JOURNAL OF GEOMETRIC MECHANICS doi:10.3934/jgm.2019018
Volume **11**, Number **3**, September **2019** pp. **325–359**

A SYMMETRY-ADAPTED NUMERICAL SCHEME FOR SDES

Francesco C. De Vecchi

Institute for Applied Mathematics and HCM
Rheinische Friedrich-Wilhelms-Universität Bonn
Endenicher Allee 60, 53115 Bonn, Germany

Andrea Romano and Stefania Ugolini*

Dipartimento di Matematica
Università degli Studi di Milano
via Saldini 50, 20133 Milano, Italy

(Communicated by Josef Teichmann)

Abstract. We propose a geometric numerical analysis of SDEs admitting Lie symmetries which allows us to individuate a symmetry adapted coordinates system where the given SDE puts in evidence notable invariant properties. An approximation scheme preserving the symmetry properties of the equation is introduced. Our algorithmic procedure is applied to the family of general linear SDEs for which two theoretical estimates of the numerical forward error are established.

1. **Introduction.** The exploitation of special geometric structures in numerical integration of both ordinary and partial differential equations (ODEs and PDEs) is nowadays a mature subject of numerical analysis often called *geometric numerical integration* (see e.g. [19, 26, 30, 42]). The importance of this research topic is due to the fact that many differential equations in mathematical applications have some particular geometrical features such as for example a conservation law, a variational principle giving rise to the equations, an Hamiltonian or symplectic structure and more general symmetry structures (for example see [38], [2]). The development of geometrically adapted numerical algorithms permits to obtain suitable integration methods which both preserve the qualitative properties of the integrated equations and have a more efficient numerical behaviour with respect to the corresponding standard discretization schemes.

In comparison the study of geometric numerical integration of stochastic differential equations (SDEs) is not so well developed. In the current literature the principal aims consist in producing numerical stochastic integrators which are able to preserve the symplectic structure (see e.g. [3, 37, 45]), some conserved quantities (see e.g. [7, 25, 34]) or the variational structure (see e.g. [4, 5, 24, 47]) of the considered SDEs. For the study of the algebraic structure of stochastic expansions in order to achieve optimal efficient stochastic integrators at all orders see [13].

2010 *Mathematics Subject Classification.* Primary: 65C30, 58D19; Secondary: 60H10.

Key words and phrases. Lie symmetry analysis, symmetries of stochastic differential equations, numerical methods for stochastic differential equations, geometric numerical integration.

The first author is supported by the German Research Foundation (DFG) via CRC 1060.

* Corresponding author: stefania.ugolini@unimmi.it.

Although the exploitation of Lie symmetries of ODEs and PDEs (see e.g. [40]) to obtain better numerical integrators is an active research topic (see e.g. [6, 12, 33, 32] and references therein), the application of the same techniques in the stochastic setting to the best of our knowledge is not yet pursued, probably because the concept of symmetry of a SDE has been quite recently developed (see e.g. [8, 11, 10, 9, 18, 29, 31, 35]).

In this paper we introduce two different numerical methods taking advantage of the presence of Lie symmetries in a given SDE in order to provide a more efficient numerical integration of it.

We first propose the definition of an invariant numerical integrator for a symmetric SDE as a natural generalization of the corresponding concept for an ODE. When one tries to construct general invariant numerical methods in the stochastic framework, in fact, a non trivial problem arises. Since both the SDE solution as well as the Brownian motion driving it are continuous but not differentiable processes, it can happen that the finite differences discretization does not converge to the SDE solution. We give some necessary and sufficient conditions in order that the two standard numerical methods for SDEs (the Euler and the Milstein schemes) are also invariant numerical methods. By using these results, in particular, we are able to identify a class of privileged coordinates systems where it is convenient to make the discretization procedure.

Our second numerical method, based on a well-defined change of the coordinates system, is inspired by the standard techniques of reduction and reconstruction of an SDE with a solvable Lie algebra of symmetries (see [10, 28]). Indeed an SDE with a solvable Lie algebra of symmetries can be reduced to a triangular system and, when the number of symmetries is sufficiently high, the latter can be explicitly integrated. In the stochastic setting the explicit integration concept is of course a quite different notion with respect to the deterministic one. Indeed the evaluation of an Ito integral, a necessary step in the reconstruction of a reduced SDE, can only be numerically implemented.

We apply our two proposed numerical techniques to the general linear SDEs, being the first non-trivial class of symmetric equations. In this case the two algorithmic methods can be harmonized in such a way as to produce the same simple family of best coordinates systems for the discretization procedure. Interestingly, the identified coordinate changes are closely related to the explicit solution formula of linear SDEs. Although the integration formula of linear SDEs is widely known, our results are original in showing that the proposed numerical scheme for linear SDEs is a particular case of implementation of a general procedure for SDEs with Lie symmetries. We finally point out that the SDE with affine drift and diffusion coefficients plays an important role since any SDE with real analytic drift and diffusion coefficients can be seen locally as such as an affine SDE.

Moreover we theoretically investigate the numerical advantages of the new numerical scheme for linear SDEs. More precisely we obtain two estimates for the forward numerical error which, in presence of an equilibrium distribution, guarantee that the constructed method is numerically stable for any size of the time step $h > 0$. This means that for any $h > 0$ the error does not grow exponentially with the maximum-integration-time T, in fact it remains finite for $T \to +\infty$. This property is not shared by standard explicit or implicit Euler and Milstein methods. The obtained estimates are new mainly because the coordinate changes involved in the formulation of our numerical scheme have strongly non-Lipschitz features, and so the standard convergence theorems can not be applied. We also illustrate our theoretical results through numerical calculations.

It is interesting to note that the main part of the theory, in particular the definitions of strong symmetry of an SDE and of a numerical scheme, can be easily extended to Stratonovich type SDEs driven by general noises ([8]), for example by exploiting rough paths theory. Unfortunately, since the proofs of Theorem 5.1 and Theorem 5.2 use in an essential way the (forward and backward) Ito formula, the long terms estimates obtained here cannot be straightforwardly generalized to the rough paths driven SDEs framework. At the same time we think that some ideas in the proof of Theorem 5.2 can be suitable exploited to obtain pathwise estimates of the long term error in the rough paths setting.

The article is structured as follows: in Section 2 we recall the notion of strong symmetry of an SDE and we describe the two standard discretization schemes used in the rest of the paper. In Section 3 we present two numerical procedures adapted with respect to the Lie symmetries of an SDE. We apply the proposed integration methods to general one and two-dimensional linear SDEs in Section 4 . In Section 5 some theoretical estimates showing the stability and efficiency of our adapted-to-symmetries numerical schemes in linear SDEs are proved. In the last section we present some numerical experiments confirming the theoretical estimates obtained in the previous section.

2. Preliminaries.

2.1. Strong symmetries of SDEs.

For simplicity in the following we take $M = \mathbb{R}^n$. If $F : M \to \mathbb{R}^k$ we denote by $\nabla(F)$ the Jacobian of F i.e. the matrix-valued function

$$\nabla(F) = (\partial_{x^i}(F^j))|_{\substack{j=1,\dots,k \\ i=1,\dots,n}}.$$

Furthermore we can identify the vector fields $Y \in TM$ with the functions $Y : M \to \mathbb{R}^n$, and if $\Phi : M \to M$ is a diffeomorphism we introduce the pushforward

$$\Phi_*(Y) = (\nabla(\Phi) \cdot Y) \circ \Phi^{-1}.$$

Definition 2.1. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space. Let μ and σ be two smooth functions defined on M and taking values in an n-dimensional vector space respectively in the vector space of $n \times m$ matrices. A solution to an SDE(μ, σ) is a pair (X, W) of adapted processes such that

i) W is a $\mathcal{F}_t$-Brownian motion in $\mathbb{R}^m$;

ii) For $i = 1, 2, ..., n$

$$X_t^i = X_0^i + \int_0^t \mu^i(X_s) ds + \int_0^t \sum_{\alpha=1}^m \sigma_\alpha^i(X_s) dW_s^\alpha, \quad t \geq 0. \tag{1}$$

Remark 1. In particular all the integrals are meaningful if a.s.:

$$\int_0^t \sum_{i,\alpha} (\sigma_\alpha^i)^2(X_s) ds < +\infty, \quad \int_0^t \sum_i |\mu^i(X_s)| ds < +\infty$$

Definition 2.2. A solution (X, W) to an SDE(μ, σ) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is said to be strong solution if X is adapted to the filtration $\mathcal{F}_t^W$ generated by the Brownian motion W and completed with respect to $\mathbb{P}$.

Of course a solution (X, W) is called a weak solution when it is not strong. In the weak solution case the Brownian motion is not given a priori but it is part of the solution (see [43]).

In this paper we fix a Brownian motion W, that is we consider only strong solutions of an SDE(μ, σ) and, consequently, we denote them simply by X. For a symmetry analysis focused on weak solutions of SDEs see [11],[10],[1].

A solution X to an SDE(μ, σ) is a diffusion process admitting as infinitesimal generator:

$$L = \sum_{\alpha=1}^{m} \sum_{i,j=1}^{n} \frac{1}{2} \sigma_\alpha^i \sigma_\alpha^j \partial_{x^i x^j} + \sum_{i=1}^{n} \mu^i \partial_{x^i}.$$

The following celebrated formula is particularly useful for obtaining stochastic differentials (see, e.g., [39],[43]).

Lemma 2.3 (Ito formula)**.** *Let X be a solution to the SDE (μ, σ) and let $f : M \to \mathbb{R}$ be a smooth function. Then $f(X)$ has the following stochastic differential*

$$df(X_t) = L(f)(X_t)dt + \nabla(f)(X_t) \cdot \sigma(X_t) \cdot dW_t.$$

We recall important definitions of symmetries of an SDE.

Definition 2.4 (strong finite symmetry)**.** We say that a diffeomorphism Φ is a (strong) finite symmetry of the SDE (μ, σ) if for any solution X to the SDE (μ, σ) also $\Phi(X)$ is a solution to the SDE (μ, σ).

By using Ito's formula it is immediate to prove the following result.

Theorem 2.5. *A diffeomorphism Φ is a symmetry of the SDE (μ, σ) if and only if*

$$\begin{aligned} L(\Phi) &= \mu \circ \Phi \\ \nabla(\Phi) \cdot \sigma &= \sigma \circ \Phi. \end{aligned}$$

where $(L(\Phi))^i = L(\Phi^i)$.

Proof. See [11], Theorem 17. □

It is well-known that vector fields acting as infinitesimal generators of one parameter transformation groups are the most important tools in Lie group theory.

Definition 2.6 (strong infinitesimal symmetry)**.** A vector field Y is said to be a (strong) infinitesimal symmetry of the SDE (μ, σ) if the group of the local diffeomorphism Φ_a generated by Y is a symmetry of the SDE (μ, σ) for any $a \in \mathbb{R}$.

The following determining equations for (any) infinitesimal symmetries are well-known (see, e.g., [18]). For their generalization to a weak solution case see [11], [10], and [1] for SDEs driven by semimartingales with jumps.

Theorem 2.7 (Determining equations)**.** *A vector field Y is an infinitesimal symmetry of the SDE (μ, σ) if and only if*

$$Y(\mu) - L(Y) = 0 \quad (2)$$

$$[Y, \sigma_\alpha] = 0. \quad (3)$$

where σ_α is the α-column of σ $(\alpha = 1, ..., m)$ and $[\cdot, \cdot]$ are the standard Lie brackets between vector fields.

Proof. For a proof with the above notations see [11], Theorem 19. □

2.2. **Numerical integration of SDEs.** For reader's convenience, we recall the two main numerical methods for simulating an SDE as well as a theorem on the strong convergence of these methods (for a detailed description see e.g. [27]).

Consider the SDE having coefficients (μ, σ), driven by the Brownian motion W, and let $\{t_n\}_n$ be a partition of $[0, T]$. The *Euler scheme* for the equation (μ, σ) with respect to the given partition is provided by the following sequence of random variables $X_n \in M$

$$X_n^i = X_{n-1}^i + \mu^i(X_{n-1})\Delta t_n + \sum_{\alpha=1}^{m} \sigma_\alpha^i(X_{n-1})\Delta W_n^\alpha,$$

where $\Delta t_n = t_n - t_{n-1}$ and $\Delta W_n^\alpha = W_{t_n}^\alpha - W_{t_{n-1}}^\alpha$. The *Milstein scheme* for the same equation (μ, σ) is instead constituted by the sequence of random variables $\bar{X}_n \in M$ such that

$$\begin{aligned} \bar{X}_n^i &= \bar{X}_{n-1}^i + \mu^i(\bar{X}_{n-1})\Delta t_n + \sum_{\alpha=1}^{m} \sigma_\alpha^i(\bar{X}_{n-1})\Delta W_n^\alpha + \\ &\quad + \frac{1}{2}\sum_{j=1}^{n}\sum_{\alpha,\beta=1}^{m} \sigma_\alpha^j(\bar{X}_{n-1})\partial_{x^j}(\sigma_\beta^i)(\bar{X}_{n-1})\Delta \mathbb{W}_n^{\alpha,\beta}, \end{aligned}$$

where $\Delta \mathbb{W}_n^{\alpha,\beta} = \int_{t_{n-1}}^{t_n}(W_s^\beta - W_{t_{n-1}}^\beta)dW_s^\alpha$. We recall that when $m = 1$ we have that

$$\Delta \mathbb{W}_n^{1,1} = \frac{1}{2}((\Delta W_n)^2 - \Delta t_n).$$

Theorem 2.8. *Let us denote by X_t the exact solution of an SDE (μ, σ) and by X_N and $\bar{X}_N$ the N-step approximations according to the Euler and Milstein schemes respectively. Suppose that the coefficients (μ, σ) are C^2 with bounded derivatives and put $t_n = \frac{nT}{N}$ and $h = \frac{T}{N}$. Then there exists a constant $C(T, \mu, \sigma)$ such that*

$$\epsilon_N = \left(\mathbb{E}[\|X_T - X_N\|^2]\right)^{1/2} \leq C(T, \mu, \sigma)h^{1/2}.$$

Furthermore when the coefficients (μ, σ) are C^3 with bounded derivatives then there exists a constant $\bar{C}(T, \mu, \sigma)$ such that

$$\bar{\epsilon}_N = \left(\mathbb{E}[\|X_T - \bar{X}_N\|^2]\right)^{1/2} \leq \bar{C}(T, \mu, \sigma)h.$$

Proof. See Theorem 10.2.2 and Theorem 10.3.5 in [27]. □

Theorem 2.8 states that X_N and $\bar{X}_N$ strongly converge in $L^2(\Omega)$ to the exact solution X_T of the SDE (μ, σ), where the order of the convergence with respect to the step size variation $h = \frac{T}{N}$ is $\frac{1}{2}$ in the Euler case and 1 in the Milstein one.

