THE PRIME GRAPH ON CLASS SIZES OF A FINITE GROUP HAS A BIPARTITE COMPLEMENT

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Abstract. Let G be a finite group, and let $\mathrm{cs}(G)$ denote the set of sizes of the conjugacy classes of G. The prime graph built on $\mathrm{cs}(G)$, that we denote by $\Delta(G)$, is the (simple undirected) graph whose vertices are the prime divisors of the numbers in $\mathrm{cs}(G)$, and two distinct vertices p, q are adjacent if and only if pq divides some number in $\mathrm{cs}(G)$. A rephrasing of the main theorem in [8] is that *the complement $\overline{\Delta}(G)$ of the graph $\Delta(G)$ does not contain any cycle of length* 3. In this paper we generalize this result, showing that $\overline{\Delta}(G)$ *does not contain any cycle of odd length*, i.e., it is a bipartite graph. In other words, the vertex set $\mathrm{V}(G)$ of $\Delta(G)$ is covered by two subsets, each inducing a complete subgraph (a clique). As an immediate consequence, setting $\omega(G)$ to be the maximum size of a clique in $\Delta(G)$, the inequality $|\mathrm{V}(G)| \leq 2\omega(G)$ holds for every finite group G.

1. Introduction

An intriguing aspect of Finite Group Theory is the relationship between the structure of a (finite) group G and the arithmetical structure of certain sets of positive integers associated to G. This general issue has attracted the interest of many authors, who investigated several variations on the theme over the past few decades; however, among the arithmetical data that can be considered in connection with a finite group G, one of the most significant and most studied is certainly the set $\mathrm{cs}(G)$, whose elements are the sizes of the conjugacy classes of G (we refer the reader to the survey [4]).

A useful tool that has been introduced in order to analyze the arithmetical structure of $\mathrm{cs}(G)$, is the prime graph built on this set: this is the simple undirected graph $\Delta(G)$ whose vertex set $\mathrm{V}(G)$ consists of the prime divisors of the numbers in $\mathrm{cs}(G)$, and whose edge set $\mathrm{E}(G)$ contains $\{p, q\} \subseteq \mathrm{V}(G)$ if and only if pq divides some number in $\mathrm{cs}(G)$. One of the relevant problems in this context is to understand which graph-theoretical properties of $\Delta(G)$ are reflected and influenced by the group structure of G.

As a general remark, the graph $\Delta(G)$ tends to have "many" edges, an instance of this fact being highlighted by Theorem A of [8]: *given a finite group G, for every choice of three (pairwise distinct) vertices of $\Delta(G)$, two among them are adjacent in $\Delta(G)$*. Recalling now that the complement graph $\overline{\Delta}(G)$ of $\Delta(G)$ is defined as the graph having the same vertex set, in which two vertices are adjacent if and only

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if they are not adjacent in $\Delta(G)$, the aforementioned statement can be expressed as follows: *for every finite group G, the graph $\overline{\Delta}(G)$ does not contain any cycle of length* 3. Note that several features of the graph $\Delta(G)$, such as the bound of 3 for the diameter in the connected case, or the structure of the non-connected case as the union of two complete subgraphs, follow at once by the above result (though they were known before [8] was published).

In the present paper, we prove the following extension of the main theorem of [8].

Theorem A. *Let G be a finite group. Then the graph $\overline{\Delta}(G)$ does not contain any cycle of odd length.*

A first consequence of Theorem A is that the graph $\Delta(G)$ cannot contain any pentagon as an induced subgraph; this follows from the fact that the vertices of such an induced subgraph would be vertices of a cycle of length five in the complement graph $\overline{\Delta}(G)$.

Furthermore, the conclusion of Theorem A is equivalent to the fact that $\overline{\Delta}(G)$ is a bipartite graph. Going back to the graph of our interest, that is $\Delta(G)$, we can therefore conclude that two cliques (i.e., subsets of $\mathrm{V}(G)$ inducing complete subgraphs) are always enough to "cover" all the vertices of $\Delta(G)$.

