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THE HALO CONJECTURE

A Dissertation in Mathematics by

Simone Panozzo

Matr. R11211

Under the supervision of Professor

Fabrizio Andreatta

Phd School Supervisor:

Prof. Vieri Mastropietro

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Contents

Introduction.

The story of p-adic modular forms mainly started during the International Summer School on "Modular functions of one variable and arithmetic applications" which took place at Antwerp University, from July 17 to August 3, 1972. It appeared immediately clear that the definition of a p-adic modular form should have taken into account the peculiarities and the arithmetic content of the p-adic topology. In particular, the first attempt to give a definition of p-adic modular forms was made by Serre in [Se72], which defined p-adic modular forms as formal power series with $\mathbb{Q}_p$ coefficients which are limits of classical modular forms with rational coefficients. This approach was very direct, and had the advantage that a single p-adic modular form could be seen as a limit of classical forms of different integral weights. In particular, in [Se72] it is proved that a p-adic modular form may have a weight which is a character of $\mathbb{Z}_p^\times$, generalizing greatly the classical idea of weight as an integer.

Parallel to the work of Serre, also Nicholas Katz proposed a definition of p-adic modular forms in his fundamental paper [Ka73]. The advantage of Katz's definition is in his purely geometrical character. In fact, instead of starting from the classical q-expansion principle, Katz used a different interpretation of modular forms. Classically, a modular form can be interpreted as a function defined over moduli spaces of elliptic curves over a suitable ring. This idea is the starting point of Katz's definition of p-adic modular forms as functions defined over moduli spaces of elliptic curves whose reductions are not supersingular. In [Ka73] there are two really important improvement in the theory of p-adic modular forms. First, even if in the first chapters Katz only deals with modular forms of integral weight (here the weight appears as the exponent in the tensor power of the sheaf of invariant differentials), in Chapter IV there is a way to see how Serre's p-adic modular forms can be incorporated in this geometric setting. In fact it is proved that the invertible sheaf which defines modular forms can be canonically reconstructed from the monodromy representation of the fundamental group of the modular curve at a geometric point. This idea allows to define geometrically also modular forms with weight $\chi \in \mathrm{End}(\mathbb{Z}_p^\times)$, simply by taking the invertible sheaf associated to the composition of the monodromy representation with the character. Another important concept introduced in [Ka73] is the notion of overconvergence. The idea of overconvergence comes to be very useful in the theory of Hecke operators, where the spaces of p-adic modular forms are really too large to have a reasonable spectral theory. In fact, p-adic modular forms are defined as functions on a line bundle over the p-adic modular curve, which parametrizes elliptic curves with a suitable level structure. In classical setting it is well-known that it is possible to introduce Hecke operators acting over the spaces of modular forms. Moreover, the spectral theory of Hecke operators gives important decomposition results and simplify the study of spaces of classical modular forms. In the p-adic setting a definition of Hecke operators is also possible, at least over the (lifting to characteristic 0) of the ordinary locus of the modular curve. The study of spectral theory of Hecke operators is then crucial also in the p-adic setting to understand the structure of the spaces of p-adic modular forms. Unfortunately, almost all the p-adic modular forms defined over the ordinary locus are eigenvectors for the operators of Hecke's algebra, hence the spectral theory is not useful to understand the structure of these spaces. Katz's idea was to consider modular forms and Hecke operators which can be extended a little bit inside the supersingular locus. This idea gave the possibility to evaluate modular forms over these neighborhoods of the ordinary locus, which are essentially described via the p-adic valuation of a suitable lifting of the Hasse invariant, whose importance in characteristic p is given by the fact that, as a modular form, it detects which points of the modular curve describe supersingular elliptic curves. Clearly the overconvergence neighborhoods of the ordinary locus are no more given by schemes, but they can be described generically as rigid analytic spaces. Inside the algebra of Hecke operators, one operator is particularly important, which is the so called U_p operator. Geometrically, in characteristic p it can be described as an operator which evaluate a modular form over all possible preimages of a given elliptic curve under Frobenius morphism. In characteristic 0, clearly, the Frobenius morphism is no more available, hence one of the object which is necessary to define Hecke operators is the canonical subgroup, which gives a lifting in characteristic 0 of the kernel of mod p Frobenius isogeny. This canonical subgroup almost trivially exists inside the ordinary locus, but the existence, unders suitable overconvergence conditions, inside the supersingular locus was first studied by Lubin, and chapter 3 of Katz gives a precise description and construction of the canonical subgroup and its properties over the supersingular locus.

During the 80's another major breakthrough was by Haruzo Hida, who constructed the first examples of p-adic families of modular forms, in [Hi1] and [Hi2]. When Hida started to work on p-adic modular forms, they were defined to be sections of an invertible sheaf defined over the p-adic modular curve with a given integer or p-adic weight. The real improvement in the theory was the discovery that ordinary modular eigenforms, which are modular forms which are simultaneous eigenvectors for all the Hecke's operators whose corresponding eigenvalue is a unit, live naturally "in families". This means that, defining the weight space as the formal spectrum of the Iwasawa algebra $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, for a given character of an ordinary modular eigenform, there exists a neighborhood of this character, and associated modular eigenforms which interpolate the given modular form. This means that under suitable hypothesis, modular eigenforms can be continuously parametrized by their p-adic weights. The problem with Hida's theory as it was formulated was that the almost unique example of p-adic family of modular forms could be given for the Eisenstein series. A key object for Hida's theory is the so called ordinary projector, which is an idempotent operator acting on the spaces of cuspidal modular forms. This object can only be defined for eigenforms whose slope, i.e. the p-adic valuation of its U_p-eigenvalue, is zero. The crucial role of the ordinary projector in Hida's theory is the main difficult to extend its approach to greater slopes eigenforms.

However, Hida's theory opened the possibility of organizing overconvergent eigenforms in a geometrical object. In particular, as a consequence of Hida's description, slope 0 overconvergent eigenforms can be parametrized by a rigid analytic curve which is finite and flat over the rigid analytic weight space. This means that each point of this rigid analytic curve corresponds to a normalized overconvergent eigenform with slope 0. At this point, the idea developed by Robert Coleman in a series of papers ([Col96], [Col97]) and Barry Mazur ([CM98]) was to generalize Hida's theory to arbitrary but finite slope overconvergent modular eigenforms. In this way Coleman and Mazur constructed a rigid analytic curve $\mathscr{E}$, called the Eigencurve, over $\mathbb{Q}_p$, whose $\mathbb{C}_p$ points parametrize normalized finite slope overconvergent eigenforms. By construction, this rigid analytic curve is fibered over the weight space, and in particular it fits into a diagram:

$$\begin{tikzcd} \mathscr{E} \arrow[r, "a_p"] \arrow[d, "w"'] & \mathbb{G}_m \\ \mathcal{W} & \end{tikzcd}$$

where $\mathcal{W}$ is the rigid analytic weight space, which is described as the Raynaud's rigid analytic fiber attached to the formal scheme $\mathrm{Spf}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]])$, w is the map sending an eigenform to its p-adic weight, and $\mathbb{G}_m$ is the p-adic torus which is the target of the map a_p sending an eigenform to its eigenvalue for the U_p operator. Coleman's construction of the Eigencurve starts with the construction of the spectral curve attached to the U_p-operator. This curve is essentially defined to be the set of zeroes of the characteristic series of the U_p-operator, which is the main object of investigation of this thesis. It is proved in [CM98] that, even if the theory of the Eigencurve is rigid analytic, hence defined over $\mathbb{Q}_p$, the characteristic series of the U_p operator is integral. Moreover, there exists a universal characteristic series with coefficients in the Iwasawa algebra which interpolates all characteristic series of U_p coming from different p-adic weights. This means that it should be possible to globalize the theory using some integral models of the construction, and moreover that also operations like the mod p reduction of the characteristic series should have some geometric explanation.

Very soon after the construction of the eigencurve has been presented by Coleman and Mazur, a lot of work has been done in order to understand its geometry. In particular, the region which seems easier to investigate is the region close to the boundary. A conjecture was suggested by a computation of Buzzard and Kilford [BK05], which addresses a question asked by Coleman and Mazur. Let us be a bit more precise about this conjecture. Let p be an odd prime number (the conjecture is also formulated for the prime $p = 2$, but we assume to work with odd primes during the whole thesis in order to simplify the notation and also the description of the weight space), and let v_p and $|-|_p$ be the p-adic valuation and the p-adic absolute value normalized in such a way that $v_p(p) = 1$. Denote now by Λ the Iwasawa algebra $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ which is the union of $p-1$ open unit disks indexed by the characters of the torsion group of $\mathbb{Z}_p^\times$. Then each closed point of $\mathcal{W}$ corresponds to a continuous p-adic character of $\mathbb{Z}_p^\times$, hence we can define a parameter T on the weight disk, whose value on a given character χ of $\mathbb{Z}_p^\times$ is $T = T_\chi = \chi(\exp(p)) - 1$. For $r \in (0,1)$ we can take $\mathcal{W}^{>r}$ to be the union of annuli where $|T| > r$, which is called the "halo" of the weight space. Clearly the weight map w produces a region of the eigencurve $\mathscr{E}^{>r}$ which is the preimage under w of $\mathcal{W}^{>r}$. Then the conjecture is the following

Conjecture 1. *Let $r \in (0,1)$ be sufficiently close to 1. Then the following hold:*

i) The space $\mathscr{E}^{>r}$ is a disjoint union of countably infinitely many connected component $Z_1, Z_2, \dots$ such that the weight map $w_{|Z_n} : Z_n \to \mathcal{W}^{>r}$ is finite and flat for each n.

ii) There exist non-negative rational numbers $\alpha_1, \alpha_2, \dots \in \mathbb{Q}$ in non-decreasing order and tending to infinity such that for each n and each point $z \in Z_n$, we have

$$|a_p(z)| = |T_{w(z)}|^{\alpha_n}$$

iii) The sequence $\alpha_1, \alpha_2, \dots$ is a disjoint union of finitely many arithmetic progressions counted with their multiplicity.

This conjecture, which is known as the Halo Conjecture has been proved to be true in many particular situations. When the tame level of the modular curve involved is trivial and the prime $p = 2$ it has been verified by an explicit computation in [BK05]. More explicit computations for small prime p and small tame levels have appeared in [Ja04], [Kil08], [KM12], [Ro14], and a partial result independent on the prime and the tame level was proved in [WXZ14]. This conjecture really describes very well the geometry of the eigencurve over the boundary of weight space. Another improvement to the proof of the Halo Conjecture has been given in 2016 in the paper of Liu, Wan and Xiao [LWX], which proves the Halo Conjecture in the case of quaternion modular forms for $\mathrm{GL}_2(\mathbb{Q}_p)$. Moreover, using the Jacquet-Langlands correspondence, this result can be viewed as part of the Halo Conjecture for overconvergent modular forms, simply identifying suitably the eigenvalues coming from spectral theory of quaternion modular forms with the eigenvalues coming from overconvergent modular forms.

However, excluding the improvement given by [LWX], the situation for overconvergent modular forms seems to be more difficult, due to the geometry of modular curves. Reading carefully the paper of Liu, Wan and Xiao [LWX] it seems that two ingredients are needed in order to attach the conjecture. The first ingredient is an integral model of spaces of families of overconvergent modular forms, which makes possible some reduction operations.

The work of Andreatta, Iovita and Pilloni [AIP] precisely deals with this kind of problem. Starting with the integrality of the characteristic series of U_p operator, they produce an integral theory of overconvergent families of modular forms. In particular, instead of Coleman, they work with formal schemes, seen as adic spaces. Let me be a bit more precise about how this construction, in order to fix the notation in what follows. Even if we work with $p \geq 3$, the construction of [AIP] works well also for the even prime. Now the adic weight space is

$$\mathcal{W} := \mathrm{Spa}(\Lambda, \Lambda)^{\mathrm{an}}$$

which, thanks to the nature of Huber's adic spaces, contains both characteristic 0 and characteristic p points. As we explained before, each connected component of $\mathcal{W}$ has ring of functions $\mathbb{Z}_p[[T]]$, and there is a bijection between the set of connected components of $\mathcal{W}$ and the finite characters of the Iwasawa algebra. Inside $\mathcal{W}$ it is possible to distinguish a p-adic and a T-adic region. The first is the region containing the center of weight space, the second one contains the boundary. In particular, the T-adic region of the connected component of the trivial finite character has as coordinate ring the ring

$$B = \mathbb{Z}_p[[T]] \left\langle \frac{p}{T} \right\rangle.$$

It is also possible to consider different regions of the weight space where the topology is p or T-adic and in the regions which don't contain neither the center or the boundary, the two topologies coincide. In [AIP] formal models of these spaces are considered. In particular, there is a formal scheme which describes the largest region where the topology is T-adic, which is given by $\mathfrak{W} = \mathrm{Spf}(B)$, which is an admissible formal blowup of the connected component of tre trivial character inside the weight space. It is also possible to consider different admissible blowups which describes formal models of smaller regions of the weight space. The topology over $\mathfrak{W}$ is the T-adic topology. If we now consider $\mathfrak{X}/\mathrm{Spf}(\mathbb{Z}_p)$, the p-adic completion of the $\Gamma_1(N)$-modular curve over $\mathbb{Z}_p$, we may base change it to $\mathfrak{W}$, and define, using a lifting to $\mathbb{Z}_p$ of a suitable power of the Hasse invariant, admissible formal blowups which are formal models for strict neighborhoods of overconvergence of the ordinary locus. We call such a blowup $\mathfrak{X}_r$, where r is the radius of overconvergence. In [AIP] it is shown that over $\mathfrak{X}_r$ it is possible to define an invertible sheaf $\mathfrak{w}_r$ whose global sections are integral overconvergent modular forms. Moreover, specializing to the special fiber, it is possible to construct a different modular sheaf whose global sections are overconvergent modular forms in characteristic p. The two constructions, one in characteristic 0 and one in characteristic p may be glued together by first considering an invertible sheaf $\mathfrak{w}_\infty$ over $\mathfrak{X}_\infty$, which is the T-adic formal scheme given by a projective limit along liftings of Frobenius to characteristic 0. As we see, everything here is integral, so these spaces produce integral models of the spaces considered by Coleman, and the space of overconvergent modular forms, which is

$$\mathrm{H}^0(\mathfrak{X}_r, \mathfrak{w}_r)$$

is proved to be a projective Banach B-module. This is Proposition 6.9 in [AIP].

Few years after [AIP], the paper of Jan Vonk [Von] showed that these spaces of integral overconvergent modular forms are also orthonormalizable, i.e. they admit an orthonormal basis, at least for r large enough. The idea is to construct a T-adic version of the Eisenstein series and to use it to trivialize the line bundle of overconvergent modular forms. This is possible only when the radius is large enough. The B-basis constructed by Vonk is highly non canonical, as it depends on the choice for a splitting of the multiplication by a lifting of the Hasse invariant over the sheaf of functions of $\mathfrak{X}_{r,I}$. The explicit form of this basis is

$$\left\{ \left(\frac{T}{\widetilde{\mathrm{Ha}}^{p^{r+1}}} \right) \mathbf{E} \ b_{m,n} \right\}_{m,n}$$

where $\widetilde{\mathrm{Ha}}$ is a characteristic 0 lifting of the Hasse invariant, $\mathbf{E}$ is the T-adic Eisenstein series, and $b_{m,n}$ are classical modular forms of weight $np^{r+1}(p-1)$. The index m coincide with the dimension of the spaces of classical modular forms of weight $np^{r+1}(p-1)$. Since the spaces of Andreatta, Iovita and Pilloni give integral models for overconvergent modular forms and both those spaces and the continuous functions from $\mathbb{Z}_p$ to itself are orthonormalizable Banach spaces, it seems reasonable to compare the two as in [LWX], in order to get closer to the Halo Conjecture.

This is the idea developed in this thesis. We want to create a connection between the spaces of integral overconvergent modular forms and the space of continuous functions from $\mathbb{Z}_p$ to a suitable ring of functions for the weight space. The main result of this thesis is the following

Theorem 1. *Let $X/\mathbb{Z}_p$ be the compactified modular curve over $\mathbb{Z}_p$ of tame level $\Gamma_1(N)$. Then there exist, for r large enough, and for every supersingular point x of the special fiber of X, a T-adically complete, separated and norm decreasing $\mathbb{Z}_p$-algebra $\mathbb{B}_x^{\mathrm{perf}}$ and an homomorphism of $\mathbb{B}^{\mathrm{perf}} := \bigoplus_{x\in \mathrm{SS}} \mathbb{B}_x^{\mathrm{perf}}$-orthonormalizable Banach modules*

$$\Psi : \mathrm{H}^0(\mathfrak{X}_r, \mathfrak{w}_r)\widehat{\otimes}_B \mathbb{B}^{\mathrm{perf}} \to \mathrm{Cont}(\mathbb{Z}_p, \mathbb{B}^{\mathrm{perf}})$$

where SS *is the set of supersingular points of the special fiber of X.*

The proof of this result consists in constructing explicitly the map ψ_x. The idea behind its definition comes from the observation contained in a letter which Serre wrote to Tate in 1996 (see [Se96]) and which contains the first approach to the so called Jacquet-Langlands correspondence. In that letter Serre describes a bijection from the set of supersingular elliptic curves over $\overline{\mathbb{F}_p}$ with a suitable adelic level structure $K = GL_2(\mathbb{Q}_p)K^p$ and an $\mathbb{F}_{p^2}$ rational non-vanishing differential and the double quotient

$$D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/\ker\tau \cdot K^p$$

where D is the quaternion algebra over $\mathbb{Q}$ ramified at $\{p, \infty\}$, K^p is the level at p and $\tau : \mathcal{O}^\times \to \mathbb{F}_{p^2}^\times$ is the reduction modulo a uniformizer of the maximal order in $D(\mathbb{Q}_p)$. Now, a modular forms comes out to be simply a function on elliptic curves with non-vanishing differential, and so, evaluating on this set, Serre obtains continuous functions from this double quotient to $\overline{\mathbb{F}_p}$. The construction of this map, which is also compatible with Hecke operators, gives a correspondence between the Hecke eigenvalues appearing in the spaces of mod p modular forms and those appearing in the space of quaternionic mod p automorphic forms, which are identified with the double quotient. The definition of this map is very geometric, since it deals with an interpretation of spaces of supersingular elliptic curves as double classes of a quotient. In my thesis, the construction of the map ψ_x goes in the same direction. In fact, I first give an interpretation of the integral model for the anticanonical tower of modular forms. After that, I consider the complete partial Igusa tower over $\mathfrak{X}_\infty$ which is constructed in chapter 6 of [AIP], and I describe a moduli interpretation of this formal scheme, which easily follows from the one given for $\mathfrak{X}_\infty$. Thanks to this moduli interpretation, I can give a moduli description of perfect overconvergent modular forms, which are described as functions over the Igusa tower which transforms in a suitable way under the action of the Galois group of the Igusa tower. In particular, I prove that this moduli interpretation holds:

Proposition 1. *A perfect modular form $f \in \mathrm{H}^0(\mathfrak{X}_\infty, \mathfrak{w}_\infty)$ is a function which associates to:*

i) A T-adically complete and separated B-algebra R which is T-torsion free and normal;

ii) A character κ defined by the morphism $\mathrm{Spf}(R) \to \mathfrak{W}_I^{\mathrm{cycl}}$;

iii) An elliptic curve $f : E \to \mathrm{Spec}(R)$ equipped with a level N-structure ψ_N;

iv) A section $\eta \in \mathrm{H}^0(\mathrm{Spf}(R), f^*(\omega^{p(1-p)}))$ such that $\eta\widetilde{\mathrm{Ha}}^p = T$ modulo p^2;

v) A p-divisible group D_∞ of height 1 such that, called $E_n := E/D_n$, E_n admits a level n canonical subgroup and D_n splits generically the exact sequence of the p^n-torsion of E_n;

vi) A morphism $\beta : T_p(E) \to \mathbb{Z}_p^2$ which becomes an isomorphism generically,

an element $f(E, \psi_N, \eta, D_\infty, \beta) \in R$ such that

$$f(E, \psi_N, \eta, D_\infty, \gamma\beta) = \kappa(\gamma)^{-1} f(E, \psi_N, \eta, D_\infty, \beta)$$

for every $\gamma \in \mathbb{Z}_p^\times$, which acts over the quintuple by changing the generator of $T_p(D_\infty)$.

Now, starting from a supersingular point $\overline{x} \in X_{\mathbb{F}_p}$ the idea is to construct a suitable characteristic 0 supersingular B-point of the modular curve $\mathfrak{X}_r$, and to create from it the associated $\mathbb{B}_x^{\mathrm{perf}}$-point of the infinite Igusa tower, where $\mathbb{B}_x^{\mathrm{perf}}$ is the normalization of the ring given by taking the completion of the inductive limit of the pullbacks of the B-point along the anticanonical tower. In the construction of $\mathbb{B}_x^{\mathrm{perf}}$ it is implicit a choice of a compatible family of p^n-th roots of unity. The moduli interpretation of the anticanonical and of the Igusa tower allows to describe such a point as an elliptic curve with a trivialization of the generic Tate module. We denote such a point by

$E_{a,b}$, where the notation represents the elliptic curve E we are considering equipped with the basis a, b of its Tate module, and where a is the basis of the canonical subgroup. We then prove that for every $n \in \mathbb{N}$ and for every $\lambda \in \mathbb{Z}_p$ it is possible to define an isogeny, which we write $\pi_{n,\lambda}$ which sends the point $E_{a,b}$ to the point $E_{a,\frac{b+\lambda a}{p^n}}$. We prove that, even if this isogeny is constructed using the trivialization of the Tate module, which only appears generically, it is well-defined integrally, and so gives another point of the infinite level Igusa tower. Using this isogeny, we define, for every $n \in \mathbb{N}$ and for every perfect modular form f, a map

$$\psi_{n,x}(f) : \mathbb{Z}_p \to \mathbb{B}_x^{\mathrm{perf}}$$

which sends the modular form f to its value over $E_{a,\frac{b+\lambda a}{p^n}}$. We prove that this map is continuous, and we globalize the construction for every $n \in \mathbb{N}$ defining a map:

$$\psi_x : \mathrm{H}^0(\mathfrak{X}_r, \mathfrak{w}_r)\widehat{\otimes}_B \mathbb{B}_x^{\mathrm{perf}} \to \mathrm{Cont}(\mathbb{Z}_p, \mathbb{B}_x^{\mathrm{perf}})$$

sending f to $\sum_{n\geq 0} \psi_{n,x}(f)$. Here we implicitly use the fact that the sheaf of modular forms at infinity can be descended to an invertible sheaf over $\mathfrak{X}_r$, and in fact this sheaf is the sheaf of overconvergent modular forms over $\mathfrak{X}_r$. The sum of these morphisms ψ_x over all possible supersingular points of $X_{\mathbb{F}_p}$ is the map Ψ of Theorem 13. We can study the different maps ψ_x in order to understand the properties of the map Ψ.

After that, we study the translation of the U_p operator in the setting of continuous functions from $\mathbb{Z}_p$ to $\mathbb{B}_x^{\mathrm{perf}}$, and we prove that the action of the U_p operator over those functions which are in the image of ψ can be described by the following formula:

Proposition 2. *Let $\mathfrak{U}_p$ be the operator acting on the image of the map Ψ_x. Then $\mathfrak{U}_p$ splits into a sum of $p-1$ operators, and its action over a function g which belongs to the image of ψ_x is the following:*

$$\mathfrak{U}_p(g)(\lambda) = \sum_{\mu=0}^{p-1} (g(p\lambda + \mu) + k_g)$$

where k_g is a constant depending only on g.

We want to remark that this explicit computation of the action implies that the map ψ_x cannot be surjective. In fact, it's easy to see that the operator $\mathfrak{U}_p$ is not compact over the space of continuous functions, while it is proved in [AIP] that the action of the U_p operator over the spaces of integral overconvergent modular forms is compact. This means that the image of ψ must be contained in a subspace of $\mathrm{Cont}(\mathbb{Z}_p, \mathbb{B}_x^{\mathrm{perf}})$ where the action of $\mathfrak{U}_p$ is compact.

The kernel and the image of the map ψ will surely be argument of later studies, but what is important in the existence of the map ψ, and of the sum over different supersingular points Ψ, is related to the fact that the translation of the U_p operator in the module of continuous functions from $\mathbb{Z}_p$ to $\mathbb{B}^{\mathrm{perf}}$ gives, up to the constant k_g, exactly the same operator which is analyzed by Liu, Wan and Xiao in [LWX]. This consideration opens the possibility to use the same techniques as in [LWX] to study the action of the U_p operator over the entire space of integral overconvergent modular forms. In fact, in [LWX], the explicit computation of the action of U_p over the spaces of quaternion overconvergent modular forms gave them the possibility to make a fruitful analysis of the Newton polygon of its characteristic series, and to prove the Halo conjecture.

In order to make possible a later analysis of the properties of the map ψ_x, and in particular in order to use the Vonk's description of the basis for $\mathrm{H}^0(\mathfrak{X}_r, \mathfrak{w}_r)$, we also construct a Mahler basis for $\mathrm{Cont}(\mathbb{Z}_p, \mathbb{B}_x^{\mathrm{perf}})$. We decide to construct such a Mahler basis in a slightly more general situation than what we need. In fact, we are able to prove the following result:

Proposition 3. *Let R be a $\mathbb{Z}_p$ Banach algebra which is norm decreasing, i.e. such that the norm of the image of any element in $\mathbb{Z}_p$ via the structure morphism of R is lower or equal than its p-adic norm. Then there is an isomorphism of Banach R-modules*

$$\mathrm{Cont}(\mathbb{Z}_p, \mathbb{Z}_p)\widehat{\otimes}_{\mathbb{Z}_p} R \cong \mathrm{Cont}(\mathbb{Z}_p, R).$$

In particular, the space of continuous functions from $\mathbb{Z}_p$ to R admits a Mahler basis.

The proof of this result is strictly based on the computations of Mahler, and it is essentially a computation in p-adic analysis which uses strongly the hypothesis on the decreasing of norm. This last result says that we can treat continuous functions from $\mathbb{Z}_p$ to $\mathbb{B}_x^{\mathrm{perf}}$ by using a Mahler basis. We hope that the existence of the map Ψ, when a complete description of its kernel and image will be exhausted, will give, combined with an analysis similar to the one performed in [LWX] a complete description of the characteristic series of $\mathfrak{U}_p$, and we hope that this description will give the proof of the Coleman's Halo Conjecture.

Structure of the Thesis

The first two chapters of this thesis does not contain anything new. The first one deals with Formal Schemes, Rigid Analytic Varieties and Adic Spaces and describes the prerequisites which are necessary to understand the geometry involved in what follows. In particular, we put our attention on the notion of generic fiber of a Formal Scheme, which can be described both as a Rigid Analytic and a Huber's adic space.

In Chapter 2 we focus on the content of [AIP]. In particular we follow the paper and reproduce the most important arguments. We also give a proof of a representability result which is used in the paper to prove the existence of partial Igusa tower.

In Chapter 3 we construct our version of Mahler basis. What is essentially new here is that we construct a Mahler basis for the space of continuous functions $\mathrm{Cont}(\mathbb{Z}_p, R)$, where R is any $\mathbb{Z}_p$-algebra which is complete and norm decreasing. This hypothesis is fulfilled by the rings we are dealing with in Chapter 4.

In Chapter 4 we construct the map Ψ and we show that it is well-defined. We also study the geometry of the U_p operator, and we use this characterization to write its action over the spaces of continuous functions from $\mathbb{Z}_p$ to the ring $\mathbb{B}^{\mathrm{perf}}$.

Chapter 1

Formal Schemes and their Generic Fibers.

In this section we recall the notion of Formal Schemes and the connection with Huber's Adic Spaces. In particular, we recall the notion of adic generic fiber, which substitute the idea of Berthelot's generic fiber. The main reference for Formal Schemes is [Bo15], while for the description of the generic fiber, we follow the treatment in [SW], which gives a generalization of Huber's papers. As in classical scheme theory, we first recall the algebraic notion which gives the open covering of a formal scheme.

1.1 Formal Schemes.

In this section we see the definition of formal schemes, their fundamental properties and the Berthelot's construction of rigid generic fiber of a formal scheme.

