\|\nabla\varphi_2\|_{L^4}\|\nabla\varphi\|\\&\ge \frac{c_0}{4}\|\nabla\varphi\|^2 - C_\tau\big(1+\|\nabla\varphi_2\|^2_{L^4}+\|\nabla\varphi_2\|^4_{L^4}\big)\|\varphi\|^2.\end{aligned}$$

Observe that

$$(\nabla\widetilde{\mu},\nabla\varphi) \ge \frac{c_0}{4}\|\nabla\varphi\|^2 - C_\tau\big(1+\|\nabla\varphi_2\|^4_{L^4}\big)\|\varphi\|^2. \tag{5.12}$$

Furthermore, we have

$$|(u\cdot\nabla\varphi_2,\varphi)| \le \|u\|_{L^4}\|\nabla\varphi_2\|_{L^4}\|\varphi\| \le \frac{\nu}{4}\|\nabla u\|^2 + c\|\nabla\varphi_2\|^2_{L^4}\|\varphi\|^2. \tag{5.13}$$

Therefore, plugging (5.12) and (5.13) into (5.11), we get

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 + \frac{c_0}{4}\|\nabla\varphi\|^2 \le C_\tau\big(1+\|\nabla\varphi_2\|^4_{L^4}\big)\|\varphi\|^2 + \frac{\nu}{4}\|\nabla u\|^2.$$

Adding this last differential inequality to (3.8), we obtain

$$\frac{1}{2}\frac{d}{dt}\big(\|u\|^2+\|\varphi\|^2\big)+\frac{\nu}{4}\|\nabla u\|^2+\frac{c_0}{4}\|\nabla\varphi\|^2\le\gamma(t)\big(\|u\|^2+\|\varphi\|^2\big), \tag{5.14}$$

where

$$\gamma(t):=\alpha(t)+C_\tau\big(1+\|\nabla\varphi_2\|_{L^4}^4\big).$$

Then, thanks to Proposition 2, for every $\tau>0$ there exists $C_\tau>0$ (always depending on τ, η and on the energies $\mathcal{E}(z_{01})$, $\mathcal{E}(z_{02})$) such that the following bounds for the solutions $z_i=[u_i,\varphi_i]$ corresponding to $[u_{0i},\varphi_{0i}]$ hold

$$\|u_i\|_{L^\infty(\tau,\infty;V_{div})}+\|\varphi_i\|_{L^\infty(\tau,\infty;W^{1,p}(\Omega))}\le C_\tau, \tag{5.15}$$

$$\|u_{i,t}\|_{L^2_{tb}(\tau,\infty;G_{div})}+\|\varphi_{i,t}\|_{L^\infty(\tau,\infty;H)}\le C_\tau, \tag{5.16}$$

Thus we have $\gamma(t)\le C_\tau$, for all $t\ge\tau$ and by applying the standard Gronwall lemma to (5.14) written for $t\ge\tau$ we get

$$\|u(t)\|^2+\|\varphi(t)\|^2\le\big(\|u(\tau)\|^2+\|\varphi(\tau)\|^2\big)e^{C_\tau t},\qquad\forall t\ge\tau. \tag{5.17}$$

By integrating (5.14) between τ and t and using (5.17) we get (5.10). ∎

Lemma 4. *Let the assumptions of Lemma 3 be satisfied. Let $u_{0i}\in G_{div}$, $\varphi_{0i}\in H$ with $F(\varphi_{0i})\in L^1(\Omega)$ and $[u_i,\varphi_i]$ be the corresponding weak solutions, $i=1,2$. Then, for every $\tau>0$ there exists $C_\tau>0$ such that we have*

$$\begin{aligned}&\|u_2(t)-u_1(t)\|^2+\|\varphi_2(t)-\varphi_1(t)\|^2+\Big|\int_\Omega F\big(\varphi_2(t)\big)-\int_\Omega F\big(\varphi_1(t)\big)\Big|^2\\&\le C_\tau\big(\|u_2(\tau)-u_1(\tau)\|^2+\|\varphi_2(\tau)-\varphi_1(\tau)\|^2\big)e^{-kt}\\&+C_\tau\int_\tau^t\big(\|u_2(s)-u_1(s)\|^2+\|\varphi_2(s)-\varphi_1(s)\|^2\big)ds,\qquad\forall t\ge\tau.\end{aligned} \tag{5.18}$$