Corollary B. *Let G be a finite group. Then the vertex set of $\Delta(G)$ can be partitioned into two subsets, each inducing a clique of $\Delta(G)$.*

The above corollary yields that, given any subset of $\mathrm{V}(G)$, at least half the elements of this subset are pairwise adjacent in $\Delta(G)$; in particular, denoting by $\omega(G)$ the clique number (i.e., the maximum size of a clique) of $\Delta(G)$, we obtain what follows.

Corollary C. *Let G be a finite group. Then the inequality $|\mathrm{V}(G)| \leq 2\omega(G)$ holds.*

Another important set of positive integers that can be linked to a finite group G, and that has been widely studied in the same spirit as here discussed, is the set $\mathrm{cd}(G)$ whose elements are the degrees of the irreducible complex characters of G. It is well known that there are deep similarities between results concerning $\mathrm{cs}(G)$ and $\mathrm{cd}(G)$, and this is the case also for the problems considered in this paper. In fact, the character-degree analog of [8, Theorem A] is a celebrated theorem by P.P. Pálfy (see [10]), but it holds only for *solvable* groups. Similarly, the results of this paper have a transposition (with identical statements, but using very different arguments and methods; see [1]) to the context of character degrees for solvable groups. On the other hand, the character-degree analogs of Theorem A and Corollary C without the solvability restriction look more articulate, and are treated in [2] and [3] respectively.

To close with, in the following discussion every group is tacitly assumed to be a finite group, and the classification of finite non-abelian simple groups is used via Theorem 9 of [5].

2. Preliminaries

If x is an element of the group G, we denote by x^G the conjugacy class of x in G, and by $\pi(x^G)$ the set of the prime divisors of $|x^G| = [G : \mathbf{C}_G(x)]$.

We list some well-known facts that we are going to use. We say that a vertex v of a graph Γ is *isolated* if there are no edges of Γ incident to v (i.e. if the degree of v in Γ is 0).

As mentioned in the Introduction, for a group G, the graph $\overline{\Delta}(G)$ is defined as follows: its set of vertices $\mathrm{V}(G)$ is the set of the prime numbers dividing some conjugacy class of G, and two vertices p and q are adjacent in $\overline{\Delta}(G)$ if there is no conjugacy class of G having size divisible by the product pq.

It is well-known that a prime divisor p does not belong to $\mathrm{V}(G)$ if and only if G has a central Sylow p-subgroup.

If N is a normal subgroup of G then, for any $g \in G$, we have $\pi((gN)^{G/N}) \subseteq \pi(g^G)$; also, for $x \in N$, we have $\pi(x^N) \subseteq \pi(x^G)$. It follows that, if p and q are in $\mathrm{V}(N)$ (resp. in $\mathrm{V}(G/N)$) and they are adjacent in $\overline{\Delta}(G)$, then they are adjacent in $\overline{\Delta}(N)$ (resp. in $\overline{\Delta}(G/N)$) as well.

The following is well known and easy to prove.

Lemma 2.1. *Let G be a group and let $x, y \in G$ be such that one of the following holds:*

(a) *Either x and y have coprime orders and they commute; or*
(b) *$x \in X$ and $y \in Y$ with X and Y normal subgroups of G such that $X \cap Y = 1$.*

Then $\pi(x^G) \cup \pi(y^G) \subseteq \pi((xy)^G)$.

Lemma 2.2. *If the Fitting subgroup of a group G is trivial, then every vertex of $\overline{\Delta}(G)$ is isolated.*

Proof. This follows from [5, Theorem 9]. □

Lemma 2.3. *Let G be a group, let p be a non-isolated vertex of $\overline{\Delta}(G)$ and P a Sylow p-subgroup of G. Then G is p-solvable, P is abelian, $[G, P]$ has a normal p-complement K and $[K, P] = K$. Furthermore, if there are no elements x in K such that $p \in \pi(x^G)$, then $K = 1$ and P is normal in G.*

Proof. Let q be a vertex adjacent to p in $\overline{\Delta}(G)$. Then by [5, Theorem B], both the Sylow p-subgroups and q-subgroups of G are abelian and G is $\{p, q\}$-solvable with $\{p, q\}$-length 1. So, G is p-solvable and, because P is abelian, the p-length of G is 1.

