

GRADIENT ESTIMATES FOR THE PERFECT CONDUCTIVITY PROBLEM IN ANISOTROPIC MEDIA

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ABSTRACT. We study the perfect conductivity problem when two perfectly conducting inclusions are closely located to each other in an anisotropic background medium. We establish optimal upper and lower gradient bounds for the solution in any dimension which characterize the singular behavior of the electric field as the distance between the inclusions goes to zero.

RESUMÉ. On étudie le problème de conductivité parfaite pour deux objets parfaitement conducteurs proches l'un de l'autre et immergés dans un milieu anisotrope. On établit des estimations optimales pour le gradient de la solution en dimension quelconque qui caractérise le comportement singulier du champ électrique quand la distance entre les objets tend vers zéro.

1. INTRODUCTION

When two perfectly conducting inclusions are located closely to each other, the electric field may become arbitrarily large as the distance between the inclusions goes to zero. We aim at establishing optimal estimates for the electric field as the distance between the inclusions goes to zero. The background medium may be anisotropic, with anisotropy determined by a norm in \mathbb{R}^N , $N \geq 2$. Hence, the governing equation is nonlinear and, differently from most of the results available in literature, the problem has to be studied by using nonlinear techniques.

1.1. Gradient estimates for the conductivity problem. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a domain representing the background medium. Denoting the two inclusions by $D_\delta^1, D_\delta^2 \subset \Omega$, where δ is the (Euclidean) distance between D_δ^1 and D_δ^2 which is assumed to be *small*, the perfectly conductivity problem is formulated as follows

$$(1.1) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega_\delta \\ |\nabla u| = 0 & \text{in } D_\delta^i, i = 1, 2, \\ \int_{\partial D_\delta^i} u_\nu = 0 & i = 1, 2, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $\varphi \in C^0(\partial\Omega)$ is some given potential prescribed on the boundary of Ω .

Problem (1.1) may be regarded as a conductivity problem in the context of electromagnetism or as an anti-plane elasticity problem in the context of elasticity, and the gradient of the solution u is either the electrical field or the stress, respectively. Furthermore, problem (1.1) may be seen as a limit case (for $k \rightarrow +\infty$) of the classical conductivity problem

$$(1.2) \quad \begin{cases} \operatorname{div}(a_k(x)\nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where

$$a_k(x) = \begin{cases} 1, & \Omega \setminus (D_\delta^1 \cup D_\delta^2), \\ k, & D_\delta^1 \cup D_\delta^2, \end{cases}$$

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with $k \in (0, +\infty)$ (see for instance [7]).

Assuming that D_δ^1 and D_δ^2 are smooth and far away from the boundary of Ω , the problem of estimating $|\nabla u|$ as δ goes to zero was first raised in [5] in relation to stress analysis of composites and many results have been obtained in the last two decades.

Regarding the classical conductivity problem (1.2) (so that $k > 0$ is finite), in [5] the authors observed numerically that $\|\nabla u_\delta\|_{L^\infty(\Omega)}$ is bounded independently of the distance δ between D_δ^1 and D_δ^2 . This result was proved rigorously by Bonnetier and Vogelius [17] for $N = 2$ and assuming D_δ^1 and D_δ^2 to be two unit balls, and it was extended by Li and Vogelius in [36] to general second order elliptic equations with piecewise smooth coefficients (see also [33] where Li and Nirenberg considered general second order elliptic systems).

When k degenerates ($k \rightarrow 0$ or $k \rightarrow +\infty$) the scenario is very different: the gradient of the solution may be unbounded as $\delta \rightarrow 0$ and the blow-up rate depends on the dimension. Indeed, it has been proved that the optimal blow-up rate of $|\nabla u|$ is $\delta^{-1/2}$ for $N = 2$, it is $(\delta|\log \delta|)^{-1}$ for $N = 3$ and δ^{-1} for $N \geq 4$, see [1, 2, 3, 6, 7, 8, 9, 10, 11, 26, 27, 28, 30, 31, 32, 35, 42, 43] and references therein.

1.2. The anisotropic conductivity problem. Our goal is to obtain gradient estimates for the perfectly conductivity problem when the background medium is anisotropic, with anisotropy described by a norm H . More precisely, the involved anisotropy arises from replacing the Euclidean norm of the gradient with an arbitrary norm in the associated variational integrals.

The kind of anisotropy considered in this paper has been widely studied in the field of anisotropic geometric functionals in the mathematical theory of crystals and composites which goes back to Wulff [44]. Indeed, variational problems in anisotropic media naturally arise in the study of crystals and whenever the microscopic environment of the interface of a medium is different from the one in the bulk of the substance so that anisotropic surface energies have to be considered. Moreover, these kinds of anisotropy are of strong interest in elasticity, noise-removal procedures in digital image processing, crystalline mean curvature flows and crystalline fracture theory. The literature is very wide and we just mention [12, 13, 14, 15, 16, 18, 21, 22, 23, 29, 34, 37, 40, 41] and references therein for an interested reader.

In order to properly state the problem, it is convenient to look at problem (1.2) from a variational point of view. More precisely, problem (1.2) can be seen as the Euler-Lagrange equation of the variational problem

$$\min_{v \in W_\varphi^{1,2}(\Omega)} I[v],$$

where

$$I[v] = \frac{1}{2} \int_\Omega a_k(x) |\nabla v|^2,$$

and

$$W_\varphi^{1,2}(\Omega) = \{v \in W^{1,2}(\Omega) : v = \varphi \text{ on } \partial\Omega\}.$$

It is well-known that there exists a unique solution $u \in W^{1,2}(\Omega)$ to (1.2), which is also the minimizer of I on $W_\varphi^{1,2}(\Omega)$ (see for instance [6]).

Analogously, the extreme conductivity problem (1.1) can be seen as the Euler-Lagrange equation of the variational problem

$$\min_{v \in W_\varphi^{1,2}(\Omega)} \left\{ \frac{1}{2} \int_\Omega |\nabla v|^2 dx : |\nabla v| = 0 \text{ in } D_\delta^i, i = 1, 2 \right\}.$$

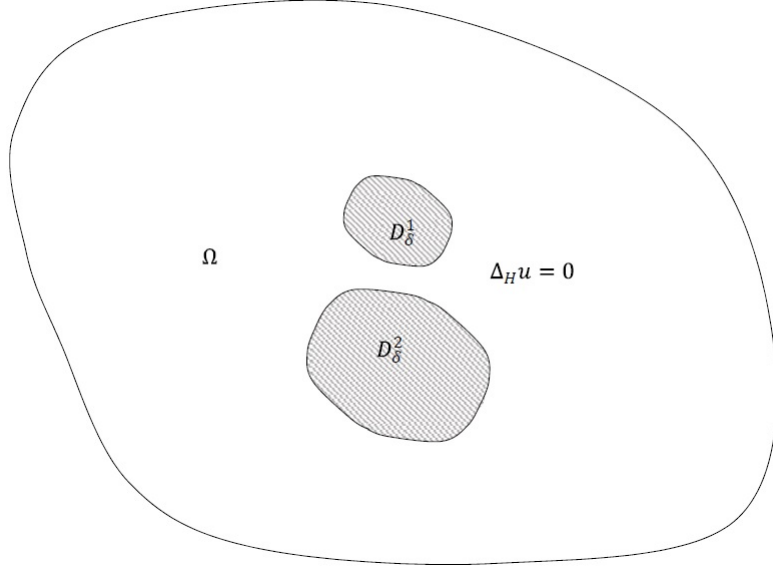


FIGURE 1. Two perfectly conducting inclusions D_δ^1 and D_δ^2 are immersed in anisotropic matrix Ω .

When the background medium is anisotropic (see Fig.1) the corresponding variational problem is given by

$$(1.3) \quad \min_{v \in W_\varphi^{1,2}(\Omega)} \left\{ \frac{1}{2} \int_\Omega H(\nabla v)^2 dx : H(\nabla v) = 0 \text{ in } D_\delta^i, i = 1, 2 \right\},$$

where H is a norm in \mathbb{R}^N , $N \geq 2$; moreover, we shall assume that H^2 is strictly convex and of class $C^3(\mathbb{R}^N \setminus \{O\})$. Since H^2 is a convex function with quadratic growth, problem (1.3) has a solution for every bounded open set Ω and, since H^2 is strictly convex and sufficiently smooth, the solution is unique. Moreover (see Appendix A), the Euler-Lagrange equation associated to (1.3) is

$$(1.4) \quad \begin{cases} \Delta_H u_\delta = 0 & \text{in } \Omega_\delta, \\ H(\nabla u_\delta) = 0 & \text{in } \overline{D}_\delta^i, i = 1, 2, \\ \int_{\partial D_\delta^i} H(\nabla u_\delta) \nabla_\xi H(\nabla u_\delta) \cdot \nu ds = 0 & i = 1, 2, \\ u_\delta = \varphi(x) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega_\delta = \Omega \setminus (\overline{D}_\delta^1 \cup \overline{D}_\delta^2)$, ν is the outward normal to ∂D_δ^i , and Δ_H denotes the Finsler Laplacian

$$\Delta_H u_\delta = \operatorname{div}(H(\nabla u_\delta) \nabla_\xi H(\nabla u_\delta)),$$

which has to be understood in the weak sense

$$\int_{\Omega_\delta} H(\nabla u_\delta) \nabla_\xi H(\nabla u_\delta) \cdot \nabla \phi dx = 0 \quad \text{for any } \phi \in C_0^1(\Omega_\delta).$$

Here and in the following, for $\nabla_\xi H(\nabla u)$ we mean the gradient of H evaluated at $\nabla u(x)$, for $x \in \overline{\Omega}$. To avoid a confusing notation, we will use the variable x for a point in the ambient space $\Omega \subset \mathbb{R}^N$, and the variable ξ for a vector in the dual space (which is the ambient space of ∇u).

Coming back to problem (1.4), we notice that u_δ is constant on each particle D_δ^i with $i = 1, 2$, i.e.

$$(1.5) \quad u_\delta = \mathcal{U}_\delta^i \quad \text{on } D_\delta^i,$$

with $\mathcal{U}_\delta^i \in \mathbb{R}$, $i = 1, 2$. We emphasize that \mathcal{U}_δ^1 and \mathcal{U}_δ^2 may be different, and their values are unknown and are determined by solving the minimization problem (1.3).

When $\delta = 0$, the corresponding perfectly conductivity problem is given by

$$(1.6) \quad \begin{cases} \Delta_H u_0 = 0 & \text{in } \Omega_0, \\ H(\nabla u_0) = 0 & \text{in } D_0^i, \quad i = 1, 2, \\ \sum_{i=1,2} \int_{\partial D_0^i} H(\nabla u_0) \nabla_\xi H(\nabla u_0) \cdot \nu ds = 0, \\ u_0 = \varphi(x) & \text{on } \partial\Omega. \end{cases}$$

We notice that the third condition in (1.6) is different from third condition in (1.4), since in (1.6) it is required that the sum of the two integrals on ∂D_0^1 and ∂D_0^2 vanishes. It is important to emphasize that the solution u_0 of (1.6) is not the limit of u_δ as $\delta \rightarrow 0^+$. Even if there is some connection between u_δ and u_0 (see discussion below on the parameter \mathcal{R}_0), the behaviour of u_δ and u_0 is very different close to the limit touching point between the two inclusions. As we will show, $H(\nabla u_0)$ is bounded in Ω_0 , while $H(\nabla u_\delta)$ may have a blow-up at the limit for $\delta \rightarrow 0^+$. Understanding this phenomenon is the main goal of this paper, and the blow-up of ∇u_δ will be characterized by the following quantity

$$(1.7) \quad \mathcal{R}_0 = \int_{\partial D_0^1} H(\nabla u_0) \nabla H(\nabla u_0) \cdot \nu ds.$$

1.3. Main result. The goal of this paper is to study the gradient blow-up for problem (1.4) under suitable regularity assumptions on the norm H . Before describing the main results, we recall some basic facts about norms in \mathbb{R}^N (see Section 2 for more details).

Given a norm H in \mathbb{R}^N , we denote by H_0 the dual norm. We recall that the sets of the form $\{H_0(x - x_0) = \text{const}\}$ are called *Wulff shapes* (or *anisotropic balls*).

