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ON A CLASS OF STOCHASTIC OPTIMAL CONTROL PROBLEMS RELATED TO BSDES WITH QUADRATIC GROWTH*

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Abstract. In this paper, we study a class of stochastic optimal control problems, where the drift term of the equation has a linear growth on the control variable, the cost functional has a quadratic growth, and the control process takes values in a closed set (not necessarily compact). This problem is related to some backward stochastic differential equations (BSDEs) with quadratic growth and unbounded terminal value. We prove that the optimal feedback control exists, and the optimal cost is given by the initial value of the solution of the related BSDE.

Key words. stochastic optimal control, backward stochastic differential equations, quadratically growing driver, unbounded final condition

AMS subject classifications. 93E20, 60H10

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1. Introduction.

In this paper, we consider a controlled equation of the form

$$\begin{cases} dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, [dW_t + r(t, X_t, u_t)\, dt], & t \in [0, T], \\ X_0 = x. \end{cases} \tag{1.1}$$

In the equation, W is an $\mathbb{R}^d$-valued Wiener process, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions; the unknown process X takes values in $\mathbb{R}^n$; x is a given element of $\mathbb{R}^n$; and u is the control process, which is assumed to be an $(\mathcal{F}_t)$-adapted process taking values in a given nonempty closed set $K \subset \mathbb{R}^m$. The control problem consists of minimizing a cost functional of the form

$$J = \mathbb{E} \int_0^T g(t, X_t, u_t)\, dt + \mathbb{E}\, \phi(X_T). \tag{1.2}$$

We suppose that r has a linear growth in u, g has quadratic growth in x and u, and ϕ has quadratic growth in x.

The main novelty of the present paper, in comparison with the existing literature, is that on the one hand we assume that neither K nor r is bounded; on the other hand we consider a degenerate control problem (since nothing is assumed on the image of σ). Moreover, we also allow ϕ to have quadratic growth.

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Nonlinear backward stochastic differential equations (BSDEs) were first introduced by Pardoux and Peng [11]. A major application of BSDEs is in stochastic control; see, e.g., [4] and [12]. We also refer the reader to [9], [5], and [7]. From several points of view, the class of control problems addressed in these papers, and in the references therein, is more general than the one considered here, but assumptions implying "bounded control image" (i.e., boundedness of K or r in our notation) are a common feature of all the above papers. The special "unbounded" case corresponding to the assumptions $K = \mathbb{R}^m$ and $g = \frac{1}{2}|u|^2 + q(t,x)$ is treated in [6] by an ad hoc exponential transform. We notice that in [6] ϕ is allowed to take the value $+\infty$. Finally, the same special case (in which the Hamiltonian is exactly the square of the norm of the gradient) was treated in [8] by analytic techniques under nondegeneracy assumptions and in an infinite-dimensional framework.

The difficulty here is that the Hamiltonian corresponding to the control problem has quadratic growth in the gradient and consequently the associated BSDE has quadratic growth in the Z variable. Well-posedness for this class of BSDEs has been proved in [10] in the case of bounded terminal value. Since we allow for unbounded terminal cost, to treat such equations we have to apply the techniques recently introduced in [2]. We notice that for such BSDEs no general uniqueness results are known: we replace uniqueness with the selection of a maximal solution. Moreover, the usual application of the Girsanov technique is not allowed (since the Novikov condition is not guaranteed), and we have to develop specific arguments both to prove the fundamental relation (see section 4) and to obtain the existence of a (weak) solution to the closed loop equation (see section 5). Our main result is to prove that the optimal feedback control exists and the optimal cost is given by the value Y_0 of the maximal solution (Y, Z) of the BSDE with the quadratic growth and unbounded terminal value mentioned above. Moreover, we show that we can construct an optimal feedback in terms of the process Z. Finally, we prove that if we fix a particular optimal feedback law, then the solution of the corresponding closed loop equation is unique; see Proposition 5.4.

An alternative approach to the control problem may consist of applying the stochastic maximum principle; see, e.g., [12] for a detailed exposition. However, this would require further differentiability conditions on the coefficients of the controlled equation and would not immediately imply existence of an optimal control, since the maximum principle is usually stated as a necessary condition for optimality.

The paper is organized as follows. In the next section, we describe the control problem; in section 3, we study the related BSDE; and in section 4, we establish the fundamental relation between the optimal control problem and BSDE. The last section is devoted to the proof of the existence of optimal feedback control.

2. The control problem. We consider the optimal control problem given by a state equation of the form

$$\begin{cases} dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, [dW_t + r(t, X_t, u_t)\, dt], \qquad t \in [0, T], \\ X_0 = x, \end{cases} \tag{2.1}$$

and given by a cost functional of the form

$$J = \mathbb{E} \int_0^T g(t, X_t, u_t)\, dt + \mathbb{E}\, \phi(X_T). \tag{2.2}$$

We work under the following assumptions.

HYPOTHESIS 2.1.

1. *The process W is a Wiener process in $\mathbb{R}^d$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $(\mathcal{F}_t)$ satisfying the usual conditions.*
2. *The set K is a nonempty closed subset of $\mathbb{R}^m$.*
3. *The functions $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$, $r : [0,T] \times \mathbb{R}^n \times K \to \mathbb{R}^d$, $g : [0,T] \times \mathbb{R}^n \times K \to \mathbb{R}$, $\phi : \mathbb{R}^n \to \mathbb{R}$ are Borel measurable.*
4. *For all $t \in [0,T]$, $x \in \mathbb{R}^n$, $r(t,x,\cdot)$ and $g(t,x,\cdot)$ are continuous functions from K to $\mathbb{R}^d$ and from K to $\mathbb{R}$, respectively.*
5. *There exists a constant $C > 0$ such that for every $t \in [0,T]$, $x, x' \in \mathbb{R}^n$, $u \in K$ it holds that*

$$|b(t,x) - b(t,x')| \le C|x - x'|, \qquad |b(t,x)| \le C(1 + |x|), \tag{2.3}$$

$$|\sigma(t,x) - \sigma(t,x')| \le C|x - x'|, \tag{2.4}$$

$$|\sigma(t,x)| \le C, \tag{2.5}$$

$$|r(t,x,u) - r(t,x',u)| \le C(1 + |u|)|x - x'|, \tag{2.6}$$

$$|r(t,x,u)| \le C(1 + |u|), \tag{2.7}$$

$$0 \le g(t,x,u) \le C(1 + |x|^2 + |u|^2), \tag{2.8}$$

$$0 \le \phi(x) \le C(1 + |x|^2). \tag{2.9}$$

6. *There exist $R > 0$ and $c > 0$ such that for every $t \in [0,T]$, $x \in \mathbb{R}^n$, and every $u \in K$ satisfying $|u| \ge R$,*

$$g(t,x,u) \ge c|u|^2. \tag{2.10}$$

We will say that an $(\mathcal{F}_t)$-adapted stochastic process $\{u_t,\ t \in [0,T]\}$ with values in K is an admissible control if it satisfies

$$\mathbb{E}\int_0^T |u_t|^2 dt < \infty. \tag{2.11}$$

This square summability requirement is justified by (2.10): a control process which is not square summable would have infinite cost.

