Modeling financial series and computing risk measures via Gram-Charlier-like expansions of the convoluted hyperbolic-secant density

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Abstract Since financial series are usually heavy-tailed and skewed, research has formerly considered well-known leptokurtic distributions to model these series and, recently, has focused on the technique of adjusting the moments of a probability law by using its orthogonal polynomials. This paper combines these approaches by modifying the moments of the convoluted hyperbolic-secant (CHS). The resulting density is a Gram-Charlier-like (GC-like) expansion capable to account for skewness and excess kurtosis. Multivariate extensions of these expansions are obtained on an argument using spherical distributions. Both the univariate and multivariate (GC-like) expansions prove to be effective in modelling heavy-tailed series and computing risk measures.

Keywords: Convoluted hyperbolic-secant distribution, orthogonal polynomials, kurtosis, skewness, Gram-Charlier-like expansion, spherical distribution.

1. Introduction

A substantial body of evidence shows that empirical distributions of returns and financial data usually exhibit accentuated peakedness, thick tails and frequent skewness. This is duly acknowledged in the financial literature (see e.g., Szego, 2004, and references therein). The well-known Gaussian law fails to accommodate these stylized facts and does not provide, accordingly, a valid paradigm for the representation and interpretation of financial data. This explains why, in several instances, research has moved towards leptokurtic distributions such as the Student-t, the Pearson type VII, the normal inverse Gaussian and stable distributions (see e.g., Mills, 1999, Rachev et al., 2010), which, by and large, maintain desirable properties like the bell-shapedness. Heavy tailed and peaked distributions have also been modelled by densities generated via a mixed approach (see e.g., Barakat, 2015 and Garcia et al., 2011), or kernel estimators obtained by appropriate techniques (see Ruppert et al., 2008).

A different stream of research has recently come to the fore (see e.g., Faliva et al., 2016, Jondeau & Rocking, 2001, Zoia, 2010). This hinges on the basic idea of the Gram-Charlier expansion (GC in short), which is used to reshape the Gaussian law by using its own orthogonal polynomials which are the Hermite ones (see Cheah et al., 1993). This paper combines the two aforementioned approaches, as it aims at reshaping a leptokurtic symmetric distribution by using its own orthogonal polynomials to meet the requirements of possibly severe kurtosis and skewness.

The distribution considered in the paper is the convoluted hyperbolic-secant distribution, (CHS henceforth), which arises from the self-convolution of the hyperbolic secant (see e.g., Baten 1934, Bracewell 1986, Fisher 2013), and we investigate its capacity to fit in with financial data once it is reshaped by means of its own orthogonal polynomials. As risk modelling applications typically require that several variables are jointly modelled, a multivariate extension of both the CHS distribution and its orthogonal polynomial expansion are provided. This latter follows from spherical distribution theory (see Fang Kai-Tai et al., 1965).

Applications to financial univariate and multivariate asset returns, characterized by substantial excess kurtosis, show the usefulness of this choice by highlighting the extent to which the polynomially-adjusted convoluted hyperbolic-secant distribution matches up with empirical evidences. The goodness of the proposed distributions in computing some risk measures, like the Value at Risk and the Expected Shortfall, by using both an unconditional and a conditional approach based on GARCH models, is also investigated.

The paper is organized as follows. In Section 2, we introduce the CHS distribution and move to its spherical extension. Then we design the polynomial shape-adapter tailored to build its Gram-Charlier-like (GC-like) expansions both for the univariate and multivariate context. In Section 3, the performance of these GC-like distributions is tested by an application involving both univariate and bivariate financial returns. Section 4 draws the conclusions. An Appendix, providing the essential about orthogonal polynomials and spherical distributions, completes the paper.

2. Univariate and multivariate Gram-Charlier-like expansions of the convoluted hyperbolic-distribution

In this Section, we will devise a GC-like expansion for the even density

$$f(x) = x\left(\sinh(\frac{\pi}{\sqrt{2}}x)\right)^{-1} \quad x \in R, \tag{1}$$

hereafter named convoluted hyperbolic-secant (CHS) distribution. The CHS distribution, which is the self-convolution of the hyperbolic secant law and the Fourier image of the logistic function as well (see, e.g., Gradshteyn and Ryzhik 1980, Johnson and Kotz, vol 2, 1994), enjoys several desirable properties like bell shapedness, leptokurtosis and existence of moments and orthogonal polynomials of every order. As far as even moments are concerned, they can be obtained from the following integral (see Gradshteyn and Ryzhik,ibidem, p. 348, formula 3.523.2)

$$\int_0^\infty \frac{x^{2h-1}}{\sinh(bx)}dx = \frac{2^{2h}-1}{2h}\left(\frac{\pi}{b}\right)^{2h}|B_{2h}|,\ h = 1, 2.... \tag{2}$$

where B_{2h} denotes the $2h$-th Bernoulli number. By setting $h = 1, 2, 3, 4, 5$ in (2) we obtain the first five even moments for this density

$$m_0 = 1, \quad m_2 = 1, \quad m_4 = 4, \quad m_6 = 34, \quad m_8 = 496\ . \tag{3}$$

These values can be used to determine the coefficients of the polynomial shape adapter of the Gram-Charlier (GC)-like expansions for the CHS distribution. The term GC-like expansion is adopted to indicate a density, $\tilde{g}(x, \alpha, \beta, ..)$ hereafter, obtained by reshaping an arbitrary distribution $f(x)$ by using its own orthogonal polynomials, that is polynomials with coefficients built from the moments of this law. The term can be traced back to the well known Gram-Charlier (GC) expansion referred to the Gaussian law and Hermite polynomials. The GC-like expansion based on the j-th orthogonal polynomial, $p_j(x)$, associated to a density $f(x)$ takes the form

$$\tilde{g}(x, \alpha) = q(x, \alpha) f(x) \tag{4}$$

Here $q(x, \alpha) = \left(1 + \frac{\alpha}{\gamma_j} p_j(x)\right)$ is a shape adapter whose role is that to increase the j-th moment of the density $f(x)$ by a quantity equal to α and γ_j is the squared norm of $p_j(x)$ (see Appendix A1 for more details on orthogonal polynomials and GC-like expansions).

