# ON THE UNIQUENESS OF A SHEAR-VORTICITY-ACCELERATION-FREE VELOCITY FIELD IN SPACE-TIMES

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ABSTRACT. We prove that in space-times a velocity field that is shear, vorticity and acceleration-free, if any, is unique up to reflection, with these exceptions: generalized Robertson-Walker space-times whose space sub-manifold is warped, and twisted space-times (the scale function is space-time dependent) whose space sub-manifold is doubly twisted. In space-time dimension n=4, the Ricci and the Weyl tensors are specified, and the Einstein equations yield a mixture of two perfect fluids.

## 1. Introduction and statement of results

In a space-time of dimension n > 3, let  $u_k$  be a smooth velocity field that is shear-free, vorticity-free and acceleration-free:

(1) 
$$u^k u_k = -1, \qquad \nabla_i u_j = \varphi(u_i u_j + g_{ij}),$$

in other words,  $u_k$  is a time-like unit torse-forming vector field, with scalar field  $\varphi$ . We enquire whether the space-time may admit other velocity fields that are time-like unit and torse-forming,

(2) 
$$w^k w_k = -1, \qquad \nabla_i w_j = \lambda (w_i w_j + g_{ij})$$

besides the trivial twin vector  $-u_k$  with scalar field  $-\varphi$ .

The existence of the vector field  $u_k$  ensures that the space-time is Twisted [10], i.e. there is a reference frame where the metric has the form:

(3) 
$$ds^{2} = -dt^{2} + a(t,x)^{2} g_{\mu\nu}^{*}(x) dx^{\mu} dx^{\nu}$$

where t is the time, the scale function a(t,x) > 0 depends on time and space coordinates, and  $g_{\mu\nu}^*(x)$  is the metric tensor of a Riemannian sub-manifold  $M^*$  with space coordinates  $x^{\mu}$ . Twisted space-times were introduced by B.-Y. Chen, to generalise the notion of warped manifolds [2][3].

In the locally "comoving" frame (3) it is  $u^0 = 1$ ,  $u^{\mu} = 0$  and  $\varphi = \dot{a}/a$ .

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With the Christoffel symbols listed in the Appendix, the normalisation and torse-forming conditions for another vector  $w_i$  are the equations:

$$-w_0^2 + a^{-2}g^{*\mu\nu}w_{\mu}w_{\nu} = -1, \qquad \partial_t w_0 = \lambda(w_0^2 - 1),$$
(4) 
$$\partial_{\mu}w_0 - (\dot{a}/a)w_{\mu} = \lambda w_0 w_{\mu}, \quad \partial_t w_{\mu} - (\dot{a}/a)w_{\mu} = \lambda w_0 w_{\mu}$$

$$\nabla_{\mu}^{*} w_{\nu} - \frac{a_{\mu}}{a} w_{\nu} - \frac{a_{\nu}}{a} w_{\mu} + g_{\mu\nu}^{*} (\frac{a_{\rho}}{a} w_{\sigma} g^{*\rho\sigma} - a\dot{a}w_{0}) = \lambda (w_{\mu} w_{\nu} + a^{2} g_{\mu\nu}^{*})$$

where  $\dot{a} = \partial_t a$ ,  $a_\mu = \partial_\mu a$ , and  $\nabla^*$  is the connection of the sub-space  $(M^*, g^*)$ . To investigate the existence of a non-trivial solution (besides  $u_0 = -1$ ,  $u_\mu = 0$ ,  $\lambda = \varphi$ ) we distinguish three cases:

Case A)  $\nabla_k \varphi = 0$  and  $\varphi \neq 0$ ,  $(\varphi = 0$  factors space and time);

Case B) 
$$\nabla_k \varphi = -u_k u^m \nabla_m \varphi$$
;

Case C) 
$$\nabla_k \varphi = -u_k u^m \nabla_m \varphi + v b_k$$
, with  $v \neq 0$ ,  $b_k b^k = 1$ ,  $u^k b_k = 0$ .

In Cases A and B the existence of  $u_k$  implies that the space-time is warped, i.e. there is a reference frame where the metric (3) has the simpler form

(5) 
$$ds^{2} = -dt^{2} + a(t)^{2} g_{\mu\nu}^{*}(x) dx^{\mu} dx^{\nu}$$

where the scale function only depends on the time. Such space-times are also named generalised Robertson-Walker (GRW) (see [9] for a review, and [4] for another covariant characterisation). They are Robertson-Walker (RW) if the Weyl tensor is zero.

For each case we prove:

**Theorem A.** In a warped space-time (5) with constant non-zero  $\varphi$ , the torse-forming velocity is unique unless  $(M^*, g^*)$  is a warped sub-manifold, i.e.  $(M^*, g^*)$  admits a unit vector field  $n_{\mu}^*(x)$  such that  $\nabla_{\mu}^* n_{\nu}^* = \theta(x)(n_{\mu}^* n_{\nu}^* - g_{\mu\nu}^*)$  with  $\nabla_{\mu}^* \theta$  proportional to  $n_{\mu}^*$ .