1.1.1 Definition of Formal Schemes.

Definition 1. *An I-adic ring A for an ideal I of A is a commutative ring with unit equipped with the unique topology given by the basis*

$$\mathcal{B} = \{x + I^n \mid x \in A \text{ and } n \in \mathbb{N}\}$$

which is I-adically complete and separated, i.e.

$$A \cong \varprojlim_n A/I^n.$$

Such a ring is called f-adic if I is finitely generated. In particular, a Noetherian adic ring is always f-adic. A ring is called adic if it is I-adic for an ideal I of A, and similarly it is called f-adic if it is adic for a finitely generated ideal. We call such an ideal I ideal of definition of A. A morphism of adic rings is a ring homomorphism which is also continuous.

Geometrically, the category of I-adic rings is anti equivalent to the category of affinoid formal schemes, which are defined in this way:

Definition 2. *Given an I-adic ring A, we denote by* $\mathrm{Spf}(A)$ *the topological space given by all the prime ideals of A containing I, i.e. all the open prime ideals of A. It is a topological space with the topology induced by the inclusion* $\mathrm{Spf}(A) \subseteq \mathrm{Spec}(A)$.

Notice that in particular, the Zariski topology over $\mathrm{Spec}(A)$ induces a topology over $\mathrm{Spf}(A)$ where a basis of open subsets is given by the sets

$$D(f) = \{\mathfrak{p} \in \mathrm{Spf}(A) \mid f \neq 0 \text{ in } A/\mathfrak{p}\}$$

for $f \in A$. In particular, it is possible to equip $\mathrm{Spf}(A)$ with a structure sheaf $\mathscr{O}_{\mathrm{Spf}(A)}$ which is defined over the basis given by $D(f)$ as

$$\mathscr{O}_{\mathrm{Spf}(A)}(D(f)) = A\langle f^{-1}\rangle$$

It can be proved that $(\mathrm{Spf}(A), \mathscr{O}_{\mathrm{Spf}(A)})$ defines a locally topologically ringed space. This allows us to give the following definition

Definition 3. *Let A be an adic ring with ideal of definition I. Then the couple* $(\mathrm{Spf}(A), \mathscr{O}_{\mathrm{Spf}(A)})$ *is called the formal spectrum of A. Given a topologically locally ringed space $(X, \mathscr{O}_X)$, we say it is an affine formal scheme if there exists an adic ring A such that $(X, \mathscr{O}_X) \cong (\mathrm{Spf}(A), \mathscr{O}_{\mathrm{Spf}(A)})$.*

Remark. Topologically, the formal spectrum $\mathrm{Spf}(A)$ is homeomorphic to the affine scheme $\mathrm{Spec}(A/I)$, but their structure sheaves are very different, in fact the structure sheaf of $\mathrm{Spec}(A/I)$ is described by the ring A/I, while the structure sheaf of $\mathrm{Spf}(A)$ is given by

$$\mathscr{O}_{\mathrm{Spf}(A)} = \varprojlim_n \mathscr{O}_{\mathrm{Spec}(A/I^n)}$$

Heuristically, this means that formal schemes describe functions which may be defined in infinitesimal neighborhood of a closed subscheme. In particular, they are related with the idea of deformations.

Now that we know what affine objects are, the following definition is clear:

Definition 4. *Let $(X, \mathscr{O}_X)$ be a locally topologically ringed space. Then it is called a formal scheme if every $x \in X$ admits an open neighborhood U, where $(U, \mathscr{O}_U)$ is isomorphic as a topologically ringed space to $(\mathrm{Spf}(A), \mathscr{O}_{\mathrm{Spf}(A)})$ for an f-adic ring A.*

Form now on, we denote by **FSch** the category of formal schemes, and by **Ad** the category of adic rings. The following result can be proved in the same way as in scheme theory.

Theorem 2. *Let X and Y be formal schemes and assume that $Y = Spf(A)$ is affine. Then there is a bijection:*

$$\mathrm{Hom}_{\mathbf{FSch}}(X, Y) \cong \mathrm{Hom}_{\mathbf{Ad}}(A, \mathscr{O}_X(X))$$

As in the classical scheme situation, formal schemes can be constructed by glueing affine pieces. In particular, the category of formal schemes is equipped with fiber product, where the fiber product is constructed by glueing the affine pieces given by completed tensor product.

Definition 5. *Let (A, I) and (B, J) be two adic rings which are topological algebras over a third adic ring (R, K), where the second component is the ideal of definition. Then we can define the fiber product of* $\mathrm{Spf}(A)$ *and* $\mathrm{Spf}(B)$ *over* $\mathrm{Spf}(R)$ *to be:*

$$\mathrm{Spf}(A) \times_{\mathrm{Spf}(R)} \mathrm{Spf}(B) = \mathrm{Spf}(A \widehat{\otimes}_R B)$$

where the completed tensor product is defined to be the usual tensor product $A \otimes_R B$ equipped with the adic topology given by the ideal $I + J$.

Remark. One can prove that the completed tensor product can be explicitly computed as

$$A \widehat{\otimes}_R B = \varprojlim_{n,m} A/I^n \otimes_R B/J^m$$

One of the crucial example of formal scheme is the one coming from the completion of a scheme along a closed subscheme.

Definition 6. *Let X be a scheme and let $Y \subseteq X$ be a closed subscheme defined by a quasi-coherent ideal $\mathscr{I} \subseteq \mathscr{O}_X$. Then $(Y, \mathscr{O}_Y)$, where $\mathscr{O}_Y = \varprojlim_n (\mathscr{O}_X/\mathscr{I}^n)_{|Y}$ is a formal scheme known as the $\mathscr{I}$-adic completion of X.*

In the case when $X = \mathrm{Spec}(A)$ is affine and Y is determined by a finitely generated ideal I of A, the I-adic completion of $\mathrm{Spec}(A)$ coincide with $\mathrm{Spf}(A)$. In order to define Raynaud's generic fiber, it is useful to introduce some finiteness conditions on formal schemes.

Definition 7. *Let R be a complete and separated adic ring with a finitely generated ideal of definition $I \subseteq R$, without I-torsion. A topological R-algebra A is called*

i) of topologically finite type if it is isomorphic to an R-algebra of type $R\langle X_1, \dots, X_n\rangle/\mathfrak{a}$ with the I-adic topology, where $R\langle X_1, \dots, X_n\rangle$ is the ring of power series with coefficients in R, whose coefficients tend to 0 for the I-adic topology, and $\mathfrak{a}$ is an ideal of $R\langle X_1, \dots, X_n\rangle$.

ii) Of topologically finite presentation if, in addition to i), $\mathfrak{a}$ is finitely generated.

iii) Admissible if i) and ii) hold true and A does not have I-torsion.

It's not so difficult to prove that the three conditions in the previous definition can be checked locally, i.e. over localizations. This means that these notions generalize to formal schemes.

Definition 8. *Let X be a formal scheme over Spf(R), where R is I-adically complete and separated. Then X is called locally of topologically finite type (resp. locally of topologically finite presentation, resp. admissible) if it can be covered with a family of affine open subsets which are of topologically finite type (resp. of topologically finite presentation, resp. admissible).*

1.1.2 Generic Fiber via Rigid Spaces.

It's clear that for formal schemes the usual notion of generic fiber fails. In fact, let us consider for example $\mathrm{Spf}(\mathbb{Z}_p)$, where $\mathbb{Z}_p$ is a p-adic ring. Clearly, since its topological space is homeomorphic to $\mathrm{Spec}(\mathbb{F}_p)$, it has no generic points, so it is not possible to define an object like $\mathrm{Spf}(\mathbb{Q}_p)$. The idea of Raynaud, suggested by Grothendieck, was to interpret the category of rigid spaces as the category of generic fibers of formal schemes, when this is possible. We first recall the notion of rigid spaces, again by first considering their affine pieces.

Definition 9. *Let K be a field equipped with a non-archimedean absolute value. Then a K-algebra A is called an affinoid K-algebra if there is an epimorphism of K-algebras $\alpha : K\langle X_1, \ldots, X_n\rangle \to A$, where $K\langle X_1, \ldots, X_n\rangle$ is the ring of power series whose coefficients tend to zero for the topology defined by the absolute value over K.*

This algebraic definition allows to define the affine pieces producing a rigid space.

Definition 10. *Let A be a K-affinoid algebra. We then define the affinoid K-space associated to A to be the set $\mathrm{Sp}(A)$ of maximal ideals of A.*

Clearly such a set can be equipped with the Zariski topology, where a basis of open subsets is given by

$$D_f = \{x \in Sp(A) \mid f(x \neq 0)\} \qquad f \in A.$$

The problem with the Zariski topology is that it is too coarse, and it does not reflect the topological properties of the field K. In fact, also in order to build rigid analytic spaces, it is a common procedure to introduce a different topology, which is the canonical topology.

Definition 11. *For any affinoid K-space $X = \mathrm{Sp}(A)$, the topology generated by all subsets of type*

$$X(f; \epsilon) = \{x \in X \mid |f(x)| \leq \epsilon\} \qquad f \in A \textit{ and } \epsilon \in \mathbb{R}_{>0}$$

is called the canonical topology of X.

It's not difficult to prove that all the subsets of X of type

$$X(f_1, \ldots, f_n) = \bigcap_{i=1}^{n} X(f_i; 1) \qquad f_1, \ldots, f_n \in A$$

give a basis for the canonical topology of X. It's not so easy to glue affinoid K-spaces, and this can be done via the notion of Grothendieck's topology. We list here the main definitions involved in the construction of rigid spaces. Again, for an exhaustive description of these objects, we quote as main references [BGR] and [Bo15].

Definition 12. *i) A Grothendieck topology τ consists of a category $\mathcal{C}$ and a set $\mathrm{Cov}(\tau)$ of families $(U_i \to U)_{i \in I}$ of morphisms in $\mathcal{C}$, called coverings, such that the following hold:*

1) If $\Phi : U \to V$ is an isomorphism in $\mathcal{C}$, then $(\Phi) \in \mathrm{Cov}(\tau)$.

2) If $(U_i \to U)_{i \in I}$ and $(V_{ij} \to U_i)_{j \in J_i}$ for $i \in I$ belong to $\mathrm{Cov}(\tau)$, then the same is true fo the compositions $(V_{ij} \to U_i \to U)_{i \in I, j \in J_i}$.

3) If $(U_i \to U)_{i \in I} \in \mathrm{Cov}(\tau)$, and if $V \to U$ is a morphism in $\mathcal{C}$, then the fiber product $U_i \times_U V$ exists in $\mathcal{C}$, and $(U_i \times_U V \to V)_{i \in I}$ belongs to $\mathrm{Cov}(\tau)$.

ii) Let τ be a Grothendieck topology and let $\mathcal{D}$ be a category with pullbacks. A presheaf on τ with values in $\mathcal{D}$ is defined to be a contravariant functor $F : \mathcal{C} \to \mathcal{D}$. We call F a sheaf if the diagram

$$F(U) \to \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \times_U U_j)$$

is an equalizer for every $(U_i \to U)_{i \in I}$ in $\mathrm{Cov}(\tau)$.

iii) Let X be an affinoid K-space. The strong Grothendieck topology on X is given as follows:

a) A subset $U \subseteq X$ is called admissible open if there is a covering $U = \bigcup_{i \in I} U_i$ of U by affinoid subdomains $U_i \subseteq X$ such that for all morphisms of affinoid K-spaces $\phi : Z \to X$ satisfying $\phi(Z) \subseteq U$, the covering $\phi^{-1}((U_i))_{i \in I}$ of Z admits a refinement that is a finite covering of Z by affinoid subdomains.

b) A covering $V = \bigcup_{j \in J} V_j$ of some admissible open subset $V \in X$ by means of admissible open sets V_j is called admissible if for each morphism of affinoid K-spaces $\phi : Z \to X$ satisfying $\phi(Z) \subseteq V$, the covering $(\phi^{-1}(V_j))_{j \in J}$ of Z admits a refinement that is a finite covering of Z by affinoid subdomains.

And finally, we give the definition of a rigid space:

Definition 13. *A rigid analytic K space is a locally ringed space $(X, \mathscr{O}_X)$, where X is a topological space with a strong Grothendieck topology such that X admits an admissible covering $(X_i)_{i \in I}$, where X_i is an affinoid K space for all $i \in I$, and $\mathscr{O}_X$ is a sheaf of affinoid K-algebras.*

Finally, as we mentioned above, it is an idea of Raynaud to see rigid spaces as generic fibers of formal schemes. In particular, the following holds:

Theorem 3. *Let R be a complete valuation ring of height 1 with field of fractions K. Then the functor $A \mapsto A \otimes_R K$ defined on the category of R-algebras of topologically finite type gives rise to a functor $X \mapsto X_{\text{rig}}$ from the category of formal R-schemes that are locally of topologically finite type, to the category of rigid K-spaces.*

This object, denoted by X_{rig} is called the rigid generic fiber of the R-formal scheme X. We remark that, in the affine situation, say $X = \text{Spf}(A)$, both the formal scheme associated to A and its rigid generic fiber can be realized as subsets of $\text{Spec}(A)$. In fact, by definition, $\text{Spf}(A)$ is the set of prime ideal of A which do not contain IA, for I the prescribed ideal of definition of R. Moreover, $(\text{Spf}(A))_{\text{rig}}$ coincides with $\text{Sp}(A \otimes_R K)$ which is the set of maximal ideals of $A \otimes_R K$, which is the set of closed points in $A \otimes K$, which gives all the closed points of the generic fiber of $\text{Spec}(A)$. We also point out the fact that the rigidification functor defined in the Theorem factors through the category of admissible formal schemes, essentially because tensoring with K kills all the I-torsion. One interesting feature of rigid and formal geometry is to describe, given a K-rigid space X_K, all the admissible formal schemes X such that $X_{\text{rig}} \cong X_K$. The solution of this question has been given, at least in a particular situation, and lead to the notion of admissible formal blow up, which will be very important in all that follows. The construction of Raynaud's generic fiber has been generalized by Berthelot, and its version of the generic fiber allows to produce rigid spaces which are understood to be generic fibers of formal schemes of the kind $R[[T_1, \ldots, T_n]]\langle X_1, \ldots, X_n \rangle$.

1.1.3 Admissible Formal Blowing-up.

The construction of admissible formal blowing-up is crucial in formal and rigid geometry. In fact, the admissible blowing-up of a formal scheme provide a class of formal schemes which have the same rigid generic fiber. Moreover, the notion of admissible blowing-up gives a new way to describe formal models of the strict neighborhoods of ordinary locus of modular curves. The idea is clearly to complete locally the usual notion of blowing-up of a scheme.

Definition 14. *Let X be a formal R-scheme that is locally of topologically finite presentation and let $\mathscr{A} \subsetneq \mathscr{O}_X$ be a coherent open ideal. Then the formal R-scheme*

$$X_{\mathscr{A}} = \varinjlim_{n} \text{Proj} \left(\bigoplus_{d=0}^{\infty} \mathscr{A}^d \otimes_{\mathscr{O}_X} (\mathscr{O}_X / I^n \mathscr{O}_X) \right)$$

together with the canonical projection $X_{\mathscr{A}} \to X$ is called the formal blowing-up of $\mathscr{A}$ on X. Any such blowing-up is referred to as an admissible formal blowing-up of X.

This construction has clearly an algebraic interpretation when we blow up an affine formal scheme. In fact, if $X = \mathrm{Spf}(A)$, it can be proved that the topological finiteness hypothesis allows to deduce that an ideal $\mathscr{A} \subseteq \mathscr{O}_X$ is coherent open if and only if it is associated to a coherent open ideal $\mathfrak{a} \subseteq A$. Then it's easy to prove the following

Proposition 4. *Let $X = \mathrm{Spf}(A)$ be an affine formal R-scheme of topologically finite presentation. Furthermore, let $\mathscr{A} = \tilde{\mathfrak{a}}$ be the coherent open ideal of $\mathscr{O}_X$, which is associated to the coherent open ideal $\mathfrak{a} \subseteq A$. Then the formal blowing-up $X_{\mathscr{A}}$ equals the I-adic completion of the scheme theoretic blowing-up $(\mathrm{Spec}(A))_{\mathfrak{a}}$ of $\mathfrak{a}$ on $\mathrm{Spec}(A)$.*

This in particular means that the notion of formal blowup is local, i.e.the blowing-up of a formal scheme can be realized as a glueing of completions of scheme theoretic blowing-up's. In the particular situation when the scheme we are starting with is admissible, it is also possible to deduce the equations for the blowing-up and a lot of good properties, like in what follows.

Proposition 5. *Let $X = \mathrm{Spf}(A)$ be an admissible formal R-scheme that is affine, and let $\mathscr{A} = \tilde{\mathfrak{a}}$ be a coherent open ideal in $\mathscr{O}_X$ associated to a coherent open ideal $\mathfrak{a} = (f_0, \dots, f_r) \subseteq A$. Then the following assertions hold for the formal blowing-up $X_{\mathscr{A}}$ of $\mathscr{A}$ on X:*

i) The ideal $\mathscr{A}\mathscr{O}_{X_{\mathscr{A}}} \subseteq \mathscr{O}_{X_{\mathscr{A}}}$ is invertible.

ii) Let U_i be the locus in $X_{\mathscr{A}}$ where $\mathscr{A}\mathscr{O}_{\mathscr{A}}$ is generated by f_i, $i = 0, \dots, r$. Then the U_i's define an open affine covering of $X_{\mathscr{A}}$.

iii) Write

$$C_i = A\left\langle \frac{f_j}{f_i} \mid j \neq i \right\rangle = A\langle Z_j \mid j \neq i\rangle / (f_i Z_j - f_j \mid j \neq i)$$

Then the I-torsion of C_i coincides with its f_i-torsion, and $U_i = Spf(A_i)$ holds for $A_i = C_i/(I - \mathrm{torsion})_{C_i}$.

Moreover, if X is an admissible formal R-scheme and $\mathscr{A} \subseteq \mathscr{O}_X$ a coherent open ideal. Then the formal blowing-up $X_{\mathscr{A}}$ of $\mathscr{A}$ on X is an admissible formal R-scheme again.

Moreover, admissible blowing-up's satisfy the following universal property:

Proposition 6. *For an admissible formal R-scheme X and a coherent open ideal $\mathscr{A} \subseteq \mathscr{O}_X$, the formal blowing-up $X_{\mathscr{A}} \to X$ is such that any morphism of formal R-schemes $\phi : Y \to X$ such that $\mathscr{A}\mathscr{O}_Y$ is an invertible ideal in $\mathscr{O}_Y$ factorizes uniquely through $X_{\mathscr{A}}$.*

The main result related to the idea of formal blowing-up's is the following theorem due to Raynaud.

Theorem 4. *Let R be a complete valuation ring of height* 1 *with field of fractions K. Then the functor* rig *induces an equivalence between the category of all admissible formal R-schemes that are quasi-paracompact, localized by the class of admissible formal blowing-up's and the category of all quasi-separated rigid K-spaces that are quasi-paracompact.*

The previous theorem essentially means two crucial facts. First that an admissible formal scheme satisfying the properties of the statement and any of its admissible blowing-up's have the same rigid generic fiber, and, moreover, that any rigid space over K admits a formal model, i.e. a formal scheme whose rigid generic fiber is the given rigid space.

1.2 Adic Spaces.

Here we define the idea of adic spaces and the notion of adic generic fiber of a formal scheme.

1.2.1 Definition of Adic Space

As we saw in the previous section, formal schemes over a valuation ring and rigid spaces over its field of fractions are strictly linked by the construction of rigid generic fiber. This suggests that it should be a good idea to work with a category which contains both formal schemes and rigid spaces, i.e. a category such that taking the generic fiber of a formal scheme is an inner operation. This naturally lead to the notion of adic spaces, which have been introduced by Huber. In this section, we follow the treatment of adic spaces as in [Hu1] and [Hu2]. We want to remark that clearly the notion of adic spaces not only constructs a category containing both formal schemes and rigid spaces, but also allows to extend greatly the notion of generic fiber. We start with the following:

Definition 15. *Let A be a topological ring, and let Γ be a totally ordered monoid which comes from a totally ordered group with a minimal element 0 added. A Γ-valued continuous valuation on A is a map $|-|: A \to \Gamma$ such that:*

i) $|1| = 1$ *and* $|0| = 0$.

ii) For any $a, b \in A$, $|ab| = |a||b|$.

iii) For any $a, b \in A$, $|a+b| \leq \max\{|a|, |b|\}$.

iv) (Continuity condition) For any $\gamma \in \Gamma$ *the set* $\{a \in A \mid |a| < \gamma\}$ *is open in* A.

Two continuous valuations over a ring A, $|-|_1$ and $|-|_2$ are called equivalent if for every $a, b \in A$

$$|a|_1 \leq |b|_1 \Leftrightarrow |a|_2 \leq |b|_2$$

The following algebraic definition gives the right notion of rings involved in Huber's theory of adic spaces.

Definition 16. *1) Let A be a topological ring.*

i) A subset $S \subseteq A$ is called bounded if, for all $U \subseteq A$ open neighborhood of 0 there exists an open neighborhood of 0, V, such that

$$V \cdot S = \{v_1 s_1 + \ldots v_n s_n \mid v_1, \ldots, v_n \in V, s_1, \ldots, s_n \in S, n \in \mathbb{N}\} \subseteq U$$

ii) An element $a \in A$ is called power bounded if the set of its positive powers $\{a^n \mid n \in \mathbb{N}\}$ is bounded. The set of power bounded elements is denoted A°.

iii) An element $a \in A$ is called topologically nilpotent if its powers converge to 0 in the topology of A. The set of topologically nilpotent elements is denoted $A^{\circ\circ}$.

iv) A subring $A^+ \subseteq A$ is called a ring of integral elements if $A^+ \subseteq A^\circ$ is open and integrally closed.

v) A topological ring A is called Huber ring if there is an open subring $A_0 \subseteq A$ which is f-adic. Such a ring is called ring of definition. A Huber ring is called Tate if it contains a topologically nilpotent element which is also a unit.

2) A pair (A, A^+) which consists of a Huber ring with a ring of integral elements is called an affinoid ring, or a Huber pair. A morphism of affinoid rings $f : (A, A^+) \to (B, B^+)$ is a continuous morphism $f : A \to B$ such that $f(A^+) \subseteq B^+$.

To every Huber pair, it is possible to associate a topological space, which gives the affinoid part of an adic space.

Definition 17. *Let (A, A^+) be an affinoid ring. Then we define the adic spectrum* $\mathrm{Spa}(A, A^+)$ *to be the set of equivalence classes of continuous valuations $|-|$ such that $|A^+| \leq 1$, equipped with the topology generated by the following subsets*

$$\{x \in \mathrm{Spa}(A, A^+) \mid |f(x)| \leq |g(x)|\} \qquad \text{for } f, g \in A$$

For elements $s_1, s_2, \ldots s_n \in A$ and finite subsets $T_1, \ldots, T_n \subseteq A$ such that $T_i A \subseteq A$ is open for all $i = 1, \ldots, n$, we define the associated rational subset to be

$$U(\{T_i/s_i\}) = U\left(\frac{T_1}{s_1}, \ldots, \frac{T_n}{s_n}\right) = \{x \in \mathrm{Spa}(A, A^+) \mid |t_i(x)| \leq |s_i(x)| \neq 0 \mid \forall t_i \in T_i\}.$$

Here we denoted by $|f(x)|$ the absolute value $x(f)$.

Proposition 7. *Let $X = \mathrm{Spa}(A, A^+)$ be an adic spectrum and let $U \subseteq X$ be a rational subset. Then there exists an affinoid ring $(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ together with a structure morphism*

$$(A, A^+) \to (\mathscr{O}_X(U), \mathscr{O}_X^+(U))$$

such that the corresponding map $\mathrm{Spa}(\mathscr{O}_X(U), \mathscr{O}_X^+(U)) \to \mathrm{Spa}(A, A^+)$ factors through U and is universal for all such maps. Moreover, if

a) A admits a Noetherian ring of definition, or *b) A is Tate strongly Noetherian, i.e. such that $A\langle X_1, \dots, X_n \rangle$ is Noetherian for every $n \in \mathbb{N}$,*

then the presheaves

$$\mathscr{O}_X(V) := \varprojlim_{V \subseteq U \text{rational}} \mathscr{O}_X(U) \qquad \mathscr{O}_X^+(V) := \varprojlim_{V \subseteq U \text{rational}} \mathscr{O}_X^+(U)$$

where, for a rational subset $U\left(\{\frac{T_i}{s_i}\}\right)$, these presheaves are defined by the previous universal property, are sheaves.

Proof. We just want to give the construction of localizations which fit into the universal property, leaving all the details to [Hu1]. So let $\mathrm{Spa}(A, A^+)$ be an affinoid adic space, and choose a ring of definition $A_0 \subseteq A$ with a finitely generated ideal of definition $I \subseteq A_0$, and write $U = U\left(\{\frac{T_i}{s_i}\}\right)$ for the rational subset U, as above. Then we can define the ring

$$A_0[\{T_i/s_i\}] := A_0\left[\frac{t_i}{s_i} \mid t_i \in T_i \text{ and } i = 1, \dots, n\right]$$

and it's not hard to see that the $IA_0[\{T_i/s_i\}]$-adic topology over the localizations $A[1/s_i]$ makes $A_0[\{T_i/s_i\}]$ to be an open subring of the localization. Take $A[\{1/s_i\}]^+$ to be the integral closure of the image of $A^+[\{T_i/s_i\}]$ inside $A[\{1/s_i\}]$. Then we define the affinoid ring $(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ to be the completion of the couple $(A\langle\{T_i/s_i\}\rangle, A\langle\{T_i/s_i\}\rangle^+)$ with respect to the topology defined above. □

The following definition coming from [Hu1] describes adic spaces.

Definition 18. *i) An affinoid ring (A, A^+) is called sheafy if the structure presheaf $\mathscr{O}_X$ on $X = \mathrm{Spa}(A, A^+)$ is a sheaf. For any sheafy affinoid ring (A, A^+) the adic space associated to (A, A^+) is the topological space $\mathrm{Spa}(A, A^+)$ together with the structure sheaf $\mathscr{O}_X$ and the induced valuation $|-|_x$ on the stalk $\mathscr{O}_{X,x}$.*

ii) Let (V) denote the category whose objects are triples $(X, \mathscr{O}_X, |-(x)|_{x\in X})$, where X is a topological space, $\mathscr{O}_X$ a sheaf of complete topological rings, and, for each $x \in X$, $|-(x)|$ is an equivalence class of valuations on $\mathscr{O}_{X,x}$, and whose morphisms between two objects $(X, \mathscr{O}_X, |-(x)|_{x\in X})$ and $(Y, \mathscr{O}_Y, |-(y)|_{y\in Y})$ are maps of locally topologically ring spaces with the obvious compatibility condition with the valuations on the *stalk. The full subcategory of adic spaces in* (V) *consists of objects* $(X, \mathscr{O}_X, (|-(x)|)_{x\in X})$ *which admit an open covering by spaces* U_i *such that the triple* $(U_i, \mathscr{O}_X|_{U_i}, (|-(x)|)_{x\in X})$ *is isomorphic to an affinoid adic space.*

iii) Let X *be an adic space. A point* $x \in X$ *is called analytic if there exists an open neighborhood* U *of* x *such that* $\mathscr{O}_X(U)$ *is Tate. The open subspace of* X *consisting of all analytic points are denoted by* X^a*. If* $X^a = X$*, we call* X *analytic.*

1.2.2 Adic Generic Fiber.

There is a way to attach to a formal scheme a different notion of generic fiber, coming from Huber's theory of adic spaces. In particular, this construction extends, in a meaning that will be clear soon, the Berthelot's construction of rigid generic fiber of a formal scheme. First, the following results are the main content of [Hu2]:

Theorem 5. *i) (Formal Geometry) Let* X *be a locally noetherian formal scheme. Then there exists an adic space* X^{ad} *and a morphism of locally and topologically ringed spaces* $\pi : (X^{\mathrm{ad}}, \mathscr{O}^+_{X^{\mathrm{ad}}}) \to (X, \mathscr{O}_X)$ *such that for every adic space* Z *and for every morphism of locally and topologically ringed spaces* $f : (Z, \mathscr{O}^+_Z) \to (X, \mathscr{O}_X)$*, there exists a unique morphism of adic spaces* $g : Z \to X^{\mathrm{ad}}$ *making the following diagram commute:*

$$\begin{tikzcd}
(Z, \mathscr{O}^+_Z) \arrow[r, "f"] \arrow[dr, dashed] & (X, \mathscr{O}_X) \\
& (X^{\mathrm{ad}}, \mathscr{O}_{X^{\mathrm{ad}}}) \arrow[u, "\pi"']
\end{tikzcd}$$

Moreover, the $(-)^{\mathrm{ad}}$*-construction defines a fully faithful functor from the category of locally noetherian formal schemes to the category of adic spaces. Finally, over any locally noetherian formal scheme* X*, there is a functor* $\mathscr{F} \mapsto \mathscr{F}^{\mathrm{ad}}$ *from the category of* $\mathscr{O}_X$*-modules to the category of* $\mathscr{O}_{X^{\mathrm{ad}}}$*-modules which is fully faithful when restricted to the category of coherent* $\mathscr{O}_X$*-modules.*

ii) (Rigid Geometry) Let K be a field which is complete wrt a valuation of rank 1. Let K^0 be the subring of its power bounded elements. Then for every rigid analytic variety V over K, there exists an adic space V^{ad} and a morphism $\pi : (V^{\mathrm{ad}}, \mathscr{O}_{V^{\mathrm{ad}}}) \to (V, \mathscr{O}_V)$ of ringed sites, where the right-hand side is the site equipped with the strong Grothendieck topology introduced above, such that V^{ad} is locally of finite type over $\mathrm{Spa}(K, K^0)$ and satisfies the following universal property. If Z is a locally of finite type adic space over $\mathrm{Spa}(K, K^0)$ and $g : (Z, \mathscr{O}_Z) \to (V, \mathscr{O}_V)$ is a morphism of rigid analytic varieties, then there exists a unique morphism $g : Z \to V^{\mathrm{ad}}$ such that the following diagram commutes:

$$\begin{tikzcd}
(Z, \mathscr{O}_Z) \arrow[r, "f"] \arrow[dr, dashed] & (V, \mathscr{O}_V) \\
& (V^{\mathrm{ad}}, \mathscr{O}_{V^{\mathrm{ad}}}) \arrow[u, "\pi"']
\end{tikzcd}$$

Moreover, this gives a fully faithful functor from the category of rigid analytic varieties over K to the category of adic spaces.

