

## ON THE REGULARIZATION OF THE INVERSE CONDUCTIVITY PROBLEM WITH DISCONTINUOUS CONDUCTIVITIES

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**ABSTRACT.** We consider the regularization of the inverse conductivity problem with discontinuous conductivities, like for example the so-called inclusion problem. We theoretically validate the use of some of the most widely adopted regularization operators, like for instance total variation and the Mumford-Shah functional, by proving a convergence result for the solutions to the regularized minimum problems.

**1. Introduction.** A conducting body is contained in a bounded domain  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 2$ , with Lipschitz boundary  $\partial\Omega$ . Let the conductivity of the body be given by  $\sigma = \sigma(x)$ ,  $x \in \Omega$ , such that  $\sigma \in L^\infty(\Omega)$  and, for some constant  $\lambda$ ,  $0 < \lambda < 1$ ,

$$0 < \lambda \leq \sigma(x) \leq \lambda^{-1} \quad \text{for a.e. } x \in \Omega.$$

If we prescribe a current density  $f$  on the boundary, where  $f \in L^2(\partial\Omega)$  with zero mean, then the electrostatic potential  $u$  in  $\Omega$  is the solution to the Neumann boundary value problem

$$(1.1) \quad \begin{cases} \operatorname{div}(\sigma \nabla u) = 0 & \text{in } \Omega \\ \sigma \nabla u \cdot \nu = f & \text{on } \partial\Omega \end{cases}$$

where  $\nu$  denotes the exterior unit normal. If we normalize  $u$  in such a way that  $u$  has zero mean on  $\partial\Omega$ , then we have existence and uniqueness of the solution. We define the *Neumann-to-Dirichlet map* associated to  $\sigma$  the operator  $\Lambda(\sigma) : {}_0L^2(\partial\Omega) \rightarrow {}_0L^2(\partial\Omega)$  such that, for any  $f \in {}_0L^2(\partial\Omega)$ ,  $\Lambda(\sigma)(f) = u|_{\partial\Omega}$ ,  $u$  solution to (1.1). Here  ${}_0L^2(\partial\Omega)$  denotes the space of  $L^2(\partial\Omega)$  functions with zero mean. We observe that  $\Lambda(\sigma)$  is a linear and bounded operator.

The inverse conductivity problem is the following. Can we determine the conductivity  $\sigma$  from the knowledge of its corresponding Neumann-to-Dirichlet map  $\Lambda(\sigma)$ ? We note that  $\Lambda(\sigma)$  can be obtained, at least in an approximate way, by performing current and voltage measurements at the boundary of our body. We refer to [25] for more realistic electrode measurements models, whose numerical investigation has been treated for instance in [20].

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In some interesting applications, the conductivity  $\sigma$  might present discontinuities. For example this is the case for the determination of *inclusions* in a conducting body. Namely, we assume that  $\sigma = k_0 + \sum_{i=1}^n (k_i - k_0)\chi_{D_i}$ , where  $k_i$ ,  $i = 0, \dots, n$ , are positive constants,  $D_i$ ,  $i = 1, \dots, n$ , are domains contained in  $\Omega$  which are pairwise internally disjoint. For any  $i = 1, \dots, n$ , the set  $D_i$  is an inclusion and  $\chi_{D_i}$  is its characteristic function. The background conductivity  $k_0$  and the conductivities of the inclusions,  $k_i$ ,  $i = 1, \dots, n$ , may be known or not. Of crucial importance in these applications is the determination of the boundaries of the inclusions. In two dimensions, uniqueness holds due to a recent result by Astala and Päiväranta, [6]. In fact they completely solved the uniqueness issue for the inverse conductivity problem in two dimensions by proving it for  $L^\infty$  conductivities, therefore also for discontinuous conductivities and the inclusion case. In three and higher dimensions, a uniqueness result for the inclusion problem may be found in [18]. We wish to mention that optimal stability estimates for the inclusion problem have been obtained in [4]. Optimality of these estimates and severely ill-posedness of the inclusion problem has been shown in [14]. We also note that there exist other uniqueness results concerning discontinuous conductivities, for example the case of piecewise analytic conductivities, with respect to a piecewise analytic partition of the domain, has been treated in [19].

Let us assume that the conductivity to be reconstructed is  $\sigma_0$  and that its Neumann-to-Dirichlet map is  $\Lambda_0 = \Lambda(\sigma_0)$ . First of all, the knowledge of the Neumann-to-Dirichlet map involves some measurements, therefore the actual data which are available are only a perturbed Neumann-to-Dirichlet map  $\Lambda_\varepsilon$ , where  $\varepsilon > 0$  is a parameter denoting the noise level, that is  $\|\Lambda_\varepsilon - \Lambda_0\| \leq \varepsilon$ . The facts that the data are noisy and that the problem is severely ill-posed have to be taken into account in order to reconstruct numerically the conductivity  $\sigma_0$  from its perturbed Neumann-to-Dirichlet map in a reasonably stable way. Following the pioneering ideas developed by Tikhonov in the 1960's, usually this is done through a regularization procedure. For a detailed account on regularization and its applications we refer to the book by Engl, Hanke and Neubauer, [16].

Roughly speaking, instead of solving a classical least squares problem in order to fit the data, we solve the following regularized minimum problem, in a suitable set  $X$  of admissible conductivities,

$$(1.2) \quad \min_{\sigma \in X} \|\Lambda(\sigma) - \Lambda_\varepsilon\|^2 + a(\varepsilon)R(\sigma)$$

where  $R$  is a so-called *regularization operator* (usually a norm or a seminorm) and the positive coefficient  $a(\varepsilon)$  is the *regularization coefficient*.

A correct choice of the regularization operator and of its coefficient should guarantee that (1.2) admits a solution, that is there exists a minimizer  $\sigma_\varepsilon$ , for any  $\varepsilon > 0$ , and that, as  $\varepsilon \rightarrow 0^+$ ,  $\sigma_\varepsilon$  converges, in a suitable norm, to the looked for conductivity  $\sigma_0$ . The minimizer  $\sigma_\varepsilon$ ,  $\varepsilon > 0$ , is usually referred to as a *regularized solution*.