**Theorem B.** In a warped space-time (5) with non-constant  $\varphi$ , the torse-forming velocity is unique.

**Theorem C.** In a twisted space-time (3) the torse-forming velocity is unique unless  $(M^*, g^*)$  is a doubly-twisted sub-manifold. This is equivalent to the requirement

(6) 
$$\nabla_i b_j = \varphi b_i u_j + (g_{ij} + u_i u_j - b_i b_j) \frac{\nabla_k b^k}{n-2}$$

The second torse-forming velocity is  $w_i = u_i \cosh \alpha + b_i \sinh \alpha$  with:

(7) 
$$\tanh \alpha = -\frac{2R_{ij}u^ib^j}{R_{ij}(u^iu^j + b^ib^j)}$$

(8) 
$$\lambda = \varphi \cosh \alpha + \frac{\nabla_k b^k}{n-2}$$

The property  $|\tanh \alpha| \le 1$  poses a restriction.

Theorems A, B and C correct Prop.2.3 in our paper [10] that claimed uniqueness in all cases. In its short proof, the sign in front of  $(\dot{f}/f) \sinh aV_{\mu}$  in the centred equation is wrong. With the correct (minus) sign, the proof by reductio ad absurdum does not work. The error does not affect the rest of the paper.

Cases A, B and C will be discussed separately, case B being simpler and preparatory for case A. For all cases, some preliminary identities that simplify the discussion are first obtained.

Lovelock's identity in n=4 and the existence of a second torse-forming vector, determine the electric component of the Weyl tensor and thus the Ricci and Weyl tensors. By the Einstein equations, an energy-momentum tensor  $T_{ij}$  is obtained, that describes a mixture of two perfect fluids with non collinear velocities, studied by Coley and McManus [5][6]. In their work, the request that one velocity is torse-forming (no restriction for the other velocity), implies that the subspace  $M^*$  admits an umbilical foliation i.e. it is doubly twisted [12][3]. In the present study, the mixture of perfect fluids arises in n=4 as a consequence of having two torse-forming vectors and only one of them, in general, is the velocity of one of the fluids.

#### 2. Preliminary results

Suppose that, besides  $u_k$  with scalar field  $\varphi$ , there exists another time-like unit torse-forming vector field  $w_k$  with scalar field  $\lambda$ , eq.(2), not collinear with  $u_k$ .

**Remark.** We are assuming  $(u^k w_k)^2 \neq 1$ , otherwise  $w_k$  would be space-like.

The following identities with the Riemann tensor,  $[\nabla_i, \nabla_j]u_k = R_{ijkm}u^m$  and  $[\nabla_i, \nabla_j]w_k = R_{ijkm}w^m$ , are evaluated with (1) and (2):

(9) 
$$R_{ijkm}u^m = (u_iu_k + g_{ik})\nabla_i\varphi - (u_iu_k + g_{ik})\nabla_i\varphi + \varphi^2(u_ig_{ik} - u_ig_{ik})$$

(10) 
$$R_{ijkm}w^m = (w_jw_k + g_{jk})\nabla_i\lambda - (w_iw_k + g_{ik})\nabla_j\lambda + \lambda^2(w_jg_{ik} - w_ig_{jk})$$

Contraction of both equations with  $g^{ik}$  gives identities with the Ricci tensor:

(11) 
$$R_{jm}u^m = (u_ju_k + g_{jk})\nabla^k\varphi + (n-1)(\varphi^2u_j - \nabla_j\varphi)$$

(12) 
$$R_{jm}w^m = (w_j w_k + g_{jk})\nabla^k \lambda + (n-1)(\lambda^2 w_j - \nabla_j \lambda)$$

Transvect (9) with  $w^i$  and (10) with  $u^i$ . In the second one use the symmetries of the Riemann tensor:  $R_{ijkm}u^iw^m = R_{mkji}u^iw^m = R_{ikjm}w^iu^m$ , then exchange k with j:

$$R_{ijkm}w^{i}u^{m} = (u_{j}u_{k} + g_{jk})w^{i}\nabla_{i}\varphi - (w^{i}u_{i}u_{k} + w_{k})\nabla_{j}\varphi + \varphi^{2}(u_{j}w_{k} - w^{i}u_{i}g_{jk})$$

$$R_{ijkm}u^{m}w^{i} = (w_{j}w_{k} + g_{jk})u^{i}\nabla_{i}\lambda - (u^{i}w_{i}w_{j} + u_{j})\nabla_{k}\lambda + \lambda^{2}(w_{k}u_{j} - u^{i}w_{i}g_{jk})$$
Subtract one equation from the other:

$$g_{jk}[w^{i}\nabla_{i}\varphi - (\varphi^{2} - \lambda^{2})w^{i}u_{i} - u^{i}\nabla_{i}\lambda] + u_{j}u_{k}(w^{i}\nabla_{i}\varphi) - (w^{i}u_{i}u_{k} + w_{k})\nabla_{j}\varphi + (\varphi^{2} - \lambda^{2})u_{j}w_{k} - w_{j}w_{k}(u^{i}\nabla_{i}\lambda) + (u^{i}w_{i}w_{j} + u_{j})\nabla_{k}\lambda = 0.$$

Contraction with a non-zero vector orthogonal to  $u_j$ ,  $w_j$  and  $\nabla_j \varphi$  gives the equation