Let $\delta \geq 0$ and let D_δ^1 and D_δ^2 be two perfectly conducting inclusions with $\overline{D_\delta^1}, \overline{D_\delta^2} \subset \Omega$ which are at distance δ one from each other. We define

$$\Omega_\delta = \Omega \setminus \left(\overline{D_\delta^1} \cup \overline{D_\delta^2} \right),$$

so that, when the inclusions touch at the limit $\delta = 0$, we write

$$\Omega_0 = \Omega \setminus \left(\overline{D_0^1} \cup \overline{D_0^2} \right).$$

We assume that at the limit the two particles touch only at the origin, so that

$$\overline{D_0^1} \cap \overline{D_0^2} = \{O\}.$$

In this paper we consider the case when D_δ^1 and D_δ^2 are two Wulff shapes of radius R_1 and R_2 , respectively, i.e.

$$D_\delta^i = B_{H_0}(x_\delta^i, R_i),$$

with $x_\delta^i = (0, \dots, 0, t_\delta^i)$, with $t_\delta^i \in \mathbb{R}$ such that

$$H_0(x_\delta^i) = R_i + \frac{\delta}{2}, \quad i = 1, 2.$$

Assuming that D_δ^1 and D_δ^2 are two Wulff shapes simplifies the calculations and the exposition. The approach can be adapted to study inclusions with boundary of class C^3 which are strictly convex close to the (unique) touching point.

Regarding the geometry of the problem, we recall that if x is a tangency point between two Wulff shapes, then x lies on the segment joining the centers of the two sets (see Fig. 2), which is parallel to $\nabla H(\nu(x))$, where ν is the Euclidean normal (see Remark 2.3 below for a proof).

We assume that

$$(1.8) \quad \text{dist}_{H_0}(\partial\Omega, \overline{D_\delta^1} \cup \overline{D_\delta^2}) \geq K,$$

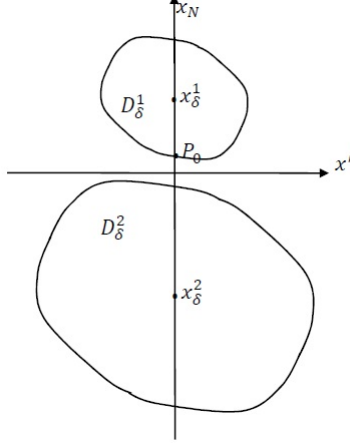


FIGURE 2. The point P_0 is such that $P_0 = R_1 \hat{P} + x_\delta^1$.

for some fixed $K > 0$ and that the distance between the two (anisotropic) balls is very small, so that

$$\text{dist}_{H_0}(D_\delta^1, D_\delta^2) = \delta,$$

for some $0 < \delta \leq \delta_0$. Here, dist_{H_0} denotes the distance in the ambient norm H_0 .

Let $\hat{P} = (0, \dots, 0, t_0)$ be such that $\hat{P} \in \partial B_{H_0}(0, 1)$ and consider the matrix $\nabla^2 H_0(\hat{P})$.¹ We denote by \mathcal{Q} the matrix obtained by considering the first $N - 1$ rows and $N - 1$ columns of $\nabla^2 H_0(\hat{P})$, i.e.

$$(1.9) \quad \mathcal{Q} = \begin{pmatrix} \partial_{\xi_1 \xi_1}^2 H_0(\hat{P}) & \dots & \partial_{\xi_1 \xi_{N-1}}^2 H_0(\hat{P}) \\ \vdots & \ddots & \vdots \\ \partial_{\xi_{N-1} \xi_1}^2 H_0(\hat{P}) & \dots & \partial_{\xi_{N-1} \xi_{N-1}}^2 H_0(\hat{P}) \end{pmatrix},$$

and recall the definition of *anisotropic normal* ν_H at a point x , which is given by

$$\nu_H(x) = \nabla_\xi H(\nu(x)),$$

where $\nu(x)$ denotes the outward Euclidean normal at x . Our main result is the following.

Theorem 1.1. *Let u_δ be the solution to (1.4) and let \mathcal{R}_0 be given by (1.7). For any fixed $\tau \in (0, 1/2]$ we have*

$$(1 - \tau)C_* \Phi_N(\delta) + o(\Phi_N(\delta)) \leq \|\nabla u_\delta\|_{L^\infty(\Omega_\delta)} \leq (1 + \tau)C_* \Phi_N(\delta) + o(\Phi_N(\delta))$$

as $\delta \rightarrow 0^+$, with

$$\Phi_N(\delta) = \begin{cases} \frac{1}{\sqrt{\delta}} & N = 2, \\ \frac{1}{\delta |\ln \delta|} & N = 3, \\ \frac{1}{\delta} & N \geq 4, \end{cases}$$

and

$$C_* = \left(\frac{R_1 + R_2}{2R_1 R_2} \right)^{\frac{N-1}{2}} (\det(\mathcal{Q}))^{\frac{N-1}{2}} \mathcal{R}_0 C,$$

where \mathcal{Q} is given by (1.9) and C depends on N and $\nu_H(\hat{P}) \cdot \nu(\hat{P})$.

¹Notice that, even if \hat{P} is not univocally determined (i.e. there are two points of $\partial B_{H_0}(0, 1)$ lying on the x_N -axis), the matrix $\nabla^2 H_0(\hat{P})$ is well defined because H_0 is centrally symmetric.

We stress that the estimates in Theorem 1.1 are optimal, in the sense that they give the optimal rate of blow up of the gradient as $\delta \rightarrow 0$. In the Euclidean case (i.e. when $H(\cdot) = |\cdot|$) we obtain the same rate of blow up as in [7]. We also obtain something more: the estimates in Theorem 1.1 almost provide a complete characterization of the leading term in the blow up. Indeed, one can choose τ arbitrarily small and get closer and closer to the sharp characterization of the blow up. The reason why we do not obtain the sharp characterization is purely technical, and how to obtain the sharp characterization is an open problem.

The strategy that we use to prove our main result has some remarkable difference compared to the one which is typically used in the Euclidean case. Indeed, in the latter case the usual approach is to use the linearity of the Laplace operator and decompose the solution u_δ in two parts:

$$(1.10) \quad u_\delta = v_\delta + w_\delta,$$

where v_δ completely characterizes the asymptotic behavior of the blow-up of the gradient of u_δ and $|\nabla w_\delta|$ is uniformly bounded independently of δ .

Since Δ_H is not linear unless H is an affine transformation of the Euclidean norm, we have to deal with a nonlinear problem and writing u_δ as in (1.10) is not helpful. Thus we first prove that the gradient is uniformly bounded away from a *small* neighborhood of the touching point and we prove, in that region, the $C^{1,\alpha}$ convergence of u_δ to u_0 (the solution of (1.6)). Then we find estimates on the gradient in a neighborhood of the touching point and we prove optimal gradient bounds by using comparison principles and a suitable P -function. Our approach is purely nonlinear, and we take inspiration from [27] where the authors study the conductivity problem in the Euclidean case for the p -Laplacian, with $p > N$. However, due to the presence of anisotropy and since $p \leq N$ in our case, there is some relevant difference between the two problems.

The paper is organized as follows. In Section 2 we recall some basic facts about norms in \mathbb{R}^N and about the Finsler (or anisotropic) Laplace operator. Section 3 is devoted to prove some maximum principle, and we introduce a P -function which is suitable for the problem. In Section 4 we prove uniform bounds on the gradient of the solution at points which are far from the touching point. Finally, in Section 5 we complete the proof of Theorem 1.1. The paper ends with two Appendixes: in the former we prove some standard facts about the perfectly conductivity problem, and in the latter we prove two technical lemmas which are crucial for the proof of Theorem 1.1.

2. NORMS AND FINSLER LAPLACIAN

About norms in \mathbb{R}^N . In this section we recall some facts about norms in \mathbb{R}^N , $N \geq 2$. Let $H : \mathbb{R}^N \rightarrow \mathbb{R}$ be a norm, i.e.

$$(2.1) \quad H \text{ is convex,}$$

$$(2.2) \quad H(\xi) \geq 0 \text{ for } \xi \in \mathbb{R}^N \text{ and } H(\xi) = 0 \text{ if and only if } \xi = 0,$$

$$(2.3) \quad H(t\xi) = |t|H(\xi) \text{ for } \xi \in \mathbb{R}^N \text{ and } t \in \mathbb{R}.$$

Since all norms in \mathbb{R}^N are equivalent, there exist two positive constants c_1, c_2 such that

$$c_1|\xi| \leq H(\xi) \leq c_2|\xi| \quad \text{for any } \xi \in \mathbb{R}^N.$$

The dual norm of H , which we denote by H_0 , is defined by

$$(2.4) \quad H_0(x) = \sup_{\xi \neq 0} \frac{x \cdot \xi}{H(\xi)} \quad \text{for } x \in \mathbb{R}^N;$$

analogously, one can define H as the dual norm of H_0 , i.e.

$$(2.5) \quad H(\xi) = \sup_{x \neq 0} \frac{x \cdot \xi}{H_0(x)} \quad \text{for } x \in \mathbb{R}^N.$$

Following our notation, H_0 is a norm in the ambient space and gives the norm of a point $x \in \Omega \subset \mathbb{R}^N$ and H is a norm in the dual space, which is identified with \mathbb{R}^N . Indeed, we notice that the gradient of a function $u : \Omega \rightarrow \mathbb{R}^N$, evaluated at $x \in \Omega$, is the element $\nabla u(x)$ of the dual space of \mathbb{R}^N , which associates to any vector $y \in \mathbb{R}^N$ the number $y \cdot \nabla u$. Unless otherwise stated, we will use the variable x to denote a point in the ambient space \mathbb{R}^N and ξ for an element in the dual space. The symbols ∇ and ∇_ξ denote the gradients with respect to the x and ξ variables, respectively.

Let $H \in C^1(\mathbb{R}^N \setminus \{0\})$, from (2.3) we have

$$(2.6) \quad \nabla_\xi H(t\xi) = \text{sign}(t) \nabla_\xi H(\xi), \quad \text{for } \xi \neq 0 \quad \text{and} \quad t \neq 0,$$

and

$$(2.7) \quad \nabla_\xi H(\xi) \cdot \xi = H(\xi), \quad \text{for } \xi \in \mathbb{R}^N,$$

where the left hand side is taken to be 0 when $\xi = 0$. If $H \in C^2(\mathbb{R}^N \setminus \{0\})$, then

$$(2.8) \quad \nabla_\xi^2 H(t\xi) = \frac{1}{|t|} \nabla_\xi^2 H(\xi), \quad \text{for } \xi \neq 0 \quad \text{and} \quad t \neq 0,$$

where ∇_ξ^2 is the Hessian operator with respect to the ξ variable; we also notice that

$$(2.9) \quad \nabla_\xi^2 H^2(t\xi) = \nabla_\xi^2 H^2(\xi), \quad \text{for } \xi \neq 0 \quad \text{and} \quad t \neq 0.$$

Hence, (2.7) implies that

$$(2.10) \quad \partial_{\xi_i \xi_k}^2 H(\xi) \xi_i = 0,$$

for every $k = 1, \dots, N$.

The following properties hold provided that $H \in C^1(\mathbb{R}^N \setminus \{0\})$ and the unitary ball $\{\xi \in \mathbb{R}^n : H(\xi) < 1\}$ is strictly convex (see [20, Lemma 3.1]):

$$(2.11) \quad H_0(\nabla_\xi H(\xi)) = 1, \quad \text{for } \xi \in \mathbb{R}^N \setminus \{0\},$$

and

$$(2.12) \quad H(\nabla H_0(x)) = 1, \quad \text{for } x \in \mathbb{R}^N \setminus \{0\};$$

furthermore, the map $H \nabla_\xi H$ is invertible with

$$(2.13) \quad H \nabla_\xi H = (H_0 \nabla H_0)^{-1}.$$

For $\xi_0 \in \mathbb{R}^N$ and $r > 0$, the ball of center ξ_0 and radius r in the norm H is denoted by

$$B_H(\xi_0, r) = \{\xi \in \mathbb{R}^N : H(\xi - \xi_0) < r\};$$

analogously,

$$B_{H_0}(x_0, r) = \{x \in \mathbb{R}^N : H_0(x - x_0) < r\}$$

denotes the ball of center x_0 and radius r in the norm H_0 . A ball in the norm H_0 is called the *Wulff shape* of H .