Remark 2.2. Some classes of linear quadratic control problems fall within the scope of our result. Consider the controlled system

$$\begin{cases} dX_t = A(t)X_t\, dt + b(t)\, dt + \Sigma(t)\, [dW_t + B(t)u_t\, dt], \qquad t \in [0,T], \\ X_0 = x \end{cases} \tag{2.12}$$

and the cost functional

$$J = \frac{1}{2}\mathbb{E}\int_0^T [\langle Q(t)X_t, X_t\rangle + \langle R(t)u_t, u_t\rangle]\, dt + \frac{1}{2}\mathbb{E}\,\langle SX_T, X_T\rangle,$$

and assume that A, b, Σ, B, Q, R are bounded Borel measurable functions with values in $\mathbb{R}^{n\times n}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times d}$, $\mathbb{R}^{d\times m}$, $\mathbb{R}^{n\times n}$, $\mathbb{R}^{m\times m}$, respectively, and that $S \in \mathbb{R}^{n\times n}$; also suppose that $Q(t) \geq 0$, $S \geq 0$, $R(t) \geq cI$ for some $c > 0$ and all t, in the sense of the usual matrix order. Then Hypothesis 2.1 is verified. We are not assuming $K = \mathbb{R}^m$ in general. If $K = \mathbb{R}^m$, then the usual linear quadratic theory applies and gives more general results in the sense that state- and control-dependent noise can be considered, as well as a more general drift coefficient, in (2.12).

Next we show that for every admissible control, the solution to (2.1) exists.

PROPOSITION 2.3. *Let u be an admissible control. Then there exists a unique, continuous, $(\mathcal{F}_t)$-adapted process X satisfying $\mathbb{E}\sup_{t\in[0,T]}|X_t|^2 < \infty$, and $\mathbb{P}$-a.s.,*

$$X_t = x + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s + \int_0^t \sigma(s, X_s)\, r(s, X_s, u_s)\, ds, \quad t \in [0, T].$$

Proof. The proof of Proposition 2.3 relies on an approximation procedure that will be used again in what follows. We introduce the sequence of stopping times

$$\tau_n = \inf\left\{t \in [0,T] \;:\; \int_0^t |u_s|^2 ds > n\right\},$$

with the convention that $\tau_n = T$ if the indicated set is empty. By (2.11), for $\mathbb{P}$-almost every $\omega \in \Omega$, there exists an integer $N(\omega)$ depending on ω such that

$$n \geq N(\omega) \qquad \Longrightarrow \qquad \tau_n(\omega) = T. \tag{2.13}$$

Let us fix $u^0 \in K$, and for every n, let us define

$$u_t^n = u_t\, 1_{t\leq\tau_n} + u^0\, 1_{t>\tau_n}$$

and consider the equation

$$\begin{cases} dX_t^n = b(t, X_t^n)\, dt + \sigma(t, X_t^n)\, [dW_t + r(t, X_t^n, u_t^n)\, dt], & t \in [0,T], \\ X_0^n = x. \end{cases} \tag{2.14}$$

We claim that (2.14) has a unique solution X^n in the class of $(\mathcal{F}_t)$-adapted processes X satisfying $\sup_{t\in[0,T]}\mathbb{E}\,|X_t|^2 < \infty$.

To prove the claim we use a classical argument: We first write (2.14) in the form

$$dX_t = b(t, X_t)\, dt + \widetilde{b}(t, X_t)\, dt + \sigma(t, X_t)\, dW_t,$$

where $\widetilde{b}(t,x) = \sigma(t, x, u_t^n) r(t, x, u_t^n)$ is a stochastic coefficient which satisfies, by (2.3)–(2.7),

$$|\widetilde{b}(t,x) - \widetilde{b}(t,x')| = |\sigma(t,x,u_t^n)r(t,x,u_t^n) - \sigma(t,x',u_t^n)r(t,x',u_t^n)| \leq C(1 + |u_t^n|)|x - x'|,$$

$$|\widetilde{b}(t,x)| = |\sigma(t,x,u_t^n)r(t,x,u_t^n)| \leq C(1 + |u_t^n|), \qquad t \in [0,T], \quad x, x' \in \mathbb{R}^n.$$

We define $K_t(\omega) = C(1 + |u_t^n(\omega)|)$, and we note that

$$\int_0^T |K_t|^2 dt \leq C\left(1 + \int_0^T |u_t^n|^2 dt\right) \leq C(1 + n + |u_0|^2) =: \widetilde{C}, \qquad \mathbb{P}\text{-a.s.},$$

by the definition of τ_n. Next we introduce the norm $\|X\|^2 := \sup_{t\in[0,T]} e^{-2\lambda t}\mathbb{E}\,|X_t|^2$ and prove that the mapping Γ defined by

$$\Gamma(X)_t = \int_0^t b(s, X_s)\, ds + \int_0^t \widetilde{b}(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s, \qquad t \in [0, T],$$

is a contraction with respect to this norm for sufficiently large $\lambda > 0$. For a pair of processes X, X', we have

$$\begin{aligned}
\mathbb{E}\left|\int_0^t [\widetilde{b}(s, X_s) - \widetilde{b}(s, X'_s)]\, ds\right|^2 &\le \mathbb{E}\left(\int_0^t K_s |X_s - X'_s|\, ds\right)^2 \\
&\le \mathbb{E}\left(\int_0^t |K_s|^2 ds \int_0^t |X_s - X'_s|^2\, ds\right) \\
&\le \widetilde{C}\mathbb{E}\int_0^t |X_s - X'_s|^2\, ds \\
&\le \widetilde{C}\|X - X'\|^2 \int_0^t e^{2\lambda s}\, ds,
\end{aligned}$$

and it follows that

$$\begin{aligned}
\sup_{t\in[0,T]} e^{-2\lambda t}\mathbb{E}\left|\int_0^t [\widetilde{b}(s, X_s) - \widetilde{b}(s, X'_s)]\, ds\right|^2 &\le \widetilde{C}\|X - X'\|^2 \int_0^t e^{-2\lambda(t-s)}\, ds \\
&\le \frac{\widetilde{C}}{2\lambda}\|X - X'\|^2.
\end{aligned}$$

The other verifications needed to prove the contraction property are standard. The claim is proved.

It is clear that the solution X^n of (2.14) is also continuous. Moreover, we have

$$X^n_t = X^{n+1}_t \qquad \text{for} \qquad t \le \tau_n;$$

therefore there exists a process X such that

$$X_t = X^n_t \qquad \text{for} \qquad t \le \tau_n,$$

and X is clearly the required solution. The property $\mathbb{E}\sup_{t\in[0,T]} |X_t|^2 < \infty$ is an immediate consequence of the following lemma, which concludes the proof. ∎

LEMMA 2.4. *Under the previous assumptions, the family of random variables*

$$\sup_{t\in[0,T]} |X^n_t|^2, \qquad n = 1, 2, \dots,$$

is uniformly integrable.

