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A variational approach to the reconstruction of cracks by boundary measurements ☆

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Abstract

We consider a conducting body which presents some (unknown) perfectly insulating defects, such as cracks or cavities, for instance. We aim to reconstruct the defects by performing measurements of current and voltage type on a (known and accessible) part of the boundary of the conductor. A crucial step in this reconstruction is the determination of the electrostatic potential inside the conductor, by the electrostatic boundary measurements performed. Since the defects are unknown, we state such a determination problem as a free-discontinuity problem for the electrostatic potential in the framework of special functions of bounded variation. We provide a characterisation of the looked for electrostatic potential and we approximate it with the minimum points of a sequence of functionals, which take also in account the error in the measurements. These functionals are related to the so-called Mumford–Shah functional, which acts as a regularizing term and allows us to prove existence of minimizers and Γ-convergence properties.

Résumé

On considère un corps conducteur qui présente des défauts (inconnus) parfaitement isolants, comme par exemple des fissures ou des cavités. On voudrait reconstituer les défauts en effectuant des mesures de type courant et tension sur une partie (connue et accessible) du bord du conducteur. Un pas crucial dans cette reconstitution est la détermination du potentiel électrostatique à l'intérieur du conducteur par les mesures électrostatiques effectuées au bord. Comme les défauts sont inconnus, on formule ce problème de détermination comme un problème aux discontinuités libres dans l'espace des fonctions spéciales à variation bornée. On donne une caractérisation du potentiel électrostatique en question et on l'approche avec les minimiseurs d'une suite de fonctionnelles, qui tiennent aussi compte des erreurs dans les mesures. Ces fonctionnelles sont liées à la fonctionnelle de Mumford–Shah, qui agit comme un terme régularisant et permet de démontrer l'existence de minimiseurs et les propriétés de Γ-convergence.

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1. Introduction

We consider a homogeneous and isotropic electrically conducting body which occupies Ω, a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, with a reasonably smooth boundary $\partial\Omega$. The conductor might present some perfectly insulating defects, such as cracks (either interior or surface breaking), cavities or material losses at the boundary, which might be caused by different phenomena, like for instance fractures or corrosion. We call K the closed set which is the union of the boundaries of these defects. If we prescribe a current density $f \in L^2(\partial\Omega)$, with $\int_{\partial\Omega} f = 0$ and such that its support is contained in $\tilde{\gamma}$, a part of the boundary of Ω which is accessible, known and disjoint from K, then the electrostatic potential $u = u(f, K)$ inside the conductor solves the following Neumann type boundary value problem, whose precise formulation will be discussed in Section 3,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\backslash K, \\ \nabla u \cdot \nu = f & \text{on } \tilde{\gamma}, \\ \nabla u \cdot \nu = 0 & \text{on } \partial(\Omega\backslash K)\backslash\tilde{\gamma}. \end{cases} \tag{1.1}$$

We are interested in the following inverse problem. We would like to determine an unknown defect K by performing boundary measurements of voltage and current type. In practice, we prescribe one or more currents f and we measure on γ, an accessible and known part of $\partial\Omega$, the value of the corresponding potentials u. Through these measurements we obtain additional information with which we would like to recover the unknown defect K. For what concerns the determination of cracks we refer to the recent review paper [12], where uniqueness, stability and reconstruction procedures, in two and three dimensions, are discussed. For the determination of other defects, such as cavities or material losses at the boundary, we refer to the following papers and to the references therein. The uniqueness and stability issues are treated in [30], for the two-dimensional case, and in [3], for the higher-dimensional case. Various reconstruction procedures and numerical methods have been suggested, see for example [7,8,21,24].

A two-steps procedure is usually employed to deal with these kinds of inverse problems, see for instance [18] and [4]. In the first step, the potential is recovered from the boundary measurements of voltage and current type. Subsequently, in the second step, features of the potential such as singularities, level sets or critical points are used to determine the unknown defect K. For instance, in our case, that is when the defects are perfectly insulating, the jump set of u, $S(u)$, is contained in K. Thus $S(u)$ would identify at least a part of our defect. Repeating the procedure for different and suitable choices of f, the union of the jump sets of the corresponding potentials would cover the whole K. The uniqueness results which are available in the literature give us information on how many and which kind of measurements we need to take in order to identify uniquely, at least in a suitable class of admissible defects, the unknown K. Here we limit ourselves to notice that in many interesting cases a finite number (usually one or two) of suitably chosen measurements is enough. We recall that, on the other hand, a single measurement is not enough, in general, to determine cracks, see for instance [18]. Furthermore, in dimension higher than 2, for what concerns insulating cracks, still a general uniqueness result with a finite number of measurements is missing, the only available result, [4], deals with planar cracks only.

In this paper we develop a reconstruction procedure for the first step of the previous scheme. If K is the unknown defect and f is the prescribed current density, we measure $g = u(f, K)|_\gamma$, where $u(f, K)$ solves (1.1). Given f and g, we would like to reconstruct the electrostatic potential inside the conductor. In [31], the unique identification of the potential from the Cauchy data (g, f) and corresponding stability results have been proven. Here, instead, we prove a characterisation of the looked for potential $u(f, K)$ in a suitable subset of the space of special functions of bounded variation in terms of the Cauchy data (g, f), see Proposition 5.6. However, in order to implement a reconstruction procedure from a numerical point of view, some further issues need to be considered. In fact, one usually has to deal with noisy measurements, that is f, the prescribed current density, and in particular g, the measured potential at the boundary, are known up to some noise which is due to the errors the measurements are subject to. Then, as usual with inverse boundary value problems of this kind, the problem is severely ill-posed. Therefore, any numerical procedure must contain some kind of regularization. Hence, we construct a family of functionals which depend on the level of noise on the measurements and contain as a regularization term the so-called Mumford–Shah functional which has been introduced in [26] as an image segmentation method. We show that these functionals admit a minimum and are such that their minimizers converge in a suitable sense to $u(f, K)$, as the level of noise goes to zero. Here the Mumford–Shah functional term has several uses. It guarantees existence of a minimum in a common compact set, see Proposition 5.2, it acts as a regularizing term and allows us to prove the Γ-convergence of the functionals, Theorem 5.3, which is the key step to prove the required convergence of the minimizers. We recall that the Mumford–Shah functional has already been used in the context of inverse problems as a regularization term for the determination of discontinuous conductivities, see [32].

The main advantages of our method are the following. In Newton's type methods, see for instance [34,11], one has to solve the direct problem at each iteration, usually in a different domain, whereas in our case we do not need to solve the direct problem. However, in order to deal with overdetermined data in a variational framework, some of them have to be prescribed as constraints, therefore we use a minmax approach in the formulation of the functionals. Furthermore, we do not make use of many a priori information on the topology of the defects, in particular we do not know a priori how many defects are present, whether they are cracks, cavities or material losses at the boundary. With respect to this issue our procedure presents some similarities with level set methods, see for instance [33,22,28]. Moreover, we can treat the case in which defects of different types are present in the conductor simultaneously. These defects may in fact include at the same time cavities, material losses at the boundary, interior and surface breaking cracks. Furthermore, we do not impose strong a priori conditions on the defects (like assuming that the cracks are linear, for example) which might be required by some of the reconstruction methods which employ particular properties satisfied by the potential if the defect is of special type, like for instance the reciprocity gap principle, [6,9], the use of Schwarz–Christoffel formula, [16,13], and the analysis of crack tip singularities, [20]. In fact, our approach is closer in spirit to shape optimization problems, see for instance [29] for applications of shape optimization techniques to the inverse crack problem. Finally, the method is developed for any dimension $N \geqslant 2$.

The plan of the paper is the following. After a section of preliminaries, Section 2, we discuss the direct problem (1.1) in Section 3. In Section 4 we discuss the classes of admissible defects we shall use. The reconstruction method is developed in Section 5, a final discussion is contained in Section 6.