In the following we will focus on GC-like expansions which make use of the third and fourth orthogonal polynomials of a given density $f(x)$ in order to account for skewness and excess-kurtosis. The expansions at stake take the form

$$\tilde{g}(x, \alpha, \beta) = q(x, \alpha, \beta) f(x), \tag{5}$$

where $q(x, \alpha, \beta)$ is the trinomial

$$q(x, \alpha, \beta) = \left(1 + \frac{\alpha}{\gamma_3} p_3(x) + \frac{\beta}{\gamma_4} p_4(x)\right). \tag{6}$$

depending on the 3rd and 4th orthogonal polynomials, $p_3(x)$ and $p_4(x)$, associated to $f(x)$ and the parameters α, β, γ_3 and γ_4. The parameters α and β represent the increase in skewness and kurtosis attainable with the polynomial expansion (5), γ_3 and γ_4 are the squared norms of $p_3(x)$ and $p_4(x)$, respectively (see formula (53) in Appendix A1).

The following theorem shows how to compute the third and fourth degree orthogonal polynomials associated to CHS density which paves the way to obtaining the family of the GC-like expansion, defined as in (5), for this density.

Theorem 1 *The family of the GC-like expansions, defined as in (5), for the CHS density is given by*

$$\tilde{g}(x,\alpha,\beta) = \left(1 + \frac{\alpha}{18} p_3(x) + \frac{\beta}{180} p_4(x)\right) x \left(\sinh(\frac{\pi}{\sqrt{2}} x)\right)^{-1}, \tag{7}$$

where α and $\beta > 0$ are parameters and

$$p_3(x) = x^3 - 4x, \quad p_4(x) = x^4 - 10x^2 + 6 \tag{8}$$

are the third and fourth degree orthogonal polynomials associated to the CHS density. Under suitable conditions on α and β, formula (7) defines a set of densities with skewness and kurtosis differing from those of the parent CHS density by an extent equal to α and β, respectively.

Proof. As the CHS density is a symmetric law, its third and fourth-order orthogonal polynomials, $p_3(x)$ and $p_4(x)$, are specified as in (57) of Appendix A1. The coefficients of these polynomials and their squared norms, γ_3, γ_4 in the trinomial (6), are obtained from formulas (54) and (56) of Appendix A1. The proof that the skewness and kurtosis of the GC-like expansion in (7) are modified by a quantity equal to α and β respectively, is based on Theorem A1 in Appendix A1.

The positiveness of $q(x,\alpha,\beta)$ is mandatory in order for $\tilde{g}(x,\alpha,\beta)$ to be a density function. It is worth distinguishing the case when only extra kurtosis has to be accounted for, from the case when both excess kurtosis and skewness are at work. Let's start with the former case.

Lemma 1 *For the binomial*

$$q(x,0,\beta) = \left(1 + \frac{\beta}{180}\left(x^4 - 10x^2 + 6\right)\right) \tag{9}$$

to be non-negative for all x, it is required that the parameter β satisfies

$$0 \le \beta \le \frac{180}{19}. \tag{10}$$

Proof. The proof rests on the argument that $p_4(x) = x^4 - ex^2 + g$ is bounded from below, that is

$$\inf_x p_4(x) = \left(\frac{4g - e^2}{4}\right) = -19. \tag{11}$$

This entails that $q(x,0,\beta) = \left(1 + \frac{\beta}{\gamma_4}\left(x^4 - ex^2 + g\right)\right)$ is non-negative provided

$$\beta \le \frac{4\gamma_4}{(e^2 - 4g)}. \tag{12}$$

Since $q(x, 0, \beta)$ is not bounded from above, negative values of β are not allowed.

In the light of the foregoing theorem, the family of distributions

$$\tilde{g}(x, 0, \beta) = \left(1 + \frac{\beta}{180} p_4(x)\right) x \left(\sinh(\frac{\pi}{\sqrt{2}} x)\right)^{-1} \tag{13}$$

proves suitable to model data with kurtosis K varying in the range

$$4 \leq K \leq 13,4737. \tag{14}$$

The first graph in Figure 1 compares the Gaussian law with the GC-like expansion of a CHS distribution, GCCHS henceforth, when $\beta = 4,735$, which is half of the maximum feasible value for this parameter (see Lemma 1). The second graph in Figure 1 shows the difference between the two aforementioned distributions. In Figure 2 the said GCCHS density is compared with the parent CHS law.

As it is well known, kurtosis measures the movement of probability mass from the shoulders of a distribution into its center and tails (see e.g., Balanda et al., 1988, Finucan, 1964). The greater the kurtosis, the lower the concentration of probability mass in the shoulders (see, e.g., Groenveld et al., 1984) with an outplacement of the concentration of probability mass near the mean (whence peakedness) and in the tails (whence thickness) of the distribution. The re-balance of probability which occurs in the center, shoulders and tails of a CHS density after a reshaping by using its orthogonal polynomials can be evaluated through some indexes. Following Darlington 1970 and Moors 1986, denoting with $\pm x_1$ and $\pm x_2$ the inner and the outer crossing points of the Gaussian and the GCCHS density, the notion of shoulders turns out to tally with the area between $\pm x_1$ and $\pm x_2$ and the notion of tails with the area outside $\pm x_2$. The crossing points $\pm x_1$ and $\pm x_2$ are obtained by solving the equation

$$h(x) - l(x) = 0, \tag{15}$$

Upon the argument put forward by Finucan 1964, the ratios considered here below may prove useful as long as the distributions cross four times.

$$\text{Index of peak up-thrust} = \frac{\int_0^{0.74} h(x)dx}{\int_0^{0.74} l(x)dx} \tag{16}$$

$$\text{Index of tails up-thrust} = \frac{\int_{2.8}^{\infty} h(x)dx}{\int_{2.8}^{\infty} l(x)dx} \tag{17}$$

$$\text{Index of shoulders down-thrust} = \frac{\int_{0.74}^{2.8} h(x)dx}{\int_{0.74}^{2.8} l(x)dx}. \tag{18}$$

Table 1 provides the percentages of (up or down) thrust effects on peak, tails and shoulders, respectively, registered by the above indexes when we move from the Gaussian to the CHS law, from the the CHS law to its GC-like expansion, obtained by setting $\beta = 4,735$ and from the Gaussian to the said GC-like expansion, respectively. Looking at the values of this table, we see that -on the one hand- the polynomial expansion has increased significantly the mass of probability located in the tails and around the peak and - on the other hand- has reduced the probability in the shoulders of the GC-like expansion with respect both its parent density and the Gaussian law. Then, we can conclude that the orthogonal polynomial transformation proves effective to face empirical evidence of thick tails and accentuated peakedness (see e.g., Ruppet et al., 1992). As for the more general case, when both extra kurtosis and skewness are involved, we have the following

Lemma 2 *The trinomial*

$$q(x,\alpha,\beta) = 1 + \frac{\alpha}{18} p_3(x) + \frac{\beta}{180} p_4(x) \tag{19}$$

is positive for all x if the pair of parameters α, β satisfy

$$\left[(\lambda - 0,8)^2 + 0,84\right]\beta \quad < 21,6 \tag{20}$$

$$25\left[\lambda\left(39 + \frac{810}{\beta}\right) - 49\right]^2 - \left[40\lambda + 39 + \frac{540}{\beta} - 37\right]^3 \quad < 0, \tag{21}$$

where $\lambda = \frac{15\alpha^2}{2\beta^2} + 2$.

