Density scaling in the mechanics of a disordered mechanical meta-material

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Nature provides examples of self-assemble lightweight disordered network structures with remarkable mechanical properties which are desirable for many applications purposes but challenging to reproduce artificially. Previous experimental and computational studies investigated the mechanical responses of random network structures focusing on topological and geometrical aspects in terms of variable connectivity or probability to place beam elements. However for practical purposes an ambitious challenge is to design new materials with the possibility to taylor their mechanical features such as stiffness. Here, we design a two dimensional disordered mechanical meta-material exhibiting unconventional stiffness-density scaling in the regime where both bending and stretching are relevant for deformation. In this regime, the mechanical meta-material covers a wide interval of the Young modulus - density plane, simultaneously exhibiting high critical stress and critical strain. Our results, supported by finite element simulations, provides the guiding principles to design on demand disordered metamaterials, bridging the gap between artificial and naturally occurring materials.

The mechanical properties of lattice materials of many types, such as foams[1], cellular solids[2–4], microlattices[5], trusses[6], made of connected elements, have been intensively studied mainly due to their lightweight structures and remarkable mechanical properties[7,8]. The high strength to weight ratio of bone[9,10] or balsa[11], the elastic properties of spider silk[12,13] and the fracture resistance of nacre[14,15] are just few of the naturally occurring structures that derive their mechanical properties from their underlying geometry. These types of materials attract growing interest in many fields ranging from commercial products such as those related to food industry, to architectural applications such as energy absorption and management[1] and in modern technologies where their geometrical features are exploited to achieve a myriade of performances.

In the recent years composite structures started to be rationally architected with the aim to achieve targeted properties[16], and emerging significant breakthroughs[17] are favoured alongside with the advances in digital manufacturing technologies i.e. 3D printing and automated assembly. Artificially designed materials are recently termed meta-materials[18–22], specifically, mechanical meta-materials indicate a class of structures whose mechanical properties are a consequence of their underlying geometry rather than their constituent material[23–25]. Through the prudent choice of a meta-materials underlying architecture, it is possible to create geometries whose structural performance far exceeds that of the material from which it is made[24]. These structures can be designed to exhibit a wide range of beneficial properties, including high strength to weight ratio[26,27], auxetic behaviour[16,28–30], energy trapping[20,31] and fracture resistance[32,33], among many others[23].

Typically artificil meta-materials are designed with a single motif repeated periodically throughout the material[21], in contrast with the disordered structures encountered in self-assembled natural objects[34,35]. Random elastic networks are widely studied[36–38], but there are still large gaps between theoretical investigations and practical purposes such as real solid materials. Mechanical meta-materials with increasing degree of disorder are just emerging[39–43]. Whether disorder is merely the price to pay for self-assembly or whether it provides some advantage is still an open issue.

The structural behaviour of lattice based meta-materials, made up of beams with non-hinged nodes, is generally divided into two regimes: stretching-dominated and bending-dominated[44]. Which of these two regimes a lattice falls into is dependent on its connectivity: When a lattice's connectivity is less that is required for rigidity in its rigid link freely hinged analogous system, the lattice will exhibit bending dominated behaviour, for lattices with connectivity above this threshold, stretching dominated behaviour will be observed[44]. The two classes of lattice exhibit dissimilar mechanical properties. For example, the relative stiffness of a two-dimensional lattice ($\tilde{Y}/Y$) is related to its relative density ($\tilde{\rho}/\rho$) via the expression

$$\frac{\tilde{Y}}{Y} \sim \left(\frac{\tilde{\rho}}{\rho}\right)^n \qquad (1)$$

where, in two-dimensions, n is one or three for stretching-dominated and bending dominated architectures respectively[45] (in $d = 3$, n takes the values 1 or 2 for stretching and bending dominated architectures respectively[44]). Due to this scaling behavior, stretching-dominated lattices of low relative density are stiffer than their bending-dominated counterparts. In the above expressions Y and ρ are the Young's modulus and density of the construction material respectively, while $\tilde{X}$ denotes the property X of the meta-material. Density dependent scaling laws similar to the one reported in Eq. 1 can be also be established for other mechanical properties, such

as buckling or strength, or even electrical properties[44].

Better understanding of stochastic lattice based structures has importance for a range of applications from performance of lightweight structures[46–48] to early diagnosis of diseases. For example, osteoporosis, a disease with major socio-economic consequence, is linked both with changes in bone mineral density (BMD) and also changes in connectivity of the trabecular bone latticework[49]. Improving the comprehension of the implications of density and connectivity in stochastic lattices is thus a key step in the early diagnosis of this disease where currently BMD is the principle diagnostic tool[49].