Proof. We set $M^n_t = \int_0^t \sigma(s, X^n_s)\, dW_s$. By (2.14), we have

$$X^n_t = x + \int_0^t b(s, X^n_s)\, ds + \int_0^t \sigma(s, X^n_s) r(s, X^n_s, u^n_s)\, ds + M^n_t.$$

First, we claim that the family $\{\sup_{t\in[0,T]} |M^n_t|^2; n = 1, 2, \dots\}$ is uniformly integrable; indeed it is uniformly bounded in $L^p(\Omega, \mathcal{F}, \mathbb{P})$ for every $p \in [1, \infty)$, since by the Burkholder–Davis–Gundy inequalities and the boundedness of σ (see (2.5)),

$$\mathbb{E}\sup_{t\in[0,T]} |M^n_t|^{2p} \le C\mathbb{E}\left(\int_0^T |\sigma(s, X^n_s)|^2\, ds\right)^p \le C$$

for some constant independent of n. Next we note that, by (2.7),

$$\left|\int_0^t \sigma(s,X_s^n)r(s,X_s^n,u_s^n)\,ds\right|^2 \le C\int_0^t (1+|u_s^n|)^2\,ds \le C\left(1+\int_0^T |u_s|^2\,ds\right).$$

Then, setting $l_t^n = \sup_{r\in[0,t]}|X_r^n|^2$, we obtain by (2.3),

$$\begin{aligned} l_t^n &\le C\left(1+\int_0^T |u_s|^2\,ds + \sup_{t\in[0,T]}|M_t^n|^2\right) + C\int_0^t |X_s^n|^2\,ds \\ &\le C\left(1+\int_0^T |u_s|^2\,ds + \sup_{t\in[0,T]}|M_t^n|^2\right) + C\int_0^t l_s^n\,ds. \end{aligned}$$

From the Gronwall lemma we deduce

$$l_t^n \le C\left(1+\int_0^T |u_s|^2\,ds + \sup_{t\in[0,T]}|M_t^n|^2\right),$$

and the required uniform integrability follows. ∎

The stochastic control problem associated with (2.1)–(2.2) consists of minimizing the cost functional $J(x,u)$ among all the admissible controls.

3. The forward-backward system. We consider again the functions b,σ,g,ϕ satisfying the assumptions in Hypothesis 2.1. We define the Hamiltonian function

$$\psi(t,x,z) = \inf_{u\in K}[g(t,x,u)+z\cdot r(t,x,u)], \qquad t\in[0,T], \quad x\in\mathbb{R}^n, \quad z\in\mathbb{R}^d, \tag{3.1}$$

where $\cdot$ denotes the usual scalar product in $\mathbb{R}^d$. We collect some immediate properties of the function ψ.

LEMMA 3.1. *The map ψ is a Borel measurable function from $[0,T]\times\mathbb{R}^n\times\mathbb{R}^d$ to $\mathbb{R}$. There exists a constant $C>0$ such that*

$$-C(1+|z|^2) \le \psi(t,x,z) \le g(t,x,u) + C|z|(1+|u|) \qquad \forall u\in K. \tag{3.2}$$

Moreover, the infimum in (3.1) is attained in a ball of radius $C(1+|x|+|z|)$, that is,

$$\psi(t,x,z) = \min_{u\in K,|u|\le C(1+|x|+|z|)}[g(t,x,u)+z\cdot r(t,x,u)], \tag{3.3}$$

$$t\in[0,T], \quad x\in\mathbb{R}^n, \quad z\in\mathbb{R}^d,$$

and

$$\psi(t,x,z) < g(t,x,u)+z\cdot r(t,x,u) \quad \text{if} \quad |u| > C(1+|x|+|z|). \tag{3.4}$$

Finally, for every $t\in[0,T]$ and $x\in\mathbb{R}^n$, $z\to\psi(t,x,z)$ is continuous on $\mathbb{R}^d$.

Proof. The measurability of ψ is straightforward since, by the continuity of r and g with respect to u, we have $\psi(t,x,z) = \inf_{u\in\widehat{K}}[g(t,x,u)+z\cdot r(t,x,u)]$, $t\in[0,T]$, $x\in\mathbb{R}^n$, $z\in\mathbb{R}^d$, where $\widehat{K}$ is any countable dense subset of K.

Coming now to the proof of (3.2), we notice that, since g is nonnegative and satisfies (2.10), we have $g(t,x,u) \geq c(|u|^2 - R^2)$ (c and R are the same as in (2.10)). By (2.7) and (2.8), we have

$$g(t,x,u) + z \cdot r(t,x,u) \geq c(|u|^2 - R^2) - C|z|(1+|u|), \tag{3.5}$$

and it follows that

$$\begin{aligned}\psi(t,x,z) \geq \inf_{u\in\mathbb{R}^m}[g(t,x,u) + z\cdot r(t,x,u)] &\geq \inf_{u\in\mathbb{R}^m}[c(|u|^2 - R^2) - C|z|(1+|u|)]\\ &= -C_1|z|^2 - C_2,\end{aligned}$$

by direct computation, for suitable constants C_1 and C_2. This proves the left-hand side of (3.2). The right-hand side of (3.2) is immediate, since by (2.7),

$$\psi(t,x,z) \leq g(t,x,u) + z\cdot r(t,x,u) \leq g(t,x,u) + |z|C(1+|u|).$$

We come now to the second assertion. By (3.5) we get

$$g(t,x,u) + z\cdot r(t,x,u) \geq c|u|\left(|u| - \frac{C}{c}|z|\right) - cR^2 - C|z|. \tag{3.6}$$

On the other hand, if we fix an arbitrary $u^0 \in K$, then

$$g(t,x,u^0) + z\cdot r(t,x,u^0) \leq C(1+|x|^2+|u^0|^2) + C|z|(1+|u^0|) \leq C_3(1+|x|^2+|z|). \tag{3.7}$$

Hence, there exists a constant $C > 0$ such that, if $|u| > C(1+|x|+|z|)$, then

$$g(t,x,u) + z\cdot r(t,x,u) > g(t,x,u^0) + z\cdot r(t,x,u^0),$$

and (3.3) follows from the continuity of g and r with respect to u.

Finally, the continuity of $\psi(t,x,\cdot)$ can be easily proved, taking into account (3.3). □

Next we take an arbitrarily complete probability space $(\Omega, \mathcal{F}, \mathbb{P}^\circ)$ and a Wiener process W° in $\mathbb{R}^d$ with respect to $\mathbb{P}^\circ$. We denote by $(\mathcal{F}_t^\circ)$ the associated Brownian filtration, i.e., the filtration generated by W° and augmented by the $\mathbb{P}^\circ$-null sets of $\mathcal{F}$; $(\mathcal{F}_t^\circ)$ satisfies the usual conditions.

We introduce the forward equation

$$\begin{cases} dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dW_t^\circ, & t\in[0,T],\\ X_0 = x, \end{cases} \tag{3.8}$$

whose solution is a continuous $(\mathcal{F}_t^\circ)$-adapted process, which exists and is unique by classical results. Next we consider the associated backward equation

$$\begin{cases} dY_t = -\psi(t,X_t,Z_t)\,dt + Z_t\,dW_t^\circ, & t\in[0,T],\\ Y_T = \phi(X_T). \end{cases} \tag{3.9}$$

The solution of (3.9) exists in the sense specified by the following proposition.

PROPOSITION 3.2. *Assume that b, σ, g, ϕ satisfy Hypothesis* 2.1. *Then there exist Borel measurable functions*

$$v : [0,T]\times\mathbb{R}^n \to \mathbb{R}, \quad \zeta : [0,T]\times\mathbb{R}^n \to \mathbb{R}^d$$

with the following property: For an arbitrarily chosen complete probability space $(\Omega, \mathcal{F}, \mathbb{P}^\circ)$ *and Wiener process* W° *in* $\mathbb{R}^d$*, denoting by* X *the solution of* (3.8)*, the processes* Y, Z *defined by*

$$Y_t = v(t, X_t), \quad Z_t = \zeta(t, X_t)$$

satisfy

$$\mathbb{E}^\circ \sup_{t\in[0,T]} |Y_t|^2 < \infty, \quad \mathbb{E}^\circ \int_0^T |Z_t|^2 \, dt < \infty;$$

moreover, Y *is continuous and nonnegative, and* $\mathbb{P}^\circ$*-a.s.,*

$$Y_t + \int_t^T Z_s \, dW_s^\circ = \phi(X_T) + \int_t^T \psi(s, X_s, Z_s) \, ds, \quad t \in [0, T].$$