Nevertheless, the theorem gives no information on the behaviour of the numerical approximations when we fix the step size h and we vary the final time T. In the standard proof of Theorem 2.8 one estimates the constants $C(T, \mu, \sigma)$ and $\bar{C}(T, \mu, \sigma)$ by proving that by the Gronwall Lemma there exist two positive constants $K(\mu, \sigma), K'(\mu, \sigma)$ such that $C(T, \mu, \sigma) = \exp(T \cdot K(\mu, \sigma))$ and $\bar{C}(T, \mu, \sigma) = \exp(T \cdot K'(\mu, \sigma))$. In some cases the exponential growth of the error is a correct prediction (see for example [36]).

Of course this is not always the case. In fact, if the SDE (μ, σ) admits an equilibrium distribution, it could happen that the two errors remain bounded with respect to the time T. Unfortunately this desired behaviour only happens for a restricted set of step sizes h. The phenomenon just described is known as the *stability* problem for a discretization method applied to an SDE. This problem, and the corresponding definition, is usually stated and tested for some specific SDEs (see e.g. [21, 46] for the geometric Brownian motion, e.g. [20, 44] for the Ornstein-Uhlenbeck process, e.g. [22, 23] for non-linear equations with a Dirac delta equilibrium distribution, and see e.g. [48] for more general situations). In Section 6 we shall provide some numerical examples for the stability problem and phenomenon for general linear SDEs.

3. Numerical integration via symmetries.

3.1. **Invariant numerical algorithms.** When a system of ODEs admits Lie-point symmetries then invariant numerical algorithms can be constructed (see e.g. [33, 32, 12, 6]). For the sake of exposition we first recall the definition of an invariant numerical scheme for a system of ODEs, in the simple case of one-step algorithms. The obvious extension for multi-step numerical schemes is immediate. The discretization of a system of ODEs is a function $F : M \times \mathbb{R} \to M$ such that if $x_n, x_{n-1} \in M$ are the $n, n-1$ steps respectively and Δt_n is the step size of our discretization we have that

$$x_n = F(x_{n-1}, \Delta t_n).$$

If $\Phi : M \to M$ is a diffeomorphism we say that the discretization defined by the map F is *invariant* with respect to the map Φ if

$$\Phi(x_n) = F(\Phi(x_{n-1}), \Delta t_n).$$

Requiring that such property holds for any $x_n \in \mathbb{R}^n$ and for any $\Delta t_n \in \mathbb{R}_+$ we get

$$\Phi^{-1}(F(\Phi(x), \Delta t)) = F(x, \Delta t) \tag{4}$$

for any $x \in M$ and $\Delta t \in \mathbb{R}$. If Φ_a is an one-parameter group generated by the vector field $Y = Y^i(x)\partial_{x^i}$, by deriving the relation $\Phi_{-a}(F(\Phi_a(x), \Delta t)) = F(x, \Delta t)$ with respect to a, we get

$$Y^i(F(x, \Delta t)) - Y^k \partial_{x^k}(F)(x, \Delta t) = 0 \tag{5}$$

which guarantees that the discretization F is invariant with respect to the semigroup Φ_a generated by Y.

We can extend the previous definition to the case of an SDE in the following way. Consider an integration scheme depending only on the time Δt and on the Brownian motion $\Delta W^\alpha_n, \alpha = 1, \ldots, m$ (e.g. the Euler method). Extension of this approach to integration methods depending also on $\Delta \mathbb{W}^{\alpha,\beta}_n$ or other random variables (such as the Milstein method) is immediate. In the stochastic case the discretization is a map $F : M \times \mathbb{R} \times \mathbb{R}^m \to M$ and we have

$$x_n = F(x_{n-1}, \Delta t, \Delta W^1, ..., \Delta W^m).$$

Equations (4) and (5) become

$$\begin{aligned} \Phi^{-1}(F(\Phi(x), \Delta t, \Delta W^\alpha)) &= F(x, \Delta t, \Delta W^\alpha), & (6) \\ Y^i(F(x, \Delta t, \Delta W^\alpha)) - Y^k \partial_{x^k}(F)(x, \Delta t, \Delta W^\alpha) &= 0. & (7) \end{aligned}$$

We stress again that, since the Ito integral strongly depends on the approximation being backward (and not forward), it is not easy to prove that a given discretization X_n converges to the real solution of the SDE (μ, σ). Indeed, different kinds of discretization (or smooth approximation like in Wong-Zakai theorem) of the Ito integral converge to different stochastic integrals. In the case of a backward approximation one obtains an Ito integral, with a symmetric approximation a Stratonovich integral and so on. This phenomenon is peculiar to the stochastic framework. Since in the case of deterministic ODEs backward, forward and symmetric approximations converge to the same result, the formulation of invariant numerical schemes in a deterministic setting is easier (see the study of symplectic scheme for SDEs in [24], [37],[36]). The following theorem provides a sufficient (and necessary) condition in order that the Euler and Milstein discretizations of an SDE are invariant with respect to any strong symmetries $Y_1, ..., Y_r$.

Theorem 3.1. *Let $Y_1, ..., Y_r$ be strong symmetries of an SDE (μ, σ). If $Y_j^i = Y_j(x^i)$ are polynomials of first degree in $x^1, ..., x^n$, then the Euler discretization (or the Milstein discretization) of the SDE (μ, σ) is invariant with respect to $Y_1, ..., Y_r$. Additionaly, If $\operatorname{span}\{\sigma_1(x_0), \dots, \sigma_m(x_0)\} = \mathbb{R}^n$ for some $x_0 \in M$, then the converse holds.*

Proof. We limit ourselves to the proof for the Euler discretization because for the Milstein discretization the proof is very similar. In the case of the Euler discretization we have that

$$F^i(x) = x^i + \mu^i(x)\Delta t + \sigma^i_\alpha(x)\Delta W^\alpha.$$

The discretization is invariant if and only if

$$\begin{aligned} 0 &= Y_j(F^i)(x) - Y_j^i(F(x)) \\ &= +Y_j^k \partial_{x^k}(F^i)(x) - Y_j^i(F(x)) \\ &= Y_j^i(x) + Y_j^k(x)\partial_{x^k}(\mu^i)(x)\Delta t \\ &\quad +Y_j^k(x)\partial_{x^k}(\sigma^i_\alpha)(x)\Delta W^\alpha - Y_j^i(x + \mu\Delta t + \sigma_\alpha \Delta W^\alpha). \end{aligned}$$

Recalling that Y_j is a symmetry for the SDE (μ, σ) and therefore it has to satisfy the determining equations (2) and (3), we have that the Euler discretization is invariant if and only if

$$\begin{gathered} Y_j^i(x) + \mu^k(x)\partial_{x^k}(Y_j^i)(x)\Delta t + \tfrac{1}{2}\sum_\alpha \sigma^k_\alpha \sigma^h_\alpha \partial_{x^k x^h}(Y_j^i)(x)\Delta t \\ +\sigma^k_\alpha(x)\partial_{x^k}(Y_j^i)(x)\Delta W^\alpha = Y_j^i(x + \mu\Delta t + \sigma_\alpha\Delta W^\alpha). \end{gathered} \tag{8}$$

Suppose that $Y_j^i = B_j^i + C^i_{j,k}x^k$, then

$$\begin{gathered} Y_j^i(x) + \mu^k(x)\partial_{x^k}(Y_j^i)(x)\Delta t + \tfrac{1}{2}\sum_\alpha \sigma^k_\alpha \sigma^h_\alpha \partial_{x^k x^h}(Y_j^i)(x)\Delta t \\ +\sigma^k_\alpha(x)\partial_{x^k}(Y_j^i)(x)\Delta W^\alpha = B_j^i + C^i_{j,k}x^k + C^i_{j,k}\mu^k(x)\Delta t + C^i_{j,k}\sigma^k_\alpha(x)\Delta W^\alpha \\ = B_j^i + C^i_{j,k}(x^k + \mu^k(x)\Delta t + \sigma^k_\alpha(x)\Delta W^\alpha) \\ = Y_j^i(x + \mu\Delta t + \sigma_\alpha\Delta W^\alpha). \end{gathered}$$

Conversely, suppose that the Euler discretization is invariant and so equality (8) holds. Let x_0 be as in the hypotheses of the theorem and choose $\Delta t = 0$. Then

$$Y_j^i(x_0 + \sigma_\alpha \Delta W^\alpha) = Y_j^i(x_0) + \sigma^k_\alpha(x_0)\partial_{x^k}(Y_j^i)(x_0)\Delta W^\alpha.$$

Since ΔW^α are arbitrary and $\operatorname{span}\{\sigma_1(x_0), ...\sigma_m(x_0)\} = \mathbb{R}^n$, Y_j^i must be of first degree in $x^1, ..., x^n$. □

Remark 2. The affinity of the coefficients Y_j^i in Theorem 3.1 is a direct consequence of the Euler and Milstein numerical approximations' affine dependence from the noise $\Delta t, \Delta W^\alpha, \Delta \mathbb{W}^{\alpha,\beta}$. Non affine numerical approximations could admit non affine symmetries $Y_1, ..., Y_r$ (see the discussion below).

Theorem 3.1 can be fruitfully applied in the following way. If $Y_1, ..., Y_r$ are strong symmetries of an SDE we look for a diffeomorphism $\Phi : M \to M' = \mathbb{R}^n$ (i.e. a coordinate change) such that $\Phi_*(Y_1), ..., \Phi_*(Y_r)$ have coefficients of first degree in the new coordinates $x'^1, ..., x'^n$ .Applying the Euler scheme to the transformed SDE $\Phi(\mu, \sigma)$ we obtain a discretization $\tilde{F}(x', \Delta t, \Delta W^\alpha)$ which is invariant with respect to $\Phi_*(Y_1), ..., \Phi_*(Y_r)$. As a consequence, the discretization $F = \Phi(\tilde{F}(\Phi^{-1}(x), \Delta t, \Delta W^\alpha)$ is invariant with respect to $Y_1, ..., Y_r$. It is easy to prove that if the map Φ is Lipschitz the constructed discretization converges in L^1 to the solution, while if the map Φ is only locally Lipschitz, the weaker convergence in probability can be established.

The existence of the diffeomorphism Φ allowing the application of Theorem 3.1 for general $Y_1, ..., Y_r$ is not guaranteed. Furthermore, even when the map Φ exists, unfortunately it is in general not unique. Consider for example the following one-dimensional SDE

$$dX_t = \left(a \tanh(X_t) - \frac{b^2}{2} \tanh^3(X_t) \right) dt + b \tanh(X_t) dW_t, \quad a, b \in \mathbb{R} \tag{9}$$

which has

$$Y = \tanh(x)\partial_x$$

as a strong symmetry. There are many transformations Φ which are able to reduce Y to differentials with coefficients of first degree, for example the following two transformations:

$$\begin{aligned} \Phi_1(x) &= \sinh(x) \\ \Phi_2(x) &= \log|\sinh(x)|. \end{aligned}$$

Indeed we have that

$$\Phi_{1,*}(Y) = x'_1 \partial_{x'_1}, \Phi_{2,*}(Y) = \partial_{x'_2}.$$

While the map Φ_1 transforms equation (9) into a geometrical Brownian motion, the transformation Φ_2 reduces equation (9) to a Brownian motion with drift. By applying Euler method by means of Φ_1 we obtain a poor numerical result (in fact Φ_1 is not a Lipschitz function and in this circumstance errors are amplified). By exploiting Φ_2 to make the discretization we obtain instead an exact simulation. The example shows that this first approach strongly depends on the choice of the diffeomorphism Φ (which has to be invertible in terms of elementary functions). So it is better to have another procedure able to individuate the best coordinate system for performing the SDE discretization.

3.2. **Adapted coordinates and triangular systems.** We now introduce another possible application of Lie's symmetries in the numerical simulation of an SDE, one that has no analogue in the context of ODEs. Indeed, in the deterministic setting, one can obtain a completely explicit result.