Let $L = \mathbf{O}^{p'}(G)$; then L has a normal p-complement K, $L = KP$ and $[K, P] = K$. As $[G, P] \le [G, L] \le L$, then K is the p-complement of $[G, P]$.

Finally, if $p \notin \pi(x^G)$ for all $x \in K$, then we get $K = \bigcup_{k \in K} \mathbf{C}_K(P^k)$, which yields $K = [K, P] = 1$ and hence P is normal in G. □

In the next lemma, the assumption 'abelian' could in fact be weakened to 'nilpotent'. However, this extra generality will not be relevant for our purposes.

Lemma 2.4. *Let G be a group such that $G/\mathbf{F}(G)$ is abelian. Then there exists an element $g \in G$ such that the set of all prime divisors of $G/\mathbf{F}(G)$ is contained in $\pi(g^G)$.*

Proof. We observe first that, by factoring out the Frattini subgroup $\mathbf{\Phi}(G)$ of G, we can assume that $F = \mathbf{F}(G)$ is a direct product of elementary abelian subgroups; in fact, F can be viewed as a faithful and completely reducible G/F-module (possibly in "mixed characteristic"). Consider a direct decomposition of F into irreducible constituents F_i: it will be enough to show that, for every i, there exists $x \in F_i$ such that $\mathbf{C}_{G/F}(x) = \mathbf{C}_{G/F}(F_i)$. But this is easily seen to be true for every non-trivial $x \in F_i$, because G/F is abelian. □

Proposition 2.5. *Let G be a group. Assume that p and q are adjacent vertices of $\overline{\Delta}(G)$, and denote by P and Q a Sylow p-subgroup and a Sylow q-subgroup of G, respectively. Assume further that $M = \mathbf{O}_{p'}([G,P])$ is a minimal normal subgroup of G, and that Q is not normal in G. Then M is abelian and the following conclusions hold.*

(a) $\overline{G} = G/\mathbf{C}_G(M)$ *is metacyclic,* $\overline{P} \leq \mathbf{F}(\overline{G})$ *and* $\overline{Q} \cap \mathbf{F}(\overline{G}) = 1$.

(b) $\overline{G} = \overline{Q}\,\overline{N}$, *where* $\overline{N} = \langle \overline{x} \in \overline{G} \mid q \in \pi(\overline{x}^{\overline{G}}) \rangle$.

Proof. We first show that M is abelian. Assume the contrary and let R be the solvable radical of G. Note that, by Lemma 2.3, there is an element x in the normal p-complement of $[G,P]$ (i.e., in M) such that $p \in \pi(x^G)$, and there is also an element y in the normal q-complement of $[G,Q]$ such that $q \in \pi(y^G)$. Since $M \cap R = 1$, Lemma 2.1 implies that $[G,Q]$ is not contained in R. Also, clearly $[G,P] \not\leq R$, hence both p and q are vertices of $\overline{\Delta}(G/R)$. But $\mathbf{F}(G/R) = 1$, so, by Lemma 2.2, $\overline{\Delta}(G/R)$ has no edges, a contradiction. We conclude that M is an elementary abelian r-group, where $r \neq p$ is a suitable prime.

Now, since $[M,P] = M$, coprimality yields $\mathbf{C}_M(P) = 1$. We observe also that M is not contained in $\mathbf{\Phi}(G)$; in fact PM is normal in G, and $M \leq \mathbf{\Phi}(G)$ would imply $P \trianglelefteq G$, which in turn yields the contradiction $M = [M,P] \leq M \cap P = 1$. Hence we have $M \cap \mathbf{\Phi}(G) = 1$, and therefore M is complemented in G.

