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A Quantitative Analysis of Metrics on $\mathbb{R}^n$ with Almost Constant Positive Scalar Curvature, with Applications to Fast Diffusion Flows

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We prove a quantitative structure theorem for metrics on $\mathbb{R}^n$ that are conformal to the flat metric, have almost constant positive scalar curvature, and cannot concentrate more than one bubble. As an application of our result, we show a quantitative rate of convergence in relative entropy for a fast diffusion equation in $\mathbb{R}^n$ related to the Yamabe flow.

1 Introduction

1.1 The prescribed scalar curvature problem on the sphere

Given a n-dimensional Riemannian manifold (M, g_0), $n \geq 3$, the problem of finding a metric g conformal to g_0 whose scalar curvature R_g is equal to a prescribed function R boils down to showing the existence of a positive solution u to the nonlinear partial differential equation (PDE)

$$-\Delta_{g_0} u + \frac{n-2}{4(n-1)} R_{g_0} u = \frac{n-2}{4(n-1)} R\, u^p, \tag{1.1}$$

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where $\Delta_{g_0} = \mathrm{div}\,(\nabla_{g_0}\cdot)$ and R_{g_0} denote, respectively, the Laplace-Beltrami operator and the scalar curvature of (M, g_0), and

$$p = \frac{n+2}{n-2} = 2^\star - 1, \qquad 2^\star = \frac{2n}{n-2}.$$

Indeed, if u solves (1.1), then the metric $g = u^{p-1} g_0$ satisfies $R_g = R$.

When (M, g_0) is the round sphere then $R_{g_0} = n(n-1)$ and (1.1) can be read on $\mathbb{R}^n$ by means of the stereographic projection. More precisely, consider the inverse stereographic projection $F : \mathbb{R}^n \to \mathbb{S}^n$ defined by

$$F(x) = \left(\frac{2x}{1+|x|^2}, \frac{|x|^2 - 1}{1+|x|^2} \right). \tag{1.2}$$

Then $v : \mathbb{S}^n \to \mathbb{R}$ solves (1.1) if and only if

$$u(x) = \left(\frac{2}{1+|x|^2} \right)^{(n-2)/2} v(F(x))$$

solves

$$-\Delta u = K\, u^p \quad \text{on } \mathbb{R}^n, \tag{1.3}$$

where $K(x) = \frac{n-2}{4(n-1)} R(F(x))$.

When looking for solutions of (1.3), it is natural to impose that u satisfies

$$u > 0 \text{ on } \mathbb{R}^n \qquad \text{and} \qquad \int_{\mathbb{R}^n} |\nabla u|^2 < \infty. \tag{1.4}$$

In general, for a given function K (and even when K is just a small perturbation of a constant) there may exist no solution to (1.3) and (1.4) (see [23]), and indeed there is a vast literature dedicated to finding necessary and sufficient conditions on K in order to guarantee the solvability of (1.3) and (1.4), see for example [1, 2, 4, 5, 7, 11–13, 19, 20, 24–29]. At the same time, when K is constantly equal to some $\kappa > 0$, the problem is completely rigid. Indeed, by [21, 30], if u solves (1.3) and (1.4) with $K \equiv \kappa > 0$, then there exist $\lambda > 0$ and $z \in \mathbb{R}^n$ such that

$$u(x) = \lambda^{(n-2)/2}\, v_\kappa(\lambda(x-z)) \qquad \forall\, x \in \mathbb{R}^n,$$

where

$$v_\kappa(x) = \left(\frac{n(n-2)}{\kappa} \right)^{(n-2)/4} \frac{1}{(1+|x|^2)^{(n-2)/2}} \qquad \forall\, x \in \mathbb{R}^n. \tag{1.5}$$

1.2 Main result

The goal of this article is to give a quantitative description of solutions u to the *prescribed scalar curvature equation* (1.3) and (1.4) in the regime when K is close (in a suitable sense) to a positive constant.

In order to identify the natural space in which the distance of K to a constant should be measured, we make the following observation:

$$-\Delta u = K\,u^p \qquad \Longleftrightarrow \qquad \int_{\mathbb{R}^n} \nabla u \cdot \nabla \varphi = \int_{\mathbb{R}^n} K\,u^p \varphi \qquad \forall\, \varphi \in \dot{W}^{1,2}(\mathbb{R}^n),$$

where $\dot{W}^{1,2}(\mathbb{R}^n)$ denotes the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm $\|\nabla \cdot\|_{L^2(\mathbb{R}^n)}$. Since

$$\varphi \in \dot{W}^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n),$$

$\int_{\mathbb{R}^n} K\,u^p\varphi$ is well defined provided $K\,u^p$ belongs to the dual of $L^{2^*}(\mathbb{R}^n)$, namely $L^{2n/(n+2)}(\mathbb{R}^n)$.