Proof. By using the Poincaré inequality for u and the Poincaré-Wirtinger inequality for φ, i.e.,

$$\lambda_1\|u\|^2\le\|\nabla u\|^2,\qquad\|\varphi-\overline{\varphi}\|^2\le c_\Omega\|\nabla\varphi\|^2, \tag{5.19}$$

from (5.14) we have

$$\frac{d}{dt}\big(\|u\|^2+\|\varphi\|^2\big)+\frac{\nu\lambda_1}{2}\|u\|^2+\frac{c_0}{2c_\Omega}\|\varphi\|^2\le2\gamma(t)\big(\|u\|^2+\|\varphi\|^2\big)+\frac{c_0|\Omega|}{2c_\Omega}\overline{\varphi}^2,$$

which yields

$$\frac{d}{dt}\big(\|u\|^2+\|\varphi\|^2\big)+k\big(\|u\|^2+\|\varphi\|^2\big)\le C_\tau\big(\|u\|^2+\|\varphi\|^2\big), \tag{5.20}$$

where $k:=\min(\lambda_1\nu,c_0/c_\Omega)/2$ and C_τ is a positive constant such that $2\gamma(t)+c_0/2c_\Omega\le C_\tau$ for all $t\ge\tau$. By using Gronwall's lemma we immediately see from (5.20) that $\|u\|^2+\|\varphi\|^2$ is controlled by the right-hand side of (5.18). Furthermore, we also have

$$\Big|\int_\Omega F\big(\varphi_2(t)\big)-\int_\Omega F\big(\varphi_1(t)\big)\Big|\le C_\tau\|\varphi(t)\|,\qquad\forall t\ge\tau.$$

Hence, the proof of (5.18) is complete. ∎

Lemma 5. *Let the assumptions of Lemma 3 be satisfied. Let $u_{0i} \in G_{div}$, $\varphi_{0i} \in H$ with $F(\varphi_{0i}) \in L^1(\Omega)$ and $[u_i, \varphi_i]$ be the corresponding weak solutions, $i = 1, 2$. Then, for every $\tau > 0$ there exists $C_\tau > 0$ such that*

$$\|u_{2,t} - u_{1,t}\|^2_{L^2(\tau,t;V'_{div})} + \|\varphi_{2,t} - \varphi_{1,t}\|^2_{L^2(\tau,t;D(B_N)')} \\ \leq C_\tau e^{C_\tau t}\big(\|u_2(\tau) - u_1(\tau)\|^2 + \|\varphi_2(\tau) - \varphi_1(\tau)\|^2\big), \qquad \forall t \geq \tau. \tag{5.21}$$

Proof. Consider the variational formulation of (3.2) and (3.3), namely,

$$\langle \varphi_t, \psi \rangle = -(\nabla\widetilde{\mu}, \nabla\psi) - (u \cdot \nabla\varphi_1, \psi) - (u_2 \cdot \nabla\varphi, \psi), \qquad \forall \psi \in V, \tag{5.22}$$

and take $\psi \in D(B_N)$. Then, for every $\tau > 0$ we see that there exists $C_\tau > 0$ such that

$$|(\nabla\widetilde{\mu}, \nabla\psi)| = |(\widetilde{\mu}, B_N\psi)| \leq \|\widetilde{\mu}\|\|\psi\|_{D(B_N)} \leq C_\tau\|\varphi\|\|\psi\|_{D(B_N)}, \qquad \forall t \geq \tau. \tag{5.23}$$

Moreover, we have

$$|(u \cdot \nabla\varphi_1, \psi)| = |(u \cdot \nabla\psi, \varphi_1) \leq c\|\nabla u\|\|\varphi_1\|\|\psi\|_{D(B_N)} \leq C\|\nabla u\|\|\psi\|_{D(B_N)},$$

where in this case it is enough to use the dissipative estimate (2.8) and therefore the constant C does not depend on τ but depends on h, $\mathcal{E}(z_{01})$ and η only. Concerning the last term on the right-hand side of (5.22) we have

$$|(u_2 \cdot \nabla\varphi, \psi)| = |(u_2 \cdot \nabla\psi, \varphi)| \leq c\|\nabla u_2\|\|\varphi\|\|\psi\|_{D(B_N)} \leq C_\tau\|\varphi\|\|\psi\|_{D(B_N)}, \qquad \forall t \geq \tau. \tag{5.24}$$

Plugging (5.23)–(5.24) into (5.22), we get

$$\|\varphi_t\|_{D(B_N)'} \leq C_\tau\big(\|\varphi\| + \|\nabla u\|\big), \qquad \forall t \geq \tau. \tag{5.25}$$

Therefore, taking also (5.10) into account, we have

$$\|\varphi_t\|_{L^2(\tau,t;D(B_N)')} \leq C_\tau e^{C_\tau t}\big(\|u(\tau)\| + \|\varphi(\tau)\|\big), \qquad \forall t \geq \tau. \tag{5.26}$$

In order to obtain an estimate for $u_{2,t} - u_{1,t}$ let us consider the difference of the Navier-Stokes equations written for two weak solutions in the variational formulation, i.e.,

$$\langle u_t, v \rangle = -\nu(\nabla u, \nabla v) - b(u_2, u_2, v) + b(u_1, u_1, v) \\ - \frac{1}{2}\big(\nabla a\varphi(\varphi_1 + \varphi_2), v\big) - \big((J * \varphi)\nabla\varphi_2, v\big) - \big((J * \varphi_2)\nabla\varphi, v\big), \qquad \forall v \in V_{div}. \tag{5.27}$$