Proof. The trinomial $q(x,\alpha,\beta)$ is a quartic

$$\frac{\beta}{180}x^4 + \frac{\alpha}{18}x^3 - \frac{\beta}{18}x^2 - \frac{2\alpha}{9}x + \frac{\beta}{30} + 1 \tag{22}$$

whose signature is the same as its leading coefficient (positive in this case) provided its roots are complex conjugate in pairs. This occurs when both the coefficients of the linear term and the discriminant of the cubic resolvent are negative (see, e.g., Bronshtein et al., 1998, p. 119-125). Formulas (20) and (21) provide convenient algebraic representations of the said conditions.

Figure 3 depicts the region of admissible values of skewness α vs. extra-kurtosis β for the GCCHS density.

So far the analysis has focused on a scalar random variable. In the following it will be extended to the vector case by using the powerful argument of the so called spherical distributions (see Appendix A2). First we will devise the spherical representation of the CHS density, that is a multivariate symmetric distribution whose marginals are CHS densities, and then the same representation for its GC-like expansion. The following theorem establishes the form of a multivariate extension of a CHS density (SCHS henceforth) which depends on its modular variable. The second one proves that the GC-like expansion of the SCHS extension (GCSCHS henceforth) follows from the polynomial extension of the same modular variable.

Theorem 2 *The n-dimensional spherical extension of the CHS law, (SCHS) hereafter, has the representation*

$$g_n(\boldsymbol{x}) = \frac{2^{\frac{(n-3)}{2}}\pi^{\frac{n}{2}+1}}{z_{\frac{n}{2}+1}\zeta(n+1)(2^{n+1}-1)}(\boldsymbol{x}'\boldsymbol{x})^{\frac{1}{2}}\left(\sinh\left(\frac{\pi}{\sqrt{2}}(\boldsymbol{x}'\boldsymbol{x})^{\frac{1}{2}}\right)\right)^{-1},\ (\boldsymbol{x}'\boldsymbol{x}) \in [0,\infty), \tag{23}$$

where z_θ and $\zeta(\cdot)$ denote the Pochhammer symbol and the Riemann zeta function, respectively.

Proof. The spherical extension of a CHS density hinges on the density, f_R, of its modular variable, R, which, in turn, depends on the density generator (see formulas (62) and (64), Appendix A2). Now, taking

$$g(y) = y^{\frac{1}{2}}\left(\sinh\left(\frac{\pi}{\sqrt{2}}y^{\frac{1}{2}}\right)\right)^{-1}, \qquad y \geq 0 \tag{24}$$

as density generator, whose affiliation from the CHS density is apparent, and by using the following integral representation of the Riemann zeta function (see, e.g., Gradshteyn and Ryzhik, p. 348, 1980)

$$\zeta(\lambda) = \alpha^\lambda\Gamma^{-1}(\lambda)\frac{2^{\lambda-1}}{2^\lambda-1}\int_0^\infty x^{\lambda-1}(\sinh(\alpha x))^{-1}dx, \tag{25}$$

where α and λ are positive parameters and $\Gamma(\cdot)$ is the Euler-Gamma function, simple computations yield the following expression for f_R

$$f_R(r) = \frac{2^{\frac{(n-1)}{2}}r^n\pi^{n+1}}{\zeta(n+1)\Gamma(n+1)(2^{n+1}-1)}\left(\sinh\left(\frac{\pi}{\sqrt{2}}r\right)\right)^{-1}, r \in [0,\infty). \tag{26}$$

This, bearing in mind formulas (62) and (64) in Appendix A2, leads to (23). Trivially, $g_1(\boldsymbol{x})$ tallies with the CHS distribution.

The graphs in Figure 4 compare the (standard) bivariate Gaussian and the bivariate SCHS density.The first graph in Figure 4 shows the SCHS (in dark) overlapped to a Gaussian law (in light). As a SCHS density is more peaked and it has fatter tails than a Gaussian law, the former distribution covers the peak and the tails of the latter. Conversely, the shoulders of a SCHS density, being slimmer than those of a Gaussian low, are covered by those of the latter density.

The following theorem provides GC-like expansions of SCHS densities (GCSCHS densities henceforth) intended to model heavy-tailed bivariate series.

Theorem 3 *Let $g_n(\boldsymbol{x})$ be as in (23) and let $q_{n,4}(\boldsymbol{x},\beta)$ be specified in term of the vector argument $\boldsymbol{x}' = [x_1, x_2, \ldots, x_n]$ as follows*

$$q_{n,4}(\boldsymbol{x},\beta) = \left(1 + \frac{\beta}{\gamma_4}p_4((\boldsymbol{x}'\boldsymbol{x})^{\frac{1}{2}})\right) \tag{27}$$

Here

$$p_4((\boldsymbol{x}'\boldsymbol{x})^{\frac{1}{2}}) = (\boldsymbol{x}'\boldsymbol{x})^2 - e(\boldsymbol{x}'\boldsymbol{x}) + g \tag{28}$$

is the fourth-order orthogonal polynomial in the Euclidean norm $(\boldsymbol{x}'\boldsymbol{x})^{1/2}$.

Then, as $g_n(\boldsymbol{x})$ *is spherical and* $q_{n,4}(\boldsymbol{x},\beta)$ *is even, the product*

$$\tilde{g}_n(\boldsymbol{x},\beta) = q_{n,4}(\boldsymbol{x},\beta)g_n(\boldsymbol{x}) \tag{29}$$

defines a family of GCSCHS densities whose kurtosis is increased by a quantity equal to

$$\tilde{\beta} = n^2 \frac{\beta}{\left(E(R)^2\right)^2} \tag{30}$$

with respect that of the parent SCHS density.

Proof. Following Gomez et al 2003, the Mardia's kurtosis index for a spherical variable is

$$K_n = n^2 \frac{E(R^4)}{(E(R^2))^2}. \tag{31}$$

According to (31), an increase in the kurtosis of a spherical variable can be attained by pushing up the fourth moment of the modular variable R. This, as proved in Theorem A3 in Appendix A2, can be achieved through a GC-like expansion of this latter variable with a shape adapter specified as in (27). The coefficients, e , g of the fourth-order polynomial $p_4(.)$ and its square norm γ_4 are, as usual, functions of the moments of R. Finally, formula (30) can be read a by-product of (31), upon noting that the effect of the shape adapter (27) is to increase the fourth moment of the modular variable by a quantity equal to β.

