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On Fairness of Systemic Risk Measures

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Abstract In our previous paper "A Unified Approach to Systemic Risk Measures via Acceptance Set" we have introduced a general class of systemic risk measures that allow random allocations to individual banks before aggregation of their risks. In the present paper, we address the question of fairness of these allocations and propose a fair allocation of the total risk to individual banks. We show that the dual formulation of the minimization problem identifying the systemic risk measure provides a valuation of the random allocations, which is fair both from the point of view of the society/regulator and from the individual financial institutions. The case with exponential utilities which allows for explicit computation is treated in details.

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1 Introduction

Consider a vector $\mathbf{X} = (X^1, \ldots, X^N) \in \mathcal{L}^0(\mathbb{R}^N)$ of N random variables denoting a configuration of risky factors at a future time T associated to a system of N entities/banks.

In the framework of Risk Measures, one of the first proposals, see [16], to measure the systemic risk of $\mathbf{X}$ was to consider the map

$$\rho(\mathbf{X}) := \inf\{m \in \mathbb{R} \mid \Lambda(\mathbf{X}) + m \in \mathbb{A}\}\,, \tag{1}$$

where

$$\Lambda : \mathbb{R}^N \to \mathbb{R},$$

is an aggregation rule that aggregates the N-dimensional risk factors into a univariate risk factor, and

$$\mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R}),$$

is a one-dimensional acceptance set. Systemic risk can again be interpreted as the minimal cash amount that secures the system when it is added to the total aggregated system loss $\Lambda(\mathbf{X})$. The interpretation of (1) is that the systemic risk is the minimal capital needed to secure the system *after aggregating individual risks.*

It might be more relevant to measure systemic risk as the minimal capital that secures the aggregated system by injecting the capital into the single institutions *before aggregating their individual risks.* This way of measuring systemic risk can be expressed by

$$\rho(\mathbf{X}) := \inf\left\{\sum_{i=1}^{N} m_i \mid \mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{R}^N,\ \Lambda(\mathbf{X} + \mathbf{m}) \in \mathbb{A}\right\}. \tag{2}$$

Here, the amount m_i is added to the financial position X^i of institution $i \in \{1, \cdots, N\}$ before the corresponding total loss $\Lambda(\mathbf{X}+\mathbf{m})$ is computed (we refer to [3], [7] and [24]).

The main novelty of our paper [7] was the possibility of adding to $\mathbf{X}$ not merely a vector $\mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{R}^N$ of cash, but, more generally, a random vector $\mathbf{Y} \in \mathcal{C}$ in a class $\mathcal{C}$ such that

$$\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}} \cap \mathcal{L}, \text{ where } \mathcal{C}_{\mathbb{R}} := \left\{\mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N) \mid \sum_{n=1}^{N} Y^n \in \mathbb{R}\right\},$$

where the subspace $\mathcal{L} \subseteq \mathcal{L}^0(\mathbb{R}^N)$ will be specified later. Here, the notation $\sum_{n=1}^{N} Y^n \in \mathbb{R}$ means that $\sum_{n=1}^{N} Y^n$ is equal to some deterministic constant in $\mathbb{R}$, even though each single Y^n, $n = 1, \cdots, N$, is a random variable.

Then, the general systemic risk measure considered in [7] can be written as

$$\rho(\mathbf{X}) := \inf\left\{\sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C},\ \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}\right\}, \tag{3}$$

and can still be interpreted as the minimal total cash amount $\sum_{n=1}^{N} Y^n \in \mathbb{R}$ needed today to secure the system by distributing the cash at the future time T among the components of the risk vector $\mathbf{X}$. However, contrary to (2), in general the allocation $Y^i(\omega)$ to institution i does not need to be decided today but depends on the scenario ω that has been realized at time T. This corresponds to the situation of a lender of last resort who is equipped with a certain amount of cash today and who will allocate it according to where it serves the most depending on the scenario that has been realized at T. Of course, in general, the use of scenario dependent allocation $\mathbf{Y}$ as in (3) reduces, in comparison to the deterministic case in (2), the minimal amount of capital $\rho(\mathbf{X})$ needed to secure the system. Restrictions on the possible distributions of cash are given by the class $\mathcal{C}$, as shown in the Example 1.

Definition 1 (i) We say that the scenario dependent allocation $\mathbf{Y}_{\mathbf{X}} = (Y^n_{\mathbf{X}})_n \in \mathcal{C}$ is a systemic *optimal allocation* for $\rho(\mathbf{X})$, defined in (3), if it satisfies $\Lambda(\mathbf{X} + \mathbf{Y}_{\mathbf{X}}) \in \mathbb{A}$ and $\rho(\mathbf{X}) = \sum_{n=1}^N Y^n_{\mathbf{X}}$.
(ii) We say that a vector $(\rho^n(\mathbf{X}))_n \in \mathbb{R}^N$ is a systemic *risk allocation* of $\rho(\mathbf{X})$ if $\sum_{n=1}^N \rho^n(\mathbf{X}) = \rho(\mathbf{X})$.

Even though, as mathematicians, we like well defined and sharp definitions, the analysis of a system of financial institutions suggests that the concept of *fairness* is a multi-faceted notion.

The aim of this paper is to analyze in detail the systemic risk measure in (3). In addition to several technical aspects regarding such systemic risk measures, we will answer the following main questions about fairness of risk allocations:

1. When is the systemic *valuation* $\rho(\mathbf{X})$ *and its optimal allocation* $\mathbf{Y}_{\mathbf{X}}$ fair from the point of view of the *whole system*?
2. When is a systemic *risk allocation* $(\rho^n(\mathbf{X}))_n \in \mathbb{R}^N$ of $\rho(\mathbf{X})$ fair from the point of view of the *whole system*?
3. When are the systemic *optimal allocation* $\mathbf{Y}_{\mathbf{X}} \in \mathcal{C}$ and the systemic *risk allocation* $(\rho^n(\mathbf{X}))_n \in \mathbb{R}^N$ associated to $\rho(\mathbf{X})$, fair from the point of view of *each individual bank*?

We provide answers to these questions in the following introductory section without entering in the mathematical details of our analysis which will be provided in the subsequent sections. The optimal solution to the dual problem of the primal problem (3) will play a crucial role. It is a vector of probability measures $\mathbf{Q}_{\mathbf{X}} = (Q^1_{\mathbf{X}}, \cdots Q^N_{\mathbf{X}})$ which will provide the fair valuation of the optimal random allocations through the formula $\rho(\mathbf{X}) = \sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]$. Existence and uniqueness of $\mathbf{Q}_{\mathbf{X}}$ is proved in Proposition 4 and Corollary 4. In Section 3 we introduce the setting of the paper and the main assumptions, and we show that our optimization problems are well posed. The main results are collected in Section 4, where we first present our results in the Orlicz space setting (introduced in Section 3.2) and in Section 5, where the existence of the optimal solution to $\rho(\mathbf{X})$ (Theorem 7), as well as other technical existence results, are provided. To guarantee existence, we need to enlarge the environment and consider appropriate spaces of integrable random variables. For this reason, we point out, in the course of the paper, those results that admit an extension to the larger setting. The case with *exponential utilities* and *grouping of banks* will be treated in details in Section 6, where meaningful sensitivity properties will be established as well.

In the rest of the paper, we shall assume that the aggregation function Λ is of the form $\Lambda(\mathbf{x}) = \sum_{n=1}^N u_n(x_n)$ for utility functions u_n, $n = 1, \cdots, N$.

2 Fairness of systemic risk measures and allocations

The main objective of this paper is to discuss various aspects of fairness of the systemic risk measures $\rho(\mathbf{X})$, random allocations $\mathbf{Y} \in \mathcal{C}$, and risk allocations of the total systemic risk among individual banks. In this introductory section, we explain and motivate the various fairness properties, both from the point of view of the society/regulator and from the individual financial institutions. Precise definitions and statements, as well as detailed proofs, will be given in the course of the paper. For the remaining of this section, we assume that the infimum of the systemic risk measure

$$\rho(\mathbf{X}) := \inf_{\mathbf{Y}\in\mathcal{C}\subset\mathcal{C}_{\mathbb{R}}} \left\{ \sum_{n=1}^{N} Y^n \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] \geq B \right\}, \tag{4}$$

is attained for an optimal (random) allocation $\mathbf{Y}_{\mathbf{X}} = (Y^1_{\mathbf{X}}, \cdots, Y^N_{\mathbf{X}}) \in \mathcal{C}$, which will turn out to be unique. Existence of such minimizer is proved in Section 5. Note that (4) is a particular case of (3), where the function Λ is the sum of the utility functions u_n and $\mathbb{A}$ is the particular acceptance set $\mathbb{A} = \{Z \in \mathcal{L}^0(\mathbb{R}), \mathbb{E}[Z] \geq B\}$ for a given constant B.

We first introduce the related optimization problem

$$\pi(\mathbf{X}) := \sup_{\mathbf{Y}\in\mathcal{C}\subset\mathcal{C}_{\mathbb{R}}} \left\{ \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] \mid \sum_{n=1}^{N} Y^n \leq A \right\}, \tag{5}$$

so that, if we interpret $\sum_{n=1}^{N} u_n(X^n + Y^n)$ as the aggregated utility of the system after allocating $\mathbf{Y}$, then $\pi(\mathbf{X})$ can be interpreted as the maximal expected utility of the system over all random allocations $\mathbf{Y} \in \mathcal{C}$ such that the aggregated budget constraint $\sum_{n=1}^{N} Y^n \leq A$ holds for a given constant A. In the following, we may write $\rho(\mathbf{X}) = \rho_B(\mathbf{X})$ and $\pi(\mathbf{X}) = \pi_A(\mathbf{X})$ in order to express the dependence on the minimal level of expected utility $B \in \mathbb{R}$ and on the maximal budget level $A \in \mathbb{R}$, respectively. We will see in Section 4.2 that

$$B = \pi_A(\mathbf{X}) \text{ if and only if } A = \rho_B(\mathbf{X}), \tag{6}$$

and, in these cases, the two problems $\pi_A(\mathbf{X})$ and $\rho_B(\mathbf{X})$ have the same unique optimal solution $\mathbf{Y}_{\mathbf{X}}$. From this, we infer that once a level $\rho(\mathbf{X})$ of total systemic risk has been determined, the optimal allocation $\mathbf{Y}_{\mathbf{X}}$ of ρ maximizes the expected system utility among all random allocations of cost less or equal to $\rho(\mathbf{X})$.

Once the total systemic risk has been identified as $\rho(\mathbf{X})$, the second essential question is how to allocate the total risk to the individual institutions. Recall that a vector $(\rho^1(\mathbf{X}), \cdots, \rho^N(\mathbf{X})) \in \mathbb{R}^N$ is called a systemic risk allocation (SRA) of $\rho(\mathbf{X})$ if $\sum_{n=1}^{N} \rho^n(\mathbf{X}) = \rho(\mathbf{X})$. For deterministic allocation, this property is known as the *"Full Allocation"* property, see for example [13].

In the case of deterministic allocations $\mathbf{Y} \in \mathbb{R}^N$, i.e. $\mathcal{C} = \mathbb{R}^N$, the optimal deterministic $\mathbf{Y}_{\mathbf{X}}$ represents a canonical risk allocation $\rho^n(\mathbf{X}) := Y^n_{\mathbf{X}}$. For general (random) allocations $\mathbf{Y} \in \mathcal{C} \subset \mathcal{C}_{\mathbb{R}}$, we then follow the natural approach to consider risk allocations of the form

$$\rho^n(\mathbf{X}) := \mathbb{E}_{Q^n}[Y^n_{\mathbf{X}}] \quad \text{for } n = 1, \cdots, N, \tag{7}$$

where $\mathbf{Q} = (Q^1, \cdots, Q^N)$ is a vector of probability measures with $\sum_{n=1}^{N} \mathbb{E}_{Q^n}[Y^n_{\mathbf{X}}] = \rho(\mathbf{X})$. In that way, $\rho^n(\mathbf{X}) = \mathbb{E}_{Q^n}[Y^n_{\mathbf{X}}]$ can be understood as a *systemic risk valuation* of $Y^n_{\mathbf{X}}$. Note that in our setting, besides providing a ranking in terms of systemic riskiness, a risk allocation $\rho^n(\mathbf{X})$ can be interpreted as a capital requirement for institution n in order to fund the total amount $\rho(\mathbf{X})$ of cash needed. In this sense, the vector $\mathbf{Q}$

allows for the monetary interpretation of a systemic pricing operator to determine the price (or cost) of (future) random allocations of the individual institutions. Obviously, it is of high interest to identify fairness criteria, acceptable both by the society and by the individual financial institutions, for such systemic valuation measures and their corresponding risk allocations.

Now, consider the situation where a valuation (or cost) operator $\mathbf{Q} = (Q^1, \cdots, Q^N)$ is given for the system. Then, a natural alternative formulation of the systemic risk measure and the related utility maximization problem in terms of the valuation provided by $\mathbf{Q}$ is

$$\rho^{\mathbf{Q}}(\mathbf{X}) = \rho^{\mathbf{Q}}_B(\mathbf{X}) := \inf_{\mathbf{Y}\in M^{\Phi}} \left\{ \sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] \mid \mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n)\right] \geq B \right\}, \quad (8)$$

$$\pi^{\mathbf{Q}}(\mathbf{X}) = \pi^{\mathbf{Q}}_A(\mathbf{X}) := \sup_{\mathbf{Y}\in M^{\Phi}} \left\{ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n)\right] \mid \sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] \leq A \right\}. \quad (9)$$

Note that in (8) and (9) the allocation $\mathbf{Y}$ is not required to belong to $\mathcal{C}_{\mathbb{R}}$ (that is adding up to a deterministic quantity) but to a vector space $\mathcal{L} = M^{\Phi}$ of random variables introduced later. Thus, for the systemic risk measure $\rho^{\mathbf{Q}}(\mathbf{X})$, we look for the minimal (systemic) cost $\sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n]$ among all $\mathbf{Y} \in M^{\Phi}$ satisfying the acceptability (utility) constraint $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n)\right] \geq B$. Analogously, for $\pi^{\mathbf{Q}}(\mathbf{X})$ we maximize the expected systemic utility among all $\mathbf{Y} \in M^{\Phi}$ satisfying the budget constraint $\sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] \leq A$. Similarly as in (6), we will see in Section 4.2 that

$$B = \pi^{\mathbf{Q}}_A(\mathbf{X}) \text{ if and only if } A = \rho^{\mathbf{Q}}_B(\mathbf{X}), \quad (10)$$

and the two problems $\pi^{\mathbf{Q}}_A(\mathbf{X})$ and $\rho^{\mathbf{Q}}_B(\mathbf{X})$ have the same unique optimal solution.

The specific choice of a systemic valuation is the central question of this paper. It will turn out that the optimizer $\mathbf{Q_X} = (Q^1_{\mathbf{X}}, \cdots Q^N_{\mathbf{X}})$ of the dual problem of (4), presented in detail in Section 4.1 and in Corollary 4, provides a systemic risk allocation $(\mathbb{E}_{Q^1_{\mathbf{X}}}[Y^1_{\mathbf{X}}], \cdots, (\mathbb{E}_{Q^N_{\mathbf{X}}}[Y^N_{\mathbf{X}}])$, see (74), with

$$\sum_{n=1}^N \mathbb{E}_{Q_{\mathbf{X}^n}}[Y^n_{\mathbf{X}}] = \rho(\mathbf{X}),$$

satisfying

$$\rho_B(\mathbf{X}) = \rho^{\mathbf{Q_X}}_B(\mathbf{X}), \quad (11)$$

$$\pi_A(\mathbf{X}) = \pi^{\mathbf{Q_X}}_A(\mathbf{X}). \quad (12)$$

Furthermore, by Proposition 13, $\mathbf{Y_X}$ is the unique optimal allocation to $\rho_B(\mathbf{X})$ and $\rho^{\mathbf{Q_X}}_B(\mathbf{X})$ in (11). Similarly, $\pi_A(\mathbf{X})$ and $\pi^{\mathbf{Q_X}}_A(\mathbf{X})$ in (12) have the same optimal solution $\mathbf{Y_X}$, see Section 5.3.2.

We now discuss fairness properties of the systemic risk measure $\rho(\mathbf{X})$, the optimal allocation $\mathbf{Y_X}$, and the systemic probability measure $\mathbf{Q_X}$ with corresponding risk allocations $\mathbb{E}_{Q^1_{\mathbf{X}}}[Y^1_{\mathbf{X}}]$,.. ..,$\mathbb{E}_{Q^N_{\mathbf{X}}}[Y^N_{\mathbf{X}}]$, both from the perspective of the society/regulator and from the individual institutions.

Fairness from the perspective of the society/regulator. Consider the systemic risk measure $\rho(\mathbf{X})$ with $\mathcal{C} = \mathbb{R}^N$. In this case not only the total amount of cash $\rho(\mathbf{X})$ but also the individual cash amounts $\mathbf{Y} \in \mathbb{R}^N$ allocated to the institutions are already known today (i.e., they are deterministic). Such a risk measure only depends on the marginal distributions of $\mathbf{X}$ as can be seen from the constraint $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \geq B$ in (4) with Y^n deterministic. However, ignoring potential dependencies among the banks might be over-conservative and too costly. By considering scenario-dependent allocations $\mathbf{Y} \in \mathcal{C} \supsetneq \mathbb{R}^N$ (and by that considering the dependencies among the banks as was shown through examples in [7]), the consequential reduction of the overall cost of securing the system is beneficial to the society. Additionally, the requirement $\mathbf{Y} \in \mathcal{C}$ and $\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$ is important from the society's perspective as it guarantees that the cash amount $\rho(\mathbf{X})$ determined today is sufficient to cover the allocations $\mathbf{Y}$ at time T in any possible scenario. There might be cross-subsidization (in the sense of a risk exchange) among the banks at time T, but $\sum_{n=1}^N Y^n = \rho(\mathbf{X})$ means that the system clears and no additional external injections (or withdrawals) are necessary at time T. In that sense, the requirement $\mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$ is fair from the society/regulator's perspective. Furthermore and most importantly, by (5) and (10), the optimal allocation $\mathbf{Y}_{\mathbf{X}}$ maximizes the expected systemic utility among all allocations with total cost less or equal to $A = \rho_B(\mathbf{X})$.

Next, consider the systemic risk valuation using $\mathbf{Q}_{\mathbf{X}}$. To explain one of the features of $\mathbf{Q}_{\mathbf{X}}$, observe first that ρ in (3) keeps the classical cash additivity property

$$\rho(\mathbf{X} + \mathbf{m}) = \rho(\mathbf{X}) - \sum_{n=1}^N m^n \text{ for all } \mathbf{m} \in \mathbb{R}^N \text{ and all } \mathbf{X}, \tag{13}$$

which is a *global* property, see Section 4.3 for details. The local version associated to (13) is

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{m})|_{\varepsilon=0} = -\sum_{n=1}^N m^n \quad \text{for } \mathbf{m} \in \mathbb{R}^N. \tag{14}$$

The expression $\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{m})|_{\varepsilon=0}$ represents the sensitivity of the risk of $\mathbf{X}$ with respect to the impact $\mathbf{m} \in \mathbb{R}^N$ and was named the *marginal risk contribution* by [3]. However, such property can not be immediately generalized to the case where $\mathbf{m} \in \mathbb{R}^N$ is replaced by random vectors $\mathbf{V}$, in particular when $\sum_{n=1}^N V^n$ is not a constant.

If the positions change from $\mathbf{X}$ to $\mathbf{X} + \varepsilon V^j \mathbf{e}_j$, where $\mathbf{e}_j$ is the jth unit vector and V^j is a random variable, then, we show in Section 4.3 that the riskiness of the entire system changes linearly by

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon V^j \mathbf{e}_j)|_{\varepsilon=0} = \mathbb{E}_{Q^j_{\mathbf{X}}}[-V^j], \tag{15}$$

which shows that $\mathbf{Q}_{\mathbf{X}}$ can be naturally introduced as a systemic risk valuation operator.

Now, given a systemic risk valuation $\mathbf{Q}$, one is naturally led to the specification (8) for a systemic risk measure. Note, however, that in (8) the clearing condition $\sum_{n=1}^N Y^n = \rho(\mathbf{X})$ is not guaranteed since the optimization is performed over all $\mathbf{Y} \in M^{\Phi}$. Using the valuation with $\mathbf{Q}_{\mathbf{X}}$ is then fair from the society/regulator's point of view since, by Proposition 13, the optimal allocation in (8) fulfills the clearing condition $\mathbf{Y} \in \mathcal{C}_{\mathbb{R}}$, and is in fact the same as the optimal allocation of the original systemic risk measure in (4). From (73) and (74) we obtain

$$\sum_{n=1}^N Y^n_{\mathbf{X}} = \rho(\mathbf{X}) = \rho^{\mathbf{Q}_{\mathbf{X}}}(\mathbf{X}) = \sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}],$$

which also shows that the selection of $\mathbb{E}_{Q_{\mathbf{X}}}[\cdot]$ as the valuation functional is as fair as computing $\rho(\mathbf{X})$ as the infimum of $\sum_{n=1}^{N} Y^n$, for admissible $\mathbf{Y}$, and supports the definition of $\mathbf{Q_X}$ as the systemic probability measure.

Fairness from the perspective of the individual institutions. The essential question for a financial institution is whether its allocated share of the total systemic risk determined by the risk allocation $(\mathbb{E}_{Q^1_{\mathbf{X}}}[Y^1_{\mathbf{X}}], \cdots, (\mathbb{E}_{Q^N_{\mathbf{X}}}[Y^N_{\mathbf{X}}])$ is fair.

For the banks, the clearing condition $\mathbf{Y} \in \mathcal{C}_{\mathbb{R}}$ is not relevant. Instead, given a vector $\mathbf{Q} = (Q^1, \cdots, Q^N)$ of valuation measures, the systemic risk measure $\rho^{\mathbf{Q}}_B(\mathbf{X})$ in (8) is more relevant. Thus, by choosing $\mathbf{Q} = \mathbf{Q_X}$, the requirements from both the society and the banks are reconciled as seen from (11). Furthermore, with the choice $\mathbf{Q} = \mathbf{Q_X}$, we have by (12)

$$\pi_A(\mathbf{X}) = \pi^{\mathbf{Q_X}}_A(\mathbf{X}) = \sup_{\sum_{n=1}^N a^n = A,} \sum_{n=1}^{N} \sup_{\mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n] = a_n} \mathbb{E}\left[u_n(X^n + Y^n)\right], \qquad (16)$$

see Lemma 3 for details. Choosing $A = \rho_B(\mathbf{X})$, we obtain by (10) and the fact that, then, $\mathbf{Y_X}$ is the optimal solution of $\pi^{\mathbf{Q_X}}_A(\mathbf{X})$, that $\mathbb{E}_{Q_{\mathbf{X}^n}}[Y^n_{\mathbf{X}}] = a_n,\ \sum_{n=1}^N \mathbb{E}_{Q_{\mathbf{X}^n}}[Y^n_{\mathbf{X}}] = A$ and (16) can be rewritten as

$$\pi_A(\mathbf{X}) = \pi^{\mathbf{Q_X}}_A(\mathbf{X}) = \sum_{n=1}^{N} \sup_{\mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n] = \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]} \mathbb{E}\left[u_n(X^n + Y^n)\right].$$

This means that by using $\mathbf{Q_X}$ for valuation, the system utility maximization in (9) reduces to individual utility maximization problems for the banks without the "systemic" constraint $\mathbf{Y} \in \mathcal{C}$:

$$\forall n, \quad \sup_{Y^n} \left\{ \mathbb{E}\left[u_n(X^n + Y^n)\right] \mid \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n] = \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}] \right\}.$$

The optimal allocation $Y^n_{\mathbf{X}}$ and its value $\mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]$ can thus be considered fair by the n^{th} bank, as $Y^n_{\mathbf{X}}$ maximizes the individual expected utility of bank n among all random allocations (not constrained to be in $\mathcal{C}_{\mathbb{R}}$) with value $\mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]$. This finally argues for the fairness of the risk allocation $(\mathbb{E}_{Q^1_{\mathbf{X}}}[Y^1_{\mathbf{X}}], \cdots, \mathbb{E}_{Q^N_{\mathbf{X}}}[Y^N_{\mathbf{X}}])$ as fair valuation of the optimal allocation $(Y^1_{\mathbf{X}}, \cdots, Y^N_{\mathbf{X}})$.

Another desirable fairness property is *monotonicity*. It is clear that if $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_{\mathbb{R}}$, then $\rho_1(\mathbf{X}) \geq \rho_2(\mathbf{X})$ for the corresponding systemic risk measures

$$\rho_i(\mathbf{X}) := \inf \left\{ \sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C}_i,\ \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A} \right\}, \quad i = 1, 2.$$

The two extreme cases occur for $\mathcal{C}_1 := \mathbb{R}^N$ (the deterministic case) and $\mathcal{C}_2 := \mathcal{C}_{\mathbb{R}}$ (the unconstraint scenario dependent case). Hence we know that when going from deterministic to scenario-dependent allocations the total systemic risk decreases. It is then desirable that each institution profits from this decrease in total systemic risk in the sense that also its individual risk allocation decreases:

$$\rho^n_1(\mathbf{X}) \geq \rho^n_2(\mathbf{X}) \text{ for each } n = 1, ..., N. \qquad (17)$$

The opposite would clearly be perceived as unfair. This is discussed in the exponential setting of Section 6.2, where we show that (17) holds when $\rho^n_1(\mathbf{X}) := Y^n_1$ and $\rho^n_2(\mathbf{X}) :=$

$\mathbb{E}_{Q_2^n}[Y_2^n]$ (where Y_j^n is the optimal allocation to the systemic risk measure $\rho_j(\mathbf{X})$ associated to $\mathcal{C}_j$, so that $Y_1^n \in \mathbb{R}$ is deterministic, and $\mathbf{Q}_2$ is the systemic probability measure associated to $\rho_2(\mathbf{X})$). By using a probability measure $\mathbf{R}$ different from $\mathbf{Q}_2$ to compute the risk allocation $\rho_2^n(\mathbf{X}) = \mathbb{E}_{R^n}[Y_{\mathbf{2}}^n]$, the property (17) is lost in general.

Additional fairness properties related to the systemic probability measure $\mathbf{Q}_{\mathbf{X}}$ are addressed in Section 6.1, Proposition 18.

We conclude this Section with a literature overview on systemic risk. In [19], [12] and [18] one can find empirical studies on banking networks, while interbank lending has been studied via interacting diffusions and mean field approach in several papers like [28], [26], [15], [35], [5]. Among the many contributions on systemic risk modeling, we mention the classical contagion model proposed by [23], the default model of [31], the illiquidity cascade models of [30], [34] and [37], the asset fire sale cascade model by [17] and [14], as well as the model in [42] that additionally includes cross-holdings. Further works on network modeling are [1], [40], [2], [32], [4], [21] and [22]. See also the references therein. For an exhaustive overview on the literature on systemic risk we refer the reader to the recent volumes of [33] and of [27].

3 The setting

We now introduce the setting and discuss some fundamental properties of systemic risk measures. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the space of random vectors

$$L^0 := L^0(\mathbb{P}; \mathbb{R}^N) := \{\mathbf{X} = (X^1, \ldots, X^N) \mid X^n \in L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}),\ n = 1, \cdots, N\}.$$

The measurable space $(\Omega, \mathcal{F})$ will be fixed throughout the paper and will not appear in the notations. Unless we need to specify a different probability, we will also suppress $\mathbb{P}$ from the notations and simply write $L^0(\mathbb{R}^N)$. In addition, we will sometimes suppress $\mathbb{R}^d$, $d = 1, ..., N$, in the notation of the vector spaces, when the dimension of the random vector is clear from the context. When $\mathbf{Q} = (Q^1, ..., Q^N)$ is a vector of probability measures on $(\Omega, \mathcal{F})$, we set $L^1(\mathbf{Q}) := \{\mathbf{X} = (X^1, \ldots, X^N) \mid X^n \in L^1(Q^n),\ n = 1, \cdots, N\}$. Unless differently stated, all inequalities between random vectors are meant to be $\mathbb{P}$-a.s. inequalities.
A vector $\mathbf{X} = (X^1, \ldots, X^N) \in L^0$ denotes a configuration of risky factors at a future time T associated to a system of N entities. We assume that $L^0(\mathbb{R}^N)$ is a vector lattice equipped with the order relation

$$\mathbf{X}_1 \succeq \mathbf{X}_2 \quad if \quad X_1^i \geq X_2^i \ \mathbb{P}-a.s.\ \forall i = 1, \cdots, N. \tag{18}$$

Let $\mathcal{C}_{\mathbb{R}}$ be the linear space

$$\mathcal{C}_{\mathbb{R}} := \{\mathbf{Y} \in L^0(\mathbb{R}^N) \mid \sum_{n=1}^N Y^n \in \mathbb{R}\}. \tag{19}$$

Here we use the notation $\sum_{n=1}^N Y^n \in \mathbb{R}$ to denote that $\sum_{n=1}^N Y^n$ is equal to some deterministic constant in $\mathbb{R}$, even though each single Y^n, $n = 1, \cdots, N$, is a random variable.