Suppose that $M = M_1 \times M_2$, with standard cartesian coordinates $x_1^1, ..., x_1^r, x_2^1, ..., x_2^{n-r}$ for some $1 < r < n$, and consider the following triangular SDE

$$\begin{aligned} dX_{2,t}^i &= \mu_2^i(X_{2,t})dt + \sigma_{2,\alpha}^i(X_{2,t})dW_t^\alpha \\ dX_{1,t}^j &= \mu_1^j(X_{1,t}^1, ..., X_{1,t}^{i-1}, X_{2,t})dt + \sigma_{1,\alpha}^j(X_{1,t}^1, ..., X_{1_t}^{i-1}, X_{2,t})dW_t^\alpha, \end{aligned}$$

where $\mu_1^i, \sigma_{1,\alpha}^i$ do not depend on $x_1^i, ..., x_1^r$. The above SDE is triangular in the variables $(x_1^1, ..., x_1^r)$. By discretizing a triangular SDE (μ, σ) it is reasonable to expect a better behaviour than in the general case. Furthermore, if $X_{2,t}^1, ..., X_{2,t}^{n-r}$

can be exactly simulated with $\sigma^i_{2,\alpha}, \mu^i_2$ growing at most polynomially, we conjecture that the error grows polynomially with respect to the maximal integration time T.

We recall that the triangular property of stochastic systems is closely related with their symmetries and in particular to SDEs with a solvable Lie algebra of symmetries. In order to briefly explain the connection between symmetries and the triangular form of SDEs, we introduce the following definitions (for more details see [10]).

Definition 3.2. A set of vector fields $Y_1, ...Y_r$ on M is called *regular* on M if, for any $x \in M$, the vectors $Y_1(x), ..., Y_r(x)$ are linearly independent.

Definition 3.3. Let $Y_1, ..., Y_r$ be a set of regular vector fields on M which are generators of a solvable Lie algebra $\mathcal{G}$. We say that $Y_1, ..., Y_r$ are in *canonical form* if there are $i_1, ..., i_l$ such that $i_1 + ... + i_l = r$ and

$$(Y_1|...|Y_r) = \left(\begin{array}{c|c|c|c} I_{i_1} & G^1_1(x) & ... & G^1_l(x) \\ \hline 0 & I_{i_2} & ... & G^2_l(x) \\ \hline \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & ... & I_{i_l} \\ \hline 0 & 0 & 0 & 0 \end{array}\right),$$

where $G^h_k : M \to \mathrm{Mat}(i_h, i_k)$ are smooth functions.

Theorem 3.4. *Let an SDE (μ, σ) admit $Y_1, ..., Y_r$ as strong symmetries and suppose that $Y_1, ..., Y_r$ constitute a solvable Lie algebra in canonical form. Then the SDE (μ, σ) assumes a triangular form with respect to $x^1, ..., x^r$.*

Proof. The proof is an application of the determining equations and Definition 3.3 (see [10]). □

As a notable consequence we can apply a methodology similar to the one proposed in the previous subsection to any SDE (μ, σ) that admits a solvable regular Lie algebra $Y_1, ..., Y_r$ of strong symmetries. We begin by searching a map $\Phi : M \to M'$ such that $\Phi(Y_1), ..., \Phi(Y_r)$ constitute a solvable Lie algebra in canonical form, implying that $\Phi(\mu, \sigma)$ is a triangular SDE. We can then apply to $\Phi(\mu, \sigma)$ one of the standard methods obtaining a discretization $\tilde{F}$. By composing $\tilde{F}$ with Φ we obtain another discretization $F(x, \Delta t, \Delta W^\alpha) = \Phi^{-1}(\tilde{F}(\Phi(x), \Delta t, \Delta W^\alpha)$ which, when Φ is Lipschitz, will be a simpler triangular discretization scheme. Differently from Theorem 3.1, in the present situation we can always construct the diffeomorphism Φ, as the following proposition states.

Proposition 1. *Let $\mathcal{G}$ be an r-dimensional solvable Lie algebra on M such that $\mathcal{G}$ has constant dimension r as a distribution of TM. Then, for any $x_0 \in M$, there exist a set of generators $Y_1, ..., Y_r$ of $\mathcal{G}$ and a local diffeomorphism $\Phi : U(x_0) \to M'$, such that $\Phi_*(Y_1), ..., \Phi_*(Y_r)$ are generators in canonical form for $\Phi_*(\mathcal{G})$.*

Proof. See [10]. □

We conclude by pointing out that for a general solvable Lie algebra $Y_1, ..., Y_r$, the map Φ, whose existence is guaranteed by Proposition 1, does not transform $\Phi_*(Y_1), ..., \Phi_*(Y_r)$ into a set of vector fields with coefficients of first degree in $x'^1, ..., x'^n$. For this reason and by Theorem 3.1, the discretization F constructed by using the diffeomorphism Φ and the usual Euler's scheme is not invariant with respect to $Y_1, ..., Y_r$.

However, if we consider solvable Lie algebras satisfying a special relation, then $\Phi_*(Y_1),...,\Phi_*(Y_r)$ will have coefficients of first degree in $x'^1,...,x'^r$, as we shall show in the following:

Proposition 2. *Suppose that the Lie algebra $\mathcal{G}=\text{span}\{Y_1,...,Y_r\}$ is such that $[[\mathcal{G},\mathcal{G}],[\mathcal{G},\mathcal{G}]]=0$. Then the coefficients of $\Phi_*(Y_1),...,\Phi_*(Y_r)$ are of first degree in $x'^1,...,x'^r$. Moreover one can choose Φ such that the coefficients of $\Phi_*(Y_1),\dots,\Phi_*(Y_r)$ are of first degree in all the variables $x'^1,...,x'^n$.*

Proof. Suppose that $Y_1,...,Y_k$ generates $\mathcal{G}^{(1)}=[\mathcal{G},\mathcal{G}]$. Then $\Phi^*(Y_i)=(\delta^l_i)$ for $i=1,...,k$. Using the fact that $[Y_i,\mathcal{G}^{(1)}]\subset\mathcal{G}^{(1)}$ and the fact that $\Phi_*(Y_1),...,\Phi_*(Y_r)$ are in canonical form, we must have that $\Phi_*(Y_{k+1}),...,\Phi_*(Y_r)$ do not depend on $x'^{k+1},...,x'^r$ and their coefficients must be of first degree in $x'^1,...,x'^r$.

The second part of the proposition follows from the well known fact that when the vector fields $Z_1,...,Z_r$ generate an integrable distribution, it is possible to choose a local coordinate system such that the coefficients of $Z_1,...,Z_r$ do not depend on $x'^{r+1},...,x'^n$. □

4. General linear SDEs.

We first consider the one-dimensional linear SDE

$$dX_t=(aX_t+b)dt+(cX_t+d)dW_t, \tag{10}$$

where $a,b,c,d\in\mathbb{R}$ and we apply the procedure previously presented in order to obtain a symmetry adapted discretization scheme.

Although it is possible to prove that equation (10) for $ad-bc\neq0$ does not admit strong symmetries (see [11]), we can look at equation (10) as a part of a two dimensional system admitting Lie symmetries.

Let us consider the system

$$\begin{pmatrix} dX_t \\ dZ_t \end{pmatrix}=\begin{pmatrix} aX_t+b \\ aZ_t \end{pmatrix}dt+\begin{pmatrix} cX_t+d \\ cZ_t \end{pmatrix}dW_t, \tag{11}$$

on $\mathbb{R}\times\mathbb{R}_+=M$, consisting of the original linear equation and the associated homogeneous one. It is simple to prove, by solving the determining equations (2) and (3), that the system (11) admits the following two strong symmetries:

$$\begin{aligned} Y_1 &= \begin{pmatrix} z \\ 0 \end{pmatrix} \\ Y_2 &= \begin{pmatrix} 0 \\ z \end{pmatrix}. \end{aligned}$$

The more general adapted coordinate system system for the symmetries Y_1,Y_2 is given by

$$\Phi(x,z)=\begin{pmatrix} \frac{x}{z}+f(z) \\ \log(z)+l \end{pmatrix},$$

where $l\in\mathbb{R}$ and $f:\mathbb{R}_+\to\mathbb{R}$ is a smooth function. Indeed in the coordinate system $(x',z')^T=\Phi(x,z)$ we have that

$$\begin{aligned} Y'_1 &= \Phi_*(Y_1)=\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ Y'_2 &= \Phi_*(Y_2)=\begin{pmatrix} -x'+e^{z'-l}\partial_z(f)(e^{z'-l})+f(e^{z'-l}) \\ 1 \end{pmatrix}. \end{aligned}$$

In order to apply 3.1, thus guaranteeing the invariance of Euler and Milstein discretization schemes, we require that the coefficients of the infinitesimal symmetries are affine functions of the coordinates. A possible choice to satisfy this condition is $f(z) = -\frac{k}{z}$ for some real constant k, for any $z \neq 0$.

In the new coordinates the original two dimensional SDE becomes

$$dX'_t = \left(\left(b - cd + ak - c^2k\right) e^{-Z'_t+l}\right) dt + (d + ck)e^{-Z'_t+l}dW_t \tag{12}$$

$$dZ'_t = \left(a - \frac{c^2}{2}\right) dt + cdW_t. \tag{13}$$

In the following, for simplicity, we consider the discretization scheme only for $l = 0$. The Euler integration scheme becomes:

$$\begin{pmatrix} Z'_n \\ X'_n \end{pmatrix} = \begin{pmatrix} Z'_{n-1} \\ X'_{n-1} \end{pmatrix} + \begin{pmatrix} \left(a - \frac{c^2}{2}\right) \\ \left(b - cd + ak - c^2k\right) e^{-Z'_{n-1}} \end{pmatrix} \Delta t_n + + \begin{pmatrix} c \\ (d + ck)e^{-Z'_{n-1}} \end{pmatrix} \Delta W_n,$$

and the Milstein scheme:

$$\begin{pmatrix} Z'_n \\ X'_n \end{pmatrix} = \begin{pmatrix} Z'_{n-1} \\ X'_{n-1} \end{pmatrix} + \begin{pmatrix} \left(a - \frac{c^2}{2}\right) \\ \left(b - \frac{1}{2}cd + ak - \frac{c^2k}{2}\right) e^{-Z'_{n-1}} \end{pmatrix} \Delta t_n + + \begin{pmatrix} c \\ (d + ck)e^{-Z'_{n-1}} \end{pmatrix} \Delta W_n + \begin{pmatrix} 0 \\ -(cd + c^2k)e^{-Z'_{n-1}} \end{pmatrix} \frac{(\Delta W_n)^2}{2}$$

We note that when $c \neq 0$ and $k = -\frac{d}{c}$ the two discretization schemes coincide.

Coming back to the original problem, in the Euler case we get:

$$\begin{aligned} X_n = &\exp\left(\left(a - \frac{c^2}{2}\right) \Delta t_n + c\Delta W_n\right) \cdot \\ &\cdot [X_{n-1} + (b - cd + ak - c^2k)\Delta t_n + (d + ck)\Delta W_n - k] + k \end{aligned} \tag{14}$$

and in the Milstein case we obtain:

$$\begin{aligned} X_n = &\exp\left(\left(a - \tfrac{c^2}{2}\right) \Delta t_n + c\Delta W_n\right) \cdot \Big[X_{n-1} + \left(b + ak - \tfrac{cd+c^2k}{2}\right) \Delta t_n + \\ &+(d + ck)\Delta W_n - \tfrac{(cd+c^2k)}{2}(\Delta W_n)^2 - k\Big] + k. \end{aligned} \tag{15}$$

Remark 3. There is a deep connection between equations (14) and (15) and the well-known integration formula for scalar linear SDEs. Indeed the equation (10) admits as solution

$$X_t = \Phi_t \left(X_0 + \int_0^t \frac{b - cd}{\Phi_s} ds + \int_0^t \frac{d}{\Phi_s} dW_s\right) \tag{16}$$

where

$$\Phi_t = \exp\left(\left(a - \frac{c^2}{2}\right) t + cW_t\right).$$

Equation (14) and (15) can also be obtained by expanding the integrals in formula (16) by applying the stochastic Taylor's Theorem (see [27]). Indeed expanding the Ito integral to first order (w.r.t. W_t) is equivalent to applying the Euler scheme (which is a first order numerical scheme), while expanding up to second order we recover the same result as in the Milstein scheme. This fact should not surprise since the adapted coordinates obtained in Subsection 3.2 were introduced exactly to obtain formula (16) from equation (11). Since the discretizations schemes (14) and (15) are closely linked with the exact solution formula of linear SDEs we call them *exact methods* (or exact discretizations) for the numerical simulation of linear SDEs.