The case $n = 2$, which will be dealt with in the following, is worth considering in some detail.

Corollary 1 *The density of a bivariate GCSCHS is*

$$\tilde{g}_2(\boldsymbol{x}) = \left(1 + \frac{\beta}{854.8198}\left[(\boldsymbol{x}'\boldsymbol{x})^2 - 14.6345(\boldsymbol{x}'\boldsymbol{x}) + 19.9269\right]\right) g_2(x)\,, \tag{32}$$

where $\boldsymbol{x}$ *is a two dimensional vector and*

$$g_2(x) = \frac{\pi}{\sqrt{2}7\Gamma(3)\zeta(3)} \frac{(\boldsymbol{x}'\boldsymbol{x})^{\frac{1}{2}}}{\sinh\left(\frac{\pi}{\sqrt{2}}(\boldsymbol{x}'\boldsymbol{x})^{\frac{1}{2}}\right)} \tag{33}$$

is the density of the parent bivariate SCHS. The kurtosis, $K_2(\beta)$*, of a GCSCHS lies within the following range*

$$10,4276 \leq K_2(\beta) \leq 29,2854 \tag{34}$$

where the lower and upper bounds correspond to $\beta = 0$ *and to its admissible maximum value, respectively.*

Proof. As proved in Theorem A3 in Appendix A2, the coefficients of the orthogonal polynomials needed to obtain the GC-like expansion of a SCHS density are functions of the moments m_j of its modular variable.

These latter can be found by using formula (65) in Appendix A2 with a density generator g specified as in (24) and can be computed by using the integral (25). Some computations yield

$$m_j = \frac{\Gamma(n+j+1)\zeta(n+j+1)(2^{n+j+1}-1)}{2^j\Gamma(n+1)\zeta(n+1)(2^{n+1}-1)}\left(\frac{\sqrt{2}}{\pi}\right)^j \tag{35}$$

which for $n = 2$ formula becomes

$$m_j = \frac{\Gamma(3+j)\zeta(3+j)(2^{3+j}-1)}{2^j\Gamma(3)\zeta(3)(2^3-1)}\left(\frac{\sqrt{2}}{\pi}\right)^j. \tag{36}$$

The values of the lower-order even moments follows accordingly

$$m_2 = 2,3224, \quad m_4 = 14,0616, \quad m_6 = 159,5036, \quad m_8 = 2908,864. \tag{37}$$

As a by-product, the coefficients of the polynomial in brackets of formula (32) can be computed by using formulas (54) and (56) in Appendix A1. The same moments can be used to evaluate the Mardia's kurtosis index. The lower bound of (34) is computed in accordance to formula (31). As far as the upper bound of (34) is concerned, note that, in light of formula (12), β turns out to be bounded from above by $25,4298$. This entails that, in light of formula (30), the maximum admissible increase of kurtosis is $18,8578$.

3. Application to financial returns data

In the application which follows, we have considered three daily financial series: the Nikkei 225 index, the ESTX50 EUR P index and the FTSE MIB index. The returns from these series, computed as difference between the adjusted closing prices of two consecutive periods divided by the adjusted closing price of the first period, will be denoted by $\hat{N}225$, $\hat{S}TOXX50E$ and $FTSEMIB.MI$ respectively, from now on. All these series are recorded from 2009/01/01 to 2014/12/31. In the following we will prove the capability of GC-like expansions of both CHS and SCHS distributions in fitting these returns and in assessing some risk measures like the Value at Risk (VaR) and the expected shortfall (ES). Let start with an analysis focused on univariate financial series.

3.1 Univariate approach

Table 2 reports the main descriptive statistics of all series. As we can see, all data exhibit excess kurtosis and skewness. The Jarque-Bera test (JB) shows that the null hypothesis of normality is strongly rejected for all returns. In addition, the Ljung-Box (LB) statistics on the squared series denotes significant presence of volatility clustering or time dependent heteroskedasticity. The skewness and extra-kurtosis estimates, α and β, reported in Table 2, have been used to build GCCHS distributions which have been fitted to the return series. Figure 5 shows the Gaussian kernel densities[1] of the financial returns superimposed by their empirical distributions (the histogram of the empirical frequencies), the (standard) normal and the fitted GCCHS. Figure 6 shows the QQplots of the quantiles of the financial returns against, respectively, the Gaussian (in the first column) and the GCCHS quantiles (in the second column). In order to evaluate the goodness of fit of both GCCHS and Gaussian distributions to data, reference can be made to some indexes based on the absolute differences between the frequencies $\hat{f}_i$ of the empirical kernels and the corresponding frequencies $f(x_i)$ estimated by using the GCCHS and the Gaussian densities, namely

$$A_1 = \frac{1}{2} h \sum_{i=1}^{N} |f(x_i) - \hat{f}_i| \tag{38}$$

$$A_2 = \frac{1}{N} \sum_{i=1}^{N} \frac{|f(x_i) - \hat{f}_i|}{\hat{f}_i + f(x_i)} \tag{39}$$

In the above formulas h represents the width of the histogram rectangles, $\hat{f}_i$ is the height of the i-th rectangle and $f(x_i)$ is the ordinate of the GCCHS or Gaussian distribution at the midpoint of the basis of the i-th rectangle. The latter index has the advantage to be bounded, that is

$$0 \leq A_2 \leq 1 \tag{40}$$

with the lower value corresponding to a perfect fit. The fit worsens as the index moves towards the upper bound. Table 3 gives the values taken by these indexes for the series under examination which prove the better fit of GCCHS to data in comparison with the Gaussian law.

