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Minimal cyclic random motion in $\mathbb{R}^n$ and hyper-Bessel functions

A. Lachal [a,*], S. Leorato [b], E. Orsingher [b]

[a] *INSA de Lyon, Centre de Mathématiques, Bât. Léonard de Vinci, 20, avenue A. Einstein, 69621 Villeurbanne cedex, France*
[b] *Università di Roma "La Sapienza", Piazzale A. Moro, 5, 00185 Roma, Italy*

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Abstract

We obtain the explicit distribution of the position of a particle performing a cyclic, minimal, random motion with constant velocity c in $\mathbb{R}^n$. The $n+1$ possible directions of motion as well as the support of the distribution form a regular hyperpolyhedron (the first one having constant sides and the other expanding with time t), the geometrical features of which are here investigated.

The distribution is obtained by using order statistics and is expressed in terms of hyper-Bessel functions of order $n+1$. These distributions are proved to be connected with $(n+1)$th order p.d.e. which can be reduced to Bessel equations of higher order.

Some properties of the distributions obtained are examined. This research has been inspired by a conjecture formulated in Orsingher and Sommella [E. Orsingher, A.M. Sommella, A cyclic random motion in R^3 with four directions and finite velocity, Stochastics Stochastics Rep. 76 (2) (2004) 113–133] which is here proved to be false.

Résumé

Dans ce travail, on étudie l'évolution dans l'espace $\mathbb{R}^n$ d'une particule animée d'un mouvement aléatoire cyclique à vitesse constante. Le mouvement est supposé minimal au sens où les différentes directions prises sont au nombre de $n+1$; de plus, ces directions forment un hyper-polyèdre régulier fixe. Le support de la distribution de la position de la particule est également un hyper-polyèdre régulier (de taille évolutive au cours du temps).

Faisant appel aux statistiques d'ordre, on a pu obtenir explicitement la loi de probabilité de la position de la particule à un instant donné. Le résultat s'exprime au moyen de fonctions de Bessel généralisées d'ordre $n+1$ et montre que cette étude est liée à des équations aux dérivées partielles hyperboliques d'ordre $n+1$.

Ce travail a été inspiré par une conjecture formulée par Orsingher et Sommella [E. Orsingher, A.M. Sommella, A cyclic random motion in R^3 with four directions and finite velocity, Stochastics Stochastics Rep. 76 (2) (2004) 113–133], laquelle se révèle finalement être fausse.

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* Corresponding author.
E-mail address: aime.lachal@insa-lyon.fr (A. Lachal).

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1. Introduction

We here study the cyclic motion in $\mathbb{R}^n$ of a particle (with position $(\underline{X}(t), t \geqslant 0)$) which can take $n+1$ directions $\vec{v}_j$, $j=0,\dots,n$, and moves with speed $c<+\infty$. The extremities of the vectors representing the directions form a regular n-dimensional polyhedron with $n+1$ vertices (for short, we say $(n+1)$-hedron) inscribed in the unit sphere.

The particle can initially choose one of the possible directions with probability $\frac{1}{n+1}$. The directions are taken successively at Poisson paced times, that is the particle moving with direction $\vec{v}_j$, after a Poisson event, takes the direction $\vec{v}_{j+1}$, $j=0,\dots,n$ (with $\vec{v}_{n+1}=\vec{v}_0$).

The minimality of the number of directions, together with the cyclicity make the derivation of the explicit distributions of motion possible.

Motions of particles in a turbulent medium, for example in the presence of a vortex, can be adequately described by the models studied below.

Usually, the analysis of random motions with finite velocity is either performed by means of analytic arguments, essentially based on partial differential equations, or by a more probabilistic approach, based on order statistics.

The analytic method has been implemented in the study of cyclic motions on the plane (with three directions) and in space $\mathbb{R}^3$ (with four directions associated with a regular tetrahedron) by Orsingher [6] and Orsingher and Sommella [7]. Random motions in the plane with three directions and Erlang distributed interarrival times is considered in Di Crescenzo [1].

The approach based on order statistics has proved to be more suitable for generalizations on higher order spaces (as will be applied here) as well as on non-cyclic motions; the cases of planar motions – symmetrically deviating and with uniform choice of directions – is examined in Leorato and Orsingher [4].

This work proves that the conjecture formulated in the paper by Orsingher and Sommella [7] is false and we are now able to obtain the exact distributions of the position of a randomly moving particle in $\mathbb{R}^n$ and to show that it matches the necessary requirements, including the connection with the partial differential equations governing the probability laws. These equations, derived by different authors (Kolesnik [3], Samoilenko [8,9] and others) are related to hyper-Bessel functions analyzed by Kiryakova [2] and Turbin and Plotkin [10].

Our main result is the derivation of the distribution of $\underline{X}(t)$:

$$\begin{aligned}\tilde{p}_r(\underline{x},t)\,\mathrm{d}\underline{x} &= \Pr\bigl\{\underline{X}(t)\in \mathrm{d}\underline{x},\ \text{complete cycle} + r \text{ directions}\bigr\}\\ &= \frac{\mathrm{d}\underline{x}\,\mathrm{e}^{-\lambda t}}{(n+1)^r n! V_n}\left(\frac{\lambda}{c}\right)^{n+r}\mathcal{H}_{r,n+1}(\underline{x},t)\mathcal{I}_{r,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n} h_j(\underline{x},t)}\right)\end{aligned}\tag{1.1}$$

where $\underline{x}=(x_0,\dots,x_{n-1})$, $\mathrm{d}\underline{x}=\mathrm{d}x_0\cdots\mathrm{d}x_{n-1}$ and $h_j(\underline{x},t)=ct+n\sum_{i=0}^{n-1} v_{i,j}x_i=0$ are the equations of the hyperfaces of a $(n+1)$-hedron $\mathfrak{T}_{ct}$ with volume $V_n(ct)^n$, and $(v_{0,j},\dots,v_{n-1,j})$ are the coordinates of the vectors $\vec{v}_j$ for $j=0,\dots,n$.

The functions $\mathcal{H}_{r,n+1}$ are defined as

$$\mathcal{H}_{r,n+1}(\underline{x},t)=\frac{1}{n+1}\sum_{j=0}^{n}\prod_{l=j}^{j+r-1} h_l(\underline{x},t)$$

while the hyper-Bessel functions $\mathcal{I}_{r,n}(x)$ are

$$\mathcal{I}_{r,n}(x)=\sum_{q=0}^{\infty}\frac{1}{(q!)^{n-r}((q+1)!)^r}\left(\frac{x}{n}\right)^{nq},\quad 0\leqslant r\leqslant n.$$

Our paper is organized as follows. The second section is devoted to the geometrical description of the directions of motion together with that of the support of the distributions.

We then turn our attention to the probabilistic analysis of the cyclic motion as well as to the analysis of the related governing equations (Section 3). In particular, the functions $q_j(\underline{u})$ for $0\leqslant j\leqslant n$, defined as

$$p_j(\underline{x},t)=\mathit{const}\cdot q_j(\underline{u})=\Pr\bigl\{\underline{X}(t)\in\mathrm{d}\underline{x},\ \text{the current direction is } \vec{v}_j\bigr\}/\mathrm{d}\underline{x}$$

where $\underline{u}=(u_0,\dots,u_n)=(h_0(\underline{x},t),\dots,h_n(\underline{x},t))$, are governed by the $(n+1)$th order partial differential equations

$$\frac{\partial^{n+1}q_j}{\partial u_0\cdots\partial u_n}=\left(\frac{\lambda}{(n+1)c}\right)^{n+1}q_j. \tag{1.2}$$

We shall see that the p.d.e. (1.2) is related to the $(n+1)$th order Bessel equation:

$$\left(w\frac{\partial}{\partial w}\right)^{n+1}q=\left(\frac{\lambda w}{c}\right)^{n+1}q.$$

The fourth part of the paper is concerned with the explicit derivation of the conditional probabilities below by means of order statistics:

$$\Pr\{\underline{X}(t)\in \underline{\mathrm{d}x}\mid N(t)=(n+1)q+r-1\},\quad r=0,\dots,n,\ q\geqslant 1, \tag{1.3}$$

where $N(t)$ denotes the number of Poisson events up to time t.

Our analysis shows that the conditional distributions (1.3) coincide with the corresponding components of (1.12) of Orsingher and Sommella [7] for $r=0,n$ and arbitrary values of $q=1,2,\dots$, while for $r=1,\dots,n-1$ the conditional distributions and, a fortiori, the absolutely continuous component differ from those conjectured. By summing (1.1) we get a complete expression for the absolutely continuous part of the distribution of motion in $\mathbb{R}^n$. Let us point out that the singular component of the distribution of $\underline{X}(t)$ is spread on the $\sum_{k=1}^{n}\binom{n+1}{k}$ subspaces of dimension $d=k-1$, composing the boundary of $\mathfrak{T}_{ct}$, with $0\leqslant d\leqslant n-1$.

The concluding section is devoted to checking that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathfrak{T}_{ct}}\mathcal{I}_{0,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n}h_j(\underline{x},t)}\right)\underline{\mathrm{d}x}\right]=\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Vol}(\mathfrak{T}_{ct})+\int_{\mathfrak{T}_{ct}}\frac{\partial}{\partial t}\left[\mathcal{I}_{0,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n}h_j(\underline{x},t)}\right)\right]\underline{\mathrm{d}x}.$$

For higher-order derivatives, a formula similar to that above does not hold and this is the reason why the conjecture formulated in Orsingher and Sommella [7] is not true.

In Subsection 5.2, we have shown that the distributions obtained (suitably simplified) satisfy an $(n+1)$th order p.d.e. related to the hyper-Bessel equation.

2. The directional vectors and the geometry of $\mathfrak{T}_{ct}$

2.1. The vectors of directions

Let $\vec{v}_j$, $j=0,\dots,n$, be the vectors representing the possible directions of the cyclic motion. Let $\vec{v}_0=(1,0,\dots,0)$ and let $\vec{v}_j=(v_{0,j},v_{1,j},\dots,v_{j,j},0,\dots,0)$ and $\vec{v}_n=(v_{0,n},v_{1,n},\dots,v_{n-1,n})$. In order to evaluate the numbers $v_{i,j}$, $0\leqslant i\leqslant n-1$, $0\leqslant j\leqslant n$, we consider the following symmetry conditions:

$$\vec{v}_j\cdot\vec{v}_k=const\quad\text{if } j\neq k, \tag{2.1}$$

$$|\vec{v}_j|^2=1\quad\text{for all } j, \tag{2.2}$$

$$\sum_{j=0}^{n}\vec{v}_j=\vec{\mathbf{0}}. \tag{2.3}$$

The constant in (2.1) is equal to $-\frac{1}{n}$ as can be obtained in the following manner:

$$0=\left(\sum_{j=0}^{n}\vec{v}_j\right)\cdot\vec{v}_k=|\vec{v}_k|^2+\sum_{\substack{j=0\\ j\neq k}}^{n}\vec{v}_j\cdot\vec{v}_k=1+n\,const.$$

Thus, the conditions (2.1) and (2.2) can be rewritten as

$$\vec{v}_j\cdot\vec{v}_k=\begin{cases}-\frac{1}{n} & \text{if } j\neq k,\\ 1 & \text{if } j=k.\end{cases} \tag{2.4}$$

The coordinates of the vector $\vec{v}_1 = (-\frac{1}{n}, v_{1,1}, 0, \dots, 0)$ can immediately be calculated because of (2.2): $v_{1,1} = \pm\sqrt{1 - \frac{1}{n^2}}$. By convention we take, throughout the paper, the positive signs. We also get, from (2.4), that $v_{0,j} = -\frac{1}{n}$, for $1 \leqslant j \leqslant n$, that is, all the first components of the vectors $\vec{v}_j$, $j \geqslant 1$, are identical.