Notice that the Theorem says precisely that we can view schemes, formal schemes and rigid analytic varieties as subcategories of the same category, the one of adic spaces. In particular, it is possible to define also in the context of adic spaces, the generic fiber of a locally noetherian formal scheme. In order to construct the adic generic fiber, we need a notion of fiber product in the category of adic spaces. This notion doesn't behave very well. In fact, for general adic spaces, the fiber product is not well-defined. By the way, it is possible to construct it in some useful situations.

Definition 19. *Let $f : X \to Y$ be a morphism of adic spaces. Then f is called locally of finite type if for every $x \in X$ there exists an open affinoid neighborhood $U = \mathrm{Spa}(B, B^+)$ of x in X and an open affinoid subspace $V = \mathrm{Spa}(A, A^+)$ of Y with $f(U) \subseteq V$ such that the induced morphism of complete Huber pairs $(A, A^+) \to (B, B^+)$ is topologically of finite type, i.e. if it factors through an isomorphism of Huber pairs*

$$B \cong A\langle X_1, \ldots, X_n \rangle_{T_1, \ldots, T_n} / \mathfrak{a}$$

for some $n \in \mathbb{N}$, finite subsets $T_1, \ldots, T_n \subseteq A$ with $T_i \cdot A$ open in A, and $\mathfrak{a}$ a closed ideal.

Here the notation $A\langle X_1, \dots, X_n\rangle_{T_1,\dots,T_n}$ denotes the ring of formal power series with coefficients definitely contained in the product $T_1 \cdot T_n$. For details about this ring, we refer to Chapter 3 of [Hu2].

Proposition 8. *Let $f : X \to Y$ and $g : Z \to Y$. Then the fiber product $X \times_Y Z$ exists in the category of adic spaces if f or g is locally of finite type.*

This allows to construct the generic fiber of a locally noetherian formal scheme. Let now $\mathcal{O}$ be a complete discrete valuation ring with uniformizer ϖ, field of fractions K and residue field k. Then, the adic space $S = \mathrm{Spa}(\mathcal{O}, \mathcal{O})$ attached to $\mathrm{Spf}(\mathcal{O})$ consists of an open point η and a closed point s, with $\kappa(\eta) = K$ and $\kappa(s) = k$. The canonical morphism $\mathrm{Spa}(K, \mathcal{O}) \to \mathrm{Spa}(\mathcal{O}, \mathcal{O})$ is an open immersion onto the open point, and it is a morphism of adic spaces locally of finite type. Now, let us consider a formal scheme $\mathfrak{X}$ which is locally formally of finite type over $\mathrm{Spf}(\mathcal{O})$, i.e. $\mathfrak{X}$ is such that for every point $x \in \mathfrak{X}$ there exists an open affinoid neighborhood $\mathrm{Spf}(A)$ such that there is a continuous open surjective homomorphism $\mathcal{O}[[T_1, \dots, T_n]]\langle X_1, \dots, X_m\rangle \to A$, where the topology of $\mathcal{O}[[T_1, \dots, T_n]]\langle X_1, \dots, X_m\rangle$ is given by the ideal $(\varpi, T_1, \dots, T_n)$. Then the adic generic fiber of $\mathfrak{X}$ is defined as

$$\mathfrak{X}_\eta^{\mathrm{ad}} := \mathfrak{X}^{\mathrm{ad}} \times_{\mathrm{Spa}(\mathcal{O},\mathcal{O})} \mathrm{Spa}(K, \mathcal{O})$$

It can be proved, see chapter 4 of [Ka], that the adic generic fiber is isomorphic to the adification of the Berthelot's generic fiber. This means that, given a formal scheme $\mathfrak{X}$, there is an isomorphism of adic spaces:

$$\mathfrak{X}_\eta^{\mathrm{ad}} \cong (\mathfrak{X}^{\mathrm{rig}})^{\mathrm{ad}}$$

where $\mathfrak{X}^{\mathrm{rig}}$ is the Berthelot's rigid generic fiber of $\mathfrak{X}$.

Chapter 2

The Spectral Halo.

In this first chapter, we follow the main constructions of [AIP]. Essentially nothing new appears in this chapter, which only recalls the foundational material about the halo conjecture. In this chapter, we organize the material contained in [AIP] following first the "characteristic 0", and then the "characteristic p" construction, and we finish with the very beautiful glueing construction in [AIP]. In order to get a self-contained exposition, we also recall the proofs of the main results.

2.1 The Weight Space.

Since p-adic modular forms can be seen as functions over the Igusa tower attached to the modular curve with an associated p-adic weight, the first object to discuss when one speaks about families of p-adic or overconvergent modular forms is the space where weights live. This is a kind of base space over which the modular curve is fibered, and parametrizes all the possible weights, integer or not, associated to modular forms. In Coleman's work, the weight space is a rigid analytic space à la Tate, while in [AIP] the authors prefer to work with Huber's analytic adic spaces. The main advantage of Huber's geometry is that it allows to talk about formal schemes and their generic fiber without changing the category. Moreover, the use of Huber's adic spaces allows to treat both "characteristic 0" and "characteristic p"-points. This improvement is fundamental in the construction of the boundary of weight space which is a completely new object introduced in [AIP].

Convention. From now on, for all this essay, we assume that p denote an odd prime. Most of the constructions are possible also in characteristic 2 with suitable modifications, but we choose to avoid the even prime in order to simplify the statements.

2.1.1 Compactification of the Weight Space.

Let $\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ be the Iwasawa algebra associated to the group $\mathbb{Z}_p^\times$, and let $\mathfrak{W} := \mathrm{Spf}(\Lambda)$ be the associated formal scheme, where the topology on Λ is (p, T)-adic. The rigid analytic generic fiber of $\mathfrak{W}$ is the weight space considered by Coleman.

Lemma 1. *$\mathfrak{W}$ is a union of $p-1$ connected components isomorphic to copies of $\mathrm{Spf}(\mathbb{Z}_p[[T]])$.*

Proof. The map $\mathbb{Z}_p[[\mathbb{Z}_p^\times]] \to \mathbb{Z}_p[(\mathbb{Z}/p\mathbb{Z})^\times][[T]]$ given by sending $\exp(p)$ to $1+T$ is a topological isomorphism, hence we get the decomposition. □

In particular, every connected component parametrizes characters of $1+p\mathbb{Z}_p$, with a fixed finite character. Notice that every connected component is the formal spectrum of a regular local ring of dimension 2 with maximal ideal (p, T).

Associated to the formal scheme $\mathfrak{W}$, one can define, via the Huber's functor in [Hu2], the adic space $\mathfrak{W}^{\mathrm{ad}} = \mathrm{Spa}(\Lambda, \Lambda)$, whose points are equivalence classes of continuous valuations over Λ. This space is equipped with a sheaf of topological algebras, denoted $\mathscr{O}_{\mathfrak{W}^{\mathrm{ad}}}$, and, for every point $x \in \mathfrak{W}^{\mathrm{ad}}$, there exists a valuation v_x over the stalk of $\mathscr{O}_{\mathfrak{W}^{\mathrm{ad}},x}$. The space $\mathfrak{W}^{\mathrm{ad}}$ contains all the points of $\mathfrak{W}$, but we are interested precisely in points not appearing in $\mathfrak{W}$. The following Lemma clarifies the meaning of this remark, but first we have to recall the notion of analytic point.

Definition 20. *Let (A, A^+) be an affinoid algebra, and let $x \in Spa(A, A^+)$. We call support of x the prime ideal of A of elements $a \in A$ whose valuation associated to x is zero. We say that $x \in Spa(A, A^+)$ is analytic if its support is not open.*

Using the definition it is easy to prove the following characterization of non analytic points of $\mathfrak{W}^{\mathrm{ad}}$.

Lemma 2. *Non-analytic points of $\mathfrak{W}^{ad}$ are in bijection with $\mathfrak{W}$.*

We then define $\mathcal{W}$ to be the open given by analytic points of $\mathfrak{W}^{\mathrm{ad}}$. It's not hard to see that $\mathcal{W}$ is quasi-compact but not affinoid. By the way it is possible to produce affinoid subspaces of $\mathcal{W}$ simply by a comparison, for every point $x \in \mathcal{W}$ of the valuation of p and T attached to x. In particular, we can state the following definition:

Definition 21. *Let $\frac{r}{s} \in \mathbb{Q}$ be a rational number. Then define:*

$$\mathcal{W}_{\leq \frac{r}{s}} := \{x \in \mathcal{W} \mid |T^r|_x \leq |p^s|_x \neq 0\} \quad \mathcal{W}_{\geq \frac{r}{s}} := \{x \in \mathcal{W} \mid |p^s|_x \leq |T^r|_x \neq 0\}$$
$$\mathcal{W}_{\geq 0} := \mathcal{W} \qquad \mathcal{W}_{\leq \infty} := \mathcal{W}$$
$$\mathcal{W}_{\leq 0} := \{x \in \mathcal{W} \mid |T|_x = 0\} \qquad \mathcal{W}_{\{\infty\}} := \{x \in \mathcal{W} \mid |p|_x = 0\}$$

Then, let $I = [a, b]$, with $a, b \in \mathbb{Q}_{>0} \cup \{\infty\}$ and define:

$$\mathcal{W}_I := \mathcal{W}_{\leq b} \cap \mathcal{W}_{\geq a}$$

For every I of this kind, properly contained in $[0, \infty]$, $\mathcal{W}_I$ is affinoid, and notice that if $I = [0, t]$ with $t \in \mathbb{Q}_{\geq 0}$, the topology over the ring of functions of $\mathcal{W}_I$ is p-adic, and p is a topologically nilpotent unit, while if I is of the form $[t, \infty]$, the topology is T-adic and T is a topologically nilpotent unit. In particular, in both situations, the ring of functions of $\mathcal{W}_I$ is a Tate ring. Moreover, if we consider $I = [a, b]$, with rational a and b, we see that the T and p-adic topologies coincide and both p and T are topologically nilpotent units. This means that the choice of I parametrizes a kind of *glissando* of the topology, from p to T-adic. In particular, if $t \in [0, \infty)$, then $\mathcal{W}_{[0,t]}$ is a finite union of balls centered in 0 and with radius $p^{-\frac{1}{t}}$. So, if we denote $\mathcal{W}^{\mathrm{rig}} := \mathcal{W}_{[0,\infty)}$, then this is the adic space associated to Berthelot's generic fiber of $\mathfrak{W}$.

So let us have a look to $\mathcal{W}_{\{\infty\}}$ which can be interpreted, at least set theoretically, as a kind of "boundary" of the weight space, in fact we may write the heuristic equality:

$$\mathcal{W}_{\{\infty\}}\text{''} = \text{''}\mathcal{W} - \mathcal{W}^{\mathrm{rig}}$$

The equality is not precisely an equality, since the right-hand side might not have the structure of an adic space. So we now define which is its structure. Notice that $\mathcal{W}_{\{\infty\}}$ has an interpretation in terms of characteristic p-points. In fact, consider a point $x \in \operatorname{Spec}(\mathbb{F}_p[(\mathbb{Z}/p\mathbb{Z})^\times]((T)))$. It corresponds to a morphism:

$$x : \mathbb{F}_p[(\mathbb{Z}/p\mathbb{Z})^\times]((T)) \to \mathbb{F}_p((T))$$

and it's easy to see that every point of $\mathcal{W}_{\{\infty\}}$ is given by composing the mod-p reduction of Λ with the morphism given by x, with the T-adic valuation of $\mathbb{F}_p((T))$. This says that $\mathcal{W}_{\{\infty\}}$ is affinoid, given by

$$\mathcal{W}_{\{\infty\}} = \mathrm{Spa}(\mathbb{F}_p[(\mathbb{Z}/p\mathbb{Z})^\times]((T)), \mathbb{F}_p[(\mathbb{Z}/p\mathbb{Z})^\times[[T]]). \tag{2.1}$$

We introduce now another notation. We denote by $\mathcal{W}^0 := \mathrm{Spa}(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]])^{\mathrm{an}}$ the analytic adic space given by the closed subspace of $\mathcal{W}$ corresponding to the trivial character of $(\mathbb{Z}/p\mathbb{Z})^\times$. We also denote by $\mathcal{W}_I^0 = \mathcal{W}^0 \cap \mathcal{W}_I$ for any interval I. This provides a decomposition of the weight space into product of $\mathcal{W}_I^0$ with the finite character part.

We want to end this section recalling how to produce integral models of the adic spaces introduced above. From now on, we essentially restrict our considerations to the connected component given by trivial finite character. For a given interval I, denote by $B_I = \mathrm{H}^0\left(\mathcal{W}_I, \mathscr{O}^+_{\mathcal{W}_I}\right)$. Then we define:

$$\mathfrak{W}_I^0 := \mathrm{Spf}(B_I) \tag{2.2}$$

Notice that $\mathfrak{W}_I^0$ is just an admissible blowup of the formal weight space $\mathfrak{W}$. In fact, in the most useful cases, we have:

$$\begin{array}{ll}
I = [p^k, p^{k'}] & B_I = \mathbb{Z}_p[[T]]\langle u, v\rangle/(T^{p^k}v - p, uv - T^{p^{k'}-p^k}) \\
I = [p^k, \infty] & B_I = \mathbb{Z}_p[[T]]\langle u\rangle/(p - uT^{p^k}) \\
I = \{\infty\} & B_I = \mathbb{F}_p[[T]] \\
I = [0,1] & B_I = \mathbb{Z}_p[[T]]\langle v\rangle/(T - vp)
\end{array}$$

and we can see that each B_I is the coordinate ring of a blowup.

2.1.2 Analyticity of the universal character.

Clearly, we have a universal character $\kappa^{\mathrm{un}} : \mathbb{Z}_p^\times \to \Lambda^\times$, simply given by the inclusion of $\mathbb{Z}_p^\times$ inside the Iwasawa algebra. This character gives a pairing of presheaves:

$$\mathcal{W} \times \mathbb{Z}_p^\times \longrightarrow \mathbb{G}_m^+$$

where $\mathbb{Z}_p^\times$ is the constant sheaf of value $\mathbb{Z}_p^\times$. The pairing is defined, for (A, A^+)-points, as:

$$(\kappa, x) \mapsto \kappa(x).$$

The main fact is that, if we restrict κ^{un} to certain opens inside weight space, then the character becomes locally analytic, i.e. it is possible to extend it to a character of the group $\mathbb{Z}_p^\times(1 + p^n\mathbb{G}_a^+) \subseteq \mathbb{G}_m^+$. We reprove the following result which will be useful in what follows:

Lemma 3. *For every $n \geq 1$, we have*

$$\kappa^{un}(1 + qp^{n-1}\mathbb{Z}_p) - 1 \subset (T^{p^{n-1}}, T^{p^{n-2}}p, \ldots, p^{n-1}T)\Lambda.$$

Proof. First, we choose once and for all the non canonical isomorphism $\mathbb{Z}_p[[\mathbb{Z}_p^\times]] \cong \mathbb{Z}_p[\mathbb{Z}/q\mathbb{Z}^\times][[T]]$ sending the topological generator $\exp(q)$, with $q = p$ if p is odd and $q = 4$ if $p = 2$, to $1 + T$. Moreover, $\exp(q)$ is a topological generator of $1 + \mathbb{Z}_p$, hence we may compute $\kappa^{\mathrm{un}}(\exp(qp^{n-1}))$. Remember that the p-adic valuation of the binomial coefficient $C_{p^n}^k$ is $n - v_p(k)$. We then have:

$$\kappa^{\mathrm{un}}(\exp(qp^{n-1})) = (1+T)^{p^{n-1}} = \sum_{k=0}^{p^{n-1}} C_{p^{n-1}}^k T^k = 1$$

modulo $(T^{p^{n-1}}, T^{p^{n-2}}p, \ldots, p^{n-1}T)$. □

This allows to prove the local analytic behaviour of the universal character.

Proposition 9. *For every $n \geq 1$, the universal character gives a pairing :*

$$\mathcal{W}_{[0,p^nq^{-1}]} \times \mathbb{Z}_p^\times(1 + qp^{n-1}\mathbb{G}_a^+) \longrightarrow \mathbb{G}_m^+$$

which restricts to a pairing:

$$\mathcal{W}_{[0,p^nq^{-1}]} \times (1 + qp^{n-1}\mathbb{G}_a^+) \longrightarrow 1 + q\mathbb{G}_a^+.$$

2.2 Modular Curves.

Now that we discussed the weight space, we are ready to present the general procedure used in [AIP] to produce formal models for strict neighborhoods of the ordinary locus of modular forms. Since in all the construction, the Hasse invariant and Hodge ideal play a crucial role, we first recall their general definition, and then we pass to the construction in [AIP].

2.2.1 Hasse invariant and Hodge ideal.

We define the Hodge ideal and the Hasse invariant at the level of generality that we need here. We first briefly recall the construction of Illusie's cotangent complex. For any ring R and a set S, we denote by $R[S]$ the polynomial algebra with coefficients in R and variables x_s with $s \in S$. It's easy to see that the functor $S \mapsto R[S]$ is left adjoint to the forgetful functor from R-algebras to sets, and this in particular means that for any R-algebra B, we have a canonical surjective map $f_B : R[B] \to B$. Iterating this map, we get a simplicial R-algebra, usually denoted $\mathcal{P}^\bullet_{B/R}$, which admits an augmentation over B and which looks like:

$$\mathcal{P}^\bullet_{B/R} := (\dots R[R[B]] \rightrightarrows R[B]) \to B$$

This map is a resolution of B in a suitable category of R-algebras and it is called the canonical simplicial R-algebra resolution of B.

Definition 22. *For any map $R \to B$ of commutative ring, we define its cotangent complex $L_{B/R}$, which is a complex of B-modules as*

$$L_{B/R} := \Omega_{\mathcal{P}^\bullet/R} \otimes_{\mathcal{P}^\bullet} B$$

where $\mathcal{P}^\bullet$ is a simplicial resolution of B by polynomial R-algebras

It is clear that the zeroth cohomology of the cotangent complex gives the sheaf of Kahler differentials of the given map. Moreover, this idea can be clearly globalized to schemes, simply by working locally and then glueing. Once we define the cotangent complex, we can give another definition.

Definition 23. *Let $H \to Spec(R)$ be a group scheme. We define its co-Lie complex to be $l_{H/R} := e^*(L_{H/R})$, the pullback of the cotangent complex along the counit of H. Finally, we define its conormal sheaf to be:*

$$\omega_H = H^0(l_{H/R})$$

Notice that, in the case when the group scheme is smooth, then the cotangent complex is only concentrated in degree 0, and so the definition of conormal sheaf gives back the usual one.

Now, if R is a characteristic p-ring, and G is a truncated Barsotti-Tate group over R, we have Frobenius and Verschiebung morphisms, and we can consider the co-Lie complex of the kernel of the Verschiebung, which is represented by the complex:

$$[\omega_G \xrightarrow{HW(G)} \omega_{G^{(p)}}]$$

where $HW(G)$ is called the Hasse-Witt matrix. Then, its determinant is what we call the Hasse invariant. Again notice that in the particular case of the Barsotti-Tate group attached to an elliptic curve, the definition gives back the usual one, where the Hasse invariant is constructed via the kernel of Verschiebung.

Moreover, if R is a p-adically complete ring, we call Hodge ideal of G, and we denote it by $\mathrm{Hdg}(G)$ the inverse image inside R of the ideal $\mathrm{Ha}(G)(\Lambda^d\omega_G)^{\otimes(1-p)} \subset R/pR$, where d is the dimension of G.

Proposition 10. *The Hodge ideal* $\mathrm{Hdg}(G)$ *is locally for the Zariski topology generated by two elements. Moreover, if* $p \in \mathrm{Hdg}(G)^2$*, then* $\mathrm{Hdg}(G)$ *is an invertible ideal, locally generated by a lifting of the Hasse invariant.*

Proof. Up to replacing $\mathrm{Spec}(R)$ by a Zarisky open, we can suppose that the ideal $\mathrm{Ha}(G)(\Lambda^d\omega_G)^{\otimes(1-p)} \subset R/pR$ is principal. If we denote by $\tilde{\mathrm{Ha}}$ a lifting of the Hasse invariant to R, then the Hodge ideal is given by $(p, \tilde{\mathrm{Ha}})$. If $p \in \mathrm{Hdg}(G)^2$, then $p = \tilde{Ha}u + p^2v$, for suitable $u, v \in R$, and so $p(1-pv) = \tilde{\mathrm{Ha}}u$, and since $1-pv$ is invertible, we conclude that $\mathrm{Hdg}(G)$ is generated by $\tilde{\mathrm{Ha}}$. □

2.2.2 An Admissible Blowup of the Modular Curve.

In this section we will recall the construction of formal strict neighborhoods of the ordinary locus. In fact, they are not really open subspaces of the modular form, since here everything is integral, and so all the formal schemes appearing here are blowup, living in different spaces. The idea is that, using Huber's theory of generic fiber, if we compute their adic analytic fiber, we find that the blowup becomes an inclusion of open subspace, which can be really thought as open neighborhoods of the ordinary locus. In this section we will introduce all the notation used in the following. We try to follow as much as possible the exposition and notation of [AIP]. Let $N \geq 4$ be an integer coprime with p, and let $X \to \mathrm{Spec}(\mathbb{Z}_p)$ be the compactified modular curve of level $\Gamma_1(N)$ parametrizing generalized elliptic curves with level N structure. Let us denote $\mathfrak{X}$ the formal scheme over $\mathrm{Spf}(\mathbb{Z}_p)$ given by p-adic completion of X. We write $\mathcal{E}$ for the universal generalized elliptic curve over $\mathfrak{X}$. We let $\omega_{\mathcal{E}} := e^*(\Omega_{\mathcal{E}/\mathfrak{X}})$ to be the conormal sheaf of $\mathcal{E}$. Clearly it is an invertible sheaf. We also take $\mathrm{Hdg}(E) \subseteq \mathscr{O}_{\mathfrak{X}}$ the Hodge ideal generated by p and a lifting of $\mathrm{Ha}(\mathcal{E})\omega^{1-p}$, where $\mathrm{Ha} \in \mathrm{H}^0\left(\mathfrak{X}, \omega_{\mathcal{E}}^{p-1} \otimes_{\mathbb{Z}_p} \mathbb{F}p\right)$.

Now, let us consider B_I, the ring introduced in the first section, for $I \subseteq [1, \infty]$. It is p-adically complete, so we can base change $\mathfrak{X}$ to $\mathfrak{X}_I$, via the structure map $\mathrm{Spf}(B_I) \to \mathrm{Spf}(\mathbb{Z}_p)$. This produces a modular curve which is fibered over a suitable region of the weight space. In particular, it means that a suitable class of functions defined over $\mathfrak{X}_I$ should represent p-adic families of modular forms. Notice that, when we base change to B_I, the topology shifts to the T-adic one. Now, we want to construct p-adic families of overconvergent modular forms, and so we need to introduce a formal scheme which parametrizes formal neighborhoods of the ordinary locus. First, we recall that, said $\tilde{\mathrm{Ha}}$ a lifting of the Hasse invariant, the ordinary locus is defined as the formal subscheme where $\tilde{\mathrm{Ha}}$ is invertible.

Definition 24. *For every integer $r \in \mathbb{N}$, let $\mathfrak{X}_{r,I}$ be the functor which associates to any T-adically complete B_I-algebra without T-torsion, the set of equivalence classes of couples $\left(f : Spf(R) \to \mathfrak{X}, \eta \in H^0\left(Spf(R), f^*\omega^{(1-p)p^{r+1}}\right)\right)$ such that*

$$Ha^{p^{r+1}}\eta = T \qquad mod \quad p^2 \tag{2.3}$$

where two couples (f, η) and (f', η') are equivalent if $f = f'$ and $\eta = \eta'\left(1 + \frac{p^2}{T}u\right)$ for some $u \in R$.

Proposition 11. *The functor $\mathfrak{X}_{r,I}$ is representable by a formal scheme without T-torsion which is an open inside an admissible blowup of $\mathfrak{X}_I$.*

Proof. We can work locally over affine open formal subschemes of $\mathfrak{X}_I$, where $\omega_{\mathcal{E}}$ is trivial. We choose $\mathrm{Spf}(B)$ to be such an open, we identify Ha with a scalar and we denote by $\tilde{\mathrm{Ha}} \in B$ a lifting of the Hasse invariant. Then, the inverse image of $\mathrm{Spf}(B)$ inside $\mathfrak{X}_{r,I}$ is $\mathrm{Spf}\ B\langle X\rangle/(\tilde{\mathrm{Ha}}^{p^{r+1}}X - T)$, which is an open inside the blowup along $(\tilde{\mathrm{Ha}}, T)$. $\square$

Notice that the advantage of working with $\mathfrak{X}_{r,I}$ is that the Hodge ideal is locally free over it, since $T \in \mathrm{Hdg}$. Moreover, the theory of canonical subgroup, as it is developed in appendix A of [AIP] allows to state the following

Proposition 12. *Let $k \in \mathbb{Z}_{\geq 0}$, and assume that $p \in T^{p^k}B_I$. Then:*

i) Over $\mathfrak{X}_{r,I}$, for every $n \leq k + r$, there exists a canonical subgroup $H_n \subseteq \mathcal{E}[p^n]$, and $H_n \subseteq H_{n'}$ for $n \leq n'$.

ii) H_n is locally free of rank p^n and it lifts the kernel of F^n, where F denotes Frobenius, modulo $p\mathrm{Hdg}(\mathcal{E})^{-\frac{p^n-1}{p-1}}$.

iii) Let $\mathcal{E}' := \mathcal{E}/H_n$. Then $\mathrm{Hdg}(\mathcal{E}') = \mathrm{Hdg}(\mathcal{E})^{p^n}$, $\mathcal{E}'$ is equipped with a canonical subgroup H'_{k+r-n} of echelon $r+k-n$ and we have an exact sequence

$$0 \longrightarrow H_n \longrightarrow H_{r+k} \longrightarrow H'_{k+r-n} \longrightarrow 0.$$

iv) Let $\mathcal{E}^D$ be the dual of $\mathcal{E}$. Then $\mathrm{Hdg}(\mathcal{E}) = \mathrm{Hdg}(\mathcal{E}')$, and for every $n \leq k+r$, the Weil pairing $\mathcal{E}[p^n] \times \mathcal{E}[p^n]^D \to \mu_{p^n}$ induces an exact sequence:

$$0 \longrightarrow H_n \longrightarrow \mathcal{E}[p^n] \longrightarrow H_n^D \longrightarrow 0$$

v) $\mathcal{E}[p^n]/H_n$ is étale over $\mathrm{Spec}(R[1/T])$, locally isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$.