In Section 3 we shall restate and prove, using  $\Gamma$ -convergence terminology and techniques, some classical results on the regularization of nonlinear operators. We remark that we state our results for metric spaces. Namely, in Corollary 3.5, we shall show that, setting  $a(\varepsilon) = \tilde{a}\varepsilon^\beta$  with constants  $\tilde{a} > 0$  and  $0 < \beta \leq 2$ , existence and convergence to  $\sigma_0$  of regularized solutions are achieved provided three conditions are satisfied. For a suitable metric on the set of admissible conductivities  $X$ , we require first that the map  $\sigma \rightarrow \Lambda(\sigma)$  is continuous. Second, that  $R$  is lower semicontinuous and the set  $\{\sigma : R(\sigma) \leq C\}$  is compact, for any  $C > 0$ . Finally, uniqueness for

the inverse problem should hold on the set  $\{\sigma : R(\sigma) < +\infty\}$ . Clearly we need to assume also that  $R(\sigma_0)$  is finite.

Moreover, in Theorem 3.4, we observe that, even when uniqueness is not guaranteed, if  $0 < \beta < \alpha$  then the first two conditions give us compactness properties of the family of regularized solutions and convergence, up to subsequences, to a conductivity  $\tilde{\sigma}$  such that  $\Lambda(\tilde{\sigma}) = \Lambda(\sigma_0) = \Lambda_0$  and  $\tilde{\sigma}$  minimizes  $R$  among all conductivities in  $X$  whose corresponding Neumann-to-Dirichlet maps coincide with  $\Lambda_0$ . Therefore the regularization procedure selects conductivities which fit the data and with minimal value of  $R$ .

For what concerns the inverse conductivity problem with discontinuous conductivities, for example the inclusion problem, a careful choice of the metric on  $X$  and of the regularization operator has to be made. The metric usually used for sets of conductivities, that is the one induced by the  $L^\infty$  norm, which guarantees continuity of the map  $\sigma \rightarrow \Lambda(\sigma)$ , is not suited to treat discontinuous conductivities. In fact, two inclusions may have a constant positive distance in the  $L^\infty$  norm no matter how close they are. We shall prove, Theorem 4.2 and Corollary 4.3, that the map  $\sigma \rightarrow \Lambda(\sigma)$  is continuous also with respect to the metric induced by the  $L^1$  norm, which is much better suited for discontinuous conductivities. The proof of the stability result relies on the higher integrability properties of the gradients of solutions to elliptic equations, which is a consequence of a classical result by Meyers, [21].

As a regularization operator, various options have been considered in the literature for these kinds of inverse problems. The regularization operators and the numerical implementations are often borrowed from corresponding techniques developed in imaging problems. The efficiency of the reconstruction method is usually validated by numerical experiments. For example, as a regularization operator  $R$  we may choose the total variation of  $\sigma$ . Dobson and Santosa, [15], treated the total variation regularization with an implementation through a discretized problem. Later on, [10, 12], Chan and its collaborators used the total variation regularization in connection with level set methods. Another possible choice of  $R$  is the so-called Mumford-Shah functional, developed in [22] for image segmentation problems. The Mumford-Shah functional has been used in [24] with an implementation exploiting approximation of the Mumford-Shah functional with simpler functionals defined on sets of smooth functions.

Both these choices satisfy the second requirement of our abstract result, that is the assumptions on  $R$ , with respect to the  $L^1$  norm. Therefore we may conclude that the use of these regularization operators is validated also through a convergence result, which we shall state in Theorem 4.6.

Let us note that the total variation regularization operator (along with some of its variants) has been extensively studied also for the regularization of linear ill-posed problems when nonsmooth solutions are looked for. We mention the papers by Acar and Vogel, [1], and by Chavent and Kunisch, [11], and the work by Vasin, see his review papers [27, 28] and the references therein.

We finally wish to mention that higher integrability of gradients of solutions and  $\Gamma$ -convergence techniques have already been used, although in a different way, to prove convergence of a regularization technique for another inverse problem involving discontinuous functions, namely the inverse crack problem, see [23].

The plan of the paper is the following. After a section containing some notation and preliminaries, Section 2, we consider an abstract approach to the regularization problem, which is carried out in Section 3. Finally, in Section 4, we present

the application of the abstract results to the inverse conductivity problem with discontinuous conductivities.

**2. Preliminaries.** Throughout the paper the integer  $N \geq 2$  will denote the space dimension. For every  $x \in \mathbb{R}^N$  and any  $r > 0$ , we shall denote by  $B_r(x)$  the open ball in  $\mathbb{R}^N$  centred at  $x$  of radius  $r$ .

We recall that a bounded domain  $\Omega \subset \mathbb{R}^N$  is said to have a *Lipschitz boundary* if for every  $x \in \partial\Omega$  there exist a Lipschitz function  $\varphi : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$  and a positive constant  $r$  such that for any  $y \in B_r(x)$  we have, up to a rigid transformation,

$$y \in \Omega \quad \text{if and only if} \quad y_N < \varphi(y').$$

Let us observe that in this case  $\partial\Omega$  has finite  $(N-1)$ -dimensional Hausdorff measure, that is  $\mathcal{H}^{N-1}(\partial\Omega) < +\infty$ . Here and in the sequel, for any non negative integer  $k$  we denote by  $\mathcal{H}^k$  the  $k$ -dimensional Hausdorff measure. We recall that for Borel subsets of  $\mathbb{R}^N$  the  $N$ -dimensional Hausdorff measure coincides with  $\mathcal{L}^N$ , the  $N$ -dimensional Lebesgue measure. Furthermore, if  $\gamma \subset \mathbb{R}^N$  is a smooth manifold of dimension  $k$ , then  $\mathcal{H}^k$  restricted to  $\gamma$  coincides with its  $k$ -dimensional surface measure. For any Borel  $E \subset \mathbb{R}^N$  we let  $|E| = \mathcal{L}^N(E)$ .

Given an open bounded set  $\Omega \subset \mathbb{R}^N$ , we denote by  $BV(\Omega)$  the Banach space of *functions of bounded variation*. We recall that  $u \in BV(\Omega)$  if and only if  $u \in L^1(\Omega)$  and its distributional derivative  $Du$  is a bounded vector measure. We endow  $BV(\Omega)$  with the standard norm as follows. Given  $u \in BV(\Omega)$ , we denote by  $|Du|$  the total variation of its distributional derivative and we set  $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$ . We shall also use the notation  $TV(u)$  to denote the *total variation* of  $u$  on  $\Omega$ , that is  $TV(u) = |Du|(\Omega)$ .