Inspired by naturally designed disordered structures, in this letter, we consider the linear response and elastic limit of a disordered lattice made up of beam elements. In contrast to periodic structures previously investigated, disordered lattices exhibit an intermediate regime characterized by an alternative relationship linking relative stiffness and relative density of the lattice. Furthermore, we establish the effect of diluting the lattice on the buckling resistance of the meta-material and show that in the intermediate regime, a novel scaling law linking critical strain (strain at buckling) and relative density exists. This mixed regime exhibits many advantageous properties. At low densities the lattice far exceeds the bending dominated lattice in terms of stiffness. At higher densities, the mixed regime simultaneously shows high buckling load and critical strain. With increasing manufacturing freedom to design structures on the micro and nano scale[50,51], and the possibility of self assembly of complex architectures[52,53], the theoretical work presented here is of increasing technological relevance.

To investigate the density scaling of disordered meta-materials, we consider a triangular lattice whereby each linkage element (beam of length L and thickness t) is placed with a given probability p. Equivalently, we can create the same structures by removing the beam with the probability $(1-p)$ from the perfect lattice. The designed geometries are subsequently deformed under a strain in the negative y direction (in all cases, the imposed strain is less than the critical strain). The thickness of the beams was varied in order to vary the relative density. Each beam element is made of a linear elastic material with Young's modulus Y and Poisson's ratio ν, we define the aspect ratio of the beam to be $a = L/t$. From the resultant architecture we take the largest connected component and investigate the mechanical properties resultant meta-material. The system size is such that the entire lattice measures $N_x l$ horizontally, and $\sqrt{3}l N_y/2$ vertically.

All simulations presented here have been performed using COMSOL Multiphysics[54] and COMSOL with MATLAB[54] in $d = 2$ with a plane strain approximation through use of the structural mechanics module. All studies assume a linear elastic material with Young's modulus of 50MPa and Poisson's ratio of 0.3; the length of the beams used was 0.01m. Mesh refinement studies were undertaken to ensure convergence of results. Re-

sults pertaining to stiffness have been obtained using Euler-Bernoulli beam elements using the in-built "stationary studies" study (a quasi-static solver). Here, a small displacement was applied to the upper boundary, while the lower boundary was fixed in place inducing a reduction in total height of the structure, the remaining boundaries were left free; the reaction force required to induce this displacement was then calculated and from this, an effective Young's modulus was calculated. For studies into buckling strength and critical strain the lower boundary was again fixed, while the upper boundary had an applied force or displacement imposed (for buckling stress and strain respectively), the remaining boundaries were left free. For these studies, the solid mechanics module used alongside the in-built linear buckling solvers (linear eigenvalue solver). For studies using the solid mechanics module, mesh density required were highly dependent on the aspect ratio of the beams, approximately 200-1500 elements were used per unit cell.
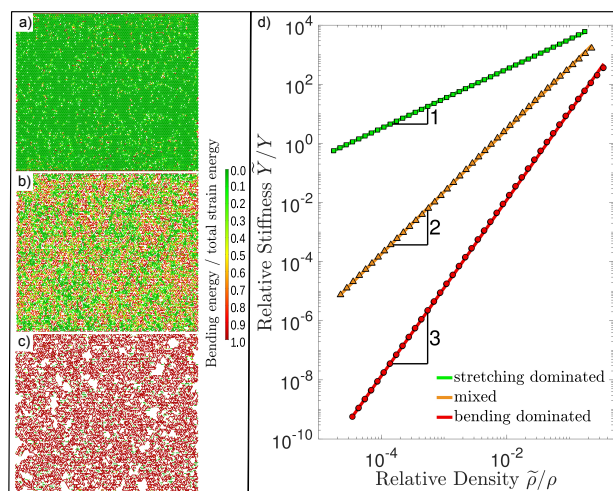

FIG. 1. Different scaling regimes of stiffness vs relative density. Examples of typical 128×128 node networks with different bond placing probability a) $p = 0.95$, b) $p = 0.642$ and c) $p = 0.4$ with a small strain imposed in the y direction. The colors show the proportion of the local strain energy that is stored as bending energy with red indicating bending energy dominates, yellows showing mixed and green indicating stretching behavior. d) Stretching ($n = 1$), mixed ($n = 2$) and bending ($n = 3$) dominated lattices are characterised by power law behaviors, fitting of the data shown gives scaling of $n = 2.97 \pm 0.02$, 2.1 ± 0.05 and 1.00 ± 0.01 for bending, mixed and stretching respectively.