2. Preliminaries

Throughout the paper the integer $N \geqslant 2$ will denote the dimension of the space. For every $x \in \mathbb{R}^N$, we shall set $x = (x', x_N)$, where $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$, and, for any $r > 0$, we shall denote by $B_r(x)$ and $B'_r(x')$, respectively, the open ball in $\mathbb{R}^N$ centred at x of radius r and the open ball in $\mathbb{R}^{N-1}$ centred at x' of radius r. Usually we shall write B_r and B'_r instead of $B_r(0)$ and $B'_r(0)$, respectively. Furthermore, for any $r > 0$ and $t > 0$, we set $Q'_r(x') = \prod_{i=1}^{N-1}(x_i - r, x_i + r) \subset \mathbb{R}^{N-1}$ and $Q_{r,t}(x) = \{y = (y', y_N) \in \mathbb{R}^N\colon\ y' \in Q'_r(x'),\ y_N \in (x_N - t, x_N + t)\}$. Again, Q'_r and $Q_{r,t}$ shall denote $Q'_r(0)$ and $Q_{r,t}(0)$, respectively.

For any non-negative integer k we denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure. We recall that for Borel subsets of $\mathbb{R}^N$ the N-dimensional Hausdorff measure coincides with $\mathcal{L}^N$, the N-dimensional Lebesgue measure. Furthermore, if $\gamma \subset \mathbb{R}^N$ is a smooth manifold of dimension k, then $\mathcal{H}^k$ restricted to γ coincides with its k-dimensional surface measure. For any Borel $E \subset \mathbb{R}^N$ we let $|E| = \mathcal{L}^N(E)$ and $[E] = \mathcal{H}^{N-1}(E)$.

We recall that a bounded domain $\Omega \subset \mathbb{R}^N$ is said to have a *Lipschitz boundary* if for every $x \in \partial\Omega$ there exist a Lipschitz function $\varphi : \mathbb{R}^{N-1} \mapsto \mathbb{R}$ and a positive constant r such that for any $y \in B_r(x)$ we have, up to a rigid transformation,

$$y \in \Omega \quad \text{if and only if} \quad y_N < \varphi(y').$$

We observe that the boundary of Ω, $\partial\Omega$, has finite $(N-1)$-dimensional Hausdorff measure, that is $[\partial\Omega] < +\infty$.

We say that a function $\varphi : A \mapsto B$, A and B being metric spaces, is *bi-Lipschitz* if it is invertible and φ and $\varphi^{-1} : \varphi(A) \mapsto A$ are both Lipschitz functions. If both the Lipschitz constants of φ and φ^{-1} are bounded by $L > 0$, then we say that φ is *bi-Lipschitz* with constant L.

We recall some basic notation and properties of functions of bounded variation and sets of finite perimeter. For a more comprehensive treatment of these subjects see, for instance, [5,17].

Given an open bounded set $\Omega \subset \mathbb{R}^N$, we denote by $BV(\Omega)$ the Banach space of *functions of bounded variation*. We recall that $u \in BV(\Omega)$ if and only if $u \in L^1(\Omega)$ and its distributional derivative Du is a bounded vector measure. We endow $BV(\Omega)$ with the standard norm as follows. Given $u \in BV(\Omega)$, we denote by $|Du|$ the total variation of its distributional derivative and we set $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$. We say that a sequence of $BV(\Omega)$ functions

$\{u_h\}_{h=1}^{\infty}$ converges *weakly** in $BV(\Omega)$ if and only if u_h converges to u in $L^1(\Omega)$ and Du_h weakly* converges to Du in Ω, that is

$$\lim_{h\to\infty}\int_{\Omega} v\,\mathrm{d}Du_h=\int_{\Omega} v\,\mathrm{d}Du \quad \text{for any } v\in C_0(\Omega). \tag{2.1}$$

We denote by $SBV(\Omega)$ the space of *special functions of bounded variation* that is the space of functions $u\in BV(\Omega)$ so that Du has a singular part, with respect to the N-dimensional Lebesgue measure, concentrated on $S(u)$, $S(u)$ being the *approximate discontinuity set* (or *jump set*) of u. The density of the absolutely continuous part of Du with respect to the N-dimensional Lebesgue measure will be denoted by ∇u, the *approximate gradient* of u.

The special functions of bounded variation satisfy the following compactness and semicontinuity theorem, see for instance [5, Theorems 4.7 and 4.8].

Theorem 2.1 *(SBV compactness and semicontinuity). We fix a constant p, $1<p<+\infty$. If $\{u_h\}_{h=1}^{\infty}$ is a sequence of functions belonging to $SBV(\Omega)$ satisfying for a given constant $C>0$,*

$$\|u_h\|_{L^\infty(\Omega)}\leqslant C, \quad \textit{for any } h, \tag{2.2}$$

and

$$\int_{\Omega}|\nabla u_h|^p+\big[S(u_h)\big]\leqslant C, \quad \textit{for any } h, \tag{2.3}$$

then we may extract a subsequence, which we relabel $\{u_k\}_{k=1}^{\infty}$, such that u_k converges weakly in $BV(\Omega)$ to a function $u\in SBV(\Omega)$ and the following lower semicontinuity properties hold*:

$$\big[S(u)\big]\leqslant\liminf_k\big[S(u_k)\big]; \qquad \int_{\Omega}|\nabla u|^p\leqslant\liminf_k\int_{\Omega}|\nabla u_k|^p. \tag{2.4}$$

Let E be a bounded Borel set contained in $\mathbb{R}^N$ and let $r>0$ be such that E is compactly contained in B_r. We say that E is a *set of finite perimeter* if its characteristic function χ_E belongs to $BV(B_r)$ and we call the number $P(E)=|D\chi_E|(B_r)$ its *perimeter*.

For any set E of finite perimeter, let ∂^*E be the *reduced boundary* in the De Giorgi sense, that is the set of $x\in\mathbb{R}^N$ such that $|D\chi_E|(B_\rho(x))>0$ for any $\rho>0$ and there exists $\nu(x)$ with $|\nu(x)|=1$ such that

$$\lim_{\rho\to0^+}\frac{D\chi_E(B_\rho(x))}{|D\chi_E|(B_\rho(x))}=-\nu(x).$$

We call the function $\nu:\partial^*E\mapsto\mathbb{S}^{N-1}$ the *exterior normal* to E. Let us also note that $\partial^*E\subset\partial\widetilde{E}$ for any $\widetilde{E}$ such that $\chi_{\widetilde{E}}=\chi_E$ almost everywhere.

The following Gauss–Green formula holds true for sets of finite perimeter, see for instance [17, Section 5.8, Theorem 1].

Theorem 2.2. *Let E be a bounded Borel set of finite perimeter. Then ∂^*E is $\mathcal{H}^{N-1}$-measurable with $[\partial^*E]$ finite and*

$$\int_E\operatorname{div}(f)=\int_{\partial^*E} f\cdot\nu\,\mathrm{d}\mathcal{H}^{N-1} \quad \textit{for any } f\in C_0^1\big(\mathbb{R}^N,\mathbb{R}^N\big). \tag{2.5}$$

Let us further remark that the intersection of two sets of finite perimeter is still a set of finite perimeter. Moreover, whenever E is open and $[\partial E]$ is finite, then E is a set of finite perimeter, see for instance [17, Section 5.11, Theorem 1].

We recall the definition and some basic properties of Γ-convergence. For a more detailed introduction we refer to [14].

Let (X,d) be a metric space. Then a sequence $F_h:X\mapsto[-\infty,+\infty]$ Γ-converges as $h\to\infty$ to a function $F:X\mapsto[-\infty,+\infty]$ if for every $x\in X$ we have:

$$\text{for every sequence } x_h \text{ converging to } x \text{ we have } F(x) \leqslant \liminf_h F_h(x_h); \tag{2.6}$$

$$\text{there exists a sequence } x_h \text{ converging to } x \text{ such that } F(x) = \lim_h F_h(x_h). \tag{2.7}$$

The function F will be called the Γ-limit of F_h as $h \to \infty$ with respect to the metric d and we denote it by $F = \Gamma\text{-}\lim_h F_h$.