Let us now consider the following two dimensional SDE

$$\begin{aligned}\begin{pmatrix} dX_t \\ dY_t \end{pmatrix} &= \left[\alpha \begin{pmatrix} X_t \\ Y_t \end{pmatrix} + \beta \begin{pmatrix} -Y_t \\ X_t \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right] dt + \\ &+ \left[\sigma \begin{pmatrix} X_t \\ Y_t \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}\right] dW_t^1 + \left[\sigma' \begin{pmatrix} -Y_t \\ X_t \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}\right] dW_t^2\end{aligned}$$

where $\alpha, \beta, c_1, c_2, \sigma, \sigma', d_1, d_2, e_1, e_2$ are real coefficients and $W_t^i, i = 1, 2$ are independent standard Brownian motions. The previous equation can be solved explicitly. In particular the homogeneous linear part has solution given by (see, e.g. [14])

$$\begin{aligned}\Phi_{t,t'} &= e^{\left(\mu - \frac{\sigma^2}{2}\right)(t-t') + \sigma(W_t^1 - W_{t'}^1)} \left(\begin{matrix} \cos(\beta(t-t') + \sigma'(W_t^2 - W_{t'}^2)) \\ \sin(\beta(t-t') + \sigma'(W_t^2 - W_{t'}^2)) \end{matrix} \right. \\ &\left. \begin{matrix} -\sin(\beta(t-t') + \sigma'(W_t^2 - W_{t'}^2)) \\ \cos(\beta(t-t') + \sigma'(W_t^2 - W_{t'}^2)) \end{matrix} \right),\end{aligned}$$

where $\mu = \alpha + \frac{\sigma'^2}{2}$. Thus the solution of the initial equation is

$$\begin{aligned}\begin{pmatrix} X_t \\ Y_t \end{pmatrix} &= \Phi_{t,0} \cdot \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} + \Phi_{t,0} \cdot \left(\int_0^t (\Phi_{s,0})^{-1} \cdot \begin{pmatrix} c_1 - \sigma d_1 + \sigma' e_2 \\ c_2 - \sigma d_2 - \sigma' e_1 \end{pmatrix} ds + \right. \\ &\left. + \int_0^t (\Phi_{s,0})^{-1} \cdot \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} dW_t^1 + \int_0^t (\Phi_{s,0})^{-1} \cdot \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} dW_t^2 \right)\end{aligned}$$

The Euler discretization of the previous equation becomes:

$$\begin{aligned}\begin{pmatrix} X_{t_n} \\ Y_{t_n} \end{pmatrix} &= \Phi_{t_n, t_{n-1}} \cdot \left(\left(\begin{pmatrix} X_{t_{n-1}} \\ Y_{t_{n-1}} \end{pmatrix} + \begin{pmatrix} c_1 - \sigma d_1 + \sigma' e_2 \\ c_2 - \sigma d_2 - \sigma' e_1 \end{pmatrix} \Delta t_n + \right. \right. \\ &\left. + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \Delta W_n^1 + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \Delta W_n^2 \right),\end{aligned}$$

where $\Delta t_n = t_n - t_{n-1}$ and $\Delta W_n^i = W_{t_n}^i - W_{t_{n-1}}^i$.

5. Theoretical estimation of the numerical forward error for linear SDEs.

We provide an explicit estimation of the forward error associated with the exact numerical schemes proposed in the previous section for simulating a general linear SDE. The explicit solution of linear SDEs is well known and widely applied in their simulation but, in the literature, to the best of our knowledge, there is no explicit estimation of the forward error.

5.1. Statements of the theorems.

Dividing $[0, T], T > 0$ in N parts we obtain $N+1$ instants $t_0 = 0, t_n = nh, t_N = T$, with $h = \frac{T}{N}$. We denote by $X_t^{N,T}$ the approximate solution given by exact Euler method, $\bar{X}_t^{N,T}$ the approximate solution with respect to exact Milstein method and by X_t the exact solution of the linear SDE (10). In the following we will omit T where it is possible.

Theorem 5.1. *For all $t, T \in \mathbb{R}, t \in [0, T]$, we have*

$$\epsilon_N = \left(\mathbb{E}[(X_t - X_t^{N,T})^2] \right)^{1/2} \leq f(T) g(h) h^{1/2},$$

where $h=\frac{T}{N}$, g is a continuous function and f is a strictly positive continuous function such that for $x\to+\infty$

$$\begin{aligned} f(x) &= O(1) \quad \text{if} \quad a<-c^2/2\\ f(x) &= O(x) \quad \text{if} \quad a=-c^2/2\\ f(x) &= O(e^{C(a,c)x}) \quad \text{if} \quad a>-c^2/2, \end{aligned}$$

for some positive $C(a,c)$.

Theorem 5.2. *For all $t,T\in\mathbb{R}, t\in[0,T]$, we have*

$$\bar{\epsilon}_N=\mathbb{E}[|X_t-\bar{X}_t^{N,T}|]\leq \bar{f}(T)\bar{g}(h)h^{1/2},$$

where $h=\frac{T}{N}$, $\bar{g}$ is a continuous function and f is a strictly positive continuous function such that for $x\to+\infty$

$$\begin{aligned} \bar{f}(x) &= O(1) \quad \text{if} \quad a<0\\ \bar{f}(x) &= O(e^{C'(a,c)x}) \quad \text{if} \quad a\geq 0, \end{aligned}$$

with a constant $C'(a,c)\in\mathbb{R}_+$.

Before giving the proof of the two previous theorems we propose some remarks. We recall that a linear SDE with $ad-bc\neq 0$ has an equilibrium distribution if and only if $a-\frac{c^2}{2}<0$. Furthermore the equilibrium distribution admits a finite first moment if and only if $a<0$ and a finite second moment if and only if $a+\frac{c^2}{2}<0$. Since we approximate the Ito integral up to the order $h^{1/2}$, the three cases in Theorem 5.1 follow from the fact that for giving an estimate of the error in the Euler discretization a bound for the second moment is needed. More precisely we can expect a bounded error with respect to T only when the second moment is finite as $T\to+\infty$.

Since in the Milstein case a finite first moment suffices, from the second theorem we deduce that the error does not grow with T when $a<0$. We can obtain an analogous estimate for the Euler method when $d=0$, i.e. in the case where the Milstein and Euler discretizations coincide (this is a situation similar to the one in an additive-noise-SDEs setting). Using only the first moment finitess for estimating the error has a price: we obtain an $h^{1/2}$ dependence of the error. We remark that the techniques used in the proof of Theorem 5.2 exploit some ideas from the recent *rough path* integration theory (see e.g. [17]), and in particular this circumstance explains the $\frac{1}{2}$ order of convergence. Due to this we conjecture that our results are also valid in the general rough path framework (for example for fractional Brownian motion by following [16]). If in Theorem 5.2 we do not require an uniform-in-time estimate, we can apply the methods used in the proof of Theorem 5.1 to obtain an error convergence of order 1.

Essentially, the above theorems prove that for $a+\frac{c^2}{2}<0$ and for $a<0$ respectively, our symmetry adapted discretization methods are stable for any value of h. In Section 6 we give a comparison between the stability of the adapted-coordinates schemes with respect to the standard Euler and Milstein ones, via numerical simulations.

We conclude by noting that Theorem 5.1 and Theorem 5.2 cannot be deduced in a trivial way from the standard theorems about the convergence of Euler and Milstein methods (such as Theorem 2.8). Indeed the Euler and Milstein discretizations of equations (12) and (13) do not have Lipschitz coefficients. Furthermore, even if a given discretization (X'_n, Z'_n) of the system composed by (12) and (13) should converge to the exact solution in $L^2(\Omega)$, the coordinate change Φ (introduced in Section 4) being not globally Lipschitz, this convergence does not imply that the transformed discretization (X_n, Z_n) converges to the exact solution (X, Z) of the equation (11) in $L^2(\Omega)$. Finally, as pointed out in Subsection 2.2, Theorem 2.8 does not guarantee an uniform-in-time convergence as the one stated in Theorem 5.1 and Theorem 5.2.

For proving the theorems we need the following two lemmas. The second allows to avoid very long calculations (see Appendix A).

Lemma 5.3. *Let W_t be a standard real Brownian motion, $\alpha, \beta \in \mathbb{R}$ and $n \in \mathbb{N}$ then for any $t \in \mathbb{R}_+$*

$$\mathbb{E}[\exp(\alpha t + \beta W_t) W_t^n],$$

is a continuous function of t and in particular it is locally bounded. Moreover we have that

$$\mathbb{E}[\exp(\alpha t + \beta W_t)] = \exp\left(\alpha + \frac{\beta^2}{2}\right) t.$$

Proof. The proof is based on the fact that W_t is a normal random variable with zero mean and variance equal to t. □

Lemma 5.4. *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $F(0,0) = 0$ and such that*

$$\mathbb{E}\left[|\partial_t(F)(h, W_h)|^\alpha\right], \mathbb{E}[\partial_w(F)(h, W_h)], \mathbb{E}[|\partial_{ww}(F)(h, W_h)|^\alpha] < L(h),$$

for some $\alpha \in 2\mathbb{N}$, for any h and for some continuous function $L : \mathbb{R} \to \mathbb{R}_+$. Then there exists an increasing function $C : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}[|F(h, W_h)|^\alpha] \leq C(h) h^{\alpha/2}.$$

If furthermore $\partial_w(F)(0,0) = 0$ and

$$\mathbb{E}\left[|\partial_{www}(F)(h, W_h)|^\alpha]\right] \leq L(h), \mathbb{E}[|\partial_{tw}(F)(h, W_h)|^\alpha] \leq L(h),$$

then there exists an increasing function $C' : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}[|F(h, W_h)|^\alpha] \leq C'(h) h^\alpha.$$

Proof. The statements of the lemma derive as special cases from Lemma 5.6.4 and Lemma 5.6.5 in [27]. □

5.2. **Proof of Theorem 5.1.** We consider the case $t = T$. In fact we will find that our estimate is uniform for $t \leq T$. Using the notations in Remark 3 we can write $X_T = I_1 + I_2$ where

$$\begin{aligned} I_1 &= \Phi_T \int_0^T (b - cd) \Phi_s^{-1} ds \\ I_2 &= \Phi_T \int_0^T (d) \Phi_s^{-1} dW_s. \end{aligned}$$

Also the approximation X_T^N can be written as the sums of two *integrals* of the form $X_T^N = I_1^N + I_2^N$ where

$$I_1^N = (b - cd) \sum_{i=1}^N \Phi_T \Phi_{t_{i-1}}^{-1} \Delta t_i, \quad I_2^N = d \sum_{i=1}^N \Phi_T \Phi_{t_{i-1}}^{-1} \Delta W_i.$$

Obviously the strong error ϵ_N can be estimated by $\|I_1 - I_1^N\|_2 + \|I_2 - I_2^N\|_2$, where hereafter $\|\cdot\|_\alpha = (\mathbb{E}[|\cdot|^\alpha])^{1/\alpha}$.

5.2.1. *Estimate of* $\|I_1 - I_1^N\|_2$. Setting $\Psi_{s,t} = \Phi_t(\Phi_s)^{-1}$ for any $s < t$, we obtain (with $\Delta t_i = h$)

$$\begin{aligned}
\|I_1 - I_1^N\|_2 &= \mathbb{E}\left[\left|\int_0^T (b-cd)\Psi_{t,T}dt - \sum_{i=1}^N (b-cd)\Psi_{t_{i-1},T}h\right|^2\right]^{1/2} \\
&= \mathbb{E}\left[\left|\sum_{i=1}^N \int_{t_{i-1}}^{t_i} (b-cd)(\Psi_{t,T} - \Psi_{t_{i-1},T})dt\right|^2\right]^{1/2} \\
&\leq |b-cd| \left(\sum_{i=1}^N \mathbb{E}\left[\left(\int_{t_{i-1}}^{t_i} |\Psi_{t,T} - \Psi_{t_{i-1},T}|dt\right)^2\right]^{1/2}\right).
\end{aligned}$$

By Jensen's inequality

$$\begin{aligned}
&\sum_{i=1}^N \mathbb{E}\left[\left(\int_{t_{i-1}}^{t_i} |\Psi_{t,T} - \Psi_{t_{i-1},T}|dt\right)^2\right]^{1/2} \\
&\leq h^{1/2} \sum_{i=1}^N \left(E\left[\int_{t_{i-1}}^{t_i} (\Psi_{t,T} - \Psi_{t_{i-1},T})^2 dt\right]\right)^{1/2}.
\end{aligned}$$

and by Fubini theorem we have to calculate $\mathbb{E}[(\Psi_{t,T} - \Psi_{t_{i-1},T})^2]$. Since

$$\Psi_{s,t} = \exp\left(\left(a - \frac{c^2}{2}\right)(t-s) + c(W_t - W_s)\right).$$

and $\Psi_{s,t} = \Psi_{s,u}\Psi_{u,t}$ for any $s \leq u \leq t$ we obtain that

$$\mathbb{E}[(\Psi_{t,T} - \Psi_{t_{i-1},T})^2] = \mathbb{E}[(\Psi_{t,T})^2]\mathbb{E}[(1 - \Psi_{t_{i-1},t})^2] \tag{17}$$

because $\Psi_{t,T}$ and $\Psi_{t_{i-1},t}$ are independent as a consequence of independence of the Brownian increments.

It is simple to note that the function

$$F_1(t - t_i, W_t - W_{t_i}) = 1 - e^{(t-t_i)\left(a - \frac{c^2}{2}\right) + c(W_t - W_{t_i})}, \quad t \geq t_i$$

satisfies $F_1(0,0) = 0$ and, by Lemma 5.3,

$$\begin{gathered}
\mathbb{E}[\partial_t(F_1)(t - t_i, W_t - W_{t_i})], \mathbb{E}[\partial_w(F_1)(t - t_i, W_t - W_{t_i})], \\
\mathbb{E}[\partial_{ww}(F_1)(t - t_i, W_t - W_{t_i})] < +\infty
\end{gathered}$$

Thus, by Lemma 5.4, there exists an increasing function C_1 such that, for all $t \geq t_i$:

$$\mathbb{E}\left[(F_1(t - t_i, W_t - W_{t_i}))^2\right] \leq C_1(t - t_i)(t - t_i).$$

Using Lemma 5.3 we get

$$\mathbb{E}\left[\Psi_{t,T}^2\right] = \exp((2a + c^2)(T - t)),$$

obtaining

$$\begin{aligned}
\|I_1 - I_1^N\|_2 &\leq |b-cd|\sqrt{C_1(h)}h^{1/2}\textstyle\sum_{i=1}^N \exp\left(\left(a + \frac{c^2}{2}\right)(T - t_i)\right)h \\
&\leq |b-cd|\sqrt{C_1(h)}G_1(T)h^{1/2},
\end{aligned} \tag{18}$$

where

$$G_1(T) := \int_0^T \exp\left(\left(a + \frac{c^2}{2}\right)(T-t)\right) dt = \frac{1}{a+\frac{c^2}{2}}(\exp((a+c^2/2)T) - 1). \quad (19)$$

5.2.2. *Estimate of* $\|I_2 - I_2^N\|_2$. We first consider $I_2 = (d)\Phi_T \int_0^T (\Phi_t)^{-1} dW_t$, where (d) is the coefficient in (10). Since the Ito integral involves adapted processes we cannot bring Φ_T under the integral sign. However it is possible to take advantage of the backward integral formulation which allows to int egrate processes that are measurable with respect to the (future) filtration $\mathcal{F}^t = \sigma\{W_s | s \in [t,T]\}$. In particular when X_s is $\mathcal{F}^t$-measurable then

$$\int_0^T X_s d^+ W_s = \lim_{n\to+\infty}\left(\sum_{i=1}^n X_{t_i^n}(W_{t_i^n} - W_{t_{i-1}^n})\right),$$

where $\{t_i^n\}|_i$ is a sequence of n points partitions of the interval $[0,T]$, having amplitude decreasing to 0 and the limit is understood in probability.