This suggests the following definition:

$$\delta(u) := \|K\,u^p - K_0(u)\,u^p\|_{L^{2n/(n+2)}} = \left(\int_{\mathbb{R}^n} \left| \frac{\Delta u}{u^p} + K_0(u) \right|^{2n/(n+2)} u^{2^*} \right)^{(n+2)/2n}, \tag{1.6}$$

where

$$K_0(u) := \frac{\int_{\mathbb{R}^n} K u^{2^\star}}{\int_{\mathbb{R}^n} u^{2^\star}} = \frac{\int_{\mathbb{R}^n} \frac{-\Delta u}{u^p} u^{2^\star}}{\int_{\mathbb{R}^n} u^{2^\star}} = \frac{\int_{\mathbb{R}^n} |\nabla u|^2}{\int_{\mathbb{R}^n} u^{2^\star}}. \tag{1.7}$$

Hence, the question becomes: if $\delta(u)$ is small, can we say that u is close to a translation/dilation of $v_{K_0(u)}$?

A negative answer is given by the following simple example: given a function $v : \mathbb{R}^n \to \mathbb{R}$, set

$$v[z,\lambda](x) := \lambda^{(n-2)/2}\, v(\lambda(x-z)), \qquad x \in \mathbb{R}^n$$

and consider

$$u = \sum_{i=1}^m v_1[z_i,\lambda_i],$$

where the functions $v_1[z_i,\lambda_i]$ are supported far away from each other. Then

$$-\Delta u = -\sum_{i=1}^m \Delta v_1[z_i,\lambda_i] = \sum_{i=1}^m v_1[z_i,\lambda_i]^p = K\left(\sum_{i=1}^m v_1[z_i,\lambda_i]\right)^p = Ku^p$$

with

$$K = \frac{\sum_{i=1}^{m} v_1[z_i,\lambda_i]^p}{\left(\sum_{i=1}^{m} v_1[z_i,\lambda_i]\right)^p},$$

and it is easy to check that by taking the point z_i sufficiently far from each other one can make $\|K\,u^p - u^p\|_{L^{2n/(n+2)}}$ arbitrarily small.

As shown by Struwe's [31] (see also [22, Theorem 3.3]), this bubbling phenomenon is the only possible "bad" case. More precisely, whenever $\delta(u)$ is small, u is close to a sum of bubbles as above. Luckily, in many applications, this phenomenon can be avoided by some preliminary study of the PDE under investigation, and one can usually localize the problem in a suitable way so that, in the region under investigation, the "energy" of u is strictly less than the energy of two bubbles. Hence, we shall focus on the latter situation.

Before stating our result, we recall the definition of the Sobolev constant on $\mathbb{R}^n$,

$$S = \inf\left\{\frac{\|\nabla v\|_{L^2(\mathbb{R}^n)}}{\|v\|_{L^{2^\star}(\mathbb{R}^n)}} : v \neq 0,\ |\{|v| > t\}| < \infty\ \ \forall\, t > 0\right\}. \tag{1.8}$$

By [3, 15, 32], the family of functions $\{v_\kappa[z,\lambda]\}_{\kappa,\lambda,z}$ corresponds to the minimizers in (1.8). Hence, since $-\Delta v_\kappa = \kappa\, v_\kappa^p$, one can easily check that

$$\int_{\mathbb{R}^n} |\nabla v_\kappa|^2 = S^2 \left(\int_{\mathbb{R}^n} v_\kappa^{2^\star}\right)^{2/2^\star} = \frac{S^n}{\kappa^{(n-2)/2}} \qquad \int_{\mathbb{R}^n} v_\kappa^{2^\star} = \frac{S^n}{\kappa^{n/2}}. \tag{1.9}$$

To simplify the notation (and because for most applications this is not a real restriction), we shall assume that the energy of u is bounded by $3/2$ the energy of a single bubble. Of course $3/2$ does not play any essential role, and the proof holds when $3/2$ is replaced by any constant strictly less than 2. Also, the assumption $K_0(u) = 1$ is not restrictive, since it can always be guaranteed by rescaling u.

Theorem 1.1. Given $n \geq 3$, there exists $C_0 = C_0(n) > 0$ with the following property. Let $u \in C^\infty(\mathbb{R}^n) \cap \mathring{H}^1(\mathbb{R}^n)$ be a positive function satisfying

$$K_0(u) = 1 \qquad \text{and} \qquad \int_{\mathbb{R}^n} |\nabla u|^2 \leq \frac{3}{2} S^n.$$

Then there exist $z \in \mathbb{R}^n$ and $\lambda \in (0,\infty)$ such that

$$u = v_1[z,\lambda] + \rho,$$

where

$$\|\nabla\rho\|_{L^2(\mathbb{R}^n)} \le C_0\delta(u). \tag{1.10}$$

□

Remark 1.2. Theorem 1.1 is easily seen to be optimal. Indeed, set $v := v_1[0,1]$, let $\varepsilon > 0$ be small, and consider $u_\varepsilon := v_1 + \varepsilon\phi$ for some $\phi \in C_c^\infty(\mathbb{R}^n)$, with $\phi \ge 0$. Then

$$\Delta u_\varepsilon = \Delta v + \varepsilon\Delta\phi = v^p + \varepsilon\Delta\phi = K_\varepsilon u^p$$

with

$$K_\varepsilon := \frac{v^p + \varepsilon\Delta\phi}{u_\varepsilon^p} = 1 + \frac{[v^p - (v+\varepsilon\phi)^p] + \varepsilon\Delta\phi}{u_\varepsilon^p} = 1 + O(\varepsilon u_\varepsilon^{-p}).$$