By following [7], we consider systemic risk measures

$$\rho : \mathcal{L} \to \mathbb{R} \cup \{\infty\} \cup \{-\infty\},$$

of the form

$$\rho(\mathbf{X}) := \inf\left\{\sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C},\ \Lambda(\mathbf{X}+\mathbf{Y}) \in \mathbb{A}\right\}, \tag{20}$$

where the map

$$\Lambda : \mathbb{R}^N \to \mathbb{R},$$

is an *aggregation rule* that aggregates the N-dimensional risk factor into a univariate risk factor, $\mathbb{A} \subseteq L^0(\mathbb{R})$ is a one dimensional *acceptance set* and the set $\mathcal{C}$ of *admissible* random elements satisfies $\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}} \cap \mathcal{L}$, where

$$\mathcal{L} \subseteq L^1(\mathbb{P}; \mathbb{R}^N),$$

is a vector subspace containing $\mathbb{R}^N$, that will be specified in the sequel.

Example 1 We now introduce one relevant example for the set of admissible random elements, which we denote $\mathcal{C}^{(\mathbf{n})}$.

Definition 2 Set $n_0 = 0$. For $h \in \{1, \cdots, N\}$, let $\mathbf{n} := (n_1, \cdots, n_h) \in \mathbb{N}^h$, with $n_{m-1} < n_m$ for all $m = 1, \cdots, h$ and $n_h := N$, represent some partition of $\{1, \cdots, N\}$. We set $I_m := \{n_{m-1}+1, \cdots, n_m\}$ for each $m = 1, \cdots, h$. The cardinality of each group is denoted with $N_m := n_m - n_{m-1}$. We introduce the following family of allocations $\mathcal{C}^{(\mathbf{n})} = \mathcal{C}_0^{(\mathbf{n})} \cap \mathcal{L}$, where

$$\mathcal{C}_0^{(\mathbf{n})} = \left\{\mathbf{Y} \in L^0(\mathbb{R}^N) \mid \ \exists\, d = (d_1, \cdots, d_h) \in \mathbb{R}^h \ :\ \sum_{i \in I_m} Y^i = d_m \text{ for } m = 1, \cdots, h\right\} \subseteq \mathcal{C}_{\mathbb{R}}. \tag{21}$$

For a given $\mathbf{n} := (n_1, \cdots, n_h)$, the values $(d_1, \cdots, d_h)$ may change, but the number of elements in each of the h groups I_m is fixed by the partition $\mathbf{n}$. It is then easily seen that $\mathcal{C}^{(\mathbf{n})}$ is a linear space containing $\mathbb{R}^N$ and closed with respect to convergence in probability. Beside the obvious interpretation of the restrictions imposed to the elements $\mathbf{Y} \in \mathcal{C}^{(\mathbf{n})}$, we point out that the family $\mathcal{C}^{(\mathbf{n})}$ admits two extreme cases:

(i) the strongest restriction occurs when $h = N$, i.e. we consider exactly N groups, and in this case $\mathcal{C}^{(\mathbf{n})} = \mathbb{R}^N$ corresponds to the deterministic case;
(ii) on the opposite side, we have only one group $h = 1$ and $\mathcal{C}^{(\mathbf{n})} = \mathcal{C}_{\mathbb{R}} \cap \mathcal{L}$ is the largest possible class, corresponding to arbitrary random injection $\mathbf{Y} \in \mathcal{L}$ with the only constraint $\sum_{n=1}^{N} Y^n \in \mathbb{R}$.

3.1 Assumptions and properties of ρ

We now specify further properties of systemic risk measures of the form (20) under some additional, but still general hypotheses. In the sequel we will always work under the following

Assumption 1

1. *$\mathcal{L} \subseteq L^1(\mathbb{P}; \mathbb{R}^N)$; $\mathcal{C}_0 \subseteq \mathcal{C}_{\mathbb{R}}$ and $\mathcal{C} = \mathcal{C}_0 \cap \mathcal{L}$ is a convex cone satisfying $\mathbb{R}^N \subseteq \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$.*
2. *$\Lambda(\mathbf{x}) = \sum_{n=1}^{N} u_n(x^n)$ where $u_n : \mathbb{R} \to \mathbb{R}$ is increasing, concave, and $\lim_{x \to -\infty} \frac{u_n(x)}{x} = +\infty$.*
3. *$B < \Lambda(+\infty)$, i.e. there exists $\mathbf{M} \in \mathbb{R}^N$ such that $\Lambda(\mathbf{M}) \geq B$.*

4. $\mathbb{A} := \left\{ Z \in L^1(\mathbb{P};\mathbb{R}) \mid \mathbb{E}[Z] \geq B \right\}.$

As $\mathcal{C}$ is a convex cone containing $\mathbb{R}^N$, $\mathbf{Y}+\delta \in \mathcal{C}$ for every $\mathbf{Y} \in \mathcal{C}$ and any deterministic $\delta \in \mathbb{R}^N$.
Under Assumption 1, a systemic risk measure of the form (20) can be written as

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C},\, \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] \geq B \right\}. \tag{22}$$

Note that there is no loss of generality in assuming that $u_n(0) = 0$ (simply replace B with $B - \sum_{n=1}^N u_n(0)$), and that a natural selection for B is $B := \sum_{n=1}^N u_n(0)$. In this case $\rho(\mathbf{0}) \leq 0$. The proof of the following proposition, which exploits the behavior of u_n at $-\infty$, is given in Appendix A.1.

Proposition 1 *For all* $\mathbf{X} \in \mathcal{L}$ *we have* $\rho(\mathbf{X}) > -\infty$.

The *domain* of ρ is defined by

$$dom(\rho) := \{\mathbf{X} \in \mathcal{L} \mid \rho(\mathbf{X}) < +\infty\}.$$

Proposition 2 *The map* $\rho : \mathcal{L} \to \mathbb{R} \cup \{+\infty\}$ *in* (22) *is convex, monotone decreasing and satisfies*

$$\{\mathbf{X} \in \mathcal{L} \mid \mathbb{E}[\Lambda(\mathbf{X})] > -\infty\} \subset dom(\rho),$$

and so $(L^\infty(\mathbb{R}^N) \cap \mathcal{L}) \subset dom(\rho)$. *If* $\mathbb{E}[u_n(X^n)] > -\infty$ *for each* n *and all* $\mathbf{X} \in \mathcal{L}$, *then*

$$\rho(\mathbf{X}) = \rho^{=}(\mathbf{X}) := \inf \left\{ \sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C},\, \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] = B \right\},\ \mathbf{X} \in dom(\rho).$$

If, in addition, for each n, $u_n : \mathbb{R} \to \mathbb{R}$ *is strictly concave and there exists an optimal allocation* $\mathbf{Y_X} = \{Y^n_{\mathbf{X}}\}_n \in \mathcal{C}_0 \cap \mathcal{L}$ *of* $\rho(\mathbf{X})$, *then it is unique.*

Proof By Proposition 1, we know that $\rho : \mathcal{L} \to \mathbb{R} \cup \{+\infty\}$ and then convexity and monotonicity are straightforward. Let $\mathbf{X} \in \mathcal{L}$ such that $\mathbb{E}[\Lambda(\mathbf{X})] > -\infty$, $m \in \mathbb{R}$ and set $\mathbf{1} = (1, \cdots, 1)$. Then $\mathbf{X} + m\mathbf{1} \uparrow +\infty$ $\mathbb{P}$-a.s. if $m \to +\infty$. As $\mathbb{E}[\Lambda(\mathbf{X})] > -\infty$, we have that $\mathbb{E}[\Lambda(\mathbf{X} + m\mathbf{1})] > -\infty$ for $m > 0$, and by monotone convergence it follows that $\mathbb{E}[\Lambda(\mathbf{X} + m\mathbf{1})] \uparrow \Lambda(+\infty) > B$. Since $\mathbb{R}^N \subseteq \mathcal{C}$, then $m\mathbf{1} \in \mathcal{C}$ and $\{\mathbf{Y} \in \mathcal{C},\, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}\} \neq \emptyset$, so that $\rho(\mathbf{X}) < +\infty$.

We now claim that if $\mathbb{E}[\Lambda(\mathbf{X} + \mathbf{Y})] > B$ then $\mathbf{Y} \in \mathcal{C}$ can not be optimal:

$$\mathbf{Y} \in \mathcal{C} \text{ and } \mathbb{E}[\Lambda(\mathbf{X} + \mathbf{Y})] > B \Longrightarrow \sum_{n=1}^{N} Y^n > \rho^{=}(\mathbf{X}). \tag{23}$$

Indeed, the continuity of u_n and $\mathbb{E}[u_n(Z^n)] > -\infty$ for all $\mathbf{Z} \in \mathcal{L}$ imply the existence of $\delta \in \mathbb{R}^N_+$, $\delta \neq \mathbf{0}$, such that $\mathbb{E}[\Lambda(\mathbf{X} + \mathbf{Y} - \delta)] = B$ and so, as $\mathbf{Y} - \delta \in \mathcal{C}$, $\rho^{=}(\mathbf{X}) \leq \sum_{n=1}^N (Y^n - \delta^n) < \sum_{n=1}^N Y^n$. This readily implies $\rho(\mathbf{X}) = \rho^{=}(\mathbf{X})$, otherwise if $\rho(\mathbf{X}) < \rho^{=}(\mathbf{X})$, then by definition of $\rho(\mathbf{X})$, there would exist $\varepsilon > 0$ and $\mathbf{Y} \in \mathcal{C}$ satisfying $\mathbb{E}[\Lambda(\mathbf{X} + \mathbf{Y})] > B$ and $\sum_{n=1}^N Y^n \leq \rho(\mathbf{X}) + \varepsilon < \rho^{=}(\mathbf{X})$, which contradicts (23).

We now show uniqueness by contradiction. Suppose that $\rho(\mathbf{X})$ is attained by two distinct $\mathbf{Y}_1 \in \mathcal{C}$ and $\mathbf{Y}_2 \in \mathcal{C}$, so that $\mathbb{P}(\mathbf{Y}_1^j \neq \mathbf{Y}_2^j) > 0$ for some j. Then we have

$$\rho(\mathbf{X}) = \sum_{n=1}^{N} Y_1^n = \sum_{n=1}^{N} Y_2^n \quad \text{and} \quad \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y_k^n)\right] = B \text{ for } k = 1, 2.$$

For $\lambda \in [0, 1]$ set $\mathbf{Y}_\lambda := \lambda \mathbf{Y}_1 + (1-\lambda)\mathbf{Y}_2$. Then $\mathbf{Y}_\lambda \in \mathcal{C}$, as $\mathcal{C}$ is convex. This implies

$$\sum_{n=1}^{N} Y_\lambda^n = \lambda \sum_{n=1}^{N} Y_1^n + (1-\lambda) \sum_{n=1}^{N} Y_2^n = \rho(\mathbf{X}), \, \forall \lambda \in [0, 1],$$

and for $\lambda \in (0, 1)$

$$B = \lambda \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y_1^n)\right] + (1-\lambda)\mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y_2^n)\right] <$$
$$< \mathbb{E}\left[\sum_{n=1}^{N} u_n(\lambda X^n + \lambda Y_1^n + (1-\lambda)X^n + (1-\lambda)Y_2^n)\right] = \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y_\lambda^n)\right],$$

where we used that u_j is strictly concave and $\mathbb{P}(Y_1^j \neq Y_2^j) > 0$. This is a contradiction with $\rho(\mathbf{X}) = \rho^{=}(\mathbf{X})$ and (23).

Remark 1 (Extension to $L^1(\mathbf{Q})$) The extension of Proposition 2 to the case where $\mathbf{Y} \in \mathcal{C}_0 \cap L^1(\mathbf{Q})$ (instead of $\mathbf{Y} \in \mathcal{C}_0 \cap \mathcal{L}$) would a priori require Assumption (85), as in this case we can not guarantee $\mathbb{E}[u_n(Z^n)] > -\infty$ for all $\mathbf{Z} \in L^1(\mathbf{Q})$, see Remark 14. However, we will obtain uniqueness also for $\mathbf{Y} \in \mathcal{C}_0 \cap L^1(\mathbf{Q})$, based on the uniqueness of the solution to $\rho^{\mathbf{Q_X}}(\mathbf{X})$, see Remark 5, and on Remark 11.

3.2 Orlicz setting

We now study some important properties of systemic risk measures of the form (22) in a Orlicz space setting, see [38] for further details on Orlicz spaces. This presents several advantages. From a mathematical point of view, it is a more general setting than L^∞, but at the same time it simplifies the analysis, since the topology is order continuous and there are no singular elements in the dual space. Furthermore, it has been shown in [10] that the Orlicz setting is the natural one to embed utility maximization problems, as the natural integrability condition $\mathbb{E}[u(X)] > -\infty$ is implied by $\mathbb{E}[\phi(X)] < +\infty$.

Let $u : \mathbb{R} \to \mathbb{R}$ be a concave and increasing function satisfying $\lim_{x \to -\infty} \frac{u_n(x)}{x} = +\infty$. Consider $\phi(x) := -u(-|x|) + u(0)$. Then $\phi : \mathbb{R} \to [0, +\infty)$ is a strict Young function, i.e., it is finite valued, even and convex on $\mathbb{R}$ with $\phi(0) = 0$ and $\lim_{x \to +\infty} \frac{\phi(x)}{x} = +\infty$. The Orlicz space L^ϕ and Orlicz Heart M^ϕ are respectively defined by

$$L^\phi := \left\{ X \in L^0(\mathbb{R}) \mid \mathbb{E}[\phi(\alpha X)] < +\infty \text{ for some } \alpha > 0 \right\}, \tag{24}$$

$$M^\phi := \left\{ X \in L^0(\mathbb{R}) \mid \mathbb{E}[\phi(\alpha X)] < +\infty \text{ for all } \alpha > 0 \right\}, \tag{25}$$

and they are Banach spaces when endowed with the Luxemburg norm. The topological dual of M^{ϕ} is the Orlicz space L^{ϕ^*}, where the convex conjugate ϕ^* of ϕ, defined by

$$\phi^*(y) := \sup_{x\in\mathbb{R}} \{xy - \phi(x)\}, \; y \in \mathbb{R},$$

is also a strict Young function. Note that

$$\mathbb{E}[u(X)] > -\infty \text{ if } \mathbb{E}[\phi(X)] < +\infty. \tag{26}$$

Remark 2 It is well known that $L^{\infty}(\mathbb{P};\mathbb{R}) \subseteq M^{\phi} \subseteq L^{\phi} \subseteq L^1(\mathbb{P};\mathbb{R})$. In addition, from the Fenchel inequality $xy \leq \phi(x) + \phi^*(y)$ we obtain

$$(\alpha|X|)\left(\lambda\frac{dQ}{d\mathbb{P}}\right) \leq \phi(\alpha|X|) + \phi^*\left(\lambda\frac{dQ}{d\mathbb{P}}\right),$$

and we immediately deduce that $\frac{dQ}{d\mathbb{P}} \in L^{\phi^*}$ implies $L^{\phi} \subseteq L^1(Q;\mathbb{R})$.

Given the utility functions $u_1, \cdots, u_N : \mathbb{R} \to \mathbb{R}$ satisfying Assumption 1, with associated Young functions $\phi_1, \cdots, \phi_N$, we define

$$M^{\Phi} := M^{\phi_1} \times \cdots \times M^{\phi_N}, \quad L^{\Phi} := L^{\phi_1} \times \cdots \times L^{\phi_N} \tag{27}$$

and consider

$$\mathcal{L} = M^{\Phi},$$

i.e. $\rho : M^{\Phi} \to \mathbb{R}\cup\{+\infty\}$. Under Assumption 1, M^{Φ} coincides with the domain of ρ and the systemic risk measures of the form (22) have good properties if restricted to M^{Φ}. Recall also that

$$\mathcal{C} = \mathcal{C}_0 \cap M^{\Phi}.$$

Proposition 3 *The map $\rho : M^{\Phi} \to \mathbb{R}\cup\{+\infty\}$ defined in (22) is finitely valued, monotone decreasing, convex, continuous and subdifferentiable on the Orlicz Heart $M^{\Phi} = dom(\rho)$.*

Proof The equality $M^{\Phi} = dom(\rho)$, so that $\rho : M^{\Phi} \to \mathbb{R}$, follows from Proposition 2, the definition of M^{Φ} in (25), and (26). The remaining properties are a consequence of Proposition 2, Theorem 9 in Appendix and the fact that M^{Φ} is a Banach space.

In the following section we will start presenting our results in the Orlicz space setting. When the utility functions u_n are of the exponential type, the Orlicz Heart M^{Φ} is sufficiently large and it contains the optimal allocation $\mathbf{Y}_{\mathbf{X}}$ to $\rho(\mathbf{X})$, see Section 6. This of course also happens in the case of general utility functions on a finite probability space. However, for arbitrary utility functions and a general probability space, the existence technical results established in Section 5 require a larger space of integrable random variables.

4 Main results

4.1 Dual representation of ρ

We now investigate the dual representation of systemic risk measures of the form (22). When $\mathbf{Z} \in M^{\Phi}$ and $\xi \in L^{\Phi^*}$, we set $\mathbb{E}[\xi\mathbf{Z}] := \sum_{n=1}^{N} \mathbb{E}[\xi^n Z^n]$ and, for $\frac{d\mathbf{Q}}{d\mathbb{P}} \in L_+^{\Phi^*}$, $\mathbb{E}_{\mathbf{Q}}[\mathbf{Z}] = \sum_{n=1}^{N} \mathbb{E}_{Q^n}[Z^n]$. We will frequently identify the density $\frac{dQ}{d\mathbb{P}}$ with the associated probability measure $Q \ll \mathbb{P}$.

Proposition 4 *For any* $\mathbf{X} \in M^{\Phi}$,

$$\rho(\mathbf{X}) = \max_{\mathbf{Q}\in\mathcal{D}} \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-X^n] - \alpha_{\Lambda,B}(\mathbf{Q}) \right\}, \tag{28}$$

where the penalty function is given by

$$\alpha_{\Lambda,B}(\mathbf{Q}) := \sup_{\mathbf{Z}\in\mathcal{A}} \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-Z^n] \right\}, \ \mathbf{Q} \in \mathcal{D}, \tag{29}$$

with $\mathcal{A} := \left\{ \mathbf{Z} \in M^{\Phi} \mid \sum_{n=1}^{N} \mathbb{E}[u_n(Z^n)] \geq B \right\}$ *and*

$$\mathcal{D} := dom(\alpha_{\Lambda,B}) \cap \left\{ \frac{d\mathbf{Q}}{d\mathbb{P}} \in L_+^{\Phi^*} \mid Q^n(\Omega) = 1 \ \forall n \ and \ \sum_{n=1}^{N} (\mathbb{E}_{Q^n}[Y^n] - Y^n) \leq 0 \ for \ all \ \mathbf{Y} \in \mathcal{C}_0 \cap M^{\Phi} \right\}. \tag{30}$$

(i) Suppose that for some $i, j \in \{1, \cdots, N\}$, $i \neq j$, *we have* $\pm(e_i 1_A - e_j 1_A) \in \mathcal{C}$ *for all* $A \in \mathcal{F}$. *Then*

$$\mathcal{D} = dom(\alpha_{\Lambda,B}) \cap \left\{ \frac{d\mathbf{Q}}{d\mathbb{P}} \in L_+^{\Phi^*} \mid Q^n(\Omega) = 1 \ \forall n, \ Q^i = Q^j \ and \ \sum_{n=1}^{N} (\mathbb{E}_{Q^n}[Y^n] - Y^n) \leq 0 \ for \ all \ \mathbf{Y} \in \mathcal{C} \right\}.$$

(ii) Suppose that $\pm(e_i 1_A - e_j 1_A) \in \mathcal{C}$ *for all* i, j *and all* $A \in \mathcal{F}$. *Then*

$$\mathcal{D} = dom(\alpha_{\Lambda,B}) \cap \left\{ \frac{d\mathbf{Q}}{d\mathbb{P}} \in L_+^{\Phi^*} \mid Q^n(\Omega) = 1, \ Q^n = Q, \ \forall n \right\}.$$

Proof The dual representation (28) is a consequence of Proposition 3, Theorem 9 and of Propositions 3.9 and 3.11 in [29], taking into consideration that $\mathcal{C}$ is a convex cone, the dual space of the Orlicz Heart M^{Φ} is the Orlicz space L^{Φ^*} and $M^{\Phi} = dom(\rho)$. Note that from Theorem 9 we know that the dual elements $\xi \in L_+^{\Phi^*}$ are positive but a priori not normalized. However, we obtain $\mathbb{E}[\xi^n] = 1$ by taking as $\mathbf{Y} = \pm e_j \in \mathbb{R}^N$, and using $\sum_{n=1}^{N}(\xi^n(Y^n) - Y^n) \leq 0$ for all $\mathbf{Y} \in \mathcal{C}$, so that $\xi^j(1) - 1 \leq 0$ and $\xi^j(-1) + 1 \leq 0$ imply $\xi^j(1) = 1$. This shows the form of the domain $\mathcal{D}$ in (30). Furthermore:

(i) Take $\mathbf{Y} := e_i 1_A - e_j 1_A \in \mathcal{C}$. From $\sum_{n=1}^{N}(Q^n(Y^n) - Y^n) \leq 0$ we obtain $Q^i(1_A) - 1_A + Q^j(-1_A) + 1_A \leq 0$, i.e., $Q^i(A) - Q^j(A) \leq 0$ and similarly taking $\mathbf{Y} := -e_i 1_A + e_j 1_A \in \mathcal{C}$, we get $Q^j(A) - Q^i(A) \leq 0$.

(ii) From (i), we obtain $Q^i = Q^j$. In addition, we get $\sum_{n=1}^{N}(\mathbb{E}_Q[Y^n] - Y^n) = \mathbb{E}_Q[\sum_{n=1}^{N} Y^n] - \sum_{n=1}^{N} Y^n) = 0$, as $\sum_{n=1}^{N} Y^n \in \mathbb{R}$.

Proposition 4 guarantees the existence of a maximizer $\mathbf{Q_X}$ to the dual problem (28) and that $\alpha_{\Lambda,B}(\mathbf{Q_X}) < +\infty$. Uniqueness will be proved in Corollary 4.

Definition 3 Let $\mathbf{X} \in M^{\Phi}$. An optimal solution of the dual problem (28) is a vector of probability measures $\mathbf{Q_X} = (Q^1_{\mathbf{X}}, \cdots, Q^N_{\mathbf{X}})$ verifying $\frac{d\mathbf{Q_X}}{d\mathbb{P}} \in \mathcal{D}$ and

$$\rho(\mathbf{X}) = \sum_{n=1}^{N} \mathbb{E}_{Q^n_{\mathbf{X}}}[-X^n] - \alpha_{\Lambda,B}(\mathbf{Q_X}). \tag{31}$$

The probability measures $\mathbf{Q}$ having density in $\mathcal{D}$ could be viewed, in the systemic N-dimensional one period setting, as the counterpart of the notion of ($\mathbb{P}$-absolutely continuous) martingale measures. Indeed, as $\mathbf{Y} \in \mathcal{C}_0 \subseteq \mathcal{C}_{\mathbb{R}}$, $\sum_{n=1}^{N} Y^n \in \mathbb{R}$ is the total amount to be allocated to the N institutions and then the total cost or value $\sum_{n=1}^{N} \mathbb{E}_{Q^n}[Y^n]$ should at most be equal to $\sum_{n=1}^{N} Y^n$, for any "fair" valuation operator $\mathbf{Q}$, that is $\frac{d\mathbf{Q}}{d\mathbb{P}} \in \mathcal{D}$.

There exists a simple relation among ρ_B, $\rho_B^{\mathbf{Q}}$ and $\alpha_{\Lambda,B}(\mathbf{Q})$ defined in (22), (8), and (29), respectively.

Proposition 5 *We have*

$$\rho_B^{\mathbf{Q}}(\mathbf{X}) = -\sum_{n=1}^{N} \mathbb{E}_{Q^n}[X^n] - \alpha_{\Lambda,B}(\mathbf{Q}), \tag{32}$$

and

$$\rho_B^{\mathbf{Q_X}}(\mathbf{X}) = \rho_B(\mathbf{X}), \tag{33}$$

where $\mathbf{Q_X}$ is an optimal solution of the dual problem (28).

Proof We have

$$\begin{aligned}
-\alpha_{\Lambda,B}(\mathbf{Q}) &= \inf\left\{\sum_{n=1}^{N} \mathbb{E}_{Q^n}[Z^n] \mid \mathbf{Z} \in M^{\Phi} \text{ and } \sum_{n=1}^{N} \mathbb{E}[u_n(Z^n)] \geq B\right\} \\
&= \inf\left\{\sum_{n=1}^{N} \mathbb{E}_{Q^n}[X^n + Y^n] \mid \mathbf{Y} \in M^{\Phi} \text{ and } \sum_{n=1}^{N} \mathbb{E}[u_n(X^n + Y^n)] \geq B\right\} \\
&= \sum_{n=1}^{N} \mathbb{E}_{Q^n}[X^n] + \rho_B^{\mathbf{Q}}(\mathbf{X}),
\end{aligned}$$

which proves (32). Then from (32) and (31) we deduce

$$\rho_B^{\mathbf{Q_X}}(\mathbf{X}) = -\sum_{n=1}^{N} \mathbb{E}_{Q^n_{\mathbf{X}}}[X^n] - \alpha_{\Lambda,B}(\mathbf{Q_X}) = \rho_B(\mathbf{X}).$$

Remark 3 If $\frac{d\mathbf{Q}}{d\mathbb{P}} \in \mathcal{D}$ then $\sum_{n=1}^{N} \mathbb{E}_{Q^n}[Y^n] \leq \sum_{n=1}^{N} Y^n$ for all $\mathbf{Y} \in \mathcal{C}$, so that

$$\begin{aligned}
\rho_B^{\mathbf{Q}}(\mathbf{X}) &= \inf\left\{\sum_{n=1}^{N} \mathbb{E}_{Q^n}[Y^n] \mid \mathbf{Y} \in M^{\Phi} \text{ and } \sum_{n=1}^{N} \mathbb{E}[u_n(X^n + Y^n)] \geq B\right\} \\
&\leq \inf\left\{\sum_{n=1}^{N} \mathbb{E}_{Q^n}[Y^n] \mid \mathbf{Y} \in \mathcal{C} \text{ and } \sum_{n=1}^{N} \mathbb{E}[u_n(X^n + Y^n)] \geq B\right\} \\
&\leq \inf\left\{\sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C} \text{ and } \sum_{n=1}^{N} \mathbb{E}[u_n(X^n + Y^n)] \geq B\right\} = \rho_B(\mathbf{X}).
\end{aligned}$$

Then (33) shows that

$$\rho_B(\mathbf{X}) = \max_{\frac{d\mathbf{Q}}{d\mathbb{P}} \in \mathcal{D}} \rho_B^{\mathbf{Q}}(\mathbf{X}) = \rho_B^{\mathbf{Q_X}}(\mathbf{X}),$$

which means that ρ_B is the most conservative among those risk measures $\rho_B^{\mathbf{Q}}$ defined through fair valuation operators $\frac{d\mathbf{Q}}{d\mathbb{P}} \in \mathcal{D}$. In this respect, the probability measure $\mathbf{Q_X}$ plays, in the theory of systemic risk measure, an analogous role played by the minimax martingale measure in the theory of contingent claim valuation in incomplete markets, see [6] for details.

We now turn our attention to the uniqueness of the optimal solution to the problem (29). The proof employs the same arguments used in the proof of Proposition 2.