Figure 1a) shows an example of three geometries for $p = 0.4, 0.642, 0.95$ deformed along negative y direction. Beams are colored according to the proportion of bending energy with respect to the total[55]. Increasing p, the designed random meta-materials cover different ranges from bending to stretching behavior. Figure 1b shows that for a dilute triangular lattice, close to the transition from the stretching dominated to the bending dominated

regime ($p = p_{\text{T}}$, in this work p_{T} is taken to be 0.642 in-line with previous estimates[56–58]) an intermediate scaling regime is observed where contributions to stiffness from both stretching and bending cannot be ignored. This results in a new exponent $n \simeq 2$ that lies between the value $n = 1$ expected for stretching dominated elasticity and $n = 3$ for bending dominated elasticity.
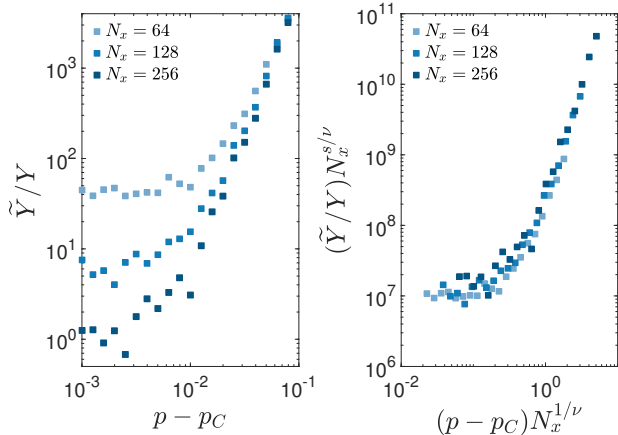

FIG. 2. a) The Young modulus $\tilde{Y}$ as a function of $p - p_c$ for different values of N_x. The data shown is for beams with an aspect ratio of 16. Each data point shown is an average over a number of simulations N_s such that the product of N_s and the number of elements in the perfect system is greater than 10^6. b) Rescaling of the data in a) according to Eq. 3.

It is noted that according to effective-medium theory (EMT), in this regime the contribution to stiffness will follow[59]:

$$\tilde{Y} \sim \mu^{(1-x)}\kappa^x, \tag{2}$$

where μ and κ are the stretching and bending stiffness of the lattice elements, respectively. EMT predicts that $x = 0.5$ in $d = 2$ and $x = 0.4$ in $d = 3$[59]. For a lattice made up of beam elements of thickness t and length L, we note that $\mu \sim (t/L)^q$ and $\kappa \sim (t/L)^r$, where in $d = 2, q = 1$ and $r = 3$, while for $d = 3, q = 2$ and $r = 4$. Since $\tilde{\rho}/\rho \propto (t/L)$ in $d = 2$ and $\tilde{\rho}/\rho \propto (t/L)^2$ in $d = 3$ within the intermediate regime, we expect to observe the scaling, $\tilde{Y}/Y \sim (\tilde{\rho}\rho)^n$, with $n = 2$ in $d = 2$ and $n = 1.4$ in $d = 3$. This scaling is plotted against the results of simulation for probabilities close to the critical probability p_{T} in 1b. This intermediate scaling close to the isostatic point is reminiscent observed in stochastic fibre networks where stretching and bending energies can be varied arbitrarily. In contrast, in this work the geometric parameters of the beam elements set the ratio of bending to stretching energies in the beam elements[60], thus here we observe a dependence of the stiffness of the meta-material on its relative density.

In contrast to the behaviour of fibre networks which exhibit a "bending rigidity percolation" probability which can be calculated through Maxwell counting and mean field arguments[61], here we observe that connectivity of the upper and lower boundaries is sufficient to yield a non-zero stiffness in a stochastic meta-material constructed from beam elements. Thus we see non-zero effective Young's modulus for probabilities greater than the geometric percolation probability p_c that for the triangular lattice is given by[62] $p_c = 2\sin(\pi/18)$.

For $p - p_c \ll 1$ we find that the Young's modulus obeys standard finite size scaling expected for percolation

$$\tilde{Y} \sim N_x^{-s/\nu} f((p - p_c)N_x^{1/\nu}) \tag{3}$$

where $\nu = 4/3$ is the correlation length exponent, while s is the elasticity exponent that can be expressed in terms of the conductivity exponent $t \simeq 0.98\nu$[63] as $s = t + 2\nu \simeq 4$[64]. Eq. 3 is verified by the data collapse reported in Fig. 2 for three different lattice size with N_x=64, 128, 256.