Assumptions on H . We shall consider norms such that the unitary balls are uniformly convex. More precisely, we are considering a uniformly elliptic norm of class C^3 outside the origin, i.e. a function $H \in C^3(\mathbb{R}^N \setminus \{0\})$ for which there exists $\lambda_*, \lambda^* > 0$ such that

$$(2.14) \quad \frac{\lambda_*}{|v|} \left| \tau - \left(\tau \cdot \frac{v}{|v|} \right) \frac{v}{|v|} \right|^2 \leq \langle \nabla^2 H(v) \tau, \tau \rangle \leq \frac{\lambda^*}{|v|} \left| \tau - \left(\tau \cdot \frac{v}{|v|} \right) \frac{v}{|v|} \right|^2,$$

for every $v, \tau \in \mathbb{R}^N$, $v \neq 0$. We recall that, under these hypotheses, the boundary of the Wulff shape is uniformly convex (see, for instance, [39] p.111).

Finsler Laplacian. The Finsler Laplacian (associated to H) of the function u is given by

$$\Delta_H u = \operatorname{div} (H(\nabla u) \nabla_\xi H(\nabla u)).$$

We recall the maximum and comparison principles for the Finsler Laplacian (see [25, Theorem 4.1] and [25, Theorem 4.2]).

Theorem 2.1. *If $-\Delta_H u \leq 0$ in Ω and $u = g \leq M$ on $\partial\Omega$, then u attains its maximum on the boundary; that is, $u(x) \leq M$ a.e. in Ω .*

Theorem 2.2. *Suppose that $-\Delta_H u \leq -\Delta_H v$ in Ω and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ a.e. in Ω .*

Let $B_{H_0}(r)$ and $B_{H_0}(R)$ be two Wulff shapes centered at the origin, with $r < R$. It will be useful to have at hand the explicit solution to the problem

$$(2.15) \quad \begin{cases} \Delta_H v = 0 & \text{in } B_{H_0}(R) \setminus \overline{B_{H_0}(r)}, \\ v = C_r & \text{on } \partial B_{H_0}(r), \\ v = C_R & \text{on } \partial B_{H_0}(R), \end{cases}$$

which is given by

$$(2.16) \quad v(x) = \begin{cases} (C_r - C_R) \frac{H_0(x)^{2-N} - R^{2-N}}{r^{2-N} - R^{2-N}} + C_R & \text{if } N \geq 3, \\ (C_r - C_R) \frac{\ln(R^{-1}H_0(x))}{\ln(R^{-1}r)} + C_R & \text{if } N = 2, \end{cases}$$

for any $x \in \overline{B_{H_0}(R)} \setminus B_{H_0}(r)$. It is readily seen that $v \in C^3(\overline{B_{H_0}(R)} \setminus B_{H_0}(r))$, it satisfies (2.15), and $H(\nabla v) \neq 0$ (see also [25, Theorem 3.1]). Moreover, the following bounds

$$(2.17) \quad \frac{(N-2)|C_r - C_R|}{[(r/R)^{2-N} - 1]R} \leq H(\nabla v(x)) \leq \frac{(N-2)|C_r - C_R|}{[(r/R)^{2-N} - 1]r} \quad \text{for } N \geq 3,$$

and

$$(2.18) \quad \frac{|C_r - C_R|}{R \ln(Rr^{-1})} \leq H(\nabla v(x)) \leq \frac{|C_r - C_R|}{r \ln(Rr^{-1})} \quad \text{for } N = 2,$$

hold for any $x \in \overline{B_{H_0}(R)} \setminus B_{H_0}(r)$.

Definition of the neck. It will be useful to introduce the following notation. For a fixed $w > 0$ sufficiently small we define the *neck* of width w as the set

$$(2.19) \quad \mathcal{N}_\delta(w) = \{x = (x', x_N) \in \Omega_\delta \text{ such that } |\mathcal{Q}^{\frac{1}{2}}x'| < w, H_0(x) < \max(R_1, R_2)\},$$

where $\mathcal{Q}^{\frac{1}{2}}$ is the square root of the matrix \mathcal{Q} defined in (1.9). Notice that, if w and δ are small enough, $\mathcal{N}_\delta(w)$ is as in Fig. 3.

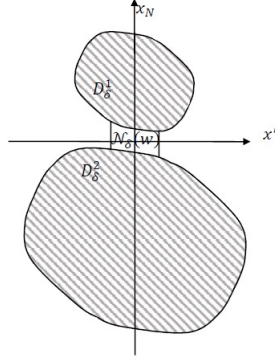
Remark 2.3. Let $B_{H_0}^1$ and $B_{H_0}^2$ be two anisotropic balls which are tangent to some point. Let x^1 and x^2 be the centers of $B_{H_0}^1$ and $B_{H_0}^2$, respectively. Then the touching point lies on the segment joining the two centers x^1 and x^2 .

Indeed, since

$$B_{H_0}^1 = \{y : H_0(y - x^1) < r_1\},$$

where r_1 is the radius of $B_{H_0}^1$, then

$$\nabla H_0(x - x^1) = \gamma_1 \nu(x),$$

FIGURE 3. The neck $\mathcal{N}_\delta(w)$.

for some $\gamma_1 > 0$. We apply H to both sides of the above equation and from (2.12) we find

$$1 = \gamma_1 H(\nu(x)),$$

which yields

$$\nabla H_0(x - x^1) = \frac{\nu(x)}{H(\nu(x))}.$$

An analogous argument shows that

$$\nabla H_0(x^2 - x) = \frac{\nu(x)}{H(\nu(x))}.$$

We apply ∇H in the last two equations and, by using the properties of the norms, we find

$$\frac{x - x^1}{H_0(x - x^1)} = \nabla H(\nu(x)) = \frac{x^2 - x}{H_0(x - x^2)},$$

as claimed.

3. MAXIMUM PRINCIPLES

In this section we prove some maximum principles for u_δ , $H(\nabla u)$ and for a P -function which is suitable for our purposes.

We first notice that the maximum and minimum of u_δ are attained at the boundary of Ω .

Lemma 3.1. *Let u_δ be the solution of problem (1.4). The maximum and the minimum of u_δ are attained on $\partial\Omega$. In particular, we have that*

$$\max_{\overline{\Omega}_\delta} |u_\delta| = \max_{\partial\Omega} |\varphi|.$$

Proof. The maximum principle for the Finsler Laplacian yields that $|u_\delta|$ attains its maximum $\partial\Omega_\delta$. We show that the maximum of u_δ can not be attained at ∂D_δ^i , with $i \in \{1, 2\}$. Indeed, assume by contradiction that $\max u_\delta = \mathcal{U}_\delta^1$. From Hopf's lemma we have that $|\nabla u_\delta| > 0$ on ∂D_δ^1 , which contradicts the third condition of (1.4). Analogously, the maximum can not be attained at ∂D_δ^2 . \square

Before giving other maximum principles, we set some notation and prove some basic inequalities for the Finsler Laplacian. In order to avoid heavy formulas, we use the following notation:

$$u_i = \frac{\partial u}{\partial x_i}, \quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

and

$$\partial_{\xi_i} H = \frac{\partial H}{\partial \xi_i}, \quad \partial_{\xi_i \xi_j}^2 H = \frac{\partial^2 H}{\partial \xi_i \partial \xi_j}.$$

Since

$$\operatorname{div}(H(\nabla u) \nabla_{\xi} H(\nabla u)) = \left(\partial_{\xi_i} H(\nabla u) \partial_{\xi_j} H(\nabla u) + H(\nabla u) \partial_{\xi_i \xi_j}^2 H(\nabla u) \right) u_{ij}$$

where $\nabla u \neq 0$, by setting

$$(3.1) \quad a_{ij} := \partial_{\xi_i} H(\nabla u) \partial_{\xi_j} H(\nabla u) + H(\nabla u) \partial_{\xi_i \xi_j}^2 H(\nabla u) = \frac{1}{2} \partial_{\xi_i \xi_j}^2 H(\nabla u)^2.$$

the Finsler Laplacian can be written as

$$(3.2) \quad \Delta_H u = a_{ij} u_{ij} = \operatorname{tr}(A \nabla^2 u)$$

at points where $\nabla u \neq 0$, where A is the symmetric matrix with entries a_{ij} , $i, j = 1, \dots, N$. We notice that from (2.7) and (2.10) we have that

$$(3.3) \quad a_{ij} u_i u_j = H(\nabla u)^2$$

at points where $\nabla u \neq 0$. It will be useful to set

$$(3.4) \quad a_{ijk} := \frac{1}{2} \partial_{\xi_i \xi_j \xi_k}^3 H(\nabla u)^2,$$

and notice that if u is a solution to $\Delta_H u = 0$ then

$$(3.5) \quad a_{ijl} u_l = 0,$$

where $\nabla u \neq 0$. Indeed, (3.5) can be proved by noticing that

$$\begin{aligned} a_{ijl} u_l &= \partial_{\xi_j} H(\nabla u) \partial_{\xi_i \xi_l}^2 H(\nabla u) u_l + \partial_{\xi_i} H(\nabla u) \partial_{\xi_j \xi_l}^2 H(\nabla u) u_l \\ &\quad + \partial_{\xi_i \xi_j}^2 H(\nabla u) \partial_{\xi_l} H(\nabla u) u_l + H(\nabla u) \partial_{\xi_i \xi_j \xi_l}^3 H(\nabla u) u_l. \end{aligned}$$

By using (2.10) we obtain

$$a_{ijl} u_l = H(\nabla u) \partial_{\xi_i \xi_j}^2 H(\nabla u) + H(\nabla u) \partial_{\xi_i \xi_j \xi_l}^3 H(\nabla u) u_l = H(\xi) \partial_{\xi_i} \left(\partial_{\xi_j \xi_l}^2 H(\xi) \xi_l \right) \Big|_{\xi = \nabla u},$$

and from (2.10) we find (3.5).

Since $\nabla_{\xi}^2 H^2(\xi)$ is 0-homogeneous, the matrix $A = (a_{ij})$ satisfies

$$|a_{ij}| \leq C_0 \quad \text{and} \quad 2\mu_1 Id \leq A \leq 2\mu_N Id,$$

where C_0 depends only on $\|\nabla^2 H^2\|_{C^0(\partial B_H(0,1))}$, and where μ_1 and μ_N are the minimum and maximum eigenvalues of $\nabla_{\xi}^2 H^2$.

Let \mathcal{L} to be the second order elliptic operator given by

$$(3.6) \quad \mathcal{L}v := \partial_i (a_{ij} v_j) = a_{ij} v_{ij} + a_{ijl} u_{ij} v_l.$$

We recall that if u is a solution to $\Delta_H u = 0$, then u is of class $C^{1,\alpha} \cap W_{loc}^{1,2}$ and, by elliptic regularity, $u \in C^{2,\alpha}$ where $\nabla u \neq 0$.

We first prove that if u is a solution to $\Delta_H u = 0$ then $H(\nabla u)$ satisfies a maximum principle and we also give a useful pointwise formula for $\mathcal{L}u^2$.

Lemma 3.2. *Let $E \subset \mathbb{R}^N$ be a bounded domain and let u be such that $\Delta_H u = 0$ in E . We have*

$$(3.7) \quad \mathcal{L}(H(\nabla u)^2) \geq \frac{2n}{n-1} \left(\partial_{\xi_i} H(\nabla u) \partial_{\xi_j} H(\nabla u) u_{ij} \right)^2,$$

and

$$(3.8) \quad \mathcal{L}(u^2) = 2H(\nabla u)^2.$$

In particular, $H(\nabla u)$ satisfies the maximum principle, i.e.

$$(3.9) \quad \max_E H(\nabla u) = \max_{\partial E} H(\nabla u).$$

Proof. We first prove (3.8). At points where $\nabla u \neq 0$ we have

$$\operatorname{div}(a_{ij}\nabla u^2) = 2a_{ij}u_iu_j + 2ua_{ij}u_{ij} + 2u\partial_i(a_{ij})u_j$$

and, since $a_{ij}u_{ij} = \Delta_H u = 0$, we obtain

$$\operatorname{div}(a_{ij}\nabla u^2) = 2a_{ij}u_iu_j + 2uu_ja_{ijk}u_{ki}.$$

From (3.3) and (3.5) we find

$$(3.10) \quad \operatorname{div}(a_{ij}\nabla u^2) = 2H(\nabla u)^2,$$

at points where $\nabla u \neq 0$. By continuity, (3.10) can be extended to zero where $\nabla u = 0$.