Lemma 1 *If each u_n is strictly concave and $\alpha_{\Lambda,B}(\mathbf{Q}) < +\infty$, then there exists at most one $\mathbf{Z} \in M^{\Phi}$ satisfying*

$$\alpha_{\Lambda,B}(\mathbf{Q}) = \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-Z^n] \quad \textit{and} \quad \sum_{n=1}^{N} \mathbb{E}[u_n(Z^n)] \geq B. \tag{34}$$

Proof Set

$$c^{=}(\mathbf{Q}) := \inf \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[Z^n] \mid \mathbf{Z} \in M^{\Phi}, \, \mathbb{E}[\Lambda(\mathbf{Z})] = B \right\}.$$

First we show that if $\mathbb{E}[\Lambda(\mathbf{Z})] > B$ then $\mathbf{Z} \in M^{\Phi}$ can not be optimal:

$$\mathbf{Z} \in M^{\Phi} \text{ and } \mathbb{E}[\Lambda(\mathbf{Z})] > B \Longrightarrow \sum_{n=1}^{N} \mathbb{E}_{Q^n}[Z^n] > c^{=}(\mathbf{Q}). \tag{35}$$

Indeed, the continuity of u_n, $\mathbb{E}[u_n(Z^n)] > -\infty$ for all $\mathbf{Z} \in M^{\Phi}$ and $\mathbb{E}[\Lambda(\mathbf{Z})] > B$ imply the existence of $\delta \in \mathbb{R}_+^N$, $\delta \neq \mathbf{0}$, such that $\mathbb{E}[\Lambda(\mathbf{Z} - \delta)] = B$ and therefore $c^{=}(\mathbf{Q}) \leq \sum_{n=1}^{N} \mathbb{E}_{Q^n}[Z^n - \delta^n] < \sum_{n=1}^{N} \mathbb{E}_{Q^n}[Z^n]$. Then

$$c(\mathbf{Q}) := -\alpha_{\Lambda,B}(\mathbf{Q}) = \inf \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[Z^n] \mid \mathbf{Z} \in M^{\Phi}, \, \mathbb{E}[\Lambda(\mathbf{Z})] \geq B \right\} = c^{=}(\mathbf{Q})$$

Indeed, $-\infty < c(\mathbf{Q}) \leq c^{=}(\mathbf{Q})$ and assume by contradiction that $c(\mathbf{Q}) < c^{=}(\mathbf{Q})$. By definition of $c(\mathbf{Q})$, there exist $\varepsilon > 0$ and $\mathbf{Z} \in M^{\Phi}$ such that $\mathbb{E}[\Lambda(\mathbf{Z})] > B$ and $\sum_{n=1}^{N} \mathbb{E}_{Q^n}[Z^n] \leq c(\mathbf{Q}) + \varepsilon < c^{=}(\mathbf{Q})$, which contradicts (35). For the uniqueness, let us suppose that $c(\mathbf{Q})$ is attained by two distinct $\mathbf{Z}_1 \in M^{\Phi}$ and $\mathbf{Z}_2 \in M^{\Phi}$, so that $\mathbb{P}(Z_1^j \neq Z_2^j) > 0$ for some j. Then we have

$$c(\mathbf{Q}) = \sum_{n=1}^{N} \mathbb{E}_{Q^n}[Z_1^n] = \sum_{n=1}^{N} \mathbb{E}_{Q^n}[Z_2^n] \quad \text{and} \quad \sum_{n=1}^{N} \mathbb{E}\left[u_n(Z_k^n)\right] \geq B \text{ for } k = 1, 2.$$

For $\lambda \in [0,1]$ set $\mathbf{Z}_\lambda := \lambda \mathbf{Z}_1 + (1-\lambda)\mathbf{Z}_2 \in M^{\Phi}$. Then $\sum_{n=1}^{N} \mathbb{E}_{Q^n}[Z_\lambda^n] = c(\mathbf{Q})$, $\forall \lambda \in [0,1]$, and for $\lambda \in (0,1)$

$$B \leq \lambda \mathbb{E}\left[\sum_{n=1}^{N} u_n(Z_1^n)\right] + (1-\lambda)\mathbb{E}\left[\sum_{n=1}^{N} u_n(Z_2^n)\right] < \mathbb{E}\left[\sum_{n=1}^{N} u_n(Z_\lambda^n)\right],$$

where we used the strict concavity of u_j and $\mathbb{P}(Z_1^j \neq Z_2^j) > 0$. This is a contradiction with $c(\mathbf{Q}) = c^{=}(\mathbf{Q})$ and (35).

Remark 4 (Extension to $L^1(\mathbf{Q})$) It is possible to extend Lemma 1 to the case where $\mathbf{Z} \in L^1(\mathbf{Q})$ by applying the simple argument stated in Remark 14. So, if each u_n is strictly concave, there exists at most one $\mathbf{Z} \in L^1(\mathbf{Q})$ satisfying (34).

Remark 5 (Uniqueness) Suppose that each u_n is strictly concave. The existence of an optimizer $\mathbf{Y_Q}$ for the problem $\rho_B^{\mathbf{Q}}(\mathbf{X})$ will be proved in Section 5.2. The uniqueness of $\mathbf{Z} \in M^{\Phi}$ (or $\mathbf{Z} \in L^1(\mathbf{Q})$) for $\alpha_{\Lambda,B}(\mathbf{Q})$, shown in Lemma 1, also implies the uniqueness of the optimizer $\mathbf{Y_Q} \in M^{\Phi}$ (or $\mathbf{Y_Q} \in L^1(\mathbf{Q})$) for $\rho_B^{\mathbf{Q}}(\mathbf{X})$, as $\rho_B^{\mathbf{Q}}(\mathbf{X}) = -\sum_{n=1}^N \mathbb{E}_{Q^n}[X^n] - \alpha_{\Lambda,B}(\mathbf{Q})$, thanks to Proposition 5. With a similar proof to the one of Lemma 1, we may replace the inequality with an equality sign in the budget constraint in the definition of $\rho_B^{\mathbf{Q}}(\mathbf{X})$.

Example 2 Consider the grouping Example 1. As $\mathcal{C}^{(\mathbf{n})}$ is a linear space containing $\mathbb{R}^N$, the dual representation (28) applies. In addition in each group we have $\pm(e_i 1_A - e_j 1_A) \in \mathcal{C}^{(\mathbf{n})}$ for all i, j *in the same group* and for all $A \in \mathcal{F}$. Therefore, in each group the components Q^i, $i \in I_m$, of the dual elements are all the same, i.e., $Q^i = Q^j$, for all $i, j \in I_m$, and the representation (28) becomes

$$\rho(\mathbf{X}) = \max_{\mathbf{Q} \in \mathcal{D}} \left\{ \sum_{m=1}^h \sum_{k \in I_m} \left(\mathbb{E}_{Q^m}[-X^k] \right) - \alpha_{\Lambda,B}(\mathbf{Q}) \right\} = \max_{\mathbf{Q} \in \mathcal{D}} \left\{ \sum_{m=1}^h \mathbb{E}_{Q^m}[-\overline{X}_m] - \alpha_{\Lambda,B}(\mathbf{Q}) \right\}, \tag{36}$$

with

$$\mathcal{D} := dom(\alpha_{\Lambda,B}) \cap \left\{ \frac{d\mathbf{Q}}{d\mathbb{P}} \in L_+^{\Phi^*} \mid Q^i = Q^j \, \forall i, j \in I_m, Q^i(\Omega) = 1 \right\}$$

and $\overline{X}_m := \sum_{k \in I_m} X^k$. Indeed,

$$\sum_{n=1}^N (\mathbb{E}_{Q^n}[Y^n] - Y^n) = \sum_{m=1}^h \sum_{k \in I_m} (\mathbb{E}_{Q^m}[Y^k] - Y^k) = \sum_{m=1}^h \left(\mathbb{E}_{Q^m}\left[\sum_{k \in I_m} Y^k \right] - \sum_{k \in I_m} Y^k \right) = 0,$$

as $\sum_{k \in I_m} Y^k \in \mathbb{R}$. If we have only one single group, all components of a dual element $\mathbf{Q} \in \mathcal{D}$ are the same.

Remark 6 Consider the grouping Example 1. Let $\mathbf{Q} = (Q^1, \cdots, Q^n)_{n=1,\cdots,N}$ be a vector of probability measures with the property that in each group the components Q^i, $i \in I_m$, satisfy $Q^i = Q^m$ for all $i \in I_m$. Then $(\mathbb{E}_{\mathbf{Q}_1}[Y_{\mathbf{X}}^1], ..., \mathbb{E}_{\mathbf{Q}_N}[Y_{\mathbf{X}}^N])$ is a systemic risk allocation as in Definition (1), i.e.,

$$\rho(\mathbf{X}) = \sum_{n=1}^N \mathbb{E}_{Q^n}[Y_{\mathbf{X}}^n] = \sum_{m=1}^h \sum_{k \in I_m} \mathbb{E}_{Q^m}[Y_{\mathbf{X}}^k] = \sum_{m=1}^h d_m.$$

Indeed, for *such* a vector $(Q^1, \cdots, Q^m)$ of probability measures we have

$$\sum_{k \in I_m} \mathbb{E}_{Q^m}[Y_{\mathbf{X}}^k] = \mathbb{E}_{Q^m}\left[\sum_{k \in I_m} Y_{\mathbf{X}}^k \right] = \mathbb{E}_{Q^m}[d_m] = d_m.$$

Returning to our general setting, from now on, we work under the following two assumptions, with the understanding that Assumption 3 will hold with respect to the probability measures ($\mathbf{Q}$ or $\mathbf{Q_X}$) involved in the statements of the results.

Assumption 2 *In addition to Assumptions 1, we assume that for any $n = 1, \cdots, N$, $u_n : \mathbb{R} \to \mathbb{R}$ is increasing, strictly concave, differentiable and satisfies the Inada conditions*

$$u_n'(-\infty) := \lim_{x\to-\infty} u_n'(x) = +\infty,\ u_n'(+\infty) := \lim_{x\to+\infty} u_n'(x) = 0.$$

Some useful properties on the convex conjugate function $v_n(y) := \sup_{x\in\mathbb{R}} \{u_n(x) - xy\}$ are collected in Lemma 10. The following additional Assumption 3 is related to the Reasonable Asymptotic Elasticity condition on utility functions, which was introduced in [41]. This assumption, even though quite weak (see [8] Section 2.2), is fundamental to guarantee the existence of the optimal solution to classical utility maximization problems (see [41] and [8]).

Assumption 3 *For any $n = 1, \cdots, N$, v_n and $Q^n \ll \mathbb{P}$ satisfy*

$$\mathbb{E}\left[v_n\left(\frac{dQ^n}{d\mathbb{P}}\right)\right] < \infty \quad \textit{iff} \quad \mathbb{E}\left[v_n\left(\lambda\frac{dQ^n}{d\mathbb{P}}\right)\right] < \infty,\ \ \forall\lambda > 0.$$

From the Fenchel inequality

$$u_n(X^n) \le X^n\frac{dQ^n}{d\mathbb{P}} + v_n\left(\frac{dQ^n}{d\mathbb{P}}\right)\ \ \mathbb{P}\text{ a.s.}$$

we immediately deduce that if $X^n \in L^1(Q^n)$ and $\mathbb{E}\left[v_n\left(\frac{dQ^n}{d\mathbb{P}}\right)\right] < \infty$ then $\mathbb{E}\left[u_n\left(X^n\right)\right] < +\infty$.

Proposition 6 *When $\alpha_{\Lambda,B}(\mathbf{Q}) < +\infty$, then the penalty function in* (29) *can be written as*

$$\alpha_{\Lambda,B}(\mathbf{Q}) := \sup_{\mathbf{Z}\in\mathcal{A}}\left\{\sum_{n=1}^N \mathbb{E}_{Q^n}[-Z^n]\right\} = \inf_{\lambda>0}\left(-\frac{1}{\lambda}B + \frac{1}{\lambda}\sum_{n=1}^N \mathbb{E}\left[v_n\left(\lambda\frac{dQ^n}{d\mathbb{P}}\right)\right]\right), \tag{37}$$

and $\mathbb{E}\left[v_n\left(\lambda\frac{dQ^n}{d\mathbb{P}}\right)\right] < \infty$ for all n and all $\lambda > 0$.

Proof In Appendix A.

Proposition 7 *When $\alpha_{\Lambda,B}(\mathbf{Q}) < +\infty$, the infimum is attained in* (37)*, i.e.,*

$$\alpha_{\Lambda,B}(\mathbf{Q}) = \sum_{n=1}^N \mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}}v_n'\left(\lambda^*\frac{dQ^n}{d\mathbb{P}}\right)\right], \tag{38}$$

where $\lambda^* > 0$ is the unique solution of the equation[1]

$$-B + \sum_{n=1}^N \mathbb{E}\left[v_n\left(\lambda\frac{dQ^n}{d\mathbb{P}}\right)\right] - \lambda\sum_{n=1}^N \mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}}v_n'\left(\lambda\frac{dQ^n}{d\mathbb{P}}\right)\right] = 0. \tag{39}$$

[1] Note that λ^* will depend on B, $(u_n)_{n=1,\cdots,N}$ and $\left(\frac{dQ_n}{d\mathbb{P}}\right)_{n=1,\cdots,N}$.

Proof Set $\xi_n := \frac{dQ^n}{d\mathbb{P}} \geq 0$ a.s.. Recall from Lemma 10 that v_n is strictly convex with $v_n(+\infty) = +\infty$, $v_n(0^+) = u_n(+\infty)$, $\lim_{z\to+\infty} \frac{v_n(z)}{z} = +\infty$ because of Assumption 2, and v_n is continuously differentiable. As $u_n'(+\infty) = 0$ and $u_n'(-\infty) = +\infty$, we get $v_n'(0) = -\infty$ and $v_n'(+\infty) = +\infty$.
Set $\eta = \frac{1}{\lambda} \in (0, +\infty)$ and consider the differentiable function $F : (0, +\infty) \to \mathbb{R}$ defined by

$$F(\eta) := -B\eta + \eta \sum_{n=1}^{N} \mathbb{E}\left[v_n\left(\frac{1}{\eta}\xi_n\right)\right].$$

Then $\alpha_{\Lambda,B}(\xi) = \inf_{\eta>0} F(\eta)$ and (39) can be rewritten as

$$F'(\eta) = 0 \tag{40}$$

with

$$F'(\eta) = -B + \sum_{n=1}^{N} \mathbb{E}\left[v_n\left(\frac{1}{\eta}\xi_n\right)\right] - \frac{1}{\eta}\sum_{n=1}^{N} \mathbb{E}\left[\xi_n v_n'\left(\frac{1}{\eta}\xi_n\right)\right].$$

Note that if $\eta^* > 0$ is the solution to (40), then by replacing such η^* into $F(\eta)$ we immediately obtain (38).
Next, thanks to the integrability conditions provided by Lemma 9, we show the existence of the solution $\eta^* > 0$ of (40). First we consider $\eta \to +\infty$. Since $\sum_{n=1}^{N} v_n(0^+) = \sum_{n=1}^{N} u_n(+\infty) > B$ by Assumption 1, we have that

$$\liminf_{\eta\to+\infty} -B + \sum_{n=1}^{N} \mathbb{E}\left[v_n\left(\frac{1}{\eta}\xi_n\right)\right] > 0.$$

Moreover, $v_n'(0) = -\infty$ shows that

$$\liminf_{\eta\to+\infty} -\frac{1}{\eta}\sum_{n=1}^{N} \mathbb{E}\left[\xi_n v_n'\left(\frac{1}{\eta}\xi_n\right)\right] \geq 0.$$

Hence $\liminf_{\eta\to+\infty} F'(\eta) > 0$. We now look at $\eta \to 0$:

$$\begin{aligned}
\lim_{\eta\to 0} F'(\eta) &= -B + \lim_{\eta\to 0} \sum_{n=1}^{N} \mathbb{E}\left[v_n\left(\frac{1}{\eta}\xi_n\right)\right] - \frac{1}{\eta}\sum_{n=1}^{N} \mathbb{E}\left[\xi_n v_n'\left(\frac{1}{\eta}\xi_n\right)\right] \\
&= -B + \lim_{t\to+\infty} \sum_{n=1}^{N} \mathbb{E}\left[v_n\left(t\xi_n\right)\right] - t\sum_{n=1}^{N} \mathbb{E}\left[\xi_n v_n'\left(t\xi_n\right)\right] \\
&= -B + \sum_{n=1}^{N} \lim_{t\to+\infty} \mathbb{E}\left[v_n\left(t\xi_n\right) - t\xi_n v_n'\left(t\xi_n\right)\right].
\end{aligned}$$

The convexity of v_n implies that for any fixed $z_0 > 0$ and $z > z_0$

$$v_n(z) - v_n(z_0) \leq v_n'(z)(z - z_0).$$

From $\lim_{z\to+\infty} \frac{v(z)}{z} = +\infty$, $v_n'(z) \to +\infty$ as $z \to +\infty$ and

$$v_n(z) - zv_n'(z) \leq v_n(z_0) - z_0 v_n'(z) \downarrow -\infty \text{ as } z \to +\infty,$$

we have by monotone convergence

$$\lim_{t\to+\infty}\mathbb{E}\left[v_n\left(t\xi_n\right)-t\xi_n v_n'\left(t\xi_n\right)\right]=-\infty,$$

so that $\liminf_{\eta\to 0}F'(\eta)=-\infty$. By the continuity of F' we obtain the existence of the solution $\eta^*>0$ for (40). Uniqueness follows from the strict convexity of F.

Example 3 Let $\Lambda=\sum_{n=1}^N u_n$ with $u_n:\mathbb{R}\to\mathbb{R}$, $u_n(x)=-e^{-\alpha_n x}$, $\alpha_n>0$, for each n, and let $B<0$. Then, $v_n'(y)=\frac{1}{\alpha_n}\ln(\frac{y}{\alpha_n})$. From the first order condition (39) we obtain that the minimizer is $\lambda^*=-\frac{B}{\beta}$, with $\beta:=\sum_{n=1}^N\frac{1}{\alpha_n}$. Therefore, from (38) we have

$$\alpha_{\Lambda,B}(\mathbf{Q})=\sum_{n=1}^N\mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}}v_n'\left(\lambda^*\frac{dQ^n}{d\mathbb{P}}\right)\right]=\sum_{n=1}^N\frac{1}{\alpha_n}\left(H\left(Q^n,\mathbb{P}\right)+\ln\left(-\frac{B}{\beta\alpha_n}\right)\right),\tag{41}$$

where $H\left(Q^n,\mathbb{P}\right):=\mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}}\ln\left(\frac{dQ^n}{d\mathbb{P}}\right)\right]$ is the relative entropy.

4.2 Fairness in the details

We now turn to the details of the introductory Section 2 and establish important relations between primal problems (4) and (5), and problems (8) and (9).

Note that in this section, we do not assume the existence of an optimizer for problems (4) or (5). We work under Assumptions 2 and 3.
Let $A\in\mathbb{R}$, $B\in\mathbb{R}$. As u_n is increasing, in both problems (4) and (5) we may replace the inequality in the constraints with an equality, and due to strict concavity the solution, if it exists, is unique (see Proposition 2 and Remark 5). Recall that under Assumptions 1, $\mathcal{C}$ is a convex cone and therefore, if $\mathbf{Y}\in\mathcal{C}$, then $\mathbf{Y}+\delta\in\mathcal{C}$ for every deterministic $\delta\in\mathbb{R}^N$.

Proposition 8 *$B=\pi_A(\mathbf{X})$ if and only if $A=\rho_B(\mathbf{X})$, and in this case the unique optimal solution, if it exists, is the same for the two problems $\pi_A(\mathbf{X})$ and $\rho_B(\mathbf{X})$.*

Proof $\Leftarrow$) Let $A=\rho_B(\mathbf{X})$ and suppose first that $\pi_A(\mathbf{X})>B$. Then there must exist $\tilde{\mathbf{Y}}\in\mathcal{C}$ such that $\sum_{n=1}^N\tilde{Y}^n\le A$ and $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+\tilde{Y}^n)\right]>B$. By continuity of u_n, then, there exists $\varepsilon>0$ and $\widehat{\mathbf{Y}}:=\tilde{\mathbf{Y}}-\varepsilon\mathbf{1}$ such that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+\widehat{Y}^n)\right]\ge B$ and $\sum_{n=1}^N\widehat{Y}^n<A$. This is in contradiction with $A=\rho_B(\mathbf{X})$.

Suppose now that $\pi_A(\mathbf{X})<B$. Then there exists $\delta>0$ such that

$$\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n)\right]\le B-\delta,$$

for all $\mathbf{Y}\in\mathcal{C}$ such that $\sum_{n=1}^N Y^n\le A$. As $A=\rho_B(\mathbf{X})$, for all $\varepsilon>0$, there exists $\mathbf{Y}_\varepsilon\in\mathcal{C}$ such that $\sum_{n=1}^N Y_\varepsilon^n\le A+\varepsilon$ and $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y_\varepsilon^n)\right]\ge B$. For any $\eta\ge\varepsilon\ge\sum_{n=1}^N Y_\varepsilon^n-A$, we get $\sum_{n=1}^N(Y_\varepsilon^n-\frac{\eta}{N})\le A+\varepsilon-\eta\le A$. By continuity of u_n, we may select $\varepsilon>0$ and $\eta\ge\varepsilon$ small enough so that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y_\varepsilon^n-\frac{\eta}{N})\right]>B-\delta$. As $\widehat{\mathbf{Y}}:=(Y_\varepsilon^n-\frac{\eta}{N})_n\in\mathcal{C}$, we obtain a contradiction.

Suppose that there exists $\mathbf{Y} \in \mathcal{C}$ that is the optimal solution of problem (4). As $A := \rho_B(\mathbf{X})$, then $\sum_{n=1}^N Y^n = A$ and the constraint in problem (5) is fulfilled for $\mathbf{Y}$. Hence, $B = \pi_A(\mathbf{X}) \geq \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \geq B$ and we deduce that $\mathbf{Y}$ is an optimal solution of problem (5).

$\Rightarrow$) Let $B = \pi_A(\mathbf{X})$ and suppose first that $\rho_B(\mathbf{X}) < A$. Then, there must exist $\tilde{\mathbf{Y}} \in \mathcal{C}$ such that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + \widetilde{Y}^n)\right] \geq B$ and $\sum_{n=1}^N \widetilde{Y}^n < A$. Then, there exists $\varepsilon > 0$ and $\widehat{\mathbf{Y}} := \tilde{\mathbf{Y}} + \varepsilon \mathbf{1} \in \mathcal{C}$ such that $\sum_{n=1}^N \widehat{Y}^n \leq A$ and $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + \widehat{Y}^n)\right] > B$. This is in contradiction with $B = \pi_A(\mathbf{X})$.

Suppose now that $\rho_B(\mathbf{X}) > A$. Then, there exists $\delta > 0$ such that $\sum_{n=1}^N Y^n \geq A + \delta$ for all $\mathbf{Y} \in \mathcal{C}$ such that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \geq B$. As $B = \pi_A(\mathbf{X})$, for all $\varepsilon > 0$ there exists $\mathbf{Y}_\varepsilon \in \mathcal{C}$ such that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n_\varepsilon)\right] > B - \varepsilon$ and $\sum_{n=1}^N Y^n_\varepsilon \leq A$. Define $\eta_\varepsilon := \inf\left\{a > 0 : \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n_\varepsilon + \frac{a}{N})\right] \geq B\right\}$ and note that $\eta_\varepsilon \downarrow 0$ if $\varepsilon \downarrow 0$. Select $\varepsilon > 0$ such that $\eta_\varepsilon < \delta$. Then, for any $0 < \beta < \delta - \eta_\varepsilon$ we have

$$\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n_\varepsilon + \frac{\eta_\varepsilon + \beta}{N})\right] \geq B,$$

and $\sum_{n=1}^N (Y^n_\varepsilon + \frac{\eta_\varepsilon + \beta}{N}) \leq A + \eta_\varepsilon + \beta < A + \delta$. As $(Y^n_\varepsilon + \frac{\eta_\varepsilon + \beta}{N}) \in \mathcal{C}$, we obtain a contradiction.

Suppose that there exists $\mathbf{Y} \in \mathcal{C}$ that is the optimal solution of problem (5) and set $B := \pi_A(\mathbf{X})$. Then $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] = B$ and the constraint in problem (4) is fulfilled for $\mathbf{Y}$. Hence, $A = \rho_B(\mathbf{X}) \leq \sum_{n=1}^N Y^n \leq A$ and we deduce that $\mathbf{Y}$ is an optimal solution of problem (4). As $\rho_B(\mathbf{X})$ admits at most one solution by Proposition 2, the same must be true for $\pi_A(\mathbf{X})$.

Remark 7 (Extension to $L^1(\mathbf{Q})$) The extension of Proposition 8 to the case where $\mathbf{Y} \in \mathcal{C}_0 \cap L^1(\mathbf{Q})$ requires Assumption (85), see Remark 14.

Now consider the situation where a valuation operator $\mathbf{Q} = (Q^1, \cdots, Q^N)$ such that $\frac{d\mathbf{Q}}{d\mathbb{P}} \in L^{\Phi^*}$ is given for the system. Note that $\rho^{\mathbf{Q}}_B(\mathbf{X}) < +\infty$ and $\pi^{\mathbf{Q}}_A(\mathbf{X}) > -\infty$. Then, similarly as in Proposition 8, we obtain

Proposition 9 $B = \pi^{\mathbf{Q}}_A(\mathbf{X}) < +\infty$ *if and only if* $A = \rho^{\mathbf{Q}}_B(\mathbf{X}) > -\infty$*, and the two problems have the same optimal solution* $\mathbf{Y}_{\mathbf{Q}}$.

Proof $\Leftarrow$) Let $\mathbf{Y}$ be an optimal solution of problem (8) and set $A := \rho^{\mathbf{Q}}_B(\mathbf{X}) > -\infty$. Then $\sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] = A$ and the constraint in problem (9) is fulfilled for $\mathbf{Y}$. Hence, $\pi^{\mathbf{Q}}_A(\mathbf{X}) \geq \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \geq B$. If $\pi^{\mathbf{Q}}_A(\mathbf{X}) > B$, then there exists $\widetilde{\mathbf{Y}} \in M^\Phi$ such that $\sum_{n=1}^N \mathbb{E}_{Q^n}\left[\widetilde{Y}^n\right] \leq A$ and $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + \widetilde{Y}^n)\right] > B$. Then, there exist $\varepsilon > 0$ and $\widehat{\mathbf{Y}} := \widetilde{\mathbf{Y}} - \varepsilon \mathbf{1} \in M^\Phi$ such that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + \widehat{Y}^n)\right] \geq B$ and $\sum_{n=1}^N \mathbb{E}_{Q^n}\left[\widehat{Y}^n\right] < A$. This is in contradiction with $A = \rho^{\mathbf{Q}}_B(\mathbf{X})$. Hence, $\pi^{\mathbf{Q}}_A(\mathbf{X}) = B$ and $\pi^{\mathbf{Q}}_A(\mathbf{X}) = \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right]$, and therefore, $\mathbf{Y}$ is an optimal solution of problem (9).

$\Rightarrow$) Let $\mathbf{Y}$ be an optimal solution of problem (9) and set $B := \pi_A^{\mathbf{Q}}(\mathbf{X}) < +\infty$. Then we have $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] = B$ and the constraint in problem (8) is fulfilled for $\mathbf{Y}$. Hence, $\rho_B^{\mathbf{Q}}(\mathbf{X}) \leq \sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] \leq A$. If $\rho_B^{\mathbf{Q}}(\mathbf{X}) < A$, then, there exists $\widetilde{\mathbf{Y}} \in M^\Phi$ such that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + \widetilde{Y}^n)\right] \geq B$ and $\sum_{n=1}^N \mathbb{E}_{Q^n}\left[\widetilde{Y}^n\right] < A$. Then, there exist $\varepsilon > 0$ and $\widehat{\mathbf{Y}} := \widetilde{\mathbf{Y}} + \varepsilon \mathbf{1} \in M^\Phi$ such that $\sum_{n=1}^N \mathbb{E}_{Q^n}\left[\widehat{Y}^n\right] \leq A$ and $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + \widehat{Y}^n)\right] > B$. This is in contradiction with $B = \pi_A^{\mathbf{Q}}(\mathbf{X})$. Hence, $\rho_B^{\mathbf{Q}}(\mathbf{X}) = A$ so that $\rho_B^{\mathbf{Q}}(\mathbf{X}) = \sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n]$ and $\mathbf{Y}$ is an optimal solution of problem (8).

Remark 8 (Extension to $L^1(\mathbf{Q})$) By applying the simple argument stated in Remark 14, Proposition 9 also holds when $\mathbf{Y} \in M^\Phi$ is replaced by $\mathbf{Y} \in L^1(\mathbf{Q})$.

Recall that we denote by $\mathbf{Q}_{\mathbf{X}} = (Q^1_{\mathbf{X}}, \cdots Q^N_{\mathbf{X}})$ an optimizer of the dual problem of (4), presented in detail in Section 4.1. The key relation (11) was shown in Proposition 5. We now prove the other key relation (12).

Corollary 1 *Let* $A := \rho_B(\mathbf{X})$. *Then* $\pi_A(\mathbf{X}) = \pi_A^{\mathbf{Q}_{\mathbf{X}}}(\mathbf{X})$.

Proof As $A = \rho_B(\mathbf{X}) \in \mathbb{R}$,

$$\begin{aligned} A = \rho_B(\mathbf{X}) = \rho_B^{\mathbf{Q}_{\mathbf{X}}}(\mathbf{X}), &\ \text{(by Proposition 5)}, \\ B = \pi_A(\mathbf{X}), &\ \text{(by Proposition 8)}, \\ B = \pi_A^{\mathbf{Q}_{\mathbf{X}}}(\mathbf{X}), &\ \text{(by Proposition 9)}, \end{aligned}$$

and therefore, $\pi_A(\mathbf{X}) = \pi_A^{\mathbf{Q}_{\mathbf{X}}}(\mathbf{X})$.