(13) 
$$w^{i}\nabla_{i}\varphi - u^{i}\nabla_{i}\lambda = (u^{i}w_{i})(\varphi^{2} - \lambda^{2})$$

and, after simplification, the following one:

(14) 
$$u_j u_k (w^i \nabla_i \varphi) - (w^i u_i u_k + w_k) \nabla_j \varphi + (\varphi^2 - \lambda^2) u_j w_k - w_j w_k (u^i \nabla_i \lambda) + (u^i w_i w_j + u_j) \nabla_k \lambda = 0.$$

The trace of the latter is:  $(w^i u_i)(\varphi^2 - \lambda^2 + w^j \nabla_j \lambda - u^j \nabla_j \varphi) + 2(u^i \nabla_i \lambda - w^j \nabla_j \varphi) = 0$ , with the aid of eq. (13) we obtain, after cancellation of  $u^j w_j \neq 0$ .

(15) 
$$u^{j}\nabla_{i}\varphi + \varphi^{2} = w^{j}\nabla_{i}\lambda + \lambda^{2}$$

Hereafter we denote

(16) 
$$\xi = (n-1)(u^j \nabla_j \varphi + \varphi^2)$$

Contraction of (14) with  $u^j$  or with  $w^k$  and use of (15), give:

(17)

$$(18)$$

$$(18)$$

$$u_j[u^i\nabla_i\varphi + w^ku_k(w^i\nabla_i\varphi)] + w_j[u^i\nabla_i\lambda + u^iw_i(w^k\nabla_k\lambda)] = [(u^iw_i)^2 - 1]\nabla_j\varphi$$

# 3. Proof of Theorem B

**Lemma 3.1.** If 
$$\nabla_i \varphi = -u_i u^k \nabla_k \varphi$$
, then  $\nabla_i \lambda = -w_i w^k \nabla_k \lambda$ .

Proof. Eq.(17) simplifies to  $w_k[w^j\nabla_j\lambda + u^iw_i(u^j\nabla_j\lambda)] = [(u^iw_i)^2 - 1]\nabla_k\lambda$ , showing that  $\nabla_k\lambda$  is collinear to  $w_k$ . It follows that  $\nabla_k\lambda = -w_k(w^j\nabla_j\lambda)$ .  $\square$ 

This result simplifies eqs.(11) and (12), showing that  $u_k$  and  $w_k$  are both eigenvectors of the Ricci tensor with eigenvalue  $\xi$ :

$$R_{ij}u^j = \xi u_i, \qquad R_{ij}w^j = \xi w_i$$

Now, the problem is about degeneracy of the eigenvalue of the Ricci tensor:

**Proposition 3.2.** In a warped space-time, for the eigenvalue  $\xi$  of the Ricci tensor to be degenerate, it is necessary that  $\dot{a}(t)/a(t) = At + B$ , with constants A and B.

*Proof.* Let us consider the eigenvalue equation  $R_{ij}w^j = \xi w_i$  in the warped frame (5). The components of the Ricci tensor can be read in [10].

The equation  $R_{00}w^0 = \xi w_0$  is  $-(n-1)(\ddot{a}/a)w^0 = \xi w_0$ , then  $\xi = (n-1)(\ddot{a}/a)$  as  $w_0 \neq 0$ . In the equation  $R_{\mu 0}w^0 + R_{\mu \nu}w^{\nu} = \xi w_{\mu}$  one has  $R_{\mu 0} = 0$  and  $R_{\mu \nu} = R_{\mu \nu}^* + g_{\mu \nu}^*[(n-2)\dot{a}^2 + a\ddot{a}]$ . A solution is always  $u^0 = 1$ ,  $u^\mu = 0$  (the

given vector). Other solutions have non-zero space components  $w^{\mu}$  solving the eigenvalue equation:

$$R_{\mu\nu}^* w^{\nu} = \xi w_{\mu} - [(n-2)\dot{a}^2 + a\ddot{a}]g_{\mu\nu}^* w^{\nu}$$
$$= (n-2) \left(\frac{d}{dt}\frac{\dot{a}}{a}\right) w_{\mu}$$

where we lowered an index:  $a^2 g_{\mu\nu}^* w^{\nu} = w_{\mu}$ . In the warped frame the Ricci tensor  $R_{\mu\nu}^*$  of  $M^*$  does not depend on time, and so must the eigenvalue. Then  $(\dot{a}/a) = At + B$  where A and B do not depend on space coordinates, as the warping function does not.

**Lemma 3.3.** If 
$$\nabla_i \varphi = -u_i u^k \nabla_k \varphi$$
 then:  $\nabla_k \xi = -u_k (u^j \nabla_i \xi)$ .

Proof. Evaluate:  $\nabla_k \varphi^2 = 2\varphi \nabla_k \varphi = 2\varphi (-u_k u^j \nabla_j \varphi) = -u_k u^j \nabla_j \varphi^2$ . Next:  $\nabla_k (u^j \nabla_j \varphi) = \varphi (u_k u^j + \delta_k^j) \nabla_j \varphi + u^j \nabla_j \nabla_k \varphi$ ; the first term is zero, the second one is:  $u^j \nabla_j (-u_k u^\ell \nabla_\ell \varphi) = -u_k [u^j \nabla_j (u^\ell \nabla_\ell \varphi)]$ . Add results and multiply by (n-1).