Thanks to (5.15) the last three terms on the right-hand side can be easily estimated as follows

$$\frac{1}{2}\big|\big(\nabla a\varphi(\varphi_1 + \varphi_2), v\big)\big| \leq c\|\nabla a\|_{L^\infty}\|\varphi\|\|\varphi_1 + \varphi_2\|_{L^\infty}\|v\| \leq C_\tau\|\varphi\|\|v\|_{V_{div}}, \\ \big|\big((J * \varphi)\nabla\varphi_2, v\big)\big| = \big|\big((\nabla J * \varphi)\varphi_2, v\big)\big| \leq c\|\nabla J\|_{L^1}\|\varphi\|\|\varphi_2\|_{L^\infty}\|v\| \leq C_\tau\|\varphi\|\|v\|_{V_{div}}, \\ \big|\big((J * \varphi_2)\nabla\varphi, v\big)\big| = \big|\big((\nabla J * \varphi_2)\varphi, v\big)\big| \leq c\|\nabla J\|_{L^1}\|\varphi_2\|_{L^\infty}\|\varphi\|\|v\| \leq C_\tau\|\varphi\|\|v\|_{V_{div}},$$

for all $t \geq \tau$. Furthermore, the trilinear form can be controlled as follows:

$$|b(u_2, u_2, v) - b(u_1, u_1, v)| = |b(u_2, u, v) + b(u, u_1, v)| \\ \leq c\big(\|\nabla u_1\| + \|\nabla u_2\|\big)\|\nabla u\|\|\nabla v\| \leq C_\tau\|\nabla u\|\|\nabla v\|, \qquad \forall t \geq \tau.$$

Combining the last four estimates with (5.27) we obtain

$$\|u_t\|_{V'_{div}} \leq C_\tau\big(\|\nabla u\| + \|\varphi\|\big), \qquad \forall t \geq \tau,$$

Thus, recalling (5.10), we deduce

$$\|u_t\|_{L^2(\tau,t;V'_{div})} \leq C_\tau e^{C_\tau t}\big(\|u(\tau)\| + \|\varphi(\tau)\|\big), \qquad \forall t \geq \tau. \tag{5.28}$$

Finally, (5.26) and (5.28) yield (5.21). ∎

Lemma 6. *Let the assumptions of Lemma 3 be satisfied. Let $u_{0i} \in G_{div}$, $\varphi_{0i} \in H$ with $F(\varphi_{0i}) \in L^1(\Omega)$ $i = 1,2$. Then, for every $\tau > 0$ and every $T > 0$ there exists $C_{\tau,T} > 0$ depending also on T such that*

$$\rho_{\mathcal{X}_\eta}(S_\eta(t_2)z_{02}, S_\eta(t_1)z_{01}) \leq C_{\tau,T}\big(\rho_{\mathcal{X}_\eta}(S_\eta(\tau)z_{02}, S_\eta(\tau)z_{01}) + |t_2 - t_1|^{1/2}\big), \tag{5.29}$$

for all $t_1, t_2 \in [\tau, \tau + T]$, where $z_{0i} := [u_{0i}, \varphi_{0i}]$, $i = 1,2$.

Proof. Setting $S_\eta(t)z_{0i} := [u_i(t), \varphi_i(t)]$, $i = 1,2$, we have

$$\begin{aligned}
&\rho_{\mathcal{X}_\eta}(S_\eta(t_2)z_{01}, S_\eta(t_1)z_{01}) \\
&= \|u_1(t_2) - u_1(t_1)\| + \|\varphi_1(t_2) - \varphi_1(t_1)\| + \Big|\int_\Omega F(\varphi_1(t_2)) - \int_\Omega F(\varphi_1(t_1))\Big| \\
&\leq \|u_{1,t}\|_{L^2(t_1,t_2;G_{div})}|t_2 - t_1|^{1/2} + \|\varphi_{1,t}\|_{L^\infty(\tau,\infty;H)}|t_2 - t_1| + C_\tau\|\varphi_{1,t}\|_{L^\infty(\tau,\infty;H)}|t_2 - t_1| \\
&\leq C_{\tau,T}|t_2 - t_1|^{1/2}, \qquad \forall t_1, t_2 \in [\tau, \tau + T],
\end{aligned} \tag{5.30}$$

where we have used (5.16). Furthermore we have

$$\begin{aligned}
&\rho_{\mathcal{X}_\eta}(S_\eta(t_2)z_{02}, S_\eta(t_2)z_{01}) \\
&= \|u_2(t_2) - u_1(t_2)\| + \|\varphi_2(t_2) - \varphi_1(t_2)\| + \Big|\int_\Omega F(\varphi_2(t_2)) - \int_\Omega F(\varphi_1(t_2))\Big| \\
&\leq C_\tau e^{C_\tau(\tau+T)}\big(\|u_2(\tau) - u_1(\tau)\| + \|\varphi_2(\tau) - \varphi_1(\tau)\|\big) \leq C_{\tau,T}\rho_{\mathcal{X}_\eta}(S_\eta(\tau)z_{02}, S_\eta(\tau)z_{01}).
\end{aligned} \tag{5.31}$$

From (5.30) and (5.31) we get (5.29). ∎

We now recall the following abstract result on the existence of exponential attractors [12, Proposition 3.1]. This result, together with the lemmas above, will be used to prove Theorem 9.