4.3 On local cash additivity and marginal risk contribution

We now show when the systemic risk measures of the form (20) are cash additive and local cash additive.

Lemma 2 *Define*

$$\mathcal{W}_{\mathcal{C}} := \{\mathbf{Z} \in \mathcal{C}_{\mathbb{R}} \mid \mathbf{Y} \in \mathcal{C} \Longleftrightarrow \mathbf{Y} - \mathbf{Z} \in \mathcal{C}\} \cap \mathcal{L}.$$

Then the risk measure ρ *defined in* (20) *is cash additive on* $\mathcal{W}_{\mathcal{C}}$, *i.e.,*

$$\rho(\mathbf{X} + \mathbf{Z}) = \rho(\mathbf{X}) - \sum_{n=1}^N Z^n \text{ for all } \mathbf{Z} \in \mathcal{W}_{\mathcal{C}} \text{ and all } \mathbf{X} \in \mathcal{L}.$$

Proof Let $\mathbf{Z} \in \mathcal{W}_{\mathcal{C}}$. Then $\mathbf{W} := \mathbf{Z} + \mathbf{Y} \in \ \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$ for any $\mathbf{Y} \in \mathcal{C}$. For any $\mathbf{X} \in \mathcal{L}$ it holds

$$\begin{aligned}\rho(\mathbf{X}+\mathbf{Z}) &= \inf\{\sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C},\ \Lambda(\mathbf{X}+\mathbf{Z}+\mathbf{Y}) \in \mathbb{A}_B\} \\ &= \inf\{\sum_{n=1}^{N} W^n - \sum_{n=1}^{N} Z^n \mid \mathbf{W}-\mathbf{Z} \in \mathcal{C},\ \Lambda(\mathbf{X}+\mathbf{W}) \in \mathbb{A}_B\} \\ &= \inf\{\sum_{n=1}^{N} W^n - \sum_{n=1}^{N} Z^n \mid \mathbf{W} \in \mathcal{C},\ \Lambda(\mathbf{X}+\mathbf{W}) \in \mathbb{A}_B\} \\ &= \rho(\mathbf{X}) - \sum_{n=1}^{N} Z^n.\end{aligned}$$

Example 4 In case of the set $\mathcal{C}^{(\mathbf{n})}$ in Example 1, ρ is cash additive on

$$\mathcal{W}_{\mathcal{C}^{(\mathbf{n})}} = \mathcal{C}^{(\mathbf{n})}. \tag{42}$$

Note that equality (42) holds since we are assuming no restrictions on the vector $d = (d, \cdots, d_m) \in \mathbb{R}^m$, which determines the grouping. If for example, we restrict d to have non negative components, then it is no longer true that $\mathcal{W}_{\mathcal{C}^{(\mathbf{n})}} = \mathcal{C}^{(\mathbf{n})}$.

Corollary 2 *For the systemic risk measures of the form* (20) *we have:*

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{V})|_{\varepsilon=0} = -\sum_{n=1}^{N} V^n \tag{43}$$

for all $\mathbf{V}$ *such that* $\varepsilon\mathbf{V} \in \mathcal{W}_{\mathcal{C}}$ *for all* $\varepsilon \in (0,1]$.

Proof It follows from Lemma 2 which gives $\rho(\mathbf{X}+\varepsilon\mathbf{V}) = \rho(\mathbf{X}) - \varepsilon\sum_{n=1}^{N} V^n$.

Remark 9 Note that Lemma 2 and Corollary 2 hold for systemic risk measures of the general form (20), without Assumption 1. Under Assumption 1 we have $\mathbb{R}^N \subseteq \mathcal{W}_{\mathcal{C}}$ and (43) holds for all $\mathbf{V} \in \mathbb{R}^N$.

The expression $\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{V})|_{\varepsilon=0}$ represents the sensitivity of the risk $\mathbf{X}$ with respect to the impact $\mathbf{V} \in L^0(\mathbb{R}^N)$. In the case of a deterministic $\mathbf{V} := \mathbf{m} \in \mathbb{R}^N$, it was called *marginal risk contribution* in [3]. Such property cannot be immediately generalized to the case of random vectors $\mathbf{V}$, also because in general $\sum_{n=1}^{N} V^n \notin \mathbb{R}$. In the following, we obtain the general local version of cash additivity, which extends the concept of marginal risk contribution to a random setting. In particular, (44) shows how the change in one component affects the change of the systemic risk measure.

Proposition 10 *Let* $\mathbf{V} \in M^{\Phi}$ *and* $\mathbf{X} \in M^{\Phi}$. *Let* $\mathbf{Q}_{\mathbf{X}}$ *be the optimal solution to the dual problem (28) associated to* $\rho(\mathbf{X})$ *and assume that* $\rho(\mathbf{X}+\varepsilon\mathbf{V})$ *is differentiable with respect to* ε *at* $\varepsilon = 0$, *and* $\frac{d\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}}}{d\mathbb{P}} \to \frac{d\mathbf{Q}_{\mathbf{X}}}{d\mathbb{P}}$ *in* $\sigma^*(L^{\Phi^*}, M^{\Phi})$, *as* $\varepsilon \to 0$. *Then,*

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{V})|_{\varepsilon=0} = -\sum_{n=1}^{N} \mathbb{E}_{Q^n_{\mathbf{X}}}[V^n]. \tag{44}$$

Proof As the penalty function $\alpha_{\Lambda,B}$ does not depend on $\mathbf{X}$, by (31) we deduce

$$\begin{aligned}\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{V})|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\left\{\sum_{n=1}^{N}\mathbb{E}_{Q^n_{\mathbf{X}+\varepsilon\mathbf{V}}}[-X^n-\varepsilon V^n]-\alpha_{\Lambda,B}(\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}})\right\}|_{\varepsilon=0}\\ &= \frac{d}{d\varepsilon}\left\{\sum_{n=1}^{N}\mathbb{E}_{Q^n_{\mathbf{X}+\varepsilon\mathbf{V}}}[-X^n]-\alpha_{\Lambda,B}(\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}})\right\}|_{\varepsilon=0}\\ &\quad+\sum_{n=1}^{N}\frac{d}{d\varepsilon}\left(\varepsilon\mathbb{E}_{Q^n_{\mathbf{X}+\varepsilon\mathbf{V}}}[-V^n]\right)|_{\varepsilon=0} \qquad (45)\\ &= 0+\sum_{n=1}^{N}\lim_{\varepsilon\to 0}\mathbb{E}_{Q^n_{\mathbf{X}+\varepsilon\mathbf{V}}}[-V^n]=\sum_{n=1}^{N}\mathbb{E}_{Q^n_{\mathbf{X}}}[-V^n], \qquad (46)\end{aligned}$$

where the equality between (45) and (46) is justified by the optimality of $\mathbf{Q_X}$ and the differentiability of $\rho(\mathbf{X}+\varepsilon\mathbf{V})$, while the last equality is guaranteed by the convergence of $\frac{d\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}}}{d\mathbb{P}}$.

Remark 10 We emphasize that the generalization (44) of (43) holds because we are computing the expectation with respect to the systemic probability measure $\mathbf{Q_X}$. A relevant example where the assumptions of Proposition 10 hold is provided in Section 6.

5 Existence of the optimal solutions

Throughout the entire Section 5, we assume $\mathbf{X} \in M^{\Phi}$ and that $\mathbf{Q} = (Q^1, ..., Q^N)$ satisfies $Q^n \ll \mathbb{P}$, $\frac{d\mathbf{Q}}{d\mathbb{P}} \in L^{\Phi^*}$ and $\alpha_{\Lambda,B}(\mathbf{Q}) < +\infty$, or equivalently $\rho_B^{\mathbf{Q}}(\mathbf{X}) > -\infty$. Recall from Proposition 6 that this implies $\mathbb{E}\left[v_n\left(\lambda\frac{dQ^n}{dP}\right)\right] < +\infty$ for all n and all $\lambda > 0$. Set

$$L^1(\mathbb{P};\mathbf{Q}) := (L^1(\mathbb{P};\mathbb{R}^N)\cap L^1(\mathbf{Q};\mathbb{R}^N)) \supseteq L^{\Phi} \supseteq M^{\Phi}, \qquad (47)$$

where the inclusions follows from Remark 2 and $\frac{d\mathbf{Q}}{d\mathbb{P}} \in L^{\Phi^*}$.
In order to prove the existence of the optimal solutions for $\rho_B^{\mathbf{Q_X}}(\mathbf{X})$ and $\rho_B(\mathbf{X})$, we will proceed in several steps. As shown in Section 5.2, in general, we can not expect to find the optimal solution $\mathbf{Y_Q}$ to the problem $\rho_B^{\mathbf{Q}}(\mathbf{X})$ in the space M^{Φ}, but only in the larger space $L^1(\mathbf{Q})$. We first prove the existence of $\mathbf{Y} \in L^1(\mathbb{P})$, which is the candidate solution, as specified in Theorem 5, to an extended problem. We already know that the optimal allocation to $\rho_B(\mathbf{X})$, when it exists, coincides with $\mathbf{Y}_{\mathbf{Q_X}} \in L^1(\mathbf{Q_X})$. So in a second step (see Theorem 7 and Corollary 5) we show that the optimal solution $\mathbf{Y_X}$ to the extended problem $\widetilde{\rho}_B^{\mathbf{Q_X}}(\mathbf{X}) = \rho_B(\mathbf{X})$ exists and $\mathbf{Y}_{\mathbf{Q_X}} = \mathbf{Y_X} \in L^1(\mathbb{P};\mathbf{Q_X})$.

W.l.o.g. we may assume that $u_i(0) = 0$, $1 \le i \le N$ and observe that then

$$u_i(x_i) = u_i(x_i^+) + u_i(-x_i^-). \qquad (48)$$

5.1 On the utility maximization problem

For $a^n \in \mathbb{R}$ consider the problem:

$$U_n(a^n) := \sup\left\{\mathbb{E}\left[u_n(X^n+W)\right] \mid W \in M^{\phi_n},\, \mathbb{E}_{Q^n}[W] \leq a^n\right\}. \tag{49}$$

If we need to emphasize the dependence on Q^n we write $U_n^{Q_n}(a^n)$. Note that $\mathbb{E}[u_n(X^n+W)] \leq u_n(\mathbb{E}[X^n+W]) < +\infty$ for all $X^n, W \in M^{\phi_n} \subseteq L^1(\mathbb{P};\mathbb{R})$. The conditions $X^n, W \in M^{\phi_n}$ imply that $\mathbb{E}[u(X^n+W)] > -\infty$, which in turn implies that $U(a^n) > -\infty$. As $\frac{d\mathbf{Q}}{dP} \in L^{\Phi^*}$, then $W \in M^{\phi_n}$ implies $W \in L^1(Q^n)$ and the problem (49) is well posed. Due to the monotonicity and concavity of u_n, U_n is monotone increasing, concave and continuous on $\mathbb{R}$ and we may replace, in the definition of U_n, the inequality with the equality sign. However, in general the optimal solution to (49) will only exist on a larger domain, as suggested by the well known result reported in Proposition 20. This leads to introduce the auxiliary problems:

$$\begin{aligned}\widehat{U}_n(a^n) &: = \sup\left\{\mathbb{E}\left[u_n(X^n+W)\right] \mid W \in L^1(Q^n),\, \mathbb{E}_{Q^n}[W] \leq a^n\right\},\\ \widetilde{U}_n(a^n) &: = \sup\left\{\mathbb{E}\left[u_n(X^n+W)\right] \mid W \in L^1(\mathbb{P},Q^n),\, \mathbb{E}_{Q^n}[W] \leq a^n\right\},\end{aligned} \tag{50}$$

where $L^1(\mathbb{P},Q^n)$ is defined as in (47).

Proposition 11

$$U_n(a^n) = \widetilde{U}_n(a^n) = \widehat{U}_n(a^n) < +\infty. \tag{51}$$

If $U_n(a^n) < u_n(+\infty)$ then

$$U^n : \mathbb{R} \to \mathbb{R} \ \ \textit{is differentiable, } U_n(-\infty) = -\infty, U_n' > 0, U_n'(-\infty) = +\infty, U_n'(+\infty) = 0 \tag{52}$$

and

$$\begin{aligned}U_n(a^n) &= \inf_{\lambda>0}\left\{\lambda\left(\mathbb{E}_{Q^n}[X^n]+a^n\right) + \mathbb{E}\left[v_n\left(\lambda\frac{dQ^n}{dP}\right)\right]\right\} &(53)\\ &= \mathbb{E}\left[u_n(X^n+\widehat{Y}^n_{\mathbf{Q}})\right], &(54)\end{aligned}$$

where the optimal solution $\widehat{Y}^n_{\mathbf{Q}} \in L^1(Q^n)$ is given by

$$\widehat{Y}^n_{\mathbf{Q}} := -X^n - v_n'\left(\lambda_n\frac{dQ^n}{d\mathbb{P}}\right), \tag{55}$$

$u_n(X^n+\widehat{Y}^n_{\mathbf{Q}}) \in L^1(\mathbb{P})$ and $\lambda_n > 0$ is the unique solution of

$$\mathbb{E}_{Q^n}[X^n] + a^n + \mathbb{E}_{Q^n}\left[v_n'\left(\lambda_n\frac{dQ^n}{dP}\right)\right] = 0. \tag{56}$$

Proof From $M^{\phi_n} \subseteq L^1(\mathbb{P},Q^n) \subseteq L^1(Q^n)$ we clearly have: $U_n(a^n) \leq \widetilde{U}_n(a^n) \leq \widehat{U}_n(a^n) \leq u_n(+\infty)$, so that

$$\text{if } U_n(a^n) = u(+\infty) \text{ then } U_n(a^n) = \widetilde{U}_n(a^n) = \widehat{U}_n(a^n) = u_n(+\infty). \tag{57}$$

By the Fenchel inequality we get

$$\mathbb{E}\left[u_n(X^n+W)\right] \leq \lambda\left(\mathbb{E}_{Q^n}[X^n]+\mathbb{E}_{Q^n}[W]\right)+\mathbb{E}\left[v_n\left(\lambda\frac{dQ^n}{dP}\right)\right],$$

and hence

$$U_n(a^n) \leq \widetilde{U}_n(a^n) \leq \widehat{U}_n(a^n) \leq \inf_{\lambda>0}\left\{\lambda\left(\mathbb{E}_{Q^n}[X^n]+a^n\right)+\mathbb{E}\left[v_n\left(\lambda\frac{dQ^n}{dP}\right)\right]\right\} < +\infty, \tag{58}$$

as $\mathbb{E}\left[v_n\left(\lambda\frac{dQ^n}{dP}\right)\right] < +\infty$. Therefore (51) is a consequence of (57) and (53). To show (53), consider the integral functional $I : M^{\phi_n} \to \mathbb{R}$ defined by $I(X^n)=\mathbb{E}\left[u_n(X^n)\right]$. It is finite valued, monotone increasing and concave on M^{ϕ_n} (as $\mathbb{E}[u_n(X^n)] \leq u_n(\mathbb{E}[X^n]) < +\infty$), and therefore, by the Theorem 9, it is norm-continuous on M^{ϕ_n}. We can then follow the well known duality approach (see for example [11]). Consider the convex cone $D^0 := \left\{W \in M^{\phi_n} \mid \mathbb{E}_{Q^n}[W] \leq 0\right\}$ which is the polar cone of the one dimensional cone $D := \left\{\lambda\frac{dQ^n}{dP} \mid \lambda \geq 0\right\}$, so that the bipolar D^{00} coincide with D. Let $\delta_{D^0} : M^{\phi_n} \to \mathbb{R}\cup\{+\infty\}$ be the support functional of D^0. By [36], or directly by hand, the concave conjugate $I^* : L^{\phi_n^*} \to \mathbb{R}\cup\{-\infty\}$ is given by $I^*(\xi^n)=\mathbb{E}\left[-v_n(\xi^n)\right]$ and so, by Fenchel duality Theorem,

$$\begin{aligned}
U_n(a^n) &= \sup_{W\in D^0}\mathbb{E}\left[u_n(X^n+a^n+W)\right] = \sup_{Z\in D^0+X^n+a^n}\mathbb{E}\left[u_n(Z)\right] \\
&= \sup_{Z\in M^{\phi_n}}\left\{\mathbb{E}\left[u_n(Z)\right]-\delta_{D^0+X^n+a^n}(Z)\right\} = \min_{\xi^n\in L^{\phi_n^*}}\left\{\delta^*_{D^0+X^n+a^n}(\xi^n)-\mathbb{E}\left[-v_n(\xi^n)\right]\right\} \\
&= \min_{\xi^n\in L^{\phi_n^*}}\left\{\mathbb{E}[\xi^n(X^n+a^n)]+\delta_{D^{00}}(\xi^n)+\mathbb{E}\left[v_n(\xi^n)\right]\right\} \\
&= \min_{\xi^n\in D^{00}}\left\{\mathbb{E}[\xi^n(X^n+a^n)]+\mathbb{E}\left[v_n(\xi^n)\right]\right\} = \min_{\lambda>0}\left\{\lambda\left(\mathbb{E}_{Q^n}[X^n]+a^n\right)+\mathbb{E}\left[v_n\left(\lambda\frac{dQ^n}{dP}\right)\right]\right\},
\end{aligned}$$

where we used $\delta^*_{D^0}=\delta_{D^{00}}$, $D^{00}=D$ and the fact that the minimizer is obtained at $\lambda>0$, otherwise if $\lambda=0$ then $U_n(a^n)=\mathbb{E}\left[v_n(0)\right]=u_n(+\infty)$. The statements (54), (55) and (56) are immediate consequence of Proposition 20, replacing c with $\mathbb{E}_{Q^n}[X^n]+a^n$ in (107). We conclude the proof by proving (52). From the inequality (58), it is clear that $U_n(-\infty)=-\infty$. Define

$$V_n(\lambda) := \mathbb{E}\left[v_n\left(\lambda\frac{dQ_n}{d\mathbb{P}}\right)\right]+\lambda\mathbb{E}_{Q_n}[X^n].$$

When $U_n(a^n) < u_n(+\infty)$, from (53) we have that

$$U_n(a^n) = \inf_{\lambda>0}\left\{V_n(\lambda)+\lambda a^n\right\},$$

which shows that U_n and V_n are conjugate of each other, i.e., $V_n(\lambda)=\sup_{a^n>0}\left\{U_n(a^n)-\lambda a^n\right\}$. From Lemmas 9 and 10 we know that the convex function V_n is differentiable on $(0,+\infty)$ and therefore U_n is differentiable on $(-\infty,+\infty)$ and

$$U_n'(a) = (V_n')^{-1}(-a) > 0.$$

We only need to show the last two conditions. As $v_n(0^+) = u_n(+\infty) = +\infty$ then $V_n(0^+) = +\infty$. Since $v_n'(0^+) = -\infty$ we get $V_n'(0^+) = -\infty$ and $U_n'(+\infty) = 0$. Moreover

$$\begin{aligned} V_n'(+\infty) &= \lim_{\lambda\to+\infty} \frac{V_n(\lambda)}{\lambda} = \lim_{\lambda\to+\infty} \frac{1}{\lambda}\mathbb{E}\left[v_n\left(\lambda\frac{dQ_n}{d\mathbb{P}}\right)\right] + \mathbb{E}_{Q_n}[X^n] \\ &\overset{Jensen}{\geq} \lim_{\lambda\to+\infty} \frac{1}{\lambda} v_n(\lambda) + \mathbb{E}_{Q_n}[X^n] = v_n'(\infty) + \mathbb{E}_{Q_n}[X^n] = +\infty \end{aligned}$$

which implies $U_n'(-\infty) = +\infty$.

Define

$$\begin{aligned} \widetilde{\pi}_A^{\mathbf{Q}}(\mathbf{X}) &:= \sup_{\mathbf{Y}\in L^1(\mathbb{P};\mathbf{Q})} \left\{ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n)\right] \mid \sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] \leq A \right\}, \\ \widehat{\pi}_A^{\mathbf{Q}}(\mathbf{X}) &:= \sup_{\mathbf{Y}\in L^1(\mathbf{Q})} \left\{ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n)\right] \mid \sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] \leq A \right\}, \end{aligned}$$

and similarly for $\widetilde{\rho}_B^{\mathbf{Q}}(\mathbf{X})$ and $\widehat{\rho}_B^{\mathbf{Q}}(\mathbf{X})$. As shown in (60), the extension to $L^1(\mathbf{Q})$ does not increment the optimal value of $\pi_A^{\mathbf{Q}}(\mathbf{X})$. In addition (60) justifies equation (16) in Section 2.

Lemma 3 *Let $A := \rho_B^{\mathbf{Q}}(\mathbf{X})$ and $\pi_A^{\mathbf{Q}}(\mathbf{X}) < +\infty$. Then*

$$\pi_A^{\mathbf{Q}}(\mathbf{X}) = \sup_{\mathbf{Y}\in M^\Phi} \left\{ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n)\right] \mid \sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] = A \right\} := \pi_A^{\mathbf{Q},=}(\mathbf{X}). \quad (59)$$

and

$$\pi_A^{\mathbf{Q}}(\mathbf{X}) = \sup_{\sum_{n=1}^N a^n = A} \sum_{n=1}^N U_n(a^n) = \widetilde{\pi}_A^{\mathbf{Q}}(\mathbf{X}) = \widehat{\pi}_A^{\mathbf{Q}}(\mathbf{X}), \quad (60)$$

$$\rho_B^{\mathbf{Q}}(\mathbf{X}) = \widetilde{\rho}_B^{\mathbf{Q}}(\mathbf{X}) = \widehat{\rho}_B^{\mathbf{Q}}(\mathbf{X}). \quad (61)$$

Proof Clearly, $+\infty > \pi_A^{\mathbf{Q}}(\mathbf{X}) \geq \pi_A^{\mathbf{Q},=}(\mathbf{X})$. By contradiction suppose that $\pi_A^{\mathbf{Q}}(\mathbf{X}) > \pi_A^{\mathbf{Q},=}(\mathbf{X})$ and take $\varepsilon > 0$ such that $\pi_A^{\mathbf{Q}}(\mathbf{X}) - \varepsilon > \pi_A^{\mathbf{Q},=}(\mathbf{X})$. By definition of $\pi_A^{\mathbf{Q}}(\mathbf{X})$ there exists $\mathbf{Y} \in M^\Phi$ satisfying $\sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] < A$ and $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n)\right] > \pi_A^{\mathbf{Q}}(\mathbf{X}) - \varepsilon$. Take $\widetilde{Y}^n = Y^n + \delta$, $\delta \in \mathbb{R}_+$, such that $\sum_{n=1}^N \mathbb{E}_{Q^n}\left[\widetilde{Y}^n\right] = A$. Then $\pi_A^{\mathbf{Q},=}(\mathbf{X}) \geq \mathbb{E}\left[\sum_{n=1}^N u_n(X^n+\widetilde{Y}^n)\right] \geq \mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n)\right] > \pi_A^{\mathbf{Q}}(\mathbf{X}) - \varepsilon > \pi_A^{\mathbf{Q},=}(\mathbf{X})$, a contradiction. Hence (59) holds true. Note that

$$M^\Phi = \left\{ \mathbf{Y} = \mathbf{a} + \mathbf{Z} \mid \mathbf{a} \in\mathbb{R}^N \text{ and } \mathbf{Z} \in M^\Phi \text{ such that } \mathbb{E}_{Q^n}[Z^n] = 0 \text{ for each } n \right\}.$$

Indeed, just take $\mathbf{Y} \in M^{\Phi}$ and let $a^n := \mathbb{E}_{Q^n}[Y^n] \in \mathbb{R}$ and $Z^n := Y^n - a^n \in M^{\phi_n}$. Then

$$\begin{aligned}\pi_A^{\mathbf{Q}}(\mathbf{X}) &= \sup_{\mathbf{Y}\in M^{\Phi}} \left\{ \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] \mid \sum_{n=1}^{N} \mathbb{E}_{Q^n}\left[Y^n\right] = A \right\} \\ &= \sup_{\sum_{n=1}^{N} a^n = A,\ Z^n \in M^{\phi_n},\ \mathbb{E}_{Q^n}[Z^n]=0\ \forall n} \left\{ \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + a^n + Z^n)\right]\right\} \\ &= \sup_{\sum_{n=1}^{N} a^n = A} \sum_{n=1}^{N} \sup_{\mathbb{E}_{Q^n}[Y^n]=a^n} \mathbb{E}\left[u_n(X^n+Y^n)\right] = \sup_{\sum_{n=1}^{N} a^n = A} \sum_{n=1}^{N} U_n(a^n),\end{aligned} \tag{62}$$

which shows the first equality in (60). Then $\pi_A^{\mathbf{Q}}(\mathbf{X}) = \widetilde{\pi}_A^{\mathbf{Q}}(\mathbf{X}) = \widehat{\pi}_A^{\mathbf{Q}}(\mathbf{X})$ are consequences of (51) and the decompositions analogous to the one just obtained for $\pi_A^{\mathbf{Q}}(\mathbf{X})$ in (62). From Proposition 9, its extension to $L^1(\mathbf{Q})$ in Remark 8 and $\rho_B^{\mathbf{Q}}(\mathbf{X}) > -\infty$, we easily deduce (61).

Lemma 4 *For arbitrary constant A, $B \in \mathbb{R}$ set*

$$K := \left\{ \mathbf{a} \in \mathbb{R}^N \mid \sum_{n=1}^{N} a^n \leq A,\ \sum_{n=1}^{N} U_n(a^n) \geq B \right\}.$$

Then K is a bounded closed set in $\mathbb{R}^N$.

Proof For $N = 1$ it is true. Let $N > 1$. First we prove that, for all $j = 1, ..., N$,

$$U_j(a)\left\{1 + \frac{\sum_{n\neq j} U_n(A-(N-1)a)}{U_j(a)}\right\} \to -\infty \text{ as } a \downarrow -\infty. \tag{63}$$

Recall that $U_n(-\infty) = -\infty$ and $U_n(+\infty) \leq u_n(+\infty)$ for all n. Suppose that for some $k \in \{1, ..., N\}$, $u_k(+\infty) < +\infty$. Then $U_k(+\infty) < +\infty$ and for all $j = 1, ..., N$

$$\lim_{a\to-\infty} \left\{ \frac{U_k(A-(N-1)a)}{U_j(a)} \right\} = 0. \tag{64}$$

Now suppose that for some $k \in \{1, ..., N\}$, $u_k(+\infty) = +\infty$. Then Proposition 11 shows that $U_k(a^k) < +\infty = u_k(+\infty), U_k' > 0$, $U_k'(-\infty) = +\infty$, $U_k'(+\infty) = 0$. By l'Hopital's rule, for all $j = 1, ..., N$ we obtain again

$$\lim_{a\to-\infty} \left\{ \frac{U_k(A-(N-1)a)}{U_j(a)} \right\} = \lim_{a\to-\infty} \frac{-(N-1)U_k'(A-(N-1)a)}{U_j'(a)} = 0. \tag{65}$$

From (64) and (65) we deduce that (63) holds true.

We conclude that for any constant B there exists a constant R such that for all $j = 1, ..., N$ and $a < R$

$$U_j(a)\left\{1 + \frac{\sum_{n\neq j} U_n(A-(N-1)a)}{U_j(a)}\right\} < B.$$

Let $\mathbf{a} \in K$ and let i be such that $a^i = \min\{a^1, ..., a^N\}$. Note that $a^j \leq A - (N-1)a^i$ for all $j = 1, ..., N$ because $\sum_{n=1}^N a^n \leq A$ holds. Assume that $a^i < R$. Then

$$B \leq \sum_{n=1}^N U_n(a^n) \leq U_i(a^i) \left\{ 1 + \frac{\sum_{n \neq i} U_n(A - (N-1)a^i)}{U_i(a^i)} \right\}, \tag{66}$$

which is a contradiction. Thus $a^j \geq R$ for all $j = 1, ..., N$, and then also $a^j \leq A - (N-1)R$ for all $j = 1, ..., N$ because $\sum_{n=1}^N a^n \leq A$ holds. This proves the claim.