In order to prove (3.7), we first notice that the following Bochner formula holds (see also [38, Lemma 2.1]):

$$(3.11) \quad a_{ij}\partial_{ij}^2 H(\nabla u)^2 = 2a_{ij}a_{kl}u_{ik}u_{jl} - \partial_l H(\nabla u)^2 a_{ijl}u_{ij},$$

where $\nabla u \neq 0$. Indeed, (3.11) follows from the following argument. Owing to (3.1) and (3.2) and since $\Delta_H u = 0$ we have

$$\begin{aligned} a_{ij}\partial_{ij}^2 (H(\nabla u)^2) &= a_{ij}\partial_j(\partial_{\xi_k} H(\nabla u)^2 u_{ik}) = a_{ij}\underbrace{\partial_{\xi_k \xi_l}^2 H(\nabla u)^2}_{2a_{kl}} u_{ik}u_{jl} + a_{ij}\partial_{\xi_k} H(\nabla u)^2 u_{ijk} \\ &= 2a_{ij}a_{kl}u_{ik}u_{jl} + \partial_{\xi_k} H(\nabla u)^2 \partial_k \underbrace{(a_{ij}u_{ij})}_{\Delta_H u = 0} - \partial_{\xi_k} H(\nabla u)^2 \partial_k (a_{ij})u_{ij} \\ &= 2a_{ij}a_{kl}u_{ik}u_{jl} - 2H(\nabla u)\partial_{\xi_k} H(\nabla u)a_{ijl}u_{lk}u_{ij}, \end{aligned}$$

where $\nabla u \neq 0$, which proves (3.11).

Since

$$\mathcal{L}H(\nabla u)^2 = a_{ij}\partial_{ij}^2 H(\nabla u)^2 + a_{ijl}u_{ij}\partial_l H(\nabla u)^2,$$

from (3.11) we have

$$\mathcal{L}H(\nabla u)^2 = 2a_{ij}a_{kl}u_{ik}u_{jl},$$

and from

$$a_{ij}a_{kl}u_{ik}u_{jl} \geq \frac{(a_{ij}u_{ij})^2}{n} + \frac{n}{n-1} \left(\frac{a_{ij}u_{ij}}{n} - \partial_{\xi_i} H \partial_{\xi_j} H u_{ij} \right)^2,$$

(see [38, Lemma 2.3]) we obtain

$$(3.12) \quad \mathcal{L}H(\nabla u)^2 \geq \frac{2n}{n-1} \left(\partial_{\xi_i} H(\nabla u) \partial_{\xi_j} H(\nabla u) u_{ij} \right)^2,$$

at point where $\nabla u \neq 0$. We set $E_0 = \{x \in E : \nabla u = 0\}$; since $u \in C^{1,\alpha}$ then E_0 is closed. From (3.12) we have that $H(\nabla u)^2$ satisfies a maximum principle in $E \setminus E_0$ and hence $\max H(\nabla u)^2$ is attained at $\partial E \cup \partial E_0$. Since $H(\nabla u) = 0$ in E_0 , we have that $H(\nabla u)$ attains the maximum at ∂E and (3.9) follows. \square

Now we prove a maximum principle for a P -function which is suitable for our problem, which take care of the presence of the neck $\mathcal{N}_\delta(w)$, $w > 0$ (see formula (2.19) for its definition).

In the following we write $x \in \mathbb{R}^N$ as $x = (x', x_N)$, where $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$. We will need to introduce a cut-off function $f \in C^2(\bar{\Omega})$ such that

$$(3.13) \quad |f| = 1 \text{ in } \bar{\Omega}_\delta \setminus \mathcal{N}_\delta(w), \quad f = 0 \text{ in } \mathcal{N}_\delta\left(\frac{w}{2}\right).$$

Moreover we choose f such that

$$(3.14) \quad \frac{f}{w} \leq |\nabla f|^2 \quad \text{and} \quad |\nabla^2 f| \leq \frac{1}{w^2} \quad \text{in} \quad \mathcal{N}_\delta(w) \setminus \mathcal{N}_\delta\left(\frac{w}{2}\right).$$

(see [27] for an explicit example in the Euclidean case).

Theorem 3.3. *Let u_δ be such that $\Delta_H u_\delta = 0$ in Ω_δ . Let f satisfy (3.13) and (3.14).*

There exists $\lambda_0 = \lambda_0(\|f\|_{C^2}, \|H\|_{C^3(\partial B_H(0,1))})$, with $\lambda_0 = O(w^{-2})$ as $w \rightarrow 0^+$, such that the function

$$(3.15) \quad P(x) = f(x)H(\nabla u_\delta)^2 + \lambda u_\delta^2$$

satisfies the maximum principle for any $\lambda \geq \lambda_0$, i.e.

$$(3.16) \quad \max_{x \in \overline{\Omega}_\delta} P(x) = \max_{x \in \partial \Omega_\delta} P(x)$$

for $\lambda \geq \lambda_0$.

Proof. We first prove the assertion when the maximum is attained at a point x_0 where $\nabla u_\delta(x_0) = 0$, and then we consider the case when $\nabla u_\delta(x_0) \neq 0$.

Step 1. Suppose that P attains the maximum at point x_0 such that $\nabla u_\delta(x_0) = 0$. Then $P(x_0) = \lambda u_\delta(x_0)^2$ and

$$f(x)H(\nabla u_\delta(x))^2 + \lambda u_\delta(x)^2 \leq \lambda u_\delta(x_0)^2$$

for any $x \in \Omega_\delta$. In particular $|u_\delta(x)| \leq |u_\delta(x_0)|$, and Lemma 3.1 yields that $x_0 \in \partial \Omega$.

Step 2. Suppose that P attains the maximum at a point x_0 such that $\nabla u_\delta(x_0) \neq 0$. We shall use the operator \mathcal{L} defined in (3.6) where the coefficients a_{ij} are given by (3.1) with u_δ in place of u . From (3.7) and (3.8) we have

$$(3.17) \quad \begin{aligned} \mathcal{L}P &\geq a_{ij}\partial_{ij}f(x)H(\nabla u_\delta)^2 + 2a_{ij}\partial_i f(x)\partial_j H(\nabla u_\delta)^2 + a_{ijl}\partial_l f(x)H(\nabla u_\delta)^2 u_{ij} \\ &+ f(x)\frac{2n}{n-1}(\partial_{\xi_i}H(\nabla u_\delta)\partial_{\xi_j}H(\nabla u_\delta)u_{ij})^2 + 2\lambda H(\nabla u_\delta)^2. \end{aligned}$$

Since H is 1-homogeneous, the quantities a_{ij} , $a_{ijl}H(\nabla u_\delta)$ and $\partial_{\xi_i}H$ are 0-homogeneous. Hence there exists a constant C_0 depending only on $\|H\|_{C^3(\partial B_H(0,1))}$ such that

$$(3.18) \quad |a_{ij}|, |a_{ijl}H(\nabla u_\delta)| \leq C_0, \text{ and } C_0^{-1} \leq |\partial_{\xi_i}H(\nabla u_\delta)| \leq C_0.$$

From (3.17), (3.18) and by using Cauchy-Schwarz inequality, we have

$$\begin{aligned} \mathcal{L}P &\geq (\lambda - C_0\|\nabla^2 f\|_{C^0})H(\nabla u_\delta)^2 + 2\sqrt{\frac{2n\lambda f}{n-1}}H(\nabla u_\delta)|\partial_{\xi_i}H(\nabla u_\delta)\partial_{\xi_j}H(\nabla u_\delta)u_{ij}| \\ &- 4C_0^2\|\nabla^2 f\|_{C^0}|\nabla^2 u_\delta|H(\nabla u_\delta) - C_0\|\nabla f\|_{C^0}|\nabla^2 u_\delta|H(\nabla u_\delta) \\ &= (\lambda - C_0\|\nabla^2 f\|_{C^0})H(\nabla u_\delta)^2 + \left(2C_0^{-2}\sqrt{\frac{2n\lambda f}{n-1}} - 4C_0^2\|\nabla^2 f\|_{C^0}\right)|\nabla^2 u_\delta|H(\nabla u_\delta). \end{aligned}$$

We can choose λ_0 large enough such that

$$\lambda - C_0\|\nabla^2 f\|_{C^0} \geq 0,$$

and

$$2C_0^{-2}\sqrt{\frac{2n\lambda f}{n-1}} - 4C_0^2\|\nabla f\|_{C^0} \geq 0,$$

for $\lambda \geq \lambda_0$. The constant λ_0 depends only on $\|H\|_{C^3(\partial B_H(0,1))}$ and $\|f\|_{C^2}$, and from (3.14) we have that $\lambda_0 = O(w^{-2})$. From step 1 and step 2 we conclude. \square

4. UNIFORM BOUNDS FOR THE GRADIENT

In this section we give estimates in the region where the gradient remains uniformly bounded. In the next lemma we show that, since the inclusions are far away from the boundary of Ω , we have that the gradient of u_δ is uniformly bounded on $\partial\Omega$ independently of δ .

Lemma 4.1. *Let u_δ be the solution of (1.4). There exists a constant $C > 0$ independent of δ such that*

$$(4.1) \quad \max_{\partial\Omega} H(\nabla u_\delta) \leq C.$$

Proof. Let $A \subset \Omega$ be a smooth set such that $\{x \in \Omega : \text{dist}(x, \partial\Omega) > K/2\} \subset A$, with K given by (1.8), and $\bar{A} \subset \Omega$. It is clear that D_δ^1 and D_δ^2 are contained in A for any $\delta \leq \delta_0$.

Let v_* and v^* be the solutions to

$$\begin{cases} \Delta_H v_* = 0 & \text{in } \Omega \setminus \bar{A}, \\ v_* = \varphi & \text{on } \partial\Omega, \\ v_* = \min_{\partial\Omega} \varphi & \text{on } \partial A, \end{cases}$$

and

$$\begin{cases} \Delta_H v^* = 0 & \text{in } \Omega \setminus \bar{A}, \\ v^* = \varphi & \text{on } \partial\Omega, \\ v^* = \max_{\partial\Omega} \varphi & \text{on } \partial A, \end{cases}$$

respectively. From Lemma 3.1, it is clear that v_* and v^* are, respectively, a lower and an upper barrier for u_δ at any point on $\partial\Omega$. Hence, the normal derivative of u_δ can be bounded in terms of the gradient of v_* , v^* , and thus $H(u_\delta)$ can be bounded by some constant C which depends only on K and φ , which implies (4.3). \square

Now we show that the gradient is uniformly bounded on the boundary of the inclusions at the points which are not in the neck.

Lemma 4.2. *Let u_δ be the solution of (1.4) and let $w > 0$ be fixed. There exists a constant $C > 0$ independent of δ such that*

$$(4.2) \quad \max_{\partial D_\delta^i \setminus \partial N_\delta(w)} H(\nabla u_\delta) \leq C, \quad i = 1, 2.$$

Proof. Let $z \in \partial D_\delta^1 \setminus \partial N_\delta(w)$ and, for $r_1 \ll R_1$, denote by $B_{H_0}(z_0, r_1)$ the interior anisotropic ball of radius r_1 and center z_0 tangent to ∂D_δ^1 at z , i.e. $B_{H_0}(z_0, r_1) \subset D_\delta^1$ and $\partial B_{H_0}(z_0, r_1) \cap \partial D_\delta^1 = \{z\}$ (as follows from the uniform convexity of the norm, see Fig. 4).

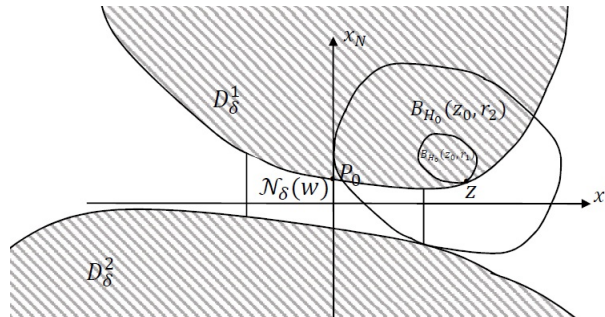


FIGURE 4. $B_{H_0}(z_0, r_1)$ and $B_{H_0}(z_0, r_2)$ are used to construct upper and lower barriers for u_δ at z .

Let r_2 be the (anisotropic) distance of z_0 from ∂D_δ^2 ; notice that $r_2 > r_1$ and the (anisotropic) ball $B_{H_0}(z_0, r_2)$ is exterior and tangent to D_δ^2 at some point $z_1 \in \partial D_\delta^2$.