Proposition 3. *Let $\mathcal{H}$ be a metric space (with metric $\rho_{\mathcal{H}}$) and let $\mathcal{V}, \mathcal{V}_1$ be two Banach spaces such that the embedding $\mathcal{V}_1 \hookrightarrow \mathcal{V}$ is compact. Let $\mathbb{B}$ be a bounded subset of $\mathcal{H}$ and let $\mathcal{S} : \mathbb{B} \to \mathbb{B}$ be a map such that*

$$\rho_{\mathcal{H}}\big(\mathcal{S}w_{02}, \mathcal{S}w_{01}\big) \leq \gamma\rho_{\mathcal{H}}(w_{02}, w_{01}) + K\|\mathcal{T}w_{02} - \mathcal{T}w_{01}\|_{\mathcal{V}}, \qquad \forall w_{01}, w_{02} \in \mathbb{B}, \tag{5.32}$$

where $\gamma \in (0, \frac{1}{2})$, $K \geq 0$ and $\mathcal{T} : \mathbb{B} \to \mathcal{V}_1$ is a globally Lipschitz continuous map, i.e.,

$$\|\mathcal{T}w_{02} - \mathcal{T}w_{01}\|_{\mathcal{V}_1} \leq L\rho_{\mathcal{H}}(w_{02}, w_{01}), \qquad \forall w_{01}, w_{02} \in \mathbb{B}, \tag{5.33}$$

for some $L \geq 0$. Then, there exists a (discrete) exponential attractor $\mathcal{M}_d \subset \mathbb{B}$ for the (time discrete) semigroup $\{\mathcal{S}^n\}_{n=0,1,2,\ldots}$ on $\mathbb{B}$ (with the topology of $\mathcal{H}$ induced on $\mathbb{B}$).

Proof of Theorem 9. Let $\mathcal{B}_0$ be a bounded absorbing set in $\mathcal{X}_\eta$. The existence of such a bounded absorbing set has been proven in [14]. Indeed, it is immediate to check that the argument of [14, Proposition 4] still applies with our choice for the metric $\rho_{\mathcal{X}_\eta}$. Let $t_0 = t_0(\mathcal{B}_0) \geq 0$ be a time such that $S_\eta(t)\mathcal{B}_0 \subset \mathcal{B}_0$ for all $t \geq t_0$. Due to (5.4) we can fix $t^* = t^*(\mathcal{B}_0) \geq t_0$ such that $S_\eta(t)\mathcal{B}_0 \subset B_{\mathcal{Z}^p_\eta}(0, \Lambda(\eta))$ for all $t \geq t^*$, where $B_{\mathcal{Z}^p_\eta}(0, \Lambda(\eta))$ is the closed ball in $\mathcal{Z}^p_\eta$ with radius $\Lambda(\eta)$ and $\Lambda(\eta)$ a positive constant which depends only on η. The (complete) metric space $\mathcal{Z}^p_\eta$ is given by

$$\mathcal{Z}^p_\eta := V_{div} \times \{\varphi \in W^{1,p}(\Omega) : \ |\overline{\varphi}| \leq \eta\}, \tag{5.34}$$

endowed with the metric

$$d_{\mathcal{Z}^p_\eta}(z_2, z_1) = \|\nabla u_2 - \nabla u_1\| + \|\varphi_2 - \varphi_1\|_{W^{1,p}(\Omega)}, \quad \forall z_i := [u_i, \varphi_i] \in \mathcal{Z}^p_\eta, \qquad i = 1, 2.$$

Note that the terms in the integrals of $F(\varphi_1), F(\varphi_2)$ are omitted in the metric since, for $p > 2$, we have the embedding $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$.