Proposition 12 *Let $A := \rho_B^{\mathbf{Q}}(\mathbf{X})$ and $\pi_A^{\mathbf{Q}}(\mathbf{X}) < +\infty$. There exists an optimal solution $a^* \in \mathbb{R}^N$ to the problem (60), namely*

$$\pi_A^{\mathbf{Q}}(\mathbf{X}) = \sup_{a \in \mathbb{R}^N \text{ s.t. } \sum_{n=1}^N a^n = A,} \sum_{n=1}^N U_n(a^n) = \sum_{n=1}^N U_n(a_*^n) \quad \textit{and} \quad \sum_{n=1}^N a_*^n = A. \tag{67}$$

Proof Let $\mathbf{a}_m = (a_m^1, \cdots, a_m^N)_{m \in \mathbb{N}}$ be the approximating sequence of the supremum in (67). Then $\sum_{n=1}^N U_n(a_m^n) \geq \pi_A^{\mathbf{Q}}(\mathbf{X}) - \delta := C$ and $\sum_{n=1}^N a_m^n = A$ for each m. Then (67) is a consequence of the continuity of U_n and of Lemma 4, which guarantees that $\mathbf{a}_m$ belongs to a bounded closed set in $\mathbb{R}^N$.

Corollary 3 *Let $A := \rho_B^{\mathbf{Q}}(\mathbf{X})$ and suppose that for each n, $U_n(a_*^n) < u_n(+\infty)$, with the notation of Proposition 12. Then*

$$B = \pi_A^{\mathbf{Q}}(\mathbf{X}) = \sum_{n=1}^N U_n(a_*^n) = \sum_{n=1}^N \mathbb{E}\left[u_n(X^n + \widehat{Y}_{\mathbf{Q}}^n)\right]$$

$$\rho_B^{\mathbf{Q}}(\mathbf{X}) = A = \sum_{n=1}^N a_*^n = \sum_{n=1}^N \mathbb{E}_{Q^n}\left[\widehat{Y}_{\mathbf{Q}}^n\right]$$

where $\widehat{Y}_{\mathbf{Q}}^n \in L^1(Q^n)$ is given by (55). Therefore, under the assumption $U_n(a_*^n) < u_n(+\infty)$, $\widehat{\mathbf{Y}}_{\mathbf{Q}}$ is the optimal solution to both extended problems $\widehat{\pi}_A^{\mathbf{Q}}(\mathbf{X})$ and $\widehat{\rho}_B^{\mathbf{Q}}(\mathbf{X})$.

Proof It follows directly from Propositions 12 and 11 and from the equality

$$a_*^n = \mathbb{E}_{Q^n}\left[\widehat{Y}_{\mathbf{Q}}^n\right],$$

which can be easily shown from (55) and (56):

$$a_*^n = -\mathbb{E}_{Q^n}[X^n] - \mathbb{E}_{Q^n}\left[v_n'\left(\lambda_n \frac{dQ^n}{dP}\right)\right] = \mathbb{E}_{Q^n}\left[\widehat{Y}_{\mathbf{Q}}^n\right],$$

where $\lambda_n > 0$ is the unique solution of (56). Proposition 9 concludes the proof, observing that $+\infty > \pi_A^{\mathbf{Q}}(\mathbf{X}) = \widehat{\pi}_A^{\mathbf{Q}}(\mathbf{X})$ and $-\infty < \rho_B^{\mathbf{Q}}(\mathbf{X}) = \widehat{\rho}_B^{\mathbf{Q}}(\mathbf{X})$, due to (60) and (61).

Due to $u(-v'(y)) = v(y) - yv'(y)$ (see Lemma 10) we also deduce from (55) that

$$B = \sum_{n=1}^{N} \mathbb{E}\left[u_n\left(X^n + \widehat{Y}^n_{\mathbf{Q}}\right)\right] = \sum_{n=1}^{N} \mathbb{E}\left[u_n\left(-v'_n\left(\lambda_n \frac{dQ^n}{d\mathbb{P}}\right)\right)\right]$$
$$= \sum_{n=1}^{N} \mathbb{E}\left[v_n\left(\lambda_n \frac{dQ^n}{d\mathbb{P}}\right)\right] - \lambda_n \sum_{n=1}^{N} \mathbb{E}_{Q^n}\left[v'_n\left(\lambda_n \frac{dQ^n}{d\mathbb{P}}\right)\right],$$

so that the vector $\{\lambda_n\}_{n=1,\dots,N}$ of positive numbers solves the equation above, that should be compared with (39). We will show in Theorem 4 that all components λ_n are equal, when in (56) the value a^n is replaced by the optimal a^n_*.

5.2 On the optimal solution of $\rho^{\mathbf{Q}}$ and comparison of optimal solutions

Theorem 4 *Suppose that $\alpha_{\Lambda,B}(\mathbf{Q}) < +\infty$. Then the random vector $\mathbf{Y}_{\mathbf{Q}}$ given by*

$$Y^n_{\mathbf{Q}} := -X^n - v'_n\left(\lambda^* \frac{dQ^n}{d\mathbb{P}}\right),$$

where λ^* is the unique solution to (39), *satisfies $Y^n_{\mathbf{Q}} \in L^1(Q^n)$, $u_n(X^n + Y^n_{\mathbf{Q}}) \in L^1(\mathbb{P})$, $\mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n_{\mathbf{Q}})\right] = B$ and*

$$\rho^{\mathbf{Q}}_B(\mathbf{X}) = \inf_{\mathbf{Y}\in M^{\Phi}} \left\{\sum_{n=1}^{N} \mathbb{E}_{Q^n}[Y^n] \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] \geq B\right\} = \sum_{n=1}^{N} \mathbb{E}_{Q^n}\left[Y^n_{\mathbf{Q}}\right] \tag{68}$$

$$= \inf_{\mathbf{Y}\in L^1(\mathbf{Q})} \left\{\sum_{n=1}^{N} \mathbb{E}_{Q^n}[Y^n] \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] \geq B\right\} := \widehat{\rho}^{\mathbf{Q}}_B(\mathbf{X}), \tag{69}$$

so that $\mathbf{Y}_{\mathbf{Q}} = \widehat{\mathbf{Y}}_{\mathbf{Q}}$ is the optimal solution to the extended problem $\widehat{\rho}^{\mathbf{Q}}_B(\mathbf{X})$.

Proof Note that $\rho^{\mathbf{Q}}_B(\mathbf{X}) > -\infty$, as $\alpha_{\Lambda,B}(\mathbf{Q}) < +\infty$. The integrability conditions hold thanks to the results stated in Appendix A.3. From (32) and the expression (38) for the penalty, we compute:

$$\rho^{\mathbf{Q}}_B(\mathbf{X}) = -\sum_{n=1}^{N} \mathbb{E}_{Q^n}[X^n] - \alpha_{\Lambda,B}(\mathbf{Q}) =$$
$$= \sum_{n=1}^{N} \mathbb{E}_{Q^n}\left[-X^n - v'_n\left(\lambda^* \frac{dQ^n}{d\mathbb{P}}\right)\right] = \sum_{n=1}^{N} \mathbb{E}_{Q^n}\left[Y^n_{\mathbf{Q}}\right].$$

We show that $Y^n_{\mathbf{Q}}$ satisfies the budget constraint:

$$\sum_{n=1}^{N} \mathbb{E}\left[u_n\left(X^n + Y^n_{\mathbf{Q}}\right)\right] = \sum_{n=1}^{N} \mathbb{E}\left[u_n\left(-v'_n\left(\lambda^* \frac{dQ^n}{d\mathbb{P}}\right)\right)\right]$$
$$= \sum_{n=1}^{N} \mathbb{E}\left[v_n\left(\lambda^* \frac{dQ^n}{d\mathbb{P}}\right)\right] - \lambda^* \sum_{n=1}^{N} \mathbb{E}_{Q^n}\left[v'_n\left(\lambda^* \frac{dQ^n}{d\mathbb{P}}\right)\right] = B$$

due to $u(-v'(y)) = v(y) - yv'(y)$ (see Lemma 10) and (39). Finally, $\rho_B^{\mathbf{Q}}(\mathbf{X}) = \widehat{\rho}_B^{\mathbf{Q}}(\mathbf{X})$ follows from (61) and the uniqueness shown in Remark 5 proves that $\mathbf{Y_Q} = \widehat{\mathbf{Y}}_{\mathbf{Q}}$.

When both solutions to the problems $\rho_B(\mathbf{X})$ and $\rho_B^{\mathbf{Q_X}}(\mathbf{X})$ exist, then they coincide.

Proposition 13 *Let* $\mathbf{Y_X} \in \mathcal{C}_0 \cap M^{\Phi}$ *be the optimal allocation to* $\rho_B(\mathbf{X})$*,* $\mathbf{Q_X}$ *be an optimal solution to the dual problem (28). Then:*

$$\mathbf{Y_X} = \mathbf{Y_{Q_X}} := -X^n - v_n'\left(\lambda^* \frac{dQ^n_{\mathbf{X}}}{d\mathbb{P}}\right).$$

Proof Note that $\mathbf{Y_X}$ satisfies:

$$\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n_{\mathbf{X}})\right] \geq B, \tag{70}$$

$$\sum_{n=1}^N Y^n_{\mathbf{X}} = \rho_B(\mathbf{X}), \tag{71}$$

$$\sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}] \leq \sum_{n=1}^N Y^n_{\mathbf{X}}, \tag{72}$$

as $\mathbf{Y_X} \in \mathcal{C}$ and $\mathbf{Q_X} \in \mathcal{D}$. From the definition of $\mathbf{Y_X}$, from (68), (32) and (31) we deduce that

$$\sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{Q_X}}] = \rho_B^{\mathbf{Q_X}}(\mathbf{X}) = -\sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[X^n] - \alpha_{\Lambda,B}(\mathbf{Q_X}) = \rho_B(\mathbf{X}) = \sum_{n=1}^N Y^n_{\mathbf{X}}. \tag{73}$$

As $\mathbf{Y_X}$ satisfies (70), by definition of $\rho_B^{\mathbf{Q_X}}(\mathbf{X})$ we have

$$\sum_{n=1}^N Y^n_{\mathbf{X}} = \rho_B(\mathbf{X}) = \rho_B^{\mathbf{Q_X}}(\mathbf{X}) \leq \sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}],$$

which shows, together with (72), that

$$\sum_{n=1}^N Y^n_{\mathbf{X}} = \sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]. \tag{74}$$

From (73) and (74) we then deduce

$$\alpha_{\Lambda,B}(\mathbf{Q_X}) = -\sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}\left[X^n + Y^n_{\mathbf{Q_X}}\right], \quad \text{and}$$

$$\alpha_{\Lambda,B}(\mathbf{Q_X}) = -\sum_{n=1}^N \left(\mathbb{E}_{Q^n_{\mathbf{X}}}\left[X^n\right] + Y^n_{\mathbf{X}}\right) = -\sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}\left[X^n + Y^n_{\mathbf{X}}\right].$$

As both $(\mathbf{X} + \mathbf{Y_X})$ and $(\mathbf{X} + \mathbf{Y_{Q_X}})$ satisfy the budget constraints associated to $\alpha_{\Lambda,B}(\mathbf{Q_X})$ in equation (34), this implies that $\alpha_{\Lambda,B}(\mathbf{Q_X})$ is attained by both $(\mathbf{X}+\mathbf{Y_X})$ and $(\mathbf{X} + \mathbf{Y_{Q_X}})$. The uniqueness shown in Lemma 1 allows us to conclude that $\mathbf{Y_X} = \mathbf{Y_{Q_X}}$.

Remark 11 (Extension to $L^1(\mathbf{Q_X})$) We will show in Section 5 the existence of an optimal solution $\mathbf{Y_X}$ to the problem $\widetilde{\rho}_B^{\mathbf{Q}}(\mathbf{X})$, namely $\mathbf{Y_X} \in \mathcal{C}_0 \cap L^1(\mathbb{P}, \mathbf{Q_X})$ satisfies (70), (71) and (72). Then the above proof and Remark 4 show that $\mathbf{Y_X} = \mathbf{Y_{Q_X}}$, even for $\mathbf{Y_X} \in \mathcal{C}_0 \cap L^1(\mathbb{P}, \mathbf{Q_X})$. Similarly, the following Corollary holds also for such $\mathbf{Y_X} \in \mathcal{C}_0 \cap L^1(\mathbb{P}, \mathbf{Q_X})$.

We now show that the maximizer of the dual representation is unique.

Corollary 4 *Suppose that there exists an optimal allocation $\mathbf{Y_X}$ to $\rho_B(\mathbf{X})$. Then the optimal solution $\mathbf{Q_X} = (Q^1_{\mathbf{X}}, \cdots, Q^N_{\mathbf{X}})$ of the dual problem (28) is unique.*

Proof Suppose that $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are two optimizers of the dual problem (28). Then $\alpha_{\Lambda,B}(\mathbf{Q}_1) < +\infty$, $\alpha_{\Lambda,B}(\mathbf{Q}_2) < +\infty$ and, by Proposition 13, we have, for each n :

$$-X^n - v'_n\left(\lambda_1^* \frac{dQ_1^n}{d\mathbb{P}}\right) = Y^n_{\mathbf{Q}_1} = Y^n_{\mathbf{X}} = Y^n_{\mathbf{Q}_2} = -X^n - v'_n\left(\lambda_2^* \frac{dQ_2^n}{d\mathbb{P}}\right), \quad \mathbb{P} \text{ a.s.}$$

As v'_n is invertible, we conclude that $\lambda_1^* \frac{dQ_1^n}{d\mathbb{P}} = \lambda_2^* \frac{dQ_2^n}{d\mathbb{P}}$, $\mathbb{P}$ a.s., which then implies $Q_1^n = Q_2^n$, as $\mathbb{E}\left[\frac{dQ_1^n}{d\mathbb{P}}\right] = \mathbb{E}\left[\frac{dQ_2^n}{d\mathbb{P}}\right] = 1$.

5.3 On the existence of the optimal allocation to $\rho(X)$

5.3.1 A first step

Theorem 5 *For $\mathcal{C} \subseteq \mathcal{C}_\mathbb{R} \cap M^\Phi$ and for any $\mathbf{X} \in M^\Phi$ there exists $\mathbf{Y} \in L^1(\mathbb{P}; \mathbb{R}^N)$ such that*

$$\sum_{n=1}^N Y^n \in \mathbb{R}, \quad \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \geq B,$$

$$\rho_B(\mathbf{X}) := \inf\left\{\sum_{n=1}^N Z^n \mid \mathbf{Z} \in \mathcal{C}, \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Z^n)\right] \geq B\right\} = \sum_{n=1}^N Y^n,$$

and a sequence $\{\mathbf{Y}_k\}_{k\in\mathbb{N}} \subset \mathcal{C}$ such that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y_k^n)\right] \geq B$ and

$$\mathbf{Y}_k \to \mathbf{Y} \ \mathbb{P}\text{-a.s.}$$

Remark 12 Recall that $\mathcal{C} := \mathcal{C}_0 \cap M^\Phi$ and that $\mathcal{C}_0 \subseteq \mathcal{C}_\mathbb{R}$ represents the effective constraint on the admissible injections, except for the integrability restriction expressed by M^Φ. Assume further that $\mathcal{C}_0$ is closed in $L^0(\mathbb{P})$, which is a reasonable assumption and holds true if $\mathcal{C} = \mathcal{C}^{(\mathbf{n})}$, in which case $\mathcal{C}_0^{(\mathbf{n})}$ is defined in (21). Then the random vector $\mathbf{Y}$ in Theorem 5 would also belong to $\mathcal{C}_0$, but in general not to $\mathcal{C}$ (as M^Φ is in general not closed for $\mathbb{P}$-a.s. convergence). The conclusion is that $\mathbf{Y}$ satisfies all the conditions for being the optimal allocation to $\rho_B(\mathbf{X})$, with the only exception for the integrability condition $\mathbf{Y} \in M^\Phi$, which is replaced by $\mathbf{Y} \in L^1(\mathbb{P}; \mathbb{R}^N)$. In the next subsection we will show when such $\mathbf{Y}$ also belongs to $\mathcal{C}_0 \cap L^1(\mathbf{Q_X}; \mathbb{R}^N)$.

It is now evident that when the cardinality of Ω is finite and the set $\mathcal{C}$ is closed for $\mathbb{P}$-a.s. convergence, then the random vector $\mathbf{Y}$ in Theorem 5 belongs to $\mathcal{C}$ and $\mathbf{Y} = \mathbf{Y_X} = \mathbf{Y_{Q_X}}$.

Proof Take a sequence of vectors $(\mathbf{V}_k)_{k\in\mathbb{N}} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}} \cap M^{\Phi} \subseteq L^1(\mathbb{P};\mathbb{R}^N)$ such that $\mathbb{R} \ni c_k := \sum_{n=1}^{N} V_k^n \downarrow \rho_B(\mathbf{X})$ as $k \to +\infty$ and $\mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n+V_k^n)\right] \geq B$. The sequence $(\mathbf{V}_k)_{k\in\mathbb{N}}$ is bounded for the $L^1(\mathbb{P};\mathbb{R}^N)$ norm if and only if so is the sequence $(\mathbf{X}+\mathbf{V}_k)_{k\in\mathbb{N}}$. Given the following decomposition in positive and negative part

$$\sum_{n=1}^{N} \mathbb{E}[|X^n+V_k^n|] = \sum_{n=1}^{N} \mathbb{E}[(X^n+V_k^n)^+] + \sum_{n=1}^{N} \mathbb{E}[(X^n+V_k^n)^-], \tag{75}$$

we define the index sets:

$$N_\infty^+ = \left\{ n \in \{1,...,N\} \mid \lim \sup_{k\to+\infty} \mathbb{E}[(X^n+V_k^n)^+] = +\infty \right\},$$
$$N_b^+ = \left\{ n \in \{1,...,N\} \mid \lim \sup_{k\to+\infty} \mathbb{E}[(X^n+V_k^n)^+] < +\infty \right\},$$

and, similarly, N_∞^- and N_b^- for the negative part. We can split the expression (75) as

$$\sum_{n\in N_\infty^+} E_{\mathbb{P}}[(X^n+V_k^n)^+] + \sum_{n\in N_b^+} E_{\mathbb{P}}[(X^n+V_k^n)^+] + \sum_{n\in N_\infty^-} E_{\mathbb{P}}[(X^n+V_k^n)^-] + \sum_{n\in N_b^-} E_{\mathbb{P}}[(X^n+V_k^n)^-].$$

If the sequence $(\mathbf{X}+\mathbf{V}_k)_{k\in\mathbb{N}}$ is not $L^1(\mathbb{P};\mathbb{R}^N)$ -bounded, then one of the sets N_∞^+ or N_∞^- must be nonempty and therefore, because of the constraint $\sum_{n=1}^{N} V_k^n = c_k$, both N_∞^+ and N_∞^- must be nonempty. From Lemma 11 and from Lemma 8 with $M := 2A$, by Jensen inequality and (48) we obtain

$$\begin{aligned}
B &\leq \sum_{n=1}^{N} \mathbb{E}[u_n(X^n+V_k^n)] \leq \sum_{n=1}^{N} u_n\left(\mathbb{E}[X^n+V_k^n]\right) \\
&= \sum_{n=1}^{N} u_n\left(\mathbb{E}[(X^n+V_k^n)^+]\right) + \sum_{n=1}^{N} u_n\left(-\mathbb{E}[(X^n+V_k^n)^-]\right) \\
&\leq A\left(\sum_{n\in N_\infty^+} \mathbb{E}[(X^n+V_k^n)^+] + \sum_{n\in N_b^+} \mathbb{E}[(X^n+V_k^n)^+]\right) \\
&\quad -2A\left(\sum_{n\in N_\infty^-} \mathbb{E}[(X^n+V_k^n)^-] + \sum_{n\in N_b^-} \mathbb{E}[(X^n+V_k^n)^-]\right) + const \\
&= A\left(c_k + \sum_{n=1}^{N} \mathbb{E}[X^n]\right) + const - A\left(\sum_{n\in N_\infty^-} \mathbb{E}[(X^n+V_k^n)^-] + \sum_{n\in N_b^-} \mathbb{E}[(X^n+V_k^n)^-]\right)
\end{aligned}$$

which is a contradiction, as the second term that multiplies A in not bounded from above. Hence we exclude that our minimizing sequence $(\mathbf{V}_k)_{k\in\mathbb{N}}$ has unbounded $L^1(\mathbb{P};\mathbb{R}^N)$ norm and we may apply a Komlós compactness argument, as stated below in Theorem 6, with $E = \mathbb{R}^N$. Applying this result to the sequence $(\mathbf{V}_k)_{k\in\mathbb{N}} \in \mathcal{C}$, we can find a sequence $\mathbf{Y}_k \in conv(\mathbf{V}_i, i \geq k) \in \mathcal{C}$, as $\mathcal{C}$ is convex, such that

$$\mathbf{Y}_k \text{ converges } \mathbb{P}\text{-}a.s. \text{ to } \mathbf{Y} \in L^1(\mathbb{P};\mathbb{R}^N).$$

Observe that by construction $\sum_{n=1}^N Y_k^n$ is $\mathbb{P}$-a.s. a real number and, as a consequence, so is $\sum_{n=1}^N Y^n$. As $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+V_k^n)\right] \geq B$, also the $\mathbf{Y}_k$ satisfy such constraint and therefore $\rho_B(\mathbf{X}) \leq \sum_{n=1}^N Y_k^n$.

Let $\mathbf{Y}_k = \sum_{i\in J_k} \lambda_i^k \mathbf{V}_i \in conv(\mathbf{V}_i, i \geq k)$, for some finite convex combination $(\lambda_i^k)_{i\in J_k}$ such that $\lambda_i^k > 0$ and $\sum_{i\in J_k} \lambda_i^k = 1$, where J_k is a finite subset of $\{k, k+1, ...\}$. For any fixed k we compute

$$\sum_{n=1}^N Y_k^n = \sum_{n=1}^N \left(\sum_{i\in J_k} \lambda_i^k V_i^n\right)_j = \sum_{i\in J_k} \lambda_i^k \left(\sum_{n=1}^N V_i^n\right) = \sum_{i\in J_k} \lambda_i^k c_i \leq c_k \left(\sum_{i\in J_k} \lambda_i^k\right) = c_k \tag{76}$$

and from $\rho_B(\mathbf{X}) \leq \sum_{n=1}^N Y_k^n \leq c_k$, we then deduce that $\sum_{n=1}^N Y^n = \rho_B(\mathbf{X})$.

We now show that $\mathbf{Y}$ also satisfies the budget constraint. In case that all utility functions are bounded from above, this is an immediate consequence of Fatou Lemma, since

$$\begin{aligned}\sum_{n=1}^N \mathbb{E}[-u_n(X^n+Y^n)] &= \sum_{n=1}^N \mathbb{E}[\underline{\lim}_{k\to\infty} (-u_n(X^n+Y_k^n))] \\ &\leq \underline{\lim}_{k\to\infty} \sum_{n=1}^N \mathbb{E}[-u_n(X^n+Y_k^n)] \leq B.\end{aligned}$$

In the general case, recall first that the sequence $\mathbf{V}_k$ is bounded in $L^1(\mathbb{P};\mathbb{R}^N)$, and the argument used in (76) shows that

$$\| \mathbf{X} + \mathbf{Y}_k \|_1 \leq \| \mathbf{X} \|_1 + \sup_k \| \mathbf{V}_k \|_1,$$

hence $\sup_k \| \mathbf{X} + \mathbf{Y}_k \|_1 < \infty$.

Now we need to exploit the Inada condition at $+\infty$. Applying the Lemma 12 to the utility functions u_n, assumed null in 0, we get

$$-u_n(x) + \varepsilon x^+ + b(\varepsilon) \geq 0 \ \forall x \in \mathbb{R}.$$

Replacing $\mathbf{X} + \mathbf{Y}$ in the expression above, applying Fatou Lemma we have

$$\begin{aligned}&\mathbb{E}\left[\sum_{n=1}^N -u_n(X^n+Y^n) + \varepsilon(X^n+Y^n)^+ + b(\varepsilon)\right] \\ &= \mathbb{E}\left[\underline{\lim}_{k\to\infty}\left(\sum_{n=1}^N -u_n(X^n+Y_k^n) + \varepsilon(X^n+Y_k^n)^+ + b(\varepsilon)\right)\right] \\ &\leq \underline{\lim}_{k\to\infty} \sum_{n=1}^N \mathbb{E}\left[-u_n(X^n+Y_k^n) + \varepsilon(X^n+Y_k^n)^+ + b(\varepsilon)\right] \\ &\leq -B + \varepsilon\left(\sup_k \| \mathbf{X} + \mathbf{Y}_k \|_1\right) + b(\varepsilon).\end{aligned}$$

As the term $b(\varepsilon)$ simplifies in the above inequality, we conclude that for all $\varepsilon > 0$

$$\mathbb{E}\left[\sum_{n=1}^N -u_n(X^n+Y^n)\right] \leq -B + \varepsilon\left(\sup_k \| \mathbf{X} + \mathbf{Y}_k \|_1 - \sum_{n=1}^N \mathbb{E}\left[(X^n+Y^n)^+\right]\right),$$

and since $\sup_k \| \mathbf{X} + \mathbf{Y}_k \|_1 < \infty$ we obtain

$$\mathbb{E}\left[\sum_{n=1}^{N} -u_n(X^n+Y^n)\right] \leq -B,$$

so that $\mathbf{Y}$ satisfies the constraint.