From (2.4) we obtain that

$$v_{1,j} = -\frac{1}{n}\sqrt{\frac{n+1}{n-1}} \quad \text{for } 2 \leqslant j \leqslant n$$

because

$$-\frac{1}{n} = v_{0,1}v_{0,j} + v_{1,1}v_{1,j} = \frac{1}{n^2} + \frac{\sqrt{n^2-1}}{n} v_{1,j} \quad \text{for } j \geqslant 2.$$

By applying the same procedure we can check that

$$\vec{v}_2 = \left(-\frac{1}{n}, -\frac{1}{n}\sqrt{\frac{n+1}{n-1}}, \sqrt{\frac{n+1}{n}}\sqrt{\frac{n-2}{n-1}}, 0, \dots, 0\right).$$

This easily permits us to find that $v_{2,j} = -\sqrt{\frac{n+1}{n}}\frac{1}{\sqrt{(n-1)(n-2)}}$ for $j > 2$ because $\vec{v}_2 \cdot \vec{v}_j = \sum_{i=0}^{2} v_{i,2}v_{i,j} = -\frac{1}{n}$.

We write down the general form of the coordinates $v_{i,j}$ of the vectors $\vec{v}_j$ for $j \leqslant n-1$

$$v_{i,j} = \begin{cases} -\sqrt{\frac{n+1}{n}}\frac{1}{\sqrt{(n-i+1)(n-i)}} & \text{if } i < j, \\ \sqrt{\frac{n+1}{n}\frac{n-j}{n+1-j}} & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases} \tag{2.5}$$

The components $v_{i,j}$ for $i < j$ can be easily evaluated by induction by using (2.4) as for $v_{1,j}$ and $v_{2,j}$. Then, in view of (2.2), we find that

$$v_{j,j} = \sqrt{1 - \sum_{i=0}^{j-1} v_{i,j}^2} = \sqrt{1 - \sum_{i=0}^{j-1} \frac{n+1}{n}\frac{1}{(n-i+1)(n-i)}}. \tag{2.6}$$

The sum inside (2.6) can be evaluated without effort:

$$\sum_{i=0}^{j-1} \frac{1}{(n-i+1)(n-i)} = \sum_{i=0}^{j-1}\left[\frac{1}{n-i} - \frac{1}{n-i+1}\right] = \frac{j}{(n+1)(n-j+1)}.$$

Finally, the last vector $\vec{v}_n$, thanks to (2.3) writes

$$v_{i,n} = -\sqrt{\frac{n+1}{n}}\frac{1}{\sqrt{(n-i+1)(n-i)}}, \quad \text{for } 0 \leqslant i \leqslant n-1.$$

The matrix $\mathbf{V}$ of the components of the directional vectors $\vec{v}_j$, $j = 0, \dots, n$, is given in Table 1. The reader can check that conditions (2.3) are fulfilled.

2.2. *The* $(n+1)$*-hedron* $\mathfrak{T}_a$

Let us introduce the points A_j, $j = 0, \dots, n$, defined by $\overrightarrow{OA_j} = a\vec{v}_j$, for a fixed $a > 0$. The points A_j, $j = 0, \dots, n$, are the vertices of a regular $(n+1)$-hedron $\mathfrak{T}_a$ with center O. The length of the edges of $\mathfrak{T}_a$ can clearly be obtained by observing that, for the edge $\overrightarrow{A_jA_k}$, with endpoints A_j and A_k, we have that $\overrightarrow{A_jA_k} = a(\vec{v}_k - \vec{v}_j)$ and thus, by (2.4),

$$\left|\overrightarrow{A_jA_k}\right|^2 = a^2\left(|\vec{v}_k|^2 + |\vec{v}_j|^2 - 2\vec{v}_j \cdot \vec{v}_k\right) = 2\frac{n+1}{n}a^2.$$

Table 1
The matrix $\mathbf{V}$ of directions

$\vec{v}_0$	$\vec{v}_1$	$\vec{v}_2$	$\cdots$	$\vec{v}_{n-1}$	$\vec{v}_n$
1	$-\frac{1}{n}$	$-\frac{1}{n}$	$\cdots$	$-\frac{1}{n}$	$-\frac{1}{n}$
0	$\sqrt{\frac{n+1}{n}\frac{n-1}{n}}$	$-\frac{1}{n}\sqrt{\frac{n+1}{n-1}}$	$\cdots$	$-\frac{1}{n}\sqrt{\frac{n+1}{n-1}}$	$-\frac{1}{n}\sqrt{\frac{n+1}{n-1}}$
0	0	$\sqrt{\frac{(n+1)(n-2)}{n(n-1)}}$	$\cdots$	$-\sqrt{\frac{n+1}{n(n-1)(n-2)}}$	$-\sqrt{\frac{n+1}{n(n-1)(n-2)}}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
0	0	0	$\cdots$	$-\sqrt{\frac{n+1}{n(n-i+2)(n-i+1)}}$	$-\sqrt{\frac{n+1}{n(n-i+2)(n-i+1)}}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
0	0	0	$\cdots$	$-\sqrt{\frac{n+1}{n}\frac{1}{2\cdot 3}}$	$-\sqrt{\frac{n+1}{n}\frac{1}{2\cdot 3}}$
0	0	0	$\cdots$	$\sqrt{\frac{n+1}{n}\frac{1}{2}}$	$-\sqrt{\frac{n+1}{n}\frac{1}{2}}$

2.2.1. Analytic representation of $\mathfrak{T}_a$

We now derive the equations of the hyperfaces of $\mathfrak{T}_a$, which play an important role in the distribution of $\underline{X}(t)$.

Let A_k be an arbitrary vertex of the hyperface F_j orthogonal to the vector $\vec{v}_j$ for $j \neq k$. It is clear that

$$\overrightarrow{OM} = \overrightarrow{OA_k} + \overrightarrow{A_kM} = a\vec{v}_k + \overrightarrow{A_kM}, \tag{2.7}$$

where M is an arbitrary point of F_j (and thus $\overrightarrow{A_kM}$ is orthogonal to $\vec{v}_j$) with coordinates $(x_0, \dots, x_{n-1})$. By taking the scalar product of (2.7) by $\vec{v}_j$ we get

$$\sum_{i=0}^{n-1} v_{i,j}x_i = \vec{v}_j \cdot \overrightarrow{OM} = \vec{v}_j \cdot \bigl(a\vec{v}_k + \overrightarrow{A_kM}\bigr) = -\frac{a}{n}.$$

The equation of the hyperface F_j for $j = 0, \dots, n$ is thus $\sum_{i=0}^{n-1} v_{i,j}x_i + \frac{a}{n} = 0$. The $(n+1)$-hedron $\mathfrak{T}_a$ is then analytically defined as

$$\mathfrak{T}_a = \left\{(x_0, \dots, x_{n-1}) \in \mathbb{R}^n\colon \sum_{i=0}^{n-1} v_{i,j}x_i + \frac{a}{n} \geqslant 0 \text{ for } j = 0, \dots, n\right\} \tag{2.8}$$

and we shall see that $\mathfrak{T}_{ct}$ is the set of all possible positions of the moving particle at time t.

Remark 2.1. Observe that the inequality $\sum_{i=0}^{n-1} v_{i,j}x_i + \frac{a}{n} > 0$ represents the half-space containing the vertex $A_j = (av_{0,j}, \dots, av_{n-1,j})$. Indeed, in A_j we have that $a\sum_{i=0}^{n-1} v_{i,j}^2 + \frac{a}{n} = a(1 + \frac{1}{n}) > 0$.

2.2.2. Volume of $\mathfrak{T}_a$

For our further analysis, it is useful to evaluate the volume of the $(n+1)$-hedron $\mathfrak{T}_a$. This can be split up into $n+1$ not regular $(n+1)$-hedrons of equal volume obtained from $\mathfrak{T}_a$ with each vertex successively being replaced by the origin O. In this way we have that

$$\operatorname{Vol}(\mathfrak{T}_a) = (n+1)\operatorname{Vol}\bigl(\mathfrak{T}_a^O\bigr) \tag{2.9}$$

where $\mathfrak{T}_a^O$ is a $(n+1)$-hedron with one vertex in O. By means of a well-known formula and by using Table 1, we get

$$\operatorname{Vol}\bigl(\mathfrak{T}_a^O\bigr) = \frac{a^n}{n!}\det(\vec{v}_0, \dots, \vec{v}_{n-1}) = \frac{a^n}{n!}\begin{vmatrix} 1 & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ 0 & \sqrt{\frac{n+1}{n}\frac{n-1}{n}} & \cdots & -\frac{1}{n}\sqrt{\frac{n+1}{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\frac{n+1}{2n}} \end{vmatrix}$$

$$= \frac{a^n}{n!}\left(\sqrt{\frac{n+1}{n}}\right)^{n-1} \cdot \frac{1}{\sqrt{n}} = \frac{(n+1)^{(n-1)/2}}{n^{n/2}\,n!}\,a^n.$$

Therefore, from (2.9) we have that

$$\mathrm{Vol}(\mathfrak{T}_a) = \frac{(n+1)^{(n+1)/2}}{n^{n/2}\,n!}\,a^n. \tag{2.10}$$

In the rest of the paper we shall use the results of this section applied to the case where $a = ct$. We also put $V_n = \mathrm{Vol}(\mathfrak{T}_1) = \frac{(n+1)^{(n+1)/2}}{n^{n/2} n!}$.

3. Analytic description of random motion

3.1. The support of the random position

We consider the randomly moving point $\underline{X}(t) = (X_0(t), X_1(t), \ldots, X_{n-1}(t))$ which, at time $t = 0$, was at the origin $O = (0, \ldots, 0)$ and initially took one of the directions $\vec{v}_j$, $j = 0, \ldots, n$, with probability $\frac{1}{n+1}$.

The motion is cyclic in the sense that at each Poisson event the particle switches from direction $\vec{v}_j$ to $\vec{v}_{j+1}$ (we set $\vec{v}_{n+1} = \vec{v}_0$ and, in general, $\vec{v}_{(n+1)k+j} = \vec{v}_j$ for any positive integer k).