Let us now set

$$\mathcal{B}_1 := \bigcup_{t \geq t^*} S_\eta(t)\mathcal{B}_0.$$

Then, $\mathcal{B}_1$ is bounded in $\mathcal{Z}^p_\eta$ and positively invariant for $S_\eta(t)$. It is easy to see that it is also absorbing in $\mathcal{X}_\eta$. Indeed, if B is a bounded subset of $\mathcal{X}_\eta$ and $t_0 = t_0(B)$ is such that $S_\eta(t_0)B \subset \mathcal{B}_0$, then we have $S_\eta(t)B \subset \cup_{\tau \geq t^*} S_\eta(\tau + t_0)B \subset \cup_{\tau \geq t^*} S_\eta(\tau)\mathcal{B}_0 =: \mathcal{B}_1$, for all $t \geq t_0 + t^*$. Furthermore, we set $\mathbb{B} := S_\eta(1)\mathcal{B}_1$. Then, $\mathbb{B} \subset B_{\mathcal{Z}^p_\eta}(0, \Lambda(\eta))$ is positively invariant and still absorbing in $\mathcal{X}_\eta$.

By choosing $\tau = 1$ in Lemma 4, then (5.18) can be written as follows

$$\begin{aligned} &\rho_{\mathcal{X}_\eta}\big(S_\eta(t)z_{02}, S_\eta(t)z_{01}\big) \leq C_1 e^{-kt/2} \rho_{\mathcal{X}_\eta}\big(S_\eta(1)z_{02}, S_\eta(1)z_{01}\big) \\ &\quad + C_1 \|S_\eta(\cdot)z_{02} - S_\eta(\cdot)z_{01}\|_{L^2(1,t;G_{div} \times H)}, \qquad \forall t \geq 1, \qquad \forall z_{01}, z_{02} \in \mathcal{X}_\eta, \end{aligned} \tag{5.35}$$

where $C_1 > 0$ depends only on $\mathcal{E}(z_{01})$, $\mathcal{E}(z_{02})$ and η. From (5.35) we therefore get

$$\begin{aligned} &\rho_{\mathcal{X}_\eta}\big(S_\eta(t-1)w_{02}, S_\eta(t-1)w_{01}\big) \leq C_1 e^{-kt/2} \rho_{\mathcal{X}_\eta}\big(w_{02}, w_{01}\big) \\ &\quad + C_1 \|S_\eta(\cdot)w_{02} - S_\eta(\cdot)w_{01}\|_{L^2(0,t-1;G_{div} \times H)}, \qquad \forall t > 1, \qquad \forall w_{01}, w_{02} \in \mathbb{B}. \end{aligned} \tag{5.36}$$

Observe that, since $w_{0i} = S(1)z_{0i}$, with $z_{0i} \in \mathcal{B}_1$, $i = 1, 2$, and $\mathcal{B}_1$ is bounded in $\mathcal{Z}^p_\eta$, then C_1 does not depend on w_{01}, w_{02}.

Choosing $\tau = 1$ also in Lemma 3 and in Lemma 5, and combining (5.10) with (5.21) we can write

$$\begin{aligned} &\|S_\eta(\cdot)z_{02} - S_\eta(\cdot)z_{01}\|^2_{L^2(1,t;V_{div} \times V)} + \|\partial_t S_\eta(\cdot)z_{02} - \partial_t S_\eta(\cdot)z_{01}\|^2_{L^2(1,t;V'_{div} \times D(B_N)')} \\ &\leq C_1 e^{C_1 t} \rho^2_{\mathcal{X}_\eta}(S_\eta(1)z_{02}, S_\eta(1)z_{01}), \qquad \forall t \geq 1, \qquad \forall z_{01}, z_{02} \in \mathcal{X}_\eta. \end{aligned} \tag{5.37}$$

Thus we find

$$\begin{aligned} &\|S_\eta(\cdot)w_{02} - S_\eta(\cdot)w_{01}\|^2_{L^2(0,t-1;V_{div} \times V)} + \|\partial_t S_\eta(\cdot)w_{02} - \partial_t S_\eta(\cdot)w_{01}\|^2_{L^2(0,t-1;V'_{div} \times D(B_N)')} \\ &\leq C_1 e^{C_1 t} \rho^2_{\mathcal{X}_\eta}(w_{02}, w_{01}), \qquad \forall t \geq 1, \qquad \forall w_{01}, w_{02} \in \mathbb{B}, \end{aligned} \tag{5.38}$$

where, as pointed out above, the constant C_1 does not depend on w_{01} and w_{02}.

Let us now introduce the following spaces

$$\mathcal{H} := \mathcal{X}_\eta = G_{div} \times \mathcal{Y}_\eta$$

$$\mathcal{V}_1 := L^2(0,T;V_{div}\times V)\cap H^1(0,T;V_{div}'\times D(B_N)')$$
$$\mathcal{V} := L^2(0,T;G_{div}\times H),$$

with $T>0$ fixed such that $C_1e^{-k(T+1)/2}<1/2$, where C_1 and k are the same constants that appear in the first term on the right-hand side of (5.36). Notice that, due to the Aubin-Lions lemma, $\mathcal{V}_1$ is compactly embedded into $\mathcal{V}$.