We denote by  $SBV(\Omega)$  the space of *special functions of bounded variation* that is the space of functions  $u \in BV(\Omega)$  so that  $Du$  has a singular part, with respect to the  $N$ -dimensional Lebesgue measure, concentrated on  $J(u)$ ,  $J(u)$  being the *approximate discontinuity set* (or *jump set*) of  $u$ . The density of the absolutely continuous part of  $Du$  with respect to the  $N$ -dimensional Lebesgue measure will be denoted by  $\nabla u$ , the *approximate gradient* of  $u$ . That is,  $Du$  may be written as follows

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \nu \mathcal{H}^{N-1} \llcorner J(u)$$

where, in a measure theoretical sense,  $\nu$  denotes the normal to  $J(u)$  and  $u^+$  and  $u^-$  denote the traces of  $u$  on the sides of  $J(u)$ . In other words,  $u \in SBV(\Omega)$  belongs to  $SBV(\Omega)$  if and only if the Cantor part of  $Du$  is zero.

For a more comprehensive treatment of  $BV$  and  $SBV$  functions see, for instance, [5].

We recall the definition and basic properties of  $\Gamma$ -convergence. We recall that  $\Gamma$ -convergence is a type of variational convergence, introduced by De Giorgi in the 1970's. A thorough reference to  $\Gamma$ -convergence may be found in [13]. For a simple introduction we refer to [9], whereas for general variational convergence techniques we refer to [8]. Let  $(X, d)$  be a metric space. Then a sequence  $F_n : X \rightarrow [-\infty, +\infty]$ ,  $n \in \mathbb{N}$ ,  $\Gamma$ -converges as  $n \rightarrow \infty$  to a function  $F : X \rightarrow [-\infty, +\infty]$  if for every  $x \in X$

we have

(2.1) for every sequence  $\{x_n\}_{n \in \mathbb{N}}$  converging to  $x$  we have

$$F(x) \leq \liminf_n F_n(x_n);$$

(2.2) there exists a sequence  $\{x_n\}_{n \in \mathbb{N}}$  converging to  $x$  such that

$$F(x) = \lim_n F_n(x_n).$$

The function  $F$  will be called the  $\Gamma$ -limit of the sequence  $\{F_n\}_{n \in \mathbb{N}}$  as  $n \rightarrow \infty$  with respect to the metric  $d$  and we denote it by  $F = \Gamma\text{-lim}_n F_n$ .

The following theorem, usually known as the Fundamental Theorem of  $\Gamma$ -convergence, illustrates the motivations for the definition of such a kind of convergence. For its proof we refer, for instance, to [9, Theorem 1.21].

**Theorem 2.1.** *Let  $(X, d)$  be a metric space and let  $F_n : X \rightarrow [-\infty, +\infty]$ ,  $n \in \mathbb{N}$ , be a sequence of functions defined on  $X$ . If there exists a compact set  $K$  such that  $\inf_K F_n = \inf_X F_n$  for any  $n \in \mathbb{N}$  and  $F = \Gamma\text{-lim}_n F_n$ , then  $F$  admits a minimum over  $X$  and we have*

$$\min_X F = \liminf_n \min_X F_n.$$

Furthermore, if  $\{x_n\}_{n \in \mathbb{N}}$  is a sequence of points in  $X$  which converges to a point  $x \in X$  and satisfies  $\lim_n F_n(x_n) = \lim_n \inf_X F_n$ , then  $x$  is a minimum point for  $F$ .

The definition of  $\Gamma$ -convergence may be extended in a natural way to families depending on a continuous parameter. For instance we say that the family of functions  $F_\varepsilon$ , defined for every  $\varepsilon > 0$ ,  $\Gamma$ -converges to a function  $F$  as  $\varepsilon \rightarrow 0^+$  if for every sequence  $\{\varepsilon_n\}_{n \in \mathbb{N}}$  of positive numbers converging to 0 as  $n \rightarrow \infty$ , we have  $F = \Gamma\text{-lim}_n F_{\varepsilon_n}$ .

**3. An abstract regularization result.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be two metric spaces. Let  $\Lambda : X \rightarrow Y$  be a continuous function. Let us fix  $x_0 \in X$  and  $\Lambda_0 = \Lambda(x_0) \in Y$ .

Let  $\varepsilon_0$  be a positive constant and let us take, for any  $\varepsilon$ ,  $0 < \varepsilon \leq \varepsilon_0$ ,  $\Lambda_\varepsilon \in Y$  such that

$$d_Y(\Lambda_\varepsilon, \Lambda_0) \leq \varepsilon.$$

We assume that  $\Lambda_\varepsilon$ ,  $0 < \varepsilon \leq \varepsilon_0$ , is kept fixed throughout this section.

We say that  $R : X \rightarrow \mathbb{R} \cup \{+\infty\}$  is a *regularization operator* for the metric space  $X$  if  $R \not\equiv +\infty$  and, with respect to the metric induced by  $d_X$ ,  $R$  is a lower semicontinuous function such that for any constant  $C > 0$  the set  $\{x \in X : R(x) \leq C\}$  is a compact subset of  $X$ .

A simple application of the direct method allows us to prove the following.

**Theorem 3.1.** *For any  $\varepsilon$ ,  $0 \leq \varepsilon \leq \varepsilon_0$ , any  $\alpha > 0$ , and any  $a > 0$ , we have that the minimization problem*

$$(3.1) \quad \min_{x \in X} (d_Y(\Lambda(x), \Lambda_\varepsilon))^\alpha + aR(x)$$

*admits a solution provided  $\Lambda$  is continuous and  $R$  is a regularization operator for  $X$ .*

*Proof.* Clearly, there exists  $x \in X$  such that  $(d_Y(\Lambda(x), \Lambda_\varepsilon))^\alpha + aR(x)$  is finite. Hence, if we take a minimizing sequence  $\{x_n\}_{n \in \mathbb{N}}$ , we may assume that  $R(x_n)$ ,  $n \in \mathbb{N}$ , is uniformly bounded. Without loss of generality, by the properties of  $R$ , we may assume that there exists  $\tilde{x} \in X$  such that  $\lim_n x_n = \tilde{x}$ . By the continuity

properties of  $\Lambda$  and the semicontinuity of  $R$ , we immediately obtain that  $\tilde{x}$  is a minimizer.  $\square$

We remark that  $\alpha = 2$  corresponds to the regularization of a least squares problem.