Theorem 6 (Theorem 1.4 [20]) *Let E be a Banach reflexive space and $(f_k)_{k\in\mathbb{N}} \subseteq L^1((\Omega,\mathcal{F},\mathbb{P});E) := L^1$ be a sequence with bounded L^1 norms. Then there exists a sequence $(g_k)_{k\in\mathbb{N}}$ and g_0 in L^1 such that $g_k \in conv(f_i, i \geq k)$ and $\| g_k - g \|_E \to 0$ $\mathbb{P}-a.s.$, as $k \to \infty$.*

5.3.2 Second Step: The optimal allocation to $\rho(\mathbf{X})$ in $L^1(\mathbf{Q_X})$

Lemma 5 *The random vector $\mathbf{Y}$ in Theorem 5 satisfies $\mathbf{Y}^{-}\in L^1(\mathbf{Q_X})$.*

Proof Applying (48) and $\phi_j(x) := -u_j(-|x|)$, note that for each fixed $1 \leq j \leq N$

$$\begin{aligned}
0 \leq \mathbb{E}\left[\phi_j((X^j+Y^j)^-)\right] &\leq \sum_{n=1}^{N} \mathbb{E}\left[\phi_n((X^n+Y^n)^-)\right] = \sum_{n=1}^{N} \mathbb{E}\left[-u_n(-(X^n+Y^n)^-)\right] \\
&= \sum_{n=1}^{N} \mathbb{E}\left[u_n(X^n+Y^n)^+\right] - \sum_{n=1}^{N} \mathbb{E}\left[u_n(X^n+Y^n)\right] \\
&\leq \sum_{n=1}^{N} u_n\left(\mathbb{E}\left[(X^n+Y^n)^+\right]\right) - B < \infty,
\end{aligned}$$

where we used Jensen inequality and $\mathbf{X}+\mathbf{Y} \in L^1(\mathbb{P};\mathbb{R}^N)$. This yields $(X^j+Y^j)^- \in L^{\phi_j} \subseteq L^1(Q^j_{\mathbf{X}})$. From $Y^j = (X^j+Y^j)^+ - (X^j+Y^j)^- - X^j \geq -(X^j+Y^j)^- - X^j$ we get

$$0 \leq (Y^j)^- \leq (-(X^j+Y^j)^- - X^j)^- = ((X^j+Y^j)^- + X)^+.$$

Since, by assumption, $X^j \in M^{\phi_j} \subseteq L^1(Q^j_{\mathbf{X}})$, then also $((X^j+Y^j)^-+X^j)^+ \in L^1(Q^j_{\mathbf{X}})$ and so

$$(Y^j)^- \in L^1(Q^j_{\mathbf{X}}),\ 1 \leq j \leq N.$$

Lemma 6 *The random vector $\mathbf{Y}$ in Theorem 5 satisfies $\mathbf{Y}^{+}\in L^1(\mathbf{Q_X})$.*

Proof We proved in Theorem 5 the existence of $\mathbf{Y}$ satisfying $\rho_B(\mathbf{X}) = \sum_{n=1}^N Y^n \in \mathbb{R}$ with $\mathbf{Y} \in L^1(\mathbb{P};\mathbb{R}^N)$, $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n)\right] \geq B$ and $\mathbf{Y}$ is the $\mathbb{P}$-a.s. limit of a sequence $\{\mathbf{Y}_k\}_k$ in $\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}} \cap M^{\Phi}$ such that $\sum_{n=1}^N Y^n_k \to \rho_B(\mathbf{X})$, as $k \uparrow +\infty$, $\sum_{n=1}^N \mathbb{E}\left[u_n(X^n+Y^n_k)\right] \geq B$ and $\sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_k] \leq \sum_{n=1}^N Y^n_k$. By passing to a subsequence, w.l.o.g we may assume $\sum_{n=1}^N Y^n_k \downarrow \rho_B(\mathbf{X})$. Let $j \in \{1,...,N\}$. By Fatou's Lemma we get

$$\mathbb{E}_{Q^j_{\mathbf{X}}}[(Y^j)^+] \leq \liminf_k \mathbb{E}_{Q^j_{\mathbf{X}}}[(Y^j_k)^+] \leq \sup_k \mathbb{E}_{Q^j_{\mathbf{X}}}[Y^j_k] + \sup_k \mathbb{E}_{Q^j_{\mathbf{X}}}[(Y^j_k)^-]. \tag{77}$$

First we show that $\sup_k \mathbb{E}_{Q^j_{\mathbf{X}}}[Y^j_k] < \infty$. Put $a^n_k = \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_k]$. Then $\sum_{n=1}^N a^n_k \leq \widetilde{A} := \sum_{n=1}^N Y^n_k \leq \sum_{n=1}^N Y^n_1$ and $\sum_{n=1}^N U_n^{Q^n_{\mathbf{X}}}(a^n_k) \geq \sum_{n=1}^N \mathbb{E}\left[u_n(X^n + Y^n_k)\right] \geq B$ for all $k \in \mathbb{N}$. Thus by Lemma 4, $\{\mathbf{a}_k\}_{k\in\mathbb{N}}$ lies in a bounded set in $\mathbb{R}^N$ and thus

$$\sup_k \mathbb{E}_{Q^j_{\mathbf{X}}}[Y^j_k] < \infty. \tag{78}$$

Next we show $\sup_k \mathbb{E}_{Q^j_{\mathbf{X}}}[(Y^j_k)^-] < \infty$. For all $k \in \mathbb{N}$ it holds that

$$\begin{aligned} 0 &\leq \mathbb{E}\left[\phi_j((X^j + Y^j_k)^-)\right] \leq \sum_{n=1}^N \mathbb{E}\left[\phi_n((X^n + Y^n_k)^-)\right] = \sum_{n=1}^N \mathbb{E}\left[-u_n(-(X^n + Y^n_k)^-)\right] \\ &= \sum_{n=1}^N \mathbb{E}\left[u_n(X^n + Y^n_k)^+\right] - \sum_{n=1}^N \mathbb{E}\left[u_n(X^n + Y^n_k)\right] \leq \sum_{n=1}^N u_n\left(\mathbb{E}\left[(X^n + Y^n_k)^+\right]\right) - B, \end{aligned}$$

where we used Jensen inequality and the fact that $\mathbf{Y}_k$ satisfies $\sum_{n=1}^N \mathbb{E}\left[u_n(X^n + Y^n_k)\right] > B$. From the proof of Theorem 5 we know that $(X^n + Y^n_k)_{k\in\mathbb{N}}$ is $L^1(\mathbb{P})$-bounded for all $n = 1, ..., N$, and thus

$$0 \leq \sup_k \mathbb{E}\left[\phi_j((X^j + Y^j_k)^-)\right] \leq \sum_{n=1}^N u_n\left(\sup_k \mathbb{E}\left[(X^n + Y^n_k)^+\right]\right) - B < \infty\,.$$

By Remark 2 it then follows that $(X^j + Y^j_k)^-_{k\in\mathbb{N}}$ is $L^1(Q^j_{\mathbf{X}})$-bounded. From $Y^j_k = (X^j + Y^j_k)^+ - (X^j + Y^j_k)^- - X^j \geq -(X^j + Y^j_k)^- - X^j$ we get

$$0 \leq (Y^j_k)^- \leq (-(X^j + Y^j_k)^- - X^j)^- = ((X^j + Y^j_k)^- + X^j)^+,$$

and thus

$$\sup_k \mathbb{E}_{Q^j_{\mathbf{X}}}[(Y^j_k)^-] \leq \sup_k \mathbb{E}_{Q^j_{\mathbf{X}}}[(X^j + Y^j_k)^-] + \mathbb{E}_{Q^j_{\mathbf{X}}}[|X^j|] < \infty\,, \tag{79}$$

where we recall that by assumption $X^j \in M^{\phi_j} \subseteq L^1(Q^j_{\mathbf{X}})$. From (78) and (79) together with (77) the claim follows.

For our final result on the existence we need one more assumption.

Definition 4 We say that $\mathcal{C}_0$ is closed under truncation if for each $\mathbf{Y} \in \mathcal{C}_0$ there exists $m_Y \in \mathbb{N}$ and $\mathbf{c}_Y = (c^1_Y, ..., c^N_Y) \in \mathbb{R}^N$ such that $\sum_{n=1}^N c^n_Y = \sum_{n=1}^N Y^n := c_Y \in \mathbb{R}$ and for all $m \geq m_Y$

$$\mathbf{Y}_m := \mathbf{Y} I_{\{\cap_{n=1}^N \{|Y^n| < m\}\}} + \mathbf{c}_Y I_{\{\cup_{n=1}^N \{|Y^n| \geq m\}\}} \in \mathcal{C}_0. \tag{80}$$

In Definition 2, the set $\mathcal{C}_0^{(\mathbf{n})}$ is closed under truncation.

Theorem 7 *Let $\mathcal{C} = \mathcal{C}_0 \cap M^{\Phi}$ and suppose that $\mathcal{C}_0 \subseteq \mathcal{C}_{\mathbb{R}}$ is closed for the convergence in probability and closed under truncation. For any $\mathbf{X} \in M^{\Phi}$ there exists $\mathbf{Y_X} \in \mathcal{C}_0 \cap L^1(\mathbb{P}; \mathbf{Q_X})$ such that*

$$\sum_{n=1}^{N} Y^n_{\mathbf{X}} \in \mathbb{R}, \quad \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n_{\mathbf{X}})\right] \geq B, \quad \sum_{n=1}^{N} \left(\mathbb{E}_{Q^n_X}\left[Y^n_{\mathbf{X}}\right] - Y^n_{\mathbf{X}}\right) = 0,$$

$$\begin{aligned}\rho_B(\mathbf{X}) &= \inf\left\{\sum_{n=1}^{N} Z^n \mid \mathbf{Z} \in \mathcal{C}_0 \cap M^{\Phi}, \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Z^n)\right] \geq B\right\} = \sum_{n=1}^{N} Y^n_{\mathbf{X}} \\ &= \inf\left\{\sum_{n=1}^{N} Z^n \mid \mathbf{Z} \in \mathcal{C}_0 \cap L^1(\mathbb{P}; \mathbf{Q_X}), \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Z^n)\right] \geq B\right\} := \widetilde{\rho}_B(\mathbf{X}),\end{aligned} \tag{81}$$

so that $\mathbf{Y_X}$ is the optimal solution to the extended problem $\widetilde{\rho}_B(\mathbf{X})$.

Proof The optimal solution $\mathbf{Y_X}$ coincides with the vector $\mathbf{Y}$ in Theorem 5, which belongs to $L^1(\mathbb{P}; \mathbf{Q_X})$, by Theorem 5, Lemma 5, Lemma 6, and to $\mathcal{C}_0$, as $\mathcal{C}_0 \ni \mathbf{Y}_m \to \mathbf{Y}$ $\mathbb{P}$-a.s. and $\mathcal{C}_0$ is closed for the convergence in probability. Comparing Theorem 7 with Theorem 5 we see that it remains to prove $\rho_B = \widetilde{\rho}_B$ (Proposition 15) and $\sum_{n=1}^{N} \left(\mathbb{E}_{Q^n_X}\left[Y^n_{\mathbf{X}}\right] - Y^n_{\mathbf{X}}\right) \leq 0$ (Proposition 14), where the truncation assumption on $\mathcal{C}_0$ is needed. The opposite inequality

$$\sum_{n=1}^{N} Y^n_{\mathbf{X}} = \rho_B(\mathbf{X}) = \rho_B^{\mathbf{Q_X}}(\mathbf{X}) \leq \sum_{n=1}^{N} \mathbb{E}_{Q^n_X}\left[Y^n_{\mathbf{X}}\right]$$

holds as $\mathbf{Y_X}$ fulfills the budget constraints of $\rho_B^{\mathbf{Q_X}}(\mathbf{X})$.

Proposition 14 *Suppose that $\mathcal{C}_0$ is closed under truncation. Then*

$$\sum_{n=1}^{N} \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n] \leq \sum_{n=1}^{N} Y^n, \ \textit{for all } \mathbf{Y} \in \mathcal{C}_0 \cap L^1(\mathbf{Q_X}; \mathbb{R}^N).$$

Proof Let $\mathbf{Y} \in \mathcal{C}_0 \cap L^1(\mathbf{Q_X}; \mathbb{R}^N)$ and consider $\mathbf{Y}_m$ for $m \in \mathbb{N}$ as defined in (80), where w.l.o.g. we assume $m_Y = 1$. Note that $\sum_{n=1}^{N} Y^n_m = c_Y (= \sum_{n=1}^{N} Y^n)$ for all $m \in \mathbb{N}$. By boundedness of $\mathbf{Y}_m$ and (80), we have $\mathbf{Y}_m \in \mathcal{C}_0 \cap M^{\Phi}$ for all $m \in \mathbb{N}$. Further, $\mathbf{Y}_m \to \mathbf{Y}$ $\mathbf{Q_X}$-a.s. for $m \to \infty$, and thus, since $|\mathbf{Y}_m| \leq \max\{|\mathbf{Y}|, |\mathbf{c}_Y|\} \in L^1(\mathbf{Q_X}; \mathbb{R}^N)$ for all $m \in \mathbb{N}$, also $\mathbf{Y}_m \to \mathbf{Y}$ in $L^1(\mathbf{Q_X}; \mathbb{R}^N)$ for $m \to \infty$ by dominated convergence. We then obtain

$$\sum_{n=1}^{N} \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n] = \lim_{m\to\infty} \sum_{n=1}^{N} \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_m] \leq \lim_{m\to\infty} \sum_{n=1}^{N} Y^n_m = c_Y = \sum_{n=1}^{N} Y^n.$$

The map $\widetilde{\rho}_B$ is defined on M^{Φ} but the admissible claims $\mathbf{Y}$ belongs to the set $\mathcal{C}_0 \cap L^1(\mathbb{P}; \mathbf{Q_X})$, not included in M^{Φ}. As $L^1(\mathbb{P}; \mathbf{Q_X}) \subseteq L^1(\mathbb{P}; \mathbb{R}^N)$ with the same argument used in the proof of Proposition 1, we can show that $\widetilde{\rho}_B(\mathbf{X}) > -\infty$ for all

$\mathbf{X} \in M^{\Phi}$. By the same argument in the proof of Proposition 2 and by (26) we also deduce that $\widetilde{\rho}_B(\mathbf{X}) < +\infty$ for all $\mathbf{X} \in M^{\Phi}$, so that

$$\widetilde{\rho}_B : M^{\Phi} \to \mathbb{R}$$

is convex and monotone decreasing on its domain $dom(\widetilde{\rho}) = M^{\Phi}$. From Theorem 9, we then know that the penalty functions of ρ_B and $\widetilde{\rho}_B$ are defined as:

$$\alpha_{\Lambda,B}(\mathbf{Q}) := \sup \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-X^n] - \rho_B(\mathbf{X}) \mid \mathbf{X} \in M^{\Phi} \right\},$$

$$\widetilde{\alpha}_{\Lambda,B}(\mathbf{Q}) := \sup \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-X^n] - \widetilde{\rho}_B(\mathbf{X}) \mid \mathbf{X} \in M^{\Phi} \right\}.$$

In order to prove $\rho_B = \widetilde{\rho}_B$ we first show that $\widetilde{\alpha}_{\Lambda,B}(\mathbf{Q}) = \alpha_{\Lambda,B}(\mathbf{Q})$. Set

$$\mathcal{D}(L) := dom(\alpha_{\Lambda,B}) \cap \left\{ \frac{d\mathbf{Q}}{d\mathbb{P}} \in L_+^{\Phi^*} \mid Q^n(\Omega) = 1 \text{ and } \sum_{n=1}^{N} (\mathbb{E}_{Q^n}[Y^n] - Y^n) \le 0 \text{ for all } \mathbf{Y} \in \mathcal{C}_0 \cap L \right\}.$$

Lemma 7 *If* $\mathbf{Q} \in \mathcal{D}(L^1(\mathbb{P}; \mathbf{Q_X}))$ *then* $\widetilde{\alpha}_{\Lambda,B}(\mathbf{Q}) = \alpha_{\Lambda,B}(\mathbf{Q})$.

Proof In the proof, we will suppress the labels Λ and B from the penalty functions. From (29) and (61), note that the penalty function can also be written as

$$\begin{aligned} \alpha(\mathbf{Q}) &= \sup_{\mathbf{Z} \in M^{\Phi}} \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-Z^n] \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(Z^n)\right] \ge B \right\} \\ &= \sup_{\mathbf{Z} \in L^1(\mathbb{P};\mathbf{Q})} \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-Z^n] \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(Z^n)\right] \ge B \right\}. \end{aligned}$$

Let $\mathbf{Q} \in \mathcal{D}(L^1(\mathbb{P}; \mathbf{Q_X}))$ and recall that $\mathbf{X} \in M^{\Phi} \subseteq L^1(\mathbb{P}; \mathbf{Q_X})$, so that $\mathbf{W} := \mathbf{X} + \mathbf{Z} \in L^1(\mathbb{P}; \mathbf{Q_X})$ for $\mathbf{X} \in M^{\Phi}$ and $\mathbf{Z} \in L^1(\mathbb{P}; \mathbf{Q_X})$. Set $\mathbb{E}[\Lambda(\mathbf{X}+\mathbf{Z})] = \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Z^n)\right]$. We then have that

$$\begin{aligned} \widetilde{\alpha}(\mathbf{Q}) &= \sup \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-X^n] - \widetilde{\rho}_B(\mathbf{X}) \mid \mathbf{X} \in M^{\Phi} \right\} \\ &= \sup_{\mathbf{X} \in M^{\Phi}} \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-X^n] + \sup \left\{ -\sum_{n=1}^{N} Z^n \mid \mathbf{Z} \in \mathcal{C}_0 \cap L^1(\mathbb{P}; \mathbf{Q_X}),\, \mathbb{E}[\Lambda(\mathbf{X}+\mathbf{Z})] \ge B \right\} \right\} \\ &= \sup \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-X^n] - \sum_{n=1}^{N} Z^n \mid \mathbf{Z} \in \mathcal{C}_0 \cap L^1(\mathbb{P}; \mathbf{Q_X}),\, \mathbf{X} \in M^{\Phi},\, \mathbb{E}[\Lambda(\mathbf{X}+\mathbf{Z})] \ge B \right\} \\ &\le \sup \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-X^n] - \sum_{n=1}^{N} Z^n \mid \mathbf{Z} \in \mathcal{C}_0 \cap L^1(\mathbb{P}; \mathbf{Q_X}),\, \mathbf{X} \in L^1(\mathbb{P}; \mathbf{Q_X}),\, \mathbb{E}[\Lambda(\mathbf{X}+\mathbf{Z})] \ge B \right\} \\ &= \sup \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-W^n] + \sum_{n=1}^{N} \mathbb{E}_{Q^n}[Z^n] - \sum_{n=1}^{N} Z^n \mid \mathbf{Z} \in \mathcal{C}_0 \cap L^1(\mathbb{P}; \mathbf{Q_X}),\, \mathbf{W} \in L^1(\mathbb{P}; \mathbf{Q_X}),\, \mathbb{E}[\Lambda(\mathbf{W})] \ge B \right\} \\ &= \sup_{\mathbf{W} \in L^1(\mathbb{P};\mathbf{Q_X})} \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-W^n] \mid \mathbb{E}[\Lambda(\mathbf{W})] \ge B \right\} + \sup \left\{ \sum_{n=1}^{N} \left(\mathbb{E}_{Q^n}[Z^n] - Z^n\right) \mid \mathbf{Z} \in \mathcal{C}_0 \cap L^1(\mathbb{P}; \mathbf{Q_X}) \right\} \\ &\le \sup_{\mathbf{W} \in L^1(\mathbb{P};\mathbf{Q_X})} \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-W^n] \mid \mathbb{E}[\Lambda(\mathbf{W})] \ge B \right\} = \alpha(\mathbf{Q}), \end{aligned}$$

because $\mathbf{Q} \in \mathcal{D}(L^1(\mathbb{P};\mathbf{Q_X}))$ implies $\sum_{n=1}^N \left(\mathbb{E}_{Q^n}[Z^n] - Z^n\right) \leq 0$ for all $\mathbf{Z} \in \mathcal{C}_0 \cap L^1(\mathbb{P};\mathbf{Q_X})$.

The opposite inequality is trivial, as $\widetilde{\rho}_B \leq \rho_B$ implies

$$\begin{aligned}\widetilde{\alpha}(\mathbf{Q}) &= \sup\left\{\sum_{n=1}^N \mathbb{E}_{Q^n}[-X^n] - \widetilde{\rho}_B(\mathbf{X}) \mid \mathbf{X} \in M^\Phi\right\}\\ &\geq \sup\left\{\sum_{n=1}^N \mathbb{E}_{Q^n}[-X^n] - \rho_B(\mathbf{X}) \mid \mathbf{X} \in M^\Phi\right\} = \alpha(\mathbf{Q}).\end{aligned}$$

Proposition 15 *If $\mathcal{C}_0$ is closed under truncation, then*

$$\rho_B(\mathbf{X}) = \widetilde{\rho}_B(\mathbf{X}) = \inf_{\mathbf{Z}\in L^1(\mathbb{P};\mathbf{Q_X})}\left\{\sum_{n=1}^N Z^n \mid \mathbf{Z} \in \mathcal{C}_0,\ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Z^n)\right] \geq B\right\}$$

Proof We know that $\widetilde{\rho}_B : M^\Phi \to \mathbb{R}$ is convex and monotone decreasing. By definition, $\widetilde{\rho}_B \leq \rho_B$. Under the truncation assumption, in Proposition 14 we proved that $\mathbf{Q_X} \in \mathcal{D}(L^1(\mathbf{Q_X})) \subseteq \mathcal{D}(L^1(\mathbb{P};\mathbf{Q_X}))$ and Lemma 7 shows that then $\widetilde{\alpha}_{\Lambda,B}(\mathbf{Q_X}) = \alpha_{\Lambda,B}(\mathbf{Q_X})$. Then, by Theorem 9,

$$\begin{aligned}\widetilde{\rho}_B(\mathbf{X}) &= \sup\left\{\sum_{n=1}^N \mathbb{E}_{Q^n}[-X^n] - \widetilde{\alpha}_{\Lambda,B}(\mathbf{Q}) \mid \frac{d\mathbf{Q}}{dP} \in L^{\Phi^*}\right\} \geq \sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[-X^n] - \widetilde{\alpha}_{\Lambda,B}(\mathbf{Q_X})\\ &= \sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[-X^n] - \alpha_{\Lambda,B}(\mathbf{Q_X}) = \rho_B(\mathbf{X}).\end{aligned}$$

From Lemma 3, Proposition 5, and Corollary 1 we already know that, for $A = \rho_B(\mathbf{X})$, the optimal values satisfy

$$\rho_B(\mathbf{X}) = \rho_B^{\mathbf{Q_X}}(\mathbf{X}) = \widetilde{\rho}_B^{\mathbf{Q_X}}(\mathbf{X}) = \widehat{\rho}_B^{\mathbf{Q}}(\mathbf{X}), \tag{82}$$

$$\pi_A(\mathbf{X}) = \pi_A^{\mathbf{Q_X}}(\mathbf{X}) = \widetilde{\pi}_A^{\mathbf{Q_X}}(\mathbf{X}) = \widehat{\pi}_A^{\mathbf{Q}}(\mathbf{X}). \tag{83}$$

From Theorem 7, Lemma 3 and by the same arguments applied in Proposition 13, Corollary 4 and Remark 11 we conclude:

Corollary 5 *Let $A = \rho_B(\mathbf{X})$. Under the same assumptions of Theorem 7, we have $\rho_B(\mathbf{X}) = \widetilde{\rho}_B(\mathbf{X})$. The unique optimal solutions to the extended problems $\widetilde{\rho}_B^{\mathbf{Q_X}}(\mathbf{X})$, $\widehat{\rho}_B^{\mathbf{Q_X}}(\mathbf{X})$, $\widetilde{\rho}_B(\mathbf{X})$ and $\widetilde{\pi}_A^{\mathbf{Q_X}}(\mathbf{X})$, $\widehat{\pi}_A^{\mathbf{Q_X}}(\mathbf{X})$ exist, coincide with*

$$\mathbf{Y_X} = \mathbf{Y_{Q_X}} = -X^n - v_n'\left(\lambda^* \frac{dQ^n_{\mathbf{X}}}{d\mathbb{P}}\right) \in \mathcal{C}_0 \cap L^1(\mathbb{P};\mathbf{Q_X}),$$

and $\mathbf{Q_X}$ is the unique optimal solution to the dual problem (28).

Remark 13 Under the Assumption (85) and if $\mathcal{C}_0$ is closed under truncation then

$$\pi_A(\mathbf{X}) = \widetilde{\pi}_A(\mathbf{X}) := \sup_{Z\in L^1(\mathbb{P};\mathbf{Q_X})}\left\{\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Z^n)\right] \mid \mathbf{Z} \in \mathcal{C}_0,\ \sum_{n=1}^N Z^n \leq A\right\}. \tag{84}$$

Indeed, (84) is a consequence of Proposition 15 and of the equivalence $B = \widetilde{\pi}_A(\mathbf{X})$ iff $A = \widetilde{\rho}_B(\mathbf{X})$, that can be shown similarly as in Proposition 8, by using the Assumption (85) and Remark 14.

Remark 14 (i) Let $\mathbf{X} \in M^{\Phi}$. If $\mathbf{Y} \in M^{\Phi}$, then the function $F(\delta) := \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n - \delta)\right]$, $\delta \in \mathbb{R}$, is finite valued and concave on $\mathbb{R}$, hence continuous on $\mathbb{R}$. However, when $\mathbf{Y} \in L^1(\mathbf{Q})$ satisfies $\mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] > B$ (with the understanding that $u_n(X^n + Y^n) \in L^1(\mathbb{P})$ for each n), it is not any more evident if F is continuous on $\mathbb{R}$, as one has to guarantee that $\mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n - \delta)\right] > -\infty$, for $\delta > 0$. Set $A_n := \{X^n + Y^n > k_n\}$ and let $k_n \in \mathbb{R}$ satisfy $\mathbb{P}(A_n) > 0$ and $Q^n(A_n) > 0$. For any $\delta > 0$, $(Y^n - \delta 1_{A_n})_n \in L^1(\mathbf{Q})$ and one has

$$\begin{aligned}
&\mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n - \delta 1_{A_n})\right] \\
&\qquad = \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) 1_{A_n^C}\right] + \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n - \delta) 1_{A_n})\right] \\
&\qquad \geq \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) 1_{A_n^C}\right] + \mathbb{E}\left[\sum_{n=1}^{N} u_n(k_n - \delta) 1_{A_n})\right] > -\infty,
\end{aligned}$$

so that it is possible to find $\delta > 0$ such that $\mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n - \delta 1_{A_n})\right] = B$ and $\mathbb{E}_{Q^n}[Y^n - \delta 1_{An}] < \mathbb{E}_{Q^n}[Y^n]$. This argument works when $(Y^n - \delta 1_{A_n})_n$ is not required to belong to $\mathcal{C}_{\mathbb{R}}$.

(ii) Consider the following assumption on the utility function u at $-\infty$:

$$\forall \delta > 0\ \exists M = M(\delta),\ K = K(\delta) \text{ and } x_0 = x_0(\delta) < 0 \text{ such that } u(x - \delta) \geq M u(x) + K\ \forall x < x_0. \tag{85}$$

Such assumption is clearly satisfied if $\overline{\lim}_{x \to -\infty} \frac{u(x-\delta)}{u(x)} < +\infty\ \forall \delta > 0$. If $\mathbf{X} \in M^{\Phi}$ and $\mathbf{Y} \in L^1(\mathbf{Q})$ satisfy $\mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] > B$, then, under (85), $F(\delta)$ is finite valued and continuous on $\mathbb{R}$ and $(Y^n - \delta)_n \in \mathcal{C}_{\mathbb{R}}$ if so is $\mathbf{Y}$.

6 The exponential case

In this section, we focus on a relevant case under Assumption 1, i.e., we set $\mathcal{C} = \mathcal{C}^{(\mathbf{n})}$, see Examples 1 and 2, and we choose $u_n(x) = -e^{-\alpha_n x}$, $\alpha_n > 0$, $n = 1, \cdots, N$, as in Example 3 . We select $B < \sum_{n=1}^{N} u_n(+\infty) = 0$. Under these assumptions, $\phi_n(x) := -u_n(-|x|) + u_n(0) = e^{\alpha_n |x|} - 1$,

$$M^{\phi_n} = M^{\phi_0} := \left\{ X \in L^0(\mathbb{R}) \mid \mathbb{E}[e^{c|X|}] < +\infty \text{ for all } c > 0 \right\},$$

the Orlicz Hearts M^{ϕ_n}, $n = 1, \cdots, N$, coincide with the single Orlicz Heart M^{ϕ_0} associated to the exponential function $\phi_0(x) := e^{|x|} - 1$ and the random variable $\overline{X} := \sum_n X^n \in M^{\phi_0}$ is well defined.

The systemic risk measure (22) becomes

$$\begin{aligned}
\rho(\mathbf{X}) &= \inf \left\{ \sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C}^{(\mathbf{n})}, \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] = B \right\}, \\
&= \inf \left\{ \sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C}^{(\mathbf{n})}, \mathbb{E}\left[-\sum_{n=1}^{N} \exp\left[-\alpha_n (X^n + Y^n)\right]\right] = B \right\}.
\end{aligned} \tag{86}$$

For a given partition $\mathbf{n}$ and allocations $\mathcal{C}^{(\mathbf{n})}$, we can explicitly compute the unique optimal allocation $\mathbf{Y}$ of (86) and the corresponding systemic risk

$$\rho(\mathbf{X}) = \sum_{i=1}^{N} Y^i = \sum_{m=1}^{h} d_m.$$

Theorem 8 *For $m = 1, \cdots, h$, and for $k \in I_m$ we have that*

$$d_m = \beta_m \log\left(-\frac{\beta}{B}\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]\right) - A_m \tag{87}$$

$$Y^k_m = -X^k + \frac{1}{\beta_m \alpha_k}\overline{X}_m + \frac{1}{\beta_m \alpha_k} d_m + \left(\frac{1}{\beta_m \alpha_k} A_m - A^k_m\right) \in M^{\phi_0}, \tag{88}$$

where $\overline{X}_m = \sum_{k \in I_m} X^k$ and

$$\beta_m = \sum_{k \in I_m} \frac{1}{\alpha_k}, \qquad \beta = \sum_{i=1}^{N} \frac{1}{\alpha_i},$$

$$A^k_m = \frac{1}{\alpha_k} \log\left(\frac{1}{\alpha_k}\right), \qquad A_m = \sum_{k \in I_m} A^k_m.$$

Proof In Appendix A.

Remark 15 Note that if we arbitrarily change the components of the vector $\mathbf{X}$, but keep fixed the components in one given subgroup, say I_{m_0}, then the risk measure $\rho(\mathbf{X})$ will of course change, but d_{m_0} and $Y^k_{m_0}$ for $k \in I_{m_0}$ remain the same.

From Propositions 3, 2 and Theorem 8 we deduce

Proposition 16 *The map ρ in* (86) *is finitely valued, monotone decreasing, convex, continuous and subdifferentiable on the Orlicz Heart $M^{\Phi} = (M^{\phi_0})^N$, and it has a unique optimal solution.*

Define:

$$\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}} := \frac{e^{-\frac{1}{\beta_m}\overline{X}_m}}{\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]} \quad m = 1, \cdots, h. \tag{89}$$

Proposition 17 *The vector $\mathbf{Q_X}$ of probability measures with densities given by (89) is the optimal solution of the dual problem (36), i.e.,*

$$\rho(\mathbf{X}) = \sum_{m=1}^{h} \mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{X}_m] - \alpha_{\Lambda,B}(\mathbf{Q_X}),$$

and $\mathbb{E}_{Q^m_{\mathbf{X}}}[Y^n_{\mathbf{X}}]$, $m = 1, \cdots, h$, $n \in I_m$, is a systemic risk allocation, as in Definition 1.