The same assertion holds for the torse-forming velocity  $w_k$ :  $\nabla_k \xi = -w_k w^j \nabla_j \xi$ . Comparison of assertions gives: if  $\xi$  is not a constant scalar, then the torse-forming velocities  $u_j$  and  $w_j$  are collinear i.e.  $u_j$  is unique.

What remains to discuss is the case that  $\xi$  is a constant scalar and is a degenerate eigenvalue. Proposition 3.2 imposes  $\varphi = At + B$  i.e.  $\xi = (n-1)[(At+B)^2 + A]$ . Then  $\xi$  is constant if A=0 i.e.  $\varphi$  is constant, which is case A.

This proves Theorem B.

## 4. Proof of theorem A

**Lemma 4.1.** If  $\varphi$  is a non-zero constant, and if  $w_k$  exists not parallel to  $u_k$ , then  $\lambda = \varphi$ .

*Proof.* If  $\lambda$  is constant, eq.(15) implies  $\lambda = \varphi$ .

Now suppose that  $\lambda$  is not a constant. Eq.(17) with constant  $\varphi$  gives:  $w_k[w^j\nabla_j\lambda+u^iw_i(u^j\nabla_j\lambda)]=[(u^iw_i)^2-1]\nabla_k\lambda$ . Then  $\nabla_k\lambda$  is proportional to  $w_k$  i.e.  $\nabla_k\lambda=-w_kw^j\nabla_j\lambda$ .

Then both  $u_j$  and  $w_j$  are eigenvectors of the Ricci tensor with the same eigenvalue  $\xi = (n-1)\varphi^2$ . Given  $w_j$  not collinear with  $u_j$ , there is a warped frame where  $w^0 = 1$ ,  $w^\mu = 0$ , and scale factor  $\tilde{a}(t)$  such that  $\dot{\tilde{a}}/\tilde{a} = \lambda$ . As in Proposition 3.2 the condition that  $\xi$  is degenerate and constant puts  $\dot{\tilde{a}}/\tilde{a} = \lambda$  constant in space-time, and this is against the hypothesis.

Being  $\dot{a}/a = \varphi$  a non-zero constant, we can set  $a(t) = \exp(\varphi t)/\varphi$ . The torse-forming conditions (4) simplify

$$\partial_t w_0 = \varphi(w_0^2 - 1) \quad \partial_\mu w_0 = \varphi w_\mu(w_0 + 1) \quad \partial_0 w_\mu = \varphi w_\mu(w_0 + 1)$$
$$\nabla^*_\mu w_\nu = \varphi[w_\mu w_\nu + a(t)^2 (w_0 + 1) g_{\mu\nu}^*]$$

The first three equations are solved by

$$w_0(x,t) = \frac{1 + C^2(x) \exp(2\varphi t)}{1 - C^2(x) \exp(2\varphi t)}, \qquad w_{\mu}(x,t) = \frac{1}{\varphi} \frac{(\partial_{\mu} C^2) \exp(2\varphi t)}{1 - C^2(x) \exp(2\varphi t)}$$

where the function  $C^2(x)$  is determined by the last differential equation:

$$\partial_\mu \partial_\nu C^2(x) - \Gamma^{*\rho}_{\mu\nu} \partial_\rho C^2(x) = 2g^*_{\mu\nu}(x) \quad i.e. \quad \nabla^*_\mu \nabla^*_\nu C^2 = 2g^*_{\mu\nu}$$

The normalization condition  $-1 = -w_0^2 + \varphi^2 e^{-2\varphi t} g^{*\mu\nu} w_\mu w_\nu$  gives:

$$4C^2 = g^{*\mu\nu}(\partial_\mu C^2)(\partial_\nu C^2)$$

If we put  $\partial_{\mu}C = n_{\mu}^{*}$ , then:  $g^{*\mu\nu}n_{\mu}^{*}n_{\nu}^{*} = 1$  and  $\nabla_{\mu}^{*}n_{\nu}^{*} = \frac{1}{C}(g_{\mu\nu}^{*} - n_{\mu}^{*}n_{\nu}^{*})$  i.e.  $n_{\mu}^{*}$  is unit and torse-forming in  $(M^{*}, g^{*})$ . Since also  $\partial_{\mu}(1/C) = -n_{\mu}^{*}/C^{2}$  the Riemannian subspace  $(M^{*}, g^{*})$  is warped, i.e. there is a choice of space coordinates such that  $g_{\mu\nu}^{*}dx^{\mu}dx^{\nu} = (dx^{1})^{2} + f^{2}(x^{1})ds^{2}$ , where  $ds^{2}$  involves the coordinates  $x^{2}, ..., x^{n-1}$ .