We construct an upper barrier \bar{v} and a lower barrier \underline{v} for u_δ at z by considering the solutions to

$$\begin{cases} \Delta_H \bar{v} = 0 & \text{in } B_{H_0}(z_0, r_2) \setminus \bar{B}_{H_0}(z_0, r_1), \\ \bar{v} = \mathcal{U}_\delta^1 & \text{on } \partial B_{H_0}(z_0, r_1), \\ \bar{v} = \max_{\partial\Omega} \varphi & \text{on } \partial B_{H_0}(z_0, r_2), \end{cases}$$

and

$$\begin{cases} \Delta_H \underline{v} = 0 & \text{in } B_{H_0}(z_0, r_2) \setminus \bar{B}_{H_0}(z_0, r_1), \\ \underline{v} = \mathcal{U}_\delta^1 & \text{on } \partial B_{H_0}(z_0, r_1), \\ \underline{v} = \min_{\partial\Omega} \varphi & \text{on } \partial B_{H_0}(z_0, r_2), \end{cases}$$

respectively, when \mathcal{U}_δ^i are defined in (1.5). As follows from (2.16) we have that \bar{v} and \underline{v} are given by

$$\bar{v}(x) = \begin{cases} (\mathcal{U}_\delta^1 - \max_{\partial\Omega} \varphi) \frac{H_0(x - z_0)^{2-N} - r_2^{2-N}}{r_1^{2-N} - r_2^{2-N}} + \max_{\partial\Omega} \varphi & \text{if } N \geq 3, \\ (\mathcal{U}_\delta^1 - \max_{\partial\Omega} \varphi) \frac{\ln(r_2^{-1} H_0(x - z_0))}{\ln(r_2^{-1} r_1)} + \max_{\partial\Omega} \varphi & \text{if } N = 2, \end{cases}$$

and

$$\underline{v}(x) = \begin{cases} (\mathcal{U}_\delta^1 - \min_{\partial\Omega} \varphi) \frac{H_0(x - z_0)^{2-N} - r_2^{2-N}}{r_1^{2-N} - r_2^{2-N}} + \min_{\partial\Omega} \varphi & \text{if } N \geq 3, \\ (\mathcal{U}_\delta^1 - \min_{\partial\Omega} \varphi) \frac{\ln(r_2^{-1} H_0(x - z_0))}{\ln(r_2^{-1} r_1)} + \min_{\partial\Omega} \varphi & \text{if } N = 2, \end{cases}$$

respectively. In particular, by using (2.12) we have

$$H(\nabla \bar{v}(z)) = \begin{cases} (N-2) \left| \mathcal{U}_\delta^1 - \max_{\partial\Omega} \varphi \right| \frac{r_1^{1-N}}{r_1^{2-N} - r_2^{2-N}} & \text{if } N \geq 3, \\ \left| \mathcal{U}_\delta^1 - \max_{\partial\Omega} \varphi \right| \frac{r_1^{-1}}{\ln(r_2 r_1^{-1})} & \text{if } N = 2, \end{cases}$$

and

$$H(\nabla \underline{v}(z)) = \begin{cases} (N-2) \left| \mathcal{U}_\delta^1 - \min_{\partial\Omega} \varphi \right| \frac{r_1^{1-N}}{r_1^{2-N} - r_2^{2-N}} & \text{if } N \geq 3, \\ \left| \mathcal{U}_\delta^1 - \min_{\partial\Omega} \varphi \right| \frac{r_1^{-1}}{\ln(r_2 r_1^{-1})} & \text{if } N = 2. \end{cases}$$

We fix $r_1 = cw$ for some small constant $c > 0$. Since $w > 0$ is fixed, there exists a constant $\alpha > 1$ such that $r_2 \geq \alpha r_1$ for any $\delta \geq 0$, with α not depending on δ . Hence we have that

$$\frac{r_1^{1-N}}{r_1^{2-N} - r_2^{2-N}} \leq \frac{1}{cw(1 - \alpha^{2-N})} \quad \text{for } N \geq 3,$$

and

$$\frac{r_1^{-1}}{\ln(r_2 r_1^{-1})} \leq \frac{1}{cw \ln \alpha} \quad \text{for } N = 2.$$

Since the maximum and minimum of u_δ are attained on $\partial\Omega$ (see Lemma 3.1) then by comparison principle we obtain that

$$H(\nabla u_\delta(z)) \leq C$$

where C depends only on the dimension N , $\|\varphi\|_{C^0(\partial\Omega)}$ and w , and does not depend on δ . \square

Lemma 4.3. *Let u_δ be the solution of (1.4) and let $w > 0$. There exists a constant $C > 0$ independent of δ and w such that*

$$(4.3) \quad \max_{\overline{\Omega_\delta \setminus \mathcal{N}_\delta(w)}} H(\nabla u_\delta) \leq \frac{C}{w}.$$

Proof. From Lemma 3.2 we know that $H(\nabla u)$ satisfies the maximum principle, so that

$$\max_{\overline{\Omega_\delta \setminus \mathcal{N}_\delta(w)}} H(\nabla u_\delta) \leq \max_{\partial(\Omega_\delta \setminus \mathcal{N}_\delta(w))} H(\nabla u_\delta).$$

From Lemmas 4.1 and 4.2 it is enough to find uniform bounds on $H(\nabla u)$ on $\partial\mathcal{N}_\delta^\pm(w)$, where

$$(4.4) \quad \partial\mathcal{N}_\delta^\pm(w) = \partial\mathcal{N}_\delta(w) \cap \{|Qx'| = \pm w\},$$

i.e. we aim at showing that there exists a constant C independent of δ and w such that

$$(4.5) \quad \max_{\partial\mathcal{N}_\delta^\pm(w)} H(\nabla u_\delta) \leq \frac{C}{w}.$$

Let P be as in Theorem 3.3 (see formula (3.15)).

Since $0 \leq f \leq 1$ and $f = 1$ on $\mathcal{N}_\delta^\pm(w)$, we have that

$$\max_{\partial\mathcal{N}_\delta^\pm(w)} H(\nabla u_\delta)^2 = \max_{\partial\mathcal{N}_\delta^\pm(w)} f(x)H(\nabla u_\delta)^2 \leq \max_{\partial\mathcal{N}_\delta^\pm(w)} P(x) \leq \max_{\overline{\Omega_\delta}} P(x).$$

From Theorem 3.3 there exists a constant $\lambda_0 = O(w^{-2})$ such that $P(x)$ given by (3.15) satisfies the maximum principle for any $\lambda \geq \lambda_0$ and the chain of inequalities above yields

$$\max_{\partial\mathcal{N}_\delta^\pm(w)} H(\nabla u_\delta)^2 \leq \max_{\overline{\Omega_\delta}} P(x) = \max_{\partial\Omega_\delta} P(x).$$

Since $\|u_\delta\|_{C^0(\Omega_\delta)} \leq \|\varphi\|_{C^0(\partial\Omega)}$ (see Lemma 3.1) and $\lambda_0 = O(w^{-2})$ (see Theorem 3.3), we have that there exists a constant C independent of δ and w such that

$$P(x) = f(x)H(\nabla u_\delta)^2 + \lambda_0 u_\delta^2 \leq f(x)H(\nabla u_\delta)^2 + Cw^{-2},$$

and hence

$$\max_{\partial\mathcal{N}_\delta^\pm(w)} H(\nabla u_\delta)^2 \leq \max_{\partial\Omega_\delta} P \leq \max_{\partial\Omega_\delta} [f(x)H(\nabla u_\delta)^2] + Cw^{-2}.$$

Since $f = 0$ in $\mathcal{N}_\delta(w/2)$, from Lemmas 4.1 and 4.2 we find (4.5) and the proof is complete. \square

Before giving the relation between u_δ and u_0 (see Proposition 4.5 below), in the next Lemma we show that gradient of u_0 is bounded.

Lemma 4.4. *Let u_0 be the solution to (1.6). Then $H(\nabla u_0) \leq C$ in $\overline{\Omega}_0$.*

Proof. The proof is analogous to the ones of Lemmas 4.1 and 4.2, and we only give a sketch. Since $H(\nabla u_0)$ attains the maximum at the boundary (see Lemma 3.2), we have to prove that $H(\nabla u_0)$ is bounded on $\partial\Omega$ and on $\partial D_0^1 \cup \partial D_0^2$. First we recall that, in view of the third condition in (1.6), the maximum and minimum of Lemma u_0 are attained at $\partial\Omega$. Hence, the bound on $\partial\Omega$ can be obtained as in the proof of Lemma 4.1. The bound on ∂D_0^1 (and analogously the one on ∂D_0^2) can be obtained by comparison principle, more precisely by comparing u_0 and v_1 and v_2 , where v_i is the solution to $\Delta_H v_i = 0$ in $\Omega \setminus \overline{D}_0^1$, $v_i = u_0$ on ∂D_0^i , $i = 1, 2$, $v_1 = \max_{\partial\Omega} \phi$ and $v_2 = \min_{\partial\Omega} \phi$ on $\partial\Omega$. \square

We are ready to show the relation between u_δ and u_0 .

Proposition 4.5. *Let u_δ be the solution of (1.4) and u_0 be the solution of (1.6).*

There exists a constant $0 < \alpha < 1$ not depending on δ such that

$$(4.6) \quad \lim_{\delta \rightarrow 0} \|u_\delta - u_0\|_{C^{1,\alpha}(E)} = 0,$$

for any compact set $E \subset \Omega_0$. Moreover, for any $i = 1, 2$ and for any neck $\mathcal{N}_\delta(w)$ of (sufficiently small) width w we have

$$(4.7) \quad \lim_{\delta \rightarrow 0} \int_{\partial D_\delta^i \setminus \partial \mathcal{N}_\delta(w)} H(\nabla u_\delta) \nabla_\xi H(\nabla u_\delta) \cdot \nu ds = \int_{\partial D_0^i \setminus \partial \mathcal{N}_\delta(w)} H(\nabla u_0) \nabla_\xi H(\nabla u_0) \cdot \nu ds.$$

Proof. Thanks to Lemma 4.3 and [24, Theorem 2], for any fixed $w > 0$ we have that there exists $\alpha > 0$ independent of δ such that

$$(4.8) \quad \|u_\delta\|_{C^{1,\alpha}(\mathcal{K})} \leq C \quad \text{for any compact set } \mathcal{K} \subset \Omega_\delta \setminus \overline{\mathcal{N}_\delta(w)},$$

where C is a constant independent of δ .

Let E be a compact set contained in Ω_0 . We want to show that u_δ converges to u_0 in $C^{1,\alpha}(E)$. Since E is fixed, there exist $w, \delta_0 > 0$ such that $E \subset \Omega_\delta \setminus \overline{\mathcal{N}_\delta(w)}$ for any $\delta < \delta_0$. From (4.8) we have that u_δ converges to some function \bar{u} in $C^{1,\alpha}(E)$, which satisfies $\Delta_H \bar{u} = 0$ in E . In order to show that \bar{u} is the solution to (1.6), i.e. $\bar{u} = u_0$, we only need to check that \bar{u} satisfies the third line in (1.6), i.e. that

$$(4.9) \quad \int_{\partial D_0^1} H(\nabla \bar{u}) \nabla_\xi H(\nabla \bar{u}) \cdot \nu ds + \int_{\partial D_0^2} H(\nabla \bar{u}) \nabla_\xi H(\nabla \bar{u}) \cdot \nu ds = 0.$$

We prove (4.9) by approximation. Let $\varepsilon > 0$ be fixed and sufficiently small, and let

$$B_{H_0}(x_0^1, R_1 + \varepsilon) \cup B_{H_0}(x_0^2, R_2 + \varepsilon).$$

If $\delta < \varepsilon$ then $D_\delta^1 \cup D_\delta^2 \subset A$ and we have that u_δ converges to \bar{u} in $C^{1,\alpha}(\overline{\Omega} \setminus A)$. From $\Delta_H u_\delta = 0$ in $A_\varepsilon \setminus \overline{D_\delta^1} \cup \overline{D_\delta^2}$ and since $H(\nabla u_\delta) = 0$ in D_δ^1 and D_δ^2 , the divergence theorem implies that

$$\int_{\partial A_\varepsilon} H(\nabla u_\delta) \nabla_\xi H(\nabla u_\delta) \cdot \nu ds = 0.$$

By letting δ to zero and since $u_\delta \rightarrow \bar{u}$ in $C^{1,\alpha}$, we obtain that

$$\int_{\partial A_\varepsilon} H(\nabla \bar{u}) \nabla_\xi H(\nabla \bar{u}) \cdot \nu ds = 0.$$

Since $\varepsilon > 0$ is arbitrary, we obtain (4.9) and (4.6) is proved.