In order to gain a deeper insight into the effectiveness of the GCCHS distributions in fitting financial series, we have compared their performance with those of other distributions. These distributions are: the Gaussian, the Student-t, the Skew-t (Fernandez et al., 1998), the normal inverse Gaussian, NIG (Barndorff-Nielsen et al., 1983), the Fischer's generalized hyperbolic density, GH, the generalized secant hyperbolic, GSH, and the skew generalized hyperbolic secant, SGSH (see e.g., Fischer, 2010, 2013). All these distributions have been estimated via maximum likelihood with the functions fitdistrplus, and ghyp (for the NIG and GH) of the R *np* package. Table 4 reports the estimates of the parameters of all these distributions once fitted to returns. Then, the test for equality of univariate densities, proposed by Maasoumi et al., 2002, and based on metric entropy, has been implemented. The function `npunitest` in the R *np* package (Hayfield et al., 2008) has been used for testing the null of equality of two densities. The test statistic is given by

$$S_\rho = \frac{1}{2} \int_{-\infty}^{\infty} \left(f(x)^{\frac{1}{2}} - g(x)^{\frac{1}{2}} \right)^2 dx \tag{41}$$

[1] The kernel densities have been estimated by using the command density in R software which, after scattering the probability mass of the empirical distribution on a regular grid, provides a linear approximation of the discretized version of the kernel.

where $f(x)$ and $g(x)$ are the two densities under comparison, that is are the kernel density of a return series and the corresponding fitting distribution, which is one of the aforementioned densities. [2] The test has been carried out for each of the three returns by assuming as fitting distribution each of the densities introduced before (GCCHS, Gaussian, Students t, Skew t, GSH, SGSH, NIG and GH distribution). Trivially, under the null that the two distributions are equal $S_\rho = 0$, otherwise $S_\rho > 0$. Table 5 displays the test statistics and the corresponding p-values (in brackets) for the above mentioned distributions. According to the results shown in Table 5, the null hypothesis cannot be rejected for the GCCHS, GSH, SGSH, NIG and GH distributions at the level $\alpha = 0.01$. This confirms that for these series the empirical and the fitting distributions are not significantly different.

Further information on the performance of the GCCHS densities and the other distributions is provided by Table 6 which shows the indexes of peak up-thrust, tails up-thrust and shoulders down-thrust provided by formulas (16)-(18), by assuming as reference density the Gaussian law. In this regard, notice that it is not always possible to calculate both the down-thrust shoulder and the up-thrust tail indexes. This happens, when the two compared distributions cross only twice instead of four times, making impossible to distinguish between shoulders and tails. This occurs for the t-Student and the Skew-t. In these cases we can only evaluate the up-thrust index for the peak and for the tails. Looking at Table 6, we can see that, as far as the tail up-thrust and the shoulder-down-thrust are concerned, the GCCHS expansions outperform all the other distributions. The values of these indexes indicate that, with respect the Gaussian, GCCHS has an higher probability percentage in the tails and a lower percentage in the shoulders than other distributions.

Figures 7-9 show the QQplots of the empirical quantiles in the left tail, in the peak and in the right tail, respectively, against the corresponding theoretical quantiles of the CGCHS, GSH, SGSH, NIG and GH quantiles. All these plots show that all the aforementioned densities fit quite well the return kernel densities, expecially the middle parts of these densities.

To gain a better insight into the fitting of the GCCHS densities, we have also computed, for each of them, the relative densities with respect their empirical counterpart (the return kernel densities) (Handcock et al., 1999). For the case under exam, each relative density is the ratio between a given GCCHS density and a return kernel density, both evaluated in correspondence of the empirical quantiles. They have been estimated as in Handcock et al., 2014 with the command *reldist* in R *reldist* package. As a relative density is distributed as an uniform variable when the compared densities are identical, values of this distribution above (below) 1

[2] The integral $\frac{1}{2}\int_{-\infty}^{\infty}\left(f(x)^{\frac{1}{2}} - g(x)^{\frac{1}{2}}\right)^2 dx$ is known as Hellinger distance. As it satisfies the triangular inequality, it is a proper measures of distance (see Hellinger of Granger et al., 2004.

provide evidence that the fitted distribution overestimate (underestimate) the frequency of the corresponding outcome. The graphs in Figure 10 show these relative densities together with their 95% confidence intervals. Looking at Figure 10, we can see that GCCHS densities neither overestimate nor underestimate the kernel densities at hand. We can therefore draw the conclusion that the CHS distribution, once adjusted by its orthogonal polynomials, proves effective in fitting leptokurtic and skewed series.

The validity of a GCCHS distribution in computing some risk measures, like the Value at Risk (VaR) and the expected shortfall (ES), has been also evaluated. As it is well known, VaR provides the minor loss we can expect to run within a certain period for a given probability (see Jorion, 2006). The expected shortfall provides information about the size of losses exceeding VaR, namely the possible average loss (see Landsman, 2004). Table 7 shows both VaR and ES estimates, computed at different significance levels α, by using GCCHS, Normal, Student t, Skew t, GSH, SGSH NIG and GH densities. In this table the estimated VaR and ES are compared with their corresponding empirical values. Empirical VaR has been computed as α-quantile of the empirical distribution, while the empirical ES as average of losses exceeding empirical VaR. Looking at Table 7, we see that GCCHS densities provide estimates of both VaR and ES that are very close to the empirical values.Table 7 reports also the lower and upper bounds of percentage-bootstrap intervals (CI_{boot}) for VaR_{emp} and ES_{emp} which have been built by selecting 10000 bootstrap samples from the empirical density of each series. The results shown in this table confirm the validity of VaR_{α} and ES_{α} obtained by using GCCHS densities which never fall outside the bootstrap intervals for the corresponding empirical values. The same does not occur when these risk measures are computed via other densities.

3.2 Multivariate approach

Finally, in order to evaluate the goodness of fit of GCSCHS densities in a multivariate context, we have considered three bivariate daily series: ($N225$-$\hat{S}TOXX50E$), ($N225$-$FTSEMIB.MI$) and ($FTSEMIB.MI$ -$\hat{S}TOXX50E$). The scatter-plots of these series in the period 2009/01/01-2014/12/31 are reported in Figure 11. Table 8 shows the lengths (N), the Mardias kurtosis indexes of the bivariate returns and their excess kurtosis, $\tilde{\beta}$, with respect the kurtosis of the bivariate SCHS. The values of the parameter β characterizing the polynomial shape adapter (27) needed to build GC expansions of these bivariate series has been determined according to formula (30) and it is shown in the last column of this table. The function MVN of the R package has been used to evaluate the Mardia's multivariate kurtosis index. The graphs in Figure 12 show the bivariate GCSCHS densities fitted to the returns. As in the univariate case, the validity of GCSCHS densities in computing the VaR and the ES has been explored. Following Kamdem (2005), the VaR at a given level α, VaR_α, of a linear portfolio represented by a spherical variable $\boldsymbol{r}$ can be evaluated as follows

$$VaR_\alpha = q_{\alpha,n}\sqrt{\boldsymbol{\delta}'\boldsymbol{\Sigma}\boldsymbol{\delta}} \tag{42}$$

where $\boldsymbol{\delta}$ is a weighting vector that, in this case, is the unit vector and $\boldsymbol{\Sigma}$ is the variance/covariance matrix of $\boldsymbol{r}$. The scalar $q_{\alpha,n}$ is the unique positive solution of the trascendal equation

$$\alpha = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\int_{q_{\alpha,n}}^{\infty}\int_{z_1^2}^{\infty}(u - z_1^2)^{\frac{n-1}{2}} g_n(u)dudz_1 \tag{43}$$

where $g_n(u)$ is defined as in (66) and $\Gamma(\cdot)$ is the Euler-Gamma function.