Finally, this solution is the maximal solution among all the solutions (Y', Z') *of* (3.9) *satisfying*

$$\mathbb{E}^\circ \left[\sup_{t\in[0,T]} |Y_t'|^2 \right] < +\infty.$$

Proof. From Lemma 3.1, there exists a constant $C > 0$ such that

$$-C(1 + |z|^2) \le \psi(t, x, z) \le g(t, x, u^0) + C(1 + |u^0|)|z|.$$

Let us first note that

$$\mathbb{E}^\circ \sup_{t\in[0,T]} |X_t|^p < \infty \ \forall p \ge 2.$$

Next, we adopt the same strategy as that in [2] to construct a maximal solution to (3.9); i.e., for each $n \ge C$, we define the globally Lipschitz continuous function,

$$\psi_n(t, x, z) = \sup\{\psi(t, x, q) - n|q - z| : q \in \mathbb{Q}^d\},$$

which is decreasing and converges to ψ; then by (Y^n, Z^n) we denote the unique solution to the BSDE with Lipschitz coefficient ψ_n,

$$\begin{cases} dY_t^n = -\psi_n(t, X_t, Z_t^n) \, dt + Z_t^n \, dW_t^\circ, & t \in [0, T], \\ Y_T^n = \phi(X_T), \end{cases} \tag{3.10}$$

and by (Y^S, Z^S) the unique solution to the BSDE,

$$\begin{cases} dY_t^S = -[g(t, X_t, u^0) + C(1 + |u^0|)|Z_t^S|] \, dt + Z_t^S \, dW_t^\circ, & t \in [0, T], \\ Y_T^S = \phi(X_T), \end{cases} \tag{3.11}$$

where C is the same as in (3.2). We notice that, since $\psi_n(t, x, 0) \ge \psi(t, x, 0) \ge 0$, then by an application of the comparison theorem (see [4]),

$$0 \le Y_t^n \le Y_t^S.$$

Then let us introduce the following stopping time: For $k \ge 1$,

$$\tau_k = \inf\{t \in [0, T] : \max(|X_t|, Y_t^S) > k\},$$

with the convention that $\tau_k = T$ if the indicated set is empty.

Then $(Y_k^n, Z_k^n) := (Y_{t\wedge\tau_k}^n, Z_t^n 1_{t\le\tau_k})$ satisfies the following BSDE:

$$Y_k^n(t) = \xi_k^n + \int_t^T 1_{s\le\tau_k}\psi_n(s, X_s, Z_k^n(s))ds - \int_t^T Z_k^n(s)dW_s^\circ,$$

where of course $\xi_k^n = Y_k^n(T) = Y_{\tau_k}^n$.

For fixed k, Y_k^n is decreasing in n and remains bounded by k. It follows from Lemma 3 in [2] (which is a slight generalization of Proposition 2.4 in [10]) that there exists a process (Y_k, Z_k) such that Y_k is a continuous process, $E\int_0^T |Z_k(s)|^2 ds < +\infty$,

$$\lim_n \sup_{t\in[0,T]} |Y_k^n(t) - Y_k(t)| = 0, \quad \lim_n E\int_0^T |Z_k^n(t) - Z_k(t)|^2 dt = 0,$$

and (Y_k, Z_k) solves the BSDE

$$Y_k(t) = \xi_k + \int_t^T 1_{s\le\tau_k}\psi(s, X_s, Z_k(s))ds - \int_t^T Z_k(s)dW_s^\circ, \tag{3.12}$$

where $\xi_k = \inf_n Y_{\tau_k}^n$.

On the other hand, $\tau_k \le \tau_{k+1}$, and, from the definition of (Y_k^n, Z_k^n), we have

$$Y_{k+1}^n(t\wedge\tau_k) = Y_k^n(t), \quad Z_{k+1}^n(t)1_{t\le\tau_k} = Z_k^n(t).$$

Sending n to infinity, we get

$$Y_{k+1}(t\wedge\tau_k) = Y_k(t), \quad Z_{k+1}(t)1_{t\le\tau_k} = Z_k(t).$$

Now we define Y and Z on $[0,T]$ by setting

$$Y_t = Y_k(t), \quad Z_t = Z_k(t) \quad \text{if} \quad t\in[0,\tau_k].$$

For $\mathbb{P}^\circ$-a.s. ω, there exists an integer $K(\omega)$ such that for $k \ge K(\omega)$, $\tau_k(\omega) = T$. Thus Y is a continuous process, $Y_T = \phi(X_T)$, and $\int_0^T |Z_t|^2 ds < \infty$, $\mathbb{P}^\circ$-a.s.

From (3.12), (Y, Z) satisfies

$$Y_{t\wedge\tau_k} = Y_{\tau_k} + \int_{t\wedge\tau_k}^{\tau_k} \psi(s, X_s, Z(s))ds - \int_{t\wedge\tau_k}^{\tau_k} Z(s)dW_s^\circ. \tag{3.13}$$

By sending k to infinity, we deduce that (Y, Z) is a solution of (3.9) and

$$\lim_n \sup_{t\in[0,T]} |Y_t^n - Y_t| = 0, \quad \lim_n \int_0^T |Z_t^n - Z_t|^2 dt = 0, \qquad \mathbb{P}^\circ\text{-a.s.}$$

Thus $|Z^n - Z|$ converges to zero in measure $d\mathbb{P}^\circ \otimes dt$, and passing, if needed, to a subsequence (that by abuse of language we still denote Z^n), we can assume that $|Z^n - Z| \to 0$, $d\mathbb{P}^\circ \otimes dt$ almost everywhere.

Now, as ψ_n is globally Lipschitz continuous, from [4] (see also [6]) there exist Borel measurable functions

$$v^n : [0,T]\times\mathbb{R}^n \to \mathbb{R}, \quad \zeta^n : [0,T]\times\mathbb{R}^n \to \mathbb{R}^d$$

such that

$$Y_t^n = v^n(t, X_t), \quad Z_t^n = \zeta^n(t, X_t).$$

It suffices to define

$$v(t,x) = \liminf_{n\to\infty} v^n(t,x), \text{ and } \zeta(t,x) = \liminf_{n\to\infty} \zeta^n(t,x)$$

(where the second lim inf is intended coordinatewise) to get

$$Y_t = v(t, X_t), \quad Z_t = \zeta(t, X_t),$$

which implies that (v, ζ) is the Borel function we look for.

Finally, $0 \le Y_t \le Y_t^S$ implies that

$$\mathbb{E} \sup_{t\in[0,T]} |Y_t|^2 < \infty,$$

and from the equation

$$|Y_t|^2 + \int_t^{\tau_k} |Z_s|^2 ds = 2\int_t^{\tau_k} Y_s \psi(s, X_s, Z_s) ds - 2\int_t^{\tau_k} Y_s Z_s dW_s^\circ,$$

taking into consideration that

$$Y_s\psi(s, X_s, Z_s) \le Y_s(g(s, X_s, u^0) + C(1 + |u^0|)|Z_s|) \le Y_s^S(g(s, X_s, u^0) + C(1 + |u^0|)|Z_s|),$$

we deduce, by standard arguments, that

$$\mathbb{E}^\circ \int_0^T |Z_t|^2 dt < \infty.$$

Moreover, this solution is the maximal solution among all the solutions (Y', Z') satisfying

$$\mathbb{E}^\circ \left[\sup_{t\in[0,T]} |Y_t'|^2 \right] < +\infty,$$

and it suffices to apply Proposition 5 of [2] to deduce that $Y^n \ge Y'$ and then $Y \ge Y'$. ∎

4. The fundamental relation. In this section we revert to the notation introduced in the first section and still assume that Hypothesis 2.1 holds.