The existence of canonical subgroup allows to construct a lifting of suitable powers of Frobenius. In particular, the following is Proposition 3.3 in [AIP]:

Proposition 13. *The isogeny "divide by the canonical subgroup", $\mathcal{E} \mapsto \mathcal{E}/H_1$ induces a finite flat morphism of degree p, $\phi : \mathfrak{X}_{r,I} \to \mathfrak{X}_{r-1,I}$, for every $r \geq 1$. This morphism lift the relative to B_I Frobenius modulo $\frac{p}{\mathrm{Hdg}}$, and it verifies $\phi^*(\mathcal{E}|_{\mathfrak{X}_{r-1,I}}) = (\mathcal{E}|_{\mathfrak{X}_{r,I}})/H_1$.*

Before passing to the next section, we also recall the following result:

Proposition 14. *The formal scheme $\mathfrak{X}_{r,I}$ is normal*

This result, which is Proposition 3.4 in [AIP], is crucial in what follows, since it allows to define the partial Igusa tower over $\mathfrak{X}_{r,I}$ by normalizing a moduli space which can be defined only over the adic analytic fiber.

2.2.3 Partial Igusa Tower.

The existence of canonical subgroup of level n, allows to introduce Galois covering of the modular curve, which parametrize trivializations of the dual canonical subgroup. The general strategy is to introduce these coverings generically, and then to take the normalization. This can be done since we saw that the modular curve $\mathfrak{X}_{r,I}$ is normal. First we associate to $\mathfrak{X}_{r,I}$, the analytic adic space $\mathfrak{X}_{r,I}^{\mathrm{ad}}$ given by adification as in [Hu1]. Then we associate to it its adic generic fiber, defined as:

$$\mathcal{X}_{r,I} := \mathfrak{X}_{r,I}^{\mathrm{ad}} \times_{\mathrm{Spa}(B_I,B_I)} \mathrm{Spa}(B_I[1/T].B_I[1/T]^+). \tag{2.4}$$

Then, if $T^{p^k}|p$, for every $r \leq r+k$, there exists a canonical subgroup $H_n \subseteq \mathcal{E}[p^n]$ of echelon n. Moreover, by Proposition 12, we know that the dual of this canonical subgroup is locally isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. Then we can consider the finite étale covering $\mathcal{IG}_{n,r,I} \to \mathcal{X}_{r,I}$ which parametrizes choices of a generator $\mathbb{Z}/p^n\mathbb{Z} \to H_n^D$. Due to a lack of reference, we give here a proof of the existence of such an object.

Proposition 15. *Let S be a connected locally Noetherian scheme (resp. connected locally Noetherian formal scheme) and let G be a finite étale abelian group scheme over S of order n. Then the functor sending an S-scheme (resp. an S-formal scheme) to the set of trivializations of G_T over T is representable by a finite étale scheme (resp. formal scheme).*

Proof. We prove the result for schemes. The situation for formal schemes is exactly the same. As G is étale, we can take a finite étale Galois cover of S, say S', with Galois group Γ such that $G_{S'}$ is a constant group scheme, write $G_{S'} = \underline{H}$, where H denotes a finite abstract group. By the classification of finite abelian groups, we know that H can be written in an essentially unique way as a finite product of prime torsion subgroups. Hence it is enough to prove the Proposition when $H = \mathbb{Z}/l^n\mathbb{Z}$ for l a prime number, and $n \in \mathbb{N}$. Now notice that Γ acts on H. Now let $J := \{\phi_i\}_i \in I$ be the set of all possible trivializations of H. Notice that

$$|J| = |\mathrm{Aut}(\mathbb{Z}/l\mathbb{Z})| = l^{n-l}(l-1)$$

Now, Γ acts on J simply by sending ϕ_i to $\phi_i \circ \gamma$ for every $\gamma \in \Gamma$, we can write the action more succintly in this way:

$$\gamma \cdot \phi_i = \phi_{\gamma(i)}.$$

Now consider the scheme $\widetilde{S} := \bigsqcup_{i\in I} S'$. It clearly represents the functor which sends an S'-scheme T' to the set of trivializations of $G_{T'}$. Moreover, we have an action of *Gamma* over $\widetilde{S}$ which permutes the component, i.e. it sends a point $x \in S_i'$ (we now write S_i' simply to keep track of the component, but

$S'_i = S'$ for all i) to a point $\gamma(x) \in S_{\gamma(i)}$. There is also a universal trivialization over $\widetilde{S}$ equipped again with an action of the group Γ. As we are taking a finite number of components in the disjoint union, we clearly have that the structure morphism $\widetilde{S} \to S$ is finite étale and Γ-invariant. Then the quotient $Y := \widetilde{S}/\Gamma$ defines a finite étale scheme over S which fits into the following Cartesian diagram:

$$\begin{tikzcd} \widetilde{S} \arrow[r] \arrow[d] & S' \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

Analogously, the quotient of the universal trivialization defines a trivialization of G_Y and its base change to $\widetilde{S}$ defines the disjoint union of trivializations. Now, the scheme Y represents the functor which sends an S-scheme T to the set of trivializations of G_T. In fact, given an S-scheme T, if we denote $T' := T \times_S S'$, we have $\mathrm{Aut}(T'/T) \cong \Gamma$, and $T'/\Gamma = T$. Hence, a T-point of Y is equivalent to an S-morphism $T' \to X$, which is Γ-invariant, hence it is also equivalent to a Γ-invariant S-morphism $T' \to X \times_S S'$, which is the same as choosing a trivialization of $G_{T'}$ over T' which is equipped with an action of Γ which is compatible with the action over T'. But then, simply taking the quotient under the action of Γ, we find the trivialization over T. $\square$

Hence the proposition says that the analytic adic space parametrizing trivializations of the dual canonical subgroup exists over the generic fiber, and it is a finite étale Galois cover with Galois group isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^\times$. The point is that we need a formal scheme defined over $\mathfrak{X}_{r,I}$. As $\mathfrak{X}_{r,I}$ is normal, a good idea would be to take the normalization of $\mathfrak{X}_{r,I}$ inside $\mathcal{IG}_{n,r,I}$. The problem is that one has to be sure that such a normalization exists, i.e. one has to ensure that the affinoid parts given by normalization can be glued. This is proved in [AIP], and it's Lemma 3.2. Hence we get a formal scheme $\mathfrak{IG}_{n,r,I}$ for suitable n which is the normalization of the analytic adic space parametrizing trivializations of the dual canonical subgroup. Notice that, the bigger r is, the bigger n can be. Moreover, the group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ again acts over $\mathfrak{IG}_{n,r,I}$ via the composition with the morphism $\mathbb{Z}/p^n\mathbb{Z} \to H_n^D$ which generically gives the trivialization.

2.3 Analytic Modular Forms.

In this section, we recall the construction of overconvergent modular forms in characteristic zero. This construction is possible over regions of the weight space where the topology is p-adic, i.e. when $I = [0, 1]$, or $I = [p^k, p^{k'}]$, with $k' < \infty$. In the second case, as we remarked above, the T-adic topology is equivalent to the p-adic one. We only treat this situation in this exposition, even if all possible situations are considered in [AIP]. The construction strongly relies on the map dlog, whose definition and main properties are recalled in the next subsection.

2.3.1 The map dlog.

We recall the definition of the dlog map in a more general situation.

Definition 25. *Let A_0 be a $\mathbb{Z}_p$-algebra which is an integral domain, let $\alpha \in A_0 \setminus \{0\}$ be an element such that A_0 is α-adically complete and such that the structure morphism $\mathbb{Z}_p \to A_0$ is continuous. Let R be an α-adically complete, without α-torsion, A_0-algebra. Let G be a truncated Barsotti Tate group of level n, height h and dimension 1. Let $\lambda \in A_0$ be such that $\lambda^2 u = p$ with u topologically nilpotent. Suppose that $\lambda \mod p \in Ha(G)(\omega_G)^{\otimes(1-p)}$. Let H_n be the canonical subgroup of G of level 1. We then define the morphism of* fppf *sheaves:*

$$dlog_n : H_n^D \longrightarrow \omega_{H_n} \tag{2.5}$$

on points by the following procedure. Let S be a scheme and let $x \in H_n^D(S)$ be a point of H_n. Then, x, being a point of H_n^D, defines a morphism of schemes $x : H_{n,S} \to \mu_{p^n,S}$. We finally put:

$$dlog_n(x) = x^* \left(\frac{dX}{X} \right)$$

where X is the local coordinate of $\mu_{p^n,S}$.

In our situation, A_0 will be B_I, with the assumption of considering a bounded interval I, α will be T. R is the ring of functions of an affine open inside $\mathfrak{X}_{r,I}$, and $G = \mathcal{E}[p^n]$, where now $\mathcal{E}$ denotes the base change of the universal generalized elliptic curve to $\mathfrak{X}_{r,I}$. Finally, $\lambda = \mathrm{Hdg}^{-\frac{p^n-1}{p-1}}$, and H_n is the canonical subgroup of level n. Clearly, varying n, we get a compatible system of maps dlog and, in principle, if the canonical subgroup exists for every n, we can consider the projective limit, which gives a dlog map for every n. Moreover, we can precompose the dlog map with the projection $\mathcal{E}[p^n] \to H_n^D$, extending the dlog morphism to all $\mathcal{E}[p^n]$. This trivial remark will be useful in the following.

2.3.2 The torsor.

In this section, we recall the construction of the torsor $\mathcal{F}_{n,r,I}$, whose functions transforming under a suitable law give the sheaf of overconvergent modular forms in characteristic 0. Fix an interval $I = [p^k, p^{k'}]$, so that the canonical subgroup H_n exists for $n \leq r + k$. Let $\mathfrak{IG}_{n,r,I}$ be the partial formal Igusa tower of level n, and let $g_n : \mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I}$ be the structure morphism. Notice that the inclusion $H_n \subseteq \mathcal{E}$ induces a map of invertible sheaves $\omega_{\mathcal{E}} \to \omega_{H_n}$.

Lemma 4. *The kernel of the morphism $\omega_{\mathcal{E}} \to \omega_{H_n}$ is $p^n \mathrm{Hdg}^{-\frac{p^n-1}{p-1}}$.*

Proof. We have an exact sequence

$$0 \longrightarrow H_n \longrightarrow \mathcal{E}[p^n] \longrightarrow \mathcal{E}[p^n]/H_n \longrightarrow 0$$

which induces an exact sequence

$$0 \longrightarrow \omega_{\mathcal{E}[p^n]/H_n} \longrightarrow \omega_{\mathcal{E}[p^n]} \longrightarrow \omega_{H_n} \longrightarrow 0$$

Now, since by definition, $\omega_{\mathcal{E}[p^n]/H_n}$ reduces to the kernel of Verschiebung to the power n modulo $p\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}$, we get that it is annihilated by $\mathrm{Hdg}^{\frac{p^n-1}{p-1}}$, and so, by Nakayama, we conclude that the kernel is $p^n \mathrm{Hdg}^{-\frac{p^n-1}{p-1}}$. □

Then, using the previous exact sequence, we get that the map $\omega_{\mathcal{E}} \to \omega_{H_n}$ factors through an isomorphism $\omega_{\mathcal{E}}/p^n \mathrm{Hdg}^{-\frac{p^n-1}{p-1}} \cong \omega_{H_n}$. We then get a diagram of *fppf*-abelian sheaves:

$$\begin{tikzcd} & \omega_{\mathcal{E}} \arrow[d] & \\ H_n^D \arrow[r, "\mathrm{dlog}"] & \omega_{H_n} \arrow[r, "\cong"] & \omega_{\mathcal{E}}/p^n \mathrm{Hdg}^{-\frac{p^n-1}{p-1}} \end{tikzcd} \tag{2.6}$$

Moreover, if we consider all this situation over $\mathfrak{IG}_{n,r,I}$, we have, generically, a trivialization of the dual of the canonical subgroup, so in particular we can consider $P \in H_n^D$ which is the image of 1 via the universal morphism $\mathbb{Z}/p^n\mathbb{Z} \to H_n^D$. Then, the image of P generates a submodule $\mathrm{Hdg}^{\frac{1}{p-1}}\omega_{H_n}$ of ω_{H_n}.

Definition 26. *We denote by $f_n : \mathcal{F}_{n,r,I} \to \mathfrak{IG}_{n,r,I}$ the $1 + p^n Hdg^{-\frac{p^n}{p-1}}\mathbb{G}_a$-torsor defined on points by:*

$$\mathcal{F}_{n,r,I} := \left\{(\omega, P) \in \omega_{\mathcal{E}} \times \mathfrak{IG}_{n,r,I} \mid \omega = dlog(P) \text{ inside } \omega_{\mathcal{E}}/p^n Hdg^{-\frac{p^n-1}{p-1}}\right\} \tag{2.7}$$

Clearly, over this torsor, we have a compatible action of $\mathbb{Z}_p^\times$ which lifts the action of $\mathbb{Z}/p^n\mathbb{Z}$ over the partial Igusa tower. This action is simply defined by multiplying both the differential and the point P. Putting everything together, we get a torsor:

$$\mathcal{F}_{n,r,I} \xrightarrow{f_n} \mathfrak{IG}_{n,r,I} \xrightarrow{g_n} \mathfrak{X}_{r,I} \tag{2.8}$$

over which the group $\mathbb{Z}_p^\times\left(1 + p^n\mathrm{Hdg}^{-\frac{p^n}{p-1}\mathbb{G}_a}\right)$ acts compatibly. Explicitly, the map f_n forgets the differential and sends the couple (ω, P) to P, which generates the canonical subgroup and so describes completely the morphism $\mathbb{Z}/p^n\mathbb{Z} \to H_n^D$, while g_n projects to the elliptic curve fibered over $\mathfrak{X}_{r,I}$. This torsor has been generically constructed also in [Pil], but the advantage of the construction of [AIP] is that here the construction is integral, and so the authors provided a model for the sheaf giving Pilloni's families of overconvergent modular forms.

2.3.3 The Sheaf of Families of Overconvergent Modular Forms.

Now we have all the information to define the sheaf of overconvergent modular forms. We want to point out that this definition strictly depends on the fact that the universal character is locally analytic. This says that the construction in characteristic p must follow a different strategy, since over there the character is no more locally analytic. By Proposition [9], we see that there is a pairing:

$$\mathfrak{W}_I^0 \times \mathbb{Z}_p^\times(1 + p^{k'+1}\mathbb{G}_a) \longrightarrow \mathbb{G}_m$$

If we assume that $n \geq k' + 2$, we get a character $\kappa : \mathbb{Z}_p^\times(1 + p^n \mathrm{Hdg}^{-\frac{p^n}{p-1}}\mathbb{G}_a) \to \mathbb{G}_m$ of group schemes defined over $\mathfrak{IG}_{n,r,I}$ which comes from the universal pairing. Then

Definition 27. *We define* $\mathfrak{w}_{n,r,I} := (g_n \circ f_n)_* \mathscr{O}_{\mathcal{F}_{n,r,I}}[\kappa^{-1}]$*, which is the sheaf of functions over the Pilloni's torsor which transforms via the character* κ^{-1} *under the action of* $\mathbb{Z}_p^\times(1 + p^n Hdg^{-\frac{p^n-1}{p-1}}\mathbb{G}_a)$

Notice that the analiticity of the universal character is necessary to extend the action of $\mathbb{Z}_p^\times$ to an action of $\mathbb{Z}_p^\times(1 + p^n \mathrm{Hdg}^{-\frac{p^n-1}{p-1}})$, which clearly fails over the boundary.

Now we can state the main theorem of Chapter 5 in [AIP] which completely characterizes the properties of $\mathfrak{w}_{n,r,I}$.

Theorem 6. *Assume that* $r \geq 1$ *and* $r + k \geq k' + 2$*. We then have an invertible sheaf* $\mathfrak{w}_I$ *over* $\mathfrak{X}_{r,I}$ *which satisfies the following properties:*

i) Let $\mathcal{X}_{r,I}$ *be the adic generic fiber of* $\mathfrak{X}_{r,I}$*, over* $\mathbb{Q}_p$*. Then the sheaf generic fiber of* $\mathfrak{w}_I$ *is the sheaf of families of overconvergent modular forms over* $\mathcal{X}_{r,I}$ *constructed in [Pil] and [AIS].*

ii) We have a Frobenius operator:

$$\iota^* \mathfrak{w}_I \cong \phi^* \mathfrak{w}_I \tag{2.9}$$

where $\iota : \mathfrak{X}_{r+1,I} \to \mathfrak{X}_{r,I}$ *is the inclusion morphism and* $\phi : \mathfrak{X}_{r+1,I} \to \mathfrak{X}_{r,I}$ *is the Frobenius.*

iii) The construction is functorial under the operation of changing the interval I*, i.e. if* $I' \subsetneq I$ *and if* $\iota_{I',I} : \mathfrak{X}_{r',I} \to \mathfrak{X}_{r,I}$ *is the natural morphism, then* $\iota_{I',I}^* \mathfrak{w}_I \cong \mathfrak{w}_{I'}$*.*

Notice that in the Theorem we didn't specify the n. This comes from the fact that it is proved in [AIP] that the definition does not depend on n, which means that, if we change n, we get a sheaf which is isomorphic to the first one. Moreover, notice that we did not specify the radius of convergence, r. In fact, it is also proved that the operation of changing r does not affect the sheaf. This result is the formal counterpart of the following, which is purely adic analytic. We want to point out that formally we can only get the modular sheaf for characters with trivial finite part, while in the adic analytic setting, also the other connected component of the weight space can be taken into account. In particular, the following holds:

Theorem 7. *For every $r \geq 3$ we have an invertible sheaf $\omega^{\kappa}_{[0,\infty)}$ over the adic analytic fiber of $\mathfrak{X}_{r,[0,\infty)}$, call it $\mathcal{M}_{r,[0,\infty)}$ and a subsheaf $\omega^{\kappa,+}_{[0,\infty)}$ of $\mathscr{O}^{+}_{\mathcal{M}_{r,[0,\infty)}}$-modules satisfying the following properties:*

i) The restriction of the subsheaf $\omega^{\kappa,+}_{[0,\infty)}$ to the component $\mathcal{W}^{0}_{[0,\infty)}$ given by the trivial character is an invertible sheaf of $\mathscr{O}^{+}_{\mathcal{X}_{r,[0,\infty)}}$-module.

ii) For every character $\chi \cdot k : \mathbb{Z}_p^{\times} \to \mathbb{C}_p^{\times}$ locally algebraic, with χ a finite character and $k \in \mathbb{Z}$ identified to a point κ of $\mathcal{W}^{rig}$, $\omega_{[0,\infty)}|_{\{\chi\cdot k\}} = \omega^{k}(\chi)$ is the usual sheaf of modular forms of weight k and nebentypus χ.

iii) The Frobenius morphism $\phi : \mathcal{M}_{r+1} \to \mathcal{M}_r$ and the inclusion $\iota : \mathcal{M}_{r+1} \to \mathcal{M}_r$ induce integral isomorphisms $\phi^{*}\omega^{\kappa,+}_{[0,\infty)} \cong \iota^{*}\omega^{\kappa,+}_{[0,\infty)}$.

2.4 Overconvergent Modular Forms in Characteristic p.

In chapter 4 of [AIP], the authors construct directly families of overconvergent modular forms in characteristic p. In characteristic p the situation is very different than in characteristic 0. In fact the characteristic 0 construction is related to the local analiticity of the universal character, a property which is no more satisfied in characteristic p. However, in characteristic p the Frobenius isogeny is well-defined, hence the canonical subgroup of each level is well-defined, and so the Igusa tower is easily constructed, at least over the ordinary locus. The main difficulty in characteristic p in fact is to show that the natural sheaf given by Chapter IV of [Ka73] overconverges over the weight space. Let us be more specific about this direct construction in characteristic p.

First of all, the idea is that the characteristic p construction appears as a boundary phenomenon, where the adic weight space is described as $\mathrm{Spa}(\mathbb{F}_p[(\mathbb{Z}/p\mathbb{Z})^{\times}]((T)), \mathbb{F}_p[(\mathbb{Z}/p\mathbb{Z})^{\times}][[T]])$. Again, it is possible to describe the construction over the connected component of the trivial finite character, and then to extend to arbitrary finite characters, hence let us fix here the formal model for the connected component of the trivial character

$$\mathfrak{W}^{0}_{\{\infty\}} = \mathrm{Spf}(\mathbb{F}_p[[T]]).$$

If $\bar{X}$ is the compactified modular curve over $\mathrm{Spec}(\mathbb{F}_p)$ of level $\Gamma_1(N)$ and $\bar{X}_{\mathrm{ord}}$ is the ordinary locus which is locally defined by the invertibility of the Hasse invariant, we can construct the formal schemes

$$\mathfrak{X} = \bar{X} \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathfrak{W}^0_{\{\infty\}} \text{ and } \mathfrak{X}_{\mathrm{ord}} = \bar{X}_{\mathrm{ord}} \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathfrak{W}^0_{\{\infty\}}$$

and, thanks to Proposition 11 also the formal scheme $\mathfrak{X}_{r,\{\infty\}} \to \mathfrak{X}_{\{\infty\}}$ whose generic fiber is defined locally by the equation

$$|\mathrm{Ha}|^{p^{r+1}} \geq T$$

inside the generic fiber of $\mathfrak{X}_{\{\infty\}}$. As we remarked above, since we are working in characteristic p, we can construct Frobenius isogenies, which define morphisms

$$\begin{aligned} &\phi : \mathfrak{X}_{\{\infty\}} \to \mathfrak{X}_{\{\infty\}} \\ &\phi : \mathfrak{X}_{\mathrm{ord}} \to \mathfrak{X}_{\mathrm{ord}} \\ &\phi : \mathfrak{X}_{r+1,\{\infty\}} \to \mathfrak{X}_{r,\{\infty\}} \end{aligned}$$

where we used the same notation since they are all defined as the quotient of the parametrized elliptic curve over the kernel of Frobenius.

Now, over the ordinary locus, the dual of the canonical subgroup is trivialized, and so the Igusa curve $\bar{IG}_{n\mathrm{ord}}$ is defined to be the moduli space of trivializations of H_n^{D}. Then the Igusa tower over the ordinary locus is defined to be:

$$\bar{IG}_{\infty,\mathrm{ord}} := \varprojlim_n \bar{IG}_{n,\mathrm{ord}}$$

Now, via the monodromy representation over a geometric point of $\bar{X}_{\mathrm{ord}}$ induced by the Igusa tower, one can construct, for every character $\bar{\kappa}_\chi : \mathbb{Z}_p^\times \to \mathbb{F}_p[[T]]^\times$, an invertible sheaf via the following procedure. One defines

$$\mathfrak{IG}_{n,\mathrm{ord}} := \bar{IG}_{n,\mathrm{ord}} \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathfrak{W}^0_{\{\infty\}}$$

and then considers the sheaf

$$\mathfrak{w}^{\bar{\kappa}_\chi} := \mathscr{O}_{\mathfrak{IG}_{\infty,\mathrm{ord}}}[\bar{\kappa}_\chi^{-1}]$$

This is an invertible sheaf such that $\phi^*\mathfrak{w}^{\bar{\kappa}_\chi} \cong \mathfrak{w}^{\bar{\kappa}_\chi}$.

Now, the idea developed in [AIP] is to show that this construction overconverges, i.e. that this sheaf may be defined over $\mathfrak{X}_{r,I}$ instead of $\mathfrak{X}_{\mathrm{ord}}$. The point is that now the Igusa curve is defined as in section 2.2.3 as $\mathfrak{IG}_{n,r,\{\infty\}}$ which is the formal model of the adic space parametrizing generic trivializations of the dual canonical subgroup. Again, since the canonical subgroup of any level is trivially defined in characteristic p, we can take the projective limit and define the T-adic formal scheme

$$\mathfrak{IG}_{\infty,r,\{\infty\}} := \varprojlim_n \mathfrak{IG}_{n,r,\{\infty\}}$$

Now, given the universal character $\bar{\kappa}$ for the connected component of the trivial finite character, it is again possible to define a sheaf

$$\mathfrak{w}_{\{\infty\}} := \mathscr{O}_{\mathfrak{IG}_{\infty,r,\{\infty\}}}[\bar{\kappa}^{-1}]$$

as the sheaf of functions defined over the Igusa tower which transform under the action of $\mathbb{Z}_p^\times$ via the inverse of the character κ. Then the following Theorem is proved:

Theorem 8. *The sheaf $\mathfrak{w}^{\{\infty\}}$ is an invertible sheaf over $\mathfrak{X}_{r,\{\infty\}}$ for every $r \geq 3$. Moreover, denoted by $i : \mathfrak{X}_{r+1,\{\infty\}} \to \mathfrak{X}_{r,\{\infty\}}$ there is an isomorphism $i^*\mathfrak{w}_{\{\infty\}} \cong \phi^*\mathfrak{w}_{\{\infty\}}$.*

which means precisely that the sheaf $\mathfrak{w}_{\{\infty\}}$ extends the Katz's sheaf beyond the ordinary locus.

2.5 The Gluing Construction.

Now we are ready to describe the perfectified construction, which allows to give a global definition of overconvergent modular forms. We provide as many details as possible in this section, since the object which we introduce here are the ones over which we will work in the main sections of this thesis. Fix an interval $I = [p^k, p^{k'}]$ contained in $[1, \infty]$, where we also admit k' to be infinity.

2.5.1 The Infinite Igusa Tower.

By the previous construction, we know that for every r, there exists at least the first canonical subgroup, which says that we can define a finite flat Frobenius morphism:

$$\phi : \mathfrak{X}_{r+1,I} \to \mathfrak{X}_{r,I}$$

which sends the universal generalized elliptic curve to its quotient over the canonical subgroup. In this way, as the morphisms are finite, and so in particular, affine, we can compute the projective limit under Frobenius, and call it the anticanonical tower:

$$\mathfrak{X}_{\infty,I} := \varprojlim_{\phi,r} \mathfrak{X}_{r,I} \tag{2.10}$$

This is a T-adic formal scheme whose affine open are formal schemes given by completions of inductive limits along Frobenius. Moreover, recall that, for any $n \leq r+k$, the partial Igusa tower $\mathfrak{IG}_{,n,r}I$ is defined by taking the normalization of the Galois covering parametrizing trivializations of the dual of the canonical subgroup. Moreover, we know that there exists a Frobenius morphism defined over the partial Igusa tower, which commutes with the Frobenius at the level of modular curves. As an example, in the case $I=[1,\infty]$, all these formal schemes organize in this commutative diagram:

$$\begin{tikzcd}
 & & & \mathfrak{IG}_{4,4,I} \arrow[d] & \dots \arrow[l, "\phi"'] \arrow[d] \\
 & & \mathfrak{IG}_{3,3,I} \arrow[d] & \mathfrak{IG}_{3,4,I} \arrow[l, "\phi"'] \arrow[d] & \dots \arrow[l, "\phi"'] \arrow[d] \\
 & \mathfrak{IG}_{2,2,I} \arrow[d] & \mathfrak{IG}_{2,3,I} \arrow[l, "\phi"'] \arrow[d] & \mathfrak{IG}_{2,4,I} \arrow[l, "\phi"'] \arrow[d] & \dots \arrow[l, "\phi"'] \arrow[d] \\
\mathfrak{IG}_{1,1,I} \arrow[d] & \mathfrak{IG}_{1,2,I} \arrow[l, "\phi"'] \arrow[d] & \mathfrak{IG}_{1,3,I} \arrow[l, "\phi"'] \arrow[d] & \mathfrak{IG}_{1,4,I} \arrow[l, "\phi"'] \arrow[d] & \dots \arrow[l, "\phi"'] \arrow[d] \\
\mathfrak{X}_{1,I} & \mathfrak{X}_{2,I} \arrow[l, "\phi"'] & \mathfrak{X}_{3,I} \arrow[l, "\phi"'] & \mathfrak{X}_{4,I} \arrow[l, "\phi"'] & \dots \arrow[l, "\phi"']
\end{tikzcd} \tag{2.11}$$

The bigger is r, the higher is the tower, so if we fix n, we get a projective system where the transition morphisms are given by Frobenius. By taking the limit, we get for every n a T-adic formal scheme:

$$\mathfrak{IG}_{n,\infty,I} := \varprojlim_{\phi} \mathfrak{IG}_{n,r,I} \tag{2.12}$$

All these formal schemes exist over $\mathfrak{X}_{\infty,I}$, and they exist for every n. Again, these perfectified Igusa towers organize in a projective system, and so we can take the limit, getting finally a formal scheme:

$$\mathfrak{IG}_{\infty,\infty,I} := \varprojlim_{n} \mathfrak{IG}_{n,\infty,I} \tag{2.13}$$

fibered over $\mathfrak{X}_{\infty,I}$. Since the group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ acts on each $\mathfrak{IG}_{n,\infty,I}$, over the infinite Igusa tower acts a copy of $\mathbb{Z}_p^\times$.