Then, take $\mathcal{S}:=S_\eta(T)$ and define a map $\mathcal{T}:\mathbb{B}\to\mathcal{V}_1$ in the following way: for every $w_0\in\mathbb{B}$ we set $\mathcal{T}w_0:=w:=S_\eta(\cdot)w_0$, i.e., $w\in\mathcal{V}_1$ is the (strong) solution corresponding to the initial datum w_0.

It is now easy to see that choosing the spaces $\mathcal{H},\mathcal{V},\mathcal{V}_1$, the set $\mathbb{B}$, and the maps $\mathcal{S}$, $\mathcal{T}$ as above, then the conditions of Proposition 3 are satisfied. Indeed, (5.32) and (5.33) follow from (5.18) and (5.38), respectively, both written for $t=T+1$.

Therefore, Proposition 3 entails the existence of a (discrete) exponential attractor $\mathcal{M}_\eta^d\subset\mathbb{B}$ for the (time discrete) semigroup $\{\mathcal{S}^n\}_{n=0,1,2,...}$ on $\mathbb{B}$ (with the topology of $\mathcal{H}$ induced on $\mathbb{B}$). Since $\mathbb{B}$ is absorbing in $\mathcal{H}$, then the basin of attraction of $\mathcal{M}_\eta^d$ is the whole phase space $\mathcal{H}$.

In order to prove the existence of the exponential attractor $\mathcal{M}_\eta$ for $(\mathcal{X}_\eta,S_\eta(t))$ with continuous time we observe first that (5.29) written with $\tau=1$ (the time T is chosen as above) yields

$$\rho_{\mathcal{X}_\eta}(S_\eta(t_2-1)w_{02},S_\eta(t_1-1)w_{01})\le C_{1,T}\big(\rho_{\mathcal{X}_\eta}(w_{02},w_{01})+|t_2-t_1|^{1/2}\big),$$

for all $w_{01},w_{02}\in\mathbb{B}$ and for all $t_1,t_2\in[1,1+T]$. Hence

$$\rho_{\mathcal{X}_\eta}(S_\eta(t'')w_{02},S_\eta(t')w_{01})\le C_{1,T}\big(\rho_{\mathcal{X}_\eta}(w_{02},w_{01})+|t''-t'|^{1/2}\big),$$

for all $w_{01},w_{02}\in\mathbb{B}$ and for all $t'',t'\in[0,T]$. Therefore, the map $[t,z]\mapsto S_\eta(t)z$ is uniformly Hölder continuous (with exponent $1/2$) on $[0,T]\times\mathbb{B}$, where $\mathbb{B}$ is endowed with the $\mathcal{H}-$metric. Therefore, the exponential attractor $\mathcal{M}_\eta$ for the continuous time case can be obtained by the classical expression

$$\mathcal{M}_\eta=\bigcup_{t\in[0,T]}S_\eta(t)\mathcal{M}_\eta^d,$$

and this concludes the proof of the theorem. ∎

We conclude by proving the existence of exponential attractors when the viscosity ν depends on φ, that is, ν is locally Lipschitz on $\mathbb{R}$ and there exist $\nu_1>0$ such that

$$\nu(s)\ge\nu_1,\qquad\forall s\in\mathbb{R}. \tag{5.39}$$

In view of Theorems 5 and 6 we can define a dynamical system by using strong solutions. Indeed, taking $d=2$ and $h\in G_{div}$, we have that for every fixed $\eta\ge0$ system (1.3)–(1.5) generates a semigroup $\{Z_\eta(t)\}_{t\ge0}$ of *closed* operators on the metric space $\mathcal{K}_\eta$ given by

$$\mathcal{K}_\eta:=V_{div}\times\{\varphi\in H^2(\Omega):|\overline{\varphi}|\le\eta\}$$

endowed with the (weaker) metric

$$\varrho(z_2,z_1)=\|u_2-u_1\|+\|\varphi_2-\varphi_1\|,\quad\forall z_i:=[u_i,\varphi_i]\in\mathcal{K}_\eta,\ i=1,2.$$

We are now ready to state and prove the following.

Theorem 10. *Assume (H1), (H3)-(H5) and (5.39). Consider either $J\in W^{2,1}(B_\delta)$ or J admissible. The dynamical system $(\mathcal{K}_\eta,Z_\eta(t))$ possesses an exponential attractor $\mathcal{E}_\eta$ which is bounded in $V_{div}\times H^2(\Omega)$ such that the following properties are satisfied:*

- *positive invariance:* $Z_\eta(t)\mathcal{E}_\eta \subseteq \mathcal{E}_\eta$ *for all* $t \geq 0$*;*

- *finite dimensionality:* $dim_F(\mathcal{E}_\eta, G_{div} \times H) < \infty$*;*