PROPOSITION 4.1. *Let v, ζ denote the functions in the statement of Proposition* 3.2. *Then for every admissible control u and for the corresponding trajectory X starting at x, we have*

$$J(u) = v(0,x) + \mathbb{E}\int_0^T [-\psi(t, X_t, \zeta(t, X_t)) + \zeta(t, X_t)\cdot r(t, X_t, u_t) + g(t, X_t, u_t)]\, dt.$$

Proof. We introduce stopping times τ_n and control processes u^n as in the proof of Proposition 2.3, and we denote by X^n the solution to (2.14). Let us define

$$W_t^n = W_t + \int_0^t r(s, X_s^n, u_s^n)\, ds.$$

From the definition of τ_n and from (2.7), it follows that

$$\int_0^T |r(s, X_s^n, u_s^n)|^2\, ds \le C\int_0^T (1 + |u_s^n|)^2\, ds \le C\int_0^{\tau_n} (1 + |u_s|)^2\, ds + C \le C + Cn. \tag{4.1}$$

Therefore defining

$$\rho_n = \exp\left(\int_0^T -r(s, X_s^n, u_s^n)\, dW_s - \frac{1}{2}\int_0^T |r(s, X_s^n, u_s^n)|^2\, ds\right),$$

the Novikov condition implies that $\mathbb{E}\rho_n = 1$. Setting $d\mathbb{P}^n = \rho_n d\mathbb{P}$, by the Girsanov theorem W^n is a Wiener process under $\mathbb{P}^n$. Let us denote by $(\mathcal{F}_t^n)$ its natural augmented filtration. Since

$$\begin{cases} dX_t^n = b(t, X_t^n)\, dt + \sigma(t, X_t^n)\, dW_t^n, & t \in [0, T], \\ X_0^n = x \end{cases}$$

has a strong solution by classical results, the process X^n is also $(\mathcal{F}_t^n)$ adapted. Let us define

$$Y_t^n = v(t, X_t^n), \quad Z_t^n = \zeta(t, X_t^n).$$

Then by Proposition 3.2, we have

$$\begin{cases} dY_t^n = Z_t^n\, dW_t^n - \psi(t, X_t^n, Z_t^n)\, dt, & t \in [0, T], \\ Y_T^n = \phi(X_T^n) \end{cases} \tag{4.2}$$

and $\mathbb{E}^n \int_0^T |Z_t^n|^2\, dt < \infty$, where $\mathbb{E}^n$ denotes expectation with respect to $\mathbb{P}^n$. It follows that

$$Y_{\tau_n}^n = \phi(X_T^n) + \int_{\tau_n}^T \psi(t, X_t^n, Z_t^n)\, dt - \int_{\tau_n}^T Z_t^n\, dW_t - \int_{\tau_n}^T Z_t^n \cdot r(t, X_t^n, u_t^n)\, dt. \tag{4.3}$$

We note that for every $p \in [1, \infty)$ we have

$$\rho_n^{-p} = \exp\left(p\int_0^T r(s, X_s^n, u_s^n)\, dW_s^n - \frac{p^2}{2}\int_0^T |r(s, X_s^n, u_s^n)|^2\, ds\right)$$
$$\cdot \exp\left(\frac{p^2 - p}{2}\int_0^T |r(s, X_s^n, u_s^n)|^2\, ds\right).$$

By (4.1) the second exponential is bounded by a constant depending on n and p, while the first one has $\mathbb{P}^n$-expectation, equal to 1. So we conclude that $\mathbb{E}_n\rho_n^{-p} < \infty$. It follows that

$$\mathbb{E}\left(\int_0^T |Z_t^n|^2 dt\right)^{1/2} = \mathbb{E}^n\left(\rho_n^{-2}\int_0^T |Z_t^n|^2 dt\right)^{1/2} \le \left(\mathbb{E}^n\rho_n^{-2}\right)^{1/2}\left(\mathbb{E}^n\int_0^T |Z_t^n|^2 dt\right)^{1/2} < \infty.$$

We conclude that the stochastic integral in (4.3) has zero expectation, and we obtain

$$\mathbb{E}Y_{\tau_n}^n = \mathbb{E}\phi(X_T^n) + \mathbb{E}\int_{\tau_n}^T [\psi(t, X_t^n, Z_t^n) - Z_t^n \cdot r(t, X_t^n, u_t^n)]\, dt.$$

Since by definition, $\psi(t, x, z) - z \cdot r(t, x, u) - g(t, x, u) \le 0$, we have

$$\mathbb{E}Y_{\tau_n}^n \le \mathbb{E}\phi(X_T^n) + \mathbb{E}\int_{\tau_n}^T g(t, X_t^n, u_t^n)\, dt. \tag{4.4}$$

Now we let $n \to \infty$. By the definition of u^n and (2.8),

$$\mathbb{E}\int_{\tau_n}^T g(t, X_t^n, u_t^n)\, dt = \mathbb{E}\int_0^T 1_{t>\tau_n} g(t, X_t^n, u^0)\, dt$$
$$\leq C\mathbb{E}\int_0^T 1_{t>\tau_n}(1 + |X_t^n|^2 + |u^0|^2)dt \leq C\mathbb{E}\left[(T-\tau_n)\left(1 + \sup_{t\in[0,T]}|X_t^n|^2\right)\right],$$

and the right-hand side tends to 0 by Lemma 2.4 and (2.13). Next we note that, again by (2.13), for $n \geq N(\omega)$ we have $\tau_n(\omega) = T$ and

$$\phi(X_T^n) = \phi(X_{\tau_n}^n) = \phi(X_{\tau_n}) = \phi(X_T).$$

Moreover, by (2.9),

$$|\phi(X_T^n)| \leq C(1 + |X_T^n|^2) \leq C\left(1 + \sup_{t\in[0,T]}|X_t^n|^2\right),$$

and by Lemma 2.4 the right-hand side is uniformly integrable. We deduce that $\mathbb{E}\phi(X_T^n) \to \mathbb{E}\phi(X_T)$, and from (4.4) we conclude that $\limsup_{n\to\infty}\mathbb{E}Y_{\tau_n}^n \leq \mathbb{E}\phi(X_T)$. On the other hand, for $n \geq N(\omega)$ we have $\tau_n(\omega) = T$ and

$$Y_{\tau_n}^n = Y_T^n = \phi(X_T^n) = \phi(X_T).$$

Since Y^n is positive, by the Fatou lemma, $\mathbb{E}\phi(X_T) \leq \liminf_{n\to\infty}\mathbb{E}Y_{\tau_n}^n$. We have thus proved that

$$\lim_{n\to\infty}\mathbb{E}Y_{\tau_n}^n = \mathbb{E}\phi(X_T). \tag{4.5}$$

Now we return to (4.2) and write

$$Y_{\tau_n}^n = Y_0^n + \int_0^{\tau_n} -\psi(t, X_t^n, Z_t^n)\, dt + \int_0^{\tau_n} Z_t^n\, dW_t + \int_0^{\tau_n} Z_t^n \cdot r(t, X_t^n, u_t^n)\, dt.$$