When F is a regular function, $F(W_t,t)$ is a process which is measurable with respect to both the filtrations $\mathcal{F}_t$ and $\mathcal{F}^t$; therefore one can calculate either $\int_0^T F(W_t,t)dW_t$ or $\int_0^T F(W_t,t)d^+W_t$.

The next well-known lemma says that we can write I_2 in terms of a backward integral, which allows to bring Φ_T under the integral sign.

Lemma 5.5. *Let* $F : \mathbb{R}^2 \to \mathbb{R}$ *be a* C^2*-function such that*

$$\mathbb{E}[(F(W_t,t))^2] < +\infty.$$

Then

$$\int_0^T F(W_t,t)dW_t = \int_0^T F(W_t,t)d^+W_t - \int_0^T \partial_w(F)(W_t,t)dt.$$

Proof. We report the proof for convenience of the reader (see, e.g., [41]). Setting

$$\tilde{F}(w,t) = \int_0^w F(u,t)du,$$

since F is C^2 then also $\tilde{F}$ is C^2. From this fact one deduces that

$$\begin{aligned}
\tilde{F}(W_t,t) - \tilde{F}(W_s,s) &= \int_s^t F(W_\tau,\tau)dW_\tau + \int_s^t \partial_t(\tilde{F})(W_\tau,\tau)d\tau \\
&\quad + \frac{1}{2}\int_s^t \partial_w(F)(W_\tau,\tau)d\tau \\
\tilde{F}(W_t,t) - \tilde{F}(W_s,s) &= \int_s^t F(W_\tau,\tau)d^+W_\tau + \int_s^t \partial_t(\tilde{F})(W_\tau,\tau)d\tau \\
&\quad - \frac{1}{2}\int_s^t \partial_w(F)(W_\tau,\tau)d\tau.
\end{aligned}$$

By equating the two expressions one obtains the final formula. □

Since

$$(\Phi_t)^{-1} = \exp(-(a-c^2/2)t - cW_t) = F(W_t,t),$$

and $\partial_w(F)(w,t) = -cF(w,t)$, by Lemma 5.5, we can write

$$\begin{aligned} I_2 &= \Phi_T(d)\int_0^T (\Phi_t)^{-1}dW_t \\ &= \Phi_T(d)\left(\int_0^T (\Phi_t)^{-1}d^+W_t + c\int_0^T (\Phi_t)^{-1}dt\right) \\ &= (d)\left(\int_0^T \Psi_{t,T}d^+W_t + c\int_0^T \Psi_{t,T}dt\right). \end{aligned}$$

Introducing $\tilde{I}_2 = (d)\int_0^T \Psi_{t,T}d^+W_t$ and

$$\tilde{I}_2^N = (d)\sum_{i=1}^N \Psi_{t_i,T}\Delta W_i,$$

we have that

$$\|I_2 - I_2^N\|_2 \leq \|\tilde{I}_2 - \tilde{I}_2^N\|_2 + \left\|(\tilde{I}_2^N - I_2^N) + cd\int_0^T \Psi_{t,T}dt\right\|_2. \tag{20}$$

We first consider the term $\|\tilde{I}_2 - \tilde{I}_2^N\|_2$. The process $\tilde{I}_2^N$ can be written as $\int_0^T (d)H_t dW_t^+$ where H_t is the $\mathcal{F}^t-$ measurable process given by

$$H_t = \sum_{i=1}^N \Psi_{t_i,T}1_{(t_{i-1},t_i]}(t),$$

where $1_{(t_{i-1},t_i]}$ is the characteristic function of the interval $(t_{i-1},t_i]$. By Ito's isometry and Fubini's Theorem we obtain

$$\begin{aligned} \|\tilde{I}_2 - \tilde{I}_2^N\|_2^2 &= (d)^2\mathbb{E}\left[\left(\int_0^T (\Psi_{t,T} - H_t)dW_t\right)^2\right] \\ &= (d)^2\mathbb{E}\left[\int_0^T (\Psi_{t,T} - H_t)^2dt\right] \\ &= (d)^2\int_0^T \mathbb{E}[(\Psi_{t,T} - H_t)^2]dt \\ &= (d)^2\sum_{i=1}^N\int_{t_{i-1}}^{t_i} \mathbb{E}[(\Psi_{t,T} - \Psi_{t_i,T})^2]dt. \end{aligned} \tag{21}$$

Since Brownian motion has independent increments, we have that

$$\mathbb{E}[(\Psi_{t,T} - \Psi_{t_i,T})^2] = \mathbb{E}[(\Psi_{t_i,T})^2]\mathbb{E}\left[(1 - \Psi_{t,t_i})^2\right].$$

Introducing the function:

$$H(t_i - t, W_{t_i} - W_t) = 1 - \Psi_{t,t_i}$$

which satisfies $H(0,0) = 0$, by Lemma 5.4 and Lemma 5.3 we obtain

$$\|\tilde{I}_2 - \tilde{I}_2^N\|_2^2 \leq (d)^2\sum_{i=1}^N \exp((2a + c^2)(T - t_i))C_2(h)h^2$$

where $C_2(h)$ is an increasing function and, finally,

$$\|\tilde{I}_2 - \tilde{I}_2^N\|_2 \leq (d)\sqrt{(G_2(T)C_2(h))}h^{1/2} \tag{22}$$

where

$$G_2(T) = \int_0^T \exp{(2a + c^2)(T - t)}dt. \tag{23}$$

In order to estimate the other term in the right-hand side of (20) we note that by introducing

$$K_i(t, W_t) = \exp\left(\left(a - \frac{c^2}{2}\right)(T-t) + c(W_T - W_t)\right)(W_{t_i} - W_t)$$

we have

$$I_2^N \quad = \quad (d)\sum_{i=1}^{N} K_i(t_{i-1}, W_{t_{i-1}}),$$

and

$$K_i(t_i, W_{t_i}) = 0$$

By applying Lemma 5.5 to $K_i(t_i, W_{t_i})$ we can write

$$\begin{aligned} 0 - K_i(t, W_t) \quad = \quad & \int_t^{t_i} \partial_w(K_i)(s, W_s) d^+W_s + \int_t^{t_i} \partial_s(K_i)(s, W_s)ds + \\ & -c\int_t^{t_i} \Psi_{s,T} ds - \frac{c^2}{2}\int_t^{t_i} K_i(s, W_s)ds. \end{aligned}$$

From the previous equality, by Ito isometry and Minkowski's integral inequality we get

$$\begin{aligned} & \left\| \tilde{I}_2^N - I_2^N + cd\int_0^T \Psi_{t,T} dt \right\|_2 \\ = \quad & d\left\| \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \Psi_{t_i,T} d^+W_t \right. \\ & + \int_{t_{i-1}}^{t_i} \partial_w(K_i)(t, W_t)d^+W_t + \int_{t_{i-1}}^{t_i} \partial_t(K_i)(t, W_t)dt \\ & \left. -\frac{c^2}{2}\int_{t_{i-1}}^{t_i} K_i(t, W_t)dt \right\|_2 \\ \leq \quad & d\left(\left\| \int_0^T R_t d^+W_t \right\|_2 + \left\| \int_0^T M_t dt \right\|_2 \right), \\ \leq \quad & d\left(\left(\int_0^T \mathbb{E}[R_t^2]dt \right)^{1/2} + \int_0^T \left(\mathbb{E}[M_t^2]\right)^{1/2} dt \right), \end{aligned}$$

where

$$\begin{aligned} R_t \quad &= \quad \sum_{i=1}^{N} (\partial_w(K_i)(t, W_t) + \Psi_{t_i,T})1_{(t_{i\_1},t_i]}(t) \\ M_t \quad &= \quad \sum_{i=1}^{N} \left(\partial_t(K_i)(t, W_t) - \frac{c^2}{2}K_i(t, W_t) \right) 1_{[t_{i\_1},t_i]}(t) \end{aligned}$$

When $t_{i-1} < t \leq t_i$, by independence

$$\mathbb{E}[R_t^2] \quad \leq \quad 2\mathbb{E}[\Psi_{t_i,T}^2]\mathbb{E}[(c\Psi_{t,t_i}(W_{t_i} - W_t))^2 + (\Psi_{t,t_i} - 1)^2].$$

Introducing

$$F_2(t_i-t, W_{t_i}-W_t) = c\exp\left(\left(a-\frac{c^2}{2}\right)(t_i-t)+c(W_{t_i}-W_t)\right)(W_{t_i}-W_t)$$
$$F_3(t_i-t, W_{t_i}-W_t) = \exp\left(\left(a-\frac{c^2}{2}\right)(t_i-t)+c(W_{t_i}-W_t)\right)-1,$$

we have that $F_2(0,0)=F_3(0,0)=0$ and $\mathbb{E}[|\partial_w(F_i)(t,W_{t_i}-W_t)|^2]$, $\mathbb{E}[|\partial_{ww}(F_i)(t, W_{t_i}-W_t)|^2]$, $\mathbb{E}[|\partial_t(F_i)(t,W_{t_i}-W_t)|^2] \leq L(t_i-t)$. So, by Lemma 5.4, there exist two continuous increasing functions $C_3(t), C_4(t)$ such that

$$\mathbb{E}[R_t^2] \leq 2\exp\left((2a+c^2)(T-t_i)\right)(C_3(t_i-t)+C_4(t_i-t))|t_i-t|.$$

Since by independence

$$\mathbb{E}[M_t^2] = \mathbb{E}[(a\Psi_{t,T}(W_{t_i}-W_t))^2] = \mathbb{E}[(\Psi_{t_i,T})^2]\mathbb{E}[(a\Psi_{t,T}(W_{t_i}-W_t))^2]$$

analogously we can prove that there exists an increasing function C_5 such that

$$\mathbb{E}[M_t^2] \leq \exp\left(\left(2a+c^2\right)(T-t_i)\right)C_5(t-t)|t_i-t|.$$

For the second term in the right-hand side of (20), we have finally the following estimate

$$\left\|\tilde{I}_2^N - I_2^N + cd\int_0^T \Psi_{t,T}dt\right\|_2 \leq d\left\{\sqrt{G_2(T)}(\sqrt{2(C_3(h)+C_4(h))}) + G_1(T)\sqrt{C_5(h)}\right\}h^{1/2}, \tag{24}$$

where $G_1(T)$ and $G_2(T)$ are given by (19) and (23) respectively.