We can now apply [6, Proposition 3.1]. First we get $\mathbf{C}_G(M) \cap Q \trianglelefteq G$, thus Q acts non-trivially on M. So, by part (a) of that proposition, $\overline{G} = G/\mathbf{C}_G(M)$ is a subgroup of the semilinear group $\Gamma(r^n)$ (where $r^n = |M|$), and $\overline{P}$ lies in the subgroup $\Gamma_0(r^n)$ of $\Gamma(r^n)$ consisting of the multiplication maps; in particular, $\overline{G}$ is metacyclic. Observe also that q cannot divide the order of $\overline{G}_0 = \overline{G} \cap \Gamma_0(r^n)$, as otherwise, since this subgroup acts fixed-point freely on M, every non-trivial element of M would have a G-conjugacy class of size divisible by pq. Now, by part (b) of [6, Proposition 3.1], every element of M is centralized by a conjugate of $\overline{Q}$ in $\overline{G}$, and a counting argument (see [7, Lemma 3.5]) yields that $|\overline{Q}|$ divides n, $(|\overline{Q}|, r^n - 1) = 1$ and that $(r^n - 1)/(r^{n/|\overline{Q}|} - 1)$ divides the order of $\overline{G}_0$. So claim (b) follows by [6, Proposition 2.6].

Finally, we observe that $r^n - 1$ has a primitive prime divisor. In fact, if this is not the case, then either $r^n = 2^6$ or $n = 2$ and r is a Fermat prime. But, by the above paragraph, $q = 2$ implies that $r = 2$ and this is a contradiction as the Sylow q-subgroups of G are abelian. On the other hand, if $q = 3$, then $(2^6 - 1)/(2^2 - 1) = 21$ divides $|\overline{G}_0|$, again a contradiction.

Hence, an application of [7, Lemma 3.7] yields $\overline{G}_0 = \mathbf{F}(\overline{G})$, and claim (a) is proved.

□

3. Proof of Theorem A

In this section we prove Theorem A, whereas the proofs of Corollaries B and C are omitted because these results are immediate consequences of Theorem A. In our proof we will consider some extra structure on certain induced subgraphs of $\overline{\Delta}(G)$, by introducing a suitable orientation; the reader can find in [9] another context in which a similar idea has been exploited.

Theorem A. *Let G be a group. Then the complement graph on class sizes $\overline{\Delta}(G)$ does not contain any cycle of odd length.*

Proof. Assume, working by contradiction, that in $\overline{\Delta}(G)$ there exists a cycle

$$\Gamma : p_1 - p_2 - \cdots - p_{d-1} - p_d - p_1$$

whose length d is odd. We denote by $W = \{p_1, p_2, \ldots, p_d\}$ the set of vertices of Γ.

For every $i \in \{1, 2, \ldots, d\}$, we choose a Sylow p_i-subgroup P_i of G and, recalling Lemma 2.3, we denote by K_i the normal p_i-complement of the commutator subgroup $[G, P_i]$.

Let X be the subset of W consisting of vertices p_i such that $P_i \trianglelefteq G$, which is equivalent to $K_i = 1$, and we set $Y = W \setminus X$. Finally, we denote by Λ the subgraph of Γ induced on the set Y.

We will reach a contradiction in a series of steps.

Step 1. If p_a, p_b are adjacent vertices of Λ, then one of the two subgroups K_a and K_b properly contains the other.

Proof. Let $N = K_a \cap K_b$, $\overline{G} = G/N$ and assume by contradiction that both $\overline{K_a}$ and $\overline{K_b}$ are non-trivial. Applying Lemma 2.3 to the group $\overline{G}$, we see that there exist both an element $\overline{x} \in \overline{K_a}$ and an element $\overline{y} \in \overline{K_b}$ such that $p_a \in \pi(\overline{x}^{\overline{G}})$ and $p_b \in \pi(\overline{y}^{\overline{G}})$. Hence, by Lemma 2.1 the product $p_a p_b$ divides $|(\overline{xy})^{\overline{G}}|$, which implies that p_a and p_b are not adjacent vertices of $\overline{\Delta}(G)$, a contradiction.