The following theorem, usually known as the Fundamental Theorem of Γ-convergence, illustrates the motivations for the definition of such a kind of convergence.

Theorem 2.3. *Let (X, d) be a metric space and let $F_h : X \mapsto [-\infty, +\infty]$ be a sequence of functions defined on X. If there exists a compact set K such that $\inf_K F_h = \inf_X F_h$ for any h and $F = \Gamma\text{-}\lim_h F_h$, then F admits a minimum over X and we have*:

$$\min_X F = \liminf_h \inf_X F_h.$$

Furthermore, if x_h is a sequence of points in X which converges to a point $x \in X$ and satisfies $\lim_h F_h(x_h) = \lim_h \inf_X F_h$, then x is a minimum point for F.

The definition of Γ-convergence may be extended in a natural way to families depending on a continuous parameter. For instance we say that the family of functions F_ε, defined for every $\varepsilon > 0$, Γ-converges to a function F as $\varepsilon \to 0^+$ if for every sequence of positive ε_h converging to 0 we have $F = \Gamma\text{-}\lim_h F_{\varepsilon_h}$.

We conclude this section devoted to the preliminaries by recalling some results of regularity for solutions to Neumann problems.

Let D be a bounded domain contained in $\mathbb{R}^N$, $N \geqslant 2$, with Lipschitz boundary. Let $A = A(x)$, $x \in D$, be an $N \times N$ matrix such that its entries are measurable and it satisfies, for a positive constant $\lambda < 1$, the following ellipticity condition:

$$\begin{aligned} &A(x)\xi \cdot \xi \geqslant \lambda |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^N \text{ and for a.e. } x \in D, \\ &\|A\|_{L^\infty(D)} \leqslant \lambda^{-1}. \end{aligned} \tag{2.8}$$

Let $f \in L^s(\partial D)$, with $1 < s \leqslant +\infty$ if $N = 2$ or $2 - (2/N) \leqslant s \leqslant +\infty$ if $N \geqslant 3$. Then, $f \in (H^1(D))'$, see for instance [1, Theorems 7.53 and 7.57], by setting $\langle f, v\rangle_{(H^1(D))', H^1(D)} = \int_{\partial D} f v$ for any $v \in H^1(D)$. Furthermore, if $\int_{\partial D} f = 0$, then $\langle f, 1\rangle_{(H^1(D))', H^1(D)} = 0$. Let $H^1_*(D) = \{u \in H^1(D){:}\ \int_D u = 0\}$. Then, there exists a unique solution to the following problem:

$$\begin{cases} u \in H^1_*(D), \\ \int_D A\nabla u \cdot \nabla v = \int_{\partial D} f v \quad \text{for any } v \in H^1(D), \end{cases} \tag{2.9}$$

provided $f \in L^s(\partial D)$, with s as before, and $\int_{\partial D} f = 0$.

In order to obtain some regularity properties of u, we begin with the following result by Meyers and a lemma. We look for conditions upon which weak solutions to elliptic equations in divergence form in a domain D belong to $H^{1,p}_{\mathrm{loc}}(D)$ with $p > 2$. The following result by N.G. Meyers, [25], states that, for any $A \in L^\infty(D, M^{N\times N})$ satisfying (2.8) with a positive constant $\lambda < 1$, this holds for some $p > 2$ depending on λ and N only.

Theorem 2.4 *(Meyers). Let D be a bounded domain with Lipschitz boundary contained in $\mathbb{R}^N$, $N \geqslant 2$. Fixed λ, $0 < \lambda < 1$, there exists a constant Q, $2 < Q < \infty$, depending on λ and on N only, $Q \to 2$ as $\lambda \to 0$ and $Q \to \infty$ as $\lambda \to 1$, such that any $A \in L^\infty(D, M^{N\times N})$, satisfying* (2.8) *with constant λ, satisfies the following property.*

For any p, $2 < p < Q$, if $h \in L^p(D, \mathbb{R}^N)$, $h_1 \in L^p(D)$ and $u \in H^1(D)$ is a weak solution to

$$\operatorname{div}(A\nabla u) = \operatorname{div}(h) + h_1 \quad \text{in } D,$$

then $u \in H^{1,p}_{\mathrm{loc}}(D)$ and for any $D_1 \Subset D$ the following estimate holds,

$$\|u\|_{H^{1,p}(D_1)} \leqslant C\big(\|u\|_{H^1(D)} + \|h\|_{L^p(D,\mathbb{R}^N)} + \|h_1\|_{L^p(D)}\big), \tag{2.10}$$

where the constant C depends on λ, N, p, D_1 and D only.

Lemma 2.5. *Given three positive constants r, t, and L such that $Lr < t/2$, let $D = \{x \in \mathbb{R}^N\colon\ x' \in B'_r,\ -t < x_N < \varphi(x')\}$ where $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ is a Lipschitz function whose Lipschitz constant is bounded by L and such that $\varphi(0) = 0$. Let $\gamma = \{x \in \mathbb{R}^N\colon\ x' \in \overline{B'_r},\ x_N = \varphi(x')\}$.*

Let $f \in L^s(\gamma)$, with $s > N - 1$, and let $u \in H^1(D)$ satisfy:

$$\begin{cases} \operatorname{div}(A\nabla u) = 0 & \text{in } D,\\ A\nabla u \cdot \nu = f & \text{on } \gamma, \end{cases} \tag{2.11}$$

that is

$$\int_D A\nabla u \cdot \nabla v = \int_\gamma f v \quad \text{for any } v \in H^1(D) \text{ such that } v = 0 \text{ on } \partial D \backslash \gamma.$$

Then there exists β, $0 < \beta < 1$, depending on r, t, L, λ, s and N only, such that $u \in C^{0,\beta}(\overline{D} \cap (\overline{B}'_{r/2} \times [-t/2, +\infty)))$.

Proof. Let v be the solution to the following auxiliary problem:

$$\begin{cases} \operatorname{div}(A\nabla v) = 0 & \text{in } D,\\ A\nabla v \cdot \nu = f & \text{on } \gamma,\\ v = 0 & \text{on } \partial D \backslash \gamma. \end{cases} \tag{2.12}$$

Let $T : B'_r \times (-t, 0) \to D$ be the map such that for any $x \in B'_r \times (-t, 0)$ we have $T(x) = (x', h(x))$, where

$$h(x) = \begin{cases} x_N & \text{if } -t < x_N \leqslant -t/2,\\ x_N + \frac{(x_N + t/2)}{t/2}\varphi(x') & \text{if } -t/2 < x_N < 0. \end{cases}$$

We have that T is bijective and T and T^{-1} are Lipschitz with Lipschitz constants bounded by C_1, C_1 depending on r, t, L and N only. Then $w = v \circ T$ solves:

$$\begin{cases} \operatorname{div}(A_1\nabla w) = 0 & \text{in } B'_r \times (-t, 0),\\ A_1\nabla w \cdot \nu = f_1 & \text{on } \gamma_1 = \overline{B}'_r \times \{0\},\\ w = 0 & \text{on } \partial(B'_r \times (-t, 0)) \backslash \gamma_1, \end{cases} \tag{2.13}$$

where A_1 satisfies (2.8) with a constant λ_1, $0 < \lambda_1 < 1$, depending on λ, r, t, L and N only, $f_1 \in L^s(\gamma_1)$ and $\|f_1\|_{L^s(\gamma_1)} \leqslant C_2\|f\|_{L^s(\gamma)}$, with C_2 depending on r, t, L and N only. Then, by Theorem 2.9 in [27] we obtain that there exists β_1, $0 < \beta_1 < 1$, depending on r, t, λ_1, s and N only, such that $w \in C^{0,\beta_1}(\overline{B}'_r \times [-t, 0])$, consequently $v \in C^{0,\beta_1}(\overline{D})$.