Proof First note that

$$\sum_{i\in I_m}\frac{1}{\alpha_i}\ln\left(-\frac{B}{\beta\alpha_i}\right)=-\beta_m\ln\left(-\frac{\beta}{B}\right)+A_m,$$

and

$$H\left(Q^m_{\mathbf{X}},\mathbb{P}\right)=\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\ln\left(\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}}\right)\right]=\frac{1}{\beta_m}\mathbb{E}_{Q^m_{\mathbf{X}}}\left[-\overline{X}_m\right]-\ln\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]. \tag{90}$$

By (41), $\alpha_{\Lambda,B}(\mathbf{Q_X})$ can be rewritten as

$$\begin{aligned}\alpha_{\Lambda,B}(\mathbf{Q_X})&=\sum_{m=1}^{h}\sum_{i\in I_m}\left\{\frac{1}{\alpha_i}H\left(Q^m_{\mathbf{X}},\mathbb{P}\right)+\frac{1}{\alpha_i}\ln\left(-\frac{B}{\beta\alpha_i}\right)\right\}\\&=\sum_{m=1}^{h}\left(\beta_m H\left(Q^m_{\mathbf{X}},\mathbb{P}\right)+\sum_{i\in I_m}\frac{1}{\alpha_i}\ln\left(-\frac{B}{\beta\alpha_i}\right)\right)\\&=\sum_{m=1}^{h}\left(\mathbb{E}_{Q^m_{\mathbf{X}}}\left[-\overline{X}_m\right]-\beta_m\ln\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]-\beta_m\ln\left(-\frac{\beta}{B}\right)+A_m\right)\\&=\sum_{m=1}^{h}\left(\mathbb{E}_{Q^m_{\mathbf{X}}}\left[-\overline{X}_m\right]-\beta_m\log\left(-\frac{\beta}{B}\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]\right)+A_m\right)\\&=\sum_{m=1}^{h}\left(\mathbb{E}_{Q^m_{\mathbf{X}}}\left[-\overline{X}_m\right]-d_m\right)=\sum_{m=1}^{h}\mathbb{E}_{Q^m_{\mathbf{X}}}\left[-\overline{X}_m\right]-\rho(\mathbf{X}).\end{aligned}$$

Remark 6 concludes the proof.

6.1 Sensitivity analysis

Let $\mathbf{X}\in M^{\Phi}$, $\mathbf{V}\in M^{\Phi}$ and set $\overline{V}_m:=\sum_{k\in I_m}V_k$, for $m=1,\cdots,h$. We consider a perturbation $\varepsilon\mathbf{V}$, $\varepsilon\in\mathbb{R}$, and perform a sensitivity analysis in the exponential case. Consider the optimal allocations $Y^i_{\mathbf{X}+\varepsilon\mathbf{V}}$ and the optimal solution $\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}}$ of the dual problem associated to $\rho(\mathbf{X}+\varepsilon\mathbf{V})$, see (89). By (88) and (87) we have

$$Y^n_{\mathbf{X}+\varepsilon\mathbf{V}}=-X^n-\varepsilon V^n+\frac{1}{\beta_m\alpha_n}\left(\overline{X}_m+\varepsilon\overline{V}_m\right)+\frac{1}{\beta_m\alpha_n}d_m(\mathbf{X}+\varepsilon\mathbf{V})+\left(\frac{1}{\beta_m\alpha_n}A_m-A^n_m\right), \tag{91}$$

where

$$d_m(\mathbf{X}+\varepsilon\mathbf{V})=\beta_m\log\left(-\frac{\beta}{B}\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\varepsilon\overline{V}_m}{\beta_m}\right)\right]\right)-A_m. \tag{92}$$

Proposition 18 *Let ρ be the systemic risk measure defined in (86). Then*

1. *Marginal risk contribution of group m:*

$$\left.\frac{d}{d\epsilon}d_m(\mathbf{X}+\varepsilon\mathbf{V})\right|_{\varepsilon=0}=\mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m],\quad m=1,...,h.$$

2. *Local causal responsibility:*

$$\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}}}[Y^n_{\mathbf{X}+\varepsilon\mathbf{V}}]\Big|_{\varepsilon=0} = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n], \quad n \in I_m.$$

3. $\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Z]\Big|_{\varepsilon=0} = -\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m, Z]$, *for any* $Z \in M^{\Phi}(\mathbb{R})$,
4. *Marginal risk allocation of institution* $n \in I_m$:

$$\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^n_{\mathbf{X}+\varepsilon\mathbf{V}}]\Big|_{\varepsilon=0} = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n] - \frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m, Y^n_{\mathbf{X}}] \tag{93}$$

$$= \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n] + \frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m, X^n] - \frac{1}{\alpha_n}\frac{1}{\beta_m}\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m, \overline{X}_m], \tag{94}$$

5. *Sensitivity of the penalty function:*

$$\frac{d}{d\varepsilon}\alpha_{\Lambda,B}(\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}})\Big|_{\varepsilon=0} = \sum_{m=1}^{h}\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m, \overline{X}_m],$$

6. *Systemic marginal risk contribution:*

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{V})\Big|_{\varepsilon=0} = \sum_{m=1}^{h}\sum_{i\in I_m}\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^i] = \sum_{m=1}^{h}\mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m].$$

The proof is postponed to the Appendix. The interpretation of these formulas is not simple because we are *dealing with the systemic probability measure* $Q^m_{\mathbf{X}}$ *and not with the "physical" measure* $\mathbb{P}$. Indeed, $Q^m_{\mathbf{X}}$ is the "artificial" measure that emerges from the dual optimization (think of the difference between the physical measure $\mathbb{P}$ and a martingale measure). To fix the idea, let us take $\mathbf{V}$ with only one component different from 0, so that we write $\mathbf{V} = V^j\mathbf{e}_j$. From Item 1 (or Item 6), we see that

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon V^j\mathbf{e}_j)\Big|_{\varepsilon=0} = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^j] \text{ (with } j \text{ belonging to the group } m\text{)}.$$

From this, we can interpret $Q^m_{\mathbf{X}}$ as systemic risk evaluation (systemic probability measure): i.e. if the position changes from $\mathbf{X}$ to $\mathbf{X}+\varepsilon V^j\mathbf{e}_j$ then the riskiness of the entire system changes linearly by $\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^j]$. In the following discussion, we have to keep in mind that $Q^m_{\mathbf{X}}$ already represents the systemic view of the system. If we replace $\mathbf{Q}_{\mathbf{X}}$ with $\mathbb{P}$, none of the results of Proposition 18 will hold in general.

Remark 16 We now comment on the results of Proposition 18.
The first term $\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n]$ in (93) or (94) is easy to interpret: it is not a systemic contribution, as it only involves the increment V^n in the (same) bank n. If we sum over all n in the same group, we obtain from (93) or (94)

$$\sum_{n\in I_m}\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^n_{\mathbf{X}+\varepsilon\mathbf{V}}]\Big|_{\varepsilon=0} = \mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m] = \frac{d}{d\varepsilon}d_m(\mathbf{X}+\varepsilon V)\Big|_{\varepsilon=0}, \tag{95}$$

as it should be. So, this first term $\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n]$ is the contribution to the marginal risk allocation of bank n regardless of any systemic influence. When summing up we get the marginal risk allocation of the whole group. Equation (95) is the Local Casual

Responsibility for the whole group, but not for the single bank inside each group. Note that the sign of the increment V^n in the first term of (93) is here relevant: an increment (positive) corresponds to a risk reduction, regardless of the dependence structure. If $\mathbf{V}$ is deterministic, the marginal risk allocation to bank n is exactly $\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n] = -V^n$ and no other correction terms are present.

To understand the other terms in (93) or (94), take $\mathbf{V} = V^j \mathbf{e}_j$ with $j \neq n$. In this way, the first term in (93) disappears ($V^n = 0$) and we obtain

$$\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon V^j \mathbf{e}_j}}[Y^n_{\mathbf{X}+\varepsilon V^j \mathbf{e}_j}]\bigg|_{\varepsilon=0} = \frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[V^j, X^n] - \frac{1}{\alpha_n}\frac{1}{\beta_m}\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[V^j, \overline{X}_m].$$

To fix the ideas, suppose that $COV_{Q^m_{\mathbf{X}}}[V^j, X^n] < 0$, and examine for the moment only the contribution of $\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[V^j, X^n]$. This component does not depend on the "systemic relevance" of bank n (i.e. it does not depend on the specific α_n) but it depends on the dependence structure between (V^j, X^n). If the systemic risk evaluation $Q^m_{\mathbf{X}}$ attributes negative correlation to (V^j, X^n), then, from the systemic perspective this is good (independently of the sign of V^j): a decrement in bank j is balanced by bank n, and viceversa. If bank n is negatively correlated (as seen by $Q^m_{\mathbf{X}}$) with the increment of bank j, then the risk allocation of bank n should decrease. Therefore, bank n takes advantage of this, as its risk allocation is reduced ($\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[V^j, X^n] < 0$). Since the overall marginal risk allocation of the group m is fixed (equal to $\mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m] = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^j]$, from (95)), someone else has to pay for such advantage to bank n. This is the last term in (94), discussed next.

For the third component in (94), we distinguish between the systemic component $-\frac{1}{\beta_m}\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[V^j, \overline{X}_m]$, which only depends on the aggregate group $\overline{X}_m$, and the *systemic relevance* $\frac{1}{\alpha_n}$ of bank n. The systemic quantity is therefore distributed among the various banks according to $\frac{1}{\alpha_n}$. In addition, this term must compensate for the possible risk reduction term (the second term in (94)), as the overall risk allocation to group m is determined by $\mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m] = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^j]$.
Note that if $\alpha_n = \alpha$ then we may rewrite Item 4 as

$$\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon \mathbf{V}}}[Y^n_{\mathbf{X}+\varepsilon \mathbf{V}}]\bigg|_{\varepsilon=0} = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n] + \alpha COV_{Q^m_{\mathbf{X}}}\left[\frac{\overline{V}_m}{N_m}, X^n\right] - \alpha COV_{Q^m_{\mathbf{X}}}\left[\frac{\overline{V}_m}{N_m}, \frac{\overline{X}_m}{N_m}\right]$$

where N_m is the number of banks in group I_m.
Finally Items 1 and 6 express the same property (which holds in general, as shown in Proposition 10) respectively for one group or for the entire system.

6.2 Monotonicity

Consider a fixed $\mathbf{X} \in M^{\Phi}$. For a given partition $\mathbf{n}$ and $\mathcal{C} = \mathcal{C}^{(\mathbf{n})}$, let Y^k_r, $k \in I_r$, $r = 1, \cdots, h$, be the corresponding optimal allocations of the primal problem (86) and $Q^r_{\mathbf{X}}$, $r = 1, \cdots, h$, be the optimal solutions of the corresponding dual problem (36) (in this section we suppress the label $\mathbf{X}$ from the optimal allocation $\mathbf{Y}_{\mathbf{X}}$ to $\rho(\mathbf{X})$).
Consider for some $m \in \{1, \cdots, h\}$ a non empty subgroup I'_m of the group I_m. Set $I''_m := I_m \backslash I'_m$. Then the $(h+1)$ groups $I_1, I_2, \cdots, I'_m, I''_m, I_{m+1}, \cdots, I_h$ corresponds to a new partition $\mathbf{n}'$. The optimal allocations of the primal problem (86) with $\mathcal{C} = \mathcal{C}^{(\mathbf{n}')}$ coincide with Y^k_r, $k \in I_r$, for $r \neq m$.

The interpretation of the monotonicity condition (98) was already formulated at the end of Section 2. Its generalization in the context of h groups is formulated below in (96).

For $r = m$, $i \in I'_m$, we have the following.

Proposition 19 *Define with $Y^i_{m'}$, $i \in I'_m$, the optimal allocation to the primal problem with $\mathcal{C} = \mathcal{C}^{(\mathbf{n}')}$. Then*

$$\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\sum_{i\in I'_m} Y^i_m\right] \le \sum_{i\in I'_m} Y^i_{m'} := d'_m. \tag{96}$$

In particular, if the group I'_m consists of only one single element $\{i\}$, then $Y^i_{m'}$ is deterministic and

$$\mathbb{E}_{Q^m_{\mathbf{X}}}[Y^i_m] \le Y^i_{m'} \quad \text{for each } i \in I_m. \tag{97}$$

If we compare the deterministic optimal allocation $\mathbf{Y}^*$ (corresponding to $\mathcal{C} = \mathbb{R}^N$) with the (random) optimal allocations $\mathbf{Y}$ associated to one single group ($\mathcal{C} = \mathcal{C}_{\mathbb{R}} \cap M^{\Phi}$), we conclude

$$\mathbb{E}_{Q_{\mathbf{X}}}[Y^n] \le (Y^*)^n \text{ for each } n = 1, \cdots, d, \tag{98}$$

where $Q_{\mathbf{X}}$ is the unique optimal solution of the dual problem associated to $\mathcal{C} = \mathcal{C}_{\mathbb{R}} \cap M^{\Phi}$.

Proof Given the subgroup I'_m, define

$$\beta'_m := \sum_{k\in I'_m} \frac{1}{\alpha_k}; \quad A'_m := \sum_{k\in I'_m} A^k_m$$
$$A' := \sum_{k\in I'_m}\left(\frac{1}{\beta_m\alpha_k}A_m - A^k_m\right) = \frac{\beta'_m}{\beta_m}A_m - A'_m.$$

with $\beta_m = \sum_{k\in I_m}\frac{1}{\alpha_k}$. Then the optimal value with respect to $\mathcal{C}^{(\mathbf{n}')}$ is given by

$$d'_m = \beta'_m \ln\left\{-\frac{\beta}{B}\mathbb{E}\left[\exp\left(-\frac{1}{\beta'_m}\sum_{k\in I'_m} X^k\right)\right]\right\} - A'_m.$$

Summing the components of the solutions relative to $\mathcal{C}^{(\mathbf{n})}$ over $k \in I'_m$, we get

$$\sum_{k\in I'_m}\mathbf{Y}^k_m = \sum_{k\in I'_m}\left(\frac{1}{\beta_m\alpha_k}\overline{X}_m - X^k\right) + \sum_{k\in I'_m}\frac{1}{\beta_m\alpha_k}d_m + \sum_{k\in I'_m}\left(\frac{1}{\beta_m\alpha_k}A_m - A^k_m\right)$$
$$= \left(\frac{\beta'_m}{\beta_m}\overline{X}_m - \sum_{k\in I'_m} X^k\right) + \frac{\beta'_m}{\beta_m}d_m + A'.$$

Using Jensen inequality we obtain

$$\begin{aligned}
&\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\sum_{k\in I'_m}\mathbf{Y}^k_m\right]\\
&=\beta'_m \ln\left\{\exp\left(\frac{1}{\beta'_m}\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\left(\frac{\beta'_m}{\beta_m}\overline{X}_m-\sum_{k\in I'_m}X^k\right)\right]\right)\right\}+\frac{\beta'_m}{\beta_m}\beta_m\log\left(-\frac{\beta}{B}\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]\right)\\
&\quad-\frac{\beta'_m}{\beta_m}A_m+A'\\
&\leq\beta'_m\ln\left\{\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\exp\left(\frac{1}{\beta_m}\overline{X}_m-\frac{1}{\beta'_m}\sum_{k\in I'_m}X^k\right)\right]\right\}+\beta'_m\log\left(-\frac{\beta}{B}\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]\right)-A'_m\\
&=\beta'_m\ln\left\{\mathbb{E}\left[\frac{\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\exp\left(\frac{1}{\beta_m}\overline{X}_m\right)\exp\left(-\frac{1}{\beta'_m}\sum_{k\in I'_m}X^k\right)}{\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]}\right]\right\}\\
&\quad+\beta'_m\log\left(-\frac{\beta}{B}\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]\right)-A'_m\\
&=\beta'_m\ln\left\{\mathbb{E}\left[\frac{\exp\left(-\frac{1}{\beta'_m}\sum_{k\in I'_m}X^k\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}\right]\frac{\beta}{\gamma}\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]\right\}-A'_m\\
&=\beta'_m\ln\left\{-\frac{\beta}{B}\mathbb{E}\left[\exp\left(-\frac{1}{\beta'_m}\sum_{k\in I'_m}X^k\right)\right]\right\}-A'_m=d'_m.
\end{aligned}$$

We have that (97) and (98) directly follow by (96).

A Appendix

A.1 Properties

Lemma 8 *Assumption 1 implies:*
(a) there exists $c\in\mathbb{R}$ and $b\in\mathbb{R}_+$ such that $u_n(x)\leq c+bx$ for all $x\geq 0$ and all n.
(b) for all n there exists $A_n\in\mathbb{R}$ and $a_n\in\mathbb{R}_+$ such that $u_n(x)\leq A_n+a_nx$ for all $x\in\mathbb{R}$.
(c) the constants b and a_n can be selected so that $a:=\min_n a_n>b$.

Proof Note that $dom(u_n)=\mathbb{R}$ for each n. Hereafter the left derivatives of the concave increasing functions u_n are denoted by u'_n and satisfy $u'_n(x)\geq 0$ for all $x\in\mathbb{R}$.

(a) The concavity of each u_n implies that $u_n(x)\leq c_n+u'_n(0)x$ for all $x\in\mathbb{R}$ (for some c_n) and therefore, setting $b:=\max_n u'_n(0)\geq 0$ and $c:=\max_n c_n$, $u_n(x)\leq c+bx$ for all $x\geq 0$.

(b) From $\lim_{x\to-\infty}\frac{u_n(x)}{x}=+\infty$ we obtain $u'_n(x)\uparrow+\infty$ as $x\downarrow-\infty$. Therefore, for each n there exists $x_n\in\mathbb{R}$ such that $u'_n(x)>b$ for all $x\leq x_n$. Then, for $x_0:=\min\{x_1,\cdots,x_N\}$, $u'_n(x)>b$ for all $x\leq x_0$. Set $a_n:=u'_n(x_0)$. Then the concavity of u_n implies: $u_n(x)\leq A_n+a_nx$ for all $x\in\mathbb{R}$ (for some A_n).

(c) Finally the construction above guarantees that $\min_n a_n=\min_n u'_n(x_0)>b$.

Proof (Proof of Proposition 1) By contradiction, we suppose that $\rho(\mathbf{X})=-\infty$, for some $\mathbf{X}\in\mathcal{L}\subseteq L^1(\mathbb{P},\mathbb{R}^N)$. Let $\mathbf{Y}_m\in\mathcal{C}$ satisfy $\sum_{n=1}^N Y^n_m\downarrow-\infty$, as $m\to+\infty$ and $\Lambda(\mathbf{X}+\mathbf{Y}_m)\in\mathbb{A}$

for each m. The condition $\sum_{n=1}^{N} Y_m^n \downarrow -\infty$, as $m \to +\infty$ implies $\sum_{n=1}^{N} \mathbb{E}[Y_m^n] \downarrow -\infty$, as $m \to +\infty$. Note also that, by Jensen inequality,

$$B \le \mathbb{E}[\Lambda(\mathbf{X} + \mathbf{Y}_m)] \le \Lambda(\mathbb{E}[\mathbf{X} + \mathbf{Y}_m]) = \sum_{n=1}^{N} u_n(\mathbb{E}[X^n] + \mathbb{E}[Y_m^n]). \tag{99}$$

We now prove that $\sum_{n=1}^{N} u_n(\mathbb{E}[X^n] + \mathbb{E}[Y_m^n]) \downarrow -\infty$, as $m \to +\infty$, which is in contradiction with (99). Set $\mathbf{x}_m := (x_m^n)_{n=1}^N$ where $x_m^n := \mathbb{E}[Y_m^n]$. Since $\sum_{n=1}^{N} x_m^n \downarrow -\infty$, there must exist $n_0 \in \{1, \cdots, N\}$ and a subsequence $\mathbf{x}_{h_m}$ such that $x_{h_m}^{n_0} \downarrow -\infty$ as $m \to +\infty$. With an abuse of notation, denote again such subsequence $\mathbf{x}_{h_m}$ with $\mathbf{x}_m$. Then we have $x_m^{n_0} \downarrow -\infty$. If there exists another coordinate $n_1 \in \{1, \cdots, N\} \setminus n_0$ such that $\liminf_{m\to\infty} x_m^{n_1} = -\infty$, take the subsequence $\mathbf{x}_{k_m}$ such that $x_{k_m}^{n_1} \downarrow -\infty$. By diagonal procedure, we obtain one single sequence denoted again by $\mathbf{x}_m$ such that $x_m^{n_0} \downarrow -\infty$ and $x_m^{n_1} \downarrow -\infty$, as $m \to +\infty$. We may adopt this procedure (at most N times) also in the case $\limsup_{m\to\infty} x_m^{n_2} = +\infty$ for some coordinate n_2. At the end, we will obtain one single sequence $\mathbf{x}_m$ and three disjoint sets of coordinate indices N_-, N_+, N^* such that

$$\begin{aligned}
&x_m^n \downarrow -\infty \quad \text{if } n \in N_- \subseteq \{1, \cdots, N\}\,,\\
&x_m^n \uparrow +\infty \quad \text{if } n \in N_+ \subseteq \{1, \cdots, N\}\,,\\
&| x_m^n | \le K \quad \text{for all } m \text{ and all } n \in N^* = \{1, \cdots, N\} \setminus (N_- \cup N_+),
\end{aligned}$$

where K is a constant independent of m. We know that $N_- \ne \emptyset$, since $n_0 \in N_-$ (but the other two sets N_+ and N^* may be empty). Since $\sum_{n=1}^{N} x_m^n \downarrow -\infty$, we deduce that, for large m, $\sum_{n=1}^{N} x_m^n \le 0$ so that

$$\sum_{n \in N_+} x_m^n \le -\sum_{n \in N_-} x_m^n - \sum_{n \in N^*} x_m^n \le -\sum_{n \in N_-} x_m^n + NK, \text{ for each fixed (large) } m. \tag{100}$$

Now we use the inequalities of Lemma 8. From $a_n \ge a$, we get (for large m) $a_n x_m^n \le a x_m^n$ when $n \in N_-$ (as $x_m^n \le 0$); for $n \in N_+$ (and large m) we have $(\mathbb{E}[X^n] + x_m^n) \ge 0$ and we can use inequality (a) in Lemma 8. Letting $d^n := \mathbb{E}[X^n]$, we obtain, for each fixed large m, that

$$\begin{aligned}
\sum_{n=1}^{N} u_n(\mathbb{E}[X^n] + \mathbb{E}[Y_m^n]) &= \sum_{n \in N_+} u_n(d^n + x_m^n) + \sum_{n \in N_-} u_n(d^n + x_m^n) + \sum_{n \in N^*} u_n(d^n + x_m^n)\\
&\le \sum_{n \in N_+} (c + b(d^n + x_m^n)) + \sum_{n \in N_-} (A_n + a_n(d^n + x_m^n)) + \sum_{n \in N^*} u_n(d^n + K)\\
&\le C + \sum_{n \in N_+} b x_m^n + \sum_{n \in N_-} a_n x_m^n\\
&\le C + bNK + (a - b) \sum_{n \in N_-} x_m^n, \qquad (101)
\end{aligned}$$

where we use (100) in inequality (101) and $C := \sum_{n \in N_+} (c + bd^n) + \sum_{n \in N_-} (A_n + a_n d^n) + \sum_{n \in N^*} u_n(d^n + K)$ is independent of m. Then $(a - b) \sum_{n \in N_-} x_m^n \downarrow -\infty$, as $m \to +\infty$, since $a > b$ by Lemma 8 and $x_m^n \downarrow -\infty$ for each $n \in N_-$. This concludes the proof.

Remark 17 Condition $\rho(\mathbf{X}) > -\infty$ is essentially a condition on the behavior of $\Lambda : \mathbb{R}^N \to \mathbb{R}$ at $-\infty$. Note that if the condition $\lim_{x\to-\infty} \frac{u_n(x)}{x} = +\infty$ is not satisfied, there might be a problem. Take $N = 2$ and the increasing concave functions

$$u_1(x) = 3x,\ u_2(x) = x.$$

Take $x_m^1 = m$, $x_m^2 = -2m$. As

$$x_m^1 + x_m^2 = -m \to -\infty, \text{ but } \Lambda(x_m) = u_1(x_m^1) + u_2(x_m^2) = 3m - 2m = m \to +\infty,$$

we cannot control $\Lambda(x)$ as in (101).

A.2 Orlicz setting

We now recall an important result for the characterization of systemic risk measures of the form (22) on the Orlicz Heart.

Theorem 9 (Theorem 1, [9]) *Suppose that $\mathcal{L}$ is a Fréchet lattice and $\rho : \mathcal{L} \to \mathbb{R} \cup \{+\infty\}$ is convex and monotone decreasing. Then*

1. *ρ is continuous in the interior of $dom(\rho)$, with respect to the topology of $\mathcal{L}$,*
2. *ρ is subdifferentiable in the interior of $dom(\rho)$,*
3. *for all $\mathbf{X} \in int(dom(\rho))$*

$$\rho(\mathbf{X}) = \max_{Q \in \mathcal{L}^*_+} \{Q(-X) - \alpha(Q)\},$$

where $\mathcal{L}^*$ is the dual of $\mathcal{L}$ (for the topology for which $\mathcal{L}$ is a Frechet lattice), $\mathcal{L}^*_+ = \{Q \in \mathcal{L}^* \mid Q \text{ is positive}\}$ and $\alpha : \mathcal{L}^* \to \mathbb{R} \cup \{+\infty\}$, defined by

$$\alpha(Q) = \sup_{\mathbf{X} \in \mathcal{L}} \{Q(-\mathbf{X}) - \rho(\mathbf{X})\},$$

is $\sigma(\mathcal{L}^*, \mathcal{L})$-lsc and convex.