We summarize these finding in figure 3 where we show the onset of non-zero stiffness at p_c, and the region of (p, a)-space for which the bending, stretching and mixed regimes are observed. With decreasing slenderness of beams (high relative densities), a broader range of p leads to lattices where both stretching and bending energies must be considered.

In the stochastic metamaterial made up of slender beam elements, elastic instability will occur when the loading reaches a critical value. It is expected that in $d = 2$ or $d = 3$, the critical value will scale as

$$\sigma_c \sim F_c/L^{d-1} \sim YI/L^{d+1} \tag{4}$$

where σ_c is the critical stress, F_c is the Euler buckling force of a slender beam, I and L are the second moment of area and length of the component beams respectively, and Y is the Young's modulus of the construction material. Thus, through variation of the aspect ratio of the component beams with a fixed p, we expect a scaling relationship between critical strain and density given by,

$$\sigma_c \sim \left(\frac{\tilde{\rho}}{\rho}\right)^m, \tag{5}$$

where, in $m = 3$ in $d = 2$ and $m = 2$ in $d = 3$[44]. This is confirmed in figure 4 where the scaling of $m = 3$ is found irrespective of the regime (stretching, bending or mixed) to which the lattice belongs. As a result of this scaling, we calculate the dependence of critical strain on the relative density of the lattice. It is predicted here that Δ_c will follow the scaling,

$$\Delta_c \sim \left(\frac{\tilde{\rho}}{\rho}\right)^{(m-n)} \tag{6}$$

where in $d = 2$, $m = 3$ irrespective of the regime, and n, as in Eq. (1), takes the values 1, 2 or 3 when the lattice is stretching dominated, mixed, or bending dominated respectively (in $d = 3$, $m = 2$ irrespective of the regime, and n is expected to take the value 1, 1.4 or 2
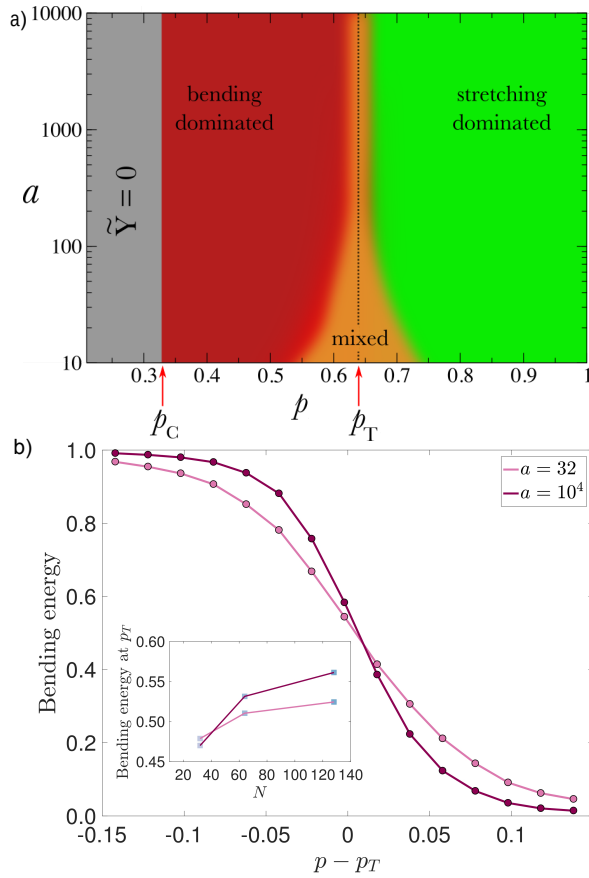
FIG. 3. a) The parameter space (p, a) and the observed regime of stretching and bending dominated alongside a mixed regime. b) The transition from bending (low p) to stretching (high p) dominated regimes for system size $N_x = 128$ for systems made up of beams with aspect ratio 32 and 10^4, a sharper transition is seen for higher aspect ratio of beams. The inset reports the variation of the bending energy as a function of three system sizes, $N_x = 32, 64, 128$. Figure shows averages over a number of simulations N_s such that to product of the total number of elements in the perfect system ($p = 1$) and N_s is greater than 2×10^5.

FIG. 4. a) The critical stress for lattices in the bending, mixed and stretching regimes. All regimes show the same scaling of σ_c vs relative density as predicted by Eq. (5). b) The critical strain for the stochastic metamaterial for the different regimes, here the scaling is dependent on the regime, and is predicted by Eq. (6). c) Stretching dominated regimes leads to high critical stress with low critical strain, bending dominated lattices exhibit high critical strain but low critical stress; Lattices with significant energy contributions from both bending and stretching simultaneously exhibit both high stress and strain at failure. The data shown is for a system size of $N_x = N_y = 20$.

for stretching dominated, mixed or bending dominated respectively); this prediction for $d = 2$ is confirmed in figure (4). We thus find that coupling high relative density and high relative strength is achieved through the use of the mixed deformation mode, that is $p \approx p_T$, this is shown in figure 4.