Let now $w_1 = (v - u) \circ T$. Since $A_1\nabla w_1 \cdot \nu = 0$ in a weak sense on γ_1, by a reflection argument and standard regularity estimates for elliptic equations in divergence form, we obtain that, for some β_2, $0 < \beta_2 < 1$, depending on r, t, λ_1 and N only, $w_1 \in C^{0,\beta_2}(\overline{B}'_{r/2} \times [-t/2, 0])$, therefore $(v - u) \in C^{0,\beta_2}(\overline{D} \cap (\overline{B}'_{r/2} \times [-t/2, +\infty)))$. The conclusion immediately follows. □

Proposition 2.6. *Let D be a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, with Lipschitz boundary. Let $A \in L^\infty(D, M^{N\times N})$ satisfy* (2.8) *with constant λ, $0 < \lambda < 1$.*

There exists a constant $Q_1 > 2$, depending on λ, N and D only, such that for any p, $2 < p < Q_1$, and any s, $p - (p/N) \leqslant s \leqslant +\infty$, there exists a constant $C(p, s)$, depending on λ, N, D, p and s only, such that for any $f \in L^s(\partial D)$ with $\int_{\partial D} f = 0$, u solution to (2.9) *satisfies*:

$$\|u\|_{H^{1,p}(D)} \leqslant C(p, s)\|f\|_{L^s(\partial D)}. \tag{2.14}$$

Furthermore, if $s > N - 1$, then there exists β, $0 < \beta < 1$, depending on λ, N, D and s only, such that $u \in C^{0,\beta}(\overline{D})$.

Proof. The first part of the proposition is contained in Theorem 2 in [19], which is an extension to Neumann problems of Meyers theorem (Theorem 2.4).

About the Hölderianity of the solution, this can be obtained in two steps. First, by standard global regularity estimates, for instance by a simple modification of arguments in [27], we can show that there exists C_3, depending on λ, N, D and s only, such that u satisfies:

$$\|u\|_{L^\infty(D)} \leqslant C_3\|f\|_{L^s(\partial D)}. \tag{2.15}$$

Then, by standard regularity estimates in the interior and by Lemma 2.5, for any $x \in \overline{D}$ there exists $r(x)$, $0 < r(x) < 1$, and $\beta(x)$, $0 < \beta(x) < 1$, such that $u \in C^{0,\beta(x)}(\overline{B}_{r(x)}(x) \cap \overline{D})$ with the $C^{0,\beta(x)}$ seminorm $C(x)$. Let $x_1, \ldots, x_n$ be points of $\overline{D}$ such that $\overline{D} \subset \bigcup_{i=1}^n B_{r(x_i)/3}(x_i)$. Let $r = \min\{r(x_i),\ i = 1, \ldots, n\}$ and $\beta = \min\{\beta(x_i),\ i = 1, \ldots, n\}$. Let $C = \max\{C(x_i),\ i = 1, \ldots, n\}$. Then, for any $y_1, y_2 \in \overline{D}$ we have that either $|y_1 - y_2| < r/3$, therefore there exists $i \in \{1, \ldots, n\}$ such that y_1 and y_2 both belong to $B_{r(x_i)}(x_i)$ and hence $|u(y_1) - u(y_2)| \leqslant C_i|y_1 - y_2|^{\beta(x_i)} \leqslant C|y_1 - y_2|^{\beta}$. Or $|y_1 - y_2| \geqslant r/3$, hence $|u(y_1) - u(y_2)| \leqslant 2\|u\|_{L^\infty(D)}(r/3)^{-\beta}|y_1 - y_2|^{\beta}$. $\square$

3. The direct problem

Let Ω, Ω_1 and $\widetilde{\Omega}_1$ be three bounded domains contained in $\mathbb{R}^N$, $N \geqslant 2$, with Lipschitz boundary such that $\Omega_1 \subset \widetilde{\Omega}_1 \subset \Omega$ and the following properties are satisfied. First, $\Omega \backslash \overline{\widetilde{\Omega}}_1$ is not empty. Then, there exists γ, an open subset of $\partial\Omega$, such that $\bar{\gamma}$ is contained in the interior of $\partial\Omega \cap \partial\Omega_1$ and $\operatorname{dist}(\overline{\Omega}_1, \partial\widetilde{\Omega}_1 \cap \Omega) > 0$. Beside γ, we also fix $\tilde{\gamma}$, a closed subset of the interior of $\partial\Omega \cap \partial\Omega_1$. We assume that $\tilde{\gamma}$ has nonempty interior, with respect to the induced topology of $\partial\Omega$.

We assume that Ω, Ω_1, $\widetilde{\Omega}_1$, γ and $\tilde{\gamma}$ are fixed throughout the paper. We observe that we shall always drop the dependence of any constant upon N, the dimension of the space.

Let K be an *admissible defect*, that is K is a compact set contained in $\overline{\Omega}$ such that $\operatorname{dist}(K, \overline{\widetilde{\Omega}}_1) > 0$. We denote with G_K the connected component of $\Omega \backslash K$ such that $\widetilde{\Omega}_1 \subset G_K$. We observe that $\bar{\gamma} \cup \tilde{\gamma} \subset \partial G_K$.

A picture of the geometric configuration may be found in Fig. 1. We observe that the grey-coloured parts correspond to the connected components of $\Omega \backslash K$ which are different from G_K. In the picture K contains cracks, surface breaking cracks, cavities and material losses at the boundary as well as other more complicated defects. The domain $\widetilde{\Omega}_1$ can be seen, from a practical point of view, as a part of the body which is known to be safe (the defects K do not intersect it) and whose exterior boundary is accessible, therefore we can prescribe current densities and perform voltage measurements there. From a technical point of view, we require some distance between the region where the current density is different from zero and Dirichlet data are available and the one where K lies. The use of the domain Ω_1 and the fact that $\tilde{\gamma}$ and γ are compactly contained in the interior of $\partial\Omega \cap \partial\Omega_1$ are due to technical reasons, for instance they allow us to prove some regularity estimates upon u, the solution to (1.1), which depend on K only through Ω, Ω_1, $\widetilde{\Omega}_1$, $\tilde{\gamma}$ and γ, see Proposition 3.1.

Let us fix a number s, $s > 1$ if $N = 2$ or $s \geqslant 2 - (2/N)$ if $N \geqslant 3$, to be chosen later. Let us prescribe $f \in L^s(\partial\Omega)$ such that $\int_{\partial\Omega} f = 0$, $f \not\equiv 0$ and $\operatorname{supp}(f) \subset \tilde{\gamma}$.

For any bounded open set $D \subset \mathbb{R}^N$, we set $L^{1,2}(D)$ as the following *Deny–Lions space*:

$$L^{1,2}(D) = \{u \in L^2_{\mathrm{loc}}(D):\ \nabla u \in L^2(D, \mathbb{R}^N)\}. \tag{3.1}$$

For basic properties of Deny–Lions spaces we refer to [15] and [23]. As a convention, we identify two elements u_1 and u_2 of $L^{1,2}(D)$ whenever $\nabla u_1 = \nabla u_2$ almost everywhere in D. We point out that if D is bounded with Lip-

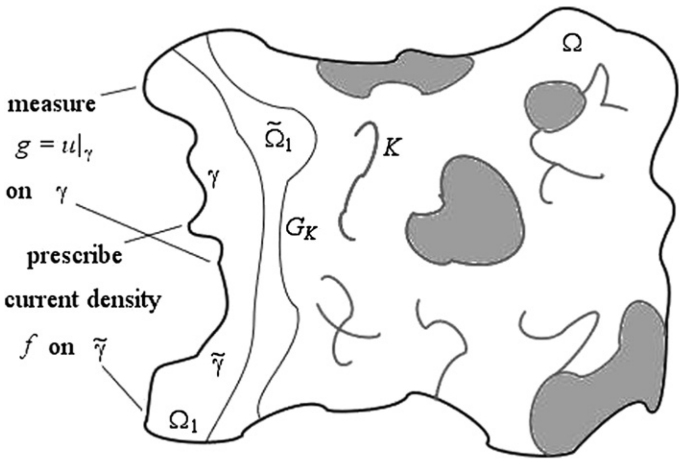

Fig. 1. Geometric configuration.

schitz boundary then any $v \in L^{1,2}(D)$ belongs to $H^1(D)$ and, obviously, vice versa. Finally, we notice that the set $\{\nabla u\colon u \in L^{1,2}(D)\}$ is a closed subspace of $L^2(D, \mathbb{R}^N)$.