Let us now suppose that  $a$  is a positive number depending on  $\varepsilon$ . For the sake of simplicity, let us assume that, for any  $\varepsilon$ ,  $0 < \varepsilon \leq \varepsilon_0$ ,  $a(\varepsilon) = \tilde{a}\varepsilon^\beta$  for some positive constants  $\tilde{a}$  and  $\beta$ . For any  $\varepsilon$ ,  $0 < \varepsilon \leq \varepsilon_0$ , we define  $G_\varepsilon : X \rightarrow \mathbb{R} \cup \{+\infty\}$  such that, for any  $x \in X$ ,  $G_\varepsilon(x) = (d_Y(\Lambda(x), \Lambda_\varepsilon))^\alpha + \tilde{a}\varepsilon^\beta R(x)$ . We rescale the functionals  $G_\varepsilon$ ,  $0 < \varepsilon \leq \varepsilon_0$ , by defining  $F_\varepsilon : X \rightarrow \mathbb{R} \cup \{+\infty\}$  such that for any  $x \in X$

$$(3.2) \quad F_\varepsilon(x) = \frac{G_\varepsilon(x)}{\varepsilon^\beta} = \frac{(d_Y(\Lambda(x), \Lambda_\varepsilon))^\alpha}{\varepsilon^\beta} + \tilde{a}R(x).$$

We recall that  $d_Y(\Lambda_\varepsilon, \Lambda_0) \leq \varepsilon$ ,  $0 < \varepsilon \leq \varepsilon_0$ .

Then we have that, for any  $\varepsilon$ ,  $0 < \varepsilon \leq \varepsilon_0$ ,  $F_\varepsilon$  admits a minimum over  $X$ . Moreover,  $F_\varepsilon$  and  $G_\varepsilon$  share the minimizers. If  $x_\varepsilon$  is one of these minimizers, then

$$\tilde{a}R(x_\varepsilon) \leq F_\varepsilon(x_\varepsilon) = \min_X F_\varepsilon \leq F_\varepsilon(x_0) \leq \varepsilon^{\alpha-\beta} + \tilde{a}R(x_0).$$

Then it is immediate to show the following equicoerciveness property.

**Proposition 3.2.** *Under the previous notation and assumptions, let  $\Lambda$  be continuous and  $R$  be a regularization operator for  $X$ . If  $R(x_0) < +\infty$  and  $\beta \leq \alpha$ , then there exists a compact set  $K \subset X$  such that  $\min_X F_\varepsilon = \min_K F_\varepsilon$  for any  $\varepsilon$ ,  $0 < \varepsilon \leq \varepsilon_0$ . Furthermore, there exists a constant  $C$ , depending on  $R(x_0)$ ,  $\varepsilon_0$ ,  $\alpha$ ,  $\tilde{a}$  and  $\beta$  only, such that*

$$\min_X F_\varepsilon = \min_K F_\varepsilon \leq C \quad \text{for any } \varepsilon, 0 < \varepsilon \leq \varepsilon_0.$$

Moreover, if we define  $F_0 : X \rightarrow \mathbb{R} \cup \{+\infty\}$  such that for any  $x \in X$  we have

$$(3.3) \quad F_0(x) = \begin{cases} \tilde{a}R(x) & \text{if } \Lambda(x) = \Lambda(x_0) = \Lambda_0 \\ +\infty & \text{otherwise} \end{cases}$$

then we can easily prove the following  $\Gamma$ -convergence result.

**Theorem 3.3.** *Under the previous notation and assumptions, let  $\Lambda$  be continuous and  $R$  be a regularization operator for  $X$ . If  $0 < \beta < \alpha$ , then, as  $\varepsilon \rightarrow 0^+$ ,  $F_\varepsilon$   $\Gamma$ -converges to  $F_0$  with respect to the metric induced by  $d_X$ .*

*Proof.* Let us fix a sequence of positive numbers  $\varepsilon_n$ ,  $n \in \mathbb{N}$ , such that  $\lim_n \varepsilon_n = 0$ . Let us call  $F_n = F_{\varepsilon_n}$  and let us prove that, as  $n \rightarrow \infty$ ,  $F_n$   $\Gamma$ -converges to  $F_0$ .

We begin by proving the  $\liminf$  inequality, that is (2.1). Let us fix  $x \in X$  and let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence in  $X$  such that  $\lim_n x_n = x$ . If  $\liminf_n F_n(x_n) = +\infty$ , then (2.1) is trivial. Otherwise, we may find a subsequence  $\{x_{n_k}\}_{k \in \mathbb{N}}$ , such that  $\liminf_n F_n(x_n) = \lim_k F_{n_k}(x_{n_k})$ . Clearly, we infer that  $\lim_k d_Y(\Lambda(x_{n_k}), \Lambda_{\varepsilon_{n_k}}) = 0$ , therefore, by the continuity of  $\Lambda$  and the properties of  $\Lambda_\varepsilon$ , we conclude that  $\Lambda(x) = \Lambda_0$ . Furthermore,  $\liminf F_n(x_n) \geq \liminf_n \tilde{a}R(x_n) \geq \tilde{a}R(x)$ , by the semicontinuity of  $R$ . Therefore (2.1) holds.

For what concerns the construction of the recovery sequence, (2.2), by (2.1) it is enough to treat the case when  $F_0(x)$  is finite. In such a case it is sufficient to take  $x_n = x$  for any  $n \in \mathbb{N}$ .  $\square$

It is an easy remark to show that either  $F_0$  is identically equal to  $+\infty$  or  $F_0$  admits a finite minimum value.

By Proposition 3.2 and Theorem 3.3, using the Fundamental Theorem of  $\Gamma$ -convergence, Theorem 2.1, we conclude that the following convergence result holds true.