5.3. **Proof of Theorem 5.2.** We make the proof only for $a<0$, since in the other case the estimate are equal to the Euler case and can be addressed by the same proof. We introduce the two integrals

$$\bar{I}_1^N = (b-cd)\sum_{i=1}^N \Phi_T\Phi_{t_{i-1}}^{-1}\Delta t_i,$$
$$\bar{I}_2^N = d\sum_{i=1}^N \Phi_T\Phi_{t_{i-1}}^{-1}\Delta W_i - \frac{cd}{2}\sum_{i=1}^N \Phi_T\Phi_{t_{i-1}}^{-1}((\Delta W_i)^2-(\Delta t_i)).$$

5.3.1. *Estimate of* $\|I_1-\bar{I}_1^N\|_1$. First we note that (with $\Delta t_i = h$)

$$\begin{aligned}\|I_1-\bar{I}_1^N\|_1 &\leq |b-cd|\sum_{i=1}^N\left\|\Phi_T\int_{t_{i-1}}^{t_i}\Phi_t^{-1}dt - \Phi_T\Phi_{t_{i-1}}^{-1}h\right\|_1\\ &\leq |b-dc|\sum_{i=1}^N\|\Psi_{t_i,T}\|_\alpha\left\|\int_{t_{i-1}}^{t_i}\Psi_{t,t_i}dt - \Psi_{t_{i-1},t_i}h\right\|_{2n}\\ &= |b-dc|\left\|\int_0^h(\Psi_{t,h}-\Psi_{0,h})dt\right\|_{2n}\left(\sum_{i=1}^N\|\Psi_{t_i,T}\|_\alpha\right)\end{aligned}$$

where we have taken, $n\in\mathbb{N}$, $\frac{1}{2n}+\frac{1}{\alpha}=1$ and $1<\alpha<2$ such that $\alpha a+\alpha(\alpha-1)\frac{c^2}{2}\leq 0$ (the last condition guarantees that when $T\to\infty$ we have $\mathbb{E}[\Psi_{t_i,T}^\alpha]\to 0$). By

Jensen's inequality and Lemma 5.4 we can derive the following estimate:

$$\begin{aligned}\left\|\int_0^h (\Psi_{t,h}-\Psi_{0,h})dt\right\|_{2n}^{2n} &\leq h^{2n-1}\int_0^h \mathbb{E}[(\Psi_{t,h}-\Psi_{0,h})^{2n}]dt\\ &\leq h^{3n}C_5(h),\end{aligned}$$

where $C_5(h)$ is an increasing function and in the last inequality we have used the fact that the function $F_4(t,W_t)=\Psi_{t,h}-\Psi_{0,h}$ is such that $F_4(0,0)=0$. By Lemma 5.3, we have that

$$\|\Psi_{t_i,T}\|_\alpha = \exp\left(\left(a+\frac{c^2}{2}(\alpha-1)\right)(T-t_i)\right),$$

and so

$$\begin{aligned}\|I_1-\bar{I}_1^N\|_1 &\leq |b-cd| \sum_{i=1}^N \exp\left(\left(a+\frac{c^2}{2}(\alpha-1)\right)(T-t_i)\right)(C_5(h))^{1/2n}h^{3/2}\\ &\leq |b-cd|G_4(T)(C_5(h))^{1/2n}h^{1/2}\end{aligned}$$

where

$$G_4(T)=\int_0^T \exp\left(\left(a+\frac{c^2}{2}(\alpha-1))(T-t)\right)\right)dt. \tag{25}$$

5.3.2. *Estimate of* $\|I_2-\bar{I}_2^N\|_1$. First we note that

$$\begin{aligned}\|I_2-\bar{I}_2^N\|_1 &\leq |d|\sum_{i=1}^N\left\|\Phi_T\int_{t_{i-1}}^{t_i}\Phi_t^{-1}dW_t-\Phi_T\Phi_{t_{i-1}}^{-1}\Delta W_i+\right.\\ &\quad\left.+\frac{c}{2}\Phi_T\Phi_{t_{i-1}}^{-1}((\Delta W_i)^2-h)\right\|_1\\ &\leq |d|\sum_{i=1}^N\|\Psi_{t_i,T}\|_\alpha\left\|\Phi_{t_i}\int_{t_{i-1}}^{t_i}\Phi_t^{-1}dW_t-\Psi_{t_{i-1},t_i}\Delta W_i+\right.\\ &\quad\left.+\frac{c}{2}\Psi_{t_{i-1},t_i}((\Delta W_i)^2-h)\right\|_{2n}\end{aligned}$$

where α, n are as in the previous subsection. We introduce the following notation

$$\begin{aligned}I_{2,t_i} &= \Phi_{t_i}\int_{t_{i-1}}^{t_i}(\Phi_t)^{-1}dW_t\\ &= \Phi_{t_i}\left(\int_{t_{i-1}}^{t_i}(\Phi_t)^{-1}d^+W_t+c\int_{t_{i-1}}^{t_i}(\Phi_t)^{-1}dt\right)\\ &= \int_{t_{i-1}}^{t_i}\Psi_{t,t_i}d^+W_t+c\int_{t_{i-1}}^{t_i}\Psi_{t,t_i}dt,\end{aligned}$$

where we have used Lemma 5.5 and the fact that $\Psi_{s,t}=\Phi_t(\Phi_s)^{-1}$. By introducing also $\hat{I}_{2,t_i}=\int_{t_{i-1}}^{t_i}\Psi_{t,t_i}d^+W_t$ and

$$\begin{aligned}\bar{I}_{2,t_i}^N &= \Psi_{t_{i-1},t_i}\Delta W_i-\frac{c}{2}\Psi_{t_{i-1},t_i}((\Delta W_i)^2-h)\\ \hat{I}_{2,t_i}^N &= \Psi_{t_i,t_i}\Delta W_i+\frac{c}{2}((\Delta W_i)^2-h),\end{aligned}$$

we have that

$$\|I_{2,t_i} - \bar{I}^N_{2,t_i}\|_{2n} \leq \|\hat{I}_{2,t_i} - \hat{I}^N_{2,t_i}\|_{2n} + \left\| (\hat{I}^N_{2,t_i} - \bar{I}^N_{2,t_i}) + c\int_{t_{i-1}}^{t_i} \Psi_{t,t_i} dt \right\|_{2n}.$$

It is simple to see that the two norms on the right-hand side of the previous expression do not depend on t_i but only on the difference $h = t_i - t_{i-1}$, so we study the functions (with $\Psi_{t_i,t_i} = 1$):

$$\begin{aligned} Z_1(h) &= \|\hat{I}_{2,h} - \hat{I}^N_{2,h}\|^{2n}_{2n} = \left\| \int_0^h (\Psi_{t,h} - 1 - c(W_h - W_t)) d^+W_t \right\|^{2n}_{2n} \\ Z_2(h) &= \left\| (\hat{I}^N_{2,t_i} - \bar{I}^N_{2,t_i}) + c\int_{t_{i-1}}^{t_i} \Psi_{t,t_i} dt \right\|^{2n}_{2n} \\ &= \left\| (1-\Psi_{0,h})W_h + \frac{c}{2}(\Psi_{0,h}+1)W_h^2 - \frac{c}{2}(\Psi_{0,h}+1)h + c\int_0^h \Psi_{t,h} dt \right\|^{2n}_{2n} \end{aligned}$$

By a well-known consequence of Ito isometry (see, e.g., [15]) we can estimate the function $Z_1(h)$ as follows:

$$Z_1(h) \leq D_n h^{n-1} \int_0^h \mathbb{E}[(\Psi_{t,h} - 1 - c(W_h - W_t))^{2n}] dt,$$

where $D_n = (n(2n-1))^n$. Since the function

$$F_5(h-t, W_h - W_t) = \exp\left((a - \frac{c^2}{2})(h-t) + c(W_h - W_t) \right) - 1 - c(W_h - W_t)$$

satisfies $F_5(0,0) = \partial_w(F_5)(0,0) = 0$, by Lemma 5.4 there exists an increasing function $C_6(h)$ such that

$$Z_1(h) \leq C_6(h) h^{3n}.$$

Concerning the function $Z_2(h)$, by introducing

$$K(t, W_t) = (1-\Psi_{t,h})(W_h - W_t) + \frac{c}{2}(\Psi_{t,h}+1)(W_h - W_t)^2 - \frac{c}{2}(\Psi_{t,h}+1)(h-t),$$

it is immediate to see that

$$Z_2(h) = \left\| K(0,0) + c\int_0^h \Psi_{t,h} dt \right\|^{2n}_{2n}.$$

By applying Lemma 5.5 to $K(h, W_h)$, and by noting that $K(h, W_h) = 0$, we obtain

$$0 - K(0,0) = \int_0^h (\partial_t(K)(t,W_t) - \frac{1}{2}\partial_{ww}(K)(t,W_t) dt + \int_0^h \partial_w K(t,W_t) d^+W_t$$

Since we have that $-\partial_t(K)(h,W_h) + \partial_{ww}(K)(h,W_h)/2 + c\Psi_{0,h} = 0$, and that $K(h,W_h) = \partial_w(K)(h,W_h) = \partial_{ww}(K)(h,W_h) = 0$, by Jensen's inequality, Lemma 5.4 and by applying the same techniques used for obtaining (24), we find that

$$Z_2(h)^{1/2n} \leq \left\{ (C_7(h))^{1/2n} + (C_8(h))^{1/2n} \right\} h^{3/2}$$

or, equivalently,

$$Z_2(h) \leq C_9(h) h^{3n},$$

with the obvious definition of the function $C_9(h)$.

Finally we have

$$\begin{aligned}\|I_2^N - \bar{I}_2^N\|_1 &\leq |d|(C_6(h)^{1/2n} + C_9(h)^{1/2n}) \cdot \\ &\cdot \sum_{i=1}^{N} \exp\left(\left(a + \frac{c^2}{2}(\alpha - 1)\right)(T - t_i)\right) h^{3/2} \\ &\leq |d|(C_6(h)^{1/2n} + C_9(h)^{1/2n}) G_4(T) h^{1/2},\end{aligned}$$

where $G_4(T)$ is given by (25).

6. **Numerical examples.**

We present some numerical results which confirm the theoretical estimates proved in Section 5 and permit to study other properties of the new discretization methods introduced in Section 4.

We simulate the linear SDE (10) with coefficients $a = -2$, $b = 10$, $c = 10$ e $d = 10$. The coefficients are such that $a + \frac{c^2}{2} > 0$ with $a < 0$. This means that the considered linear equation admits an equilibrium probability density with finite first moment and infinite second moment. The coefficient d has been chosen to be big enough in order to put in evidence the noise effect.

We make a comparison between the Euler and Milstein methods applied directly to equation (10) and the new exact methods (14) and (15) with the constants $k = 0$ and $k = \frac{-d}{c} = -1$. In particular we observe that when $k = -1$, the schemes (14) and (15) coincide. We calculate the following two errors:

- the weak error $E^w = |\mathbb{E}[X_t - X_t^N]|$,
- the strong error $E^s = \mathbb{E}[|X_t - X_t^N|]$.

The weak error is estimated trought the explicit expression

$$\mathbb{E}[X_t] = e^{at},$$

for the first moment of the linear SDE solution, and by using a Monte-Carlo method with 1000000 paths for calculating $\mathbb{E}[X_t^N]$. The strong error is estimated by exploiting a Monte-Carlo simulation of X_t and X_t^N with 1000000 paths. In order to simulate X_t we apply the Milstein method with a steps-size of $h = 0.0001$, for which we have verified that it gives a good approximation of both $\mathbb{E}[X_t]$ and the equilibrium density for $t \to +\infty$. Since we use Monte-Carlo methods for estimating E^w and E^s, the two errors include both the systematic errors of the considered schemes and the statistical errors of the Monte-Carlo estimate procedure.

In Figure 1 we report the weak and strong errors with respect to the maximum time of integration t which varies from 0.1 to 1 and stepsize $h = 0.025$. As predicted by Theorem 5.2, the error of the exact method for $k = -1$ remains bounded. It is important to note that for the exact method in the case $k = 0$ (where Theorem 5.1 and Theorem 5.2 do not apply) the errors remains bounded too, while for the Euler and Milstein methods the errors grow exponentially with t.

In Figure 2 we report the weak and strong errors with respect to the maximum time of integration t, which varies from 0.1 to 1, and stepsize $h = 0.01$. In this situation also the errors of the Mistein method remain bounded. In other words $h = 0.01$ belongs to the stability region of the Milstein method but not to the stability region of the Euler method.

In Figure 3 we plot the weak and strong errors with fixed final time $t = 0.5$ and steps number $N = 10, ..., 80$, where the stepsize is $h = \frac{t}{N}$. Here we note that the weak and strong errors for the exact methods do not change with the stepsize. This means that with a stepsize of only $h = 0.05$ the exact methods have weak and strong systematic errors less than the statistical errors. Instead for the Milstein scheme the errors grow and only with a stepsize equal to $h = 0.0125$ the systematic errors are comparable with the statistical ones. Equivalently we can say that the stability region is $[0, 0.0125]$. In the Euler case the systematic error is not comparable with the statistical one.

In Figure 4 we report the total variation distance between the empirical probabilities of X_t and of X_t^N obtained simulating 1000000 paths. We note that there is a big difference between the exact method for $k = 0$ and for $k = -1$. The discrepancy is due to the fact that the exact method with $k = 0$ tends to overestimate the points with probability less then $\frac{-d}{c}$ (for $c \neq 0$) more than the Euler scheme does.

Now we simulate the two dimensional linear SDE analized in Section 4 by where

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$B_1 = \sigma \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B_2 = \sigma' \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

choosing $\alpha = -20$, $\beta = -0.5$, $\sigma = \sigma' = 5$, $\mathbf{c} = \mathbf{e} = \begin{pmatrix} 0.1 & 0.1 \end{pmatrix}^T$ and $\mathbf{d} = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$. Our choice of the parameters guarantees the existence of an equilibrium probability density.

We compare approximated solutions obtained by the Euler method and by our exact method using $h = 0.01$. To this end we calculated both the strong and weak componentwise error

$$E_i^w = |\mathbb{E}[X_t^i - X_t^{i,N}]| \tag{26}$$

$$E_i^w = \mathbb{E}[|X_t^i - X_t^{i,N}|] \tag{27}$$

where X_t^i is the $i-$th component of the solution. This time our *true* solution is calculated using the Euler method with timestep $h = 0.0001$. As in the previous example the error are estimated using a Montecarlo simulation, this time with 10000 paths, both for the approximated and the *true* solution. Again we expect E_i^w and E_i^s to include both systematic and statistical errors.

In Figure 5 and Figure 6 we compare the strong and weak errors of both components of the simulated solutions with respect to the maximum time of integration varying from 0.1 to 1. As can be seen the error from our new method is bounded at all times while the Euler method errors show an exponential growth with respect to the maximum time.

In Figure 7 and Figure 8 we compare the errors of both approximations for solutions with $T = 1$ and timestep size varying between 0.1 to 0.01. As in the previous one-dimensional case we can see how the new exact method gives a good approximation of the *true* solution even with large timesteps, while the Euler method fails to achieve the same magnitude of error even using significative smaller timesteps.

Appendix. In the proof of Theorem 5.1, by using Lemma 5.3 and the independence of Brownian increments, we can estimate the errors in a very explicit way without exploiting Lemma 5.4. Here we show the main steps and final results of the procedure.