This shows that

$$\|K_\varepsilon u_\varepsilon^p - u_\varepsilon^p\|_{L^{2n/(n+2)}} = O(\varepsilon). \tag{1.11}$$

Also, since

$$K_0(u_\varepsilon) = \frac{\int_{\mathbb{R}^n} K_\varepsilon u_\varepsilon^{2^\star}}{\int_{\mathbb{R}^n} u_\varepsilon^{2^\star}}$$

by (1.11) and Hölder inequality,

$$|K_0(u_\varepsilon) - 1| \le \frac{|\int_{\mathbb{R}^n}(K_\varepsilon - 1)u_\varepsilon^{2^\star}|}{\int_{\mathbb{R}^n} u_\varepsilon^{2^\star}} \le \left(\frac{\int_{\mathbb{R}^n} |K_\varepsilon - 1|^{2n/(n+2)} u_\varepsilon^{2^\star}}{\int_{\mathbb{R}^n} u_\varepsilon^{2^\star}}\right)^{(n+2)/2n} = O(\varepsilon).$$

Combining this bound with (1.11) we get $\delta(u_\varepsilon) \le C\varepsilon \simeq \|\nabla(\varepsilon\phi)\|_{L^2(\mathbb{R}^n)}$, which proves the optimality of our result. □

As an application of Theorem 1.1, we investigate the behavior of solutions to a fast diffusion equation in $\mathbb{R}^n$ related to the Yamabe flow.

1.3 Convergence to equilibrium: a fast diffusion equation related to the Yamabe flow

Given $m \in (0,1)$, the Cauchy problem for the fast diffusion equation is written as

$$\frac{\mathrm{d}}{\mathrm{d}t}u = \Delta(u^m) \qquad \text{in } (0,\infty)\times\mathbb{R}^n. \tag{1.12}$$

Assuming that the initial datum u_0 is non-negative and fastly decaying at infinity, it is well known that solutions to (1.12) are smooth and positive for all times if $m > m_c = (n-2)/n$, while they vanish in finite time if $m \le m_c$. There is a huge literature on the subject, and we refer the interested reader to the monograph [33] for a comprehensive overview and more references.

A case of a special interest corresponds to the choice $m = (n-2)/(n+2)$, where the equation for u is equivalent to the (not volume-preserving) Yamabe flow

$$\frac{\mathrm{d}}{\mathrm{d}t} g = -R_g\, g$$

for the metric $g_{ij}(t) = u(t)^{p-1}\, dx_i\, dx_j$ (recall that $p = 2^\star - 1$). Hence, we consider the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t} u = \Delta(u^m) \qquad \text{in } (0,\infty)\times\mathbb{R}^n, \qquad m = \frac{n-2}{n+2} = \frac{1}{p} \tag{1.13}$$

with a continuous initial datum $u_0 \ge 0$ satisfying $u_0(x) = O(|x|^{-(n+2)})$ as $|x| \to \infty$. Because of its geometric relevance, this equation has received a lot of attention. In particular, as shown in [18, Theorem 1.1] (see also [14, 34]), under the above assumptions on u_0 there exists a vanishing time $T = T(u_0) > 0$, a point $z \in \mathbb{R}^n$, and a number $\lambda > 0$, such that $u \equiv 0$ for $t \ge T$ and

$$\left\| \frac{(T-t)^{-1/(1-m)} u(t,\cdot)}{V_{1/(1-m)}[z,\lambda]^{1/m}} - 1 \right\|_{L^\infty(\mathbb{R}^n)} \to 0 \qquad \text{as } t \to T^-. \tag{1.14}$$

Later on, in [8–10, 17], the authors investigated the asymptotic of solutions for all values of $m \in (0,1)$, and proved both qualitative and quantitative convergence results under the assumption that the initial datum is trapped in between two Barenblatt solutions with the same extinction time.

As observed in [17] (see also [16] for a different but related analysis), this trapping assumption on the initial datum is very restrictive in our setting, as it completely misses the picture given by [18] and gives rise to an extinction profile different from the one in (1.14), which is believed to be the correct one for “most” initial data. Indeed, as shown in [17, Theorem 1.4], there exists a large class of initial data of the form

$$u_0(x) = \frac{C_0}{|x|^{(n+2)/2}}(1+o(1)) \qquad \text{as } |x| \to \infty, \quad C_0 > 0,$$

that do not satisfy the trapping assumption and whose solutions behave as follows: there exists a time $t_0 \in (0,T)$ (which can be explicitly computed in terms of C_0 and is given by $t_0 = \frac{C_0(n+2)}{(n-2)^2}$) such that the behavior of the solution is governed by a Barenblatt profile $B(t,\cdot)$ up to the time t_0 when the Barenblatt vanishes. Then the solution develops a singularity at $t = t_0$, and it satisfies $u(t,x) = O(|x|^{-(n+2)})$ as $|x| \to \infty$ for all $t > t_0$ (see [17, Theorem 1.4(ii)]). In particular, as observed in [17, Corollary 1.5], this allows one to apply [18, Theorem 1.1] and deduce that $u(t,x)$ exhibits the vanishing profile of a sphere, as shown by (1.14).