Lattice based meta-materials, made up of beam elements, have been traditionally classed as either bending or stretching dominated depending on their connectivity[44]. It is notable that recent experimental work has established that some 3 dimensional lightweight nanolattices exhibit density scaling that is inconsistent with these classifications[65–67]. Here we have established, through theory and simulation, the existence of a third class of lattice architecture where both bending and stretching energies must be considered. We have es-
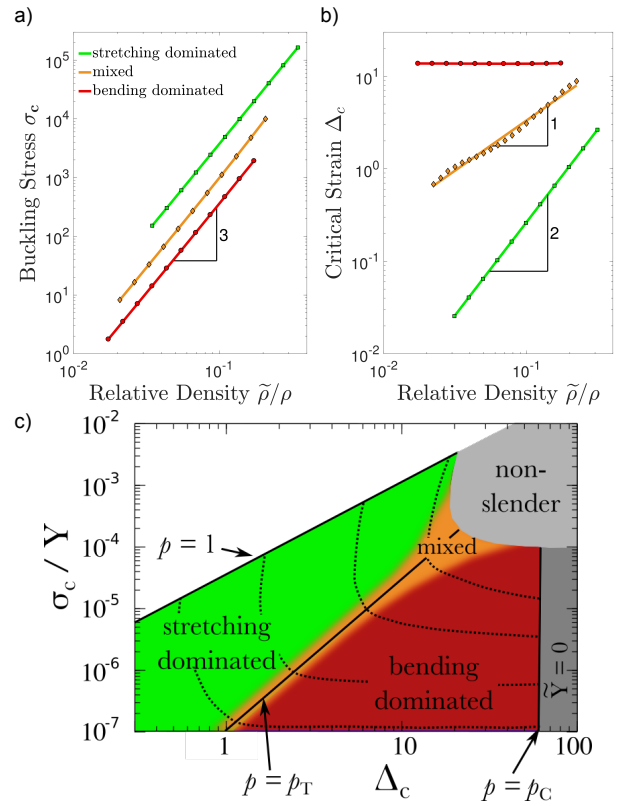
tablished that such architectures lead to meta-materials with high critical stress and critical strain. We plan to verify our predictions for the relevant scaling relationships for three dimensional geometries, which would be computationally more demanding.

It is notable that in this stochastic methodology generates structures that are closer in design to their naturally occurring counterparts. These naturally occurring structures exhibit high flaw tolerance and insensitivity to perturbations[35]. Though this flaw tolerance is linked with optimisation of the materials geometry and hierarchy[35,68], it remains a plausible possibility that flaw tolerance is aided by disorder. Such a link between randomness and robustness has been previously been found in machine resilience[69], algorithm design[70] and interdependent lattice networks[71]. Extending this link to structural mechanics has huge potential for the design of func-

tional robust structures.

[1] MF Ashby. The properties of foams and lattices. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 364(1838):15–30, 2006.

[2] Lorna J Gibson and Michael F Ashby. *Cellular solids: structure and properties*. Cambridge university press, 1999.

[3] Haydn NG Wadley. Multifunctional periodic cellular metals. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 364(1838):31–68, 2006.

[4] Michael F Ashby and RF Mehl Medalist. The mechanical properties of cellular solids. *Metallurgical Transactions A*, 14(9):1755–1769, 1983.

[5] NA Fleck, VS Deshpande, and MF Ashby. Micro-architectured materials: past, present and future. In *Proceedings of the royal society of London a: mathematical, physical and engineering sciences*, volume 466, pages 2495–2516. The Royal Society, 2010.

[6] Vikram S Deshpande, Norman A Fleck, and Michael F Ashby. Effective properties of the octet-truss lattice material. *Journal of the Mechanics and Physics of Solids*, 49(8):1747–1769, 2001.

[7] Lorna J Gibson. Biomechanics of cellular solids. *Journal of biomechanics*, 38(3):377–399, 2005.

[8] A Srikantha Phani and Mahmoud I Hussein. *Dynamics of Lattice Materials*. John Wiley & Sons, 2017.

[9] R Hodgskinson and JD Currey. Young's modulus, density and material properties in cancellous bone over a large density range. *Journal of Materials Science: Materials in Medicine*, 3(5):377–381, 1992.

[10] Jae-Young Rho, Liisa Kuhn-Spearing, and Peter Zioupos. Mechanical properties and the hierarchical structure of bone. *Medical engineering & physics*, 20(2):92–102, 1998.