We conclude that (say) $K_a = N$, whence $K_a \leq K_b$. Also, if L is a normal subgroup of G such that K_a/L is a chief factor of G, then we can apply Proposition 2.5 to the group G/L, obtaining that $\widehat{G} = G/\mathbf{C}_G(K_{p_a}/L)$ has a normal Sylow p_a-subgroup, and a non-trivial Sylow p_b-subgroup intersecting $\mathbf{F}(\widehat{G})$ trivially. In particular, the roles of p_a and p_b are not symmetric, and therefore the inclusion of K_a in K_b must be proper. □

As a consequence of Step 1, we can define an orientation of the graph Λ as follows; if K_a contains K_b, then we replace the edge $p_a - p_b$ with the arc having p_a as the first vertex, and p_b as the second vertex; thus, we write $p_a \to p_b$. We denote by $\widehat{\Lambda}$ the digraph obtained from Λ in this way.

Step 2. Let $p_a, p_b \in Y$ such that $p_a \to p_b$ is an arc of $\widehat{\Lambda}$, and consider a normal subgroup L of G with $L \leq K_b$ and such that $V = K_b/L$ is a chief factor of G. Then, setting $\overline{G} = G/\mathbf{C}_G(V)$ and $\overline{N_a} = \langle \overline{x} \in \overline{G} \mid p_a \in \pi(\overline{x}^{\overline{G}}) \rangle$, by Proposition 2.5 we get that V is elementary abelian and the following conclusions hold.
(a) $\overline{G}$ is metacyclic, $\overline{P_b} \leq \mathbf{F}(\overline{G})$ and $\overline{P_a} \cap \mathbf{F}(\overline{G}) = 1$.
(b) $\overline{G} = \overline{P_a}\ \overline{N_a}$.

Step 3. If $p_a - p_b - p_c$ is a path in Λ, then the arcs $p_a - p_b$ and $p_b - p_c$ have opposite orientations in $\widehat{\Lambda}$.

Proof. Assume, without loss of generality, that $p_a \to p_b \to p_c$ is an oriented path in $\widehat{\Lambda}$. Then, considering a chief factor $V = K_c/L$ of G, by Step 2 we have that $\overline{G} = G/\mathbf{C}_G(V)$ is metacyclic, $\overline{P_c} \leq \mathbf{F}(\overline{G})$ and $\overline{P_b} \cap \mathbf{F}(\overline{G}) = 1$. Hence Lemma 2.4 yields that $\overline{P_a} \leq \mathbf{F}(\overline{G})$ and hence $[\overline{G}, \overline{P_a}]$ is a p_a-group. It follows that $\overline{K_a}$, the p_a-complement of $[\overline{G}, \overline{P_a}]$, is trivial, a contradiction as $\overline{K_a}$ contains $\overline{K_b} \neq 1$. □

Step 4. For $i \in \{1, 2, \ldots, d\}$, if $p_i \in X$, then both p_{i-1} and p_{i+1} are in Y (i.e., p_{i-1} and p_{i+1} are vertices of Λ).

Proof. Assume that p and q are two consecutive vertices of Γ, and that $\{p, q\} \subseteq X$. So, for $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, both P and Q are normal in G and, since p and q are adjacent in $\overline{\Delta}(G)$, it follows that $G = \mathbf{C}_G(P) \cup \mathbf{C}_G(Q)$. Therefore, either P or Q is central in G, a contradiction. □

Step 5. If $q \in X$, $p_a, p_b \in Y$ and $q - p_a - p_b$ is a path in Γ, then in $\widehat{\Lambda}$ there is the arc $p_a \leftarrow p_b$ (i.e. p_a is the end-point of the arc).