2.5.2 Ramification of the Igusa tower.

In this section, we show how ramification of the Igusa tower changes when we pass from finite to infinite level. First, the morphism $h : \mathfrak{IG}_{n,r,I} \to \mathfrak{IG}_{n-1,r,I}$ which forgets the trivialization of the n-th canonical subgroup, is finite étale over the generic fiber by definition of Igusa cover. Moreover, the group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ is the Galois group of $\mathcal{IG}_{n,r,I}$, and so it acts over the normalization $\mathfrak{IG}_{n,r,I}$. In particular, the following hold:

Lemma 5. *For every $n \geq 2$,*

$$\Big((h_n)_*\mathscr{O}_{\mathfrak{IG}_{n,r,I}}\Big)^{1+p^{n-1}\mathbb{Z}/p^n\mathbb{Z}} = \mathscr{O}_{\mathfrak{IG}_{n-1,r,I}}$$

Moreover,

$$\Big((h_1)_*\mathscr{O}_{\mathfrak{IG}_{1,r,I}}\Big) = \mathscr{O}_{\mathfrak{X}_{r,I}}$$

Proof. This clearly holds generically, since the covering is Galois. So the conclusions follow by normalization. □

This allows us to define a Trace morphism $\mathrm{Tr} : (h_n)_*\mathscr{O}_{\mathfrak{IG}_{n,r,I}} \to \mathscr{O}_{\mathfrak{IG}_{n-1,r,I}}$ which is defined in this way. Let $\mathrm{Spf}(R) \subseteq \mathfrak{IG}_{n-1,r,I}$ be an open affinoid, and let $f \in \mathrm{Spf}(R)$. If $\mathrm{Spf}(R')$ is the preimage of $\mathrm{Spf}(R)$ inside $\mathfrak{IG}_{n,r,I}$, then we define:

$$\mathrm{Tr}(f) = \sum_{\sigma \in 1+p^{n-1}\mathbb{Z}/p^n\mathbb{Z}} \sigma(f)$$

As usual, the trace takes care of the ramification of the covering. In particular, the main result is the following.

Proposition 16. *For every $n \geq 2$, we have*

$$\mathrm{Hdg}^{p^{n-1}}\mathscr{O}_{\mathfrak{IG}_{n-1,r,I}} \subseteq \mathrm{Tr}((h_n)_*\mathscr{O}_{\mathfrak{IG}_{n,r,I}})$$

Moreover, for $n = 1$, $\mathrm{Tr}_{\mathfrak{IG}_{1,r,I}}(h_*\mathscr{O}_{\mathfrak{IG}_{1,r,I}}) = \mathscr{O}_{\mathfrak{X}_{r,I}}$

Proof. In the case $n = 1$ the proposition is trivial, in fact the degree of the first Igusa curve over $\mathfrak{X}_{r,I}$ is $p-1$, and so the morphism is surjective. Simply take one element in $\mathscr{O}_{\mathfrak{X}_{r,I}}$ and take its trace. Then you get the same element multiplied by $p-1$, which is invertible in $\mathbb{Z}_p$. Now, let $\mathrm{Spf}(R)$ be an affine of $\mathfrak{X}_{r,I}$, which trivializes Hdg. Assume it is generated by $\tilde{\mathrm{Ha}}$. Let $\mathrm{Spf}(B_n)$ and $\mathrm{Spf}(B_{n-1})$ be the fibers of H_n^D and H_{n-1}^D over $\mathrm{Spf}(R)$, and let $\mathrm{Spf}(C)$ be the fiber of the *fppf* quotient $(H_n/H_{n-1})^D$ over $\mathrm{Spf}(R)$. Let us consider $D_{B_n/B_{n-1}}$ and $D_{C/R}$ the different of the morphisms $B_{n-1} \to B_n$ and $R \to C$. The equality

$$D_{B_n/B_{n-1}} \otimes_{B_{n-1}} B_n = D_{C/R} \otimes_R B_n$$

holds as an equality of $B_n \otimes_{B_{n-1}} B_n = C \otimes_R B_n$-modules. Moreover, by the relation between different and discriminant, we conclude that $D_{C/R} = \mathrm{Hdg}^{p^{n-1}} C$. Then, since the morphism $H_n^D \to H_{n-1}^D$ is an homogeneous space under the action of $(H_n/H_{n-1})^D$ for the $fppf$ topology, we can apply faithfully flat descent and conclude that $D_{B_n/B_{n-1}} = \mathrm{Hdg}^{p^{n-1}} B_n$. Moreover, we have an isomorphism $D^{-1}_{B_n/B_{n-1}} \cong \mathrm{Hom}_{B_{n-1}}(B_n, B_{n-1})$ given by the morphism sending x to $\mathrm{Tr}_{B_n/B_{n-1}}(\cdot x)$. Then it remains to prove that the ideal $\mathrm{Tr}_{B_n/B_{n-1}}(D^{-1}_{B_n/B_{n-1}}) \subseteq B_{n-1}$ is the whole B_{n-1}. Since this property is local for the Zariski topology over $\mathrm{Spec}(B_{n-1})$, we may assume that B_n is free over B_{n-1}. Then there exists a surjective morphism $B_n \to B_{n-1}$ of B_{n-1}-modules, which can be written as $\mathrm{Tr}_{B_n/B_{n-1}}(\cdot x)$ for $x \in D^{-1}_{B_n/B_{n-1}}$, and so $\mathrm{Tr}_{B-n/B_{n-1}}(D^{-1}_{B_n/B_{n-1}}) = B_{n-1}$. Consequently, one can find an element $d_n \in B_n$ such that $\mathrm{Tr}_{B_n/B_{n-1}}(d_n) = \tilde{\mathrm{Ha}}^{p^{n-1}}$. Finally, using the normality of $\mathfrak{IG}_{n,r,I}$, one can conclude that there exist also sections d_n which verify the same equality integrally. $\square$

The previous proposition completely characterize the ramification of Hodge ideal along the Igusa tower. The situation is very different over the anticanonical tower, where the ramification is killed by the fact that we consider an object which is preperfectoid. In fact, the following holds:

Proposition 17. *We have* $\mathrm{Hdg}_r \mathscr{O}_{\mathfrak{IG}_{n-1,\infty,I}} \subseteq \mathrm{Tr}_{\mathfrak{IG}}(\mathscr{O}_{\mathfrak{IG}_{n,\infty,I}})$ *for every* $r \in \mathbb{Z}_{\geq 1}$*, where* Hdg_r *is the Hodge ideal associated to the* r*-th universal generalized elliptic curve.*

Proof. We know that $\mathrm{Hdg}_r^{p^{n-1}} \mathscr{O}_{\mathfrak{IG}_{n+1,r,I}} \subseteq \mathrm{Tr}(\mathscr{O}_{\mathfrak{IG}_{n,r,I}})$, and for every $r \geq n$, $\mathrm{Hdg}_{r-n+1} = \mathrm{Hdg}_r^{p^{n-1}} \subseteq \mathrm{Tr}(\mathscr{O}_{\mathfrak{IG}_{n,\infty,I}})$. $\square$

2.5.3 Modular forms at infinity.

The existence of the full Igusa tower at infinity, allows to define the sheaf of overconvergent modular forms over the perfection, i.e. over $\mathfrak{X}_{\infty,I}$. In fact, it is possible to copy the definition in characteristic p

Definition 28. *The sheaf of perfectified overconvergent modular forms is:*

$$\mathfrak{w}_I^{perf} := \mathscr{O}_{\mathfrak{IG}_{\infty,\infty,I}}[\kappa^{-1}] \tag{2.14}$$

i.e. it is the sheaf of functions defined over the Igusa tower which transform via the inverse of universal character under the action of $\mathbb{Z}_p^\times$, which acts over the tower by normalization.

As in the purely characteristic p and the purely characteristic 0 situations, it is possible to prove:

Proposition 18. *The sheaf $\mathfrak{w}_I^{perf}$ is an invertible sheaf.*

Proof. This is Proposition 6.4 in [AIP]. □

Moreover, it is possible to prove that in the case $I = [p^k, p^{k'}]$, the sheaf $\mathfrak{w}_I$ which is defined in characteristic 0, descends $\mathfrak{w}_I^{\text{perf}}$. What is difficult, but really interesting, is the fact that also in the case $I = [p, \infty]$, descent happens. In fact, we now recall the proof of the following

Theorem 9. *If $r \geq 3$, the sheaf $\mathfrak{w}_I^{perf}$ descends to an invertible sheaf $\mathfrak{w}_I$ over $\mathfrak{X}_{r,I}$, and $\mathfrak{w}_I$ can be characterized by the property of being the unique coherent subsheaf of $\mathscr{O}_{\mathfrak{X}_{r,I}}$-modules of $\mathfrak{w}_I^{perf}$ which is compatible with the restriction maps induced by $J \subseteq I$.*

We just want to point out that, considering a different interval $J \subseteq I$,it is possible to define a restriction map, which is induced by the generic inclusion of weight spaces $\mathcal{W}_J \subseteq \mathcal{W}_I$. The proof of this Theorem requires a Lemma which compares the functions defined when the interval I is $[p, \infty]$ to the cases when the interval is bounded.

Lemma 6. *There exists an isomorphism of sheaves over $\mathfrak{X}_{r,I}$:*

$$\mathscr{O}_{\mathfrak{X}_{r,I}} = \lim_{\substack{\longleftarrow \\ k+1 \geq k' \geq k \geq 1}} \mathscr{O}_{\mathfrak{X}_{r,[p^k,p^{k'}]}} \tag{2.15}$$

Moreover, there is also an isomorphism of sheaves over $\mathfrak{X}_{\infty,I}$

$$\mathscr{O}_{\mathfrak{X}_\infty} = \lim_{\substack{\longleftarrow \\ k+1 \geq k' \geq k \geq 1}} \mathscr{O}_{\mathfrak{X}_{\infty,[p^k,p^{k'}]}} \tag{2.16}$$

We can now recall the proof of the Theorem.

Proof. First, Andreatta, Iovita and Pilloni, propose a candidate for the descent. Inspired by the previous Lemma, they put:

$$\mathfrak{w}_I = \varprojlim_{k+1 \geq k' \geq k \geq 1} \mathfrak{w}_{[p^k, p^{k'}]} \tag{2.17}$$

Now, we prove it is an invertible sheaf which descends $\mathfrak{w}_I^{\text{perf}}$. In fact, it is enough to prove it in the case $r = 3$, which is the smallest possible r in the statement. For bigger r, this can simply be verified by pullback. First, fix $\text{Spf}(A) \subseteq \mathfrak{X}$ which is an affinoid open which trivializes the sheaf $\omega_{\mathcal{E}}$, and consider $\text{Spf}(B)$ its inverse image inside $\mathfrak{X}_{3,I}$. Clearly, since the sheaf of invariant differentials is trivialized, also the Hodge ideal is trivialized, and we say it is generated by $\tilde{\text{Ha}}_3$. Now, using the Trace map that we introduced in the previous section, it is not so hard to prove that there exist

$$\begin{aligned}
&c_0 = 1 \in \mathscr{O}_{\mathfrak{IG}_{0,3,I}}(\text{Spf}(B)) \\
&c_1 = \frac{1}{p-1} \in \mathscr{O}_{\mathfrak{IG}_{,1,3}I}(\text{Spf}(B)) \\
&\dots \\
&c_n \in \tilde{H}a^{-\frac{p^n - p}{p-1}} \mathscr{O}_{\mathfrak{IG}_{,n,3}I}(\text{Spf}(B))
\end{aligned}$$

for $n \leq 3$, and $c_n \in \tilde{H}a_3^{-p^3} \mathscr{O}_{\mathfrak{IG}_{n,\infty,I}}(\text{Spf}(B))$ for $n \geq r+1$ of elements such that $\text{Tr}_{\mathfrak{Ig}}(c_n) = c_{n-1}$. Now, the idea is to construct explicitly the generator of $\mathfrak{w}_I$. They do this by posing:

$$b_n = \sum_{\sigma \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \kappa(\tilde{\sigma}) \sigma . c_n$$

where $\tilde{\sigma}$ is a lifting of σ to $\mathbb{Z}_p^\times$. Clearly, $b_n - b_{n-1} \in T^{n-1} \tilde{H}a_3^{-p^3}$, hence the sequence converges T-adically, so the element $b_\infty = \lim_{n \to \infty} b_n$ exists inside $\mathscr{O}_{\mathfrak{IG}_{,\infty,\infty}I}(\text{Spf}(B))$, and it is not difficult to prove that $b_\infty = 1$ modulo $T\tilde{H}a^{-p^3}$, which implies that b_∞ is a generator of $\mathfrak{w}_I^{\text{perf}}$ over $\text{Spf}(B)$.

Moreover, if $J = [p^k, p^{k+1}]$ with $k \geq 1$ and consider the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{IG}_{\infty,\infty,I} \arrow[d] & \mathfrak{IG}_{\infty,\infty,J} \arrow[l] \arrow[r] \arrow[d] & \mathcal{F}_{k+3,3,J} \arrow[d] \\
\mathfrak{IG}_{k+3,\infty,I} \arrow[d] & \mathfrak{IG}_{k+3,\infty,J} \arrow[l] \arrow[r] \arrow[d] & \mathfrak{IG}_{k+3,3,J} \arrow[d] \\
\mathfrak{X}_{\infty,I} & \mathfrak{X}_{\infty,J} \arrow[l] \arrow[r, "h_r"] & \mathfrak{X}_{3,J}
\end{tikzcd}$$

Let $\mathrm{Spf}(C)$ to be the preimage of $\mathrm{Spf}(B)$ inside $\mathfrak{X}_{r,I}$, and let $s \in \mathscr{O}_{\mathcal{F}_{k+3,3,J}}(\mathrm{Spf}(C))$ to be a section which transforms under $\kappa^{-1}(\sigma)$ for every $\sigma \in (1+p^{k+3}\mathrm{Hdg}_3^{-\frac{p^{k+3}}{p-1}}\mathbb{G}_a)$, and $s = 1$ modulo p. Then, for every $0 \leq n \leq 3+k$, let $c'_n \in \tilde{\mathrm{Ha}}_3^{-\frac{p^n-p}{p-1}}\mathscr{O}_{\mathfrak{IG}_{n,3,J}}(\mathrm{Spf}(C))$ to be elements such that $\mathrm{Tr}_{\mathfrak{Ig}}(c'_n) = c'_{n-1}$, and $c'_n = c_n$ if $n \leq r$. We then denote:

$$f = \sum_{\sigma \in (\mathbb{Z}/p^{k+3}\mathbb{Z})^\times} \kappa(\sigma)\sigma.c'_{3+k}s$$

which is a generator of $\mathfrak{w}_J$ over $\mathrm{Spf}(C)$ by Lemma 5.4 of [AIP]. Now, since the following hold:

$$T^{p^k}|p$$
$$\tilde{\mathrm{Ha}}_3^{p^4}|T$$

inside C, we finally get that

$$p\tilde{\mathrm{Ha}}_3^{-p^{3+k}} = (pT^{-p^k})(T\tilde{Ha}_3^{-p^4})^{p^{k-1}}T^{p^k-p^{k-1}}$$

Since, modulo $p\tilde{\mathrm{Ha}}^{-p^{3+k}}$ we have $f = \sum_{\sigma\in(\mathbb{Z}/p^{k+3}\mathbb{Z})^\times} \kappa(\tilde{\sigma})\sigma.c'_{3+k}$, the same holds modulo T^2, and it is not difficult to prove, by descending recurrence, that this holds for every $3 \leq n \leq 3+k$. Finally, it is also easy to prove, using the Trace, that $\mathrm{Tr}_3(b_\infty)$ is a generator of $\mathfrak{w}_J$ over $\mathrm{Spf}(C)$, but then, using the Lemma, we get that

$$\mathfrak{w}_I(\mathrm{Spf}(B)) = \mathrm{Tr}_3(b_\infty) \varprojlim_{k+1\geq k'\geq k\geq 1} \mathscr{O}_{\mathfrak{X}_{r,[p^k,p^{k'}]}}(\mathrm{Spf}(B)) = \mathrm{Tr}_3(b_\infty)B$$

which shows that $\mathfrak{w}_I$ is an invertible sheaf. Moreover, by the fact that $\varprojlim_{k+1 \geq k' \geq k \geq 1} \mathfrak{w}^{\mathrm{perf}}_{[p^k, p^{k'}]} = \mathfrak{w}^{\mathrm{perf}}_I$, which implies that $h_{r,*}\mathfrak{w}_I = \mathfrak{w}^{\mathrm{perf}}_I$, which is the descent. The unicity is easy. □

Now, this sheaf is the sheaf which is useful to compare all the constructions. In fact, it is defined over $\mathfrak{X}_{r,I}$, and so it can be base changed to the boundary or the center of the weight space. It is proved again in [AIP] that both the base change give the modular sheaves that we introduced before. In particular, considering its restriction to the boundary of weight space, we get modulo p overconvergent modular forms, while base changing it to a region not containing the boundary, we get the usual analytic families of overconvergent modular forms.

2.6 U_p-operator and the Eigencurve.

In this section, we briefly recall the construction of the U_p-operator, and the property of the spaces of overconvergent modular forms to be Banach spaces, which is proved in the last chapter of [AIP].

2.6.1 The definition of U_p-operator

The U_p-operator is defined as usual using correspondences. In fact, in the previous sections, we recalled the construction of formal neighborhoods of the ordinary locus, which come with two morphisms, one which generically is an inclusion of open neighborhoods of the ordinary locus:

$$\iota : \mathfrak{X}_{r,I} \longrightarrow \mathfrak{X}_{r-1,I}$$

and the other one, which is a lifting of Frobenius:

$$\phi : \mathfrak{X}_{r,I} \to \mathfrak{X}_{r-1,I}.$$

Moreover, we know that Frobenius is finite and flat, and so in particular it admits a Trace morphism. Hence, for a given $I \subseteq [1, \infty]$, we can define the U_p-operator as:

$$\mathrm{H}^0(\mathfrak{X}_{r,I}, \mathfrak{m}_I) \to \mathrm{H}^0(\mathfrak{X}_{r-1,I}, \iota^*\mathfrak{w}_I) \cong \mathrm{H}^0(\mathfrak{X}_{r-1,I}, \phi^*\mathfrak{w}_I) \overset{\frac{1}{p}\mathrm{Tr}_\phi}{\to} \mathrm{H}^0(\mathfrak{X}_{r,I}, \mathfrak{w}_I). \tag{2.18}$$

The same definition can be given generically, where it is proved in [AIP] that the spaces of overconvergent modular forms are Banach spaces. In particular, the following result holds:

Theorem 10. *The T-adic Banach $B_I[1/T]$-space* $\mathrm{H}^0(\mathcal{M}_{r,I}, \omega_I^{\kappa})$ *for $I = [p, \infty]$ is a projective Banach space, hence it admits an orthonormal basis.*

Once we know how to define the U_p-operator, we can prove that it is compact, and we can apply the machinery of the usual construction of Eigenvarieties, getting at the end a spectral variety and an eigencurve, defined as the zero locus of the Fredholm determinant of U_p.

Chapter 3

A construction of a Mahler basis.

The aim of this part is to construct a Mahler basis for more general functions than in Mahler's Paper. Our main reference for this construction is [DS]. We carefully follow the various steps in the proof of the existence of Mahler basis for $\mathrm{Cont}(\mathbb{Z}_p, \mathbb{Q}_p)$, and we adapt the proof to our new situation. In particular, we want to prove the following.

Proposition 19. *Let R be a $\mathbb{Z}_p$-Banach Algebra, i.e. a complete normed ring equipped with a ring homomorphism $\mathbb{Z}_p \to R$ which is norm decreasing. Then the normed R-module $\mathrm{Cont}(\mathbb{Z}_p, R)$ of continuous R-valued functions on $\mathbb{Z}_p$ admits a Mahler basis, i.e. every continuous function $f : \mathbb{Z}_p \to R$ can be written as a series:*

$$f(z) = \sum_{n=0}^{\infty} a_n(f) \binom{z}{n}$$

where $\{a_n(f)\}_{n \in \mathbb{N}}$ is a sequence of elements of R converging to 0.

We split the proof in several Lemmas. In each Lemma, we assume the notation of the last proposition. Moreover, with a little abuse of notation, we write $r \in \mathbb{Z}_p$ for an element $r \in R$, meaning its image under the structure morphism of R as a $\mathbb{Z}_p$-algebra. Finally, during the proof, we have to deal with two different norms. First we have the p-adic norm on $\mathbb{Z}_p$, induced by the p-adic valuation, which we denote $|-|_p$, and then we have the own norm of R, which we simply denote by $|-|$. The first results are easy computations which are useful during the proof.

Lemma 7. *Let $x \in \mathbb{Z}_p$. Then, for every $n \in \mathbb{N}$, we define the binomial coefficient formally to be*

$$\binom{x}{n} = \frac{x(x-1)(x-2)\dots(x-n+1)}{n!}.$$

We then have $\binom{x}{n} \in \mathbb{Z}_p$.

Proof. This is clear because if x belongs to $\mathbb{Z}$, also $\binom{x}{n}$ is an integer. Then the Lemma follows by continuity. $\square$

Lemma 8. *Let $n \in \mathbb{N}$, and define the function $\binom{z}{n} : \mathbb{Z}_p \to R$ to be the composition of the usual binomial coefficient function from $\mathbb{Z}_p$ to $\mathbb{Z}_p$ with the structure morphism of R as a $\mathbb{Z}_p$-Banach Tate Algebra. Then $\binom{z}{n}$ is a continuous function.*

Proof. This is clear since the binomial coefficient is essentially a polynomial function. $\square$

Lemma 9. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of elements of R converging to 0. Then the series $\sum_{n\geq 0} a_n \binom{r}{n}$ converges for all $r \in \mathbb{Z}_p$.*

Proof. By the assumption on the structure morphism of R as a $\mathbb{Z}_p$-algebra, we have, for every $r \in \mathbb{Z}_p$,

$$\left|\binom{r}{n}\right| \leq \left|\binom{r}{n}\right|_p \leq 1$$

where the last inequality comes from Lemma 7. This implies the thesis, as

$$\left|a_n \binom{r}{n}\right|_p \leq |a_n|$$

which goes to 0 by the hypothesis. Therefore the sequence $a_n\binom{r}{n}$ goes to zero, and so the series $\sum_{n\geq 0} a_n \binom{r}{n}$ converges. $\square$

Lemma 10. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of elements of R converging to 0. Then the function $f : \mathbb{Z}_p \to R$ defined by*

$$f(z) = \sum_{n\geq 0} a_n \binom{z}{n}$$

is continuous.

Proof. Let $z_0 \in \mathbb{Z}_p$, we want to prove that f is continuous at z_0, then, by the arbitrarity of z_0, we will conclude that f is continuous over $\mathbb{Z}_p$. First, we know that the sequence a_n of elements of R converges to 0, from which we conclude that for every $\varepsilon > 0$, there exists $N > 0$, depending on ε such that $|a_n| < \varepsilon$ for all $n \geq N$. Moreover, by the Lemma 8, we know that all the functions $\binom{z}{n}$ are continuous everywhere, for all n between 0 and $N-1$. In particular, they are all continuous in z_0. Hence, for every n and for every $\varepsilon_n > 0$, there exists $\delta_n > 0$, depending both on n and on ε_n, such that for every $z \in \mathbb{Z}_p$ with $|z - z_0|_p < \delta_n$, we have:

$$\left|\binom{z}{n} - \binom{z_0}{n}\right| < \varepsilon_n$$

In particular, by choosing a suitable minimal $\delta > 0$, we can say that there exists a $\delta > 0$ such that for every $z \in \mathbb{Z}_p$ with $|z - z_0|_p < \delta$, we have:

$$\left|\binom{z}{n} - \binom{z_0}{n}\right| < \varepsilon$$

for every natural number n between 0 and $N-1$. Now, if $z \in \mathbb{Z}_p$ is close to z_0, we conclude that:

$$\begin{aligned}
|f(z) - f(z_0)| &= \left|\sum_{n\geq 0} a_n \binom{z}{n} - \sum_{n\geq 0} a_n \binom{z_0}{n}\right| = \\
&= \left|\sum_{n\geq 0} a_n \left(\binom{z}{n} - \binom{z_0}{n}\right)\right| = \\
&= \left|\sum_{n\in\{0,\dots,N-1\}} a_n \left(\binom{z}{n} - \binom{z_0}{n}\right) + \sum_{n\geq N} a_n \left(\binom{z}{n} - \binom{z_0}{n}\right)\right| \leq \\
&\leq \max_{n\in\{0,\dots N-1\}} |a_n| \left|\binom{z}{n} - \binom{z_0}{n}\right| + \max_{n\geq N} |a_n| \left|\binom{z}{n} - \binom{z_0}{n}\right|
\end{aligned}$$

Now, if $n \geq N$, we use the decreasing property of norm and Lemma 7 to conclude that

$$\begin{aligned}
\max_{n\geq N} |a_n| \left|\binom{z}{n} - \binom{z_0}{n}\right| &\leq \max_{n\geq N} |a_n| \left|\binom{z}{n} - \binom{z_0}{n}\right|_p \\
&\leq \max_{n\geq N} |a_n| < \varepsilon,
\end{aligned}$$

while for $n \in \{0, \ldots, N-1\}$, we use the following:

$$\max_{n \in \{0,\ldots,N-1\}} |a_n| \left| \binom{z}{n} - \binom{z_0}{n} \right| \le \max_{n \in \{0,\ldots,N-1\}} |a_n| \, \varepsilon$$

which says that, up to changing ε with a suitable ε', we get:

$$|f(z) - f(z_0)| \le \varepsilon'$$

for every $\varepsilon' > 0$ if $|z - z_0|_p < \delta$, which is the continuity of f in z_0. $\square$

Now we go on by defining the usual discrete difference operator occurring in the original proof of the existence of Mahler basis.

Definition 29. *Let $f : \mathbb{Z}_p \to R$ be any function. We define the n-th discrete derivative of f to be the function $\Delta^n f : \mathbb{Z}_p \to R$ defined by the iterated procedure:*

$$\begin{cases} (\Delta^0 f)(z) = f(z) \\ (\Delta^1 f)(z) = f(z+1) - f(z) \\ (\Delta^n f)(z) = (\Delta(\Delta^{n-1} f)(z) \end{cases}$$

The following result is crucial in the rest of the proof, as it gives an explicit description of the coefficients of Mahler expansion.

Lemma 11. *If $\{a_n\}_n$ is a sequence of elements of R converging to 0 and $f(z) = \sum_{n \ge 0} a_n \binom{z}{n}$ is the continuous function from $\mathbb{Z}_p$ to R of the previous Lemma, we have the equality*

$$a_n = (\Delta^n f)(0)$$

Proof. We first compute:

$$\Delta \binom{z}{n} = \binom{z+1}{n} - \binom{z}{n} = \left(\binom{z}{n-1} + \binom{z}{n} \right) - \binom{z}{n} = \binom{z}{n-1}$$

which in particular implies that $\Delta \binom{z}{0} = 0$. Now we apply the discrete derivative operator to a function f as in the statement. We have:

$$\begin{aligned}(\Delta f)(z) &= f(z+1) - f(z) = \\ &= \sum_{n\geq 0} a_n \binom{z+1}{n} - \sum_{n\geq 0} a_n \binom{z}{n} = \\ &= \sum_{n\geq 0} a_n \left(\binom{z+1}{n} - \binom{z}{n}\right) = \\ &= \sum_{n\geq 1} a_n \binom{z}{n-1} = \\ &= \sum_{n\geq 0} a_{n+1} \binom{z}{n}\end{aligned}$$

This says that the effect of applying the discrete derivative k-times is to move the index on the right of k-positions, hence, evaluating $(\Delta^k f)$ in zero, one gets the value a_k. $\square$

In order to prove our main result, we need a more explicit formula for the n-th discrete derivative of a function f. In particular, the following holds:

Lemma 12. *Let $f : \mathbb{Z}_p \to R$ be any function. Then, for every $n \geq 0$ and for every $z \in \mathbb{Z}_p$ the following equality holds:*

$$(\Delta^n f)(z) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(x+i).$$

Proof. We prove the result by induction. When $n = 0$, there is nothing to prove. So we prove that the case n implies the case $n+1$. Assume that the formula is true for some $n \in \mathbb{N}$ and for every $z \in \mathbb{Z}_p$. Then we have:

$$\begin{aligned}(\Delta^{n+1} f)(z) &= (\Delta(\Delta^n f))(z) = \\ &= (\Delta^n(f))(z+1) - (\Delta^n f)(z) = \\ &= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(z+1+i) - \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(z+i) = \\ &= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(z+(i+1)) + \sum_{i=0}^{n} (-1)^{n-i+1} \binom{n}{i} f(z+i) \\ &= \sum_{i=1}^{n+1} (-1)^{n-(i-1)} \binom{n}{i-1} f(z+i) + \sum_{i=0}^{n} (-1)^{n-(i-1)} \binom{n}{i} f(z+i)\end{aligned}$$

Now we take a look to the different terms appearing in the two formulas.