Let $N(t)$ be the number of Poisson events up to time t and let $T_1, \ldots, T_n, \ldots$ denote the instants where they occur. Therefore, the position $\underline{X}(t)$ of the moving particle starting with direction $\vec{v}_0$ is

$$\underline{X}(t) = c\left[I_{\{N(t)\geqslant 1\}} \sum_{k=0}^{N(t)-1} (T_{k+1} - T_k)\vec{v}_k + (t - T_{N(t)})\vec{v}_{N(t)}\right]. \tag{3.1}$$

The first sum in (3.1) refers to the case where at least one change of direction has occurred while the second one is related to the displacement along the current direction at time t. If $N(t) = 0$, then $\underline{X}(t) = c(t - T_{N(t)})\vec{v}_0 = ct\vec{v}_0$ and the particle is located in A_0 (with $a = ct$) at time t. If we put $T_{N(t)+1} = t$ the displacement (3.1) takes the form $\underline{X}(t) = c\sum_{k=0}^{N(t)}(T_{k+1} - T_k)\vec{v}_k$ and, by (2.4), for all $j = 0, \ldots, n$,

$$\begin{aligned}
\underline{X}(t)\cdot\vec{v}_j + \frac{ct}{n} &= c\sum_{\substack{k=0\\ \vec{v}_k\neq\vec{v}_j}}^{N(t)} -\frac{1}{n}(T_{k+1} - T_k) + c\sum_{\substack{k=0\\ \vec{v}_k=\vec{v}_j}}^{N(t)} (T_{k+1} - T_k) + \frac{ct}{n}\\
&= -\frac{c}{n}\sum_{k=1}^{N(t)}(T_{k+1} - T_k) + c\left(1 + \frac{1}{n}\right)\sum_{\substack{k=0\\ \vec{v}_k=\vec{v}_j}}^{N(t)} (T_{k+1} - T_k) + \frac{ct}{n}\\
&= c\left(1 + \frac{1}{n}\right)\sum_{\substack{k=0\\ \vec{v}_k=\vec{v}_j}}^{N(t)} (T_{k+1} - T_k) \geqslant 0.
\end{aligned}$$

This permits us to conclude that, in view of (2.8), the moving particle at time t is always inside or on the boundary of $\mathfrak{T}_{ct}$.

In order that the moving point $\underline{X}(t)$ be located on the boundary $\partial\mathfrak{T}_{ct}$ we must have that

$$\underline{X}(t)\cdot\vec{v}_j + \frac{ct}{n} = 0 \quad \text{for some } 0 \leqslant j \leqslant n.$$

This equality is realized if $N(t)$ is such that the identity $\vec{v}_k = \vec{v}_j$ does not hold for any value of k and thus the set on which the sum is performed is empty. If the initial direction is $\vec{v}_0$, then

$$\underline{X}(t)\cdot\vec{v}_j + \frac{ct}{n} = 0 \quad \text{for } N(t) < j \leqslant n, \tag{3.2}$$

which means that the moving particle lies on a $N(t)$-face (face of dimension $N(t)$) of $\partial\mathfrak{T}_{ct}$ at time t.

Remark 3.1. For example, for $n=2$ and $N(t)=1$ (which implies $j=2$ in (3.2)), we have

$$\underline{X}(t)\cdot\vec{v}_2+\frac{ct}{2}=0$$

which represents the side of $\partial\mathfrak{T}_{ct}$ opposite to $\vec{v}_2$. If $N(t)=0$, (3.2) holds for $j=1,2$ and therefore, the moving point is located simultaneously on the two lines $\underline{X}(t)\cdot\vec{v}_1+\frac{ct}{2}=0$ and $\underline{X}(t)\cdot\vec{v}_2+\frac{ct}{2}=0$, that is on the vertex $(ct,0)$.

3.2. About the governing equations

Let $D(t)$ be the current direction of motion at time t. Clearly $D(t)$ takes values $\vec{v}_j$, $j=0,\dots,n$. For the densities of the probabilities $p_j(\underline{x},t)\,\mathrm{d}\underline{x}=\Pr\{X(t)\in\mathrm{d}\underline{x},\,D(t)=\vec{v}_j\}$ for $\underline{x}\in\operatorname{int}(\mathfrak{T}_{ct})$ we have the following theorem.

Theorem 3.2. *The densities p_j satisfy the following differential system*

$$\frac{\partial p_j}{\partial t}(\underline{x},t)=-c\frac{\partial p_j}{\partial\vec{v}_j}(\underline{x},t)-\lambda\big(p_j(\underline{x},t)-p_{j-1}(\underline{x},t)\big)\quad \textit{for } 0\leqslant j\leqslant n, \tag{3.3}$$

where $p_{-1}=p_n$ and

$$\frac{\partial p_j}{\partial\vec{v}_j}(\underline{x},t)=\overrightarrow{\operatorname{grad}}\,p_j\cdot\vec{v}_j.$$

Proof. Suppose that the particle is located in $\underline{x}$ at time $t+\Delta t$. If no Poisson event has occurred during $[t,t+\Delta t]$ (which happens with probability $1-\lambda\Delta t+\mathrm{o}(\Delta t)$), then the particle must have been in $\underline{x}-c\Delta t\vec{v}_j$ at time t. If exactly one Poisson event has occurred (with probability $\lambda\Delta t+\mathrm{o}(\Delta t)$) during $[t,t+\Delta t]$, the direction of the particle at time t was $\vec{v}_{j-1}$. Finally, the probability that more than one Poisson event has occurred is $\mathrm{o}(\Delta t)$. This short discussion leads to the following equality:

$$p_j(\underline{x},t+\Delta t)=(1-\lambda\Delta t)p_j(\underline{x}-c\Delta t\vec{v}_j,t)+\lambda\Delta t p_{j-1}(\underline{x},t)+\mathrm{o}(\Delta t).$$

We next expand

$$p_j(\underline{x}-c\Delta t\vec{v}_j,t)=p_j(\underline{x},t)-c\Delta t\sum_{i=0}^{n-1}\frac{\partial p_j}{\partial x_i}(\underline{x},t)v_{i,j}+\mathrm{o}(\Delta t)$$

and

$$p_j(\underline{x},t+\Delta t)=p_j(\underline{x},t)+\frac{\partial p_j}{\partial t}(\underline{x},t)\Delta t+\mathrm{o}(\Delta t).$$

Some obvious simplifications and the limit with $\Delta t\to 0$ yield (3.3). $\square$

The system (3.3) can be substantially reduced as shown in the next theorem.

Theorem 3.3. *Let $p_j(\underline{x},t)=\mathrm{e}^{-\lambda t}q_j(\underline{u})$ where $\underline{u}$ is the vector with components*

$$u_j=ct+n\sum_{i=0}^{n-1}v_{i,j}x_i,\quad j=0,\dots,n. \tag{3.4}$$

Then the functions q_j satisfy the differential system

$$(n+1)c\frac{\partial q_j}{\partial u_j}=\lambda q_{j-1},\quad j=0,\dots,n. \tag{3.5}$$

Proof. The exponential transformation applied to (3.3) readily yields the system

$$\frac{\partial q_j}{\partial t}=-c\frac{\partial q_j}{\partial\vec{v}_j}+\lambda q_{j-1},\quad j=0,\dots,n. \tag{3.6}$$

In view of (3.4) we have that

$$\frac{\partial q_j}{\partial t} = c\sum_{k=0}^{n} \frac{\partial q_j}{\partial u_k} \quad \text{and} \quad \frac{\partial q_j}{\partial x_i} = n\sum_{k=0}^{n} \frac{\partial q_j}{\partial u_k}\, v_{i,k}. \tag{3.7}$$

Thus, by (2.4),

$$\begin{aligned}\frac{\partial q_j}{\partial \vec{v}_j} &= \overrightarrow{\operatorname{grad}}\, q_j \cdot \vec{v}_j = \sum_{i=0}^{n-1} \frac{\partial q_j}{\partial x_i} v_{i,j} = n\sum_{k=0}^{n} \frac{\partial q_j}{\partial u_k}\left(\sum_{i=0}^{n-1} v_{i,j} v_{i,k}\right) \\ &= n\sum_{\substack{k=0\\k\neq j}}^{n}\left(-\frac{1}{n}\frac{\partial q_j}{\partial u_k}\right) + n\frac{\partial q_j}{\partial u_j} = -\sum_{k=0}^{n} \frac{\partial q_j}{\partial u_k} + (n+1)\frac{\partial q_j}{\partial u_j}.\end{aligned} \tag{3.8}$$

By plugging (3.7) and (3.8) into (3.6) we obtain (3.5). □

We remark that Eqs. (3.5) have substantially the same form as those of the systems (3.4) of Orsingher and Sommella [7] and (2.8) of Orsingher [6], except for some constants, because of a different definition of the transformation (3.4).

Corollary 3.4. *Each function q_j satisfies the following $(n+1)$th order partial differential equation*

$$\frac{\partial^{n+1} q_j}{\partial u_0 \cdots \partial u_n} = \left(\frac{\lambda}{(n+1)c}\right)^{n+1} q_j, \quad j = 0, \ldots, n. \tag{3.9}$$

Proof. By differentiating the first equation of (3.5) with respect to u_n we get

$$\frac{\partial^2 q_0}{\partial u_0 \partial u_n} = \frac{\lambda}{(n+1)c}\frac{\partial q_n}{\partial u_n} = \left(\frac{\lambda}{(n+1)c}\right)^2 q_{n-1}. \tag{3.10}$$

By differentiating (3.10) we obtain

$$\frac{\partial^3 q_0}{\partial u_0 \partial u_n \partial u_{n-1}} = \left(\frac{\lambda}{(n+1)c}\right)^3 q_{n-2}$$

and by iterating this procedure we finally get Eq. (3.9) for $j = 0$. The other cases are quite similar. □

Proposition 3.5. *The solutions q of p.d.e.* (3.9) *depending only on the variable $w = \sqrt[n+1]{u_0 \cdots u_n}$ verify the $(n+1)$th order hyper-Bessel equation*

$$\left(w\frac{\partial}{\partial w}\right)^{n+1} q = \left(\frac{\lambda w}{c}\right)^{n+1} q. \tag{3.11}$$

Proof. Let us now consider the transformation

$$\begin{cases} v_j = u_j & \text{for } j = 0, \ldots, n-1, \\ w = \sqrt[n+1]{u_0 \cdots u_n}. \end{cases} \tag{3.12}$$

In view of (3.12), for $i = 0, \ldots, n-1$,

$$\frac{\partial q_j}{\partial u_i} = \sum_{k=0}^{n-1} \frac{\partial q_j}{\partial v_k}\frac{\partial v_k}{\partial u_i} + \frac{\partial q_j}{\partial w}\frac{\partial w}{\partial u_i} = \frac{\partial q_j}{\partial v_i} + \frac{w}{(n+1)v_i}\frac{\partial q_j}{\partial w}, \tag{3.13}$$

and

$$\frac{\partial q_j}{\partial u_n} = \sum_{k=0}^{n-1} \frac{\partial q_j}{\partial v_k}\frac{\partial v_k}{\partial u_n} + \frac{\partial q_j}{\partial w}\frac{\partial w}{\partial u_n} = \frac{v_0 \cdots v_{n-1}}{(n+1)w^n}\frac{\partial q_j}{\partial w}. \tag{3.14}$$