Let K be an admissible defect, then there exists a function $u = u(f, K) \in L^{1,2}(\Omega\backslash K)$ such that

$$\int_{\Omega\backslash K} \nabla u \cdot \nabla v = \int_{\partial\Omega\cap\partial\Omega_1} f v \quad \text{for every } v \in L^{1,2}(\Omega\backslash K). \tag{3.2}$$

Such a function is unique in the sense that the gradients of any two solutions to (3.2) coincide almost everywhere in $\Omega\backslash K$. We always take as u the solution satisfying the following two normalization conditions. First,

$$\int_{\gamma} u = 0, \tag{3.3}$$

and, second, since u is constant on any connected component of $\Omega\backslash K$ different from G_K, we pose:

$$u = 0 \quad \text{almost everywhere in } \Omega\backslash G_K. \tag{3.4}$$

In such a way, u is defined almost everywhere in Ω and is the unique solution to (3.2)–(3.4).

We wish to remark that (3.2) is the weak formulation of the following Neumann type boundary value problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\backslash K, \\ \nabla u \cdot \nu = f & \text{on } \partial\Omega \cap \partial\Omega_1, \\ \nabla u \cdot \nu = 0 & \text{on } \partial(\Omega\backslash K)\backslash(\partial\Omega \cap \partial\Omega_1). \end{cases} \tag{3.5}$$

Here $\Omega\backslash K$ might represent an electrostatic conductor in which some perfectly insulating defects, given by K, are present. In such a case u represents the electrostatic potential if the current density f is applied on the boundary of the conductor. The electrostatic potential $u = u(f, K)$ strongly depends on K, apart from clearly depending on Ω and f.

We also remark that (3.2) is equivalent to the following minimization problem:

$$\min_{u \in L^{1,2}(\Omega\backslash K)} \frac{1}{2} \int_{\Omega\backslash K} |\nabla u|^2 - \int_{\partial\Omega\cap\partial\Omega_1} f u. \tag{3.6}$$

The following regularity properties of u can be inferred.

Proposition 3.1. *Under the previous assumptions, let $s > N - 1$ and let us fix $f \in L^s(\partial\Omega)$ such that $\int_{\partial\Omega} f = 0$, $f \not\equiv 0$ and* $\operatorname{supp}(f) \subset \tilde{\gamma}$. *Let K be an admissible defect and let u be the solution to* (3.2)–(3.4).

Then there exists a constant $C_1 > 0$, depending on s, Ω, Ω_1, $\widetilde{\Omega}_1$, γ, $\tilde{\gamma}$ only, such that

$$\|\nabla u\|_{L^2(\Omega\backslash K)} \leqslant C_1 \|f\|_{L^s(\partial\Omega)}, \tag{3.7}$$

$$\|u\|_{L^\infty(\Omega)} \leqslant C_1 \|f\|_{L^s(\partial\Omega)}. \tag{3.8}$$

Furthermore, there exists a constant β, $0 < \beta < 1$, depending on s, Ω, Ω_1, $\widetilde{\Omega}_1$, γ, $\tilde{\gamma}$ and $\operatorname{dist}(K, \overline{\widetilde{\Omega}}_1)$ *only, such that $u \in C^{0,\beta}(\overline{\widetilde{\Omega}}_1)$.*

Finally, there exists a constant $r > 2$ and a constant C_2, depending on s, Ω, Ω_1, $\widetilde{\Omega}_1$, γ, $\tilde{\gamma}$ and $\operatorname{dist}(K, \overline{\widetilde{\Omega}}_1)$ *only, such that $\nabla u \in L^r(\widetilde{\Omega}_1, \mathbb{R}^N)$ and*

$$\|\nabla u\|_{L^r(\widetilde{\Omega}_1)} \leqslant C_2 \|f\|_{L^s(\partial\Omega)}. \tag{3.9}$$

Proof. The first part can be obtained as in the proof of Proposition 3.1 in [31]. The Hölderianity can be obtained by using Lemma 2.5 as we did in the proof of Proposition 2.6. The last part and (3.9) is an easy consequence of Proposition 2.6, reflection arguments on the boundary and Meyers theorem (Theorem 2.4). □

We remark that, in view of (3.8), u actually belongs to $H^1(\Omega\backslash K)$. Furthermore, under the additional assumption that $[K] < +\infty$, or equivalently that $[\partial G_K] < +\infty$, we have that u belongs to $SBV(\Omega)$, its approximate discontinuity set $S(u)$ satisfies $[S(u)\backslash\partial G_K] = 0$ and, finally, ∇u, the weak derivative of u in $\Omega\backslash K$, coincides almost everywhere in Ω with the approximate gradient of u, see for instance [5, Proposition 4.4].

4. Classes of admissible defects

We limit ourselves to the two or three-dimensional case, however it is not difficult to see how these definitions can be generalized to higher dimensions.

If $N = 2$, fixed a positive constant $L \geqslant 1$, we say that Γ is an *L-Lipschitz*, or *L-*$C^{0,1}$, *arc* if, up to a rigid transformation, $\Gamma = \{(x, y) \in \mathbb{R}^2\colon\ -a/2 \leqslant x \leqslant a/2,\ y = \varphi_1(x)\}$, where $L^{-1} \leqslant a \leqslant L$ and $\varphi_1 : \mathbb{R} \mapsto \mathbb{R}$ is a Lipschitz map with Lipschitz constant bounded by L and such that $\varphi_1(0) = 0$. For any α, $0 \leqslant \alpha \leqslant 1$, we say that Γ is an *L-*$C^{1,\alpha}$ *arc* if φ_1 is $C^{1,\alpha}$ and its $C^{1,\alpha}$ norm is bounded by L. The points $(a/2, \varphi_1(a/2))$ and $(-a/2, \varphi_1(-a/2))$ will be called the *vertices* or *endpoints* of the arc Γ.

Let us consider now the case $N = 3$. Let T be the closed equilateral triangle which is contained in the plane $\pi = \{(x, y, z) \in \mathbb{R}^3\colon\ z = 0\}$ with vertices $V_1 = (0, 1, 0)$, $V_2 = (-\sqrt{3}/2, -1/2, 0)$ and $V_3 = (\sqrt{3}/2, -1/2, 0)$ and $T' \subset \mathbb{R}^2$ be its projection on the plane π and V_i', $i = 1, 2, 3$, its vertices. Fixed a positive constant $L \geqslant 1$, we call an *L-Lipschitz*, or *L-*$C^{0,1}$, *generalized triangle* a set Γ such that, up to a rigid transformation, $\Gamma = \{(x, y, z) \in \mathbb{R}^3\colon\ (x, y) \in \varphi(T'),\ z = \varphi_1(x, y)\}$, where $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a bi-Lipschitz function with constant L such that $\varphi(0) = 0$ and $\varphi_1 : \mathbb{R}^2 \mapsto \mathbb{R}$ is a Lipschitz map with Lipschitz constant bounded by L and such that $\varphi_1(0) = 0$. For any α, $0 \leqslant \alpha \leqslant 1$, we say that Γ is an *L-*$C^{1,\alpha}$ *generalized triangle* if φ_1 is $C^{1,\alpha}$ and its $C^{1,\alpha}$ norm is bounded by L.