**Theorem 3.4.** *Under the previous notation and assumptions, let  $\Lambda$  be continuous and  $R$  be a regularization operator for  $X$ . Let us also assume that  $R(x_0) < +\infty$  and  $\beta < \alpha$ .*

*Then we have that there exists  $\min_X F_\varepsilon$ , for any  $\varepsilon$ ,  $0 \leq \varepsilon \leq \varepsilon_0$ , and*

$$\min_X F_0 = \lim_{\varepsilon \rightarrow 0^+} \min_X F_\varepsilon < +\infty.$$

*Let  $\{\varepsilon_n\}_{n \in \mathbb{N}}$  be a sequence of positive numbers converging to 0 as  $n \rightarrow \infty$ . Let  $\{\tilde{x}_n\}_{n \in \mathbb{N}}$  be such that  $\lim F_{\varepsilon_n}(\tilde{x}_n) = \lim_n \min_X F_{\varepsilon_n}$ .*

*Then, up to a subsequence,  $\tilde{x}_n$  converges to a point  $\tilde{x} \in X$  such that  $\tilde{x}$  is a minimizer of  $F_0$ , that is in particular  $\Lambda(\tilde{x}) = \Lambda(x_0)$  and  $R(\tilde{x}) = \min\{R(x) : x \in X \text{ such that } \Lambda(x) = \Lambda(x_0)\}$ .*

*Obviously the result holds if we take as  $\{\tilde{x}_n\}_{n \in \mathbb{N}}$  a sequence  $\{x_{\varepsilon_n}\}_{n \in \mathbb{N}}$  of minimizers of  $F_{\varepsilon_n}$ .*

**Corollary 3.5.** *Under the assumptions of Theorem 3.4, let  $\{\tilde{x}_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$  satisfy  $\lim_{\varepsilon \rightarrow 0^+} F_\varepsilon(\tilde{x}_\varepsilon) = \lim_{\varepsilon \rightarrow 0^+} \min_X F_\varepsilon$  (for example we may pick as  $\{\tilde{x}_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$  a family  $\{x_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$  of minimizers of  $F_\varepsilon$ ).*

*If  $\tilde{x}$  is the only solution to  $\min\{R(x) : x \in X \text{ such that } \Lambda(x) = \Lambda(x_0)\}$ , we have that*

$$\lim_{\varepsilon \rightarrow 0^+} \tilde{x}_\varepsilon = \tilde{x}.$$

*Under the assumptions of Theorem 3.4, and even if  $\beta = \alpha$ , let  $\{\tilde{x}_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$  satisfy  $\limsup_{\varepsilon \rightarrow 0^+} F_\varepsilon(\tilde{x}_\varepsilon) < +\infty$ .*

*If on the set  $\{x \in X : R(x) < +\infty\}$  the map  $\Lambda$  is injective, then we have*

$$\lim_{\varepsilon \rightarrow 0^+} \tilde{x}_\varepsilon = x_0.$$

**4. Application to the inverse conductivity problem with discontinuous conductivities.** We wish to apply the previous section analysis to inverse problems. Summarizing, in order to have convergence of the regularized solutions to the looked for solution, we need the following three properties

- 1) continuity of the forward function  $\Lambda$ ;
- 2) a regularization operator  $R$  for  $X$ ;
- 3) injectivity of the forward function (uniqueness of the inverse problem).

Clearly the first two items must be true with respect to the same metric on  $X$ .

Let us describe the inverse conductivity problem. We begin with the direct problem by describing the forward function and studying its continuity properties.

Let  $\Omega$  be a bounded domain contained in  $\mathbb{R}^N$ ,  $N \geq 2$ , with Lipschitz boundary. We assume that  $\Omega$  and a constant  $\lambda$ ,  $0 < \lambda < 1$ , are fixed throughout this section.

Let  $A = A(x)$ ,  $x \in \Omega$ , be an  $N \times N$  matrix such that its entries are real valued measurable functions and it satisfies the following ellipticity condition

$$(4.1) \quad \begin{aligned} A(x)\xi \cdot \xi &\geq \lambda|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^N \text{ and for a.e. } x \in \Omega, \\ \|A\|_{L^\infty(\Omega)} &\leq \lambda^{-1}. \end{aligned}$$

Here  $\|A\|_{L^\infty(\Omega)}$  is the essential supremum over  $\Omega$  of  $\|A\|$ , where, for any  $A \in \mathbb{M}^{N \times N}$ ,  $\|A\|$  denotes its norm as a linear operator of  $\mathbb{R}^N$  into itself.

We shall denote by  $L_\lambda^\infty(\Omega, \mathbb{M}^{N \times N})$  the set of *conductivity tensors*  $A$  such that  $A \in L^\infty(\Omega, \mathbb{M}^{N \times N})$  and  $A$  satisfies (4.1) with constant  $\lambda$ . Analogously, we shall call  $L_\lambda^\infty(\Omega)$  the set of real valued measurable functions  $\sigma = \sigma(x)$ ,  $x \in \Omega$ , such that

$$0 < \lambda \leq \sigma(x) \leq \lambda^{-1} \quad \text{for a.e. } x \in \Omega.$$

Let  $H^{1/2}(\partial\Omega)$  be the space of traces of  $H^1(\Omega)$  functions on the boundary  $\partial\Omega$ . We recall that  $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$ , with continuous immersion. We denote

$${}_0H^{1/2}(\partial\Omega) = \left\{ v \in H^{1/2}(\partial\Omega) : \int_{\partial\Omega} v = 0 \right\}.$$

As usual  $H^{-1/2}(\partial\Omega)$  is the dual of  $H^{1/2}(\partial\Omega)$ . We denote

$${}_0H^{-1/2}(\partial\Omega) = \left\{ f \in H^{-1/2}(\partial\Omega) : \langle f, 1 \rangle_{(H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega))} = 0 \right\}.$$

Let  $H_*^1(\Omega) = \{u \in H^1(\Omega) : \int_{\partial\Omega} u = 0\}$ , that is the space of  $H^1(\Omega)$  functions whose traces on  $\partial\Omega$  belong to  ${}_0H^{1/2}(\partial\Omega)$ . Then, there exists a unique solution to the following problem

$$(4.2) \quad \begin{cases} u \in H_*^1(\Omega) \\ \int_{\Omega} A \nabla u \cdot \nabla v = \langle f, v|_{\partial\Omega} \rangle_{(H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega))} \quad \text{for any } v \in H^1(\Omega) \end{cases}$$

provided  $f \in {}_0H^{-1/2}(\partial\Omega)$ . We note that (4.2) is the weak formulation of the Neumann boundary value problem (1.1) with  $\sigma$  replaced by  $A$ .

We define the *Neumann-to-Dirichlet map* associated to the conductivity tensor  $A$  as follows. We let  $\Lambda(A) : {}_0H^{-1/2}(\partial\Omega) \rightarrow {}_0H^{1/2}(\partial\Omega)$  such that for any  $f \in {}_0H^{-1/2}(\partial\Omega)$  we define

$$\Lambda(A)(f) = u|_{\partial\Omega}$$

where  $u$  solves (4.2). We have that  $\Lambda(A)$  is a bounded linear operator whose norm depends on  $N$ ,  $\lambda$  and  $\Omega$  only.