Proof. Let $Q \in \mathrm{Syl}_q(G)$; as $q \in X$, then $Q \trianglelefteq G$ and $C = \mathbf{C}_G(Q) \trianglelefteq G$. Assume, working by contradiction, that the edge $p_a - p_b$ of Λ is oriented in $\widehat{\Lambda}$ as $p_a \to p_b$. We consider a normal subgroup L of G with $L \leq K_b$ and such that $V = K_b/L$ is a chief factor of G, and we write $\overline{G} = G/\mathbf{C}_G(V)$. We observe that $\overline{N_a} \leq \overline{C}$, where $\overline{N_a}$ is the subgroup of $\overline{G}$ generated by the element having conjugacy class size multiple of p_a, because q is adjacent to p_a in $\overline{\Delta}(G)$. Hence, Step 2 yields $\overline{G} = \overline{CP_a}$. In particular, there exists an element $\overline{x} \in \overline{C}$ such that $p_a \in \pi(\overline{x}^{\overline{G}})$.

As a consequence, we claim that $\mathbf{C}_G(V)$ is contained in C. In fact, set $U = \mathbf{C}_G(V) \cap C$, and observe that p_a divides the size of the conjugacy class of xU in G/U; if now $U \neq \mathbf{C}_G(V)$, then we can choose an element $y \in \mathbf{C}_G(V) \setminus C$ and conclude, by Lemma 2.1(b), that p_a divides the size of the conjugacy class of xyU in G/U. Therefore, p_a divides $|(xy)^G|$ as well; on the other hand, clearly xy does not lie in C, hence also q divides $|(xy)^G|$. This contradiction proves our claim.

From $\overline{G} = \overline{CP_a}$ and $\mathbf{C}_G(V) \leq C$, we hence deduce $G = CP_a$. Thus $Q_0 = [Q, P_a] = [Q, G]$ is normal in G and $Q_0 \neq 1$. Moreover, for every $y \in Q_0 \setminus \{1\}$, we have $p_a \in \pi(y^G)$.

Now, as $V = K_b/L$ is abelian, we clearly have $K_b \leq \mathbf{C}_G(V)$, thus $K_b \leq C$. Furthermore, $|V|$ is a t-power for some prime $t \neq q$, since otherwise C would lie in $\mathbf{C}_G(V)$ and thus $\overline{C}$ would be trivial, against what observed in the first paragraph of the proof of this step. By Lemma 2.3 applied to G/L, there are t-elements $w \in K_b$ such that $p_b \in \pi(w^G)$. Hence, Lemma 2.1(a) yields the contradiction that $p_a p_b$ divides $|(yw)^G|$ for every $y \in Q_0 \setminus \{1\}$. □

Step 6. X is non-empty, and the connected components of Λ are paths with an odd number of vertices.

Proof. First, we show that the set X is non-empty. In fact, if $X = \emptyset$, then $\widehat{\Lambda}$ is an orientation of the full cycle Γ and hence it has an odd number of arcs. But this, by elementary counting, implies that there is in $\widehat{\Lambda}$ at least an oriented path $p_a \to p_b \to p_c$, which is impossible by Step 3. So, $X \neq \emptyset$.

By Step 4, no two vertices in X are consecutive vertices of the cycle Γ, thus, in particular, Λ is not the empty graph. In order to prove that each of its connected components has an odd number of vertices, we show that it has an even number of edges. Assuming that the connected component Λ_0 of Λ has an odd number of edges, then by Step 3 one of the two extreme vertices of $\widehat{\Lambda_0}$ (where $\widehat{\Lambda_0}$ is the corresponding subdigraph of $\widehat{\Lambda}$) cannot be the end-point of the corresponding arc of $\widehat{\Lambda_0}$, against Step 5. □

Step 7. The final contradiction.

Let $\Lambda_1, \ldots, \Lambda_c$ be the connected components of Λ. By Step 4 we have $c = |X|$, whereas Step 6 yields that $|\Lambda_i|$ is an odd number for every $i \in \{1, \ldots, c\}$. Now, we see that

$$d = |X| + |Y| = c + \sum_{i=1}^{c} |\Lambda_i|$$

is congruent to $2c$ modulo 2, i.e., d is even, against our assumptions. This is the final contradiction that completes the proof. $\square$

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