In light of (3.13) we have that

$$\frac{\partial^n q_j}{\partial u_0 \cdots \partial u_{n-1}} = \left(\prod_{i=0}^{n-1} \frac{1}{v_i}\left(v_i \frac{\partial}{\partial v_i} + \frac{w}{n+1}\frac{\partial}{\partial w}\right)\right) q_j \tag{3.15}$$

and, by using the well-known formula

$$\prod_{i=0}^{n-1}(\alpha_i + x) = \sum_{m=0}^{n} x^m \sum_{0 \leqslant l_1 < \cdots < l_{n-m} \leqslant n-1} \alpha_{l_1} \cdots \alpha_{l_{n-m}} =: \sum_{m=0}^{n} x^m \sigma_{n-m}(\alpha_0, \ldots, \alpha_{n-1}),$$

the differential operator in (3.15) can be written as

$$\frac{\partial^n}{\partial u_0 \cdots \partial u_{n-1}} = \frac{1}{v_0 \cdots v_{n-1}} \sum_{m=0}^{n} \left(\frac{w}{n+1}\frac{\partial}{\partial w}\right)^m \sigma_{n-m}\left(v_0 \frac{\partial}{\partial v_0}, \ldots, v_{n-1}\frac{\partial}{\partial v_{n-1}}\right).$$

Formula (3.14), applied to the above expression, enables us to write the differential operator in (3.9) as

$$\begin{aligned}\frac{\partial^{n+1}}{\partial u_0 \cdots \partial u_n} &= \frac{1}{(n+1)w^n}\frac{\partial}{\partial w}\left[\sum_{m=0}^{n}\left(\frac{w}{n+1}\frac{\partial}{\partial w}\right)^m \sigma_{n-m}\left(v_0 \frac{\partial}{\partial v_0}, \ldots, v_{n-1}\frac{\partial}{\partial v_{n-1}}\right)\right] \\ &= \frac{1}{w^{n+1}} \sum_{m=0}^{n-1}\left(\frac{w}{n+1}\frac{\partial}{\partial w}\right)^{m+1} \sigma_{n-m}\left(v_0 \frac{\partial}{\partial v_0}, \ldots, v_{n-1}\frac{\partial}{\partial v_{n-1}}\right) + \frac{1}{((n+1)w)^{n+1}}\left(w\frac{\partial}{\partial w}\right)^{n+1}.\end{aligned}$$

Therefore, the solutions of (3.9) depending only on w satisfy the ordinary equation (3.11) □

Remark 3.6. We check here that the hyper-Bessel function

$$\mathcal{I}_{0,n}(w) = \sum_{k=0}^{\infty} \frac{1}{(k!)^n}\left(\frac{w}{n}\right)^{nk}$$

is a solution to the hyper-Bessel equation $(w\frac{\partial}{\partial w})^n q = w^n q$. Since $(w\frac{\partial}{\partial w}) w^{nk} = (nk) w^{nk}$ and thus $(w\frac{\partial}{\partial w})^n w^{nk} = (nk)^n w^{nk}$, we have that

$$\begin{aligned}\left(w\frac{\partial}{\partial w}\right)^n \mathcal{I}_{0,n}(w) &= \sum_{k=0}^{\infty} \frac{1}{(k!)^n}\frac{(nk)^n\, w^{nk}}{n^{nk}} = \sum_{k=1}^{\infty} \frac{1}{n^{n(k-1)}}\frac{w^{nk}}{((k-1)!)^n} \\ &= w^n \sum_{k=0}^{\infty}\frac{1}{(k!)^n}\left(\frac{w}{n}\right)^{nk} = w^n \mathcal{I}_{0,n}(w).\end{aligned}$$

This implies that the function $\mathcal{I}_{0,n+1}(\frac{\lambda}{c}w)$ is a solution to Eq. (3.11).

4. Order statistics applied to cyclic motions

In this section, we introduce the number $N_j(t)$ of times the direction $\vec{v}_j$ is taken, up to time t. Of course, the random numbers $N_j(t)$ and $N(t)$ are linked by

$$\sum_{j=0}^{n} N_j(t) = N(t) + 1.$$

We can immediately write down the following relationship:

$$\Pr\{\underline{X}(t) \in \mathrm{d}\underline{x}\} = \sum_{k_0, \ldots, k_n \geqslant 0} \Pr\{N_0(t) = k_0, \ldots, N_n(t) = k_n\} \Pr\{\underline{X}(t) \in \mathrm{d}\underline{x} \mid N_0(t) = k_0, \ldots, N_n(t) = k_n\}. \tag{4.1}$$

For the cyclic motion the explicit form of $\Pr\{N_0(t) = k_0, \ldots, N_n(t) = k_n\}$ is almost straightforward and this makes the derivation of (4.1) in a closed form possible.

For the planar motion with three directions, the probabilities $\Pr\{N_0(t)=k_0,\ N_1(t)=k_1,\ N_2(t)=k_2\}$ where $N_0(t)+N_1(t)+N_2(t)=N(t)+1$, have been explicitly evaluated for the symmetrically deviating motion by means of an extension of Bose–Einstein statistics (Leorato and Orsingher [4]).

For a number of directions greater than or equal to 4, the evaluation of the distribution of $(N_0(t),\dots,N_n(t))$ becomes extremely difficult except for the uniform case, where $(N_0(t),\dots,N_n(t))$ is a multinomial random vector.

The aim of this section is the evaluation of the conditional probabilities in (4.1) by means of order statistics. This method has been successfully applied in the case of planar, cyclic motions with orthogonal directions (Leorato et al. [5]) and also for motions with three directions (Leorato and Orsingher [4]).

The results of Subsections 4.1 and 4.2 hold for any form of the chance mechanism regulating the change of directions. Since the behaviour of $(N_0(t),\dots,N_n(t))$ in the cyclic case is essentially deterministic, the relevant part of the analysis reduces to the derivation of the conditional probabilities appearing in (4.1).

We observe that the minimality of the number of directions is very important in order to obtain the conditional distributions in (4.1) in a relatively simple way as will become clear below.

4.1. Some preliminary results about the position of the particle

We start by writing the vector (3.1) in a new convenient form:

$$\begin{aligned}\underline{X}(t)&=c\left[I_{\{N(t)\geqslant 1\}}\sum_{k=0}^{N(t)-1}(T_{k+1}-T_k)\vec{v}_k+(t-T_{N(t)})\vec{v}_{N(t)}\right]\\&=c\left[I_{\{N(t)\geqslant 1\}}\sum_{j=0}^{n}\sum_{\substack{0\leqslant l\leqslant N(t)-1:\\ \vec{v}_j\text{ is taken in }[T_l,T_{l+1})}}(T_{l+1}-T_l)\vec{v}_j+\sum_{j=0}^{n}(t-T_{N(t)})\vec{v}_j I_{\{\vec{v}_j\text{ is taken in }[T_{N(t)},t]\}}\right]\\&=ct\sum_{j=0}^{n}L_j(t)\vec{v}_j\end{aligned}\tag{4.2}$$

where

$$L_j(t)=\frac{1}{t}\left[I_{\{N(t)\geqslant 1\}}\sum_{\substack{0\leqslant l\leqslant N(t)-1:\\ \vec{v}_j\text{ is taken in }[T_l,T_{l+1})}}(T_{l+1}-T_l)+(t-T_{N(t)})I_{\{\vec{v}_j\text{ is taken in }[T_{N(t)},t]\}}\right]$$

is the proportion of time spent travelling with direction $\vec{v}_j$. The r.v.'s $L_j(t)$ can also be written as

$$L_j(t)=\frac{1}{t}\sum_{m=1}^{N_j(t)}T_m^{(j)},\tag{4.3}$$

where $T_m^{(j)}$ indicates how long the particle has travelled the mth time that $\vec{v}_j$ has been taken. We remark that

$$\sum_{j=0}^{n}\sum_{m=1}^{N_j(t)}T_m^{(j)}=t\sum_{j=0}^{n}L_j(t)=t.$$

The r.v.'s $T_m^{(j)}$, $1\leqslant m\leqslant N_j(t)$, $0\leqslant j\leqslant n$, are independent and exponentially distributed with parameter λ.

Theorem 4.1. *Fix some positive integers $k_0,\dots,k_n$ such that $\sum_{j=0}^{n}k_j=k+1$. The joint conditional distribution of $(L_0(t),\dots,L_{n-1}(t))$ is given by*

$$\Pr\bigl\{L_0(t)\in \mathrm{d}l_0,\dots,L_{n-1}(t)\in \mathrm{d}l_{n-1}\mid N_0(t)=k_0,\dots,N_n(t)=k_n\bigr\}=f(l_0,\dots,l_{n-1})\,\mathrm{d}l_0\cdots \mathrm{d}l_{n-1}\tag{4.4}$$

where

$$f(l_0,\dots,l_{n-1})=\frac{k!}{\prod_{j=0}^{n}(k_j-1)!}\prod_{j=0}^{n}l_j^{k_j-1}I_{\{l_0,\dots,l_{n-1}\geqslant 0,\sum_{j=0}^{n-1}l_j\leqslant 1\}},\tag{4.5}$$

and $l_n=1-\sum_{j=0}^{n-1}l_j$.

Proof. For the instants $T_1, \dots, T_k$ of occurrence of the Poisson events we have the following well-known conditional distribution:

$$\Pr\{T_1 \in \mathrm{d}t_1, \dots, T_k \in \mathrm{d}t_k \mid N(t) = k\} = \frac{k!}{t^k} I_{\{0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant t\}} \,\mathrm{d}t_1 \cdots \mathrm{d}t_k. \tag{4.6}$$

For the r.v.'s $S_j = T_j - T_{j-1}$, $j = 1, \dots, k$, after some obvious transformations we have that

$$\Pr\{S_1 \in \mathrm{d}s_1, \dots, S_k \in \mathrm{d}s_k \mid N(t) = k\} = \frac{k!}{t^k} I_{\left\{\substack{s_1 \geqslant 0, \dots, s_k \geqslant 0,\\ s_1 + \cdots + s_k \leqslant t}\right\}} \mathrm{d}s_1 \cdots \mathrm{d}s_k.$$

From (4.6) we can extract, by direct integration, the distribution of $(T_{j_1}, \dots, T_{j_n})$, under the condition that $N(t) = k$, where $0 < j_1 < j_2 < \cdots < j_n < k$. Indeed,

$$\begin{aligned}
&\frac{\Pr\{T_{j_1} \in \mathrm{d}t_{j_1}, \dots, T_{j_n} \in \mathrm{d}t_{j_n} \mid N(t) = k\}}{\mathrm{d}t_{j_1} \cdots \mathrm{d}t_{j_n}} \\
&\quad = \frac{k!}{t^k} \int \cdots \int_{\{0 \leqslant t_1 < \cdots < t_{j_1} < \cdots < t_{j_2} < \cdots < t_{j_n} < \cdots < t_k \leqslant t\}} \mathrm{d}t_1 \cdots \mathrm{d}t_{j_1 - 1} \,\mathrm{d}t_{j_1+1} \cdots \mathrm{d}t_{j_2-1} \,\mathrm{d}t_{j_2+1} \cdots \mathrm{d}t_{j_n-1} \,\mathrm{d}t_{j_n+1} \cdots \mathrm{d}t_k \\
&\quad = \frac{k!}{t^k} \int_{\{0 \leqslant t_1 < \cdots < t_{j_1-1} < t_{j_1}\}} \mathrm{d}t_1 \cdots \mathrm{d}t_{j_1-1} \int_{\{t_{j_1} < t_{j_1+1} < \cdots < t_{j_2-1} < t_{j_2}\}} \mathrm{d}t_{j_1+1} \cdots \mathrm{d}t_{j_2-1} \cdots \int_{\{t_{j_n} < t_{j_n+1} < \cdots < t_k \leqslant t\}} \mathrm{d}t_{j_n+1} \cdots \mathrm{d}t_k \\
&\quad = \frac{k!}{t^k} \frac{t_{j_1}^{j_1 - 1}}{(j_1 - 1)!} \frac{(t_{j_2} - t_{j_1})^{j_2 - j_1 - 1}}{(j_2 - j_1 - 1)!} \cdots \frac{(t - t_{j_n})^{k - j_n}}{(k - j_n)!} I_{\{0 \leqslant t_{j_1} < \cdots < t_{j_n} \leqslant t\}}.
\end{aligned}$$

Let us now assume that the intervals of time spent travelling with the same direction are put together and let us choose

$$j_m = k_0 + \cdots + k_{m-1}, \quad 1 \leqslant m \leqslant n.$$

The time T_{j_m} can now be regarded as the instant at which the particle ends its travelling with directions $\vec{v}_j$, $0 \leqslant j \leqslant m - 1$, and switches to direction $\vec{v}_m$ (see Fig. 1).