By exploiting our Theorem 1.1, we can improve the convergence result in (1.14) and obtain a quantitative rate of convergence under the same assumptions as in [18]. More precisely, in Section 3 we prove the following:

Theorem 1.3. Let u be a solution to the fast diffusion equation (1.13) starting from a non-negative continuous initial datum u_0 satisfying $u_0(x) = O(|x|^{-(n+2)})$ as $|x| \to \infty$. Let $T = T(u_0) > 0$ denote the vanishing time of u. Then there exist $z \in \mathbb{R}^n$, $\lambda > 0$, and a dimensional constant $\kappa(n) > 0$ such that

$$\left\| \frac{(T-t)^{-1/(1-m)}u(t,\cdot)}{v_{1/(1-m)}[z,\lambda]^{1/m}} - 1 \right\|_{L^\infty(\mathbb{R}^n)} \leq C_* \, (T-t)^{\kappa(n)} \qquad \forall\, 0 < t < T,$$

where $C_* > 0$ is a constant depending on the initial datum u_0. □

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Recall the notation

$$n \geq 3, \qquad 2^\star = \frac{2n}{n-2}, \qquad p = 2^\star - 1 = \frac{n+2}{n-2} = 1+s, \qquad s = p-1 = \frac{4}{n-2}\,.$$

We begin by observing that it is enough to prove the theorem for $u : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$u \in C^\infty(\mathbb{R}^n) \cap \mathring{H}^1(\mathbb{R}^n), \qquad u > 0 \text{ on } \mathbb{R}^n$$

and such that

$$K_0(u) = 1, \qquad \int_{\mathbb{R}^n} |\nabla u|^2 \leq \frac{3}{2} S^n, \qquad \delta = \delta(u) \leq \delta_0 \tag{2.1}$$

for a suitably small constant $\delta_0 = \delta_0(n)$.

Indeed we note that if $\delta(u) > \delta_0$, then the theorem is trivially true simply by choosing $\lambda = 1$, $z = 0$, setting

$$\rho = u - v_1$$

and then simply choosing $C_0 = C_0(n)$ large enough.

Step one. Thanks to [31], for a constant $\varepsilon_0 = \varepsilon_0(n) > 0$ to be fixed later on, we can choose δ_0 depending on ε_0 in such a way that there exist $z \in \mathbb{R}^n$, $\lambda, \alpha > 0$, such that

$$\begin{aligned} \left\| \nabla u - \alpha \nabla v_1[z, \lambda] \right\|_{L^2(\mathbb{R}^n)} &\le \varepsilon_0, \\ |\alpha - 1| &\le \varepsilon_0. \end{aligned} \tag{2.2}$$

Without loss of generality, the parameters $z \in \mathbb{R}^n$ and $\lambda, \alpha \in (0, \infty)$ can be chosen in such a way that

$$\left\| \nabla u - \alpha \nabla v_1[z, \lambda] \right\|_{L^2(\mathbb{R}^n)} = \min_{w \in \mathbb{R}^n,\, \mu, a > 0} \left\| \nabla u - a \nabla v_1[w, \mu] \right\|_{L^2(\mathbb{R}^n)}. \tag{2.3}$$

In particular, if we set

$$\begin{gathered} \rho = u - \sigma, \qquad \sigma = \alpha U, \\ U = v_1[z, \lambda], \qquad V = \left.\frac{\partial v_1[w, \mu]}{\partial \mu}\right|_{w = z, \mu = \lambda}, \qquad W^j = \left.\frac{\partial v_1[w, \mu]}{\partial w_j}\right|_{w = z, \mu = \lambda}, \end{gathered}$$

then by (2.3) we find that

$$\int_{\mathbb{R}^n} \nabla U \cdot \nabla \rho = \int_{\mathbb{R}^n} \nabla V \cdot \nabla \rho = \int_{\mathbb{R}^n} \nabla W^j \cdot \nabla \rho = 0, \qquad \forall\, 1 \le j \le n\,. \tag{2.4}$$

Also, the first bound in (2.2) gives

$$\|\nabla \rho\|_{L^2(\mathbb{R}^n)} \le \varepsilon_0\,. \tag{2.5}$$

By a spectral analysis argument (see, e.g., the appendix to [6]), (2.4) implies that

$$\int_{\mathbb{R}^n} |\nabla \rho|^2 \ge \Lambda \int_{\mathbb{R}^n} U^{p-1} \rho^2, \tag{2.6}$$

where $\Lambda = \Lambda(n)$ is such that

$$\Lambda > p\,.$$

Step two. Having in mind to exploit $\Lambda > p$, we now test the equation $-\Delta u = K\, u^p$ with ρ, and using $\int_{\mathbb{R}^n} \nabla\rho \cdot \nabla U = 0$ we get

$$\int_{\mathbb{R}^n} |\nabla\rho|^2 = \int_{\mathbb{R}^n} K\, u^p\, \rho = \int_{\mathbb{R}^n} u^p\, \rho + \int_{\mathbb{R}^n} (K-1)\, u^p\, \rho\,. \tag{2.7}$$