From (17) we obtain that

$$\int_{t_{i-1}}^{t_i} \mathbb{E}[(\Psi_{t,T})^2]\mathbb{E}[(1-\Psi_{t_{i-1},t})^2]dt =: M_1(h)$$

with

$$M_1(h) = \frac{-a-c^2+h\exp\left((2a+c^2)h\right)(c^4+3ac^2+2a^2)}{c^4+3ac^2+2a^2} + \\ + \frac{(c^2+3a)\exp\left((2a+c^2)h\right)+(2c^2+4a)\exp\left(ah\right)}{c^4+3ac^2+2a^2}$$

Since $M_1(0)=\partial_h M_1(0)=0$, then $|M_1(h)| \leq M_2(h)h^2$ with $M_2(h) := max_{k\in[0,h]}$ $|\partial_h^2 M_1(k)|$, and, finally,

$$\|I_1 - I_1^N\|_2 \leq |b-cd|h^{1/2}\sqrt{M_2(h)}G_1(T)$$

where $G_1(T)$ is given by (19), in agreement with (18).

From (21) we obtain

$$\begin{aligned}\|\tilde{I}_2-\tilde{I}_2^N\|_2^2 &= (d)^2\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\mathbb{E}[(\Psi_{t_i,T})^2]\mathbb{E}\left[(\Psi_{t,t_i})^2+1-2\Psi_{t,t_i}\right] \\ &= (d)^2\sum_{i=1}^N \exp\left((2a+c^2)(T-t_i)\right)M_3(h)\end{aligned}$$

where

$$M_3(h) = \frac{3a+2c^2+a\exp\left(2a+c^2\right)+h(2a+ac^2)-(4a+2c^2)\exp\left(ah\right)}{2a^2+ac^2}$$

Since $M_3(0)=\partial_h M_3(0)=0$, we have that $|M_3(h)| \leq M_4(h)h^2$ with $M_4(h) := max_{k\in[0,h]}|\partial_h^2 M_3(k)|$, and

$$\|\tilde{I}_2-\tilde{I}_2^N\|^2 \leq (d)\sqrt{G_2(T)M_4(h)}h^{1/2},$$

that is, Inequality (22).

The second term on the right-hand side of (20) becomes

$$\begin{aligned}\left\|\tilde{I}_2^N - I_2^N + cd\int_0^T \Psi_{t,T}dt\right\|_2^2 &= d^2\mathbb{E}\left[\left(\sum_{i=1}^N \Psi_{t_i,T}(1-\Psi_{t_{i-1},t_i})(W_{t_i}-W_{t_{i-1}})\right.\right. \\ &\quad \left.\left. + \sum_{i=1}^N \Psi_{t_i,T}c\int_{t_{i-1}}^{t_i}\Psi_{t,t_i}dt\right)^2\right] \\ &= d^2\left[\sum_{i=1}^N \mathbb{E}[(\Psi_{t_i,T})^2]\mathbb{E}[(K_i+H_i)^2]+\right. \\ &\quad +2\sum_{i<j}\mathbb{E}[(\Psi_{t_j,T})^2]\mathbb{E}[\Psi_{t_{j-1},t_j}(H_j+K_j)]\cdot \\ &\quad \left.\cdot\,\mathbb{E}[\Psi_{t_i,t_{j-1}}]\mathbb{E}[(H_i+K_i)]\right]\end{aligned}$$

where we have used independence and we have set

$$K_i = (1 - \Psi_{t_{i-1},t_i})(W_{t_i} - W_{t_{i-1}}), \quad H_i = c\int_{t_{i-1}}^{t_i} \Psi_{t,t_i} dt$$

Let us consider

$$\begin{aligned} M_5(h) \;&:=\; \mathbb{E}[(H_i + K_i)^2] = \exp\left(2a + c^2\right)(4c^2h^2 + h) - 2\exp\left(ah\right)(c^2h^2 + h) + h \\ &+\frac{c^2(1 - \exp\left((2a + c^2)h\right)}{a(c^2 + 2a)} + \frac{c^2(\exp\left((2a + c^2)h\right) - \exp\left(ah\right)}{a(a + c^2)} \\ &+2\left[+\frac{2c^2[\exp\left((2a + c^2)h\right)(h(a + c^2) - 1) + \exp\left(ah\right)]}{(a + c^2)^2}\right. \\ &+\frac{c^2[\exp\left((2a + c^2)h\right) - \exp\left(ah\right)(1 + h(a + c^2))]}{(a + c^2)^2} \\ &\left.-\frac{c^2[(ah - 1)\exp\left(ah\right) + 1]}{a^2}\right]. \end{aligned}$$

Since $M_5(0) = \partial_h M_5(0) = 0$, we have that $|M_5(h)| \leq M_6(h)h^2$, where $M_6(h) := max_{k\in[0,h]}|\partial_h^2 M_5(k)|$. Being:

$$\begin{aligned} M_7(h) \;&:=\; \mathbb{E}[\Psi_{t_{j-1},t_j}(H_j + K_j)] \\ &= \frac{c\exp\left((2a + c^2)h\right) - c\exp\left(ah\right) + ch(a + c^2)\exp\left(ah\right)}{(a + c^2)} \\ &\quad -\frac{2ch\exp\left((2a + c^2)h\right)(a + c^2)}{(a + c^2)} \\ \mathbb{E}[\Psi_{t_i,t_{j-1}}] &= \exp\left(a(t_{j-1} - t_i)\right) \\ M_8(h) \;&:=\; \mathbb{E}[H_i + K_i] = -ch\exp\left(ah\right) + \frac{c(\exp\left(ah\right) - 1)}{a}, \end{aligned}$$

then by putting $M_9(h) = M_7(h)M_8(h)$, one can easily verify that

$$M_9(0) = \partial_h M_9(0) = \partial_h^2 M_9(0) = \partial_h^3 M_9(0) = 0$$

(because $M_7(0) = \partial_h M_7(0) = M_8(0) = \partial_h M_8(0) = 0$) and, therefore, $|M_9(h)| \leq M_{10}(h)h^4$, where $M_{10}(h) := max_{k\in[0,h]}|\partial_h^4 M_9(k)|$. Finally we get the following estimate:

$$\begin{aligned} &\left\|\tilde{I}_2^N - I_2^N + cd\int_0^T \Psi_{t,T} dt\right\|_2^2 \\ \leq\;& d^2\left[\sum_{i=1}^N \exp\left((2a + c^2)(T - t_i)\right)M_6(h)h^2 + \right. \\ &+2\sum_{i<j}\exp\left((2a + c^2)(T - t_j)\right)\cdot \\ &\left.\cdot\exp\left(a(t_{j-1} - t_i)\right)M_9(h)\right] \end{aligned}$$

$$\leq \quad d^2 \Bigg[G_2(T)M_6(h)h + 2M_{10}(h) \cdot \\ \cdot \Bigg[\sum_i \exp\left((2a+c^2)(T-t_{i+1})\right)h^4 + \\ + \sum_{i<j+1} \exp\left((2a+c^2)(T-t_j)\right)\exp\left(a(t_{j-1}-t_i)\right)h^4 \Bigg]$$

that is

$$\left\| \tilde{I}_2^N - I_2^N + cd\int_0^T \Psi_{t,T}dt \right\|_2^2 \leq d^2\left[G_2(T)M_6(h)h + 2M_{10}(h)(G_2(T)h^3 + \bar{G}(T)h^2)\right],$$

with

$$\bar{G}(T) = \int_0^T \int_0^t \exp\left((2a+c^2)(T-t) + a(t-s)\right)dsdt,$$

from which we get:

$$\left\| \tilde{I}_2^N - I_2^N + cd\int_0^T \Psi_{t,T}dt \right\|_2 \quad \leq \quad d\left[\sqrt{G_2(T)M_6(h) + 2M_{10}(h)\bar{G}(T)}h^{1/2} \right. \\ \left. + \sqrt{2M_{10}(h)G_2(T)}h^{3/2}\right],$$

to be compared with (24).

REFERENCES

[1] S. Albeverio, F. C. De Vecchi, P. Morando and S. Ugolini, Weak symmetries of stochastic differential equations driven by semimartingales with jumps, arXiv preprint, arXiv:1904.10963.

[2] S. Albeverio, F. C. De Vecchi and S. Ugolini, Entropy chaos and bose-einstein condensation, *J. Stat. Phys.*, **168** (2017), 483–507.

[3] C. Anton, J. Deng and Y. S. Wong, Weak symplectic schemes for stochastic Hamiltonian equations, *Electron. Trans. Numer. Anal.*, **43** (2014/15), 1–20.

[4] N. Bou-Rabee and H. Owhadi, Stochastic variational integrators, *IMA J. Numer. Anal.*, **29** (2009), 421–443.

[5] N. Bou-Rabee and H. Owhadi, Long-run accuracy of variational integrators in the stochastic context, *SIAM J. Numer. Anal.*, **48** (2010), 278–297.

[6] R. Campoamor-Stursberg, M. A. Rodríguez and P. Winternitz, Symmetry preserving discretization of ordinary differential equations. Large symmetry groups and higher order equations, *J. Phys. A*, **49** (2016), 035201, 21pp.

[7] C. Chen, D. Cohen and J. Hong, Conservative methods for stochastic differential equations with a conserved quantity, *Int. J. Numer. Anal. Model.*, **13** (2016), 435–456.

[8] F. C. De Vecchi, *Lie Symmetry Analysis and Geometrical Methods for Finite and Infinite Dimensional Stochastic Differential Equations*, PhD thesis, 2017.

[9] F. C. De Vecchi, P. Morando and S. Ugolini, A note on symmetries of diffusions within a martingale problem approach, *Stochastics and Dynamics*, **19** (2019), 1950011, 21 pp.

[10] F. C. De Vecchi, P. Morando and S. Ugolini, Reduction and reconstruction of stochastic differential equations via symmetries, *J. Math. Phys.*, **57** (2016), 123508, 22pp.

[11] F. C. De Vecchi, P. Morando and S. Ugolini, Symmetries of stochastic differential equations: A geometric approach, *J. Math. Phys*, **57** (2016), 063504, 17pp.

[12] V. Dorodnitsyn, *Applications of Lie Groups to Difference Equations*, vol. 8 of Differential and Integral Equations and Their Applications, CRC Press, Boca Raton, FL, 2011.

[13] K. Ebrahimi-Fard, A. Lundervold, S. J. A. Malham, H. Munthe-Kaas and A. Wiese, Algebraic structure of stochastic expansions and efficient simulation, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, **468** (2012), 2361–2382.

[14] N. B.-L. Eckhard Platen, *Numerical Solution of Stochastic Differential Equations with Jumps in Finance*, Springer Berlin Heidelberg, 2010.

[15] A. Friedman, *Stochastic Differential Equations and Applications. Vol. 1*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975, Probability and Mathematical Statistics, Vol. 28.

[16] P. Friz and S. Riedel, Convergence rates for the full Gaussian rough paths, *Ann. Inst. Henri Poincaré Probab. Stat.*, **50** (2014), 154–194.

[17] P. K. Friz and M. Hairer, *A Course on Rough Paths*, Universitext, Springer, Cham, 2014, With an introduction to regularity structures.

[18] G. Gaeta and N. R. Quintero, Lie-point symmetries and stochastic differential equations, *J. Phys. A*, **32** (1999), 8485–8505.

[19] E. Hairer, C. Lubich and G. Wanner, *Geometric Numerical Integration*, vol. 31 of Springer Series in Computational Mathematics, 2nd edition, Springer-Verlag, Berlin, 2006, Structure-preserving algorithms for ordinary differential equations.

[20] D. B. Hernández and R. Spigler, *A*-stability of Runge-Kutta methods for systems with additive noise, *BIT*, **32** (1992), 620–633.

[21] D. J. Higham, Mean-square and asymptotic stability of the stochastic theta method, *SIAM J. Numer. Anal.*, **38** (2000), 753–769 (electronic).

[22] D. J. Higham and P. E. Kloeden, Numerical methods for nonlinear stochastic differential equations with jumps, *Numer. Math.*, **101** (2005), 101–119.

[23] D. J. Higham, X. Mao and C. Yuan, Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations, *SIAM J. Numer. Anal.*, **45** (2007), 592–609 (electronic).

[24] D. D. Holm and T. M. Tyranowski, Variational principles for stochastic soliton dynamics, *Proc. A.*, **472** (2016), 20150827, 24pp.

[25] J. Hong, S. Zhai and J. Zhang, Discrete gradient approach to stochastic differential equations with a conserved quantity, *SIAM J. Numer. Anal.*, **49** (2011), 2017–2038.

[26] A. Iserles, *A First Course in the Numerical Analysis of Differential Equations*, 2nd edition, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2009.

[27] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, vol. 23 of Applications of Mathematics (New York), Springer-Verlag, Berlin, 1992.

[28] R. Kozlov, Symmetries of systems of stochastic differential equations with diffusion matrices of full rank, *J. Phys. A*, **43** (2010), 245201, 16pp.

[29] J.-A. Lázaro-Camí and J.-P. Ortega, Reduction, reconstruction, and skew-product decomposition of symmetric stochastic differential equations, *Stoch. Dyn.*, **9** (2009), 1–46.

[30] B. Leimkuhler and S. Reich, *Simulating Hamiltonian Dynamics*, vol. 14 of Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, 2004.

[31] P. Lescot and J.-C. Zambrini, Probabilistic deformation of contact geometry, diffusion processes and their quadratures, in *Seminar on Stochastic Analysis, Random Fields and Applications V*, vol. 59 of Progr. Probab., Birkhäuser, Basel, 2008, 203–226.