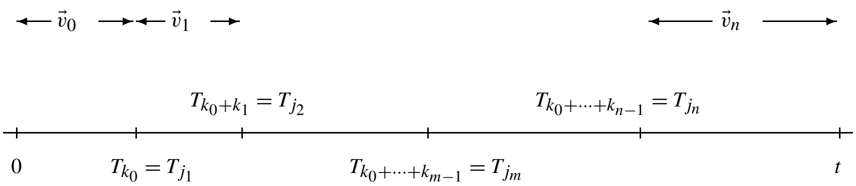

Fig. 1. How the interval $[0, t]$ is split up into subintervals.

In light of the exchangeability of the r.v.'s representing the length of time between successive changes of direction, we can write that

$$L_0(t) \stackrel{d}{=} \frac{T_{j_1}}{t}, L_1(t) \stackrel{d}{=} \frac{T_{j_2} - T_{j_1}}{t}, \dots, L_{n-1}(t) \stackrel{d}{=} \frac{T_{j_n} - T_{j_{n-1}}}{t}. \tag{4.7}$$

By means of the transformation

$$t_{j_1} = t l_0,\ t_{j_2} = t(l_0 + l_1),\ \dots,\ t_{j_n} = t(l_0 + \cdots + l_{n-1})$$

emerging from (4.7), we can extract the distribution (4.4) of $(L_0(t), \dots, L_{n-1}(t))$ from (4.7) as follows:

$$\begin{aligned}
&\Pr\{L_0(t) \in \mathrm{d}l_0, \dots, L_{n-1}(t) \in \mathrm{d}l_{n-1} \mid N(t) = k, N_0(t) = k_0, \dots, N_{n-1}(t) = k_{n-1}\} \\
&\quad = \Pr\{T_{j_1} \in t\,\mathrm{d}l_0, T_{j_2} \in t\,\mathrm{d}(l_0 + l_1), \dots, T_{j_n} \in t\,\mathrm{d}(l_0 + \cdots + l_{n-1}) \mid N(t) = k\} \\
&\quad = \frac{k!(tl_0)^{k_0 - 1} \cdots (tl_{n-1})^{k_{n-1} - 1}(t - t(l_0 + \cdots + l_{n-1}))^{k - \sum_{j=0}^{n-1} k_j} t^n \,\mathrm{d}l_0 \cdots \mathrm{d}l_{n-1}}{t^k (k_0 - 1)!(k_1 - 1)! \cdots (k - k_0 - \cdots - k_{n-1})!} I_{\{l_0, \dots, l_{n-1} \geqslant 0 \colon \sum_{j=0}^{n-1} l_j \leqslant 1\}}
\end{aligned}$$

$$= \frac{k!}{\prod_{j=0}^{n}(k_j-1)!}\prod_{j=0}^{n} l_j^{k_j-1}\,\mathrm{d}l_j\, I_{\{l_0,\dots,l_{n-1}\geqslant 0:\ \sum_{j=0}^{n-1} l_j\leqslant 1\}}$$

where $k_n = 1 + k - k_0 - \cdots - k_{n-1}$ and $l_n = 1 - l_0 - \cdots - l_{n-1}$. □

4.2. Deriving the conditional law

The connection between the position vector $\underline{X}(t)$, $t \geqslant 0$, and the random times $L_j(t)$, $0 \leqslant j \leqslant n$, spent moving with direction $\vec{v}_j$, is given by formula (4.2), which we rewrite, for our convenience, in the following manner:

$$\underline{X}(t) = ct\left[\sum_{j=0}^{n-1} L_j(t)\vec{v}_j + \left(1 - \sum_{j=0}^{n-1} L_j(t)\right)\vec{v}_n\right] = ct\left[\vec{v}_n + \sum_{j=0}^{n-1} L_j(t)(\vec{v}_j - \vec{v}_n)\right]. \tag{4.8}$$

An alternative representation of (4.8) is therefore

$$\begin{pmatrix} X_0(t)\\ X_1(t)\\ \vdots\\ X_{n-1}(t)\end{pmatrix} = ct\left[\begin{pmatrix} v_{0,n}\\ v_{1,n}\\ \vdots\\ v_{n-1,n}\end{pmatrix} + \widetilde{\mathbf{V}}\begin{pmatrix} L_0(t)\\ L_1(t)\\ \vdots\\ L_{n-1}(t)\end{pmatrix}\right]$$

where $\widetilde{\mathbf{V}}$ is the matrix of the coordinates of the vectors $\vec{v}_0 - \vec{v}_n, \vec{v}_1 - \vec{v}_n, \dots, \vec{v}_{n-1} - \vec{v}_n$ with respect to the canonical basis:

$$\widetilde{\mathbf{V}} = (v_{i,j} - v_{i,n})_{0\leqslant i,j\leqslant n-1}. \tag{4.9}$$

This means that the vectors $\underline{X}(t) = (X_0(t), \dots, X_{n-1}(t))$ and $\underline{L}(t) = (L_0(t), \dots, L_{n-1}(t))$ are linked by the relationship

$$\underline{X}(t) = \phi\big(\underline{L}(t)\big) \tag{4.10}$$

where $\phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ is defined by

$$\phi(l_0,\dots,l_{n-1}) = \left(ct\left(v_{0,n} + \sum_{j=0}^{n-1}(v_{0,j} - v_{0,n})l_j\right), \dots, ct\left(v_{n-1,n} + \sum_{j=0}^{n-1}(v_{n-1,j} - v_{n-1,n})l_j\right)\right).$$

4.2.1. Inversion of the matrix $\widetilde{\mathbf{V}}$

We need to invert the affine transformation ϕ in order to determine the distribution of $\underline{X}(t)$ from (4.5). To this aim, we must evaluate the inverse $\widetilde{\mathbf{V}}^{-1}$ of the matrix (4.9).

In light of (2.4), for all $0 \leqslant k \leqslant n-1$, we can write

$$\vec{v}_k \cdot (\vec{v}_j - \vec{v}_n) = \begin{cases} 0 & \text{if } k \neq j,\\ 1 + \frac{1}{n} & \text{if } k = j\end{cases}$$

or, equivalently,

$$\sum_{i=0}^{n-1} v_{i,k}(v_{i,j} - v_{i,n}) = \frac{n+1}{n}\delta_{k,j} \tag{4.11}$$

where $\delta_{k,j}$ is the Kronecker delta function. The relationship (4.11) can be rewritten in terms of matrices

$$\frac{n}{n+1}\begin{pmatrix} v_{0,0} & \cdots & v_{0,n-1}\\ \vdots & & \vdots\\ v_{n-1,0} & \cdots & v_{n-1,n-1}\end{pmatrix}^{\mathrm{T}}\begin{pmatrix} v_{0,0} - v_{0,n} & \cdots & v_{0,n-1} - v_{0,n}\\ \vdots & & \vdots\\ v_{n-1,0} - v_{n-1,n} & \cdots & v_{n-1,n-1} - v_{n-1,n}\end{pmatrix} = \frac{n}{n+1}\mathbf{W}\widetilde{\mathbf{V}} = \mathbf{I}.$$

Thus

$$\widetilde{\mathbf{V}}^{-1} = \frac{n}{n+1}\mathbf{W} = \frac{n}{n+1}\begin{pmatrix} v_{0,0} & \cdots & v_{0,n-1}\\ \vdots & & \vdots\\ v_{n-1,0} & \cdots & v_{n-1,n-1}\end{pmatrix}^{\mathrm{T}}. \tag{4.12}$$

For the evaluation of the Jacobian of the transformation ϕ, we need the determinant $\det(\widetilde{\mathbf{V}})$, which equals $[\det(\widetilde{\mathbf{V}}^{-1})]^{-1}$, where

$$\det\bigl(\widetilde{\mathbf{V}}^{-1}\bigr)=\left(\frac{n}{n+1}\right)^n\det(\vec{v}_0,\dots,\vec{v}_{n-1})\overset{\text{by (2.10)}}{=}\frac{n^{n/2}}{(n+1)^{(n+1)/2}}=\frac{1}{n!V_n}.$$

Since, $\mathbf{J}_\phi=ct\widetilde{\mathbf{V}}$, the Jacobian of ϕ is clearly given by

$$|\mathbf{J}_\phi|=n!\,V_n(ct)^n. \tag{4.13}$$

4.2.2. Inversion of the transformation ϕ

We are now able to invert the transformation ϕ. To do this, we write the equivalences

$$\begin{aligned}\phi(l_0,\dots,l_{n-1})=(x_0,\dots,x_{n-1})&\iff\begin{pmatrix}x_0\\ \vdots\\ x_{n-1}\end{pmatrix}=ct\left[\begin{pmatrix}v_{0,n}\\ \vdots\\ v_{n-1,n}\end{pmatrix}+\widetilde{\mathbf{V}}\begin{pmatrix}l_0\\ \vdots\\ l_{n-1}\end{pmatrix}\right]\\ &\iff\begin{pmatrix}l_0\\ \vdots\\ l_{n-1}\end{pmatrix}=\frac{1}{ct}\widetilde{\mathbf{V}}^{-1}\begin{pmatrix}x_0-ctv_{0,n}\\ \vdots\\ x_{n-1}-ctv_{n-1,n}\end{pmatrix}.\end{aligned}$$

Referring to (4.12), we obtain, for $0\leqslant k\leqslant n-1$,

$$l_k=\frac{n}{(n+1)ct}\sum_{i=0}^{n-1}v_{i,k}(x_i-ctv_{i,n})=\frac{1}{(n+1)ct}\left(n\sum_{i=0}^{n-1}v_{i,k}x_i+ct\right).$$

As a result, we get, by putting $\underline{x}=(x_0,\dots,x_{n-1})$,

$$\phi^{-1}(\underline{x})=\frac{1}{(n+1)ct}\bigl(h_0(\underline{x},t),\dots,h_{n-1}(\underline{x},t)\bigr). \tag{4.14}$$

4.2.3. The conditional law

We are now in a position to state the theorem concerning the conditional distribution of $\underline{X}(t)$.