Since $u = \sigma + \rho$, a Taylor expansion yields

$$\int_{\mathbb{R}^n} u^p\, \rho = \int_{\mathbb{R}^n} \sigma^p \rho + p \int_{\mathbb{R}^n} \sigma^{p-1}\rho^2 + \mathrm{O}\left(\int_{\mathbb{R}^n} |\nabla\rho|^2\right)^{1+\gamma}, \tag{2.8}$$

where

$$\gamma = \min\left\{\frac{1}{2}, \frac{2}{n-2}\right\}.$$

From $\sigma^p = \alpha^p U^p = -\alpha^p \Delta U$ and using again that $\int_{\mathbb{R}^n} \nabla\rho \cdot \nabla U = 0$ we get

$$\int_{\mathbb{R}^n} |\nabla\rho|^2 - p \int_{\mathbb{R}^n} \sigma^{p-1}\rho^2 = \int_{\mathbb{R}^n} (K-1)\, u^p\, \rho + \mathrm{O}\left(\int_{\mathbb{R}^n} |\nabla\rho|^2\right)^{1+\gamma}.$$

Note that, by Hölder inequality (recall that $p = 2^\star/(2^\star)'$ and $(2^\star)' = 2n/(n+2)$)

$$\left|\int_{\mathbb{R}^n} (K-1)\, u^p\, \rho\right| \le \delta(u)\, \|\rho\|_{L^{2^\star}(\mathbb{R}^n)} \le S\, \|\nabla\rho\|_{L^2(\mathbb{R}^n)}\, \delta(u)\,. \tag{2.9}$$

Also, recalling (2.6),

$$\int_{\mathbb{R}^n} \sigma^{p-1}\rho^2 = \alpha^{p-1} \int_{\mathbb{R}^n} U^{p-1}\rho^2 \le \frac{\alpha^{p-1}}{\Lambda} \int_{\mathbb{R}^n} |\nabla\rho|^2.$$

Hence

$$\left(1 - \frac{\alpha^{p-1} p}{\Lambda}\right) \int_{\mathbb{R}^n} |\nabla\rho|^2 \le S\, \|\nabla\rho\|_{L^2(\mathbb{R}^n)}\, \delta(u) + \mathrm{O}\left(\int_{\mathbb{R}^n} |\nabla\rho|^2\right)^{1+\gamma}.$$

Since $\Lambda > p$ and $|\alpha - 1| \le \varepsilon_0$, choosing $\delta(u)$ small enough we can ensure that $1 - \frac{\alpha^{p-1} p}{\Lambda} \ge c_0 > 0$ for some dimensional constant c_0, and we get

$$c_0 \int_{\mathbb{R}^n} |\nabla\rho|^2 \le S\, \|\nabla\rho\|_{L^2(\mathbb{R}^n)}\, \delta(u) + \mathrm{O}\left(\int_{\mathbb{R}^n} |\nabla\rho|^2\right)^{1+\gamma}.$$

Since $\int_{\mathbb{R}^n} |\nabla\rho|^2$ is smaller than ε_0, we can also reabsorb the last term to conclude that $\|\nabla\rho\|_2 \le C\, \delta(u)$.

Step three. We now quantitatively control $|\alpha-1|$. To this aim, we observe that assumption $K_0(u) = 1$ is equivalent to

$$\int_{\mathbb{R}^n} |\nabla u|^2 = \int_{\mathbb{R}^n} u^{2^\star}$$

(see (1.7)). Note that, since $\|\nabla\rho\|_2 \le C\,\delta(u)$ (by Step 2),

$$\int_{\mathbb{R}^n} |\nabla u|^2 = \int_{\mathbb{R}^n} |\nabla\sigma|^2 + \int_{\mathbb{R}^n} |\nabla\rho|^2 = \alpha^2 S^n + O(\delta^2).$$

On the other hand, recalling that $\int_{\mathbb{R}^n} \sigma^p \rho = 0$ (cp. with Step 2)

$$\int_{\mathbb{R}^n} u^{2^\star} = \int_{\mathbb{R}^n} \sigma^{2^\star} + p\int_{\mathbb{R}^n} \sigma^p\rho + O(\delta^2) = \alpha^{2^*} S^n + O(\delta^2).$$

Comparing these expressions, we immediately deduce that

$$\alpha^{2^*} S^n = \alpha^2 S^n + O(\delta^2),$$

hence $|\alpha-1| \le O(\delta^2)$. Thus, if we set $\rho' := \rho + (\alpha-1)U$, we proved that

$$u = U + \rho',$$

where $\|\nabla\rho'\|_2 \le C_0\delta(u)$, as desired. ■

3 An Application to Fast Diffusion Equations: Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. Thus, we consider a continuous initial datum $u_0 \ge 0$ satisfying $u_0(x) = O(|x|^{-(n+2)})$ as $|x| \to \infty$, and u a solution of (1.13) with $u(0,\cdot) = u_0$. As explained in Section 1.3, under these assumptions the qualitative convergence result (1.14) holds, and our goal is to quantify the rate of convergence.