For any two Banach spaces  $B_1, B_2$ ,  $\mathcal{L}(B_1, B_2)$  will denote the Banach space of bounded linear operators from  $B_1$  to  $B_2$  with the usual operator norm. Hence the forward function is  $\Lambda : L_\lambda^\infty(\Omega, \mathbb{M}^{N \times N}) \rightarrow \mathcal{L}({}_0H^{-1/2}(\partial\Omega), {}_0H^{1/2}(\partial\Omega))$ .

For what concerns the metric on  $L_\lambda^\infty(\Omega, \mathbb{M}^{N \times N})$ , we observe that the natural metric might be the one induced by the  $L^\infty$  norm. In fact, let us consider the following computation

Let  $A_1, A_2 \in L_\lambda^\infty(\Omega, \mathbb{M}^{N \times N})$ . For any  $f \in {}_0H^{-1/2}(\partial\Omega)$ , let  $u_1$  and  $u_2$  be the solutions to (4.2) with  $A$  replaced by  $A_1$  and  $A_2$ , respectively.

Then, for any  $v \in H^1(\Omega)$  and any  $i = 1, 2$ , we have

$$\int_{\Omega} A_i \nabla u_i \cdot \nabla v = \langle f, v|_{\partial\Omega} \rangle.$$

Therefore,

$$\begin{aligned} 0 &= \int_{\Omega} A_1 \nabla u_1 \cdot \nabla v - \int_{\Omega} A_2 \nabla u_2 \cdot \nabla v \\ &= \int_{\Omega} A_1 \nabla(u_1 - u_2) \cdot \nabla v + \int_{\Omega} (A_1 - A_2) \nabla u_2 \cdot \nabla v. \end{aligned}$$



By taking  $v = u_1 - u_2$  and using the ellipticity condition (4.1), we obtain

$$\lambda \int_{\Omega} |\nabla(u_1 - u_2)|^2 \leq \int_{\Omega} |(A_1 - A_2)\nabla u_2 \cdot \nabla(u_1 - u_2)|.$$

Then by Hölder's inequality, we have

$$\lambda \left( \int_{\Omega} |\nabla(u_1 - u_2)|^2 \right)^{1/2} \leq \|A_1 - A_2\|_{L^\infty(\Omega)} \|\nabla u_2\|_{L^2(\Omega)}.$$

We may easily conclude that

$$\|\Lambda(A_1) - \Lambda(A_2)\|_{\mathcal{L}({}_0H^{-1/2}(\partial\Omega), {}_0H^{1/2}(\partial\Omega))} \leq C\|A_1 - A_2\|_{L^\infty(\Omega)}$$

where  $C$  depends on  $N$ ,  $\lambda$  and  $\Omega$  only. In other words, the function  $\Lambda$  is Lipschitz continuous, with Lipschitz constant  $C$ , from  $L^\infty_\lambda(\Omega, \mathbb{M}^{N \times N})$ , with the metric induced by the  $L^\infty$  norm, to  $\mathcal{L}({}_0H^{-1/2}(\partial\Omega), {}_0H^{1/2}(\partial\Omega))$ , with its usual norm.

However, we have already pointed out that the  $L^\infty$  norm is not suited to treat the case of discontinuous conductivity tensors. For discontinuous conductivities one should consider a weaker norm, namely the one induced by the  $L^1$  norm (or, equivalently, given the uniform  $L^\infty$  bound, by the  $L^q$  norm for some  $q$ ,  $1 \leq q < +\infty$ ).

In order to have continuity of the forward function with respect to the distance induced by the  $L^1$  norm we need to change the spaces among which operates  $\Lambda(A)$ ,  $A \in L^\infty_\lambda(\Omega, \mathbb{M}^{N \times N})$ .

Let  $f \in L^s(\partial\Omega)$ , with  $1 < s \leq +\infty$  if  $N = 2$  or  $2 - (2/N) \leq s \leq +\infty$  if  $N \geq 3$ . Then,  $f \in H^{-1/2}(\partial\Omega)$ , see for instance [2, Theorems 7.53 and 7.57], by setting  $\langle f, v \rangle_{(H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega))} = \int_{\partial\Omega} f v$  for any  $v \in H^{1/2}(\partial\Omega)$ . We have that  $L^s(\partial\Omega) \subset H^{-1/2}(\partial\Omega)$  with continuous immersion, and, furthermore, if  $\int_{\partial\Omega} f = 0$ , then  $f \in {}_0L^s(\partial\Omega) \subset {}_0H^{-1/2}(\partial\Omega)$ . Therefore, for any  $s$  as before, we have that  $\Lambda(A) : {}_0L^s(\partial\Omega) \rightarrow {}_0H^{1/2}(\partial\Omega)$  is a bounded linear operator whose norm depends on  $N$ ,  $\lambda$ ,  $\Omega$  and  $s$  only.

By using Theorem 2 in [17], which is an extension to Neumann problems of a classical theorem by Meyers, [21], the following proposition holds true. For more details see for instance Section 2 of [23].

**Proposition 4.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ ,  $N \geq 2$ , with Lipschitz boundary. Let  $A \in L^\infty_\lambda(\Omega, \mathbb{M}^{N \times N})$  for some constant  $\lambda$ ,  $0 < \lambda < 1$ .*

*There exists a constant  $Q_1 > 2$ , depending on  $N$ ,  $\lambda$  and  $\Omega$  only, such that for any  $p$ ,  $2 < p < Q_1$ , any  $s$ ,  $p - (p/N) \leq s \leq +\infty$ , and any  $f \in {}_0L^s(\partial\Omega)$ ,  $u$  solution to (4.2) satisfies*

$$(4.3) \quad \|u\|_{W^{1,p}(\Omega)} \leq C(s,p)\|f\|_{L^s(\partial\Omega)},$$

where  $C(s,p)$  is a constant depending on  $N$ ,  $\lambda$ ,  $\Omega$ ,  $s$  and  $p$  only.

In the sequel, the constant  $Q_1$  will denote the constant appearing in Proposition 4.1, which depends on  $N$ ,  $\lambda$  and  $\Omega$ . We shall also fix constants  $p$ ,  $2 < p < Q_1$ ,  $s$ ,  $p - (p/N) \leq s \leq +\infty$ , and  $q$ ,  $2 < q < +\infty$  such that

$$(4.4) \quad \frac{1}{q} + \frac{1}{p} + \frac{1}{2} = 1.$$

Let  $W^{1-1/p,p}(\partial\Omega)$  be the space of traces of  $W^{1,p}(\Omega)$  functions on  $\partial\Omega$  and let us recall that  $W^{1-1/p,p}(\partial\Omega) \subset L^p(\partial\Omega)$ , with continuous immersion.