A.2.1 Dual representation in the Orlicz setting

Proof (of Proposition 6)
Consider the convex functional $\Theta_n : M^{\phi_n}(\mathbb{R}) \to \mathbb{R}$ defined by $\Theta_n(Z) := \mathbb{E}[-u_n(Z)]$ and let Θ_n^* be its convex conjugate. We have: $\Theta_n(Z^n) > -\infty$, as $M^{\phi_n}(\mathbb{R}) \subseteq L^1(\mathbb{P})$ and $\mathbb{E}[u_n(Z^n)] \leq u_n(\mathbb{E}[Z^n]) < +\infty$; $\Theta_n(Z^n) < +\infty$, as $Z^n \in M^{\phi_n}(\mathbb{R})$ implies $\mathbb{E}[u_n(Z^n)] > -\infty$. Then we have $\Theta_n^*(\xi) = \mathbb{E}[v_n(-\xi)]$, for $\xi \in L^{\phi_n^*}(\mathbb{R})$ by [9], Section 5.2. Let $f : M^{\Phi} \to \mathbb{R}$ be defined by $f(\mathbf{Z}) := \sum_{n=1}^N \mathbb{E}[-u_n(Z^n)] + B = \sum_{n=1}^N \Theta_n(Z^n) + B$, and observe that

$$\mathcal{A} := \left\{ \mathbf{Z} \in M^{\Phi} \mid \sum_{n=1}^N \mathbb{E}[u_n(Z^n)] \geq B \right\} = \left\{ \mathbf{Z} \in M^{\Phi} \mid f(\mathbf{Z}) \leq 0 \right\}.$$

We have that f is convex and decreasing with respect to the order relation (18). Let $f^*(\xi)$ be its convex conjugate, for $\xi \in L^{\Phi^*}$. We assume that $\xi \neq \mathbf{0}$. By the Fenchel inequality

$$\mathbb{E}[\mathbf{Z}\xi] \leq f(\mathbf{Z}) + f^*(\xi),$$

we obtain for all $\mathbf{Z} \in \mathcal{A}$ and $\lambda > 0$,

$$\mathbb{E}[-\mathbf{Z}\xi] = \lambda \mathbb{E}[\mathbf{Z}(-\frac{1}{\lambda}\xi)] \leq \lambda [f(\mathbf{Z}) + f^*(-\frac{1}{\lambda}\xi)] \leq \lambda f^*(-\frac{1}{\lambda}\xi), \ \mathbb{P}\text{-a.s.}$$

Hence

$$\alpha_{\Lambda,B}(\xi) := \sup_{\mathbf{Z} \in \mathcal{A}} \{\mathbb{E}[-\mathbf{Z}\xi]\} \leq \inf_{\lambda > 0} \lambda f^*(-\frac{1}{\lambda}\xi). \tag{102}$$

By definition of the convex Fenchel conjugate and the fact that M^{Φ} is a product space, we have

$$\begin{aligned} f^*(\xi) \ :&= \sup_{\mathbf{Z} \in M^{\Phi}} \{\mathbb{E}[\xi \mathbf{Z}] - f(\mathbf{Z})\} \\ &= -B + \sup_{\mathbf{Z} \in M^{\Phi}} \left\{ \sum_{n=1}^N \mathbb{E}[\xi_n Z^n] - \sum_{n=1}^N \Theta_n(Z^n) \right\} \\ &= -B + \sum_{n=1}^N \left(\sup_{Z \in M^{\Phi}(\mathbb{R})} \{\mathbb{E}[\xi_n Z] - \Theta_n(Z)\} \right) \\ &= -B + \sum_{n=1}^N \Theta_n^*(\xi_n), \end{aligned}$$

where we have used (27), and therefore

$$\inf_{\lambda>0} \lambda f^*(-\frac{1}{\lambda}\xi) = \inf_{\lambda>0}\left(-B\lambda + \lambda \sum_{n=1}^{N} \Theta_n^*(-\frac{1}{\lambda}\xi_n)\right) = \inf_{\lambda>0}\left(-B\lambda + \lambda \sum_{n=1}^{N} \mathbb{E}\left[v_n\left(\frac{1}{\lambda}\xi_n\right)\right]\right).$$

We need only to prove that there is no duality gap in (102), i.e., if $\alpha_{\Lambda,B}(\xi) < +\infty$ then

$$\alpha_{\Lambda,B}(\xi) = \inf_{\lambda>0} \lambda f^*(-\frac{1}{\lambda}\xi). \tag{103}$$

Observe that, by the definition of f^*, we have for each $\lambda > 0$

$$\lambda f^*(-\frac{1}{\lambda}\xi) := \sup_{\mathbf{Z}\in M^\Phi} \{\mathbb{E}[-\xi\mathbf{Z}] - \lambda f(\mathbf{Z})\}.$$

As ξ is not identically equal to $\mathbf{0}$ and M^Φ is a linear space, we have $\sup_{\mathbf{Z}\in M^\Phi} \{\mathbb{E}[-\xi\mathbf{Z}]\} = +\infty$ and therefore

$$\inf_{\lambda>0} \lambda f^*(-\frac{1}{\lambda}\xi) = \inf_{\lambda>0} \sup_{\mathbf{Z}\in M^\Phi} \{\mathbb{E}[-\xi\mathbf{Z}] - \lambda f(\mathbf{Z})\} = \inf_{\lambda\geq 0} \sup_{\mathbf{Z}\in M^\Phi} \{\mathbb{E}[-\xi\mathbf{Z}] - \lambda f(\mathbf{Z})\}.$$

We claim that

$$\inf_{\lambda\geq 0} \sup_{\mathbf{Z}\in M^\Phi} \{\mathbb{E}[-\xi\mathbf{Z}] - \lambda f(\mathbf{Z})\} = \sup_{\mathbf{Z}\in M^\Phi} \inf_{\lambda\geq 0} \{\mathbb{E}[-\xi\mathbf{Z}] - \lambda f(\mathbf{Z})\}. \tag{104}$$

Assuming (104), we may immediately conclude that

$$\begin{aligned}\inf_{\lambda>0} \lambda f^*(-\frac{1}{\lambda}\xi) &= \sup_{\mathbf{Z}\in M^\Phi} \inf_{\lambda\geq 0} \{\mathbb{E}[-\xi\mathbf{Z}] - \lambda f(\mathbf{Z})\} = \sup_{\mathbf{Z}\in M^\Phi}\left\{\mathbb{E}[-\xi\mathbf{Z}] - \sup_{\lambda\geq 0} \lambda f(\mathbf{Z})\right\}\\ &= \sup_{\mathbf{Z}\in\mathcal{A}} \{\mathbb{E}[-\xi\mathbf{Z}]\} := \alpha_{\Lambda,B}(\xi).\end{aligned}$$

We now prove (104) by showing the equivalent condition (simply multiply each side of (104) by -1):

$$\sup_{\lambda\geq 0} \inf_{\mathbf{Z}\in M^\Phi} \{\mathbb{E}[\xi\mathbf{Z}] + \lambda f(\mathbf{Z})\} = \inf_{\mathbf{Z}\in M^\Phi} \sup_{\lambda\geq 0} \{\mathbb{E}[\xi\mathbf{Z}] + \lambda f(\mathbf{Z})\}. \tag{105}$$

In order to make an easy comparison with the results in [39], let $f_0(\mathbf{Z}) := \mathbb{E}[\xi\mathbf{Z}]$. Consider the function $F : M^\Phi \times \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$, defined by

$$F(\mathbf{Z}, u) = \begin{cases} f_0(\mathbf{Z}) \text{ if } \mathbf{Z} \in M^\Phi \text{ and } f(\mathbf{Z}) \leq u, \\ +\infty \quad \text{otherwise,} \end{cases}$$

see (2.8) in [39], and the associated Lagrangian, see (4.4) in [39],

$$K(\mathbf{Z}, \lambda) = \begin{cases} f_0(\mathbf{Z}) + \lambda f(\mathbf{Z}) & \text{if } \mathbf{Z} \in M^\Phi,\ \lambda \geq 0, \\ -\infty & \text{if } \mathbf{Z} \in M^\Phi,\ \lambda < 0, \\ +\infty & \text{if } \mathbf{Z} \notin M^\Phi. \end{cases}$$

Then (105) can be rewritten as

$$\sup_{\lambda\geq 0} \inf_{\mathbf{Z}\in M^\Phi} K(\mathbf{Z}, \lambda) = \inf_{\mathbf{Z}\in M^\Phi} \sup_{\lambda\geq 0} K(\mathbf{Z}, \lambda). \tag{106}$$

As $f : M^\Phi \to \mathbb{R}$ is convex decreasing and finite valued, Theorem 9 guarantees that it is continuous on M^Φ (for the M^Φ-norm). Therefore, see Example 1 on pages 7 and 22 in [39], the function F is closed convex in $(\mathbf{Z}, u)$.

Then the absence of duality gap, expressed by (106) follows from Theorems 17 and 18 of [39], provided that the (convex) optimal value function, defined in (4.7) [39],

$$\varphi(u) := \inf_{\mathbf{Z}\in M^\Phi} F(\mathbf{Z}, u),\ u \in \mathbb{R},$$

is bounded from above in a neighborhood of 0. Clearly, it is sufficient to show the existence of an element $\mathbf{Z}_0 \in M^\Phi$ such that $u \to F(\mathbf{Z}_0, u)$ is bounded from above in a neighborhood of 0. The assumption $\Lambda(+\infty) > B$ guarantees the existence of $\mathbf{Z}_0 \in M^\Phi$ such that $\sum_{n=1}^N \mathbb{E}[u_n(Z_0^n)] > B$ (take Z_0^n equal to some large enough constant), i.e., $f(\mathbf{Z}_0) := \sum_{n=1}^N \mathbb{E}[-u_n(Z_0^n)] + B < 0$. Set $0 < \delta < |f(\mathbf{Z}_0)|$. Hence for all $u \in \mathbb{R}$ such that $|u| < \delta$ we have $f(\mathbf{Z}_0) < u$ and $F(\mathbf{Z}_0, u) = \mathbb{E}[\xi \mathbf{Z}_0] < +\infty$, as $\mathbf{Z}_0 \in M^\Phi$ and $\xi \in L^{\Phi^*}$.

Remark 18 In [25], (103) is deduced, by different means, in a $L^\infty(\mathbb{R})$ setting and in the one-dimensional case. In [3], (103) is obtained, by different means, in the multi-dimensional deterministic case, i.e. in $\mathbb{R}^N$.

A.3 Auxiliary results for existence

The following auxiliary results are standard and can be found in many articles on utility maximization. Recall that we are working under Assumptions 2 and 3.

Lemma 9 *Let $v : \mathbb{R}_+ \to \mathbb{R}$ be a strictly convex differentiable function with $v'(0^+) = -\infty$, $v'(+\infty) = +\infty$ and let $Q \ll \mathbb{P}$. Then*
(a) $v'(\lambda \frac{dQ}{d\mathbb{P}}) \in L^1(Q)\ \forall \lambda > 0$;
(b) $F(\lambda) \triangleq \mathbb{E}[\frac{dQ}{d\mathbb{P}} v'(\lambda \frac{dQ}{d\mathbb{P}})]$ defines a bijection between $(0, +\infty)$ and $(-\infty, +\infty)$.

Lemma 10 *The convex conjugate function $v : \mathbb{R} \to (-\infty, +\infty]$ of u, given by $v(y) = \sup_{x \in \mathbb{R}} \{u(x) - xy\}$, is a proper lsc convex function, equal to $+\infty$ on $(-\infty, 0)$, bounded from below on $\mathbb{R}$, finite valued strictly convex, continuously differentiable on $(0, +\infty)$ and satisfying*

$$v(+\infty) = +\infty,\ v(0^+) = u(+\infty),\ v'(0^+) = -\infty,\ v'(+\infty) = +\infty,$$
$$u'(x) = (v')^{-1}(-x), u(-v'(y)) = -yv'(y) + v(y), \quad \forall y \geq 0,$$

where the usual rule $0 \cdot \infty = 0$ is applied.

Proposition 20 (Proposition 3.6, [11]) *Let $Q \ll \mathbb{P}$. For all $c \in \mathbb{R}$ the optimizer $\lambda(c; Q)$ of*

$$\min_{\lambda > 0} \left\{ \mathbb{E}\left[v\left(\lambda \frac{dQ}{d\mathbb{P}} \right) \right] + \lambda c \right\}$$

is the unique positive solution of the first order condition

$$\mathbb{E}_Q\left[v'\left(\lambda \frac{dQ}{d\mathbb{P}} \right) \right] + c = 0. \tag{107}$$

If $\sup \left\{ \mathbb{E}[u(g)] \mid g \in L^1(Q) \text{ and } \mathbb{E}_Q[g] \leq c \right\} < u(+\infty)$, the random variable $\widehat{g} := -v'(\lambda(c; Q) \frac{dQ}{d\mathbb{P}})$ belongs to the set $\left\{ g \in L^1(Q) \mid \mathbb{E}_Q[g] = c \right\}$, satisfies $u(\widehat{g}) \in L^1(\mathbb{P})$, and

$$\min_{\lambda > 0} \left\{ \mathbb{E}\left[v\left(\lambda \frac{dQ}{d\mathbb{P}} \right) \right] + \lambda c \right\} = \sup \left\{ \mathbb{E}[u(g)] \mid g \in L^1(Q) \text{ and } \mathbb{E}_Q[g] \leq c \right\} = \mathbb{E}[u(\widehat{g})] < u(+\infty).$$

Lemma 11 *If $\lim_{x \to -\infty} \left(\frac{u_n(x)}{x} \right) = +\infty$, then for every $M > 0$ there exists a constant $d > 0$ with $u_n(x) \leq Mx + d$ for all n and $x \leq 0$.*

Proof The assumption implies that there exists $K > 0$ (which depends on M) such that for all n $u_n(x) \leq Mx$ for $x \leq -K$. Hence $Mx - u_n(x) \geq 0$ for $x \in (-\infty, -K)$. It is clear now that since the function $Mx - u_n(x)$ is continuous on $[-K, 0]$ we may add a properly chosen $d > 0$ so that $Mx + d - u_n(x) \geq 0$ for all $x \in (-\infty, 0]$ and all n.

Lemma 12 *Suppose that for every $n \in \{1, ..., N\}$ the function $u_n : \mathbb{R} \to \mathbb{R}$ is continuous increasing and satisfies*

$$\lim_{x \to +\infty} \frac{u_n(x)}{x} = 0.$$

Then for every $\varepsilon > 0$ there exists $b = b(\varepsilon) > 0$ such that $u_n(x) \leq \varepsilon x + b$ for $x \geq 0$ and all n.

Proof The assumption guarantees the existence of a constant $K > 0$, which depends on ε, such that $u_n(x) \leq \varepsilon x$ for $x \geq K$ and all n. Hence

$$u_n(x) \leq \varepsilon x + K\varepsilon + \sup_n \left(\sup_{[0,K]} u_n(s) \right) \quad \forall x \geq 0.$$

A.4 The exponential case

Proof (Proof of Theorem 8) For the sake of simplicity we start by choosing $h = 1$. We note that

$$\begin{aligned}
&\rho_B(\mathbf{X}) \\
&= \inf \left\{ \sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in M^{\Phi} : \exists d \in \mathbb{R} \text{ s.t. } \sum_{n=1}^{N} Y^n = d \text{ and } \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] = B \right\} \\
&= \inf \left\{ d \mid (d, \mathbf{Y}) \in \mathbb{R} \times M^{\Phi} \text{ s.t. } \sum_{n=1}^{N} Y^n = d \text{ and } \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] = B \right\}.
\end{aligned}$$

Let $F : \mathbb{R} \times M^{\Phi} \to \mathbb{R}$ be given by

$$F(d, \mathbf{Y}) = d$$

and $f_1 : \mathbb{R} \times M^{\Phi} \to M^{\phi_0}$ and $f_2 : \mathbb{R} \times M^{\Phi} \to \mathbb{R}$ be defined by

$$f_1(d, \mathbf{Y}) = \sum_{n=1}^{N} Y^n - d \quad \text{and} \quad f_2(d, \mathbf{Y}) = \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] - B,$$

respectively. Then we can rewrite

$$\rho_B(\mathbf{X}) = \inf_{(d,\mathbf{Y}) \in \mathbb{R} \times M^{\Phi}} \{ F(d, \mathbf{Y}) \mid f_1(d, \mathbf{Y}) = 0,\ f_2(d, \mathbf{Y}) = 0 \}$$

with associated Lagrangian $L(d, \mathbf{Y}, \mathbf{Z}, \mu) : \mathbb{R} \times M^{\Phi} \times (M^{\phi_0})^* \times \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned}
L(d, \mathbf{Y}, \mathbf{Z}, \mu) &= F(d, \mathbf{Y}) + \mathbb{E}[Z f_1(d, \mathbf{Y})] + \mu f_2(d, \mathbf{Y}) \\
&= d + \mathbb{E}\left[Z \left(\sum_{n=1}^{N} Y^n - d \right) \right] + \mu \left(\mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] - B \right).
\end{aligned}$$

The problem boils down to solve the system $\nabla L = 0$, taking derivatives with respect to each $(d, \mathbf{Y}, \mathbf{Z}, \mu)$.
Consider now the general case $h > 1$. We have

$$\begin{aligned}
L(\{d_m\}_{m=1}^h, \mathbf{Y}, \{Z^m\}_{m=1}^h, \mu) &= \sum_{m=1}^{h} d_m + \mathbb{E}\left[\sum_{m=1}^{h} Z^m (\overline{Y}_m - d_m) \right] \\
&\quad + \mu \left\{ \mathbb{E}\left[\sum_{m=1}^{h} \sum_{k \in I_m} \exp(-\alpha_k (X_k + Y_k)) \right] + B \right\},
\end{aligned}$$

with $\overline{Y}_m = \sum_{k \in I_m} Y_k$.
We compute the Gateaux derivative in the direction $\mathbf{V} \in M^{\Phi}$:

$$\begin{aligned}
&\lim_{\epsilon \to 0} \frac{\mathcal{L}(\mathbf{Y} + \epsilon \mathbf{V}) - \mathcal{L}(\mathbf{Y})}{\epsilon} \\
&= \lim_{\epsilon \to 0} \mu \mathbb{E}\left[\sum_{m=1}^{h} \sum_{k \in I_m} \exp(-\alpha_k (X_k + Y_k)) \frac{\exp(-\epsilon V_k \alpha_k) - 1}{\epsilon} \right] + \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}\left[\epsilon \sum_{m=1}^{h} \sum_{k \in I_m} V_k Z^m \right] \qquad (108) \\
&= -\mu \mathbb{E}\left[\sum_{m=1}^{h} \sum_{k \in I_m} \exp(-\alpha_k (X_k + Y_k)) V_k \alpha_k \right] + \mathbb{E}\left[\sum_{m=1}^{h} V_k Z^m \right] =: \phi_Y(\mathbf{V}),
\end{aligned}$$

where in (108) we can apply the Dominated Convergence Theorem by using estimations similar to the ones in Remark 19. We now show that $\phi_Y(\mathbf{V})$ is also the Fréchet derivative of M^{Φ}, i.e., that

$$\lim_{\|\mathbf{V}\|_{M^{\Phi}}\to 0}\frac{L(\mathbf{Y}+\mathbf{V})-L(\mathbf{Y})-\phi_Y(\mathbf{V})}{\|\mathbf{V}\|_{M^{\Phi}}}=0.$$

We have

$$L(\mathbf{Y}+\mathbf{V})-L(\mathbf{Y})-\phi_Y(\mathbf{V})=\mu\mathbb{E}\left[\sum_{m=1}^{h}\sum_{k\in I_m}\exp(-\alpha_k(X_k+Y_k))(\exp(-v^k\alpha_k)-1+v^k\alpha_k)\right],$$

and we obtain

$$\begin{aligned}
&\mathbb{E}[\exp(-\alpha_k(X_k+Y_k))(\exp(-\alpha_kV_k)-1+\alpha_kV_k)]\\
&\le \mathbb{E}[|\exp(-\alpha_k(X_k+Y_k))(\exp(-\alpha_kV_k)-1+\alpha_kV_k)|]\\
&\le K_1\mathbb{E}[\exp(-\alpha_k(X_k+Y_k-|V_k|))V_k^2]\\
&\le K_2\mathbb{E}[\exp(-2\alpha_k(X_k+Y_k-|V_k|))]^{\frac{1}{2}}\,\mathbb{E}[V_k^4]^{\frac{1}{2}}\\
&\le K_2\mathbb{E}[\exp(-4\alpha_k(X_k+Y_k))]^{\frac{1}{4}}\,\mathbb{E}[\exp(4|V_k|)]^{\frac{1}{4}}\,\mathbb{E}[V_k^4]^{\frac{1}{2}}\\
&= K_3\,\|V_k\|^2_{L^4(\mathbb{R})},
\end{aligned}$$

where we use twice the Hölder inequality. Since

$$K_3\,\|V_k\|^2_{\mathcal{L}^4(\mathbb{R})}\le K_4\,\|V_k\|^2_{M^{\phi_k}},$$

we have

$$|L(\mathbf{Y}+\mathbf{V})-L(\mathbf{Y})-\phi_Y(\mathbf{V})|\le K_4\,\|\mathbf{V})\|^2_{M^{\Phi}}.$$

To conclude the proof, it is then sufficient to substitute $\mathbf{Y}$ of the form (88) in $\phi_Y(\mathbf{V})$ to verify that $\phi_Y(\mathbf{V})=0$ for all $\mathbf{V}\in M^{\Phi}$.

Proof (Proof of Proposition 18) The following results hold because $\mathbf{X},\mathbf{V}\in M^{\Phi}$ and Remark 19.

1. By (92) we get

$$\begin{aligned}
\frac{d}{d\epsilon}d_m(\mathbf{X}+\epsilon\mathbf{V})\Big|_{\epsilon=0}&=\beta_m\frac{\frac{d}{d\epsilon}\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}=-\frac{\mathbb{E}\left[-\overline{V}_m\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}\\
&=\mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m]. \qquad (109)
\end{aligned}$$

2. By (92) and (109) we deduce

$$\begin{aligned}
\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}}}[Y^i_{\mathbf{X}+\varepsilon\mathbf{V}}]|_{\varepsilon=0}&=\lim_{\epsilon\to 0}\frac{1}{\varepsilon}\left\{\mathbb{E}_{Q^m_{\mathbf{X}}}\left[Y^i_{\mathbf{X}+\varepsilon\mathbf{V}}-Y^i_{\mathbf{X}}\right]\right\}\\
&=\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^i]+\frac{1}{\beta_m\alpha_i}\mathbb{E}_{Q^m_{\mathbf{X}}}[\overline{V}_m]+\frac{1}{\beta_m\alpha_i}\,\frac{d}{d\epsilon}d_m(\mathbf{X}+\epsilon\mathbf{V})\Big|_{\epsilon=0}\\
&=\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^i]. \qquad (110)
\end{aligned}$$

3. Note that

$$
\begin{aligned}
&\frac{d}{d\epsilon}\left(\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}\right)\\
&=\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}\left(-\frac{\overline{V}_m}{\beta_m}\right)-\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]^2}\frac{d}{d\epsilon}\left(\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]\right)\\
&=\frac{1}{\beta_m}\left\{-\overline{V}_m\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}+\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}\mathbb{E}\left[\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}\overline{V}_m\right]\right\}\\
&=\frac{1}{\beta_m}\left\{-\overline{V}_m\frac{dQ^m_{\mathbf{X}+\epsilon\mathbf{V}}}{d\mathbb{P}}+\frac{dQ^m_{\mathbf{X}+\epsilon\mathbf{V}}}{d\mathbb{P}}\mathbb{E}_{Q^m_{\mathbf{X}+\epsilon\mathbf{V}}}[\overline{V}_m]\right\}.
\end{aligned}
$$

Hence we have by Remark 19 that

$$
\begin{aligned}
\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\mathbb{E}_{Q^m_{\mathbf{X}+\epsilon\mathbf{V}}}[Z]-\mathbb{E}_{Q^m_{\mathbf{X}}}[Z]\right)&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\mathbb{E}_{\mathbb{P}}\left[\left(\frac{dQ^m_{\mathbf{X}+\epsilon\mathbf{V}}}{d\mathbb{P}}-\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}}\right)Z\right]\right)\\
&=\mathbb{E}_{\mathbb{P}}\left[\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\frac{dQ^m_{\mathbf{X}+\epsilon\mathbf{V}}}{d\mathbb{P}}-\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}}\right)Z\right] \qquad (111)\\
&=\mathbb{E}_{\mathbb{P}}\left[\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}-\frac{\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}\right)Z\right]\\
&=\mathbb{E}_{\mathbb{P}}\left[\frac{d}{d\epsilon}\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}\Bigg|_{\epsilon=0}Z\right]\\
&=\frac{1}{\beta_m}\mathbb{E}_{\mathbb{P}}\left[\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}}\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\overline{V}_m\right]Z\right]-\mathbb{E}_{\mathbb{P}}\left[\overline{V}_m\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}}Z\right]\\
&=\frac{1}{\beta_m}\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\overline{V}_m\right]\mathbb{E}_{Q^m_{\mathbf{X}}}[Z]-\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\overline{V}_m Z\right]=-\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m,Z]. \qquad (112)
\end{aligned}
$$

4. By (109) and (110) we obtain

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\varepsilon}\left(\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^i_{\mathbf{X}+\varepsilon\mathbf{V}}]-\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^i_{\mathbf{X}}]\right)\\
&=\lim_{\epsilon\to 0}\left\{\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[-V^i]+\frac{1}{\beta_m\alpha_i}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[\overline{V}_m]\right\}+\frac{1}{\beta_m\alpha_i}\left.\frac{d}{d\epsilon}d_m(\mathbf{X}+\epsilon\mathbf{V})\right|_{\epsilon=0}\\
&=\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^i]+\frac{1}{\beta_m\alpha_i}\mathbb{E}_{Q^m_{\mathbf{X}}}[\overline{V}_m]+\frac{1}{\beta_m\alpha_i}\mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m]=\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^i]. \qquad (113)
\end{aligned}
$$

By (112) and (113) we have

$$
\begin{aligned}
&\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^i_{\mathbf{X}+\varepsilon\mathbf{V}}]|_{\varepsilon=0}\\
&=\lim_{\epsilon\to 0}\frac{1}{\varepsilon}\left(\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^i_{\mathbf{X}+\varepsilon\mathbf{V}}]-\mathbb{E}_{Q^m_{\mathbf{X}}}[Y^i_{\mathbf{X}}]\right)\\
&=\lim_{\epsilon\to 0}\frac{1}{\varepsilon}\left(\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^i_{\mathbf{X}+\varepsilon\mathbf{V}}]-\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^i_{\mathbf{X}}]\right)+\lim_{\epsilon\to 0}\frac{1}{\varepsilon}\left(\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^i_{\mathbf{X}}]-\mathbb{E}_{Q^m_{\mathbf{X}}}[Y^i_{\mathbf{X}}]\right)\\
&=\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^i]-\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m,Y^i_{\mathbf{X}}]\\
&=\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^i]+\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m,X^i]-\frac{1}{\beta_m}\frac{1}{\beta_m}\frac{1}{\alpha_i}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m,\overline{X}_m],
\end{aligned}
$$

where the last equation follows from (88).

5. Set
$$\theta_m(\varphi_m) := \left(\beta_m H(Q^m, \mathbb{P}) + \sum_{i\in I_m} \frac{1}{\alpha_i} \ln\left(-\frac{B}{\beta\alpha_i}\right)\right).$$
By (90) we then have
$$\begin{aligned}
&\theta_m\left(\frac{dQ^m_{\mathbf{X}+\varepsilon\mathbf{V}}}{d\mathbb{P}}\right) - \theta_m\left(\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}}\right)\\
&= -\beta_m \mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}\left[\left(\frac{\overline{X}_m + \epsilon\overline{V}_m}{\beta_m}\right)\right] + \beta_m \mathbb{E}_{Q^m_{\mathbf{X}}}\left[\frac{\overline{X}_m}{\beta_m}\right]\\
&\quad - \beta_m \ln\left(\mathbb{E}\left[e^{-\frac{1}{\beta_m}(\overline{X}_m+\epsilon\overline{V}_m)}\right] + \beta_m \ln\left(\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]\right)\right)\\
&= -\left(\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}\left[\overline{X}_m\right] - \mathbb{E}_{Q^m_{\mathbf{X}}}\left[\overline{X}_m\right]\right) - \epsilon \mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}\left[\overline{V}_m\right] - \beta_m \ln\left(\frac{\mathbb{E}\left[e^{-\frac{1}{\beta_m}(\overline{X}_m+\epsilon\overline{V}_m)}\right]}{\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]}\right).
\end{aligned} \tag{114}$$
By De L'Hôpital it follows
$$\lim_{\varepsilon\to 0} \ln\left(\frac{\mathbb{E}\left[e^{-\frac{1}{\beta_m}(\overline{X}_m+\epsilon\overline{V}_m)}\right]}{\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]}\right) = -\frac{1}{\beta_m}\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\overline{V}_m\right], \tag{115}$$
Hence by (112), (114), and (115) we get
$$\begin{aligned}
\lim \frac{1}{\varepsilon}&\left\{\theta_m\left(\frac{dQ^m_{\mathbf{X}+\varepsilon\mathbf{V}}}{d\mathbb{P}}\right) - \theta_m\left(\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}}\right)\right\}\\
&= \frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[\overline{V}_m, \overline{X}_m] - \mathbb{E}_{Q^m_{\mathbf{X}}}\left[\overline{V}_m\right] + \mathbb{E}_{Q^m_{\mathbf{X}}}\left[\overline{V}_m\right]\\
&= \frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[\overline{V}_m, \overline{X}_m].
\end{aligned}$$
6. It follows by (109).

Remark 19 In (111) we can apply the dominated convergence theorem because of the following. We have

$$\begin{aligned}
&\left|\frac{1}{\epsilon}\left(\frac{dQ^m_{\mathbf{X}+\epsilon\mathbf{V}}}{d\mathbb{P}} - \frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}}\right)\right|\\
&= \left|\frac{1}{\epsilon}\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]} - \frac{\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}\right|\\
&= \left|\frac{1}{\epsilon}\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right] - \exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}\right|\\
&\leq \left|\frac{1}{\epsilon}\frac{\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}\frac{\exp\left(-\frac{\epsilon\overline{V}_m}{\beta_m}\right)\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right] - \mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\exp\left(-\frac{\epsilon\overline{V}_m}{\beta_m}\right)\right]}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m-|\overline{V}_m|}{\beta_m}\right)\right]}\right|\\
&\leq f(\overline{X}_m, \overline{V}_m)\underbrace{\left\{\frac{|\overline{V}_m|}{\beta_m}\exp\left(\frac{|\overline{V}_m|}{\beta_m}\right)\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right] + \mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\frac{|\overline{V}_m|}{\beta_m}\exp\left(\frac{|\overline{V}_m|}{\beta_m}\right)\right]\right\}}_{:=Z_m}
\end{aligned}$$

where we have set

$$f(\overline{X}_m, \overline{V}_m) := \frac{\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]} \frac{1}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m-|\overline{V}_m|}{\beta_m}\right)\right]},$$

and used that

$$\left|\frac{\exp(-\frac{\epsilon\overline{V}_m}{\beta_m})-1}{\epsilon}\right| \leq \frac{1}{\beta_m}|\overline{V}_m| \exp\left(\frac{|\overline{V}_m|}{\beta_m}\right).$$

Note that in this case $M^{\phi_0} \subseteq L^2(\mathbb{R})$, hence

$$f(\overline{X}_m, \overline{V}_m) = \frac{\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]} \frac{1}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m-|\overline{V}_m|}{\beta_m}\right)\right]} = K_1 \exp\left(-\frac{\overline{X}_m}{\beta_m}\right) \in L^2(\mathbb{R})$$

and

$$\begin{aligned} Z_m &\leq \exp\left(\frac{2|\overline{V}_m|}{\beta_m}\right)\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right] + \mathbb{E}\left[\exp\left(\frac{2|\overline{V}_m|-\overline{X}_m}{\beta_m}\right)\right] \\ &= K_2 + K_3 \exp\left(\frac{2|\overline{V}_m|}{\beta_m}\right) \in L^2(\mathbb{R}) \end{aligned}$$

because $\overline{X}_m, \overline{V}_m \in M^{\phi_0}$. We can conclude that $f(\overline{X}_m, \overline{V}_m)Z_m$ is in $L^1(\mathbb{R})$.

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