We first notice that (1.14) can be restated as follows (see [18, Theorem 1.1]): there exists a vanishing time $T > 0$ (depending on u_0) such that $u \equiv 0$ for $t \ge T$ and

$$\frac{u(t,x)}{(T-t)^{1/(1-m)}} = v_{1/(1-m)}[z_\infty, \lambda_\infty]^{1/m} + \theta(t,x) \quad \text{for } t < T, \tag{3.1}$$

where $\lambda_\infty > 0$, $z_\infty \in \mathbb{R}^n$, and θ satisfy

$$\sup_{x\in\mathbb{R}^n}(1+|x|^{n+2})|\theta(t,x)| \to 0 \qquad \text{as } t \to T^-. \tag{3.2}$$

For proving Theorem 1.3, we need to show the existence of $\kappa(n) > 0$ such that

$$\sup_{x\in\mathbb{R}^n}(1+|x|^{n+2})|\theta(t,x)| \le C(T-t)^{\kappa(n)} \qquad \text{for } 0<t<T\,. \tag{3.3}$$

Following [18], define

$$w(s,x) = \left.\frac{u(t,x)^m}{(T-t)^{m/(1-m)}}\right|_{t=T(1-e^{-s})}, \qquad W_\infty(x) = v_{1/(1-m)}[z_\infty,\lambda_\infty](x),$$

so that (3.1) and (3.2) imply, setting for short $w(s) = w(s,\cdot)$,

$$\lim_{s\to+\infty}\|w(s)-W_\infty\|_{L^{2^\star}(\mathbb{R}^n)} = 0\,. \tag{3.4}$$

Recalling the notation $p = 2^\star - 1$, we see that w satisfies

$$\frac{\mathrm{d}}{\mathrm{d}s}w^p = \Delta w + \frac{1}{1-m}\,w^p \qquad \text{on } (0,\infty)\times\mathbb{R}^n, \tag{3.5}$$

while of course $W = W_\infty$ is a solution to

$$\Delta W + \frac{1}{1-m}\,W^p = 0 \qquad \text{on } \mathbb{R}^n\,. \tag{3.6}$$

Let us consider the functional

$$J[v] = \int_{\mathbb{R}^n}\frac{|\nabla v|^2}{2} - \frac{1}{1-m}\int_{\mathbb{R}^n}\frac{v^{2^\star}}{2^\star}\,.$$

By (3.5) we compute

$$\frac{\mathrm{d}}{\mathrm{d}s}J[w(s)] = -\int_{\mathbb{R}^n}\left(\Delta w(s) + \frac{1}{1-m}w(s)^p\right)\frac{\mathrm{d}}{\mathrm{d}s}w(s) = -\frac{1}{p}\int_{\mathbb{R}^n}\left(\frac{\Delta w(s)}{w(s)^p} + \frac{1}{1-m}\right)^2 w(s)^{2^\star}, \tag{3.7}$$

so that $s \mapsto J[w(s)]$ is decreasing, with $J[w(s)] \ge J[W_\infty]$ by Fatou's lemma and (3.4). Exploiting the fact that

$$\mathbb{R} \ni c \mapsto \int_{\mathbb{R}^n}\left(\frac{\Delta w(s)}{w(s)^p} - c\right)^2 w(s)^{2^\star}$$

attains its minimum at

$$c = \frac{\int_{\mathbb{R}^n}\frac{\Delta w(s)}{w(s)^p}\,w(s)^{2^\star}}{\int_{\mathbb{R}^n}w(s)^{2^\star}} = -\frac{\int_{\mathbb{R}^n}|\nabla w(s)|^2}{\int_{\mathbb{R}^n}w(s)^{2^\star}} = -K_0(w(s))$$

by Hölder inequality we find that

$$\begin{aligned}\delta(w(s))^2 &\le \left(\int_{\mathbb{R}^n} w(s)^{2^\star}\right)^{2/n} \int_{\mathbb{R}^n} \left(\frac{\Delta w(s)}{w(s)^p} + K_0(w(s))\right)^2 w(s)^{2^\star} \\ &\le C(n) \int_{\mathbb{R}^n} \left(\frac{\Delta w(s)}{w(s)^p} + \frac{1}{1-m}\right)^2 w(s)^{2^\star}.\end{aligned} \tag{3.8}$$

Hence, thanks to (3.4), there exists s_0 (depending on the initial datum u_0) such that

$$c(n) \le \int_{\mathbb{R}^n} w(s)^{2^\star} \le C(n) \qquad \forall\, s \ge s_0\,. \tag{3.9}$$

Combining (3.7)–(3.9), we find

$$\delta(w(s))^2 \le -C(n)\,\frac{\mathrm{d}}{\mathrm{d}s} J[w(s)] \qquad \forall\, s \ge s_0, \tag{3.10}$$

so that

$$\int_{s_0}^{\infty} \delta(w(s))^2\,\mathrm{d}s \le C(n)\,(J[w(s_0)] - J[w(\infty)]) \le C(n)\,(J[w(s_0)] - J[W_\infty]) < \infty\,.$$

In particular we can find a sequence $s_j \to \infty$ such that $\delta(w(s_j)) \to 0$ as $j \to \infty$. We can thus apply Struwe's theorem to $\{w(s_j)\}_{j\in\mathbb{N}}$. Since (3.4) excludes bubbling, we conclude that

$$\lim_{s\to\infty} J[w(s)] = \lim_{j\to\infty} J[w(s_j)] = J[W_\infty]\,.$$

Furthermore, using again (3.4), this implies

$$\lim_{s\to\infty} \int_{\mathbb{R}^n} |\nabla w(s)|^2 = \int_{\mathbb{R}^n} |\nabla W_\infty|^2 \tag{3.11}$$

and thus also

$$\lim_{s\to\infty} \|\nabla w(s) - \nabla W_\infty\|_{L^2(\mathbb{R}^n)} = 0, \qquad \lim_{s\to\infty} K_0(w(s)) = \frac{\int_{\mathbb{R}^n} |\nabla W_\infty|^2}{\int_{\mathbb{R}^n} W_\infty^{2^\star}} = \frac{1}{1-m}\,. \tag{3.12}$$