According to (42), the theoretical ES at a given level α, (ES_α), can be evaluated as follows

$$ES_\alpha = K_{ES}\sqrt{\boldsymbol{\delta}'\boldsymbol{\Sigma}\boldsymbol{\delta}} \tag{44}$$

where

$$K_{ES} = \frac{\pi^{\frac{n-1}{2}}}{\alpha\Gamma\left(\frac{n-1}{2}\right)}\int_{q_{\alpha,n}^2}^{-\infty}(u - q_{\alpha,n}^2)^{\frac{n-1}{2}}\tilde{g}_n(u,\beta)du. \tag{45}$$

As benchmarks for VaR_α and ES_α we have evaluated the corresponding empirical measures in two different ways. Firstly, we have computed the empirical VaR as follows

$$VaR_{e_1} = \sqrt{VaR_1^2 + VaR_2^2 - 2cov(X_1, X_2)} \tag{46}$$

where VaR_j is the VaR of the j-th univariate series and cov(.) denotes the covariance of the bivariate series X_i, X_j. The function VARES of the R package has been used for this scope. Then, after computing the ES_j of each univariate series X_j by using VaR_{e_1} as critical level, we have computed the empirical ES of the bivariate series as follows

$$ES_{e_1} = ES_1 + ES_2. \tag{47}$$

Other measure of empirical VaR and ES, VaR_{e_2} and ES_{e_2} hereafter, have been obtained by applying Kandems formulas (42) and (44) with $g_n(u)$ replaced by the empirical density of the bivariate series and $q_{\alpha,n}$ by the empirical quantile. Tables 9 and 10 compare both VaR_α and ES_α with the corresponding empirical values (VaR_{e_1}, VaR_{e_2}), and (ES_{e_1},ES_{e_2}), respectively. As it happens in the univariate case, also in the bivariate case, GCSCHS densities provide good estimates for both VaR and ES. Table 9 and 10 show empirical VaR and ES respectively, compared with the corresponding theoretical risk measures - VaR_α and ES_α - obtained via GCCHS. Looking at these tables we see that the estimates of these risk measures provided by GCSCHS densities are always close to the corresponding empirical values and that they always lie inside the percentage-bootstrap intervals (CI_{boot}) built for VaR_{emp} and ES_{emp} by selecting 10000 bootstrap samples from the empirical density of each series. Hence, we can draw the conclusion that GCSCHS densities provide precautionary estimates of the risk measures here considered.

So far we have considered the unconditional approach for modelling the empirical distributions of financial asset returns which assumes constant the location and scale parameters of the distributions used to fit them. As it is well known, the presence of volatility clusters in the empirical distributions of financial asset returns is a typical feature which can be modelled by using ARCH/GARCH models (Engle, 1982). This is why in this section the validity of GCSCHS densities in estimation a risk measure, like VaR, via a GARCH model has been tested (see T. Angelidis et al., 2004). To this end we have considered a GARCH model whose innovation has a GCSCHS density. In order to reduce the number of parameters to estimate we have considered a constant conditional correlation model (CCC) (Bollerslev, 1990) specified as follows

$$\begin{aligned} r_t &= \omega + \epsilon_t \,, \\ \epsilon_t &= \boldsymbol{H}^{\frac{1}{2}} \boldsymbol{\eta}_t \end{aligned} \tag{48}$$

where r_t is a n-dimensional vector of returns, $\boldsymbol{\omega}$ denotes their conditional mean (assumed constant for simplicity) and ϵ_t is a GARCH process whose innovation $\boldsymbol{\eta}_t$ is a leptokurtic vector with a GCSCHS density

$$\boldsymbol{\eta}_t \sim GCSCHS(\mathbf{0}, \boldsymbol{I}, \beta), \tag{49}$$

and $\boldsymbol{H}$ is the positive-definite conditional variance of specified as follows

$$\boldsymbol{H} = \boldsymbol{D}_t \boldsymbol{R} \boldsymbol{D}_t. \tag{50}$$

Here $\boldsymbol{D}_t$ is a diagonal matrix whose diagonal entries are the standard conditional variances of the returns and $\boldsymbol{R}$ is their correlation matrix (assumed time invariant). The predictive performance of the model (48) has been assessed by evaluating the percentage of returns that exceed VaR_α, for a given α, via the LR unconditional coverage test (Kupiec, 1995). In order to evaluate VaR_α, and accordingly the number of exceeding returns, a recursive two-step procedure has been implemented. First, we have estimated the model (48) for each of three bivariate series by using the maximum likelihood in the period 2009/01/01-2014/12/31. These estimates have been used to evaluate the parameter β characterizing the GCSCHS density (49) which, according to formulas (42) and (43), is needed to compute VaR_α. In a second step, a set of standardized returns has been computed by using sample data from 2015/01/01 to 2016/12/31 (see Choi et al., 2008)

$$\tilde{r}_t = \frac{(\delta r_t - \omega)}{\sqrt{\delta' \boldsymbol{\Sigma} \delta}} \,. \tag{51}$$

For each t, $\tilde{r}_t$ has been compared with VaR_α computed at time $t-1$, $VaR_{\alpha,t-1}$, and if $\tilde{r}_t < VaR_{\alpha,t-1}$, the number of returns exceeding VaR_α is increased by one. The total number of returns exceeding VaR_α, has been used to perform the LR Kupiecs binomial test which verifies if the percentage of out of sample estimated losses -provided by the CCC model with a GCSCHS innovation- which exceed VaR_α is equal to α, which is the expected number of exceptions for the given confidence level. A p-value lower than α can be interpreted as evidence against the null hypothesis. Table 11 shows the Kupiecs statistics and the corresponding p-values (in brackets). Since the test does not reject the null hypothesis that frequency of exceptions correspond with the defined confidence level, we can conclude that GCSCHS distributions perform quite well also in modelling the innovation of GARCH models.