In both cases, the image through φ of any vertex or side of T' will be called a vertex or side of $\varphi(T')$, respectively. The image on the graph of φ_1 of one of the vertices of $\varphi(T')$ will be called a *vertex* of Γ, whereas the image of one of the sides of $\varphi(T')$ will be called a *side* of Γ. Let us observe that our definition of vertex or side is not an intrinsic concept. We may say that Γ is a generalized triangle with vertices $\widetilde{V}_1$, $\widetilde{V}_2$ and $\widetilde{V}_3$ (and corresponding sides) if there exist φ and φ_1 with the prescribed properties such that, up to a rigid transformation, $\Gamma = \{(x, y, z) \in \mathbb{R}^3\colon\ (x, y) \in \varphi(T'),\ z = \varphi_1(x, y)\}$ and $\widetilde{V}_i = (\varphi(V_i'), \varphi_1(\varphi(V_i')))$, $i = 1, 2, 3$. This allows us the greatest generality in the choice of vertices and sides of a generalized triangle. We also remark that there exists a constant $L_1 > 0$, depending on L only, such that we can find $\varphi_2 : \mathbb{R}^3 \mapsto \mathbb{R}^3$, a bi-Lipschitz function with constant L_1, such that $\Gamma = \varphi_2(T)$.

Definition 4.1. Let us assume that $\Omega \subset B_R \subset \mathbb{R}^N$, with $R \geqslant 1$ and $N = 2, 3$. For any positive constants $L \geqslant 1$, δ and c, $c < 1$, any $k = 0, 1$ and α, $0 \leqslant \alpha \leqslant 1$, such that $k + \alpha \geqslant 1$, we define $\mathcal{B}(N, (k, \alpha), L, \delta, c)$ in the following way. We say that $A \in \mathcal{B}(N, (k, \alpha), L, \delta, c)$ if and only if $A \subset \overline{B}_{2R}$, there exists a positive integer n, depending on A, such that $A = \bigcup_{i=1}^n \Gamma_i$, Γ_i an *L-*$C^{k,\alpha}$ arc (if $N = 2$) or generalized triangle (if $N = 3$) for any $i = 1, \ldots, n$, such that the following conditions are satisfied:

(i) for any $i, j \in \{1, \ldots, n\}$ with $i \neq j$, we have that either $\Gamma_i \cap \Gamma_j$ is not empty or $\operatorname{dist}(\Gamma_i, \Gamma_j) \geqslant \delta$;
(ii) for any $i, j \in \{1, \ldots, n\}$ with $i \neq j$, if $\Gamma_i \cap \Gamma_j$ is not empty then $\Gamma_i \cap \Gamma_j$ is a common endpoint V if $N = 2$ and either a common vertex V or a common side γ if $N = 3$. Furthermore, in such a case, for any $x \in \Gamma_i$ we have $\operatorname{dist}(x, \Gamma_j) \geqslant c|x - V|$ or $\operatorname{dist}(x, \Gamma_j) \geqslant c \operatorname{dist}(x, \gamma)$, respectively.

Let us remark that there exists an integer M, depending on N, R, L, δ and c only, such that for any $A \in \mathcal{B}(N, (k, \alpha), L, \delta, c)$ we have that $n \leqslant M$. We notice that condition (ii), through the constant c, allows us to control from below the angle with which two different arcs or generalized triangles may meet.

More importantly, we have that any of the classes $\mathcal{B}$ described in Definition 4.1 is nonempty, is composed by non-empty compact sets and it is compact with respect to the Hausdorff distance, see for a proof the analogous reasonings used to prove Lemma 6.1 in [31]. Finally, if A belongs to any of these classes, then $[A]$ is bounded by a constant depending on the class only.

Definition 4.2. For any class $\mathcal{B}$ as in Definition 4.1, we shall call $H(\mathcal{B})$ the following subset of $SBV(\Omega)$. We say that $u \in SBV(\Omega)$ belongs to $H(\mathcal{B})$ if $\nabla u \in L^2(\Omega, \mathbb{R}^N)$, $[S(u) \cap \widetilde{\Omega}_1] = 0$ and there exists $A \in \mathcal{B}$, A depending on u, such that $[S(u) \backslash A] = 0$.

In the next lemma we show that $H = H(\mathcal{B})$ is closed with respect to a suitable kind of convergence, linked to the one used in the SBV compactness and semicontinuity theorem (Theorem 2.1).

Lemma 4.3. *Let $H = H(\mathcal{B})$ for some class $\mathcal{B}$ as in Definition* 4.1. *Let $\{u_h\}_{h=1}^{\infty}$ be a sequence of functions belonging to H satisfying for a given constant $C > 0$,*

$$\|u_h\|_{L^\infty(\Omega)} \leqslant C \quad \text{and} \quad \int_\Omega |\nabla u_h|^2 \leqslant C, \quad \text{for any } h.$$

Then we may extract a subsequence, which we relabel $\{u_k\}_{k=1}^{\infty}$, such that u_k converges weakly in $BV(\Omega)$ to a function $u \in SBV(\Omega)$ such that $u \in H$. Furthermore,* (2.4) *holds.*

Proof. Let A_h be the set in the chosen class such that $[S(u_h)\backslash A_h] = 0$. Since there exists a constant C such that $[A_h] \leqslant C$ for any h, we can apply Theorem 2.1 and clearly 2.4 holds. We need only to verify that the limit u belongs to H. First of all, we notice that $u_h \in H^1(\widetilde{\Omega}_1)$ for any h and their H^1 norm are uniformly bounded, therefore, up to subsequences, u_h converges weakly in $H^1(\widetilde{\Omega}_1)$, as $h \to \infty$, and thus $u \in H^1(\widetilde{\Omega}_1)$ as well. Furthermore, we have that, again up to subsequences, A_h converges in the Hausdorff distance as $h \to \infty$ to A, A still belonging to the chosen class. Therefore, on any open D compactly contained in $\Omega\backslash A$ we have, with the same reasoning, that $u \in H^1(D)$, hence it is not difficult to conclude that $[S(u)\backslash A] = 0$. □

We shall use the following class of admissible defects.

Definition 4.4. For any class $\mathcal{B}$ as in Definition 4.1, we call $\mathcal{B}'$ the class of admissible defects K such that $\text{dist}(K, \overline{\widetilde{\Omega}}_1) \geqslant \delta$, $\mathcal{H}^{N-2}(K \cap \partial\Omega) < +\infty$ and there exists $A \in \mathcal{B}$ such that $K \subset A$ and $\mathcal{H}^{N-2}(K \cap \overline{A\backslash K}) < +\infty$. Moreover, for any p, $2 < p < +\infty$, we call $\mathcal{B}'_p$ the class of admissible defects $K \in \mathcal{B}'$ such that there exists a constant C, depending on K, such that for any $u \in H^1(\Omega\backslash K)$ satisfying:

$$\int \nabla u \cdot \nabla v = 0 \quad \text{for any } v \in H^1(\Omega\backslash K) \text{ such that } v = 0 \text{ on } \partial\Omega \cap \partial\Omega_1,$$

we have

$$\|\nabla u\|_{L^p((\Omega\backslash K)\backslash\overline{\widetilde{\Omega}}_1)} \leqslant C\|u\|_{H^1(\Omega\backslash K)}. \tag{4.1}$$

Let us remark that, if $K \in \mathcal{B}'$, then we can find κ, a closed subset of K, such that $\mathcal{H}^{N-2}(\kappa) < +\infty$ and for any $x \in \partial G_K\backslash\kappa$ there exist a Lipschitz function $\varphi : \mathbb{R}^{N-1} \mapsto \mathbb{R}$ and a positive constant r such that we have, up to a rigid transformation, $\partial G_K \cap B_r(x) = \{y = (y', y_N) \in B_r(x):\ y_N = \varphi(y')\}$.

We remark that, for any $K \in \mathcal{B}'$, the property of belonging to $\mathcal{B}'_p$, for some $p > 2$, is purely a geometric one, it depends only on the geometric properties of ∂G_K. In fact the following proposition holds.