# 5. Proof of theorem C

If  $\nabla_k \varphi$  is not collinear with  $u_k$ , the coefficient of  $w_k$  in eq.(18) cannot be zero, and the same equation shows that  $\nabla_k \varphi$  is a linear combination of  $u_k$  and  $w_k$ . Eq.(17) shows that also  $\nabla_k \lambda$  is a linear combination of  $u_k$  and  $w_k$ . If  $\nabla_k \varphi + u_k(u^i \nabla_i \varphi) = v b_k$ , where  $v \neq 0$ ,  $b^j b_j = 1$ ,  $b_j u^j = 0$ , then  $w_k$  is spanned by the vectors  $u_k$  and  $b_k$ . It is convenient to introduce the hyperbolic rotation of the orthogonal pair (u, b) to the orthogonal pair (w, c):

(19) 
$$\begin{cases} w_i = u_i \cosh \alpha + b_i \sinh \alpha \\ c_i = u_i \sinh \alpha + b_i \cosh \alpha \end{cases} \quad \alpha \neq 0$$

Then:  $w^2 = -1$ ,  $c_k w^k = 0$ ,  $c^k c_k = 1$ ,  $u_i u_j - b_i b_j = w_i w_j - c_i c_j$ . The choice that w has a component parallel to u is not a limitation: if w exists, also -w is time-like torse-forming with scalar field  $-\lambda$ .

**Proposition 5.1.** The only possible hyperbolic rotation is

(20) 
$$\tanh \alpha = -\frac{2R_{ij}u^ib^j}{R_{ij}(u^iu^j + b^ib^j)}$$

*Proof.* Contraction of (11) with  $u^i$  and of (12) with  $w^i$  give:  $R_{ij}u^iu^j = R_{ij}w^iw^j = -\xi$ . Then:  $0 = R_{ij}(w^iw^j - u^iu^j) = \sinh\alpha[R_{ij}(u^iu^j + b^ib^j)\sinh\alpha + 2(R_{ij}u^ib^j)\cosh\alpha]$ . If  $\alpha \neq 0$ , the result is obtained.

Let us write the condition (2) in terms of the hyperbolic components:

$$\nabla_{i}(u_{j}\cosh\alpha + b_{j}\sinh\alpha)$$

$$= \lambda[u_{i}u_{j}\cosh^{2}\alpha + b_{i}b_{j}\sinh^{2}\alpha + (u_{i}b_{j} + u_{j}b_{i})\cosh\alpha\sinh\alpha + g_{ij}]$$

$$(\nabla_{i}\alpha)(u_{j}\sinh\alpha + b_{j}\cosh\alpha) + \varphi\cosh\alpha(u_{i}u_{j} + g_{ij}) + (\nabla_{i}b_{j})\sinh\alpha$$

$$= \lambda[u_{i}u_{j}\cosh^{2}\alpha + b_{i}b_{j}\sinh^{2}\alpha + (u_{i}b_{j} + u_{j}b_{i})\cosh\alpha\sinh\alpha + g_{ij}]$$

Contraction with  $u^j$ , and the hypothesis  $\sinh \alpha \neq 0$  give:

$$\nabla_i \alpha = \lambda (u_i \sinh \alpha + b_i \cosh \alpha) - \varphi b_i$$

Insertion in the previous equation gives, after simple algebra,

$$\sinh \alpha (\nabla_i b_j) = (\lambda - \varphi \cosh \alpha)(u_i u_j + g_{ij} - b_i b_j) + \varphi b_i u_j \sinh \alpha$$

The trace of the equation gives the expression (8) for the parameter  $\lambda$  and, if  $\sinh \alpha \neq 0$ , the equation (6).

**Proposition 5.2.** Condition (6) is equivalent to the requirement that the space submanifold  $(M^*, g^*)$  admits a unit vector  $n_{\mu}^*(x)$  such that

(21) 
$$\nabla_{\mu}^* n_{\nu}^* = \frac{\nabla_{\rho}^* n^{*\rho}}{n-2} (g_{\mu\nu}^* - n_{\mu}^* n_{\nu}^*) + n_{\mu}^* n_{\nu}^{*\prime}$$

where  $n_{\nu}^{*\prime} = n^{*\rho} \nabla_{\rho}^* n_{\nu}^*$ .

*Proof.* In the comoving frame where  $u^0 = 1$  (and  $b^0 = 0$ ) the normalization  $b^k b_k = 1$  and the conditions (6) become:

$$\begin{split} a^{-2}g^{*\mu\nu}b_{\mu}b_{\nu} &= 1\\ \partial_{0}b_{\mu} - \Gamma^{\nu}_{0\mu}b_{\nu} &= 0\\ \partial_{\mu}b_{\nu} - \Gamma^{\rho}_{\mu\nu}b_{\rho} &= \frac{1}{n-2}(a^{2}g^{*}_{\mu\nu} - b_{\mu}b_{\nu})\frac{1}{a^{2}}g^{*\rho\sigma}(\partial_{\rho}b_{\sigma} - \Gamma^{\tau}_{\rho\sigma}b_{\tau}) \end{split}$$