Theorem 4.2. *Fix $k_0,\dots,k_n\geqslant 1$ such that $k_0+\cdots+k_n=k+1$. The conditional distribution of $\underline{X}(t)$ is given by*

$$\begin{aligned}&\Pr\bigl\{\underline{X}(t)\in \mathrm{d}\underline{x}\mid N_0(t)=k_0,\dots,N_n(t)=k_n\bigr\}\\ &\quad=\frac{(n+1)^n}{n!V_n}\frac{k!}{\prod_{j=0}^n(k_j-1)!}\frac{1}{((n+1)ct)^k}\prod_{j=0}^n h_j(\underline{x},t)^{k_j-1}\,\mathrm{d}\underline{x}\end{aligned} \tag{4.15}$$

for $\underline{x}\in\operatorname{int}(\mathfrak{T}_{ct})$, where

$$h_j(\underline{x},t)=ct+n\sum_{i=0}^{n-1}v_{i,j}x_i,\quad j=0,\dots,n. \tag{4.16}$$

Proof. From (4.5), (4.10) and (4.14) we have, for $\underline{x}\in\operatorname{int}(\mathfrak{T}_{ct})$,

$$\begin{aligned}&\Pr\bigl\{\underline{X}(t)\in \mathrm{d}\underline{x}\mid N_0(t)=k_0,\dots,N_n(t)=k_n\bigr\}=f\bigl(\phi^{-1}(\underline{x})\bigr)|\mathbf{J}_{\phi^{-1}}|\,\mathrm{d}\underline{x}\\ &\quad=\frac{k!|\mathbf{J}_{\phi^{-1}}|\,\mathrm{d}\underline{x}}{\prod_{j=0}^n(k_j-1)!}\prod_{j=0}^{n-1}\left(\frac{h_j(\underline{x},t)}{(n+1)ct}\right)^{k_j-1}\left(1-\frac{1}{(n+1)ct}\sum_{j=0}^{n-1}h_j(\underline{x},t)\right)^{k_n-1}\\ &\quad=\frac{k!}{\prod_{j=0}^n(k_j-1)!}\frac{|\mathbf{J}_{\phi^{-1}}|\mathrm{d}\underline{x}}{((n+1)ct)^{k-n}}\prod_{j=0}^n h_j(\underline{x},t)^{k_j-1}\end{aligned}$$

where in the last equality above we used

$$
\begin{aligned}
1-\frac{1}{(n+1)ct}\sum_{j=0}^{n-1}h_j(\underline{x},t) &= 1-\frac{1}{(n+1)ct}\sum_{j=0}^{n-1}\left(ct+n\sum_{i=0}^{n-1}v_{i,j}x_i\right)\\
&=\frac{ct+n\sum_{i=0}^{n-1}v_{i,n}x_i}{(n+1)ct}=\frac{h_n(\underline{x},t)}{(n+1)ct},
\end{aligned}
$$

and $\mathbf{J}_{\phi^{-1}}$ is the Jacobian matrix of the transformation ϕ^{-1}, whose determinant, in view of (4.13), is given by

$$
|\mathbf{J}_{\phi^{-1}}|=\frac{1}{n!\,V_n(ct)^n}.
$$

This concludes the proof of (4.15). □

Remark 4.3. The clever reader can ascertain that (4.15) coincides with (2.6) of Leorato and Orsingher [4] in the case $n=2$ for which $\underline{x}=(x,y)$, $h_0(x,y,t)=ct+2x$, $h_1(x,y,t)=ct-x+\sqrt{3}y$ and $h_2(x,y,t)=ct-x-\sqrt{3}y$.

4.3. Deriving the unconditional law

Formula (4.1) makes the derivation of the unconditional distribution of $\underline{X}(t)$ possible once the joint distribution of $(N_0(t),\dots,N_n(t))$ is known.

For cyclic motions this can be easily done. Notice that any integer k can be written as $(n+1)q+r-1$ with $0\leqslant r\leqslant n$. The equality $N(t)=k$, rewritten as $N(t)=(n+1)q+r-1$, means that, at time t, the particle has run q complete cycles and next has taken r consecutive directions.

If the initial direction is $\vec{v}_0$ and $N(t)=(n+1)q+r-1$, then the current direction is $\vec{v}_{r-1}$. The first cycle is complete if the current direction at time t is $\vec{v}_n$, although $\vec{v}_n$ will be in force also after t.

If the initial direction is $\vec{v}_j$ and $N(t)=(n+1)q+r-1$, then

$$
N_j(t)=\cdots=N_{j+r-1}(t)=q+1 \quad\text{and}\quad N_{j+r}(t)=\cdots=N_{j+n}(t)=q
$$

where we set $N_l(t)=N_{l-n-1}(t)$ for $l>n$. This means that the number of times each direction is taken in a cyclic motion differs at most by one unit.

In order for the particle to move inside $\mathfrak{T}_{ct}$ it is necessary and sufficient that at least one cycle be completed (i.e. $q\geqslant 1$).

Theorem 4.4. *We have the following explicit distribution for* $0\leqslant r\leqslant n$:

$$
\begin{aligned}
\tilde{p}_r(\underline{x},t)\,\mathrm{d}\underline{x} &= \Pr\{\underline{X}(t)\in\mathrm{d}\underline{x}\text{ and the last cycle has run } r \text{ directions}\}\\
&=\Pr\left\{\underline{X}(t)\in\mathrm{d}\underline{x},\ \bigcup_{q=1}^{\infty}\{q\text{ complete cycles }+r\text{ directions}\}\right\}\\
&=\frac{\mathrm{e}^{-\lambda t}\,\mathrm{d}\underline{x}}{(n+1)^r n!V_n}\left(\frac{\lambda}{c}\right)^{n+r}\mathcal{H}_{r,n+1}(\underline{x},t)\mathcal{I}_{r,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n}h_j(\underline{x},t)}\right)
\end{aligned}
\tag{4.17}
$$

for $\underline{x}\in\operatorname{int}(\mathfrak{T}_{ct})$, *where*

$$
\mathcal{H}_{r,n+1}(\underline{x},t)=\frac{1}{n+1}\sum_{j=0}^{n}h_j(\underline{x},t)\cdots h_{j+r-1}(\underline{x},t)
\tag{4.18}
$$

with $\mathcal{H}_{0,n+1}(\underline{x},t)=1$, *and*

$$
\mathcal{I}_{r,n+1}(\xi)=\sum_{q=0}^{\infty}\frac{1}{(q!)^{n+1-r}((q+1)!)^r}\left(\frac{\xi}{n+1}\right)^{(n+1)q}.
$$

Proof. The probability (4.17) can be written as

$$
\begin{aligned}
&\Pr\left\{\underline{X}(t)\in\underline{\mathrm{d}x},\ \bigcup_{q=1}^{\infty}\{N(t)=(n+1)q+r-1\}\right\}\\
&\quad=\sum_{q=1}^{\infty}\sum_{j=0}^{n}\Pr\{\underline{X}(t)\in\underline{\mathrm{d}x},\ D(0)=\vec{v}_j,\ N(t)=(n+1)q+r-1\}\\
&\quad=\sum_{q=1}^{\infty}\sum_{j=0}^{n}\Pr\{\underline{X}(t)\in\underline{\mathrm{d}x},\ N_j(t)=\cdots=N_{j+r-1}(t)=q+1,\ N_{j+r}(t)=\cdots=N_{j+n}(t)=q\}\\
&\quad=\sum_{q=1}^{\infty}\sum_{j=0}^{n}\Pr\{N(t)=(n+1)q+r-1,\ D(0)=\vec{v}_j\}\\
&\qquad\times\Pr\{\underline{X}(t)\in\underline{\mathrm{d}x}\mid N_j(t)=\cdots=N_{j+r-1}(t)=q+1,\ N_{j+r}(t)=\cdots=N_{j+n}(t)=q\}\\
&\quad\overset{\text{by (4.15)}}{=}\frac{\mathrm{e}^{-\lambda t}}{(n+1)}\frac{(n+1)^n}{n!V_n}\sum_{q=1}^{\infty}\sum_{j=0}^{n}\frac{(\lambda t)^{(n+1)q+r-1}}{(q!)^r((q-1)!)^{n-r+1}}\\
&\qquad\times\frac{[h_j(\underline{x},t)\cdots h_{j+r-1}(\underline{x},t)]^q[h_{j+r}(\underline{x},t)\cdots h_{j+n}(\underline{x},t)]^{q-1}}{((n+1)ct)^{(n+1)q+r-1}}\,\underline{\mathrm{d}x}\\
&\quad=\frac{\mathrm{e}^{-\lambda t}}{n+1}\frac{(n+1)^n}{n!V_n}\left(\frac{\lambda}{(n+1)c}\right)^{n+r}\sum_{j=0}^{n}\sum_{q=0}^{\infty}\frac{(\lambda/((n+1)c))^{(n+1)q}}{((q+1)!)^r(q!)^{n-r+1}}\\
&\qquad\times\left(\prod_{i=0}^{n}h_i(\underline{x},t)\right)^q h_j(\underline{x},t)\cdots h_{j+r-1}(\underline{x},t)\,\underline{\mathrm{d}x}\\
&\quad=\frac{\mathrm{e}^{-\lambda t}}{(n+1)^r n!V_n}\left(\frac{\lambda}{c}\right)^{n+r}\frac{1}{n+1}\sum_{j=0}^{n}h_j(\underline{x},t)\cdots h_{j+r-1}(\underline{x},t)\\
&\qquad\times\sum_{q=0}^{\infty}\frac{(\lambda/((n+1)c)\sqrt[n+1]{\prod_{j=0}^{n}h_j(\underline{x},t)})^{(n+1)q}}{(q!)^{n-r+1}((q+1)!)^r}\,\underline{\mathrm{d}x}.
\end{aligned}
\tag{4.19}
$$

This concludes the proof of the theorem. □

By summing formula (4.17) with respect to r, we explicitly obtain the absolutely continuous component of the distribution of $\underline{X}(t)$, stated below.