[11] Xiaoyu Zheng, Howon Lee, Todd H Weisgraber, Maxim Shusteff, Joshua DeOtte, Eric B Duoss, Joshua D Kuntz, Monika M Biener, Qi Ge, Julie A Jackson, et al. Ultralight, ultrastiff mechanical metamaterials. *Science*, 344(6190):1373–1377, 2014.

[12] Haijun Zhou and Yang Zhang. Hierarchical chain model of spider capture silk elasticity. *Physical review letters*, 94(2):028104, 2005.

[13] T Vehoff, A Glišović, H Schollmeyer, A Zippelius, and T Salditt. Mechanical properties of spider dragline silk: humidity, hysteresis, and relaxation. *Biophysical journal*, 93(12):4425–4432, 2007.

[14] Hideki Kakisawa and Taro Sumitomo. The toughening mechanism of nacre and structural materials inspired by nacre. *Science and technology of advanced materials*, 12(6):064710, 2012.

[15] George Mayer. Rigid biological systems as models for synthetic composites. *Science*, 310(5751):1144–1147, 2005.

[16] Roderic Lakes. Foam structures with a negative poisson's ratio. *Science*, pages 1038–1040, 1987.

[17] Tobias Frenzel, Muamer Kadic, and Martin Wegener. Three-dimensional mechanical metamaterials with a twist. *Science*, 358(6366):1072–1074, 2017.

[18] Tiemo Bückmann, Nicolas Stenger, Muamer Kadic, Johannes Kaschke, Andreas Frölich, Tobias Kennerknecht, Christoph Eberl, Michael Thiel, and Martin Wegener. Tailored 3d mechanical metamaterials made by dip-in direct-laser-writing optical lithography. *Advanced Materials*, 24(20):2710–2714, 2012.

[19] Jayson Paulose, Anne S Meeussen, and Vincenzo Vitelli. Selective buckling via states of self-stress in topological metamaterials. *Proceedings of the National Academy of Sciences*, 112(25):7639–7644, 2015.

[20] Bastiaan Florijn, Corentin Coulais, and Martin van Hecke. Programmable mechanical metamaterials. *Physical review letters*, 113(17):175503, 2014.

[21] Katia Bertoldi, Vincenzo Vitelli, Johan Christensen, and Martin van Hecke. Flexible mechanical metamaterials. *Nature Reviews Materials*, 2(11):17066, 2017.

[22] Corentin Coulais. As the extension, so the twist. *Science*, 358(6366):994–995, 2017.

[23] Muamer Kadic, Tiemo Bückmann, Robert Schittny, and Martin Wegener. Metamaterials beyond electromagnetism. *Reports on Progress in physics*, 76(12):126501, 2013.

[24] Daniel Rayneau-Kirkhope, Chengzhao Zhang, Louis Theran, and Marcelo A Dias. Analytic analysis of auxetic metamaterials through analogy with rigid link systems. *Proc. R. Soc. A*, 474(2210):20170753, 2018.

[25] Tie Jun Cui, David Smith, and Ruopeng (Eds.) Liu. *Metamaterials: Theory, Design, and Applications*. Springer US, 2010.

[26] Xianglong Yu, Ji Zhou, Haiyi Liang, Zhengyi Jiang, and Lingling Wu. Mechanical metamaterials associated with stiffness, rigidity and compressibility: a brief review. *Progress in Materials Science*, 2017.

[27] Daniel Rayneau-Kirkhope, Yong Mao, and Robert Farr. Ultralight hierarchical meta-materials on a body-centred cubic lattice. *EPL (Europhysics Letters)*, 119(1):14001, 2017.

[28] Xiaonan Hou and Vadim V Silberschmidt. Metamaterials with negative poisson's ratio: A review of mechanical properties and deformation mechanisms. In *Mechanics of Advanced Materials*, pages 155–179. Springer, 2015.

[29] Augustus Edward Hough Love. *A treatise on the mathematical theory of elasticity*. Cambridge university press, 2013.

[30] Graeme W Milton. Composite materials with poisson's ratios close to—1. *Journal of the Mechanics and Physics of Solids*, 40(5):1105–1137, 1992.

[31] Sicong Shan, Sung H Kang, Jordan R Raney, Pai Wang, Lichen Fang, Francisco Candido, Jennifer A Lewis, and Katia Bertoldi. Multistable architected materials for trapping elastic strain energy. *Advanced Materials*, 27(29):4296–4301, 2015.