The second equation is  $\partial_t(b_\mu/a) = 0$ . Then, the vector  $n_\mu^* = b_\mu/a$  is normalized and independent of time (it is a vector field of  $M^*$ ). The last equation, with some algebra and use of the Christoffel symbols in [10] becomes

$$\nabla_{\mu}^* n_{\nu}^* = \frac{\nabla_{\rho}^* n^{*\rho}}{n-2} (g_{\mu\nu}^* - n_{\mu}^* n_{\nu}^*) + n_{\mu}^* \frac{1}{a} (a_{\nu} - n_{\nu}^* n_{\rho}^* a^{\rho})$$

Contraction with  $n^{*\mu}$  gives  $n_{\nu}^{*\prime} = \frac{1}{a}(a_{\nu} - n_{\nu}^* n_{\rho}^* a^{\rho})$ , and (21) is obtained.  $\square$ 

Eq.(21) coincides with eq.(7.9) by Coley and McManus [5], in n=4. The existence of the normalized vector  $n_{\mu}^*$  with condition (21) (i.e. shear and vorticity free, but not geodesic) implies that  $(M^*, g^*)$  is a doubly twisted manifold, i.e. there are space coordinates and functions  $f_1$ ,  $f_2$  such that

$$g_{\mu\nu}^*(x)dx^{\mu}dx^{\nu} = f_1(x)^2(dx^1)^2 + f_2(x)^2ds^2$$

where  $ds^2$  only refers to coordinates  $x^2, ..., x^{n-1}$  (see Table 1 in Borowiec and Wojnar [1] and Corollary 1 in Ferrando et al. [7]). In particular, the space manifold  $(M^*, g^*)$  is twisted if and only if the vector fields  $a_{\mu}$  and  $n_{\mu}^*$  are also parallel.

In a twisted manifold, the general form of the Ricci tensor is [10]:

(22) 
$$R_{ij} = \frac{R - n\xi}{n - 1} u_i u_j + \frac{R - \xi}{n - 1} g_{ij} + (n - 2)v(u_i b_j + u_j b_i) - (n - 2)E_{ij}$$

where  $v = b^k \nabla_k \varphi$ . If another torse-forming vector  $w_i$  exists, eq.(19), the same Ricci tensor is:

$$R_{ij} = \frac{R - n\xi}{n - 1} w_i w_j + \frac{R - \xi}{n - 1} g_{ij} + (n - 2)v'(w_i c_j + w_j c_i) - (n - 2)E'_{ij}$$

where  $E'_{ij} = w^r w^s C_{rijs}$  and  $v' = c^k \nabla_k \lambda$ .

**Lemma 5.3.** v' = -v.

*Proof.* Contract of Eq.(17) with  $c^k$  and use  $c^k w_k = 0$ :

$$c^{k}u_{k}[w^{i}\nabla_{i}\varphi + w^{i}u_{i}(u^{j}\nabla_{j}\varphi)] = [(u^{i}w_{i})^{2} - 1]v'$$

It is  $c^k u_k = -\sinh \alpha \neq 0$ ,  $w^k u_k = -\cosh \alpha$ . Then:

$$(\cosh \alpha u^i + \sinh \alpha b^i)\nabla_i \varphi - \cosh \alpha (u^j \nabla_j \varphi) = -\sinh \alpha v'$$

Simplify and use  $b^i \nabla_i \varphi = v$ .

**Proposition 5.4.** If  $u_i$  and  $w_i$  are non-collinear torse-forming vector fields, then the Weyl tensor  $C_{jklm}$  has the constraint

(23) 
$$(u^r u^s + b^r b^s) C_{rijs} = (u_i u_j + b_i b_j) (b^k b^l E_{kl})$$

where  $E_{ij} = u^r u^s C_{rijs}$  is the electric component of the Weyl tensor, with the properties  $E_{ij}u^i = 0$  and  $E^i{}_i = 0$ .

*Proof.* Subtraction of the two expressions of the Ricci tensor and use of (19) with  $\alpha \neq 0$  give:

$$\left[\frac{R - n\xi}{(n-1)(n-2)}\sinh\alpha - 2v\cosh\alpha\right] \left[\sinh\alpha(u_iu_j + b_ib_j) + \cosh\alpha(u_ib_j + u_jb_i)\right]$$

$$= \sinh\alpha[\sinh\alpha(u^ru^s + b^rb^s) + \cosh\alpha(u^rb^s + u^sb^r)]C_{rijs}$$

Contraction with  $u^i u^j$  gives:

$$\frac{R - n\xi}{(n-1)(n-2)} \sinh \alpha - 2v \cosh \alpha = \sinh \alpha \left( b^r b^s E_{rs} \right)$$

Then:

(24) 
$$\sinh \alpha [(u_i u_j + b_i b_j)(b^r b^s E_{rs}) - (u^r u^s + b^r b^s) C_{rijs}]$$

$$= -\cosh \alpha [(u_i b_j + u_j b_i)(b^r b^s E_{rs}) - (u^r b^s + u^s b^r) C_{rijs}]$$

A torse-forming vector field has the property of being "Weyl-compatible" [11]:

$$(u_i C_{jklm} + u_j C_{kilm} + u_k C_{ijlm})u^m = 0.$$

It implies  $C_{iklm}u^m = u_k E_{il} - u_i E_{kl}$ . Then

$$(u^rb^s + u^sb^r)C_{rijs} = -b^sC_{jsir}u^r + b^rC_{rijs}u^s = (u_iE_{js} + u_jE_{is})b^s$$