Corollary 4.5. *We have for* $\underline{x}\in\operatorname{int}(\mathfrak{T}_{ct})$

$$
\Pr\{\underline{X}(t)\in\underline{\mathrm{d}x},\ N(t)\geqslant n\}=\frac{\underline{\mathrm{d}x}\,\mathrm{e}^{-\lambda t}}{n!V_n}\left(\frac{\lambda}{c}\right)^n\sum_{r=0}^{n}\left(\frac{\lambda}{(n+1)c}\right)^r\mathcal{H}_{r,n+1}(\underline{x},t)\mathcal{I}_{r,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n}h_j(\underline{x},t)}\right).
\tag{4.20}
$$

Remark 4.6. (i) By definition of $\tilde{p}_r$, the following relation plainly holds:

$$
\int_{\mathfrak{T}_{ct}}\tilde{p}_r(\underline{x},t)\,\underline{\mathrm{d}x}=\Pr\left\{\bigcup_{q=1}^{\infty}\{N(t)=(n+1)q+r-1\}\right\}=\sum_{q=0}^{\infty}\frac{\mathrm{e}^{-\lambda t}(\lambda t)^{(n+1)q+n+r}}{((n+1)q+n+r)!}.
\tag{4.21}
$$

Nevertheless, the reader may check it from direct integration of (4.17) using the technical result below:

$$\int_{\mathfrak{T}_{ct}} \prod_{j=0}^{n} h_j(\underline{x},t)^{k_j}\,\underline{\mathrm{d}x} = \frac{((n+1)ct)^{\sum_{j=0}^{n} k_j+n}}{(n+1)^{(n-1)/2}n^{n/2}} \frac{\prod_{j=0}^{n}\Gamma(k_j+1)}{\Gamma(\sum_{j=0}^{n}k_j+n+1)}.$$

(ii) Now, summing (4.21) with respect to r, we get that

$$\int_{\mathfrak{T}_{ct}} \Pr\{\underline{X}(t)\in\underline{\mathrm{d}x}, N(t)\geqslant n\} = \mathrm{e}^{-\lambda t}\sum_{r=0}^{n}\sum_{q=0}^{\infty}\frac{(\lambda t)^{(n+1)q+n+r}}{((n+1)q+n+r)!} = \mathrm{e}^{-\lambda t}\sum_{l=n}^{\infty}\frac{(\lambda t)^l}{l!}$$
$$= \Pr\{N(t)\geqslant n\} = \Pr\{\underline{X}(t)\in \mathrm{int}(\mathfrak{T}_{ct})\},$$

and then $\int_{\mathfrak{T}_{ct}} \Pr\{\underline{X}(t)\in\underline{\mathrm{d}x} \mid N(t)\geqslant n\} = 1$. Hence, we have verified that the support of $(\underline{X}(t)\mid N(t)\geqslant n)$ is $\mathfrak{T}_{ct}$.

(iii) In (4.21) appear the so-called generalized hyperbolic functions of order $n+1$. The nth order hyperbolic functions are defined as (*cosine hyperbolic*)

$$\mathrm{ch}_{n,j}(x) = \sum_{k=0}^{\infty}\frac{x^{nk+j}}{(nk+j)!} = \frac{1}{n}\left[\mathrm{e}^x + \sum_{k=1}^{n-1}\mathrm{e}^{(\cos\frac{2k\pi}{n})x}\cos\left(\left(\sin\frac{2k\pi}{n}\right)x - \frac{2jk\pi}{n}\right)\right]$$

for $0\leqslant j\leqslant n-1$. With this notation, the rhs of (4.21) reads $\mathrm{e}^{-\lambda t}\,\mathrm{ch}_{n+1,n+r}(\lambda t)$.

Remark 4.7. We now consider some special cases of (4.17).

(i) If $r=0$, that is if $N_0(t)=\cdots=N_n(t)=q$, we have that

$$\begin{aligned}\tilde{p}_0(\underline{x},t)\,\underline{\mathrm{d}x} &= \Pr\{\underline{X}(t)\in\underline{\mathrm{d}x},\ \text{the current cycle is complete}\}\\ &= \sum_{q=1}^{\infty}\Pr\{\underline{X}(t)\in\underline{\mathrm{d}x},\ N_0(t)=\cdots=N_n(t)=q\}\\ &= \frac{\mathrm{e}^{-\lambda t}}{n!V_n}\left(\frac{\lambda}{c}\right)^n \mathcal{I}_{0,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n}h_j(\underline{x},t)}\right)\underline{\mathrm{d}x}.\end{aligned}$$

For $n=2$ this corresponds to formula (2.8) of Leorato and Orsingher [4].

(ii) When $r=1$, a new cycle has begun and its first direction has been taken. In this case we have

$$\mathcal{H}_{1,n+1}(\underline{x},t) = \frac{1}{n+1}\sum_{j=0}^{n}h_j(\underline{x},t) = \frac{1}{n+1}\sum_{j=0}^{n}\left(ct + n\sum_{i=0}^{n-1}v_{i,j}x_i\right) = ct.$$

The hyper-Bessel function $\mathcal{I}_{1,n+1}(\xi)$ can be written as

$$\begin{aligned}\mathcal{I}_{1,n+1}(\xi) &= \sum_{q=0}^{\infty}\frac{1}{(q!)^n(q+1)!}\left(\frac{\xi}{n+1}\right)^{(n+1)q}\\ &= \left(\frac{\xi}{n+1}\right)^{-n-1}\sum_{q=0}^{\infty}\frac{1}{(q!)^{n+1}(q+1)}\left(\frac{\xi}{n+1}\right)^{(n+1)(q+1)}\\ &= \left(\frac{\xi}{n+1}\right)^{-n-1}\int_0^{\xi}\sum_{q=0}^{\infty}\frac{1}{(q!)^{n+1}}\left(\frac{u}{n+1}\right)^{(n+1)q+n}\mathrm{d}u\\ &= \left(\frac{\xi}{n+1}\right)^{-n-1}\int_0^{\xi}\left(\frac{u}{n+1}\right)^n\mathcal{I}_{0,n+1}(u)\,\mathrm{d}u\\ &= (n+1)\int_0^1 w^n\,\mathcal{I}_{0,n+1}(w\xi)\,\mathrm{d}w.\end{aligned}$$

In conclusion we have that

$$
\begin{aligned}
\tilde{p}_1(\underline{x},t)\,\underline{\mathrm{d}x} &= \Pr\bigl\{\underline{X}(t)\in\underline{\mathrm{d}x},\ \text{the last cycle has run only 1 direction}\bigr\}\\
&= \underline{\mathrm{d}x}\,\frac{\mathrm{e}^{-\lambda t}\lambda t}{(n+1)!V_n}\left(\frac{\lambda}{c}\right)^n \mathcal{I}_{1,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n}h_j(\underline{x},t)}\right)\\
&= \underline{\mathrm{d}x}\,\frac{\mathrm{e}^{-\lambda t}\lambda t}{n!V_n}\left(\frac{\lambda}{c}\right)^n \int_0^1 w^n \mathcal{I}_{0,n+1}\left(\frac{\lambda w}{c}\sqrt[n+1]{\prod_{j=0}^{n}h_j(\underline{x},t)}\right)\mathrm{d}w.
\end{aligned}
$$

(iii) If $r=n$, only the last direction must be taken to complete the cycle. In this case, formula (4.18) yields

$$
\begin{aligned}
\mathcal{H}_{n,n+1}(\underline{x},t) &= \frac{1}{n+1}\sum_{j=0}^{n}\prod_{l=j}^{j+n-1}h_l(\underline{x},t) = \frac{1}{n+1}\sum_{j=0}^{n}\frac{1}{h_{j+n}(\underline{x},t)}\prod_{l=j}^{j+n}h_l(\underline{x},t)\\
&= \frac{1}{n+1}\prod_{l=0}^{n}h_l(\underline{x},t)\sum_{j=0}^{n}\frac{1}{h_j(\underline{x},t)}.
\end{aligned}\tag{4.22}
$$

In the last step we applied the periodicity of h_l with respect to l. For the hyper-Bessel function $\mathcal{I}_{n,n+1}(\xi)$ we have that

$$
\begin{aligned}
\mathcal{I}_{n,n+1}(\xi) &= \sum_{q=0}^{\infty}\frac{1}{q!((q+1)!)^n}\left(\frac{\xi}{n+1}\right)^{(n+1)q}\\
&= \left(\frac{\xi}{n+1}\right)^{-n}\sum_{q=0}^{\infty}\frac{q+1}{((q+1)!)^{n+1}}\left(\frac{\xi}{n+1}\right)^{(n+1)(q+1)-1}\\
&= \left(\frac{\xi}{n+1}\right)^{-n}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\sum_{q=0}^{\infty}\frac{1}{((q+1)!)^{n+1}}\left(\frac{\xi}{n+1}\right)^{(n+1)(q+1)}\right]\\
&= \left(\frac{\xi}{n+1}\right)^{-n}\frac{\mathrm{d}}{\mathrm{d}\xi}\bigl(\mathcal{I}_{0,n+1}(\xi)-1\bigr) = \left(\frac{\xi}{n+1}\right)^{-n}\mathcal{I}'_{0,n+1}(\xi).
\end{aligned}\tag{4.23}
$$

In the light of (4.22) and (4.23), formula (4.17) reads

$$
\begin{aligned}
&\Pr\bigl\{\underline{X}(t)\in\underline{\mathrm{d}x},\ \text{the last cycle has run } n \text{ directions}\bigr\}\\
&\quad= \frac{\mathrm{e}^{-\lambda t}}{(n+1)^n n!V_n}\left(\frac{\lambda}{c}\right)^{2n}\frac{1}{n+1}\prod_{l=0}^{n}h_l(\underline{x},t)\sum_{j=0}^{n}\frac{1}{h_j(\underline{x},t)}\left(\frac{(n+1)c}{\lambda}\right)^n\\
&\qquad\times\left(\prod_{j=0}^{n}h_j(\underline{x},t)\right)^{-n/(n+1)}\mathcal{I}'_{0,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n}h_j(\underline{x},t)}\right)\underline{\mathrm{d}x}\\
&\quad= \frac{\mathrm{e}^{-\lambda t}}{n!(n+1)V_n}\left(\frac{\lambda}{c}\right)^n\sqrt[n+1]{\prod_{j=0}^{n}h_j(\underline{x},t)}\sum_{j=0}^{n}\frac{1}{h_j(\underline{x},t)}\mathcal{I}'_{0,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n}h_j(\underline{x},t)}\right)\underline{\mathrm{d}x}.
\end{aligned}\tag{4.24}
$$

On the other hand, we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[\mathcal{I}_{0,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n}h_j(\underline{x},t)}\right)\right]\\
&\quad= \frac{\lambda}{c}\left[\frac{\partial}{\partial t}\left(\prod_{j=0}^{n}h_j(\underline{x},t)\right)^{1/(n+1)}\right]\mathcal{I}'_{0,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n}h_j(\underline{x},t)}\right)
\end{aligned}
$$

$$= \frac{\lambda}{(n+1)c}\left(\prod_{j=0}^{n} h_j(\underline{x},t)\right)^{1/(n+1)-1} \sum_{j=0}^{n}\left(\prod_{\substack{l=0\\ l\neq j}}^{n} h_l(\underline{x},t)\right)\frac{\partial h_j}{\partial t}(\underline{x},t)\mathcal{I}'_{0,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n} h_j(\underline{x},t)}\right)$$

$$= \frac{\lambda}{n+1}\sqrt[n+1]{\prod_{j=0}^{n} h_j(\underline{x},t)}\left(\sum_{j=0}^{n}\frac{1}{h_j(\underline{x},t)}\right)\mathcal{I}'_{0,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n} h_j(\underline{x},t)}\right).$$

Therefore, probability (4.24) can be rewritten in the following simple way:

$$\tilde{p}_n(\underline{x},t)\,\underline{\mathrm{d}x} = \Pr\{\underline{X}(t)\in\underline{\mathrm{d}x},\ \text{the last cycle has run } n \text{ directions}\}$$
$$= \frac{\mathrm{e}^{-\lambda t}\,\underline{\mathrm{d}x}}{\lambda n! V_n}\left(\frac{\lambda}{c}\right)^n \frac{\partial}{\partial t}\left[\mathcal{I}_{0,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n} h_j(\underline{x},t)}\right)\right]. \tag{4.25}$$

Formula (4.25) for $n = 1, 2, 3$, coincides exactly with the second terms in the distributions (1.11), (1.9) and (1.10) of Orsingher and Sommella [7], respectively. The other terms containing higher-order derivatives in (1.9) and (1.10) therein do not correspond to what emerges from (4.17) and thus the conjecture (1.12) formulated in that paper fails.