Proposition 4.5. *Let $K \in \mathcal{B}'$ satisfy the following geometric condition. For any $x_0 \in \partial G_K\backslash(\partial\Omega \cap \partial\widetilde{\Omega}_1)$, there exists $r > 0$, depending on x_0, such that for any U connected component of $G_K \cap B_r(x_0)$ we can find $r_1 > 0$, an open set U_1, such that $U \cap B_{r_1}(x_0) \subset U_1 \subset U$, and a bijective map $T : U_1 \to (0,1)^N$ such that the following properties hold. The maps T and T^{-1} are locally Lipschitz and there exists a constant C such that $\|DT\|$ and $\|DT^{-1}\|$ are bounded by C almost everywhere. By the regularity of $Q = (0,1)^N$, we extend by continuity T^{-1} up to the boundary and we have that $T^{-1} : [0,1]^N \to \mathbb{R}^N$ is a Lipschitz map with Lipschitz constant bounded by C. Furthermore, if we set $\Gamma = [0,1]^{N-1} \times \{1\}$, we require that $T^{-1}(\Gamma) = \partial U_1 \cap \partial G_K$ and $T^{-1}(y) \in G_K$ for any $y \in [0,1]^N\backslash\Gamma$.*

Then, there exists $p > 2$ such that $K \in \mathcal{B}'_p$.

Proof. We observe that on $\partial G_K\backslash(\partial\Omega \cap \partial\Omega_1)$, u satisfies, in a weak sense, a homogeneous Neumann condition. Therefore, in view of Meyers theorem, we can proceed in the following way. We call $v = u \circ T^{-1}$ and we observe that $v \in H^1(Q)$. Furthermore, for any $w \in H^1(Q)$, such that $w = 0$ on a neighbourhood of $\partial Q\backslash((0,1)^{N-1} \times \{1\})$, we have that $\tilde{w} = w \circ T$ belongs to $H^1(U_1)$ and, if we extend $\tilde{w}$ putting it equal to zero outside U_1, we obtain that $\tilde{w} \in H^1(\Omega\backslash K)$ and it is equal to zero on $\partial\Omega \cap \partial\Omega_1$. By a change of variables, we can show that, for some $N \times N$ matrix $A \in L^\infty(Q, M^{N\times N})$, satisfying (2.8) with a positive constant $\lambda < 1$, we have that $\text{div}(A\nabla v) = 0$ in Q and $A\nabla v \cdot \nu = 0$ on any compactly contained subset of $(0,1)^{N-1} \times \{1\}$. We observe that there exist a and b, with $0 < a < b < 1$, such that if $T^{-1}(y) = x_0$, then $y \in [a,b]^{N-1} \times \{1\}$. We apply a reflection argument and Meyers

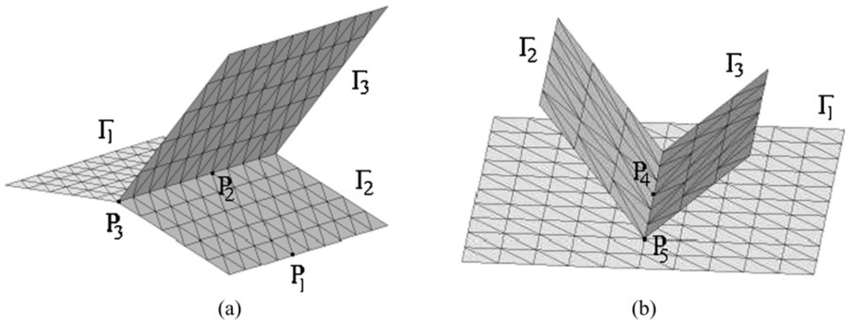

Fig. 2. (a) 3 intersecting cracks; (b) 2 surface breaking cracks.

theorem to show that there exists $p > 2$ such that $\nabla v \in L^p((a/2, (1+b)/2)^{N-1} \times (1/2, 1))$, hence $\nabla u \in L^p(U_2)$, where U_2 is an open set such that $U \cap B_{r_2}(x_0) \subset U_2 \subset U$ for some positive r_2. □

If $K \in \mathcal{B}'$, the construction of such a map T is obvious for any $x_0 \in \partial G_K \backslash \kappa$, κ as before. In the following examples we show which kinds of geometrical configurations may be allowed for the construction of such a map T in a neighbourhood of $x_0 \in \kappa$. We observe that any transformation through a bi-Lipschitz map of the configurations described in the examples would preserve the required property. We limit ourselves to examples in the three-dimensional case, which is the most interesting one.

Example 4.6. In Fig. 2(a), the intersection of three cracks Γ_1, Γ_2 and Γ_3 is illustrated. If each of the crack is locally the graph of a Lipschitz function, then any P belonging to the interior of a crack satisfies the property described before.

The boundary of a crack. Let us now consider the point P_1, which belongs to the boundary of Γ_2. Let us assume that, in some Cartesian coordinate system, $P_1 = 0$ and, locally in a neighbourhood of 0, $\Gamma_2 = \{(x, y, z) \in \mathbb{R}^3\colon x \leqslant 0, y = 0\}$. If $(\rho, \theta) \in (0, +\infty) \times (-\pi, \pi)$ are the polar coordinates in the plane xy excluding the negative x-axis, then the map T such that $(\rho, \theta, z) \to (\rho, \theta/2, z)$ satisfies the required properties and transforms $(B'_r \times (-r, r)) \backslash \Gamma_2$ into $(B'_r \times (-r, r)) \cap \{x > 0\}$.

Two or more intersecting cracks. Two kinds of interesting points can be considered. The point P_2 is such $B_r(P_2)$, with $r > 0$ sufficiently small, is decomposed by the defect into three connected components U_1, U_2 and U_3. If $\partial U_i \cap B_r(P_2)$ is a Lipschitz graph for any $i = 1, 2, 3$, as in this case, then it is easy to show that P_2 satisfies the required property. More delicate is the situation of a point like P_3. Let us construct the map T^{-1}. Let us consider the set $B_r \cap \{z > 0\}$. We consider the standard spherical coordinates $(\rho, \theta, \psi) \in (0, +\infty) \times (0, 2\pi) \times (0, \pi)$. We divide the plane xy into three cones with the same amplitude and vertex at the origin, namely, the cones are $0 < \theta < 2\pi/3$, $2\pi/3 < \theta < 4\pi/3$ and $4\pi/3 < \theta < 2\pi$. For any point $P \in B_r \cap \{z > 0\}$ such that $P \in \{x = 0 \text{ and } y = 0\}$, we pose $T^{-1}(P) = P$. Let $P \in B_r \cap \{z > 0\}$ be such that $P \notin \{x = 0 \text{ and } y = 0\}$ and let $\theta \in [0, 2\pi)$ be the angle formed by the projection on the plane xy of the halfline l, l passing through P and the origin, and the positive x-axis. If $\theta = 2k\pi/3$, with $k = 0, 1, 2$, then we pose $T^{-1}(P) = P$. Let us assume that $0 < \theta < 2\pi/3$, that is the projection is in the first cone. On the other two cones T^{-1} will be defined analogously. Let $(\rho, \theta, \psi) \in (0, r) \times (0, 2\pi/3) \times (0, \pi/2)$ be the spherical coordinates of P, then $T^{-1}(P) = (\rho, \theta, 2\psi - \frac{|\theta - \pi/3|}{\pi/3}\psi) \in (0, r) \times (0, 2\pi/3) \times (0, \pi)$. Up to different angles between the three cracks, we obtain that locally $T^{-1}(B_r \cap \{z > 0\})$ is a region like the one around P_3. Furthermore, T^{-1} is injective on $B_r \cap \{z > 0\}$ and satisfies the required regularity properties. Clearly we can adapt this example to the intersection of any number of cracks meeting in a point P_3 as in Fig. 2(a).