- The term appearing in the first sum when $i = n + 1$ is:

$$(-1)^0 \binom{n}{n} f(z+n+1) = f(z+n+1)$$

- The term appearing in the second sum when $i = 0$ is

$$(-1)^{n+1} \binom{n}{0} f(z) = (-1)^{n+1} f(z)$$

- The remaining terms run over the same set of indices, and they sum into the following:

$$\sum_{i=1}^{n} (-1)^{n-(i-1)} \left(\binom{n}{i-1} + \binom{n}{i} \right) f(z+i) = \sum_{i=0}^{n} (-1)^{n-(i-1)} \binom{n+1}{i} f(z+i)$$

  where we used the famous Pascal's identity to sum binomial coefficients.

But now notice that we can write the first two terms in this very useful way:

- The first term:

$$f(z+n+1) = (-1)^{n-(n+1-1)} \binom{n+1}{n+1} f(z+(n+1))$$

- The second one:

$$(-1)^{n+1} f(z) = (-1)^{n-(0-1)} \binom{n+1}{0} f(z)$$

which fit into the previous sum respectively as the term $i = n + 1$ and $i = 0$. This says that:

$$(\Delta^{n+1} f)(z) = \sum_{i=0}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} f(z+i)$$

which is the formula we wanted to prove. $\square$

Notice that this proof is purely combinatorial, nothing related to continuity is needed to prove it, but the point is that, even if the explicit formula can be proved for any function from $\mathbb{Z}_p$ to R, the expansion result only holds for continuous functions, as there are counterexamples to the general situation. Moreover, notice that this formula is really crucial into the proof. In fact, it essentially involves the values of f at every non-negative integer, which is enough to determine an expansion of f, since, combining the density of $\mathbb{N}$ in $\mathbb{Z}_p$ whith the assumption of continuity, this says that the function is completely described by its value on integers elements of $\mathbb{Z}_p$.

We now recall with proof some basic facts about p-adic orders of factorials and binomial coefficients. The first Lemma we use is the so called Legendre Formula, which can be written in many different ways. We choose the one which is more useful for our intent.

Lemma 13. *Let n be a positive integer, and let p be a prime. Then the following formula for the p-adic order of $n!$ holds:*

$$\mathrm{ord}_p(n!) = \frac{n - s_p(n)}{p-1}$$

where $s_p(n)$ is the sum of the valuations of the digits in the p-adic expansion of n (which is finite since n is integer).

Proof. Write n in base p (clearly such a writing exists unique):

$$n = \sum_{l=0}^{N} n_l p^l$$

where N is the maximal exponent of p appearing in the expansion of n. Then we clearly have:

$$\left\lfloor \frac{n}{p^i} \right\rfloor = \begin{cases} \sum_{l=i}^{N} n_l p^{l-i} & \text{if } i \leq N \\ 0 & \text{otherwise} \end{cases}$$

Now notice that, since $n!$ is the product of all the integers between 1 and n, we obtain at least one factor of p in $n!$ for each multiple of p in the set $\{1, \dots, n\}$, and those are precisely $\left\lfloor \frac{n}{p} \right\rfloor$. Argueing in the same way, we have that p^2 contributes with an additional factor of p, and the same is for p^3 and so on. In this way, we clearly get:

$$\mathrm{ord}_p(n!) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor .$$

which is the original statement of Legendre's formula. But then, using the previous equation, we get:

$$\begin{aligned}
\mathrm{ord}_p(n!) &= \sum_{j=0}^{N} \left\lfloor \frac{n}{p^j} \right\rfloor = \\
&= \sum_{j=0}^{N} \sum_{l=j}^{N} n_l p^{l-j} = \\
&= \sum_{l=1}^{N} \sum_{j=1}^{l} n_l p^{l-j} = \\
&= \sum_{l=1}^{N} n_l \frac{p^l - 1}{p - 1} = \\
&= \frac{1}{p-1} \left(\sum_{l=0}^{N} n_l p^l - \sum_{l=0}^{N} n_l \right) = \\
&= \frac{n - s_p(n)}{p-1}
\end{aligned}$$

which is the formula. □

Lemma 14. *For any $1 \leq i \leq p^k$, the p-adic order of $\binom{p^r}{i}$ is $k - \mathrm{ord}_p(i)$.*

Proof. Let $i \geq 1$ and let $x \in \mathbb{N}$. Then the following equation clearly holds by definition:

$$\binom{x}{i} = \frac{x}{i} \binom{x-1}{i-1}$$

If we now set $x = p^k$, we get:

$$\binom{p^k}{i} = \frac{p^k}{i} \binom{p^k - 1}{i-1}.$$

We now show that $\binom{p^k-1}{i-1}$ is not divisible by p, which says that the p-adic order of $\binom{p^k}{i}$ is precisely $k - \mathrm{ord}_p(i)$, which is the Lemma. Now, using Lemma

13, we get:

$$\begin{aligned}\operatorname{ord}_p\left(\binom{p^k-1}{i-1}\right) &= \operatorname{ord}_p\left(\frac{(p^k-1)!}{(i-1)!(p^k-i)!}\right) = \\ &= \frac{p^k-1-s_p(p^k-1)}{p-1} - \frac{i-1-s_p(i-1)}{p-1} - \frac{p^k-i-s_p(p^k-i)}{p-1} = \\ &= \frac{s_p(i-1)+s_p(p^k-i)-s_p(p^k-1)}{p-1}.\end{aligned}$$

Showing that $\binom{p^k-1}{i-1}$ is not divisible by p is the same as showing that the numerator of the previous fraction is zero. Now, if $i = p^k$, there is nothing to prove, so assume $1 \leq i \leq p^k - 1$, and write the minimal p-adic expansion of i, i.e. $i = c_m p^m + \ldots + c_{k-1}p^{k-1}$ with $c_m \neq 0$, in such a way that i is the p-adic order of i. Then we have

$$\begin{aligned}&i-1 = (p-1)+\ldots+(p-1)p^{m-1}+(c_m-1)p^m+c_{m+1}p^{m+1}+\ldots+c_{k-1}p^{k-1}\\ &p^k-i = (p-c_m)p^m+(p-1-c_{m+1})p^{m+1}+\ldots+(p-1-c_{m-1})p^{k-1}\\ &p^k-1 = (p-1)+(p-1)p+\ldots+(p-1)p^{k-1}\end{aligned}$$

and we finally get:

$$\begin{aligned}s_p(i-1)+s_p(p^k-i) &= ((p-1)m+s_p(i)-1)+(1+(p-1)(k-m)-s_p(k)) = \\ &= (p-1)k = \\ &= s_p(p^k-1)\end{aligned}$$

proving that the numerator is zero. □

We are now ready to prove the main result, i.e. Proposition 19.

Proof. Let $f : \mathbb{Z}_p \to R$ be a continuous function and we let:

$$a_n(f) := (\Delta^n)f(0) = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}f(i)$$

We first show that $a_n(f)$ converges to 0 in R. First notice that the following obvious inequality holds:

$$|(\Delta^n f)(0)| \leq \max_{z\in\mathbb{Z}_p}|(\Delta^n f)(z)| =: ||(\Delta^n f)||$$

But then, for each n and z,

$$\begin{aligned} |(\Delta^{n+1} f)(z)| &= |(\Delta(\Delta^n f))(z)| = \\ &= |(\Delta^n f)(z+1) - (\Delta^n f)(z)| \leq \\ &\leq \max_{z \in \mathbb{Z}_p}(|(\Delta^n f)(z+1)|, |(\Delta^n f)(z)|) \leq \\ &\leq ||\Delta^n f|| \end{aligned}$$

This says that $||\Delta^m f|| \leq ||\Delta^n f||$ when $m \geq n$, and so it implies that we can prove $||\Delta^n(f)|| \to 0$ using a particular diverging sequence of positive integers. In particular, we can choose a sequence given by powers of p, i.e. we prove that $||\Delta^{p^k} f|| \to 0$ as $k \to +\infty$. So, let now n be an integer greater than 0, and let $z \in \mathbb{Z}_p$. We have:

$$\begin{aligned} (\Delta^n f)(z) &= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(z+i) \\ &= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (f(z+i) - f(z)) \end{aligned}$$

where we used the fact that

$$\begin{aligned} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(z) = f(z) \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} &= \\ = f(z)(1-1)^n = 0 \end{aligned}$$

Now, notice that the term with $i = 0$ is:

$$(-1)^n (f(z) - f(z)) = 0$$

so we can rewrite the previous sum as

$$(\Delta^n f)(z) = \sum_{i=1}^{n} (-1)^{n-i} \binom{n}{i} (f(z+i) - f(z))$$

We now reduce to our subsequence, setting $n = p^k$, and we get:

$$(\Delta^{p^k} f)(z) = \sum_{i=0}^{p^k} (-1)^{p^k - i} \binom{p^k}{i} (f(z+k) - f(z))$$

Now, we want to show that for every $\varepsilon > 0$, for every $z \in \mathbb{Z}_p$ and for all suitable large k, $|(\Delta^{p^k} f)(z)| < \varepsilon$. By the previous formula,

$$|(\Delta^{p^k} f)(z)| \leq \max_{1 \leq i \leq p^k} \left| \binom{p^k}{i} \right| |f(z+i) - f(z)| \leq$$
$$\leq \max_{1 \leq i \leq p^k} \left| \binom{p^k}{i} \right|_p |f(z+i) - f(z)|$$

But now, we know that $\left|\binom{p^k}{i}\right|_p \leq 1$, for every $1 \leq i \leq p^k$ and, by definition, $|f(z+i) - f(z)| \leq ||f||$. Now, by Lemma 14, we know that

$$\left| \binom{p^k}{i} \right|_p = p^{\text{ord}_p(i) - k} = \frac{1}{p^k |i|_p}$$

which says that $\left|\binom{p^k}{i}\right|_p |i|_p = \frac{1}{p^r}$. This implies that

$$\left| \binom{p^k}{i} \right|_p \leq \frac{1}{\sqrt{p^k}} \quad \text{or} \quad |i|_p \leq \frac{1}{\sqrt{p^k}}$$

Moreover, since f is continuous, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ if $|x - y|_p < \delta$. If we now take M large enough to ensure $1/\sqrt{p^M} < \min(\delta, \varepsilon)$, we get, if we choose $m \geq M$ and $1 \leq i \leq p^m$, the following two options:

- If $\left|\binom{p^m}{i}\right|_p \leq 1/\sqrt{p^m}$, then
$$\left| \binom{p^m}{i} \right|_p |f(z+i) - f(z)| \leq \frac{||f||}{\sqrt{p^m}} \leq \varepsilon ||f|| < \varepsilon \left(||f|| + \frac{1}{2} \right).$$
- If $|i|_p \leq 1/\sqrt{p^m}$, then $|i|_p < \delta$, and so for every $z \in \mathbb{Z}_p$, we have $|(z+i) - z|_p < \delta$, which implies $|f(z+i) - f(z)|_p < \varepsilon$, and so:
$$\left| \binom{p^m}{i} \right|_p |f(z+m) - f(z)| \leq |f(z+m) - f(z)| < \varepsilon$$

  Now, recalling that $||f|| < \infty$, since f is continuous, $\mathbb{Z}_p$ is compact, and so f is bounded, we conclude that for every $\varepsilon > 0$, there exists an $M > 0$ such that for every $m \geq M$,
$$||\Delta^{p^m} f|| \leq \varepsilon \max \left(||f|| + \frac{1}{2}, 1 \right).$$

  Putting everything together, we conclude that $||\Delta^n f||$ goes to zero.

This is not the thesis yet, since we have to prove that this fact implies the Proposition, i.e. we still have to show that for every $z \in \mathbb{Z}_p$, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n(f) \binom{z}{n}$$

Now, by Lemma 10, we know that the right-hand side defines a continuous function, so, in order to prove that f admits this expansion, it is enough to prove that these two functions coincide on the dense subset of positive integers. Let $k \in \mathbb{N}$, then

$$\begin{aligned}\sum_{n=0}^{\infty} a_n(f) \binom{k}{n} &= \sum_{n=0}^{k} \left(\sum_{d=0}^{n} (-1)^{n-d} \binom{n}{d} f(d) \right) \binom{k}{n} = \\ &= \sum_{d=0}^{k} \left(\sum_{n=d}^{k} (-1)^{n-m} \binom{n}{d} \binom{k}{n} \right) f(d)\end{aligned}$$

Now, we can write:

$$\begin{aligned}\binom{n}{d}\binom{k}{n} &= \frac{n!}{d!(n-d)!} \frac{k!}{n!(n-k)!} = \\ &= \frac{k!}{d!(n-d)!(k-d)!} \\ &= \frac{k!}{d!(k-d)!} \frac{(k-d)!}{(n-d)!(k-n)!} \\ &= \binom{k}{d}\binom{k-d}{n-d}\end{aligned}$$

This allows us to write the previous formula as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} a_n(f) \binom{k}{n} &= \sum_{d=0}^{k} \left(\sum_{n=d}^{k} (-1)^{n-d} \binom{k}{d} \binom{k-d}{n-d} \right) f(d) = \\ &= \sum_{d=0}^{k} \left(\sum_{n=d}^{k} (-1)^{n-d} \binom{k-d}{n-d} \right) \binom{k}{d} f(d) = \\ &= \sum_{d=0}^{k} \left(\sum_{n=0}^{k-d} (-1)^{n} \binom{k-d}{n} \right) \binom{k}{d} f(d)\end{aligned}$$

Now, the inner sum is nothing but the Newton's expansion for $(1-1)^{k-d}$, which is zero for $d < k$ and is 1 for $d = k$, so we see that

$$\sum_{n=0}^{\infty} a_n(f)\binom{k}{n} = \binom{k}{k} f(k) = f(k)$$

which proves the equality for every integer, now continuity do the rest. □

The following is a Corollary of Proposition 19 which computes what we called $||f||$ for a continuous function.

Corollary 1. *For a continuous function $f : \mathbb{Z}_p \to R$ with Mahler expansion $\sum_{n=0}^{\infty} a_n(f)\binom{z}{n}$,*

$$||f|| = \max_{n\geq 0} |a_n(f)|$$

Proof. For every $z \in \mathbb{Z}_p$,

$$|f(z)| = \left|\sum_{n=0}^{\infty} a_n \binom{z}{n}\right| \leq \max_{n\geq 0} \left|a_n(f)\binom{z}{n}\right| \leq \max_{n\geq 0} |a_n(f)|$$

Therefore $||f|| = \max_{z\in\mathbb{Z}_p} |f(z)| \leq \max_{z\in\mathbb{Z}_p} |a_n(f)|$.

For the reverse inequality, we use the explicit formula we found for the coefficient $a_n(f)$ as $(\Delta^n f)(0)$:

$$\begin{aligned} |a_n(f)| &= \left|\sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} f(j)\right| \leq \\ &\leq \max_{0\leq j\leq n} \left|\binom{n}{j} f(j)\right| \leq \\ &\leq \max_{0\leq k\leq n} |f(k)| \leq ||f||. \end{aligned}$$

which ends the proof. □

The next Corollary comes trivially from the Proposition.

Corollary 2. *Let R be a Banach normed ring with structure morphism which is norm decreasing. Then:*

$$\mathrm{Cont}(\mathbb{Z}_p, \mathbb{Z}_p)\widehat{\otimes}_{\mathbb{Z}_p} R \cong \mathrm{Cont}(\mathbb{Z}_p, R)$$

where $\mathrm{Cont}(\mathbb{Z}_p, -)$ *are the continuous functions from* $\mathbb{Z}_p$*.*

Proof. In fact, both sides admit an orthonormal basis. The right-hand side admits the orthonormal basis that we constructed here, while the left hand side admits the basis given by $\left\{\binom{z}{n}\widehat{\otimes}1\right\}_{n\in\mathbb{N}}$. But then the two rings trivially coincide. $\square$

Chapter 4

The construction of Ψ.

4.1 Notations.

In what follows, we adopt the following notations.

- All schemes will be denoted by capital letters, like $X, Y, Z, \ldots$.
- All formal schemes will be denoted by italic letters, like $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \ldots$.
- All adic spaces will be denoted by calligraphic letters, like $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$.
- From now on, $B = \mathbb{Z}_{p^2}[[T]]\langle u \rangle/(T^p u - p)$ will be the blowing up of the connected component of the trivial finite character corresponding to the interval $[p, +\infty]$. Hence, we fix $I = [p, +\infty]$ in what follows.
- $\mathfrak{X}$ is the p-adic completion of the modular curve $X_1(N)/\mathrm{Spec}(\mathbb{Z}_{p^2})$ of level $\Gamma_1(N)$ for some $N \geq 4$ and $(p, N) = 1$. Notice that here our notation is a bit different from [AIP], because the authors don't consider any base change to $\mathbb{Z}_{p^2}$. We work over this extended base as we need to consider supersingular points which admit a canonical structure over $\mathbb{Z}_{p^2}$. We also denote by $\mathfrak{X}_I$ the T-adic formal scheme defined by base change of $\mathfrak{X}$ along the map $\mathrm{Spf}(B) \to \mathrm{Spf}(\mathbb{Z}_{p^2})$. Here we maintain the I in the notation in order to be coherent with [AIP].
- $\mathfrak{X}_{r,I}$ will be the formal scheme defined by the open formal subscheme of the admissible blowup of $\mathfrak{X}$ along the Hodge ideal, defined by the condition that $\widetilde{\mathrm{Ha}}^{p^{r+1}}$ generates the Hodge ideal.

- $\mathcal{X}_{r,I}$ is the adic generic fiber of $\mathfrak{X}_{r,I}$. It is defined as $\mathfrak{X}^{\mathrm{ad}}_{r,I} \times_{\mathrm{Spa}(B,B)} \mathrm{Spa}(B[1/T], B^+)$ and it comes equipped with an open immersion inside the adic generic fiber of $\mathfrak{X}$ which identifies it with a strict open neighborhood of the ordinary locus of $\mathcal{X}$.

- We denote by $\mathscr{E}_r$ the universal elliptic curve living over $\mathfrak{X}_{r,I}$. It is obtained by base change of the universal elliptic curve over $\mathfrak{X}$ via the blowin up universal morphism. We notice that it is a formal scheme, but when we base change it over an affine formal scheme $\mathrm{Spf}(R)$, it gives a real elliptic curve over $\mathrm{Spec}(R)$ as it is proper.

- $H_n \subseteq \mathscr{E}_r$ is the canonical subgroup (when it exists) of level n. It lifts the kernel of Frobenius morphism modulo $\frac{p}{\mathrm{Ha}^{p^{r+1}}}$

- $\mathcal{IG}_{n,r,I}$ is the adic space over $\mathcal{X}_{r,I}$ which parametrizes trivializations of the dual canonical subgroup $H_n^\vee$, which is étale over the generic fiber.

- $\mathfrak{IG}_{n,r,I}$ is the normalization of $\mathfrak{X}_{r,I}$ inside $\mathcal{IG}_{n,r,I}$. We denote again by ϕ_r the Frobenius morphism $\mathfrak{IG}_{n,r,I} \to \mathfrak{IG}_{n,r-1,I}$ induced by the one over the modular curve.

4.2 A perfection of weight space.

Here we want to define a ring which will be crucial for the definition of Ψ. As the idea is to define a valuation map for modular forms over supersingular points, we first construct the points over which our modular forms will be evaluated.

4.2.1 Supersingular points.

Here we want to choose a supersingular point of $\mathfrak{X}_{r,I}$, hence we have to say how to construct such a supersingular point. We start by choosing a supersingular point $\overline{x} \in X_1(N)(\overline{\mathbb{F}}_p)$, which corresponds to a morphism $\overline{x} : \mathrm{Spec}(\overline{\mathbb{F}}_p) \to X_1(N)$.

Proposition 20. *Every supersingular elliptic curve over $\overline{\mathbb{F}}_p$ admits a natural structure over $\mathbb{F}_{p^2}$, i.e. if $E/\overline{\mathbb{F}}_p$ is a supersingular elliptic curve, then there exists an elliptic curve $E'/\mathbb{F}_{p^2}$ such that E is the base change of E' along the inclusion $\mathbb{F}_{p^2} \subseteq \overline{\mathbb{F}}_p$.*

Proof. The proof comes essentially from chapter V of [Sil]. Elliptic curves over $\overline{\mathbb{F}}_p$ are completely classified by their j-invariant. We recall that, if E is defined (over any field K) by a Weierstrass equation $y^2 = x^3 + Ax + B$, its j-invariant is the element of K defined as

$$j(E) = 1728\frac{4A^3}{4A^3 + 27B^2}.$$

Now, a supersingular elliptic curve over $\overline{\mathbb{F}}_p$ is, by definition, an elliptic curve such that the endomorphism defined by multiplication by p is trivial, i.e. $E[p] = 0$. If we decompose the multiplication by p using Frobenius, we get

$$[p] = \widehat{\phi} \circ \phi$$

where $\widehat{\phi}$ is the dual isogeny. Then $\widehat{\phi} : E^{(p)} \to E$ has trivial kernel and must be inseparable of degree p. By the theory of isogenies of elliptic curves, see chapter 2 of [Sil] we can decompose $\widehat{\phi}$ as $\widehat{\phi} = \widehat{\phi}_{\text{sep}} \circ \phi$, where $\widehat{\phi}_{\text{sep}}$ is an is an isomorphism from $E^{(p^2)}$ to E. But then we get

$$j(E) = j(E^{\{(p^2)}) = j(E)^{p^2}$$

which says that the j-invariant of E satisfies the equation $X^{p^2} - X = 0$, which means that it's defined over $\mathbb{F}_{p^2}$. $\square$

In particular, the previous Proposition says that we can look for all supersingular points of $X_1(N)$ simply by taking morphisms $\overline{x} : \mathrm{Spec}(\mathbb{F}_{p^2}) \to X_1(N)$.

As we are now interested in taking deformations of supersingular elliptic curves, we recall a bit about Deformation Theory, and we then explain how to use it to produce the points we are interested in. We remind the well-known notation $W(k)$ which denotes the ring of Witt vectors attached to a field k.

Definition 30. *Let k be a field and let $\mathcal{C}$ be the category of complete local Noetherian $W(k)$-algebras whose residue field is k. Let E_0/k be an elliptic curve. We then call functor of deformations of E_0 the functor*

$$\mathrm{Def}_{E_0/k} : \mathcal{C} \to \mathit{Set}$$

which sends $R \in \mathcal{C}$ to the set of couples (E, ι), where E is an isomorphism class of elliptic curves over R and

$$\iota : E \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k) \cong E_0$$

is an isomorphism. If $f : R \to R'$ is a morphism in $\mathcal{C}$, then

$$\mathrm{Def}_{E_0/k}(f) : \mathrm{Def}_{E_0/k}(R) \to \mathrm{Def}_{E_0/k}(R')$$

is the map sending (E, ι) to $(E \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R'), \iota_{R'})$.

The interesting fact is that the previous functor is represented by the completed stalk of the modular curve, once we fix a level N-structure.

Proposition 21. *Let k be a field and $y = (E_0, P_0) \in X_1(N)(k)$ be a point corresponding to an elliptic curve E_0 over k with the level N-structure P_0. Let $\mathcal{C}$ be the category of complete local Noetherian $W(k)$-algebras whose residue field is k. Let $\mathrm{Def}_{E_0/k,N} : \mathcal{C} \to \mathrm{Set}$ be the functor sending an object $R \in \mathcal{C}$ to the couple $(\mathrm{Def}_{E_0/k}(R), P)$, where $\mathrm{Def}_{E_0/k}(R)$ is as in Definition 30 and P is a level N structure over E/R. Then $\mathrm{Def}_{E_0,N}$ is prorepresented by the completed local ring $\widehat{\mathscr{O}_{X_1(N),y}}$.*

Proof. We start by proving the Proposition in the case when $R \in \mathcal{C}$ is Artinian, i.e. we let R be a complete local Noetherian $W(k)$-algebra with maximal ideal $\mathfrak{m}_R$ which is nilpotent and such that $R/\mathfrak{m}_R \cong k$. We prove that

$$\mathrm{Def}_{E_0/k,N}(R) \cong \mathrm{Hom}_{W(k)}(\widehat{\mathscr{O}_{X_1(N),y}}, R)$$

Let $(E/R, \iota, P) \in \mathrm{Def}_{E_0/k,N}(R)$ be given. Then this triple corresponds by definition to a morphism

$$\begin{tikzcd}
\mathrm{Spec}(R) \arrow[r] & X_1(N) \\
\mathrm{Spec}(k) \arrow[r] \arrow[u, hook] & \{y\} \arrow[u, hook]
\end{tikzcd}$$

But the morphism $\mathrm{Spec}(R) \to X_1(N)$ localizes to a morphism of local $W(k)$-algebras $\mathscr{O}_{X_1(N),y} \to R$, where R coincide with its stalk as it is a local ring. By completing, and noting that the maximal ideal of R is nilpotent, we get a morphism $\widehat{\mathscr{O}_{X_1(N),y}} \to R$.

Vice versa, if we have a continuous morphism $\widehat{\mathscr{O}_{X_1(N),y}} \to R$ in particular it is local, hence it defines a diagram:

$$\begin{tikzcd} \mathrm{Spec}(R) \arrow[r] & \mathrm{Spec}(\widehat{\mathscr{O}_{X_1(N),y}}) \\ \mathrm{Spec}(k) \arrow[r] \arrow[u, hook] & \{y\} \arrow[u, hook] \end{tikzcd}$$

which gives by composition a morphism $\mathrm{Spec}(R) \to X_1(N)$ compatible with the reduction. But this, by definition, gives the triple $(E/R, \iota, P)$.

Now, for a general object $R \in \mathcal{C}$ we have $R = \varprojlim_n R/\mathfrak{m}_R^n$. Then a triple $(E/R, \iota, P)$ corresponds to a compatible family of such triples over $R/\mathfrak{m}_R^n$, which means that we have a compatible family of local morphisms of $W(k)$-algebras $\widehat{\mathscr{O}_{X_1(N),y}} \to R/\mathfrak{m}_R^n$, which corresponds uniquely to a continuous morphism $\widehat{\mathscr{O}_{X_1(N),y}} \to R$. $\square$

We only mention that it is possible to study deformations of elliptic curves via a careful study of deformations of p-divisible groups attached to such elliptic curve. In particular, the Serre Tate theorem states the very useful fact that the deformation theory of an elliptic curve is essentially the same as the deformation theory of its p-divisible group. We recall the statement of the result even if we decided to follow another point of view.

Theorem 11. *Let R be a ring, I and ideal of R and p a prime number. Assume that $I + (p)$ is a nilpotent ideal, and write $R_0 = R/I$. Let Ell_r denote the category of elliptic curves over R. Let $\mathcal{A}_R$ denote the category of triples $(E_0/R_0, G, \iota)$, where E_0/R_0 is an elliptic curve, G/R is a p-divisible group and*

$$\iota : E_0[p^\infty] \xrightarrow{\sim} G \otimes_R R_0$$

is an isomorphism of p-divisible groups over R_0. A morphism between two triples (E_0, G, ι) and (E_0', G', ι') is a pair (f_0, f) which satisfies the obvious compatibility condition. Then the functor $\mathrm{Ell}_R \to \mathcal{A}_R$ sending E/R to the triple $(E \otimes_R R_0, E[p^\infty], \mathrm{Id})$ is an equivalence of categories.

Now, we are interested in using these results to deform the point $\overline{x}$: $\mathrm{Spec}(\mathbb{F}_{p^2}) \to X_1(N)$ to a point, $x : \mathrm{Spec}(\mathbb{Z}_{p^2}) \to X_1(N)$. In particular, when we consider $X_1(N)$, we see that the stalk has a particularly easy form, as the modular curve is smooth over $\mathrm{Spec}(\mathbb{Z}_{p^2})$, and so its completed stalk at every point is a complete regular local ring of relative dimension 1, for which the following structure theorem holds.