[32] D. Levi, P. Olver, Z. Thomova and P. Winternitz (eds.), *Symmetries and Integrability of Difference Equations*, vol. 381 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2011, `https://doi.org/10.1007/978-3-319-56666-5`, Lectures from the Summer School (Séminaire de Máthematiques Supérieures) held at the Université de Montréal, Montréal, QC, June 8–21, 2008.

[33] D. Levi and P. Winternitz, Continuous symmetries of difference equations, *J. Phys. A*, **39** (2006), R1–R63.

[34] S. J. A. Malham and A. Wiese, Stochastic Lie group integrators, *SIAM J. Sci. Comput.*, **30** (2008), 597–617.

[35] S. V. Meleshko, Y. N. Grigoriev, N. K. Ibragimov and V. F. Kovalev, *Symmetries of Integro-Differential Equations: With Applications in Mechanics and Plasma Physics*, vol. 806, Springer Science & Business Media, 2010.

[36] G. N. Milstein, E. Platen and H. Schurz, Balanced implicit methods for stiff stochastic systems, *SIAM J. Numer. Anal.*, **35** (1998), 1010–1019 (electronic).

[37] G. N. Milstein, Y. M. Repin and M. V. Tretyakov, Numerical methods for stochastic systems preserving symplectic structure, *SIAM J. Numer. Anal.*, **40** (2002), 1583–1604 (electronic).

[38] L. M. Morato and S. Ugolini, Stochastic description of a bose-einstein condensate, *Ann. H. Poincaré*, **12** (2011), 1601–1612.

[39] B. Øksendal, *Stochastic Differential Equations*, Sixth edition, Universitext, Springer-Verlag, Berlin, 2003, An introduction with applications.

[40] P. J. Olver, *Applications of Lie Groups to Differential Equations*, vol. 107 of Graduate Texts in Mathematics, 2nd edition, Springer-Verlag, New York, 1993.

[41] É. Pardoux and P. E. Protter, A two-sided stochastic integral and its calculus, *Probab. Theory Related Fields*, **76** (1987), 15–49.

[42] G. R. W. Quispel and R. I. McLachlan, Special issue on geometric numerical integration of differential equations, *Journal of Physics A: Mathematical and General*, **39** (2006), front matter (3 pp.).

[43] L. C. G. Rogers and D. Williams, *Diffusions, Markov Processes, and Martingales,* Vol. 2, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2000, Itô calculus, Reprint of the second (1994) edition.

[44] Y. Saito and T. Mitsui, Stability analysis of numerical schemes for stochastic differential equations, *SIAM J. Numer. Anal.*, **33** (1996), 2254–2267.

[45] M. Tao, H. Owhadi and J. E. Marsden, Nonintrusive and structure preserving multiscale integration of stiff ODEs, SDEs, and Hamiltonian systems with hidden slow dynamics via flow averaging, *Multiscale Model. Simul.*, **8** (2010), 1269–1324.

[46] A. Tocino, Mean-square stability of second-order Runge-Kutta methods for stochastic differential equations, *J. Comput. Appl. Math.*, **175** (2005), 355–367.

[47] L. Wang, J. Hong, R. Scherer and F. Bai, Dynamics and variational integrators of stochastic Hamiltonian systems, *Int. J. Numer. Anal. Model.*, **6** (2009), 586–602.

[48] C. Yuan and X. Mao, Stability in distribution of numerical solutions for stochastic differential equations, *Stochastic Anal. Appl.*, **22** (2004), 1133–1150.

Received May 2017; revised July 2019.

E-mail address: francesco.devecchi@uni-bonn.de
E-mail address: andrea.romano4@studenti.unimi.it
E-mail address: stefania.ugolini@unimi.it

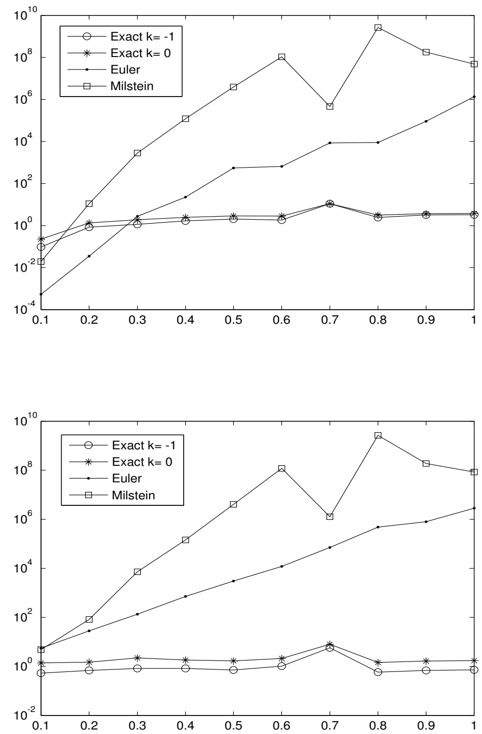


FIGURE 1. Strong and weak errors with $t \in [0.1, 1]$ and stepsize $h = 0.025$

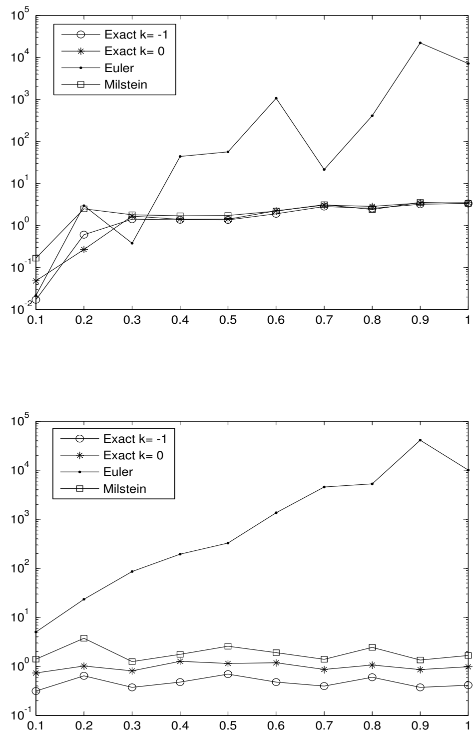


FIGURE 2. Strong and weak errors with $t \in [0.1, 1]$ and stepsize $h = 0.01$

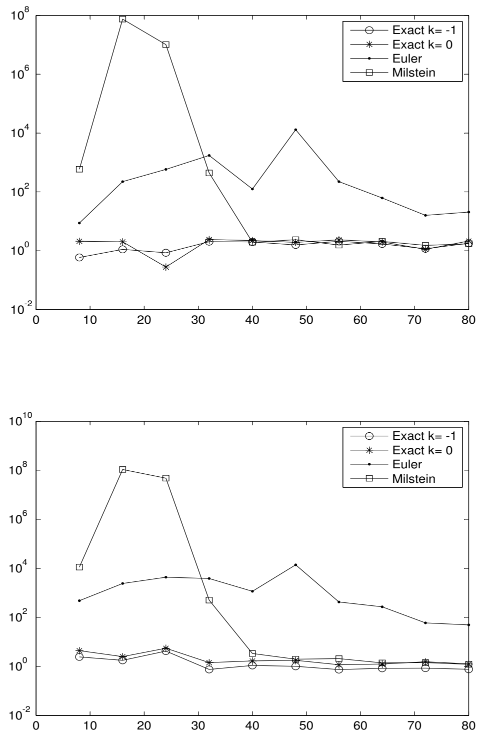


FIGURE 3. Strong and weak errors with $t = 0.5$ and step number $N = [10, 80]$

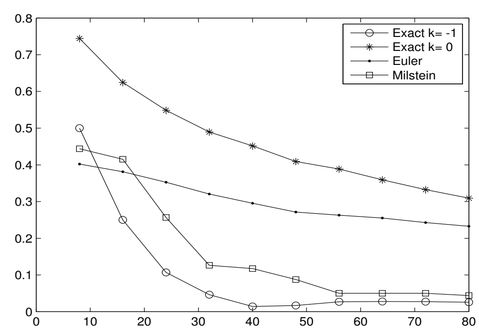
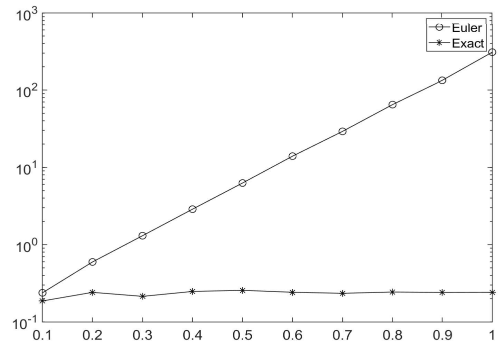


FIGURE 4. Total variation distance with $t = 0.5$ and $h \in [10, 80]$



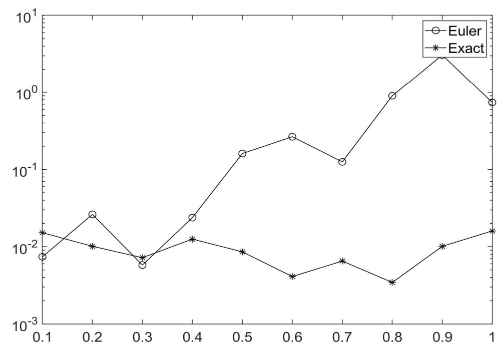


FIGURE 5. X_t strong and weak errors with $t \in [0.1, 1]$ and stepsize $h = 0.025$

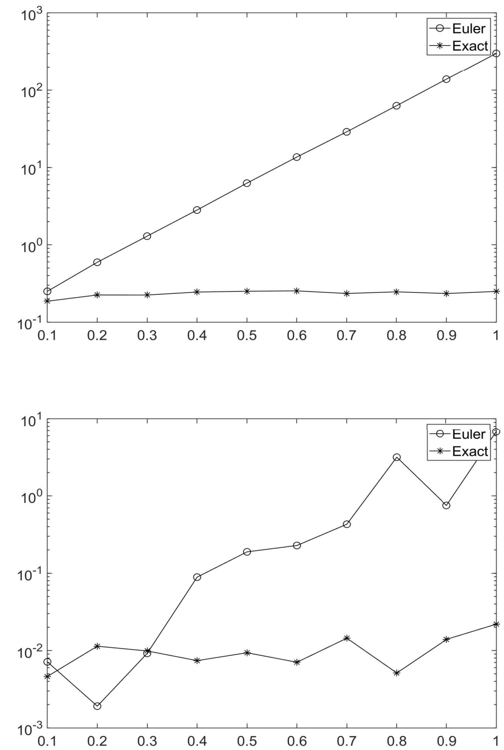


FIGURE 6. Y_t strong and weak errors with $t \in [0.1, 1]$ and stepsize $h = 0.025$

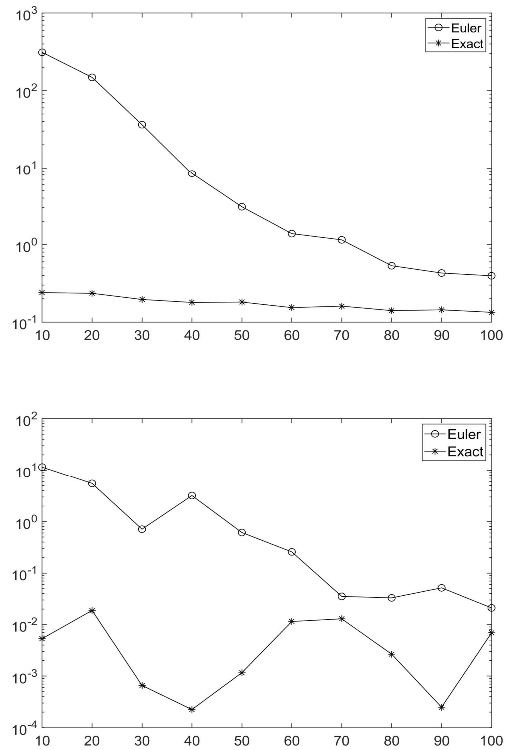


FIGURE 7. X_t strong and weak errors with $T = 1$ and step number $N = [10, 100]$

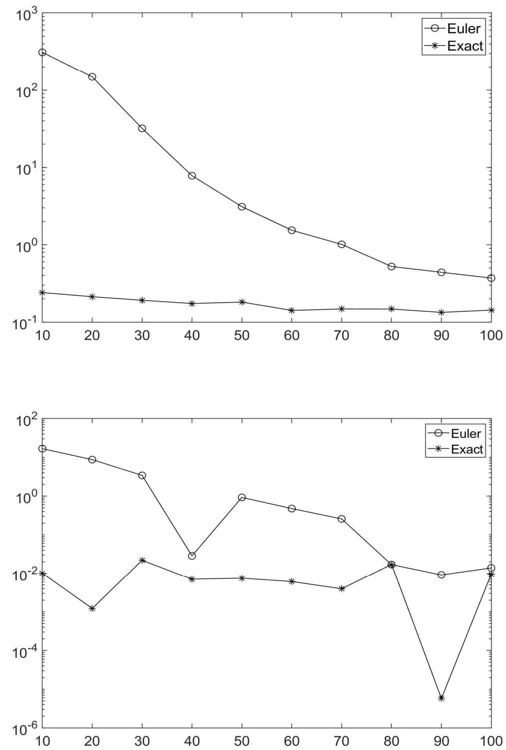


FIGURE 8. Y_t strong and weak errors with $T = 1$ and step number $N = [10, 100]$