4. Conclusion

This paper proposes a family of leptokurtic distributions obtained via a polynomial transformation of a leptokurtic density, called convoluted hyperbolic-secant (CHS). The CHS density shares several desirable properties with the logistic and hyperbolic secant laws, to which it is connected by some intriguing relationships. Reshaping the CHS by using its own orthogonal polynomials yields Gram-Charlier-like expansions (GCCHS) able to account for skewness and kurtosis found in empirical data. The multivariate extension of both CHS and its GCCHS expansions can be obtained on a spherical distribution argument. The possibility of encoding both excess kurtosis and skewness, by using the orthogonal polynomial technique, makes GCCHS densities and their spherical version, a valuable resource for modelling financial asset return distributions. This considerably broadens the application domain of the previous approach based on the transformation of the Gaussian law by Hermite polynomials (see e.g., Zoia, 2010). An application to empirical financial returns data provides practical evidence of the effectiveness of the proposed densities to fit leptokurtic, skewed both univariate and multivariate distributions. Their capability in assessing some risk measures, like the value at risk and the expected shortfall, using both an unconditional and a conditional approach based on GARCH models, is also evaluated and leads to the conclusion that they compare favorably with the alternatives considered by the extant literature.

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Appendix

A1. ***Orthogonal Polynomials***

A sequence of polynomials

$$p_n(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 \tag{52}$$

where $a_0, a_1, \ldots, a_{n-1}$ are reals and n is a non-negative integer, is orthogonal with respect to a density function $f(x)$ with finite moments if the following holds

$$\int_{-\infty}^{\infty} p_n(x)p_d(x)f(x)dx = \gamma_n\delta_{nd},\ d = 0, 1, ..., n-1,\ n \in N \cup \{0\}. \tag{53}$$

Here $\gamma_n > 0$, δ_{nd} is the Kronecker symbol ($\delta_{nd} = 1$ if $n = d$, and zero otherwise) and $p_0(x) = 1$ by convention. Condition (53) determines $p_n(x)$. The coefficients $a_0, a_1, \ldots, a_{n-1}$ of $p_n(x)$ are functions of the moments m_j of the density $f(x)$

$$a_j = (M_{n+1,n+1})^{-1}M_{n+1,j+1}, \tag{54}$$

where $M_{n+1,i}$ is the cofactor of the $(n+1, i)$ entry of the $(n+1, n+1)$ moment matrix

$$\begin{bmatrix} m_0 & m_1 & m_2 & \ldots & m_n \\ \vdots & \vdots & \vdots & & \vdots \\ m_{n-1} & m_n & m_{n+1} & \ldots & m_{2n-1} \\ 1 & x & x^2 & \ldots & x^n \end{bmatrix}. \tag{55}$$

The quantity γ_n is given by

$$\gamma_n = \frac{M_{n+2,n+2}}{M_{n+1,n+1}} \tag{56}$$

where reference is made to a moment matrix of dimensions $(n+2, n+2)$ (see, e.g., Chihara, 1978, Szego, 1967).

For even densities, odd moments are null and orthogonal polynomials $p_n(x)$ are even functions if n is even, and odd otherwise (see Szego, 1967). In particular, the third and fourth order polynomials orthogonal to an even density $f(x)$ turn out to be of the form

$$p_3(x) = x^3 - dx, \qquad p_4(x) = x^4 - ex^2 + g \tag{57}$$

where d, g and e are functions of moments of the random variable x as specified in formula (54).

For our purposes the following trinomial

$$q(x,\alpha,\beta) = 1 + \frac{\alpha}{\gamma_3} p_3(x) + \frac{\beta}{\gamma_4} p_4(x) \tag{58}$$

is of particular interest because it can be used to alter the third and fourth moments of $f(x)$ to an extent equal to α and β, respectively. To this end, consider the function

$$\tilde{g}(x,\alpha,\beta) = q(x,\alpha,\beta) f(x) \tag{59}$$

where $(q(x,\alpha,\beta))$ is subject to be positive. The function $\tilde{g}(x,\alpha,\beta)$ is called Gram-Charlier-like (GC-like) expansion of $f(x)$. The following theorem proves a fundamental result on moments of GC-like expansions defined as in (59).

Theorem A1 *The moments μ_j up to the 4-th order of the GC-like expansion in (59) are related to the moments m_j of the parent density $f(x)$ as follows*

$$\begin{cases} \mu_j = m_j & \text{for} \quad j < 3 \\ \mu_j = m_j + \alpha & \text{for} \quad j = 3 \\ \mu_j = m_j + \beta & \text{for} \quad j = 4. \end{cases} \tag{60}$$

Higher moments of $\tilde{g}(x,\alpha,\beta)$ turn out to be algebraic functions of the moments of $f(x)$ likewise.

Proof. The proof follows from the properties of orthogonal polynomials as shown in Zoia, 2010 and Faliva et al, 2016.

A2. ***Spherical Distributions***

Spherical distributions and the corresponding random vectors are also called radial (Kelker, 1970) or isotropic (Bingham et al., Kiesel, 2002) because they correspond to the class of rotationally symmetric distributions. Accordingly, any spherical random vector admits a stochastic representation of the form

$$\boldsymbol{x} = R\boldsymbol{U}^{(n)} \tag{61}$$

where $\boldsymbol{U}^{(n)}$ is a random vector uniformly distributed on the unit hypersphere with n-1 topological dimensions and $R = (\boldsymbol{x}'\boldsymbol{x})^{\frac{1}{2}}$ is a non-negative random variable, called modular variable, independent of $\boldsymbol{U}^{(n)}$. A nice property of spherical distributions is that their densities may be expressed via the density function of the modular variable, provided this is absolutely continuous.

Theorem A2 *If an n-dimensional spherical random vector* $\boldsymbol{x}$ *has a density* $g_n(\boldsymbol{x})$*, then it has the form*

$$g_n(\boldsymbol{x}) = k f_R\left((\boldsymbol{x}'\boldsymbol{x})^{\frac{1}{2}}\right), \tag{62}$$

where f_R *is the density of the modular variable* $R = (\boldsymbol{x}'\boldsymbol{x})^{\frac{1}{2}}$,

$$k = \frac{\Gamma\left(\frac{n}{2}\right)}{2(\pi)^{\frac{n}{2}}} r^{1-n} \tag{63}$$

and $\Gamma(\cdot)$ *is the Euler-Gamma function. Besides, the density* f_R *has the integral representation*

$$f_R(r) = \frac{2r^{n-1}}{\int_0^\infty y^{\frac{n}{2}-1} g(y) dy} g(r^2) \tag{64}$$

in terms of a non-negative Lebesgue measurable function $g(\cdot)$ *called density generator*.

Proof. For the proof see Fang Kai-Tai et al., 1965, and Gomez et al., 2003.

The issue of density reshaping based on orthogonal polynomials can be extended to n-dimensional spherical distributions. In fact, as in the univariate case, (even) moments of a spherical distribution can be properly modified by adjusting this latter with ad hoc orthogonal polynomials. These latter have coefficients which are built from the moments of the modular variable characterizing the spherical law. Accordingly, multiplication of a spherical variable by a polynomial shape adapter depending on these orthogonal polynomials leads to a GC-like expansions of the same variable.