- *exponential attraction:* $\exists\, Q : \mathbb{R}^+ \to \mathbb{R}^+$ *increasing and* $\kappa > 0$ *such that, for all* $R > 0$ *and for all* $\mathcal{B} \subset \mathcal{K}_\eta$ *with* $\sup_{z\in\mathcal{B}} \rho(z, 0) \leq R$ *there holds*

$$dist_{\mathcal{K}_\eta}(Z_\eta(t)\mathcal{B}, \mathcal{E}_\eta) \leq Q(R)e^{-\kappa t}, \qquad \forall t \geq 0.$$

Proof. Step 1. We will briefly show that a dissipative estimate like (5.4) still holds for the strong solution of (1.3)–(1.5) under the assumptions of the theorem. More precisely, the following estimate holds

$$\|\nabla u(t)\| + \|\varphi(t)\|_{H^2(\Omega)} + \int_t^{t+1} \|u(s)\|^2_{H^2(\Omega)^2} ds \leq \Theta(\eta), \qquad \forall t \geq t^*. \tag{5.40}$$

for some positive constant Θ independent of the initial data and time, and some time $t_\# > 0$ which depends only $\mathcal{E}(z_0)$. In order to get this estimate, first we recall estimate (2.8) by Theorem 1 which also holds for nonconstant viscosity. The proof of (5.40) follows immediately from the proof of Theorem 5. Indeed, we observe preliminarily that (5.5), (5.6) and (5.8) already hold uniformly with respect to time and initial data in the nonconstant case, i.e., there exists a time $t_\# > 0$, depending only on $\mathcal{E}(z_0)$, such that

$$\varphi \in L^\infty\left(t_\#, \infty; L^\infty(\Omega) \cap V\right) \cap W^{1,2}\left(t_\#, \infty; H\right) \tag{5.41}$$

and

$$\sup_{t\geq t_\#} \|\varphi\|_{C^{\delta/2,\delta}\left([t,t+1]\times\overline{\Omega}\right)} \leq \Theta(\eta). \tag{5.42}$$

In particular, this regularity allows us to obtain that uniformly

$$\mu \in L^\infty\left(t_\#, \infty; L^\infty(\Omega) \cap V\right), \, l \in L^2(t_\#, \infty; \left(L^2(\Omega)\right)^2).$$

This can be done by arguing exactly in the same fashion as in the derivation of estimates (4.18)-(4.21), with the exception that the constant $R > 0$ is such that

$$\operatorname*{ess\,sup}_{t\in\left(t_\#,\infty\right)} \|\varphi(t)\|_{L^\infty} \leq R.$$

Then, we can employ the same procedure as in the proof of Theorem 5 (with a function $Q = Q(R) > 0$ which is now independent of the initial data, by (5.41)-(5.42)) to deduce by virtue of the uniform Gronwall lemma (see [34, Chapter III, Lemma 1.1]) that

$$u \in L^\infty(t_*, \infty; V_{div}) \cap L^2(t_*, \infty; H^2(\Omega)^2), \quad u_t \in L^2(t_*, \infty; G_{div}), \tag{5.43}$$

for some $t_* \geq 1$ depending only on $t_\#$. Finally, arguing exactly as in the proof of Theorem 5 we deduce $\varphi \in L^\infty\left(t_*, \infty; H^2(\Omega)\right)$ uniformly with respect to time and the data. Note that estimate (5.40) entails the existence of a bounded absorbing set $\mathcal{B}_2 \subset \mathcal{K}_\eta$ for the semigroup $Z_\eta(t)$.

Step 2. As in the proof of Theorem 9, it will be sufficient to construct the exponential attractor for the restriction of $Z_\eta(t)$ on this set $\mathcal{B}_2$. Thus, it suffices to verify the validity of Lemmas 4 and 5 for the difference $u = u_2 - u_1$, $\varphi = \varphi_2 - \varphi_1$, where (u_i, φ_i) is a (given) strong solution and $i = 1, 2$. The first one is an immediate consequence of estimate (4.55) (see the proof of Theorem 6) and the application of Poincaré-type inequalities (5.19) (see the proof of Lemma 4). Indeed, in the nonconstant case we have

$$\begin{aligned}&\|u(t)\|^2+\|\varphi(t)\|^2\\&\leq C\big(\|u(\tau))\|^2+\|\varphi(\tau)\|^2\big)e^{-kt}+C\int_\tau^t\big(\|u(s)\|^2+\|\varphi(s)\|^2\big)ds,\qquad\forall t\geq\tau,\end{aligned}\tag{5.44}$$

for some constant $C=C_\tau>0$, where $(u_i(\tau),\varphi_i(\tau))\in\mathcal{B}_2$ for each $i=1,2$. For the second one, we observe that in order to estimate $u_t:=u_{2,t}-u_{1,t}$, we have