Lemma 15. *Let $(A, \mathfrak{m})$ be a complete regular local ring of dimension 2, which is a $\mathbb{Z}_p$-algebra such that $p \notin \mathfrak{m}^2$. Then A is a formal power series ring over a field or over a complete p-ring, i.e. a complete DVR with maximal ideal generated by p. Moreover, in the equal characteristic case, A is isomorphic to a power series ring in n variables, while in the unequal characteristic case, it is isomorphic to a power series ring in $n-1$-variables.*

Proof. This proof comes from [Mat]. Let R be a coefficient ring of A, i.e. a complete Noetherian local subring of A with maximal ideal generated by p and such that $A = R + \mathfrak{m}$. We know it exists by the discussion following Theorem 29.2 in [Mat]. In the case of A being equal characteristic, we also know that R is a field and we can choose $x_1, \ldots, x_n$ to be a minimal set of generators of $\mathfrak{m}$. Then we get $A \cong R[[x_1, \ldots, x_n]]$ by Theorem 29.4 of [Mat]. In the case where A is unequal characteristic with residue characteristic p, R is a complete p-ring and we can choose $\{p, x_2, \ldots, x_n\}$ to be a minimal set of generators for $\mathfrak{m}$, since $p \in \mathfrak{m} - \mathfrak{m}^2$, where, by regularity, n coincides with the Krull dimension of A. Then $R[[x_2, \ldots, x_n]]$ is a power series ring in $n-1$-variables. $\square$

In our case, since the Witt vectors over $\mathbb{F}_{p^2}$ are given by $\mathbb{Z}_{p^2}$, which is the ring of integers in the degree 2 unramified extension of $\mathbb{Q}_p$, we get:

$$\widehat{\mathscr{O}_{X_1(N),\overline{x}}} \cong \mathbb{Z}_{p^2}[[Y]].$$

and so a deformation of $\overline{x}$ to $\mathbb{Z}_{p^2}$ is completely described via a continuous morphism $\mathbb{Z}_{p^2}[[Y]] \to \mathbb{Z}_{p^2}$, i.e. by the choice of an element inside $p\mathbb{Z}_{p^2}$.

Notice that, by definition, we can consider the formal scheme given by the p-adic completion of $X_1(N)$, which we denoted by $\mathfrak{X}$. Now, the point $\overline{x}$ determines uniquely a $\mathrm{Spf}(\mathbb{Z}_{p^2})$-point of $\mathfrak{X}$, in fact the completed stalk of $X_1(N)$ at $\overline{x}$ can be identified with the completion wrt its maximal ideal, of the stalk of $\mathfrak{X}$ at the point given by the image of $\overline{x}$. We then have a morphism:

$$x : \mathrm{Spf}(\mathbb{Z}_{p^2}) \to \mathrm{Spf}(\mathbb{Z}_{p^2}[[Y]]) \cong \mathrm{Spf}(\widehat{\mathscr{O}_{\mathfrak{X},\overline{x}}}) \to \mathrm{Spf}(\mathscr{O}_{\mathfrak{X},\overline{x}}) \to \mathfrak{X}$$

Up to choosing a small enough neighborhood $\mathrm{Spf}(A)$ of $\overline{x}$ in $\mathfrak{X}$, we can trivialize a lifting of the Hasse invariant over $\mathrm{Spf}(A)$, and so in particular, we can assume that inside $\mathbb{Z}_p[[Y^2]]$ there is a scalar which trivialize a lifting of the Hasse invariant, which we still denote $\widetilde{\mathrm{Ha}}$. Then the following result gives a more explicit characterization of the point x.

Lemma 16. $\widetilde{\mathrm{Ha}}$ *is congruent to* uY *modulo* p *inside* $\mathbb{Z}_{p^2}[[Y]]$*, where* u *is a unit in* $\mathbb{Z}_{p^2}[[Y]]$*.*

Proof. By Igusa's Theorem, we know that Hasse invariant has simple zeroes modulo p around a supersingular point, see Theorem 12.6.1 of [KM85]. This says that, modulo p, the reduction of the Hasse invariant gives a uniformizer of $\mathbb{F}_{p^2}[[Y]]$, which means that $x^*(\widetilde{\mathrm{Ha}}) = uY$ for some unit $u \in \mathbb{F}_{p^2}[[Y]]$. This says that, up to changing Y with uY, where u is now a unit inside $\mathbb{Z}_{p^2}[[Y]]$, Y is congruent to $x^*(\widetilde{\mathrm{Ha}})$ modulo p. $\square$

But then, up to a change of coordinates, the point x is uniquely determined by the image of the lifting of Hasse invariant through the morphism $\mathbb{Z}_{p^2}[[T]] \to \mathbb{Z}_{p^2}$. In particular, from now on, we write $\mathbb{Z}_{p^2}[[\widetilde{\mathrm{Ha}}]]$ for the ring $\mathbb{Z}_p[[Y]]$, where $\widetilde{\mathrm{Ha}}$ is in fact treated as a variable.

Now, we can choose the point x of $\mathfrak{X}$ as one of the points of $\mathfrak{X}$ lifting $\overline{x}$. In order to deal with families, we are in fact interested in B-points of $\mathfrak{X}_I$. Then, we can base change the point x, along the structure morphism $\mathrm{Spf}(B) \to \mathrm{Spf}(\mathbb{Z}_{p^2})$, getting a point:

$$x_I : \mathrm{Spf}(B) \to \mathfrak{X}_I$$

which is still entirely described by the image of the Hasse invariant inside B. Actually we want to work with points living inside $\mathfrak{X}_{r,I}$ for some $r \in \mathbb{N}$, so we have to change $\mathrm{Spf}(B)$ in order to get a point which maps into $\mathfrak{X}_{r,I}$. The idea is to use the following result:

Lemma 17. *Let* A *be an* I*-adically complete ring, and let* C *be an adic* A*-algebra. Then there is a bijection between the set of continuous morphisms* $g : A[[X]] \to C$ *and the ideal* $\sqrt{IC}$*.*

Proof. Clearly every morphism $g : A[[X]] \to C$ is determined by the value of $g(X)$. By continuity, $(I, X)^n \subseteq g^{-1}(IC)$ for some $n \geq 0$. But this implies that $g(X)^n \in IC$, and so $g(X) \in \sqrt{IC}$. The converse is clear. $\square$

Comparing with the notations of the Lemma, we have $A = C = B$, $I = (T)$, $\sqrt{I} = (T)$, so, looking at the equation which locally defines $\mathfrak{X}_{r,I}$, which is:

$$X\widetilde{\mathrm{Ha}}^{p^{r+1}} = T$$

we see that a good idea to factor the point x through some $\mathfrak{X}_{r,I}$ is to add to B some roots of T, as it is done in chapter 6 of [AIP]. So defining

$$B_{p^{-k}} = B[T^{\frac{1}{p^k}}] \tag{4.1}$$

we see that, by choosing the image of $\widetilde{\mathrm{Ha}}$ to be, up to unit, $T^{\frac{1}{p^{r+1}}}$, we have a well-defined point:

$$x_{r,I} : \mathrm{Spf}(B_{p^{-(r+1)}}) \to \mathfrak{X}_{r,I}.$$

We want to finish this section by giving a more explicit and almost obvious characterization of the operation of adding p^k-th roots of T.

Lemma 18. *Using now the notation the notation of* [AIP]*, the map* $B_{p^k I} \to B_{I,p^{-k}}$ *defined by sending* T *to* $T^{\frac{1}{p^k}}$ *is an isomorphism.*

Proof. First, it is well-defined, as the element $T^{p^{2+k}}u - p$ goes to

$$\left(T^{\frac{1}{p^k}}\right)^{p^{2+k}} u - p = T^2 u - p = 0$$

Then surjectivity and injectivity are obvious. □

4.2.2 Points of the Anticanonical Tower.

We start the construction of the map by first recalling a bit about the anticanonical tower of modular curves. The object we are considering is the T-adic formal scheme given by the inverse limit

$$\mathfrak{X}_{\infty,I} := \varprojlim_{\phi,r} \mathfrak{X}_{r,I} \tag{4.2}$$

which indeed defines a formal scheme as Frobenius morphisms are finite, and hence affine. If $\mathrm{Spf}(R) \subseteq \mathfrak{X}$ is an affine open, and $\mathrm{Spf}(R_r)$ is its preimage inside $\mathfrak{X}_{r,I}$, then the preimage inside $\mathfrak{X}_{\infty,I}$ is given by the T-adic completion of the direct limit $R_\infty = \varinjlim_r R_r$. Moreover, by definition, we may base change each universal elliptic curve $\mathscr{E}_r$ to the anticanonical tower $\mathfrak{X}_{\infty,I}$ via the universal map given by inverse limit. In this way we clearly get a universal system of elliptic curves $\{E_r\}$, where E_{r-1} is the image of E_r under the isogeny given by the lifting of Frobenius constructed in Proposition 3.3 of [AIP].

Over $\mathfrak{X}_{\infty,I}$, we see that an infinite Igusa tower is defined. In fact, over each $\mathfrak{X}_{r,I}$ the formal scheme $\mathfrak{IG}_{n,r,I}$ is defined as the normalization of the generic partial Igusa tower which parametrizes trivialization of the dual of the n-th level of the canonical subgroup, under suitable numerical conditions on n. Moreover, a Frobenius morphism is well-defined also at the level of partial Igusa towers, and all possible diagrams commute, hence, fixing n, one can construct the T-adic formal scheme

$$\mathfrak{IG}_{n,\infty,I} := \varprojlim_{r} \mathfrak{IG}_{n,r,I}.$$

Finally, it is possible to take the limit over n, getting finally the infinite Igusa tower

$$\mathfrak{IG}_{\infty,\infty,I} = \varprojlim_{n} \mathfrak{IG}_{n,\infty,I}.$$

According to our intent, we need a moduli interpretation of all these spaces, at least far from the cusps. This is not completely possible, working with the category of T-adically complete and separated B_I-algebra which are T-torsion free, while a moduli interpretation can be given if we add the hypothesis of choosing **normal** rings. This is not a big problem, since in the following, we will construct a normal ring over which we will evaluate the anticanonical and the Igusa tower.

Proposition 22. *Let R be a complete and separated B-algebra which is T-torsion free and normal. Then there is a bijection between $\mathfrak{X}_{\infty,I}(R) - \{\text{cusps}\}$ and the set of quadruples $(E, \psi_N, \eta, D_\infty)$, where*

- *i) E is an elliptic curve over $\mathrm{Spec}(R)$;*
- *ii) ψ_N is an N-th level structure over E;*
- *iii) $\eta \in \mathrm{H}^0(\mathrm{Spf}(R), \omega_E^{p^r(1-p)})$ is a section such that $\eta\widetilde{\mathrm{Ha}}^p = T$ modulo p^2;*
- *iv) D_∞ is a p-divisible subgroup of $E[p^\infty]$ of height 1 such that, called $E_n := E/D_n$, E_n admits a level n canonical subgroup and D_n splits generically the exact sequence of the p^n-torsion of E_n.*

Proof. Let us start with a point $\mathrm{Spf}(R) \to \mathfrak{X}_{\infty,I}$. By the definition, this object corresponds to a sequence:

$$\mathbf{E} = (\psi_N, (E_0, \eta_0), (E_1, \eta_1), (E_2, \eta_2) \ldots)$$

such that $(E_r, \eta_r, \psi_N) \in \mathfrak{X}_{r,I}(R)$, $E_r = E_{r+1}/H_1(E_{r+1})$ and η_r is the unique section such that $\eta_{r+1} = \phi^*(\eta_r)$, see Prop.3.3 of [AIP]. We want to point out that each E_r is really an elliptic curve over $\mathrm{Spec}(R)$. In fact, if we have a morphism $f : \mathrm{Spf}(R) \to \mathfrak{X}_{r,I}$, we get a formal elliptic curve by base changing the universal elliptic curve over $\mathfrak{X}_{r,I}$ along f. But now recall that a formal elliptic curve is a proper formal scheme, and so it is algebraizable by formal GAGA theorem. Hence f describes an elliptic curve over $\mathrm{Spec}(R)$. We then define the quadruple:

$$E = (E_0, \psi_N, \eta_0, D_\infty = \{D_n\}_{n\in\mathbb{N}}) \qquad \text{with } D_n = \ker((\phi^n)^{\mathrm{D}} : E_0 \to E_n).$$

We claim that D_n satisfies the property in the statement. Clearly, D_n has rank p^n, since it is the kernel of the dual n-th Frobenius isogeny. Then, by definition, the family of D_n's form a p-divisible group, as it is clear that each D_n is identified with the p-torsion of D_{n+1}. Moreover, $(\phi^n)^{\mathrm{D}}$ is an isogeny, hence $E_0/D_n \cong E_n$, hence E_0/D_n admits level n canonical subgroup. We have to show that D_n splits generically the exact sequence:

$$0 \to H_n(E_n) \to E_n[p^n] \to E_n[p^n]/H_n(E_n) \to 0$$

This is clear, as generically the quotient D_{2n}/D_n splits this exact sequence, since it realizes a subgroup of $E_n[p^n]$ which is identified with the quotient $E_n[p^n]/D_n$. But as the group D_∞ is p-divisible, we have the identification (which in fact also holds integrally)

$$D_n \cong \frac{D_{2n}}{D_n}$$

which says exactly that D_n splits the torsion sequence. So, starting with $\mathbf{E}$, we constructed the quadruple $(E_0, \psi_N, \eta_0, D_\infty)$.

Vice versa, if we have a quadruple $(E_0, \psi_N, \eta_0, D_\infty)$ as in the statement, we define

$$E_n := E_0/D_n \qquad \eta_n = \phi^*(\eta_0)$$

where Frobenius exists by the hypothesis on the quadruple, since E_n admits level n canonical subgroup. We have to show that the family $\mathbf{E} = (\psi_n, (E_0, \eta_0), (E_1, \eta_1), \ldots)$ is Frobenius compatible. First notice that the property of splitting the exact sequence of the p^n-torsion identifies generically D_n with the dual of $H_n(E_n)$. Hence, generically, the compatibility under Frobenius holds true. But now notice that generically for every n, we get an

$R\left[\frac{1}{T}\right]$ point x_n of $X(\Gamma_1(N)\cap\Gamma_0(p^n))$, which parametrizes elliptic curves with tame level N and the choice of a subgroup of order p^n. But, by hypothesis, R is normal, hence the morphism $\mathrm{Spec}\left(R\left[\frac{1}{T}\right]\right)\to X(\Gamma_1(N)\cap\Gamma_0(p^n))$ lifts to a morphism from $\mathrm{Spec}(R)$ to the integral model of $X(\Gamma_1(N)\cap\Gamma_0(p))$, which exists by [DR]. But then this morphism identifies D_n with the dual of the canonical subgroup of level n of E_n, hence we get the desired compatibility also integrally. $\square$

Now, this suggests a moduli interpretation of the infinite level Igusa tower.

Proposition 23. *Let R be as above. Then there is a bijection between $\mathfrak{IG}_{\infty,\infty,I}(R)$ over point distinct from the cusps, and the set of quintuples $(E,\psi_N,\eta,D_\infty,\beta)$, where $(E,\psi_N,\eta,D_\infty)\in\mathfrak{X}_{\infty,I}(R)$, and $\beta: T_pD_\infty\to\mathbb{Z}_p$ is a morphism of sheaves, which becomes an isomorphism generically.*

Proof. This is clear, since we identify the p-divisible group D_∞ with the p-divisible group given by the dual canonical subgroup, hence we get the complete trivialization by the very definition as each finite level $\mathfrak{IG}_{n,r,I}(R)$ gives a morphism $\mathbb{Z}/p^n\mathbb{Z}\to D_n$, which becomes an isomorphism generically. $\square$

Now we still have to change the ring we are using to construct the point. First of all we base change $B_{p^{-(r+1)}}$ (see 4.1 for the definition of this ring) along the natural inclusion of $\mathbb{Z}_{p^2}$ into $\mathbb{Z}_{p^2}[\zeta^{\frac{1}{p^\infty}}]$. This corresponds to choose a compatible set of p^n-th roots of unit. We tacitly assume this base change without keeping track of it in the notation. Now, we have a point $x_{r,I}:\mathrm{Spf}(B_{p^{-(r+1)}})\to\mathfrak{X}_{r,I}$, but we will need a point of the total Igusa tower, and we cannot be sure that such a point exists with coefficients in $B_{p^{-(r+1)}}$. In order to solve this problem, we introduce the following rings.

Definition 31. *We call $\mathbb{B}_x^{(s)}$ the ring given by the pullback of the point $x_{r,I}$ along the s-th power of Frobenius, i.e. $\mathbb{B}_x^{(s)}$ is the ring which makes the following diagram cartesian:*

$$\begin{tikzcd}
\mathrm{Spf}(\mathbb{B}_x^{(s)}) \arrow[r] \arrow[d] & \mathrm{Spf}(B_{p^{-(r+1)}}) \arrow[d, "x_{r,I}"] \\
\mathfrak{X}_{r+s,I} \arrow[r, "\phi^s"] & \mathfrak{X}_{r,I}
\end{tikzcd}$$

We remark that in the previous definition, we erased the dependence on the interval I, and we inserted the dependence on the starting point x, which is more important from what follows. We also point out that, according to the notation, we have the tautological equality $B_{p^{-(r+1)}} = \mathbb{B}_x^{(0)}$ This is the main ingredient to define a point of the Igusa tower. In fact, we can state the following:

Definition 32. *We call* $\mathbb{B}_{x,a}^{\mathrm{perf}}$ *the ring given by the* T*-adic completion of* $\varinjlim_s \mathbb{B}_x^{(s)}$. *We also call* $\mathbb{B}_x^{\mathrm{perf}}$ *the normalization of the base change of* $\mathbb{B}_{x,a}^{\mathrm{perf}}$ *along the morphism* $\mathfrak{IG}_{\infty,\infty,I} \to \mathfrak{X}_{\infty,I}$.

We used the a in the notation for anticanonical, since the ring $\mathbb{B}_{x,a}^{\mathrm{perf}}$ is constructed as a point of the anticanonical tower.

Remark. All these rings are well-defined since Frobenius morphisms are affine, and so their base changes are affine.

As we introduced a lot of notations, we want to fix all of them here.

- We denote $B = \mathbb{Z}_{p^2}[[T]]\left\langle \frac{p}{T^p} \right\rangle$;
- We denote $B_{p^{-k}} = B[T^{\frac{1}{p^k}}]$ for every natural integer k. In particular, we constructed a morphism $\mathrm{Spf}(B_{p^{-(r+1)}}) \to \mathfrak{X}_{r,I}$ associated with the supersingular point $\overline{x}$.
- We denote again $B_{p^{-k}}$ the base change of $B_{p^{-k}}$ to $\mathbb{Z}_p[\zeta_{p^\infty}]$ which is the p^n-th cyclotomic ring over $\mathbb{Z}_p$. As we assume to always work over it, the notation is not misleading.
- We denote $\mathbb{B}_x^{(s)}$ the ring of functions of the base change of the morphism $\mathrm{Spf}(B_{p^{-(r+1)}}) \to \mathfrak{X}_{r,I}$ along the s-th power of Frobenius isogeny $\phi^s : \mathfrak{X}_{r+s,I} \to \mathfrak{X}_{r,I}$.
- We denote by $\mathbb{B}_{x,a}^{\mathrm{perf}}$ the ring defined as the T adic completion of the direct limit of the $\mathbb{B}_x^{(s)}$. This ring depends on r, but we assume to work with a fixed r, hence the notation does not miss anything.
- We finally denote $\mathbb{B}_x^{\mathrm{perf}}$ the normalization of the ring of functions of the base change of $\mathrm{Spf}(\mathbb{B}_{x,a}^{\mathrm{perf}}) \to \mathfrak{X}_{\infty,I}$ along the morphism $\mathfrak{IG}_{\infty,\infty,I} \to \mathfrak{X}_{r,I}$.

Lemma 19. *Let R be a ring of characteristic p and let $R^{\mathrm{perf}} := \varinjlim_{\phi} R$ be the perfection of R, where the limit is computed along Frobenius. Then R is a perfect ring, i.e. Frobenius is an isomorphism.*

Proof. In fact, by definition, a direct limit of rings can be characterized as a set as

$$\varinjlim_{\phi} R = \bigsqcup R/\sim$$

where the equivalence relation identifies two elements in the following way:

$$x \sim y \Leftrightarrow \exists k, j \in \mathbb{N} \text{ such that } \phi^k(x) = \phi^j(y)$$

Then ϕ is clearly injective as, if $x \in R^{\mathrm{perf}}$ is such that $\phi(x) = 0$, then it means that $x \sim 0$ in the relation defining the direct limit, hence $x = 0$. Moreover, it is also surjective as any element in the direct system is equivalent to all of its images. $\square$

Lemma 20. *The ring $\mathbb{B}_x^{\mathrm{perf}}/\frac{p}{\mathrm{Ha}}\mathbb{B}_x^{\mathrm{perf}}$ contains all the p^n-th roots of T.*

Proof. In fact ϕ is the classical Frobenius morphism modulo $\frac{p}{\mathrm{Ha}}$ and the point $x_{r,I}$ is defined by mapping the Hasse invariant to a root of T. In particular, there is an injection $\mathbb{F}_p[T] \subseteq B_{p^{-k}}/\frac{p}{\mathrm{Ha}}B_{p^{-k}}$ and the map given by the pullback under Frobenius acts as the absolute Frobenius over the image of $\mathbb{F}_p[T]$. This, using Lemma 19 and the exactness of the direct limit, says that $\mathbb{B}_x^{\mathrm{perf}}$ contains all the p^n-th roots of T modulo $\frac{p}{\mathrm{Ha}}$. $\square$

Notice that we also know that Frobenius is finite and flat of rank p, hence we can conclude that, modulo $\frac{p}{\mathrm{Ha}}$ each $\mathbb{B}_x^{(s)}$ is generated over $\mathbb{B}_x^{(s-1)}$ by the p elements given by the p-th roots of the $p^{(s-1)}$-th roots of T.

Remark. These rings define in a natural way morphisms from $\mathbb{B}_{x,a}^{\mathrm{perf}}$ (resp. $\mathbb{B}_x^{\mathrm{perf}}$) to $\mathfrak{X}_{\infty,I}$ (resp. $\mathfrak{IG}_{\infty,\infty,I}$). Those morphism do define points which admit a moduli interpretation. In fact, all these rings are chosen to be normal, and this motivates the choice of working with normalizations instead of working with non normal rings.

Definition 33. *We finally denote by*

$$x_{\infty,I} : \mathrm{Spf}\left(\mathbb{B}_x^{\mathrm{perf}}\right) \to \mathfrak{IG}_{\infty,\infty,I}$$

the point of $\mathfrak{IG}_{\infty,\infty,I}$ coming from $x_{r,I}$ via all the previous pullbacks and normalization of $\mathbb{B}_x^{\mathrm{perf}}$ inside $\mathbb{B}_x^{\mathrm{perf}}\left[\frac{1}{T}\right]$.

The choice of a point with values in a normal ring allows to use the Proposition 23 to describe what the point is. Finally, we state the following moduli interpretation of a perfect modular form.

Proposition 24. *A perfect modular form* $f \in \mathrm{H}^0(\mathfrak{X}_{\infty,I}, \mathfrak{w}^{\mathrm{perf}})$ *is a function which associates to*

i) A T*-adically complete and separated* $\mathbb{B}_x^{\mathrm{perf}}$*-algebra* R *which is* T*-torsion free and normal.*

ii) A character κ *defined by the morphism* $\mathrm{Spf}(R) \to \mathfrak{W}_I^{\mathrm{cycl}}$*.*

iii) An elliptic curve $f : E \to \mathrm{Spf}(R)$ *equipped with a level* N*-structure* ψ_N*.*

iv) A section $\eta \in \mathrm{H}^0(\mathrm{Spf}(R), f^*(\omega^{p(1-p)}))$ *such that* $\eta\widehat{\mathrm{Ha}}^p = T \bmod p^2$*.*

v) A p*-divisible group* D_∞ *of height* 1 *as in Proposition 22.*

vi) A morphism $\beta : T_p(E) \to \mathbb{Z}_p a \oplus \mathbb{Z}_p b$ *which becomes an isomorphism generically.*

an element $f(E, \psi_N, \eta, D_\infty, \beta) \in R$ *such that*

$$f(E, \psi_N, \eta, D_\infty, \gamma\beta) = \kappa(\gamma)^{-1} f(E, \psi_N, \eta, D_\infty, \beta)$$

for every $\gamma \in \mathbb{Z}_p^\times$*, which acts over the quintuple by changing the generator of* $T_p(D_\infty)$*.*

4.3 The map ψ_x.

Now that we have a moduli interpretation of perfect modular forms, we can construct the function which should translate the situation from modular forms to continuous functions. The construction of the map ψ essentially depends on two maps, one is purely defined over continuous p-adic functions, and the other one is constructed in a real geometric fashion. The role of the first map will be clear later, when we will construct the $\mathfrak{U}_p$-operator which acts on p-adic functions.

4.3.1 A truncation morphism.

We consider here a map which truncates and reorder a p-adic number. The following Lemma is a triviality, but we need it to define the map.

Lemma 21. *Every element $\lambda \in \mathbb{Z}_p$ can be written uniquely in the form*

$$\lambda = \sum_{i=0}^{\infty} \lambda_i p^i$$

where $0 \leq \lambda_i \leq p-1$ for every i.

Proof. First, let us start with $\lambda \in \mathbb{Z}_p$ and $n \geq 1$. Since $\mathbb{Q}$ is dense inside $\mathbb{Q}_p$, we can find a rational number $\frac{a}{b}$ such that

$$\left|\lambda - \frac{a}{b}\right|_p \leq p^{-n} < 1$$

But notice that

$$\left|\frac{a}{b}\right| \leq \max\left\{|\lambda|, \left|\lambda - \frac{a}{b}\right|\right\} \leq 1$$

which says that $\frac{a}{b} \in \mathbb{Z}_p$. Moreover, we can always choose $\frac{a}{b}$ to be between 0 and $p^n - 1$. In particular, there is always a sequence of integers $\alpha_n \in \mathbb{Z}$ converging to λ which satisfy $0 \leq \alpha_n \leq p^n - 1$ and $\alpha_n \equiv \alpha_{n+1}$ modulo p^n. Now, all these integers can be written in base p, getting

$$\begin{aligned}
&\alpha_0 = b_0 \\
&\alpha_1 = b_0 + b_1 p \\
&\alpha_2 = b_0 + b_1 p + b_2 p^2 \\
&\ldots
\end{aligned}$$

where $0 \leq b_i \leq p-1$ for every i. But then we get

$$\lambda = b_0 + b_1 p + b_2 p^2 + \ldots + b_n p^n + \ldots$$

where the equality is clear since each reduction modulo p^n of the right-hand side is just α_n and a series converges if and only if the sequence of its partial sums converges. Finally, unicity is clear by elementary congruence theory. $\square$

Definition 34. *Let $\lambda \in \mathbb{Z}_p$, and write it in the unique form*

$$\lambda = \sum_{n=0}^{\infty} \lambda_i p^i$$

Define, for every $n \in \mathbb{N}$ the map

$$\sigma_n : \mathbb{Z}_p \to \mathbb{Z}_p$$

sending λ to $\sigma_n(\lambda) = \sum_{i=0}^{n-1} \lambda_{n-1-i} p^i$, with the convention that $\sigma_0(\lambda) = 0$.

Remark. We remark that σ_n is not an endomorphism of $\mathbb{Z}_p$, in fact it is not compatible as it is easy to show. Hence σ_n is just a function from $\mathbb{Z}_p$ to itself.

Lemma 22. *The function $\sigma_n : \mathbb{Z}_p \to \mathbb{Z}_p$ is continuous for every $n \in \mathbb{N}$.*

Proof. Clearly σ_0 is continuous, so let us put $n \geq 1$. We fix $\varepsilon > 0$, which we can assume to be a negative power of p, say $\varepsilon = p^{-k}$ for some $k \geq 0$. Let $\lambda = \sum_{i=0}^{\infty} \lambda_i p^i$ and $\mu = \sum_{i=0}^{\infty} \mu_i p^i$ be two p-adic integers such that

$$|\sigma_n(\lambda) - \sigma_n(\mu)| = \left| \sum_{i=0}^{n-1} (\lambda_{n-1-i} - \mu_{n-1-i}) p^i \right| < \varepsilon$$

Now,

$$\left| \sum_{i=0}^{n-1} (\lambda_{n-1-i} - \mu_{n-1-i}) p^i \right| = p^{-\bar{i}}$$

where $\bar{i}$ is the lowest integer between 0 and $n-1$ such that $\lambda_{n-1-i} - \mu_{n-1-i}$ is different from 0, if it exists. If such an $\bar{i}$ does not exist, then $\lambda_i = \mu_i$ in the expansion for every $0 \leq i \leq n$, which means that $\lambda \equiv \mu$ modulo p^n. This means that the ball

$$B(\lambda, p^{-n}) = \{z \in \mathbb{Z}_p \text{ such that } |z - \lambda| < p^{-n}\}$$

satisfies the continuity condition for every ε, hence σ_n is continuous. $\square$

4.3.2 A fundamental isogeny.

Now we define a geometric map.