Eq.(24) becomes:

(25) 
$$\sinh \alpha [(u_i u_j + b_i b_j)(b^r b^s E_{rs}) - (u^r u^s + b^r b^s) C_{rijs}]$$

$$= -\cosh \alpha [(u_i b_j + u_j b_i)(b^r b^s E_{rs}) - (u_i E_{js} + u_j E_{is}) b^s]$$

Contraction with  $u^i$ :

$$\sinh \alpha [u_i(b^r b^s E_{rs}) + b^r b^s u^i C_{riis}] = -\cosh \alpha [b_i(b^r b^s E_{rs}) - E_{is} b^s]$$

The left-hand side of the equation is zero:  $b^r b^s C_{rijs} u^i = b^r b^s C_{jsri} u^i = b^r b^s (u_s E_{jr} - u_j E_{sr}) = -u_j (b^r b^s E_{rs})$ . Then  $b^j$  is eigenvector of  $E_{js}$ :

$$(26) E_{js}b^s = b_j(b^rb^sE_{rs})$$

and the right hand side of eq.(25) is zero.

6. 
$$n=4$$
, the two-fluid picture

We show that in a space-time of dimension n=4 the presence of two torse-forming vectors specifies, via the Einstein equations, a stress-energy tensor that describes a mixture of two perfect fluids, with velocities  $u_i$  and  $u'_i$ , studied by Coley and McManus [5].

In n = 4, as a consequence of Lovelock's identity [8], the Weyl tensor is fully determined by its electric component  $E_{ij} = u^r u^s C_{rijs}$ :

$$C_{ijkl} = 2(u_i u_l E_{jk} - u_i u_k E_{jl} + u_j u_k E_{il} - u_j u_l E_{ik}) + g_{il} E_{jk} - g_{ik} E_{jl} + g_{jk} E_{il} - g_{jl} E_{ik}$$

Contraction with  $b^i b^l$  and use of (26) give:

$$b^{i}b^{l}C_{ijkl} = (2u_{j}u_{k} - 2b_{j}b_{k} + g_{jk})E_{rs}b^{r}b^{s} + E_{jk}$$

Then (23) gives  $E_{ij}$  in terms of  $b_i$ ,  $h_{ij} = u_i u_j + g_{ij}$  and a scalar:

(27) 
$$2E_{ij} = 3\left(b_i b_j - \frac{1}{3} h_{ij}\right) (E_{rs} b^r b^s).$$

The Ricci tensor (22) becomes:

(28) 
$$R_{ij} = \frac{1}{3}(R - 4\xi)u_i u_j + \frac{1}{3}(R - \xi)g_{ij} + 2v(u_i b_j + u_j b_i) -3\left(b_i b_j - \frac{1}{3}h_{ij}\right)(E_{rs}b^r b^s)$$

Einstein's equations  $R_{ij} - \frac{1}{2}Rg_{ij} = T_{ij}$  give the corresponding energy-momentum tensor (in units that absorb the gravitational constant):

$$T_{ij} = \frac{1}{3}(R - 4\xi)u_iu_j - \frac{1}{6}(R + 2\xi)g_{ij} + 2v(u_ib_j + u_jb_i) -3\left(b_ib_j - \frac{1}{3}h_{ij}\right)\left(E_{rs}b^rb^s\right)$$

The tensor, besides the perfect fluid-like term, contains a current term with vector  $2vb_i$  orthogonal to the velocity, and a peculiar stress tensor. This expression describes a mixture of two perfect fluids [5].

Consider two perfect fluids with velocities  $u_i$  and  $u'_i = u_i \cosh \theta + t_i \sinh \theta$ , where the tilt angle  $\theta$  and the space-like unit vector  $t_i$  are yet unspecified:

$$T_{ij}^{(2)} = (p_1 + \mu_1)u_iu_j + p_1g_{ij} + (p_2 + \mu_2)u_i'u_j' + p_2g_{ij}$$

$$= [(p_1 + \mu_1) + (p_2 + \mu_2)(1 + \frac{4}{3}\sinh^2\theta)]u_iu_j + [p_1 + p_2 + \frac{1}{3}(p_2 + \mu_2)\sinh^2\theta]g_{ij}$$

$$+ (p_2 + \mu_2)\sinh\theta\cosh\theta(u_it_j + u_jt_i) + (p_2 + \mu_2)\sinh^2\theta(t_it_j - \frac{1}{3}h_{ij})$$

If we equate  $T_{ij}$  and  $T_{ij}^{(2)}$ , unicity of the decompositions with respect to the velocity field  $u_i$ , gives  $t_i = b_i$  (up to a sign) and, with little algebra:

$$3(p_1 + \mu_1) + 3(p_2 + \mu_2) = R - 4\xi + 12E_{rs}b^rb^s, \quad 6(p_1 + p_2) = -R - 2\xi + 6E_{rs}b^rb^s$$
$$(p_2 + \mu_2)\sinh\theta\cosh\theta = 2v, \quad (p_2 + \mu_2)\sinh^2\theta = -3E_{rs}b^rb^s$$

The last two equations give the tilt angle between the fluid velocities u and u', while the tilt angle bewteen the torse-forming vectors u and w is eq.(20):

$$\tanh \theta = -\frac{3E_{rs}b^rb^s}{2v}, \quad \tanh \alpha = \frac{12v}{R - 4\xi - 6E_{rs}b^rb^s}$$

Thus u' and w are, in general, different time-like vectors. By expressing  $b_i$  in terms of  $u_i$  and  $w_i$  we obtain:

$$u_i' = u_i \frac{\sinh(\alpha - \theta)}{\sinh \alpha} - w_i \frac{\sinh \theta}{\sinh \alpha}$$

It turns out that  $u_i'$  coincides with  $w_i$  if  $p_1 + \mu_1 = p_2 + \mu_2$ .