Arguing as before, we conclude that the stochastic integral has zero $\mathbb{P}$-expectation. Moreover, we have $Y_0^n = v(0,x)$, and, for $t \leq \tau_n$, also have $u_t^n = u_t$, $X_t^n = X_t$, and $Z_t^n = \zeta(t, X_t)$. Thus we obtain

$$\mathbb{E}Y_{\tau_n}^n = v(0,x) + \mathbb{E}\int_0^{\tau_n}[-\psi(t, X_t, \zeta(t, X_t)) + \zeta(t, X_t)\cdot r(t, X_t, u_t)]\, dt$$

and

$$\mathbb{E}\int_0^{\tau_n} g(t, X_t, u_t)\, dt + \mathbb{E}Y_{\tau_n}^n$$
$$= v(0,x) + \mathbb{E}\int_0^{\tau_n}[-\psi(t, X_t, \zeta(t, X_t)) + \zeta(t, X_t)\cdot r(t, X_t, u_t) + g(t, X_t, u_t)]\, dt.$$

Noting that $-\psi(t,x,z) + z\cdot r(t,x,u) + g(t,x,u) \geq 0$ and recalling that $g(t,x,u) \geq 0$, by (4.5) and the monotone convergence theorem, we obtain for $n \to \infty$,

$$\mathbb{E}\int_0^T g(t, X_t, u_t)\, dt + \mathbb{E}\phi(X_T) = v(0,x)$$
$$+ \mathbb{E}\int_0^T [-\psi(t, X_t, \zeta(t, X_t)) + \zeta(t, X_t)\cdot r(t, X_t, u_t) + g(t, X_t, u_t)]\, dt,$$

which gives the required conclusion. $\square$

COROLLARY 4.2. *For every admissible control u and any initial datum x, we have $J(u) \geq v(0,x)$, and the equality holds if and only if the following feedback law holds $\mathbb{P}$-a.s. for almost every $t \in [0,T]$:*

$$\psi(t, X_t, \zeta(t, X_t)) = \zeta(t, X_t) \cdot r(t, X_t, u_t) + g(t, X_t, u_t),$$

where X is the trajectory starting at x and corresponding to control u.

5. Existence of optimal controls: The closed loop equation. Let us consider again the functions b, σ, g, ϕ satisfying the assumptions in Hypothesis 2.1. We recall the definition of the Hamiltonian function:

$$\psi(t,x,z) = \inf_{u \in K} [g(t,x,u) + z \cdot r(t,x,u)], \qquad t \in [0,T], x \in \mathbb{R}^n, \quad z \in \mathbb{R}^d. \tag{5.1}$$

LEMMA 5.1. *There exists a Borel measurable function $\gamma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^d \to K$ such that*

$$\psi(t,x,z) = g(t,x,\gamma(t,x,z)) + z \cdot r(t,x,\gamma(t,x,z)), \qquad t \in [0,T], \quad x \in \mathbb{R}^n, \quad z \in \mathbb{R}^d. \tag{5.2}$$

Moreover, there exists a constant $C > 0$ such that

$$|\gamma(t,x,z)| \leq C(1 + |x| + |z|). \tag{5.3}$$

Proof. Consider the function $F(t,x,z,u) = g(t,x,u) + z \cdot r(t,x,u)$, $t \in [0,T]$, $x \in \mathbb{R}^n$, $z \in \mathbb{R}^d$. Clearly F is a Carathéodory map (that is, $F(t,x,z,\cdot)$ is continuous for all $t \in [0,T]$, $x \in \mathbb{R}^n$, and $z \in \mathbb{R}^d$, and $F(\cdot,\cdot,\cdot,u)$ is Borel measurable for all $u \in K$; see [1, p. 311]). By (3.3) we have

$$\psi(t,x,z) \in \{F(t,x,z,u) : u \in K\} \quad \forall t \in [0,T], \quad x \in \mathbb{R}^n, \quad z \in \mathbb{R}^d.$$

Thus by the Filippov theorem (see, e.g., [1, Thm. 8.2.10, p. 316]) there exists a Borel measurable map $\gamma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^d \to K$ such that $F(t,x,z,\gamma(t,x,z)) = \psi(t,x,z)$; see [3] as well. The fact that $|\gamma(t,x,z)| \leq C(1+|x|+|z|)$ is an immediate consequence of (3.4). $\square$

Next we address the problem of finding a weak solution to the so-called closed loop equation. We define

$$\underline{u}(t,x) = \gamma(t,x,\zeta(t,x)), \qquad t \in [0,T], \quad x \in \mathbb{R}^n,$$

where ζ is defined in Proposition 3.2. The closed loop equation is

$$\begin{cases} dX_t = b(t,X_t)\, dt + \sigma(t,X_t)\, [dW_t + r(t,X_t,\underline{u}(t,X_t))\, dt], & t \in [0,T], \\ X_0 = x. \end{cases} \tag{5.4}$$

By a weak solution we mean a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions, a Wiener process W in $\mathbb{R}^d$ with respect to $\mathbb{P}$ and $(\mathcal{F}_t)$, and a continuous $(\mathcal{F}_t)$-adapted process X with values in $\mathbb{R}^n$ satisfying, $\mathbb{P}$-a.s.,

$$\int_0^T |\underline{u}(t,X_t)|^2\, dt < \infty, \tag{5.5}$$

and such that (5.4) holds. We note that by (2.7) it also follows that

$$\int_0^T |r(t,X_t,\underline{u}(t,X_t))|\, dt < \infty, \quad \mathbb{P}\text{-a.s.},$$

so that (5.4) makes sense.

PROPOSITION 5.2. *Assume that b, σ, g, ϕ satisfy Hypothesis 2.1. Then there exists a weak solution of the closed loop equation, satisfying in addition*

$$\mathbb{E}\int_0^T |\underline{u}(t,X_t)|^2\, dt < \infty. \tag{5.6}$$

Proof. Let us take an arbitrary complete probability space $(\Omega, \mathcal{F}, \mathbb{P}^\circ)$ and a Wiener process W° in $\mathbb{R}^d$ with respect to $\mathbb{P}^\circ$. Let $(\mathcal{F}_t^\circ)$ be the associated Brownian filtration. We define the process X as the solution of the equation

$$\begin{cases} dX_t = b(t,X_t)\, dt + \sigma(t,X_t)\, dW_t^\circ, \qquad t\in[0,T],\\ X_0 = x. \end{cases} \tag{5.7}$$

The solution is a continuous $(\mathcal{F}_t^\circ)$-adapted process, which exists and is unique by classical results. Moreover, it satisfies $\mathbb{E}^\circ \sup_{t\in[0,T]} |X_t|^p < \infty$ for every $p \in [1,\infty)$.

By Proposition 3.2, setting

$$Y_t = v(t,X_t), \quad Z_t = \zeta(t,X_t),$$

the following backward equation holds:

$$\begin{cases} dY_t = -\psi(t,X_t,Z_t)\, dt + Z_t\, dW_t^\circ, \qquad t\in[0,T],\\ Y_T = \phi(X_T), \end{cases}$$

and we have

$$\mathbb{E}^\circ \int_0^T |Z_t|^2\, dt < \infty. \tag{5.8}$$

By (2.7) we have $|r(t,X_t,\underline{u}(t,X_t))| \le C(1+|\underline{u}(t,X_t)|)$, and by (5.3),

$$|\underline{u}(t,X_t)| = |\gamma(t,X_t,\zeta(t,X_t))| \le C(1+|X_t|+|\zeta(t,X_t)|) = C(1+|X_t|+|Z_t|). \tag{5.9}$$