Now, for any $s > 0$, denote by $W(s)$ the unique minimizer of

$$W \mapsto \|\nabla w(s) - \nabla W\|_{L^2(\mathbb{R}^n)} \tag{3.13}$$

among all positive functions W satisfying (3.6). Then, setting $w(s) = W(s) + \rho(s)$,

$$\int_{\mathbb{R}^n} \nabla W(s)\cdot\nabla\rho(s) = 0 = \int_{\mathbb{R}^n} W(s)^p \rho(s), \qquad \lim_{s\to\infty} \|\nabla\rho(s)\|_{L^2(\mathbb{R}^n)} = 0, \tag{3.14}$$

where the last limit follow by (3.12) and by the minimality property of $W(s)$. Since J is constant on solutions of (3.6) (so $J[W(s)] = J[W_\infty]$), we can expand J around $W(s)$ with the aid of (3.14), to find out that

$$\begin{aligned} I[w(s)] &:= J[w(s)] - J[W_\infty] = J[w(s)] - J[W(s)] \\ &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\rho(s)|^2 - \frac{1}{(1-m)\,2^\star}\int_{\mathbb{R}^n}\Big(\big(W(s)+\rho(s)\big)^{2^\star} - W(s)^{2^\star} - 2^\star\, W(s)^p\,\rho(s)\Big) \\ &\approx \int_{\mathbb{R}^n} |\nabla\rho(s)|^2 - \frac{p}{1-m}\int_{\mathbb{R}^n} W(s)^{p-1}\rho(s)^2 \\ &\approx \int_{\mathbb{R}^n} |\nabla\rho(s)|^2 = \int_{\mathbb{R}^n} |\nabla w(s) - \nabla W(s)|^2 \qquad \forall\, s \geq s_0, \end{aligned} \tag{3.15}$$

where $a \approx b$ means that $a/C(n) \leq b \leq C(n)\,a$ for some positive dimensional constant $C(n)$, the second $\approx$ follows from the spectral gap estimate (2.6).

We now notice that by (3.11), up to possibly increase the value of s_0, we can apply Theorem 1.1 to each $w(s)$ with $s \geq s_0$: in particular, for every $s \geq s_0$ there exists $\overline{W}(s)$ such that

$$-\Delta\overline{W}(s) = K_0(w(s))\,\overline{W}(s)^p, \qquad \|\nabla w(s) - \nabla\overline{W}(s)\|_{L^2(\mathbb{R}^n)} \leq C(n)\,\delta(w(s))\,. \tag{3.16}$$

By (3.9) and by (3.7),

$$\begin{aligned} \left|K_0(w(s)) - \frac{1}{1-m}\right|^2 &= \left|\frac{\int_{\mathbb{R}^n}\left(\frac{\Delta w(s)}{w(s)^p} + \frac{1}{1-m}\right) w(s)^{2^\star}}{\int_{\mathbb{R}^n} w(s)^{2^\star}}\right|^2 \\ &\leq C(n)\int_{\mathbb{R}^n}\left(\frac{\Delta w(s)}{w(s)^p} + \frac{1}{1-m}\right)^2 w(s)^{2^\star} = -C(n)\,\frac{\mathrm{d}}{\mathrm{d}s}J[w(s)]\,. \end{aligned} \tag{3.17}$$

Setting $\alpha(s) = [(1-m)\,K_0(w(s))]^{1/(p-1)}$, the function $\hat{W}(s) = \alpha(s)\,\overline{W}(s)$ satisfies (3.6), with

$$\|\nabla\hat{W}(s) - \nabla\overline{W}(s)\|_{L^2(\mathbb{R}^n)} \leq C(n)\left|K_0(w(s)) - \frac{1}{1-m}\right|,$$

and hence, by triangle inequality, (3.16), (3.10), and (3.17),

$$\|\nabla w(s) - \nabla\hat{W}(s)\|^2_{L^2(\mathbb{R}^n)} \leq C(n)\left(\delta(w(s))^2 + \left|K_0(w(s)) - \frac{1}{1-m}\right|^2\right) \leq -C(n)\,\frac{\mathrm{d}}{\mathrm{d}s}J[w(s)]\,.$$

By the minimality property of $W(s)$ and since $J[w(s)]$ and $I[w(s)]$ differ by a constant, we conclude that

$$\|\nabla\rho(s)\|^2_{L^2(\mathbb{R}^n)} = \|\nabla w(s) - \nabla W(s)\|^2_{L^2(\mathbb{R}^n)} \leq -C(n)\,\frac{\mathrm{d}}{\mathrm{d}s}I[w(s)]$$

for every $s \geq s_0$, and thus, by (3.15),

$$\frac{\mathrm{d}}{\mathrm{d}s} I[w(s)] \leq -C(n)\, I[w(s)] \qquad \forall\, s \geq s_0\,.$$