#### Appendix

We report from [10] the Christoffel symbols, and the components of the Riemann and Ricci tensors for the metric (3) of twisted space-times.  $(i, j, k, \dots = 0, 1, \dots, n-1; \mu, \nu, \rho, \dots = 1, 2, \dots, n-1)$ .

Christoffel symbols:  $\Gamma_{ij}^k = \Gamma_{ji}^k = \frac{1}{2}g^{km}(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}).$ 

(29) 
$$\Gamma_{i,0}^{0} = 0$$
,  $\Gamma_{0,0}^{k} = 0$ ,  $\Gamma_{\mu,0}^{\rho} = (\dot{a}/a)\delta_{\mu}^{\rho}$ ,  $\Gamma_{\mu,\nu}^{0} = a\dot{a}g_{\mu\nu}^{*}$ ,

(30) 
$$\Gamma^{\rho}_{\mu,\nu} = \Gamma^{*\rho}_{\mu,\nu} + (a_{\nu}/a)\delta^{\rho}_{\mu} + (a_{\mu}/a)\delta^{\rho}_{\nu} - (a^{\rho}/a)g^{*}_{\mu\nu}$$

where  $\dot{a} = \partial_t a$ ,  $a_{\mu} = \partial_{\mu} a$  and  $a^{\mu} = g^{*\mu\nu} a_{\nu}$ .

Riemann tensor:  $R_{jkl}^{\ m} = -\partial_j \Gamma^m_{k,l} + \partial_k \Gamma^m_{j,l} + \Gamma^p_{j,l} \Gamma^m_{kp} - \Gamma^p_{k,l} \Gamma^m_{jp}$ 

(31) 
$$R_{\mu 0\rho}{}^{0} = (a\ddot{a})g_{\mu\rho}^{*}$$

(32) 
$$R_{\mu\nu\rho}{}^{0} = g_{\mu\rho}^{*}(a\partial_{\nu}\dot{a} - \dot{a}a_{\nu}) - g_{\nu\rho}^{*}(a\partial_{\mu}\dot{a} - \dot{a}a_{\mu})$$

(33) 
$$R_{\mu\nu\rho}{}^{\sigma} = R_{\mu\nu\rho}^{*}{}^{\sigma} + (\dot{a}^{2} - \frac{a^{\lambda}a_{\lambda}}{a^{2}})(g_{\mu\rho}^{*}\delta_{\nu}^{\sigma} - g_{\nu\rho}^{*}\delta_{\mu}^{\sigma})$$
$$+ \frac{2}{a^{2}}(a^{\sigma}a_{\nu}g_{\mu\rho}^{*} - a^{\sigma}a_{\mu}g_{\nu\rho}^{*} + a_{\mu}a_{\rho}\delta_{\nu}^{\sigma} - a_{\nu}a_{\rho}\delta_{\mu}^{\sigma})$$
$$+ \frac{1}{a}\left[\nabla_{\mu}^{*}(a^{\sigma}g_{\nu\rho}^{*} - a_{\rho}\delta_{\nu}^{\sigma}) - \nabla_{\nu}^{*}(a^{\sigma}g_{\mu\rho}^{*} - a_{\rho}\delta_{\mu}^{\sigma})\right]$$

Ricci tensor:  $R_{il} = R_{ikl}^{\ \ k}$ 

- (34)  $R_{00} = -(n-1)(\ddot{a}/a)$
- (35)  $R_{\mu 0} = -(n-2)\partial_{\mu}(\dot{a}/a)$

(36) 
$$R_{\mu\nu} = R_{\mu\nu}^* + g_{\mu\nu}^* [(n-2)\dot{a}^2 + a\ddot{a}] + 2(n-3)\frac{a_{\mu}a_{\nu}}{a^2} - (n-4)\frac{a^{\sigma}a_{\sigma}}{a^2}g_{\mu\nu}^* - (n-3)\frac{1}{a}\nabla_{\mu}^*a_{\nu} - \frac{1}{a}g_{\mu\nu}^*\nabla_{\sigma}^*a^{\sigma}$$

Curvature scalar:  $R = R^k_k$ 

(37) 
$$R = \frac{R^*}{a^2} + (n-1) \left[ (n-2)\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} \right] - (n-2) \left[ (n-5)\frac{a^{\sigma}a_{\sigma}}{a^4} + 2\frac{\nabla_{\sigma}^*a^{\sigma}}{a^3} \right]$$

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