Once we have that $u_\delta \rightarrow u_0$ in $C^{1,\alpha}$ on compact sets, the proof of (4.7) follows straightforwardly (see for instance [27, p.736-737]). \square

5. PROOF OF THEOREM 1.1

Step 1: upper and lower bounds on the gradient in the neck. Let $w > 0$ be fixed. We are going to find upper and lower bounds on the gradient of the solution in the neck in terms of $\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2$, which we assume to be non negative (the case $\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2 \leq 0$ is completely analogous). In particular, we aim at showing that for any fixed $\tau \in (0, 1/2)$ there exists a constant C independent on δ such that

$$(5.1) \quad H(\nabla u_\delta(P)) \nabla_\xi H(\nabla u_\delta(P)) \cdot \nu(P) \leq -\frac{\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2}{\delta + (1 + \tau) \frac{R_1 + R_2}{2R_1 R_2} \mathcal{Q}P^\perp \cdot P^\perp} (1 + o(\delta^2 + |P - P_0|^2)) + C.$$

and
(5.2)

$$H(\nabla u_\delta(P))\nabla_\xi H(\nabla u_\delta(P)) \cdot \nu(P) \geq -\frac{\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2}{\delta + (1-\tau)\frac{R_1+R_2}{2R_1R_2}\mathcal{Q}P^\perp \cdot P^\perp}(1 + o(\delta^2 + |P - P_0|^2)) - C,$$

for any $P \in \partial D_\delta^1 \cap \partial \mathcal{N}_\delta(w)$, where $P_0 \in \partial D_\delta^1$ lies on the segment joining the two centers of D_δ^1 and D_δ^2 , and P^\perp is the projection of P on the orthogonal to P_0 (see Fig.5).

We start by finding a lower bound on ∇u_δ at $P \in \partial D_\delta^1 \cap \partial \mathcal{N}_\delta(w)$. We consider an anisotropic ball touching ∂D_δ^1 at P from the inside and denote it by $B_{H_0}(y_0, r_1)$, so that $B_{H_0}(y_0, r_1) \subset D_\delta^1$ and y_0 and r_1 are the center and the radius of the ball, respectively, where we let

$$r_1 = tR_1.$$

We denote by r_2 the radius of the anisotropic ball with center at y_0 which touches ∂D_δ^2 from the outside, i.e.

$$r_2 = \text{dist}_{H_0}(y_0, \partial D_\delta^2) = \min\{H_0(x - y_0) : x \in \partial D_\delta^2\}.$$

For $x \neq y_0$, let \underline{v} be given by

$$\underline{v}(x) = \begin{cases} (\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2) \frac{H_0(x - y_0)^{2-N} - r_2^{2-N}}{r_1^{2-N} - r_2^{2-N}} + \mathcal{U}_\delta^2 & \text{if } N \geq 3, \\ (\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2) \frac{\ln(r_2^{-1}H_0(x - y_0))}{\ln(r_2^{-1}r_1)} + \mathcal{U}_\delta^2 & \text{if } N = 2. \end{cases}$$

Notice that $\Delta_H \underline{v} = 0$ in $\mathbb{R}^N \setminus \{y_0\}$ and $\underline{v} = \mathcal{U}_\delta^i$ on $\partial B_{H_0}(y_0, r_i)$, $i = 1, 2$. We notice that we can find a constant M , not depending on δ , such that if the ratio

$$(5.3) \quad \mathcal{M} = \begin{cases} \frac{\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2}{r_1^{2-N} - r_2^{2-N}} & \text{if } N \geq 3, \\ \frac{\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2}{\ln(r_2 r_1^{-1})} & \text{if } N = 2. \end{cases}$$

is large enough, say $\mathcal{M} > M$, then \underline{v} is a lower barrier for u_δ .

Now assume that $\mathcal{M} > M$, so that \underline{v} is a lower barrier for u_δ . Since

$$H(\nabla \underline{v}(P)) = \begin{cases} \frac{(\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2)(N-2)}{r_1^{2-N} - r_2^{2-N}} r_1^{1-N} & \text{if } N \geq 3, \\ \frac{\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2}{\ln(r_2 r_1^{-1})} \frac{1}{r_1} & \text{if } N = 2. \end{cases}$$

from the mean value theorem we have that there exists $\bar{r} \in (r_1, r_2)$ such that

$$H(\nabla \underline{v}(P)) = \frac{\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2}{r_2 - r_1} \left(\frac{\bar{r}}{r_1} \right)^{N-1}$$

for any $N \geq 2$, and hence

$$(5.4) \quad H(\nabla \underline{v}(P)) \geq \frac{\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2}{r_2 - r_1}$$

for any $N \geq 2$.

Thanks to (5.4) we can give an upper bound on the quantity $H(\nabla u_\delta(P))\nabla_\xi H(\nabla u_\delta(P)) \cdot \nu(P)$. Indeed, since \underline{v} is a lower barrier for u_δ then

$$(5.5) \quad \nabla_\xi H(\nabla u_\delta(P)) \cdot \nu(P) = -\nabla_\xi H(\nu(P)) \cdot \nu(P) = -H(\nu(P)) = -1,$$

where the last equality holds because P lies on a Wulff shape. From (5.4) and (5.5) we find

$$(5.6) \quad H(\nabla u_\delta(P)) \nabla_\xi H(\nabla u_\delta(P)) \cdot \nu(P) \leq -\frac{\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2}{r_2 - r_1}.$$

If $\mathcal{M} \leq M$, from elliptic estimates we have $H(\nabla u_\delta) \leq C$, where C does not depend on δ . Indeed, from the mean value theorem we have

$$\frac{\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2}{r_2 - r_1} \leq \frac{N-1}{t^{N-1} R_1^{N-1}} M.$$

Since ∂D_δ^1 is of class C^3 , u_δ is constant on ∂D_δ^1 , and the distance of P from ∂D_δ^2 is of size $r_2 - r_1$, from interior regularity estimates we have that $H(\nabla u_\delta) \leq C$, where C does not depend on δ .

Hence

$$(5.7) \quad H(\nabla u_\delta(P)) \nabla_\xi H(\nabla u_\delta(P)) \cdot \nu(P) \leq -\frac{\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2}{r_2 - r_1} + C.$$

Let $r_1 = tR_1$, we have

$$(5.8) \quad r_2 - r_1 = \delta + (1-s) \frac{R_1 + R_2}{2R_1 R_2} \mathcal{Q} P^\perp \cdot P^\perp + o(\delta^2 + |\omega|^2)$$

as δ and $|P - P_0|$ go to zero, and where P^\perp is the projection of P on the orthogonal to P_0 . We do not prove (5.8) here, and we postpone its proof in the Appendix B (see Lemma B.1). From (5.7) and (5.8) we obtain (5.1).

Now we obtain the lower bound (5.2). We consider a ball $B_{H_0}(\bar{y}, \rho_2)$ touching ∂D_δ^1 at P from the outside and such that the center \bar{y} is contained in D_δ^2 and we denote by ρ_1 the radius of the concentric ball touching ∂D_δ^1 from the inside (here we are assuming that w is small enough). For $x \neq \bar{y}$, let \bar{v} be given by

$$\bar{v}(x) = \begin{cases} -(\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2) \frac{H_0(x - \bar{y})^{2-N} - \rho_2^{2-N}}{\rho_1^{2-N} - \rho_2^{2-N}} + \mathcal{U}_\delta^1 & \text{if } N \geq 3, \\ -(\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2) \frac{\ln(\rho_2^{-1} H_0(x - \bar{y}))}{\ln(\rho_2^{-1} \rho_1)} + \mathcal{U}_\delta^1 & \text{if } N = 2. \end{cases}$$

The function \bar{v} is such that $\Delta_H \bar{v} = 0$ in $\mathbb{R}^N \setminus \{\bar{y}\}$, $\bar{v} = \mathcal{U}_\delta^1$ on $\partial B_{H_0}(y_0, \rho_2)$, and $\bar{v} = \mathcal{U}_\delta^2$ on $\partial B_{H_0}(y_0, \rho_1)$.

If the ratio \mathcal{M} (defined as in (5.3)) is large enough, say $\mathcal{M} > M$ for some constant M not depending on δ , then \bar{v} is an upper barrier for u_δ and we obtain that

$$H(\nabla u_\delta(P)) \leq H(\nabla \bar{v}(P)) \leq \frac{\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2}{\rho_2 - \rho_1} \left(\frac{\bar{\rho}}{\rho_2} \right)^{N-1}$$

for some $\bar{\rho} \in (\rho_1, \rho_2)$, and we obtain

$$H(\nabla u_\delta(P)) \leq \frac{\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2}{\rho_2 - \rho_1}.$$

By arguing as for the upper bound before, if $\mathcal{M} \leq M$ then we can find a constant C such that $H(\nabla u_\delta(P)) \leq C$. Hence, we have that

$$H(\nabla u_\delta(P)) \leq \frac{\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2}{\rho_2 - \rho_1} + C.$$

By arguing as in Lemma B.2 below, we can prove that for any fixed $s \in (0, 1/2)$ we have that

$$(5.9) \quad \rho_2 - \rho_1 = \delta + (1+s) \frac{R_1 + R_2}{2R_1 R_2} \mathcal{Q} P^\perp \cdot P^\perp + o(\delta^2 + |P - P_0|^2)$$

as δ and $|P - P_0|$ go to zero, and where P^\perp is the projection of P on the orthogonal to P_0 and from (5.5) we obtain (5.2).

Step 2: Bounds on $\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2$. In this step we aim at proving that for any fixed $\tau \in (0, 1/2)$ we have that

$$(5.10) \quad (1 - \tau) \left(\frac{R_1 + R_2}{2R_1R_2} \right)^{\frac{N-1}{2}} (\det(\mathcal{Q}))^{\frac{N-1}{2}} C |\partial_{\xi_N} H_0(P_0)| \mathcal{R}_0 \Psi_N(\delta) + o(\Psi_N(\delta)) \\ \leq \mathcal{U}_\delta^1 - \mathcal{U}_\delta^2 \leq (1 + \tau) \left(\frac{R_1 + R_2}{2R_1R_2} \right)^{\frac{N-1}{2}} (\det(\mathcal{Q}))^{\frac{N-1}{2}} C |\partial_{\xi_N} H_0(P_0)| \mathcal{R}_0 \Psi_N(\delta) + o(\Psi_N(\delta))$$

where \mathcal{R}_0 is given by (1.7), $\partial_{\xi_N} H$ is the partial derivative of H with respect to the last variable ξ_N , C depends only on the dimension N and with

$$(5.11) \quad \Psi_N(\delta) = \begin{cases} \delta^{1/2} & N = 2, \\ (\log(1/\delta))^{-1} & N = 3, \\ 1 & N \geq 4. \end{cases}$$

Let $w > 0$ be fixed. From (1.4) and the divergence theorem we have that

$$(5.12) \quad 0 = \int_{\partial D_\delta^1} H(\nabla u_\delta) \nabla H(\nabla u_\delta) \cdot \nu \\ = \underbrace{\int_{\partial D_\delta^1 \cap \partial \mathcal{N}_\delta(w)} H(\nabla u_\delta) \nabla H(\nabla u_\delta) \cdot \nu}_{I_1} + \int_{\partial D_\delta^1 \setminus \partial \mathcal{N}_\delta(w)} H(\nabla u_\delta) \nabla H(\nabla u_\delta) \cdot \nu.$$

We consider the set $E = D_0^1 \cup E_0$, where E_0 is some smooth fixed set containing D_δ^1 and not containing D_δ^2 , and such that $\partial(E \cap \mathcal{N}_\delta(w)) \subset \partial D_0^1$ for w small enough. Notice that $\partial E \cap \mathcal{N}_\delta(w) = \partial E_1 \subset \partial D_0^1$ and $\partial E \setminus \mathcal{N}_\delta(w) = \partial E_2 \subset \partial E_0$.