Now let us define stopping times

$$\tau_n = \inf\left\{ t\in[0,T] \ :\ \int_0^t |\underline{u}(s,X_s)|^2 ds > n \right\},$$

with the convention that $\tau_n = T$ if the indicated set is empty. By (5.8) and (5.9), for $\mathbb{P}^\circ$-a.s. $\omega\in\Omega$, there exists an integer $N(\omega)$ depending on ω such that $\tau_n(\omega) = T$ for $n \ge N(\omega)$. Let us fix $u^0 \in K$, and for every n let us define

$$u_t^n = \underline{u}(t,X_t)\, 1_{t\le\tau_n} + u^0\, 1_{t>\tau_n},$$

$$M_t^n = \exp\left(\int_0^t r(s,X_s,u_s^n)\, dW_s^\circ - \frac{1}{2}\int_0^t |r(s,X_s,u_s^n)|^2\, ds \right),$$

$$M_t = \exp\left(\int_0^t r(s,X_s,\underline{u}(s,X_s))\, dW_s^\circ - \frac{1}{2}\int_0^t |r(s,X_s,\underline{u}(s,X_s))|^2\, ds \right),$$

$$W_t^n = W_t^\circ - \int_0^t r(s,X_s,u_s^n)\, ds,$$

$$W_t = W_t^\circ - \int_0^t r(s,X_s,\underline{u}(s,X_s))\, ds.$$

By the previous estimates, M^n, M, W^n, and W are well defined; moreover,

$$\int_0^T |r(s, X_s, u_s^n) - r(s, X_s, \underline{u}(s, X_s))|^2 \, ds \to 0, \qquad \mathbb{P}^\circ\text{-a.s.},$$

and consequently $M_T^n \to M_T$ in probability and $\sup_{t\in[0,T]} |W_t^n - W_t| \to 0$, $\mathbb{P}^\circ$-a.s. We will conclude the proof by showing that there exists a probability $\mathbb{P}$ such that W is a Wiener process with respect to $\mathbb{P}$ and $(\mathcal{F}_t^\circ)$.

The definition of τ_n and the Novikov condition implies that $\mathbb{E}^\circ M_T^n = 1$. Setting $d\mathbb{P}^n = M_T^n d\mathbb{P}^\circ$, by the Girsanov theorem W^n is a Wiener process with respect to $\mathbb{P}^n$ and $(\mathcal{F}_t^\circ)$. Writing the backward equation with respect to W^n, we obtain

$$Y_{\tau_n} = Y_0 + \int_0^{\tau_n} -\psi(t, X_t, Z_t) \, dt + \int_0^{\tau_n} Z_t \, dW_t^n + \int_0^{\tau_n} Z_t \cdot r(t, X_t, u_t^n) \, dt.$$

Arguing as in the proof of Proposition 4.1, we conclude that the stochastic integral has zero expectation with respect to $\mathbb{P}^n$. Taking into account that $u_t^n = \underline{u}(t, X_t)$ for $t \leq \tau_n$, we obtain

$$\begin{aligned}
&\mathbb{E}^n Y_{\tau_n} + \mathbb{E}^n \int_0^{\tau_n} g(t, X_t, \underline{u}(t, X_t)) \, dt \\
&\quad = Y_0 + \mathbb{E}^n \int_0^{\tau_n} [-\psi(t, X_t, Z_t) + Z_t \cdot r(t, X_t, \underline{u}(t, X_t)) + g(t, X_t, \underline{u}(t, X_t))] \, dt \\
&\quad = Y_0,
\end{aligned}$$

with the last equality coming from the definition of $\underline{u}$. Recalling that Y is nonnegative, it follows that

$$\mathbb{E}^n \int_0^{\tau_n} g(t, X_t, \underline{u}(t, X_t)) \, dt \leq C$$

for some constant C independent of n. By (2.10) we also deduce

$$\mathbb{E}^n \int_0^{\tau_n} |\underline{u}(t, X_t)|^2 \, dt \leq C. \tag{5.10}$$

Next we prove that the family $\{M_T^n,\ n = 1, 2, \dots\}$ is uniformly integrable by showing that $\mathbb{E}^\circ[M_T^n \, 1_{\{M_T^n > c\}}] \to 0$ as $c \to \infty$, uniformly with respect to n. We have

$$\mathbb{E}^\circ[M_T^n \, 1_{\{M_T^n > c\}}] = \mathbb{E}^\circ[M_T^n \, 1_{\{M_T^n > c, \tau_n = T\}}] + \mathbb{E}^\circ[M_T^n \, 1_{\{M_T^n > c, \tau_n < T\}}]. \tag{5.11}$$

The first term in the right-hand side tends to 0 uniformly with respect to n, since

$$\mathbb{E}^\circ[M_T^n \, 1_{\{M_T^n > c, \tau_n = T\}}] = \mathbb{E}^\circ[M_T \, 1_{\{M_T > c, \tau_n = T\}}] \leq \mathbb{E}^\circ[M_T \, 1_{\{M_T > c\}}] \to 0,$$

due to the fact that the equality $\mathbb{E}^\circ M_T^n = 1$ and the Fatou lemma imply that $\mathbb{E}^\circ M_T \leq 1$. The second term in the right-hand side of (5.11) can be estimated as follows:

$$\begin{aligned}
&\mathbb{E}^\circ[M_T^n \, 1_{\{M_T^n > c, \tau_n < T\}}] \leq \mathbb{E}^\circ[M_T^n \, 1_{\tau_n < T\}}] = \mathbb{P}^n(\tau_n < T) \\
&\quad \leq \mathbb{P}^n \left(\int_0^{\tau_n} |\underline{u}(t, X_t)|^2 dt > n \right) \leq \frac{1}{n} \mathbb{E}^n \int_0^{\tau_n} |\underline{u}(t, X_t))|^2 dt \leq \frac{C}{n},
\end{aligned}$$

with the last inequality coming from (5.10). The required uniform integrability follows immediately. Recalling that $M_T^n \to M_T$ in probability, we conclude that $\mathbb{E}^\circ|M_T^n - M_T| \to 0$, and in particular $\mathbb{E}^\circ M_T = 1$, and M is a $\mathbb{P}^\circ$-martingale. Thus we can define a probability $\mathbb{P}$ by setting $d\mathbb{P} = M_T d\mathbb{P}^\circ$, and by the Girsanov theorem we conclude that W is a Wiener process with respect to $\mathbb{P}$ and $(\mathcal{F}_t^\circ)$.

It remains to prove (5.6). We define stopping times

$$\sigma_n = \inf\left\{t \in [0,T] : \int_0^t |Z_s|^2 ds > n\right\},$$

with the convention that $\sigma_n = T$ if the indicated set is empty. Writing the backward equation with respect to W, we obtain

$$Y_{\sigma_n} = Y_0 + \int_0^{\sigma_n} -\psi(t, X_t, Z_t)\, dt + \int_0^{\sigma_n} Z_t\, dW_t + \int_0^{\sigma_n} Z_t \cdot r(t, X_t, \underline{u}(t, X_t))\, dt,$$

from which we deduce that

$$\begin{aligned} &\mathbb{E}Y_{\sigma_n} + \mathbb{E}\int_0^{\sigma_n} g(t, X_t, \underline{u}(t, X_t))\, dt \\ &\quad = Y_0 + \mathbb{E}\int_0^{\sigma_n} [-\psi(t, X_t, Z_t) + Z_t \cdot r(t, X_t, \underline{u}(t, X_t)) + g(t, X_t, \underline{u}(t, X_t))]\, dt \\ &\quad = Y_0, \end{aligned}$$

with the last equality coming from the definition of $\underline{u}$. Recalling that Y is nonnegative, it follows that

$$\mathbb{E}\int_0^{\sigma_n} g(t, X_t, \underline{u}(t, X_t))\, dt \le C$$

for some constant C independent of n. By (2.10) and by sending n to infinity, we finally prove (5.6). □

Corollary 5.3. *By Corollary* 4.2 *it immediately follows that if X is the solution to* (5.4) *and we set $u_s^\sharp = \underline{u}(s, X_s)$, then $J(x, u^\sharp) = v(0, x)$, and consequently X is an optimal state, $u_s^\sharp$ is an optimal control, and $\underline{u}$ is an optimal feedback.*

Next we prove uniqueness in law for the closed loop equation. We remark that condition (5.5) is part of our definition of a weak solution.