[32] Paramita Das, Jani-Markus Malho, Khosrow Rahimi, Felix H Schacher, Baochun Wang, Dan Eugen Demco, and Andreas Walther. Nacre-mimetics with synthetic nanoclays up to ultrahigh aspect ratios. *Nature communications*, 6:5967, 2015.

[33] Nifang Zhao, Miao Yang, Qian Zhao, Weiwei Gao, Tao Xie, and Hao Bai. Superstretchable nacre-mimetic graphene/poly (vinyl alcohol) composite film based on interfacial architectural engineering. *ACS nano*, 11(5):4777–4784, 2017.

[34] Jason W Nichol and Ali Khademhosseini. Modular tissue engineering: engineering biological tissues from the bottom up. *Soft matter*, 5(7):1312–1319, 2009.

[35] Peter Fratzl. Biomimetic materials research: what can we really learn from nature's structural materials? *Journal of the Royal Society Interface*, 4(15):637–642, 2007.

[36] Shechao Feng and Pabitra N Sen. Percolation on elastic networks: new exponent and threshold. *Physical review letters*, 52(3):216, 1984.

[37] Jan Wilhelm and Erwin Frey. Elasticity of stiff polymer networks. *Physical review letters*, 91(10):108103, 2003.

[38] Le Yan, Jean-Philippe Bouchaud, and Matthieu Wyart. Edge mode amplification in disordered elastic networks. *Soft matter*, 13(34):5795–5801, 2017.

[39] MJ Mirzaali, R Hedayati, P Vena, L Vergani, M Strano, and AA Zadpoor. Rational design of soft mechanical metamaterials: independent tailoring of elastic properties with randomness. *Applied Physics Letters*, 111(5):051903, 2017.

[40] Michelle M Driscoll, Bryan Gin-ge Chen, Thomas H Beuman, Stephan Ulrich, Sidney R Nagel, and Vincenzo Vitelli. The role of rigidity in controlling material failure. *Proceedings of the National Academy of Sciences*, 113(39):10813–10817, 2016.

[41] Hanifpour, Maryam, Petersen, Charlotte F., Alava, Mikko J., and Zapperi, Stefano. Mechanics of disordered auxetic metamaterials. *Eur. Phys. J. B*, 91(11):271, 2018.

[42] Weizhu Yang, Qingchang Liu, Zongzhan Gao, Zhufeng Yue, and Baoxing Xu. Theoretical search for heterogeneously architected 2d structures. *Proceedings of the National Academy of Sciences*, 115(31):E7245–E7254, 2018.

[43] Daniel R Reid, Nidhi Pashine, Justin M Wozniak, Heinrich M Jaeger, Andrea J Liu, Sidney R Nagel, and Juan J de Pablo. Auxetic metamaterials from disordered networks. *Proceedings of the National Academy of Sciences*, page 201717442, 2018.

[44] MF Ashby. The properties of foams and lattices. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 364(1838):15–30, 2006.

[45] D. Rayneau-Kirkhope. Stiff auxetics: Hierarchy as a route to stiff, strong lattice based auxetic meta-materials. *Scientific Reports*, 8(1):12437, 2018.

[46] Yang Si, Jianyong Yu, Xiaomin Tang, Jianlong Ge, and Bin Ding. Ultralight nanofibre-assembled cellular aerogels with superelasticity and multifunctionality. *Nature Communications*, 5:5802 EP –, 12 2014.

[47] Marcus A. Worsley, Sergei O. Kucheyev, Joe H. Satcher Jr., Alex V. Hamza, and Theodore F. Baumann. Mechanically robust and electrically conductive carbon nanotube foams. *Appl. Phys. Lett.*, 94:073115, 2009.

[48] Haolun Wang, Xuan Zhang, Ning Wang, Yan Li, Xue Feng, Ya Huang, Chunsong Zhao, Zhenglian Liu, Minghao Fang, Gang Ou, Huajian Gao, Xiaoyan Li, and Hui Wu. Ultralight, scalable, and high-temperature–resilient ceramic nanofiber sponges. *Science Advances*, 3(6), 2017.

[49] Chamith S. Rajapakse, Jesper S. Thomsen, Julio S. Espinoza Ortiz, Sunil J. Wimalawansa, Ebbe N. Ebbesen, Lis Mosekilde, and Gemunu H. Gunaratne. An expression relating breaking stress and density of trabecular bone. *Journal of Biomechanics*, 37(8):1241 – 1249, 2004.

[50] Jens Bauer, Lucas R Meza, Tobias A Schaedler, Ruth Schwaiger, Xiaoyu Zheng, and Lorenzo Valdevit. Nanolattices: An emerging class of mechanical metamaterials. *Advanced Materials*, 29(40):1701850, 2017.