Since $\Delta_H u_\delta = 0$ in Ω_δ we apply the divergence theorem in $E \setminus \overline{\mathcal{N}_\delta(w)}$ and we have that

$$(5.13) \quad \int_{\partial D_\delta^1 \setminus \mathcal{N}_\delta(w)} H(\nabla u_\delta) \nabla H(\nabla u_\delta) \cdot \nu = \underbrace{\int_{\partial E_2} H(\nabla u_\delta) \nabla H(\nabla u_\delta) \cdot \nu}_{I_2} \\ + \underbrace{\int_{E \cap (\partial \mathcal{N}_\delta^+(w) \cup \partial \mathcal{N}_\delta^-(w))} H(\nabla u_\delta) \nabla H(\nabla u_\delta) \cdot \nu}_{I_3},$$

where $\mathcal{N}_\delta^-(w)$ are defined by (4.4). Proposition 4.5 and Lemma 4.3 yield

$$(5.14) \quad I_2 = \int_{\partial E_2} H(\nabla u_0) \nabla H(\nabla u_0) \cdot \nu + o(1),$$

as $\delta \rightarrow 0$. We recall that by definition

$$\mathcal{R}_0 := \int_{\partial D_0^1} H(\nabla u_0) \nabla H(\nabla u_0) \cdot \nu.$$

Since $u_0 \in W^{1,\infty}$ (see Lemma 4.4), by applying the divergence theorem in the set $E \setminus \overline{D_0^1}$ we have that

$$\mathcal{R}_0 = \int_{\partial E} H(\nabla u_0) \nabla H(\nabla u_0) \cdot \nu,$$

and from (5.14) we obtain

$$(5.15) \quad \left| I_2 + \mathcal{R}_0 \right| \leq Cw^{N-1} + o(1),$$

as $\delta \rightarrow 0$, where C does not depends on w . Notice that, from Lemma 4.3 we have

$$\left| I_3 \right| \leq \frac{C\delta}{w},$$

where C does not depends on w . This last estimate together with (5.12) and (5.15) yield

$$\left| I_1 - \mathcal{R}_0 \right| \leq Cw^{N-1} + \frac{C\delta}{w} + o(1).$$

By choosing $w = \delta^{1/2}$ we have that

$$(5.16) \quad \left| I_1 - \mathcal{R}_0 \right| = o(1),$$

as $\delta \rightarrow 0^+$.

Now we estimate I_1 . Together with (5.16), this will imply upper and lower bounds on $\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2$. We recall that I_1 is given by

$$I_1 = \int_{\mathcal{I}} H(\nabla u_\delta) \nabla H(\nabla u_\delta) \cdot \nu,$$

where we set $\mathcal{I} = \partial D_\delta^1 \cap \partial \mathcal{N}_\delta(w)$ to lighten the notation. From (5.1) and (5.2), we obtain that for any $\tau \in (0, 1/2)$ we have

$$(5.17) \quad - \int_{\mathcal{I}} \frac{\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2}{\delta + (1 - \tau) \frac{R_1 + R_2}{2R_1R_2} \mathcal{Q}P^\perp \cdot P^\perp} d\sigma (1 + o(1)) - Cw \leq I_1$$

and

$$(5.18) \quad I_1 \leq - \int_{\mathcal{I}} \frac{\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2}{\delta + (1 + \tau) \frac{R_1 + R_2}{2R_1R_2} \mathcal{Q}P^\perp \cdot P^\perp} d\sigma (1 + o(1)) + Cw.$$

Hence, we have to understand the asymptotic behaviour of

$$(5.19) \quad \hat{I} = \int_{\mathcal{I}} \frac{d\sigma}{\delta + c \mathcal{Q}P^\perp \cdot P^\perp}$$

as $\delta \rightarrow 0$, where $c = (1 \pm \tau) \frac{R_1 + R_2}{2R_1R_2} > 0$. Once we have that, since I_1 is finite, the asymptotic behaviour of $\mathcal{U}_\delta^1 - \mathcal{U}_\delta^2$ follows from (5.17) and (5.18).

We notice that P^\perp lies on $\{x_N = 0\}$, and so we write $P^\perp = x' = (x_1, \dots, x_{N-1})$ for $P \in \mathcal{I}$. From the implicit function theorem, there exists a function $\phi : \{|\mathcal{Q}^{1/2}x'| < w\} \rightarrow \mathbb{R}$ such that $H_0(x', \phi(x')) = R_1$, $\phi(0) = \delta$ and $(x', \phi(x')) \in \mathcal{I}$. Hence (5.19) becomes

$$\hat{I} = \int_{\{|\mathcal{Q}^{1/2}x'| < w\}} \frac{\sqrt{1 + |\nabla_{x'} \phi(x')|^2} dx'}{\delta + c \mathcal{Q}x' \cdot x'}.$$

Since

$$1 + |\nabla_{x'} \phi(x')|^2 = \frac{|\nabla H_0(x', \phi(x'))|^2}{(\partial_{\xi_N} H_0(x', \phi(x')))^2},$$

and $(x', \phi(x'))$ lies on a Wulff shape, we find that

$$1 + |\nabla_{x'} \phi(x')|^2 = \frac{1}{(\partial_{\xi_N} H_0(x', \phi(x')))^2} = \frac{1}{(\partial_{\xi_N} H_0(P_0))^2} (1 + o(x')),$$

as $x' \rightarrow 0$, and, by letting $y' = c^{\frac{1}{2}} \delta^{-\frac{1}{2}} \mathcal{Q}^{\frac{1}{2}} x'$ we obtain

$$\widehat{I} = (c \det(\mathcal{Q}))^{-\frac{N-1}{2}} C_N |\partial_{\xi_N} H_0(P_0)|^{-1} \Psi_N^{-1}(\delta)(1 + o(1))$$

as $\delta \rightarrow 0^+$. $\Psi_N(\delta)$ is given by (5.11), where we used Remark 5.1.

From (5.17) and (5.18) we obtain (5.10). The assumption of the theorem follows from the mean value theorem. The proof of Theorem 1.1 is complete.

Remark 5.1. Let $z \in \mathbb{R}^{N-1}$, I_δ be given by

$$I_\delta = \int_{|z| < \frac{1}{\sqrt{\delta}}} \frac{dz}{1 + |z|^2}$$

and

$$\psi_N(\delta) = \begin{cases} 1 & N = 2, \\ -\log \delta & N = 3, \\ \delta^{-\frac{N-3}{2}} & N \geq 4, \end{cases}$$

Then

$$\lim_{\delta \rightarrow 0} C_N \psi_N(\delta)^{-1} I_\delta = 1,$$

where C_N is a constant depending only on the dimension N .

APPENDIX A. BASIC FACTS FOR THE ANISOTROPIC CONDUCTIVITY PROBLEM

Let Ω be a subset of \mathbb{R}^N and $\{D^i\}_{i \in \{1, \dots, m\}}$ be a family open domains, such that $\overline{D}^i \cap \overline{D}^j = \emptyset$ for $i \neq j$, with boundaries of class $C^{2,\alpha}$, with $0 < \alpha < 1$. Let

$$D = \bigcup_{i=1}^m D^i.$$

Let $\Omega_D = \Omega \setminus \overline{D}$ and let $\varphi \in C^{2,\alpha}(\Omega)$. As mentioned in the Introduction, the perfectly conductivity problem is the following

$$(E_H) \quad \begin{cases} \operatorname{div}(H(\nabla u) \nabla_\xi H(\nabla u)) = 0 & \text{in } \Omega_D, \\ u_+ = u_- & \text{in } \partial D, \\ H(\nabla u) = 0 & \text{in } D, \\ \int_{\partial D^i} H(\nabla u) \nabla_\xi H(\nabla u) \cdot \nu ds = 0 & i = 1, \dots, m, \\ u = \varphi & \text{on } \partial \Omega, \end{cases}$$

where ν denotes the outward unit normal to D and Ω .

By regularity elliptic theory we have that $u \in C^{1,\alpha}(\Omega_D)$ (see [24]) and $H(\nabla u) \nabla_\xi H(\nabla u) \in W_{\text{loc}}^{1,2}(\Omega)$ (see [4, 19]).

Theorem A.1. *There exists at most one solution $u \in H^1(\Omega_D) \cap C^{1,\alpha}(\overline{\Omega}_D)$ of problem (E_H).*

Proof. Let $u_1, u_2 \in H^1(\Omega_D)$ be two solutions of (E_H). By multiplying the first equation of (E_H) by $u_1 - u_2$ and integrating by parts, for $j \in \{1, 2\}$, we have

$$\begin{aligned} 0 &= \int_{\Omega_D} H(\nabla u_j) \nabla_\xi H(\nabla u_j) \cdot \nabla(u_1 - u_2) dx - \int_{\partial \Omega} H(\nabla u_j) \nabla_\xi H(\nabla u_j) (u_1 - u_2) \cdot \nu ds \\ &\quad + \sum_{i=1}^m \int_{\partial D^i} H(\nabla u_j) \nabla_\xi H(\nabla u_j) (u_1 - u_2) \cdot \nu ds \\ &= \int_{\Omega_D} H(\nabla u_j) \nabla_\xi H(\nabla u_j) \cdot \nabla(u_1 - u_2) dx, \end{aligned}$$

where in the last equality we used the fourth condition in (E_H) and the fact that $u_1 = u_2$ on $\partial\Omega$. Thus, by the strong convexity of H , we have

$$0 = \int_{\Omega_D} (H(\nabla u_1) \nabla_\xi H(\nabla u_1) - H(\nabla u_2) \nabla_\xi H(\nabla u_2)) \cdot \nabla(u_1 - u_2) dx \geq \lambda \int_{\Omega_D} |\nabla(u_1 - u_2)|^2 dx \geq 0.$$

Thus $\nabla u_1 = \nabla u_2$ in Ω_δ and, since $u_1 = u_2$ on ∂D^i , we have $u_1 = u_2$ in Ω_D . \square

We define the energy functional

$$I_\infty[u] = \frac{1}{2} \int_{\Omega_\delta} H(\nabla u)^2 dx,$$

where u belongs to the set

$$\mathcal{A} := \{u \in W_\varphi^{1,2}(\Omega) : H(\nabla u) = 0 \text{ on } \overline{D}\}.$$

Theorem A.2. *There exists a minimizer $u \in \mathcal{A}$ satisfying*

$$I_\infty[u] = \min_{v \in \mathcal{A}} I_\infty[v].$$

Moreover, $u \in W^{1,2}(\Omega_D) \cap C^{1,\alpha}(\overline{\Omega}_D)$ is a solution to (E_H) .

Proof. The existence of the minimizer and the Euler Lagrange equation $\Delta_H u = 0$ follows from standard methods in the calculus of variations. The only thing which we need to show is the fourth equation of (E_H) . Let $i \in \{1, \dots, m\}$ be fixed and let $\phi \in C_0^\infty(\Omega)$ be such that

$$\phi = \begin{cases} 1, & \text{on } \partial D^i, \\ 0, & \text{on } \partial D^j, \text{ for } j \neq i. \end{cases}$$

Since u is a minimizer, by integrating by parts we obtain

$$\begin{aligned} 0 &= - \int_{\Omega_D} \operatorname{div}(H(\nabla u) \nabla_\xi H(\nabla u)) \phi dx \\ &= \int_{\Omega_D} H(\nabla u) \nabla_\xi H(\nabla u) \cdot \nabla \phi dx - \int_{\partial\Omega} H(\nabla u) \nabla_\xi H(\nabla u) \phi \cdot \nu ds \\ &\quad + \sum_{j=1}^m \int_{\partial D^j} H(\nabla u) \nabla_\xi H(\nabla u) \phi \cdot \nu ds \\ &= \int_{\partial D^i} H(\nabla u) \nabla_\xi H(\nabla u) \cdot \nabla \phi, dx \end{aligned}$$

and we conclude. \square

APPENDIX B. ESTIMATES FOR THE RADII OF THE TOUCHING BALLS IN THE PROOF OF THEOREM 1.1

In this Appendix we prove two technical lemmas needed in the proof of Theorem 1.1. We recall that D_δ^1 and D_δ^2 are Wulff shapes of radii R_1 and R_2 , respectively.

In the first lemma, for a point $P \in \partial D_\delta^1$ we consider the ball of radius r_1 touching ∂D_δ^1 at P from the inside; r_2 is the radius of the concentric ball which touches D_δ^2 from the outside (see Fig. 5).

Lemma B.1. *Let $s \in (0, 1]$ and let P_0, P, R_1, R_2, r_2 and r_1 be as in the proof of Theorem 1.1, then*

$$(B.1) \quad r_2 - r_1 = \delta + (1 - s) \frac{R_1 + R_2}{2R_1 R_2} \mathcal{Q} P^\perp \cdot P^\perp + o(\delta^2 + |\omega|^2)$$

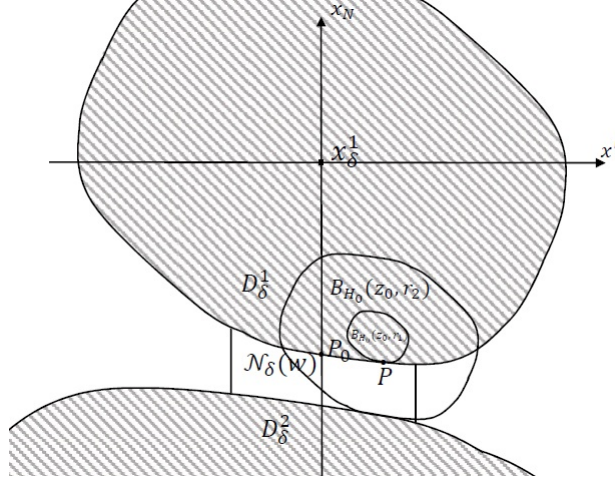


FIGURE 5.

as δ and $|P - P_0|$ go to zero, and where P^\perp is the projection of P on the orthogonal to P_0 .