This proves that, for a constant C_* depending on the initial datum u_0,

$$I[w(s)] \leq C_*\, \mathrm{e}^{-\kappa(n)\, s} \qquad \forall\, s \geq 0 \tag{3.18}$$

(note that the above inequality is trivially true on $[0, s_0]$ by choosing C_* large enough) and hence, applying (3.15) again, we deduce that

$$\int_{\mathbb{R}^n} |\nabla w(s) - \nabla W(s)|^2 \leq C_*\, \mathrm{e}^{-\kappa(n)\, s} \qquad \forall\, s \geq 0\,. \tag{3.19}$$

In other words, we have proven that for all s there exists a function $W(s)$ solving (3.6) which is exponentially close to $w(s)$. To conclude the result, we need to show that $W(s)$ is exponentially close to W_∞. To this aim we notice that, from (3.5),

$$\frac{\mathrm{d}}{\mathrm{d}s} w^{2^\star} = \frac{p+1}{p} w \frac{\mathrm{d}}{\mathrm{d}s} w^p = \frac{2n}{n+2} \left(\Delta w + \frac{1}{1-m} w^p \right) w,$$

so we obtain

$$|w(s) - w(t)|^{2^\star} \leq |w(s)^{2^*} - w(t)^{2^*}| \leq C(n) \left| \int_t^s \left(\frac{\Delta w(r)}{w(r)^p} + \frac{1}{1-m} \right) w(r)^{2^*} \mathrm{d}r \right|,$$

and Hölder inequality yields

$$\int_{\mathbb{R}^n} |w(s) - w(t)|^{2^*} \leq C(n) \int_t^s \delta(w(r)) \mathrm{d}r \quad \forall\, s > t > 0\,. \tag{3.20}$$

Now, by (3.10) and (3.18)

$$\left(\int_k^{k+1} \delta(w(r))\, \mathrm{d}r \right)^2 \leq \int_k^{k+1} \delta(w(r))^2 \leq C(n) \left(I[w(k)] - I[w(k+1)] \right) \leq C_*\, \mathrm{e}^{-\kappa(n)\, k}$$

which implies

$$\int_t^\infty \delta(w(s))\, \mathrm{d}s \leq C_* \mathrm{e}^{-c(n)t} \qquad \forall\, t > 0\,.$$

Hence from (3.20) we obtain

$$\int_{\mathbb{R}^n} |w(s) - w(t)|^{2^*} \leq C_* \mathrm{e}^{-c(n)t} \qquad \forall\, s > t > 0\,.$$

From Sobolev inequality and (3.19), it follows that $\{W(s)\}_{s>0}$ is a Cauchy family in $L^{2^*}(\mathbb{R}^n)$ satisfying the exponential bound

$$\int_{\mathbb{R}^n} |W(s) - W(t)|^{2^*} \leq C_* \mathrm{e}^{-c(n)t} \qquad \forall\, s > t > 0\,.$$

Since $\|\nabla W(s) - \nabla W_\infty\|_{L^2(\mathbb{R}^n)} \to 0$ as $s \to \infty$ (see (3.12) and (3.19)), we conclude that

$$\int_{\mathbb{R}^n} |W_\infty - W(t)|^{2^*} \leq C_* \mathrm{e}^{-c(n)t} \qquad \forall\, t > 0,$$

that combined with (3.19) and Sobolev inequality yields

$$\int_{\mathbb{R}^n} |W_\infty - w(t)|^{2^*} \leq C_* \mathrm{e}^{-c(n)t} \qquad \forall\, t > 0\,. \tag{3.21}$$

Now, to conclude the proof, we argue as follows: let $F : \mathbb{S}^n \to \mathbb{R}^n$ denote the inverse stereographic projection (see (1.2)), and let $v(s) : \mathbb{S}^n \to \mathbb{R}$ be defined from $w(s)$ via the transformation

$$w(s,x) = \left(\frac{2}{1+|x|^2}\right)^{(n-2)/2} v(s, F(x))\,. \tag{3.22}$$

Then $v(s)$ solves the equation

$$\frac{\mathrm{d}}{\mathrm{d}s} v^p = \Delta_{\mathbb{S}^n} v + \frac{1}{1-m} v^p - \frac{n(n-2)}{4} v \qquad \text{on } (0,\infty) \times \mathbb{S}^n \tag{3.23}$$

(see for instance [25] and [18, Equation (2.3)]) and (3.21) translates into

$$\int_{\mathbb{S}^n} |v(s) - v_\infty|^{2^*} \leq C_*\, \mathrm{e}^{-\kappa(n)\, s} \qquad \forall\, s \geq 0, \tag{3.24}$$

where v_∞ is the stationary solution of (3.23) corresponding to W_∞ under the transformation (3.22). Since $v(s)$ is uniformly bounded away from zero and infinity for s large (see [18, Proposition 5.1]), it follows by (3.24) and parabolic regularity that (up to replacing $\kappa(n)$ by $\kappa(n)/2$)

$$\|v(s) - v_\infty\|_{L^\infty(\mathbb{S}^n)} \leq C_*\, \mathrm{e}^{-\kappa(n)\, s} \qquad \forall\, s \geq 0\,.$$

Going back to the original variables, this implies that (3.3) holds, concluding the proof.

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