If  $B_1, B_2$  are Banach spaces such that  $B_1 \subset {}_0L^s(\partial\Omega)$  and  ${}_0W^{1-1/p,p}(\partial\Omega) \subset B_2$ , both with continuous immersion, then for any  $A \in L^\infty_\lambda(\Omega, \mathbb{M}^{N \times N})$  we have that

$\Lambda(A) : B_1 \rightarrow B_2$  is a bounded linear operator whose norm depends on  $N$ ,  $\lambda$ ,  $\Omega$ ,  $s$ ,  $p$ ,  $B_1$  and  $B_2$  only.

We now investigate the continuity of the function  $\Lambda$ . Let  $A_1, A_2 \in L^\infty_\lambda(\Omega, \mathbb{M}^{N \times N})$ , let  $f \in {}_0L^s(\partial\Omega)$ , let  $u_1$  and  $u_2$  be the solutions to (4.2) with  $A$  replaced by  $A_1$  and  $A_2$ , respectively. With the same computation we have used before, we obtain that

$$\lambda \left( \int_\Omega |\nabla(u_1 - u_2)|^2 \right)^{1/2} \leq \|A_1 - A_2\|_{L^q(\Omega)} \|\nabla u_2\|_{L^p(\Omega)}.$$

Here  $\|A_1 - A_2\|_{L^q(\Omega)} = \left( \int_\Omega \|A_1 - A_2\|^q \right)^{1/q}$ . By Proposition 4.1, we may find a constant  $C$  depending on  $N$ ,  $\lambda$ ,  $\Omega$ ,  $s$  and  $p$  only such that

$$\|u_1 - u_2\|_{H^1(\Omega)} \leq C \|A_1 - A_2\|_{L^q(\Omega)} \|f\|_{L^s(\partial\Omega)}.$$

In other words we have proved the following.

**Theorem 4.2.** *Under the previous notation and assumptions, let  $B_1$  and  $B_2$  be two Banach spaces such that  $B_1 \subset {}_0L^s(\partial\Omega)$  and  ${}_0H^{1/2}(\partial\Omega) \subset B_2$ , both with continuous immersion.*

*Then the map  $\Lambda : L^\infty_\lambda(\Omega, \mathbb{M}^{N \times N}) \rightarrow \mathcal{L}(B_1, B_2)$  is Lipschitz continuous if we take on  $L^\infty_\lambda(\Omega, \mathbb{M}^{N \times N})$  the metric induced by the  $L^q$  norm, that is for any  $A_1, A_2 \in L^\infty_\lambda(\Omega, \mathbb{M}^{N \times N})$  we have*

$$(4.5) \quad \|\Lambda(A_1) - \Lambda(A_2)\|_{\mathcal{L}(B_1, B_2)} \leq C \|A_1 - A_2\|_{L^q(\Omega)},$$

where the Lipschitz constant  $C$  depends on  $N$ ,  $\lambda$ ,  $\Omega$ ,  $s$ ,  $p$ ,  $B_1$  and  $B_2$  only.

**Corollary 4.3.** *Under the assumptions of Theorem 4.2, we have that the map  $\Lambda$  is Hölder continuous with respect to the metric induced by the  $L^1$  norm on  $L^\infty_\lambda(\Omega, \mathbb{M}^{N \times N})$ , that is for any  $A_1, A_2 \in L^\infty_\lambda(\Omega, \mathbb{M}^{N \times N})$  we have*

$$(4.6) \quad \|\Lambda(A_1) - \Lambda(A_2)\|_{\mathcal{L}(B_1, B_2)} \leq C \lambda^{1/q-1} \|A_1 - A_2\|_{L^1(\Omega)}^{1/q},$$

where  $C$  is the same constant appearing in (4.5).

**Remark 4.4.** We may choose  $p$  in such a way that  $s$  may be taken equal to 2. Therefore, we may always take  $B_1 = B_2 = {}_0L^2(\partial\Omega)$  in the previous two results.

A further result can be proven by using Meyers theorem and let us note that a similar observation has been already made in [3].

**Remark 4.5.** Let  $\Omega_1$  be compactly contained in  $\Omega$ . We fix  $\tilde{A} \in L^\infty_\lambda(\Omega, \mathbb{M}^{N \times N})$  and let us assume that  $\Lambda$  is restricted to the set

$$\tilde{X} = \{A \in L^\infty_\lambda(\Omega, \mathbb{M}^{N \times N}) : A = \tilde{A} \text{ a.e. outside } \Omega_1\}.$$

For example, in the inclusion problem this is equivalent to assume that the background conductivity is known and that all the inclusions are a priori known to be contained in  $\Omega_1$ .

Let us fix  $B_1$  and  $B_2$ , two Banach spaces such that  $B_1 \subset {}_0H^{-1/2}(\partial\Omega)$  and  ${}_0H^{1/2}(\partial\Omega) \subset B_2$ , both with continuous immersion. Then there exists  $Q > 2$ , depending on  $N$  and  $\lambda$  only, such that for any  $p$ ,  $2 < p < Q$ , and any  $q$ ,  $2 < q < +\infty$  satisfying (4.4), we have that (4.5) and (4.6) holds for any  $A_1, A_2 \in \tilde{X}$ . The constant  $C$  in this case depends on  $N$ ,  $\lambda$ ,  $\Omega$ ,  $B_1$ ,  $B_2$  and  $\Omega_1$ .

We now turn our attention to the inverse problem and its regularization. For simplicity, let us now concentrate on scalar conductivities only. In fact, for what concerns anisotropic conductivities, it is well-known that uniqueness does not hold in general since any suitable change of variable leaving the boundary fixed would lead to a different conductivity with the same Neumann-to-Dirichlet map. Let us also note that, in many interesting cases, this is the only obstruction to uniqueness, see [26] and [7].

Therefore, we denote  $X = L^\infty_\lambda(\Omega)$  and

$$d_X(\sigma_1, \sigma_2) = \int_\Omega |\sigma_1 - \sigma_2| \quad \text{for any } \sigma_1, \sigma_2 \in X.$$

In other words, the topology on  $X$  is the one induced by the  $L^1$  norm. As we have already noted, the  $L^1$  norm is the natural one to measure the distance between discontinuous conductivities. For example, if we have two different inclusions, with the same conductivity, then the  $L^1$  distance of the conductivities corresponds to the Lebesgue measure of the symmetric difference between the two inclusions.