Proof. Without loss of generality, we may assume that the ball D_δ^1 has center at the origin and D_δ^2 has center in $Z = (0, \dots, 0, Z_N)$, with $Z_N < 0$ and $H_0(Z) = R_1 + R_2 + \delta$. Let Q be the center of the ball of radius $r_1 = tR_1$, $t \in (0, 1]$, touching ∂D_δ^1 at P from the inside, and let r_2 be the radius of the ball centered at Q which is tangent to ∂D_δ^2 . In particular

$$(B.2) \quad r_2 + R_2 = H_0(Q - Z).$$

It is clear that, denoting by ν and ν_H the Euclidean and anisotropic outer unit normals, respectively, we have

$$\nu_H := \nu_H^{ext} = \nabla_\xi H(\nu(x)) = \nabla_\xi H\left(-\frac{\nabla u}{|\nabla u|}\right) = \frac{x}{H_0(x)} = \frac{x}{R_1},$$

at any point on ∂D_δ^1 . Since

$$(B.3) \quad Q - P = -tR_1\nu_H(P),$$

$$(B.4) \quad P = R_1\nu_H(P),$$

and

$$(B.5) \quad Z = (R_1 + R_2 + \delta)\nu_H(P_0),$$

then, by using (B.3), (B.4) and (B.5), we have that (B.2) can be written as

$$\begin{aligned} r_2 + R_2 &= H_0(Q - Z) = H_0(Q - P + P - Z) \\ &= H_0((1-t)R_1\nu_H(P) - (R_1 + R_2 + \delta)\nu_H(P_0)) \\ &= H_0((1-t)R_1\nu_H(P_0) + (1-t)R_1(\nu_H(P) - \nu_H(P_0)) - (R_1 + R_2 + \delta)\nu_H(P_0)) \\ &= R_1H_0\left(\left(\frac{R_2}{R_1} + t\right)\nu_H(P_0) + \psi\right) \end{aligned}$$

where

$$\psi = \frac{\delta}{R_1}\nu_H(P_0) - (1-t)(\nu_H(P) - \nu_H(P_0))$$

is small for δ small and P close to P_0 . By Taylor expansion and using the homogeneity properties of H_0 , we have

$$\begin{aligned} \frac{r_2}{R_2} + 1 &= \frac{R_1}{R_2} \left\{ \left(\frac{R_2}{R_1} + t \right) H_0(\nu_H(P_0)) + \nabla H_0(\nu_H(P_0)) \cdot \psi + \frac{1}{2} \left(\frac{R_2}{R_1} + t \right)^{-1} \nabla^2 H_0(\nu_H(P_0)) \psi \cdot \psi + o(|\psi|^2) \right\} \\ &= \frac{R_1}{R_2} \left\{ \frac{R_2}{R_1} + t + \nabla H_0(\nu_H(P_0)) \cdot \psi + \frac{1}{2} \left(\frac{R_2}{R_1} + t \right)^{-1} \nabla^2 H_0(\nu_H(P_0)) \psi \cdot \psi + o(|\psi|^2) \right\}. \end{aligned}$$

as $\delta \rightarrow 0^+$ and $P \rightarrow P_0$. Since $tR_1 = r_1$ we have

$$(B.6) \quad \frac{r_2 - r_1}{R_2} = \frac{R_1}{R_2} \left(\nabla H_0(\nu_H(P_0)) \cdot \psi + \frac{1}{2} \left(\frac{R_2}{R_1} + t \right)^{-1} \nabla^2 H_0(\nu_H(P_0)) \psi \cdot \psi + o(|\psi|^2) \right).$$

Since

$$\nabla H_0(\nu_H(P_0)) \nu_H(P_0) = 1,$$

then

$$\nabla H_0(\nu_H(P_0)) \cdot \psi = \frac{\delta}{R_1} - (1-t) \nabla H_0(\nu_H(P_0)) \cdot (\nu_H(P) - \nu_H(P_0)).$$

and since

$$\nabla^2 H_0(\nu_H(P_0)) \nu_H(P_0) = 0,$$

we find

$$\nabla^2 H_0(\nu_H(P_0)) \psi \cdot \psi = (1-t)^2 \nabla^2 H_0(\nu_H(P_0)) (\nu_H(P) - \nu_H(P_0)) \cdot (\nu_H(P) - \nu_H(P_0)).$$

From (B.6) we obtain

$$(B.7) \quad \frac{r_2 - r_1}{R_2} = \frac{\delta}{R_2} + \frac{R_1}{R_2} \left[-(1-t) \nabla H_0(\nu_H(P_0)) \cdot \zeta + \frac{(1-t)^2}{2 \left(\frac{R_2}{R_1} + t \right)} \nabla^2 H_0(\nu_H(P_0)) \zeta \cdot \zeta + o(\delta^2 + |\zeta|^2) \right].$$

where we set

$$\zeta = \nu_H(P) - \nu_H(P_0).$$

Now we observe that $1 = H_0(\nu_H(P)) = H_0(\nu_H(P_0) + \zeta)$ which gives

$$1 = H_0(\nu_H(P_0)) + \nabla H_0(\nu_H(P_0)) \cdot \zeta + \frac{1}{2} \nabla^2 H_0(\nu_H(P_0)) \zeta \cdot \zeta + o(|\zeta|^2),$$

so that, since $H_0(\nu_H(P_0)) = 1$,

$$-\nabla H_0(\nu_H(P_0)) \cdot \zeta = \frac{1}{2} \nabla^2 H_0(\nu_H(P_0)) \zeta \cdot \zeta + o(|\zeta|^2),$$

and (B.8)

$$r_2 - r_1 = \delta + R_1 \frac{1-t}{2} \frac{1 + \frac{R_2}{R_1}}{t + \frac{R_2}{R_1}} \nabla^2 H_0(\nu_H(P_0)) \zeta \cdot \zeta + o(\delta^2 + |\zeta|^2).$$

From (2.10), we notice that the range of $\nabla^2 H_0(\xi)$ lies in ξ^\perp and hence

$$(B.8) \quad r_2 - r_1 = \delta + R_1 \frac{1-t}{2} \frac{1 + \frac{R_2}{R_1}}{t + \frac{R_2}{R_1}} \nabla^2 H_0(\nu_H(P_0)) \nu_H(P)^\perp \cdot \nu_H(P)^\perp + o(\delta^2 + |\zeta|^2),$$

where $\nu_H(P)^\perp$ is the projection of $\nu_H(P)$ on the orthogonal to $\nu_H(P_0)^\perp$. Since P and P_0 are on the boundary of the Wulff shape, we have $\nu_H(P) = P/R_1$, $\nu_H(P_0) = P_0/R_1 = \hat{P}$ and from (1.9) we obtain (B.9). \square

In the following lemma, for a point $P \in \partial D_\delta^1 \cap \mathcal{N}_\delta(w)$, with w small enough, we consider a ball of radius ρ_2 touching ∂D_δ^1 at P from the outside and having center inside D_δ^2 ; ρ_1 is the radius of the concentric ball which touches D_δ^2 from the inside (see Fig. 6).

Lemma B.2. *Let P_0, P, ρ_2 and ρ_1 be as in the proof of Theorem 1.1. There exists a constant C independent of δ and w such that for any $\delta + C|w| < t < \frac{1}{2}$, with δ and $|w|$ sufficiently small, we have*

$$(B.9) \quad \rho_2 - \rho_1 = \delta + (1+t) \frac{R_1 + R_2}{2R_1R_2} \mathcal{Q}P^\perp \cdot P^\perp + o(\delta^2 + |P - P_0|^2)$$

as δ and $|P - P_0|$ go to zero, and where P^\perp is the projection of P on the orthogonal to P_0 .

Proof. By arguing as in the proof of Lemma B.1 we have that ρ_1 and ρ_2 are related by the following identity

$$R_2 - \rho_1 = H_0((R_1 + \rho_2)\nu_H(P) - (R_1 + R_2 + \delta)\nu_H(P_0)).$$

By simple manipulations we have

$$R_2 - \rho_1 = (R_2 - \rho_2)H_0(\nu_H(P_0) + \phi),$$

where

$$\phi = \frac{(R_1 + \rho_2)(\nu_H(P) - \nu_H(P_0)) - \delta\nu_H(P_0)}{\rho_2 - R_2}.$$

Hence

$$R_2 - \rho_1 = (R_2 - \rho_2) \left\{ H_0(\nu_H(P_0)) + \nabla H_0(\nu_H(P_0)) \cdot \phi + \frac{1}{2} \nabla^2 H_0(\nu_H(P_0)) \phi \cdot \phi + o(|\phi|^2) \right\},$$

as $\delta \rightarrow 0$ and $P \rightarrow P_0$, and since $H_0(\nu_H(P_0)) = 1$, we find

$$\rho_2 - \rho_1 = (R_2 - \rho_2) \left\{ \nabla H_0(\nu_H(P_0)) \cdot \phi + \frac{1}{2} \nabla^2 H_0(\nu_H(P_0)) \phi \cdot \phi + o(|\phi|^2) \right\}.$$

By setting

$$\zeta = \nu_H(P) - \nu_H(P_0),$$

as done in the previous lemma we have that

$$\begin{aligned} \nabla H_0(\nu_H(P_0)) \cdot \phi &= -\frac{R_1 + \rho_2}{R_2 - \rho_2} \nabla H_0(\nu_H(P_0)) \cdot \zeta + \frac{\delta}{R_2 - \rho_2} \nabla H_0(\nu_H(P_0)) \cdot \nu_H(P_0) \\ &= \frac{\delta}{R_2 - \rho_2} - \frac{R_1 + \rho_2}{R_2 - \rho_2} \nabla H_0(\nu_H(P_0)) \cdot \zeta \\ &= \frac{\delta}{R_2 - \rho_2} + \frac{R_1 + \rho_2}{R_2 - \rho_2} \frac{1}{2} \nabla^2 H_0(\nu_H(P_0)) \zeta \cdot \zeta + o(|\zeta|^2), \end{aligned}$$

and we find

$$\rho_2 - \rho_1 = \delta + \frac{R_1 + \rho_2}{2} \frac{R_1 + R_2}{R_2 - \rho_2} \nabla^2 H_0(\nu_H(P_0)) \nu_H(P) \cdot \nu_H(P) + o(|\zeta|^2 + \delta^2).$$

We choose $\rho_2 = \bar{t}R_1$ with $\bar{t} > 0$. Since ∂D_δ^1 and ∂D_δ^2 are smooth, there exists a constant $\bar{C} > 0$ such that $\bar{t} > \delta + \bar{C}|w|^2$. Since $\nu_H(P) = P/R_1$ and $\nu_H(P_0) = P_0/R_1$, we conclude. \square

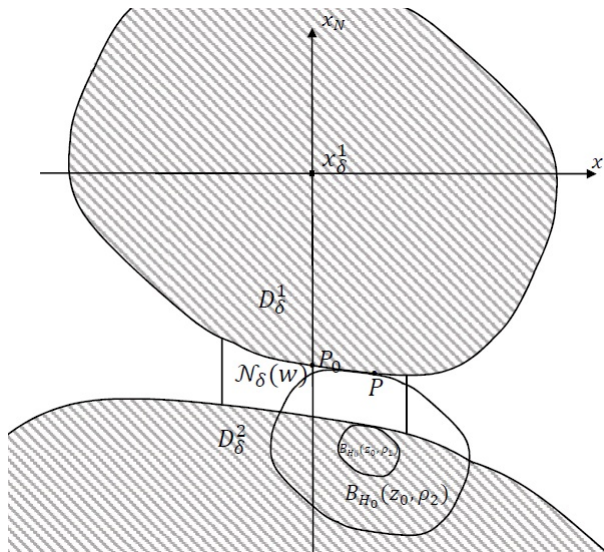


FIGURE 6.

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