Proposition 5.4. *Assume that b, σ, g, ϕ satisfy Hypothesis* 2.1*. Fix $\gamma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^d \to K$ satisfying* (5.2) (*and consequently* (5.3)) *and let $\underline{u}(t,x) = \gamma(t, x, \zeta(t,x))$. Then the weak solution of the closed loop equation* (5.4) *is unique in law.*

Proof. Let $(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{F}_t)$, W, X be a weak solution of (5.4).

Let us define

$$M_T = \exp\left(-\int_0^T r(s, X_s, \underline{u}(s, X_s))\, dW_s - \frac{1}{2}\int_0^T |r(s, X_s, \underline{u}(s, X_s))|^2\, ds\right),$$

$$W_t^\circ = W_t + \int_0^t r(s, X_s, \underline{u}(s, X_s))\, ds.$$

By (2.7) and (5.5), M_T and W° are well defined. We claim that $\mathbb{E}M_T = 1$. Assuming the claim for a moment, and setting $d\mathbb{P}^\circ = M_T d\mathbb{P}$, by the Girsanov theorem W° is a Wiener process under $\mathbb{P}^\circ$, X solves

$$\begin{cases} dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t^\circ, & t \in [0,T], \\ X_0 = x, \end{cases}$$

and

$$M_T = \exp\left(-\int_0^T r(s, X_s, \underline{u}(s, X_s))\, dW_s^\circ + \frac{1}{2}\int_0^T |r(s, X_s, \underline{u}(s, X_s))|^2\, ds \right).$$

By the Lipschitz and linear growth conditions on b and σ (see (2.3), (2.4), (2.5)) the law of (X, W°) under $\mathbb{P}^\circ$ is uniquely determined by b, σ, x. Taking into account the last displayed formula, we conclude that the law of (X, W°, M_T) under $\mathbb{P}^\circ$ is also uniquely determined, and consequently so is the law of X under $\mathbb{P}$.

To conclude the proof it remains to show that $\mathbb{E}M_T = 1$. We define stopping times

$$\tau_n = \inf\left\{ t \in [0, T] \;:\; \int_0^t |\underline{u}(s, X_s)|^2 ds > n \right\},$$

with the convention that $\tau_n = T$ if the indicated set is empty. By (5.5), for $\mathbb{P}$-almost every $\omega \in \Omega$, there exists an integer $N(\omega)$ depending on ω such that $\tau_n(\omega) = T$ for $n \geq N(\omega)$. Let us fix $u^0 \in K$, and for every n, let us define

$$u_t^n = \underline{u}(t, X_t)\; 1_{t \leq \tau_n} + u^0\; 1_{t > \tau_n},$$

$$M_T^n = \exp\left(-\int_0^T r(s, X_s, u_s^n)\, dW_s - \frac{1}{2}\int_0^T |r(s, X_s, u_s^n)|^2\, ds \right).$$

By (2.7) and the definition of τ_n, the Novikov condition shows that $\mathbb{E}M_T^n = 1$. Moreover, we have

$$\int_0^T |r(s, X_s, u_s^n) - r(s, X_s, \underline{u}(s, X_s))|^2\, ds \to 0, \qquad \mathbb{P}\text{-a.s.},$$

and consequently $M_T^n \to M_T$ in probability. In order to conclude the proof it is therefore enough to show that the family $\{M_T^n,\, n = 1, 2, \dots\}$ is uniformly integrable.

To prepare for this, let us set $d\mathbb{P}^n = M_T^n d\mathbb{P}$ and note that, by the Girsanov theorem, the process

$$W_t^n = W_t + \int_0^t r(s, X_s, u_s^n)\, ds$$

is a Wiener process under $\mathbb{P}^n$. Since X solves

$$\begin{cases} dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t^n, & t \in [0, T], \\ X_0 = x, \end{cases}$$

it follows that X is adapted to the Brownian filtration $(\mathcal{F}_t^n)$ associated to W^n, and its law under $\mathbb{P}^n$ is uniquely determined by b, σ, x. In particular, the quantities

$$C' := \mathbb{E}^n \int_0^T |X_t|^2\, dt, \qquad C'' := \mathbb{E}^n \int_0^T |\zeta(t, X_t)|^2\, dt$$

do not depend on n (here $\mathbb{E}^n$ denotes, of course, the expectation with respect to $\mathbb{P}^n$). C' is clearly finite. By Proposition 3.2, setting $Z_t = \zeta(t, X_t)$, we have

$$\mathbb{E}^n \int_0^T |\zeta(t, X_t)|^2\, dt = \mathbb{E}^n \int_0^T |Z_t|^2\, dt < \infty,$$

and it follows that C'' is also finite.

Now let us prove the uniform integrability of the family $\{M_T^n,\ n = 1, 2, \dots\}$ by showing that $\mathbb{E}[M_T^n\, 1_{\{M_T^n > c\}}] \to$ as $c \to \infty$, uniformly with respect to n. We have

$$\mathbb{E}[M_T^n\, 1_{\{M_T^n > c\}}] = \mathbb{E}[M_T^n\, 1_{\{M_T^n > c, \tau_n = T\}}] + \mathbb{E}[M_T^n\, 1_{\{M_T^n > c, \tau_n < T\}}]. \tag{5.12}$$

The first term in the right-hand side tends to 0 uniformly with respect to n, since

$$\mathbb{E}[M_T^n\, 1_{\{M_T^n > c, \tau_n = T\}}] = \mathbb{E}[M_T\, 1_{\{M_T > c, \tau_n = T\}}] \le \mathbb{E}[M_T\, 1_{\{M_T > c\}}] \to 0,$$

due to the fact that the equality $\mathbb{E}M_T^n = 1$ and the Fatou lemma imply that $\mathbb{E}M_T \le 1$. The second term in the right-hand side of (5.12) can be estimated as follows:

$$\begin{aligned} &\mathbb{E}[M_T^n\, 1_{\{M_T^n > c, \tau_n < T\}}] \le \mathbb{E}[M_T^n\, 1_{\tau_n < T}] = \mathbb{P}^n(\tau_n < T) \\ &\quad \le \mathbb{P}^n\left(\int_0^{\tau_n} |\underline{u}(t, X_t)|^2 dt > n\right) \le \frac{1}{n}\mathbb{E}^n \int_0^{\tau_n} |\underline{u}(t, X_t)|^2 dt \le \frac{1}{n}\mathbb{E}^n \int_0^T |\underline{u}(t, X_t)|^2 dt. \end{aligned}$$

By (5.3) we have

$$|\underline{u}(t, X_t)|^2 = |\gamma(t, X_t, \zeta(t, X_t))|^2 \le C(1 + |X_t|^2 + |\zeta(t, X_t)|^2)$$

for some constant C, and it follows that

$$\mathbb{E}[M_T^n\, 1_{\{M_T^n > c, \tau_n < T\}}] \le \frac{C}{n}\mathbb{E}^n \int_0^T (1 + |X_t|^2 + |\zeta(t, X_t)|^2)\, dt = \frac{C}{n}(T + C' + C''),$$

with C' and C'' defined as above. The required uniform integrability follows immediately, and this concludes the proof. $\square$

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