Theorem A3 *Let* $g_n(\boldsymbol{x})$ *be a spherical density and* $q_{n,4}(\boldsymbol{x},\beta)$ *be a shape adapter specified as in* (27) *in terms of a* 4*-th order orthogonal polynomial. If the coefficients of this polynomial are based on the moments* m_j

$$m_j = \frac{\int_0^\infty y^{\frac{n+j}{2}-1} g(y) dy}{\int_0^\infty y^{\frac{n}{2}-1} g(y) dy}. \tag{65}$$

of the modular variable R, then the expansion

$$\tilde{g}_n(\boldsymbol{x},\beta) = q_{n,4}(\boldsymbol{x},\beta) g_n(\boldsymbol{x}) \tag{66}$$

is the GC-like expansion associated with the spherical density $g_n(\boldsymbol{x})$.

Proof. Upon noting that, in light of (62) and (64), the density of a spherical variable is an even function in $||\boldsymbol{x}||^2 = R^2$,

$$g_n(\boldsymbol{x}) = \frac{\Gamma\left(\frac{n}{2}\right)}{(\pi)^{\frac{n}{2}} \int_0^\infty y^{\frac{n}{2}-1} g(y) dy} g(\boldsymbol{x}'\boldsymbol{x}) = \frac{\Gamma\left(\frac{n}{2}\right)}{(\pi)^{\frac{n}{2}} \int_0^\infty y^{\frac{n}{2}-1} g(y) dy} g(||\boldsymbol{x}||^2) \tag{67}$$

the argument of density reshaping by orthogonal polynomials can be extended to n dimensional spherical distributions by using the polynomials in the variable $\|\boldsymbol{x}\|^2$, that is polynomials which are orthogonal to the density of R^2, $f_{R^2}(r)$ hereafter. In this connection note that, in light of the following relationship

$$f_R(r) = 2r f_{R^2}(r), \tag{68}$$

the density of a spherical variable can be directly expressed in terms of $f_{R^2}(r)$ as follows

$$g_n(\boldsymbol{x}) = 2rk f_{R^2}\left((\boldsymbol{x}'\boldsymbol{x})^{1/2}\right) \tag{69}$$

Accordingly, the GC-like expansions of $g_n(\boldsymbol{x})$ can be obtained by reshaping $f_{R^2}(r)$ with the binomial $q_{n,4}(\boldsymbol{x},\beta)$. To this end, consider the following GC-like expansion of the density f_{R^2}

$$f_{\tilde{R}^2}(r,\beta) = q_{n,2}(r^2,\beta) f_{R^2}(r) \tag{70}$$

where $q_{n,2}(r^2,\beta) = \left(1 + \frac{\beta}{\gamma_4} p_2(r^2)\right)$ can be read either as binomial depending on a complete second order polynomial, $p_2\left(\boldsymbol{x}'\boldsymbol{x}\right)$, in the variable $\boldsymbol{x}'\boldsymbol{x} = r^2$, or as an incomplete fourth-order polynomial in the variable $(\boldsymbol{x}'\boldsymbol{x})^{1/2} = r$, that is

$$p_2(r^2) = [(r^2)^2 - er^2 + g] = [r^4 - er^2 + g] = p_4(r^4) \tag{71}$$

This entails that the following identity

$$q_{n,2}(r^2,\beta) = q_{n,4}(r,\beta) \tag{72}$$

holds true. Now, according to formula (54), the coefficients, e and g, of the second order polynomial $p_2(r^2)$ can be expressed n terms of the moments $\widetilde{m}_j$ of the variable R^2 as follows

$$e = \frac{M_{3,2}}{M_{3,3}} \frac{\widetilde{m}_3\widetilde{m}_0 - \widetilde{m}_1\widetilde{m}_2}{\widetilde{m}_2\widetilde{m}_0 - \widetilde{m}_1^2}, \qquad g = \frac{M_{3,1}}{M_{3,3}} \frac{\widetilde{m}_3\widetilde{m}_1 - \widetilde{m}_2^2}{\widetilde{m}_2\widetilde{m}_0 - \widetilde{m}_1^2} \tag{73}$$

Upon noting that $\widetilde{m}_0 = 1$, the above coefficients can be expressed in terms of the moments, m_j, of the modular variable R as follows

$$e = \frac{m_6 - m_2 m_4}{m_4 - m_2^2}, \qquad g = \frac{m_6 m_2 - m_4^2}{m_4 - m_2^2} \tag{74}$$

Now, simple computations prove that the spherical variable defined in terms of the reshaped modular variable $\tilde{R}$

$$\hat{g}_n(\boldsymbol{x}) = k f_{\tilde{R}}\left((\boldsymbol{x}'\boldsymbol{x})^{1/2}\right) \tag{75}$$

coincides with the GC-like expansion (66). In fact, taking into account (68), (69) and (70), some computations yield

$$\hat{g}_n(\boldsymbol{x}) = k f_{\tilde{R}}\left((\boldsymbol{x}'\boldsymbol{x})^{1/2}\right) = 2rk f_{\tilde{R}^2}\left((\boldsymbol{x}'\boldsymbol{x})^{1/2}\right) = \tag{76}$$

$$= 2rk q_{n,2}(r^2,\beta) f_{R^2}\left((\boldsymbol{x}'\boldsymbol{x})^{1/2}\right) = k q_{n,4}(r,\beta) f_R\left((\boldsymbol{x}'\boldsymbol{x})^{1/2}\right) = \tilde{g}_n(\boldsymbol{x},\beta) \tag{77}$$

It follows that the GC-like expansion $\tilde{g}_n(\boldsymbol{x},\beta)$ can be obtained by reshaping the modular variable, R with the binomial $q_{n,4}(r,\beta)$. Accordingly, the coefficients of the polynomial $p_4(\boldsymbol{x}'\boldsymbol{x})^{1/2}$characterizing this binomial are functions of the moments of R which are specified as in (65), as proved in Theorem 3 on page 349 in Gomez et al.,2003.

The following graph compares the density f_{R^2} of a bivariate convoluted hyperbolic secant distribution with the polynomially modified distribution $f_{\widetilde{R}^2} = \left(1 + \frac{\beta}{\gamma_4}[(r^2)^2 - er^2 + g]\right) f_{R^2}$ for $\beta = 4, 7, 9$

This other graph shows the spherical densities $g_n(\boldsymbol{x}) = k f_R$ and $\widetilde{g}_n(\boldsymbol{x}) = q_{n,4}(r,\beta) k f_{\widetilde{R}}$ with $\beta = 7$.