[51] Jae-Hwang Lee, Jonathan P Singer, and Edwin L Thomas. Micro-/nanostructured mechanical metamaterials. *Advanced materials*, 24(36):4782–4810, 2012.

[52] C Nadir Kaplan, Wim L Noorduin, Ling Li, Roel Sadza, Laura Folkertsma, Joanna Aizenberg, and L Mahadevan. Controlled growth and form of precipitating microsculptures. *Science*, 355(6332):1395–1399, 2017.

[53] Kun Liu, Nana Zhao, and Eugenia Kumacheva. Self-assembly of inorganic nanorods. *Chemical Society Reviews*, 40(2):656–671, 2011.

[54] COMSOL Multiphysics. Comsol multiphysics user guide (version 4.3 a). *COMSOL, AB*, pages 39–40, 2012.

[55] D Rayneau-Kirkhope. Stiff auxetics: Hierarchy as a route to stiff, strong lattice based auxetic meta-materials. *Sci Rep*, 8(1):12437, Aug 2018.

[56] S. Roux and A. Hansen. Transfer-matrix study of the elastic properties of central-force percolation. *EPL (Europhysics Letters)*, 6(4):301, 1988.

[57] Alex Hansen and Stéphane Roux. Universality class of central-force percolation. *Phys. Rev. B*, 40:749–752, Jul 1989.

[58] Sepehr Arbabi and Muhammad Sahimi. Mechanics of disordered solids. i. percolation on elastic networks with central forces. *Phys. Rev. B*, 47:695–702, Jan 1993.

[59] Chase P Broedersz, Xiaoming Mao, Tom C Lubensky, and Frederick C MacKintosh. Criticality and isostaticity in fibre networks. *Nature Physics*, 7(12):983, 2011.

[60] Stephen P Timoshenko and James M Gere. Theory of elastic stability second edition. *Dover Civil and Mechanical Engineering. New York, NY: Dover Publications.*, 2009.

[61] Chase P Broedersz and Fred C MacKintosh. Modeling semiflexible polymer networks. *Reviews of Modern Physics*, 86(3):995, 2014.

[62] MF Sykes and JW Essam. Some exact critical percolation probabilities for bond and site problems in two dimensions. *Physical Review Letters*, 10(1):3, 1963.

[63] Peter Grassberger. Conductivity exponent and backbone dimension in 2-d percolation. *Physica A: Statistical Mechanics and its Applications*, 262(3):251 – 263, 1999.

[64] M Sahimi. Relation between the critical exponent of elastic percolation networks and the conductivity and geometrical exponents. *Journal of Physics C: Solid State Physics*, 19(4):L79, 1986.

[65] Lucas R. Meza, Gregory P. Phlipot, Carlos M. Portela, Alessandro Maggi, Lauren C. Montemayor, Andre Comella, Dennis M. Kochmann, and Julia R. Greer. Reexamining the mechanical property space of three-dimensional lattice architectures. *Acta Materialia*, 140:424 – 432, 2017.

[66] Lucas R. Meza, Satyajit Das, and Julia R. Greer. Strong, lightweight, and recoverable three-dimensional ceramic nanolattices. *Science*, 345(6202):1322–1326, 2014.

[67] Xuan Zhang, Jiahao Yao, Bin Liu, Jun Yan, Lei Lu, Yi Li, Huajian Gao, and Xiaoyan Li. Three-dimensional high-entropy alloy–polymer composite nanolattices that overcome the strength–recoverability trade-off. *Nano Letters*, 18(7):4247–4256, 07 2018.

[68] Huajian Gao, Baohua Ji, Ingomar L Jäger, Eduard Arzt, and Peter Fratzl. Materials become insensitive to flaws at nanoscale: lessons from nature. *Proceedings of the national Academy of Sciences*, 100(10):5597–5600, 2003.

[69] Martin Randles, David Lamb, E Odat, and Azzelarabe Taleb-Bendiab. Distributed redundancy and robustness in complex systems. *Journal of Computer and System Sciences*, 77(2):293–304, 2011.

[70] Peter Hertling, Christoph M Hoffmann, Wolfram Luther, and Nathalie Revol. *Reliable Implementation of Real Number Algorithms: Theory and Practice: International Seminar Dagstuhl Castle, Germany, January 8-13, 2006, Revised Papers*, volume 5045. Springer, 2008.

[71] Jing Yuan, Lixiang Li, Haipeng Peng, Jürgen Kurths, Jinghua Xiao, and Yixian Yang. The effect of randomness for dependency map on the robustness of interdependent lattices. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 26(1):013105, 2016.