$$\begin{aligned}\langle u_t,v\rangle=&-(\nu(\varphi_2)\nabla u,\nabla v)-((\nu(\varphi_1)-\nu(\varphi_2))\nabla u_1,\nabla v)\\&-b(u_2,u_2,v)+b(u_1,u_1,v)\\&-\frac{1}{2}\big(\nabla a\varphi(\varphi_1+\varphi_2),v\big)-\big((J*\varphi)\nabla\varphi_2,v\big)-\big((J*\varphi_2)\nabla\varphi,v\big),\end{aligned}\tag{5.45}$$

for all $v\in W:=\left(H^{2+\varepsilon}(\Omega)\right)^2\cap V_{div}$ and some $\varepsilon>0$ (such that the embedding $H^{2+\varepsilon}\subset W^{1,\infty}$ holds). While all the terms on the right-hand side of (5.45), with the exception of the first two, can be word by word estimated exactly as in the proof of Lemma 5, we notice that assumption (5.39) and the essential L^∞-bound on φ yield

$$\begin{aligned}|(\nu(\varphi_2)\nabla u,\nabla v)|&\leq C\|\nabla u\|\|\nabla v\|,\\|((\nu(\varphi_1)-\nu(\varphi_2))\nabla u_1,\nabla v)|&\leq C\|\varphi\|\|\nabla u_1\|\|v\|_{H^{2+\varepsilon}}.\end{aligned}\tag{5.46}$$

Thus, we easily get

$$\|u_t\|_{W'}\leq C\left(\|\nabla u\|+\|\varphi\|\right),\quad\forall t\geq\tau,\tag{5.47}$$

which together with (4.55) and (5.25) yields the following estimate

$$\|u_t(t)\|^2_{L^2(\tau,t;W')}+\|\varphi_t\|^2_{L^2(\tau,t;D(B_N)')}\leq Ce^{Ct}\big(\|u(\tau)\|^2+\|\varphi(\tau)\|^2\big),\quad\forall t\geq\tau.\tag{5.48}$$

Estimates (5.44) and (5.48) convey that a certain smoothing property holds for the difference of any two strong solutions associated with any two given initial data in $\mathcal{B}_2$.

Step 3. It is now not difficult to finish the proof of the theorem, using the abstract scheme of Proposition 3 by arguing in a similar fashion as in the proof of Theorem 9. The differences are quite minor and so we leave them to the interested reader. ∎

Remark 5. On account of [16, Proofs of Proposition 1 and Lemma 3] and (4.37), using uniform Gronwall's lemma (see [34, Chapter III, Lemma 1.1]), it is possible to show that any weak solution becomes a strong solution in finite time. We remind that this property is based on the validity of the energy identity (2.7). Indeed, estimate (5.40) ensures that, given a weak trajectory z starting from $z_0\in\mathcal{X}_\eta$ (cf. (5.1)), there exists a time $t^*=t^*(z_0)\geq 0$ such that $z(t)\in\mathcal{B}_1(\Lambda(\eta))$ for all $t\geq t^*$, where $\mathcal{B}_1(\Lambda(\eta))$ is the closed ball in the space $V_{div}\times H^2(\Omega)$ with radius $\Lambda(\eta)$ and constraint $|\overline{\varphi}|\leq\eta$. Let us briefly mention some consequences of this property. First, the global attractor of the generalized semiflow on $\mathcal{X}_\eta$ generated by the problem with nonconstant viscosity (see [14]) is bounded in $V_{div}\times H^2(\Omega)$. Therefore we can show the validity of a smoothing property (cf. (5.44) and (5.48)) on the global attractor and deduce that it has finite fractal dimension. Moreover, the regularizing effect also allows us to prove the precompactness of (weak) trajectories (see [16, Lemma 3]). This is an essential ingredient to establish the convergence of a weak solution to a single equilibrium which can be done along the lines of [16, Section 5].

6 Conclusions

Uniqueness of a weak solution was proven for the nonlocal Cahn-Hilliard-Navier-Stokes in two dimensions with constant viscosity. This result holds either for a regular or a singular potential and also for singular potentials and degenerate mobility. Uniqueness of weak solutions seems out of reach if viscosity in the Navier-Stokes equations depends on φ. Therefore we established first the existence of a strong solution, a nontrivial result in itself. Then we show weak-strong uniqueness. This was done by assuming constant mobility and regular potential. In the case of constant viscosity and singular potential, the existence of a strong solution seems difficult to obtain. However, this can be achieved when the mobility is degenerate, provided some natural assumptions are satisfied (though we gave no proof here). On account of this, weak-strong uniqueness can also be demonstrated for nonconstant viscosity, degenerate mobility and singular potential. In the last section we investigated the global longtime behavior of the corresponding dynamical system. Uniqueness of weak solutions allowed us to prove the connectedness of the global attractor whose existence was obtained elsewhere. Then we established the existence of an exponential attractors for weak solutions (constant mobility and regular potential). Finally, in the case of variable viscosity, we showed that an exponential attractor can be still constructed by using strong solutions. These last two results essentially depend on the continuous dependence estimates which entail uniqueness.

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