5. Some properties of the distributions obtained

5.1. About the derivatives of the integrated distributions

The following theorem generalizes the relationships (3.25) of Orsingher and Sommella [7] and (3.8) of Orsingher [6]. In view of (4.25), formula (5.1) connects the distribution of $\underline{X}(t)$ when the cycle is complete ($r = 0$) with the case where the cycle is *almost* complete ($r = n$).

This result suggests the misleading idea that the distributions $\tilde{p}_r(\underline{x},t)$ can be obtained by successive derivations with respect to t of $\tilde{p}_0(\underline{x},t)$ also for all $n > r \geqslant 1$. Unfortunately, this is not true because formula (5.1) cannot be extended to the case of higher-order derivatives (i.e. by replacing $\mathrm{d}/\mathrm{d}t$ and $\partial/\partial t$ respectively by $\mathrm{d}^k/\mathrm{d}t^k$ and $\partial^k/\partial t^k$ for $k \geqslant 2$) and this is the reason why the conjecture formulated in Orsingher and Sommella [7] is false.

We remark that in the one-dimensional case (that is for the telegraph process) the idea underlying the conjecture works because only the first derivative of $\tilde{p}_0(x,t)$, $x\in\mathbb{R}$ is involved.

Theorem 5.1. *The following identity holds*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathfrak{T}_{ct}}\mathcal{I}_{0,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n} h_j(\underline{x},t)}\right)\underline{\mathrm{d}x}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Vol}(\mathfrak{T}_{ct}) + \int_{\mathfrak{T}_{ct}}\frac{\partial}{\partial t}\left[\mathcal{I}_{0,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n} h_j(\underline{x},t)}\right)\right]\underline{\mathrm{d}x} \tag{5.1}$$

where $\operatorname{Vol}(\mathfrak{T}_{ct}) = V_n(ct)^n = \frac{(n+1)^{(n+1)/2}}{n!n^{n/2}}(ct)^n$.

Proof. We give a heuristic proof of the result (5.1) based on a geometrical approach. Indeed, the function F defined by

$$F(t) := \int_{\mathfrak{T}_{ct}}\mathcal{I}_{0,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{\prod_{j=0}^{n} h_j(\underline{x},t)}\right)\underline{\mathrm{d}x} := \int_{\mathfrak{T}_{ct}}\mathfrak{I}(\underline{x},t)\,\underline{\mathrm{d}x} \tag{5.2}$$

can be written as

$$F(t) = \int_0^{ct}\mathrm{d}s\int_{\partial\mathfrak{T}_s}\mathfrak{I}(\underline{x},t)\,\mathrm{d}\sigma(\underline{x}) \tag{5.3}$$

where $\mathrm{d}\sigma(\underline{x})$ stands for the infinitesimal element of the hyper-surface $\partial\mathfrak{T}_s$ which is the boundary of the $(n+1)$-hedron $\mathfrak{T}_s$. By performing the time-derivative of (5.3) we have that

$$F'(t) = c \int_{\partial \mathfrak{T}_{ct}} \mathfrak{I}(\underline{x}, t)\, \mathrm{d}\sigma(\underline{x}) + \int_0^{ct} \mathrm{d}s \int_{\partial \mathfrak{T}_s} \frac{\partial \mathfrak{I}}{\partial t}(\underline{x}, t)\, \mathrm{d}\sigma(\underline{x})$$
$$= c \int_{\partial \mathfrak{T}_{ct}} \mathfrak{I}(\underline{x}, t)\, \mathrm{d}\sigma(\underline{x}) + \int_{\mathfrak{T}_{ct}} \frac{\partial \mathfrak{I}}{\partial t}(\underline{x}, t)\, \underline{\mathrm{d}x}. \tag{5.4}$$

Since, for $\underline{x} \in \partial \mathfrak{T}_{ct}$, we have that $\prod_{j=0}^{n} h_j(\underline{x}, t) = 0$, we deduce $\mathfrak{I}(\underline{x}, t) = 1$ and thus

$$\int_{\partial \mathfrak{T}_{ct}} \mathfrak{I}(\underline{x}, t)\, \mathrm{d}\sigma(\underline{x}) = \int_{\partial \mathfrak{T}_{ct}} \mathrm{d}\sigma(\underline{x}) = \mathrm{Vol}_{n-1}(\partial \mathfrak{T}_{ct}) = c^{n-1}\, \mathrm{Vol}_{n-1}(\partial \mathfrak{T}_t). \tag{5.5}$$

In order to evaluate the $(n-1)$-dimensional volume $\mathrm{Vol}_{n-1}(\partial \mathfrak{T}_t)$, we observe that this latter can be viewed as the volume of the region comprised between the boundaries of the $(n+1)$-hedrons $\mathfrak{T}_t$ and $\mathfrak{T}_{t+\mathrm{d}t}$ divided by the infinitesimal depth $\mathrm{d}t$. More precisely, we can write that

$$\mathrm{Vol}_{n-1}(\partial \mathfrak{T}_t) = \frac{\mathrm{Vol}_n(\mathfrak{T}_{t+\mathrm{d}t}) - \mathrm{Vol}_n(\mathfrak{T}_t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\, \mathrm{Vol}_n(\mathfrak{T}_t)$$

and then

$$\mathrm{Vol}_{n-1}(\partial \mathfrak{T}_{ct}) = c^{n-1}\, \mathrm{Vol}_{n-1}(\partial \mathfrak{T}_t) = c^{n-1} \frac{\mathrm{d}}{\mathrm{d}t}\big[c^{-n}\, \mathrm{Vol}_n(\mathfrak{T}_{ct})\big] = \frac{1}{c} \frac{\mathrm{d}}{\mathrm{d}t}\, \mathrm{Vol}_n(\mathfrak{T}_{ct}).$$

Finally, putting this last relation into (5.5) and next into (5.4) leads to (5.1). □

Remark 5.2. An alternative (and more rigorous) proof of Theorem 5.1 can be worked out by applying the derivation rule to the n-fold integral:

$$F(t) = \int_{\mathfrak{T}_{ct}} \mathfrak{I}(\underline{x}, t)\, \underline{\mathrm{d}x}$$
$$= \int_{\varphi_0(t)}^{\psi_0(t)} \mathrm{d}x_0 \int_{\varphi_1(x_0,t)}^{\psi_1(x_0,t)} \mathrm{d}x_1 \cdots \int_{\varphi_{n-1}(x_0,\ldots,x_{n-2},t)}^{\psi_{n-1}(x_0,\ldots,x_{n-2},t)} \mathfrak{I}(x_0, \ldots, x_{n-1}, t)\, \mathrm{d}x_{n-1}, \tag{5.6}$$

where $\varphi_0(t) = -\frac{ct}{n}$, $\psi_0(t) = ct$ and, for $1 \leqslant j \leqslant n-1$

$$\varphi_j(x_0, \ldots, x_{j-1}, t) = \frac{1}{(n-j)a_j}\left(\sum_{i=0}^{j-1} a_i x_i + \frac{ct}{n}\right),$$
$$\psi_j(x_0, \ldots, x_{j-1}, t) = -\frac{1}{a_j}\left(\sum_{i=0}^{j-1} a_i x_i + \frac{ct}{n}\right),$$

and where $a_j = -\sqrt{\frac{n+1}{n}}\, \frac{1}{\sqrt{(n-j+1)(n-j)}}$.

5.2. *The distributions $\tilde{p}_r$ and the related hyper-Bessel equations*

We now show that the probability (4.17) purged of the exponential factor and reduced to a simplified form, by means of the geometrical transformation (3.4) are solutions to the partial differential equation

$$\frac{\partial^{n+1} q}{\partial u_0 \cdots \partial u_n} = \left(\frac{\lambda}{c(n+1)}\right)^{n+1} q. \tag{5.7}$$

It should be noted that in Section 3 we proved that the joint distributions $p_j(\underline{x}, t)$, after the exponential and geometrical reductions sketched above, also resolve the same p.d.e. (see Corollary 3.4).

Let us write the probability (4.17) as $\tilde{p}_r(\underline{x},t) = const \cdot \tilde{q}_r(\underline{u})$ where $\underline{u} = (u_0,\dots,u_n) = (h_0(\underline{x},t),\dots,h_n(\underline{x},t))$ and

$$\begin{aligned}\tilde{q}_r(\underline{u}) &= \sum_{j=0}^{n}(u_j u_{j+1}\cdots u_{j+r-1})\mathcal{I}_{r,n+1}\left(\frac{\lambda}{c}\sqrt[n+1]{u_0\cdots u_n}\right)\\ &= \sum_{q=0}^{\infty}\frac{(\lambda/(n+1)c)^{(n+1)q}}{(q!)^{n+1-r}((q+1)!)^r}\sum_{j=0}^{n}(u_0\cdots u_n)^q(u_j\cdots u_{j+r-1}).\end{aligned}$$

Since $\frac{\partial^{n+1}}{\partial u_0\cdots\partial u_n}(u_0^{k_0}\cdots u_n^{k_n}) = (k_0\cdots k_n)u_0^{k_0-1}\cdots u_n^{k_n-1}$, we have that

$$\frac{\partial^{n+1}}{\partial u_0\cdots\partial u_n}\left[(u_0\cdots u_n)^q(u_j\cdots u_{j+r-1})\right] = q^{n+1-r}(q+1)^r(u_0\cdots u_n)^{q-1}(u_j\cdots u_{j+r-1}).$$

Therefore,

$$\begin{aligned}\frac{\partial^{n+1}\tilde{q}_r}{\partial u_0\cdots\partial u_n}(\underline{u}) &= \sum_{q=0}^{\infty}\left(\frac{\lambda}{(n+1)c}\right)^{(n+1)q}\frac{q^{n+1-r}\,(q+1)^r}{(q!)^{n+1-r}((q+1)!)^r}\sum_{j=0}^{n}(u_0\cdots u_n)^{q-1}(u_j\cdots u_{j+r-1})\\ &= \sum_{q=1}^{\infty}\frac{(\lambda/(n+1)c)^{(n+1)q}}{((q-1)!)^{n+1-r}(q!)^r}\sum_{j=0}^{n}(u_0\cdots u_n)^{q-1}(u_j\cdots u_{j+r-1})\\ &= \left(\frac{\lambda}{(n+1)c}\right)^{n+1}\sum_{q=0}^{\infty}\frac{(\lambda/(n+1)c)^{(n+1)q}}{(q!)^{n+1-r}((q+1)!)^r}\sum_{j=0}^{n}(u_0\cdots u_n)^q(u_j\cdots u_{j+r-1})\\ &= \left(\frac{\lambda}{(n+1)c}\right)^{n+1}\tilde{q}_r(\underline{u})\end{aligned}$$

and thus $\tilde{q}_r$ solves (5.7).

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