Notation. From now on, we denote an R-point of $\mathfrak{IG}_{\infty,\infty,I}$, for R a T-adically complete and separated B-algebra which is T-torsion free and normal by the symbol $E_{a,b}$, meaning, via Proposition, that we are considering a quintuple $(E, \psi_N, \eta, D_\infty, \beta)$ with β defining the basis $\{a, b\}$ of the generic Tate module of E.

Definition 35. *Let $n \in \mathbb{N}$ be an integer, and let $\lambda \in \mathbb{Z}_p$ be a p-adic integers whose p-expansion has non zero digits only in the first n terms. Then define the map:*

$$\pi_{n,\lambda} : \mathfrak{IG}_{\infty,\infty,I}\left(\mathbb{B}_x^{\mathrm{perf}}\right) \to \mathfrak{IG}_{\infty,\infty,I}\left(\mathbb{B}_x^{\mathrm{perf}}\right)$$

sending an element $E_{a,b}$ to the element $E_{a,\frac{b+\lambda a}{p^n}}$.

In order to state a meaningful definition, we have to say what $E_{a,\frac{b+\lambda a}{p^n}}$ is. In fact, at the moment, the map is just defined via the generic inclusion of Tate modules:

$$T_p\left(E_{a,\frac{b+\lambda a}{p^n}}\right) \to T_p(E_{a,b})$$

induced by the matrix

$$A_{n,\lambda} = \begin{pmatrix} 1 & -\lambda \\ 0 & p^n \end{pmatrix}$$

We have to prove that this map induces an isogeny of elliptic curves of degree p^n.

Proposition 25. *Given $E_{a,b} = (E, \psi_n, \eta, D_\infty, \beta)$, for every $n \in \mathbb{N}$ and for every λ as in the previous definition, $\pi_{n,\lambda}$ induces an isogeny of elliptic curves over $\mathbb{B}_x^{\mathrm{perf}}$, which we call $\varpi_{n,\lambda}$. Moreover, ϖ has degree p^n. In particular, the map $\varpi_{n,\lambda}$ is described by the quotient of E modulo a subgroup $H_{n\lambda} \subseteq E[p^n]$ of order p^n, and $\pi_{n,\lambda}$ sends the quintuple $E_{a,b}$ to the quintuple $E_{a,\frac{b+\lambda a}{p^n}} = (E/H_{n,\lambda}, \psi_N, \varpi_{n,\lambda *}(\eta), \varpi_{n,\lambda *}D_\infty, \pi_{n,\lambda} \circ \beta)$.*

Proof. First, we can reduce to the case when $n = 1$. In fact, if $\lambda \in \mathbb{Z}_p$ has non zero digits only in the first n terms, then we can write it as $\lambda = \sum_{i=0}^n \lambda_i p^i$ for some $0 \leq \lambda_i \leq p-1$. But then the matrix which realizes the inclusion can be written as

$$A_{n,\lambda} = \prod_{i=0}^{n} \begin{pmatrix} 1 & -\lambda_i \\ 0 & p \end{pmatrix}$$

Hence, we prove that the morphism induces an isogeny at integral level for $n = 1$ and then we iterate the procedure.

Fix $0 \leq \lambda \leq p-1$ and consider the mod p reduction of the previous morphism:

$$\mathbb{F}_p a \oplus \mathbb{F}_p b \hookrightarrow \mathbb{F}_p a \oplus \mathbb{F}_p \frac{b+\lambda a}{p}$$

induced by the reduced matrix

$$\overline{A_{n,\lambda}} = \begin{pmatrix} 1 & -\overline{\lambda} \\ 0 & 0 \end{pmatrix}$$

In order to define an isogeny of degree p, we need to construct a finite flat subgroup scheme $H_\lambda \subseteq E[p]$ such that the map is described as the quotient over this subgroup. Clearly, the kernel of this morphism is the subgroup generated by the element $\overline{\lambda}a + b$, which induces generically a subgroup of the p-torsion, which we denote by $H_\lambda[\frac{1}{T}]$. Hence, generically the morphism which realizes the inclusion in the quotient over this subgroup. Notice that this subgroup, for every choice of λ is disjoint from the canonical subgroup and the quotient over $H_\lambda\left[\frac{1}{T}\right]$ preserves the canonical subgroup as the image of a is a itself. This subgroup corresponds to a morphism

$$\rho'_\lambda : \mathrm{Spec}\left(\mathbb{B}_x^{\mathrm{perf}}[1/T]\right) \to X(\Gamma_0(p) \cap \Gamma_1(N))_{\mathbb{B}_x^{\mathrm{perf}}[1/T],\mathrm{a}}$$

where $X(\Gamma_0(p) \cap \Gamma_1(N))_a$ is the modular curve over $\mathbb{B}_x^{\mathrm{perf}}[1/T]$ which parametrizes elliptic curve with level N-structure equipped with a subgroup of order p disjoint from the canonical subgroup. This modular curve admits a model over $\mathbb{B}_x^{\mathrm{perf}}$ by [DR], which is a smooth scheme, hence normal, denoted simply by $X(\Gamma_0(p) \cap \Gamma_1(N))$. In particular, this model parametrizes elliptic curves over $\mathbb{B}_x^{\mathrm{perf}}$ equipped with a level N structure and a level p subgroup disjoint from the canonical subgroup over $\mathbb{B}_x^{\mathrm{perf}}$. Moreover, since this model is normal and $\mathbb{B}_x^{\mathrm{perf}}$ is normal too, there is a morphism

$$\rho_\lambda : \mathrm{Spec}\left(\mathbb{B}_x^{\mathrm{perf}}\right) \to X(\Gamma_0(p) \cap \Gamma_1(N))_a$$

which defines a subgroup scheme H_λ of E over $\mathbb{B}_x^{\mathrm{perf}}$ which has trivial intersection with the canonical subgroup. Moreover, this subgroup has order p since $\mathrm{Spec}(\mathbb{B}_x^{\mathrm{perf}})$ is connected and the generic order is p. Hence we conclude that the isogeny we were looking for is the quotient map $E \to E/H_\lambda$. $\square$

Remark. The isogenies $\varpi_{n,\lambda}$ are all the possible isogenies of order p^n of E, except the Frobenius one. In fact the isogenies of order p^n correspond to finite flat subgroup schemes of $E[p^n]$ of order p^n, and letting λ vary between 0 and p^n we describe p^n subgroups of $E[p^n]$, which are all distinct, hence describe all possible subgroups except the canonical subgroup.

Hence, we constructed a family of isogenies.

Corollary 3. *The isogeny $\varpi_{n,\lambda}$ induces an isomorphism of level n-canonical subgroup H_n.*

Proof. This is implicit in the proof of Proposition 25 by the characterization of the subgroup which realizes the isogeny. □

4.3.3 The map.

We now define the following map:

Definition 36. *Let $x \in \mathfrak{IG}_{\infty,\infty,I}\left(\mathbb{B}_x^{\mathrm{perf}}\right)$ be a supersingular point corresponding to a quintuple $E_{a,b}$. For every $n \in \mathbb{N}$ we define the following function:*

$$\psi_{n,x} : \mathrm{H}^0(\mathfrak{X}_{\infty,I}, \mathfrak{w}^{\mathrm{perf}})\widehat{\otimes}_{B_I}\mathbb{B}_x^{\mathrm{perf}} \to \mathrm{Cont}\left(\mathbb{Z}_p, \mathbb{B}_x^{\mathrm{perf}}\right)$$

by the rule

$$\psi_{n,x}(f)(\lambda) = \pi^*_{n,\sigma_n(\lambda)}(f)(E_{a,b})$$

for every $\lambda \in \mathbb{Z}_p$.

We now denote all the spaces of modular forms by $H^0(\mathfrak{X}_{\infty,I}, \mathfrak{w}^{\mathrm{perf}})$ leaving implicit the base change to $\mathbb{B}_x^{\mathrm{perf}}$. More explicitly given a modular form $f \in \mathrm{H}^0(\mathfrak{X}_{\infty,I}, \mathfrak{w}^{\mathrm{perf}})$ and for every $\lambda \in \mathbb{Z}_p$:

$$\psi_{n,x}(f)(\lambda) = f\left(E_{a,\frac{b+\sigma_n(\lambda)a}{p^n}}\right)$$

A priori, it is not clear why such a map should define a continuous function from $\mathbb{Z}_p$ to $\mathbb{B}_x^{\mathrm{perf}}$, but notice that the truncation given by σ_n tells precisely that if two p-adic integers are equal modulo p^n, then their images under $\psi_{n,x}(f)$ are the same for every f, which means that $\psi_{n,x}(f)$ is continuous for every n.

Moreover, recall that there is a canonical way to see modular forms over $\mathfrak{X}_{r,I}$ inside $\mathrm{H}^0(\mathfrak{X}_{\infty,I}, \mathfrak{w}^{\mathrm{perf}})$. In fact, by Theorem 6.4 of [AIP], we have an isomorphism of invertible sheaves:

$$h_r^*(\mathfrak{w}_{r,I}) \cong \mathfrak{w}_I^{\mathrm{perf}}$$

where $\mathfrak{w}_{r,I}$ is the sheaf of modular forms over $\mathfrak{X}_{r,I}$ and $h_r : \mathfrak{X}_{\infty,I} \to \mathfrak{X}_{r,I}$ is the canonical projection given by the inverse limit. This implies that there exists a map

$$h_r^* : \mathrm{H}^0(\mathfrak{X}_{r,I}, \mathfrak{w}_{r,I}) \to \mathrm{H}^0(\mathfrak{X}_{\infty,I}, \mathfrak{w}^{\mathrm{perf}}).$$

This map is clearly injective, so we will identify modular forms over $\mathfrak{X}_{r,I}$ with a submodule of the module of perfect modular forms. In particular, the following result holds:

Proposition 26. *For every $n \in \mathbb{N}$ and for every $f \in \mathrm{H}^0(\mathfrak{X}_{r,I}, \mathfrak{w}_{r,I})$, the map $\psi_{n,x}(f)$ factors as a function*

$$\psi_{n,x}(f) : \mathbb{Z}/p^n\mathbb{Z} \to B^{(n)}_{I,p^{-(r+1)}}$$

Proof. Clearly the map $\psi_{n,x}(f)$ factors through the reduction $\mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$, as the truncation induced by σ_n tells exactly that two p-adic integers which are congruent modulo p^n have the same image under $\psi_{n,x}(f)$. Moreover, if we start with $f \in \mathrm{H}^0(\mathfrak{X}_{r,I}, \mathfrak{w}_{r,I})$, we compute, for every $\lambda \in \mathbb{Z}_p$:

$$\begin{aligned}
\psi_{n,x}(f)(\lambda) &= \psi_{n,x}(h_r^* f)(\lambda) = \\
&= \pi^*_{n,\sigma_n(\lambda)}(h_r^* f)(E_{a,b}) = \\
&= \pi^*_{n,\sigma_n(\lambda)}(h_{r+n}^*((\phi^n)^* f))(E_{a,b}) = \\
&= (h^*_{r+n}(\phi^n)^*(f)) \left(E_{a,\frac{b+\sigma_n(\lambda)a}{p^n}}\right) = \\
&= (\phi^n)^*(f) \left(E_{a_n,\left(\frac{b+\sigma_n(\lambda)a}{p^n}\right)_n}\right)
\end{aligned}$$

where we used Proposition 6.7 of [AIP] for the last equality, and we denoted by $E_{a_n,\left(\frac{b+\sigma_n(\lambda)a}{p^n}\right)_n}$ the elliptic curve $E_{a,\frac{b+\sigma_n(\lambda)a}{p^n}}$ where we forget the terms in the trivialization of Tate module greater than n. Now, this means exactly that the effect of computing $\psi_{n,x}(f)$ is to lift the modular form f to $\mathrm{H}^0(\mathfrak{X}_{r+n,I}, \mathfrak{w}_{r+n,I})$ and to compute its value on a truncated trivialization of the Tate module. This, by the construction of $\mathbb{B}_x^{\mathrm{perf}}$ as an inverse limit along Frobenius says exactly that the value of the modular form $(\phi^n)^*(f)$ belongs to $\mathbb{B}^{(n)}_{x,p^{(-r+1)}}$, which is the thesis. $\square$

We are now ready to state the main definition of this work, which is the definition of the map which should realize the correspondence.

Definition 37. *Let $r \geq 5$ be fixed and let $x \in \mathfrak{IG}_{\infty,\infty,I}\left(\mathbb{B}_x^{\text{perf}}\right)$ be a supersingular point corresponding to a quintuple $E_{a,b}$. We define the morphism:*

$$\psi_x : \mathrm{H}^0(\mathfrak{X}_{r,I}, \mathfrak{w}_{r,I})\widehat{\otimes}_{B_I}\mathbb{B}_x^{\text{perf}} \to \mathrm{Cont}\left(\mathbb{Z}_p, \mathbb{B}_x^{\text{perf}}\right)$$

via the formula

$$\psi_x(f)(\lambda) = \sum_{n=0}^{\infty} \psi_{n,x}(f)(\lambda) T^{\frac{n}{p^{r+1}}}$$

Remark. We want to point out that a priori the image of ψ_x does not live inside $\mathbb{B}_x^{\text{perf}}$. In fact, we know that each $\psi_{n,x}(f)$ maps into $B^{(n)}_{p^{-(r+1)}}$, and we also know that $\mathbb{B}^{(n)}_{x,p^{-(r+1)}}$ admits a canonical map to $\mathbb{B}_x^{\text{perf}}$ which is given by the direct limit. Hence we see the image of $\psi_{n,x}(f)(\lambda)$ inside $\mathbb{B}_x^{\text{perf}}$ via this map.

Lemma 23. *The map ψ_x is well-defined, i.e. for every $f \in \mathrm{H}^0(\mathfrak{X}_{r,I}, \mathfrak{w}_{r,I})$, $\psi_x(f)$ defines a continuous function from $\mathbb{Z}_p$ to $\mathbb{B}_x^{\text{perf}}$.*

Proof. We have to prove that, for every $f \in \mathrm{H}^0(\mathfrak{X}_{r,I}, \mathfrak{w}_{r,I})$, the function $\psi(f) : \mathbb{Z}_p \to \mathbb{B}_x^{\text{perf}}$ is continuous. This means that for every $\lambda_0 \in \mathbb{Z}_p$ and for every $N \in \mathbb{N}$, we can always choose a sufficiently small ball around λ_0 such that the image of this ball is contained inside $T^N\mathbb{B}_x^{\text{perf}}$. So for fixed N, choose the ball $B(\lambda_0, p^{-Np^{r+1}})$. Then the image only depends on $\psi_h(f)(\lambda)$ for $0 \leq h \leq Np^{r+1}$. But we know from Proposition 26 that $\psi_h(f)$ factors through $\mathbb{Z}/p^h\mathbb{Z}$, hence we are done. $\square$

4.3.4 The map Ψ.

What we have done till now clearly works for any supersingular point of the modular curve. In particular, we can consider a version of the map ψ which sums over all the possible supersingular points. First of all we notice that this sum is well-defined as:

Proposition 27. *For every prime p there is only a finite number of supersingular elliptic curves over $\overline{\mathbb{F}}_p$.*

Proof. This is clear thanks to Proposition 20. In fact we know that over $\overline{\mathbb{F}}_p$ elliptic curves are completely characterized by their j-invariant and, since the j-invariant of an elliptic curve must be an element of $\mathbb{F}_{p^2}$, we know that there are only finite possible supersingular j-invariants. $\square$

This easy result allows to state the last definition:

Definition 38. *Let $\{x_i\}_{i\in \mathrm{SS}}$ be a set of points of $\mathfrak{IG}_{\infty,\infty,I}$ which lift different supersingular points modulo p. Call $\mathbb{B}^{\mathrm{perf}} := \bigoplus_{x\in\mathrm{SS}} \mathbb{B}_x^{\mathrm{perf}}$. Then for every $r \geq 3$ we define the following map*

$$\Psi : \mathrm{H}^0(\mathfrak{X}_{r,I}, \mathfrak{w}_{r,I})\widehat{\otimes}_B \mathbb{B}^{\mathrm{perf}} \to \bigoplus_{i\in\mathrm{SS}} \mathrm{Cont}(\mathbb{Z}_p, \mathbb{B}_x^{\mathrm{perf}})$$

as the function sending a modular form f to the sum

$$\Psi(f) := \sum_{x\in\mathrm{SS}} \psi_x(f)$$

where each ψ_x is constructed as in Definition 37, associated to the corresponding supersingular point x.

In particular, the map Ψ, which is defined as the finite direct sum of the maps ψ_x takes into account the behaviour of the modular form f at every supersingular point of the given modular curve. This definition is strictly related to the definition of the maps realizing the first instance of Jacquet-Langlands correspondence in [Se96]. Moreover, this map realizes a kind of translation between the world of modular forms and the world of p-adic continuous functions, exactly in the same way as in [LWX]. We just want to point out here that the map Ψ is not the zero map, since modular forms are global sections of an invertible sheaf, and if we reduce each component ψ_x of Ψ modulo a suitable root of T, we see that, up to changing the starting point, the reduction cannot be zero, as we are working with overconvergent modular forms.

4.4 The U_p operator.

In this section we study the U_p-operator. First we give a description via correspondences which allows to split its effect on modular forms into the sum of p elementary operators. In the second part we compute the action of $\mathfrak{U}_p$ on the image of Ψ, where $\mathfrak{U}_p$ is the translation of U_p under the map Ψ itself. In this way we get, like in [LWX] an operator acting on spaces of p-adic continuous functions, with a very explicit description.

4.4.1 Geometry of the U_p operator.

Following Section 6.9 of [AIP], we see that the U_p operator acts on modular forms via the correspondence:

$$\begin{tikzcd} & \mathfrak{X}_{r+1,I} \arrow[dl, "i"'] \arrow[dr, "\phi"] & \\ \mathfrak{X}_{r,I} & & \mathfrak{X}_{r,I} \end{tikzcd}$$

and its action on global sections can be described via the notion of Tate trace introduced in Corollary 6.3 of [AIP] as

$$U_p : \mathrm{H}^0(\mathfrak{X}_{r,I}, \mathfrak{w}_{r,I}) \xrightarrow{i^*} \mathrm{H}^0(\mathfrak{X}_{r,I}, i^*\mathfrak{w}_{r,I}) \cong \mathrm{H}^0(\mathfrak{X}_{r+1,I}, \phi^*\mathfrak{w}_{r,I}) \xrightarrow{\phi_*}$$
$$\xrightarrow{\phi_*} \mathrm{H}^0(\mathfrak{X}_r, \phi_*\phi^*\mathfrak{w}_{r,I}) \xrightarrow{\frac{1}{p}\mathrm{Tr}_\phi} \mathrm{H}^0(\mathfrak{X}_{r,I}, \mathfrak{w}_{r,I})$$

Proposition 28. *The U_p operator can be written as a sum of p operators u_μ, where each u_μ acts on $f \in \mathrm{H}^0(\mathfrak{X}_{r,I}, \mathfrak{w}_{r,I})$ via the formula*

$$u_\mu(f)(E_{a,b}) = f(E_{a,\frac{b+\mu a}{p}})$$

Proof. By Proposition 3.3 of [AIP], we have the following cartesian diagram

$$\begin{tikzcd} \mathscr{E}_{r+1}/H_1(\mathscr{E}_{r+1}) \arrow[r] \arrow[d] & \mathscr{E}_r \arrow[d] \\ \mathfrak{X}_{r+1,I} \arrow[r, "\phi"'] & \mathfrak{X}_{r,I} \end{tikzcd}$$

When we consider a point $x \in \mathfrak{X}_{\infty,I}(\mathbb{B}^{\mathrm{perf}}_{x,a})$, it then defines a $\mathbb{B}^{\mathrm{perf}}_{x,a}$-point in $\mathfrak{X}_{r,I}$ simply forgetting part of the p-divisible group associated to x. This says that in order to compute $\phi^*(x)$ we need to compute all possible elliptic curves (in fact they must be p as Frobenius has degree p) in $\mathfrak{IG}_{\infty,\infty,I}(\mathbb{B}^{\mathrm{perf}}_x)$ whose image via Frobenius is the given point x. Working as in Proposition 25, we can characterize these elliptic curves by computing the action of Frobenius over the trivialization of Tate module. In particular, if we denote E_μ one of the p preimages of E, we see that the matrix representing Frobenius must be:

$$A_\mu = \begin{pmatrix} p & \mu \\ 0 & 1 \end{pmatrix}$$

where $\mu \in \{0, \ldots, p-1\}$. In fact the mod p reduction of A_μ admits precisely the vector space over $\mathbb{F}_p$ generated by the image of a as kernel. This says that the points of $\mathfrak{IG}_{\infty,\infty,I}(\mathbb{B}_x^{\mathrm{perf}})$ whose image via Frobenius is a given curve $E_{a,b}$ are all the curves $E_{\frac{a}{p}, b-\mu p \frac{a}{p}}$, hence we can write:

$$\phi^*(E_{a,b}) = \bigsqcup_{\mu=0}^{p-1} E_{\frac{a}{p}, b-\mu p \frac{a}{p}}$$

Now, recalling that the Trace map acts on a modular form simply as a sum of the modular form evaluated in all the preimages of a given elliptic curve, we get:

$$U_p(f)(E_{a,b}) = \frac{1}{p}\mathrm{Tr}_\phi(\phi^*(f))(E_{a,b}) = \frac{1}{p}\sum_{\mu=0}^{p-1} f\left(E_{a,\frac{b+\mu a}{p}}\right)$$

Now, simply define

$$u_\mu(f)(E_{a,b}) := f\left(E_{a,\frac{b+\mu a}{p}}\right)$$

Then

$$U_p = \frac{1}{p}\sum_{\mu=0}^{p-1} u_\mu$$

□

4.4.2 Numerology of the U_p operator.

By the previous section, we know that the U_p operator splits in a sum of p operators $U_p = \sum_{\mu=0}^{p-1} u_\mu$. We now want to investigate the action of each elementary operator u_μ under the map ψ. In particular, we define the operator:

Definition 39. *Given $\mu \in \{0, \ldots, p-1\}$, we call $\mathfrak{u}_\mu$ the operator acting on $\psi_x(\mathrm{H}^0(\mathfrak{X}_{r,I}, \mathfrak{w}_{r,I})\widehat{\otimes}_B \mathbb{B}_x^{\mathrm{perf}})$, defined by the rule:*

$$\mathfrak{u}_\mu(\psi_x(f)) = \psi_x(u_\mu(f)).$$

Theorem 12. *We have:*

$$\mathfrak{u}_\mu(g)(\lambda) = \frac{1}{T}\left(g(\mu + p\lambda) - k_g\right).$$

where k_g is a constant depending only on g.

Proof. In fact we have:

$$\begin{aligned}
\psi_{n,\lambda}(u_\mu(f)) &= \pi^*_{n,\sigma_n(\lambda)}(u_\mu(f))(E_{a,b}) = \\
&= u_\mu(f)\left(E_{a,\frac{b+\sigma_n(\lambda)a}{p^n}}\right) = \\
&= f\left(E_{a,\frac{\frac{b+\sigma_n(\lambda)a}{p^n}+\mu a}{p}}\right) = \\
&= f\left(E_{a,\frac{b+\sigma_n(\lambda)a+p^n\mu a}{p^{n+1}}}\right) = \\
&= f\left(E_{a,\frac{b+\sigma_{n+1}(\mu+p\lambda)a}{p^{n+1}}}\right) = \\
&= \pi^*_{n,\sigma_{n+1}(\mu+p\lambda)}(E_{a,b}) = \\
&= \psi_{n+1,\mu+p\lambda}(f)
\end{aligned}$$

where the fifth equality holds true as if $\lambda = \sum_{i=0}^{\infty}\lambda_i p^i$, then

$$p\lambda = \sum_{i=0}^{\infty}\lambda_i p^{i+1}$$

from which

$$\begin{aligned}
\sigma_{n+1}(\mu + p\lambda) &= \sigma_{n+1}(\mu + \sum_{i=0}^{\infty}\lambda_i p^{i+1}) = \\
&= \lambda_{n-1} + \lambda_{n-2}p + \lambda_{n-3}p^2 + \ldots + \lambda_0 p^{n-1} + \mu p^n = \\
&= \sigma_n(\lambda) + p^n\mu
\end{aligned}$$

which proves the equality. But then, denoting $\mathfrak{u}_\mu$ the operator acting on

$\mathrm{Cont}(\mathbb{Z}_p, B_I^{\mathrm{perf}})$ corresponding to u_μ, we get:

$$\begin{aligned}\mathfrak{u}_\mu(\psi(f))(\lambda) &= \sum_{n=0}^{\infty} \mathfrak{u}_\mu(\psi_{n,\lambda}(f))T^n = \\ &= \sum_{n=0}^{\infty} \psi_{n+1,\mu+p\lambda}(f)T^n = \\ &= \sum_{n=1}^{\infty} \psi_{n,\mu+p\lambda}(f)T^{n-1} = \\ &= \frac{1}{T}\left(\psi(f)(\mu+p\lambda) - \psi_0(f)(\lambda)\right)\end{aligned}$$

We point out that the term $\psi_0(f)(\lambda)$ in fact does not depend on λ. In fact we know that by definition $\sigma_0(\lambda) = 0$, hence the term $\psi_0(\lambda)$ is simply the valuation of f on the point $E_{a,b}$, which only depends on f, since the point is fixed. Hence, denoting $k_f := \psi_0(f)(\lambda)$ we end the proof. $\square$

Corollary 4. *The action of the U_p operator corresponds to the action of the $\mathfrak{U}_p$ operator on $\mathrm{Cont}(\mathbb{Z}_p, \mathbb{B}_x^{\mathrm{perf}})$ described by the formula*

$$(\mathfrak{U}_p(g))(\lambda) = \sum_{\mu=0}^{p-1} \frac{1}{T}\left(g(\mu+p\lambda) - k_g\right)$$

where k_g is a constant depending on g.

In this way, we proved our main result, which is the following:

Theorem 13. *Let $X/\mathbb{Z}_p$ be the compactified modular curve over $\mathbb{Z}_p$ of tame level $\Gamma_1(N)$. Then there exist, for r large enough, and for every supersingular point x of the special fiber of X, a T-adically complete, separated and norm decreasing $\mathbb{Z}_p$-algebra $\mathbb{B}_x^{\mathrm{perf}}$ and an homomorphism of $\mathbb{B}^{\mathrm{perf}} := \bigoplus_{x\in \mathrm{SS}} \mathbb{B}_x^{\mathrm{perf}}$-orthonormalizable Banach modules*

$$\Psi : \mathrm{H}^0(\mathfrak{X}_r, \mathfrak{w}_r)\widehat{\otimes}_B \mathbb{B}^{\mathrm{perf}} \to \mathrm{Cont}(\mathbb{Z}_p, \mathbb{B}^{\mathrm{perf}})$$

where SS is the set of supersingular points of the special fiber of X. Moreover, the operator $\mathfrak{U}_p$, associated to U_p via the map Ψ which acts on the image of the map Ψ, splits into a sum of $p-1$ operators, and its action over a function g which belongs to the image of ψ_x is described by the following rule:

$$\mathfrak{U}_p(g)(\lambda) = \sum_{\mu=0}^{p-1}(g(p\lambda+\mu) + k_g)$$

where k_g is a constant depending only on g.

Proposition 29. *Let $\mathfrak{U}_p$ be the operator acting on the image of the map Ψ_x. Then $\mathfrak{U}_p$ splits into a sum of $p-1$ operators, and its action over a function g which belongs to the image of ψ_x is the following:*

$$\mathfrak{U}_p(g)(\lambda) = \sum_{\mu=0}^{p-1}(g(p\lambda + \mu) + k_g)$$

where k_g is a constant depending only on g.

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