We fix  $Y = \mathcal{L}(B_1, B_2)$ , with the usual operator norm, where  $B_1$  and  $B_2$  are two Banach spaces satisfying  $B_1 \subset {}_0L^s(\partial\Omega)$  and  ${}_0H^{1/2}(\partial\Omega) \subset B_2$ , both with continuous immersion. We recall that we may take  $B_1 = B_2 = {}_0L^2(\partial\Omega)$ , if  $p$  is sufficiently close to 2.

The forward function is the map  $\Lambda : X \rightarrow Y$  such that, for any  $\sigma \in X$ ,  $\Lambda(\sigma)$  is the Neumann-to-Dirichlet map associated to  $\sigma$ . Corollary 4.3 guarantees that  $\Lambda : X \rightarrow Y$  is continuous.

As a regularization operator for  $X$ , with respect to the  $L^1$  metric, we have many different choices. We illustrate two of them, which have been already used for this kind of inverse problems.

The first one is the following. We set

$$(4.7) \quad R(\sigma) = TV(\sigma) \quad \text{for any } \sigma \in X.$$

Clearly  $R(\sigma) = +\infty$  if  $\sigma$  does not belong to  $BV(\Omega)$ . We know that  $R$  is lower semicontinuous with respect to the  $L^1$  norm and, as a corollary of Theorem 3.23 in [5], we also have that for any  $C > 0$  the set  $\{u \in BV(\Omega) : \|u\|_{BV(\Omega)} \leq C\}$  is a compact subset of  $L^1(\Omega)$ . Therefore,  $R$  is a regularization operator for  $X$ . The total variation regularization have been used in [15], with a discretization method, and in [10, 12], with level set methods.

Another possible choice of a regularization operator is

$$(4.8) \quad R(\sigma) = \begin{cases} \int_\Omega |\nabla\sigma|^2 + \mathcal{H}^{N-1}(J(\sigma)) & \text{if } \sigma \in SBV(\Omega) \cap X, \\ +\infty & \text{otherwise.} \end{cases}$$

Here the functional defining  $R$  is the so-called Mumford-Shah functional introduced in the context of image segmentation in [22]. The compactness and semicontinuity theorem for special functions of bounded variation due to Ambrosio, see for instance [5, Theorem 4.7 and Theorem 4.8], guarantees that also in this case  $R$  is a regularization operator for  $X$ . The Mumford-Shah functional has been used as a regularization operator for the inverse conductivity problem in [24], with an approximation method with smoother functionals.

We are now in the position to state the main result of the paper.

**Theorem 4.6.** *Under the previous notation and assumptions, let  $\Lambda : X \rightarrow Y$  be the forward function, where, for any  $\sigma \in X$ ,  $\Lambda(\sigma)$  is the Neumann-to-Dirichlet map associated to  $\sigma$ . Let  $R$  be defined either as in (4.7) or in (4.8).*

*Let  $\sigma_0 \in X$  be such that  $R(\sigma_0) < +\infty$  and  $\Lambda_0 = \Lambda(\sigma_0)$ . For any  $\varepsilon$ ,  $0 < \varepsilon \leq \varepsilon_0$ , let  $\Lambda_\varepsilon \in Y$  be such that  $d_Y(\Lambda_\varepsilon, \Lambda_0) = \|\Lambda_\varepsilon - \Lambda_0\|_{\mathcal{L}(B_1, B_2)} \leq \varepsilon$ .*

*Let us fix positive constants  $\alpha$ ,  $\beta$  and  $\tilde{\alpha}$ , such that  $0 < \beta < \alpha$ . For any  $\varepsilon$ ,  $0 < \varepsilon \leq \varepsilon_0$ , let  $F_\varepsilon$  be defined as in (3.2) and  $F_0$  be defined as in (3.3).*

*Then we have that there exists  $\min_X F_\varepsilon$ , for any  $\varepsilon$ ,  $0 \leq \varepsilon \leq \varepsilon_0$ , and*

$$\min_X F_0 = \lim_{\varepsilon \rightarrow 0^+} \min_X F_\varepsilon < +\infty.$$

*Let  $\{\varepsilon_n\}_{n \in \mathbb{N}}$  be a sequence of positive numbers converging to 0 as  $n \rightarrow \infty$ . Let  $\{\tilde{\sigma}_n\}_{n \in \mathbb{N}}$  be such that  $\lim F_{\varepsilon_n}(\tilde{\sigma}_n) = \lim_n \min_X F_{\varepsilon_n}$ .*

*Then, up to a subsequence,  $\tilde{\sigma}_n$  converges in the  $L^1$  norm to  $\tilde{\sigma} \in X$  such that  $\tilde{\sigma}$  is a minimizer of  $F_0$ , that is in particular  $\Lambda(\tilde{\sigma}) = \Lambda(\sigma_0)$  and  $R(\tilde{\sigma}) = \min\{R(\sigma) : \sigma \in X \text{ such that } \Lambda(\sigma) = \Lambda(\sigma_0)\}$ .*

*Let us further assume that the space dimension is 2, that is  $N = 2$ . Let  $\{\tilde{\sigma}_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$  satisfy  $\limsup_{\varepsilon \rightarrow 0^+} F_\varepsilon(\tilde{\sigma}_\varepsilon) < +\infty$ . Then, even if  $\beta = \alpha$ , we have that*

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} |\tilde{\sigma}_\varepsilon - \sigma_0| = 0.$$

*Proof.* It is an immediate consequence of Corollary 4.3 and the properties of the regularization operators, which allow us to use Theorem 3.4.

For the case  $N = 2$ , we may apply Corollary 3.5 by exploiting the uniqueness result by Astala and Päiväranta, [6].  $\square$

The use of many  $BV$ -related regularization operators may suggest the following

**Open problem** For  $N \geq 3$ , prove a uniqueness result for the inverse problem for conductivities belonging to some class of  $BV$  or  $SBV$  functions.

Such a result is still missing and we believe this to be an extremely challenging but very interesting task. Until this problem remains unsolved, in order to have that the conclusion of Corollary 3.5 holds true, we need to choose a different regularization operator  $R$  for  $X$  which also guarantees that the set  $\{\sigma \in X : R(\sigma) < +\infty\}$  is contained in some class of conductivities for which we have uniqueness results. For example, for the inclusion problem, in the class of inclusions for which Isakov proved unique determination, [18].

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