One or more surface breaking cracks. In Fig. 2(b), we may think that Γ_1 is a part of $\partial\Omega$, a part of the boundary of a cavity inside Ω, or a crack inside Ω and Γ_2 and Γ_3 are surface breaking cracks with respect to Γ_1. The point P_4 can be treated as the point P_2. Let us concentrate our attention to point P_5. We argue in a similar way as for point P_3. Let us consider the set $B_r \cap \{y > 0\}$ and we describe the map T^{-1}. We consider the standard spherical coordinates $(\rho, \theta, \psi) \in (0, +\infty) \times (0, 2\pi) \times (0, \pi)$. In these coordinates $B_r \cap \{y > 0\} = (0, r) \times (0, \pi) \times (0, \pi)$. First of all we define the map T_1^{-1} as the map such that, if $(\rho, \theta, \psi) \in (0, r) \times (0, \pi) \times (0, \pi)$ are the spherical coordinates of a point P in $B_r \cap \{y > 0\}$, then $T_1^{-1}(P) = (\rho, \theta, \psi/2)$. Then for any $P \in T_1^{-1}(B_r \cap \{y > 0\})$, whose spherical

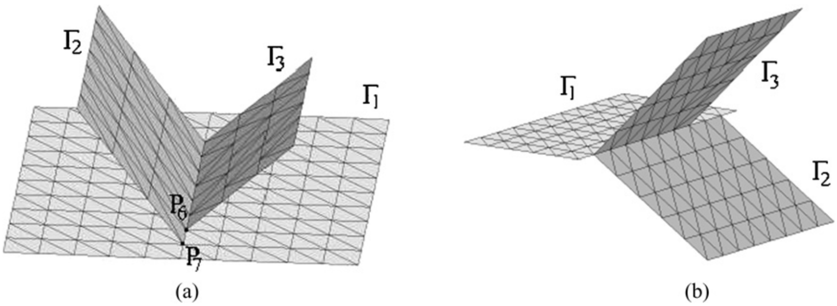

Fig. 3. (a) Surface breaking cracks; (b) Intersecting cracks.

coordinates are $(\rho,\theta,\psi)\in(0,r)\times(0,\pi)\times(0,\pi/2)$, we pose $T_2^{-1}(P)$ as follows. If ψ is such that $0<\psi\leqslant\pi/4$, then $T_2^{-1}(P)=P$. If $\pi/4<\psi<\pi/2$, then if $0<\theta\leqslant\pi/3$ we set $T_2^{-1}(P)=(\rho,\theta,\pi/4+(3-2\frac{|\theta-\pi/6|}{\pi/6})(\psi-\pi/4))$, whereas if $\pi/3\leqslant\theta\leqslant 2\pi/3$ we set $T_2^{-1}(P)=(\rho,\theta,\pi/4+(3-2\frac{|\theta-\pi/2|}{\pi/6})(\psi-\pi/4))$, and, finally, if $2\pi/3\leqslant\theta<\pi$ then $T_2^{-1}(P)=(\rho,\theta,\pi/4+(3-2\frac{|\theta-5\pi/6|}{\pi/6})(\psi-\pi/4))$. Posing $T^{-1}=T_2^{-1}\circ T_1^{-1}$, we have that T^{-1} is injective on $B_r\cap\{y>0\}$, satisfies the required regularity properties and $T^{-1}(B_r\cap\{y>0\})$, up to different angles between the cracks, is locally a region like the one around P_5. We can easily modify this example to deal with any number $1,2,\ldots$ of surface breaking cracks with respect to Γ_1.

In Fig. 3(a) and (b) some more complicated situations are illustrated. However they can be treated in a similar way as before. For instance, in Fig. 3(a), again the illustration of some surface breaking cracks, the point P_6 can be seen as the intersection of a surface breaking crack Γ_3 with respect to the surface given by the union of Γ_1 and Γ_2, thus it can be treated as the point P_5. The same reasoning applies to the point P_7. In Fig. 3(b), we can see the intersection of three cracks. If we assume that locally $\Gamma_1=\{(x,y,z)\in\mathbb{R}^3\colon\ x\leqslant 0,\ y=0\}$ and $(\rho,\theta)\in(0,+\infty)\times(-\pi,\pi)$ are the polar coordinates in the plane xy, then the map $(\rho,\theta,z)\to(\rho,\theta/2,z)$ allows us to transform the configuration of Fig. 3(b) into the one of Fig. 3(a).

5. The reconstruction method

We assume that Ω, Ω_1, $\widetilde{\Omega}_1$, γ and $\tilde\gamma$ are fixed. We assume that $\Omega\subset B_R$, with $R\geqslant 1$, and we also fix positive constants δ, L, $L\geqslant 1$, c, $0<c<1$, an integer $k=0,1$ and α, $0\leqslant\alpha\leqslant 1$, such that $k+\alpha\geqslant 1$. Let us also choose $s>N-1$ and p, $2<p<+\infty$. Let $\mathcal{B}$ be the class corresponding to these constants, as in Definition 4.1, and let $H=H(\mathcal{B})$.

Let K_0 be an admissible defect such that $K_0\in\mathcal{B}'_p$. We recall that K_0 represents our unknown defect.

Let $f_0\in L^s(\partial\Omega)$ be such that f_0 is not identically equal to zero, $\int_{\partial\Omega}f_0=0$ and $\operatorname{supp}(f_0)\subset\tilde\gamma$. Let $u_0=u(f_0,K_0)$ be the solution to (3.2)–(3.4) with f replaced by f_0 and K replaced by K_0. Let $g_0=u_0|_\gamma$.

Let us assume that the Cauchy data (g_0,f_0) are known up to some error due to noise. Let us fix ε, $0<\varepsilon\leqslant 1$, then the noisy Cauchy data are given by f_ε and g_ε. Here f_ε belongs to $L^s(\partial\Omega)$ and satisfies $\operatorname{supp}(f_\varepsilon)\subset\tilde\gamma$ and $\int_{\partial\Omega}f_\varepsilon=0$, whereas g_ε belongs to $L^2(\gamma)$ and satisfies $\int_\gamma g_\varepsilon=0$. We assume that

$$\|f_0-f_\varepsilon\|_{L^s(\partial\Omega\cap\partial\Omega_1)}\leqslant\varepsilon\quad\text{and}\quad\|g_0-g_\varepsilon\|_{L^2(\gamma)}\leqslant\varepsilon. \tag{5.1}$$

Therefore ε estimates from above the noise level of the measurements.

We recall that the only information which is available to us in order to reconstruct the potential u_0, or, better, an approximation to u_0, is the following. We only know the domain Ω and the noisy Cauchy data $(g_\varepsilon,f_\varepsilon)$. No other information is available, in particular K_0 is not known, therefore we may not use, for instance, (3.6), the variational characterization of (3.2), since the space $L^{1,2}(\Omega\backslash K_0)$ is not known. Since K_0 is related to the discontinuity set of u_0, we investigate in the sequel the possibility to formulate our problem in the framework of free-discontinuity problems in a suitable space of functions of bounded variation.

By Proposition 3.1, we can infer the following estimates. Let $q>2$ be the minimum among p and the constant r appearing in Proposition 3.1. For any ε, $0\leqslant\varepsilon\leqslant 1$, we have that $\tilde{u}_\varepsilon=u(f_\varepsilon,K_0)$ satisfies:

$$\begin{aligned}&\|\tilde{u}_\varepsilon\|_{L^\infty(\Omega)}\leqslant C_1\big(\|f_0\|_{L^s(\partial\Omega)}+1\big); \qquad \|\nabla\tilde{u}_\varepsilon\|_{L^2(\Omega)}\leqslant C_1\big(\|f_0\|_{L^s(\partial\Omega)}+1\big);\\ &\|\nabla\tilde{u}_\varepsilon\|_{L^q(\Omega)}\leqslant C_2\big(\|f_0\|_{L^s(\partial\Omega)}+1\big).\end{aligned}\tag{5.2}$$

Here C_1 is as in Proposition 3.1, whereas C_2 might depend on K_0. Let us call $E=2C_1(\|f_0\|_{L^s(\partial\Omega)}+1)$ and $F=F(K_0)=C_2(\|f_0\|_{L^s(\partial\Omega)}+1)$. As a consequence, we have that for any ε, $0\leqslant\varepsilon\leqslant 1$, $\tilde{u}_\varepsilon=u(f_\varepsilon,K_0)$ belongs to H. We observe that $\tilde{u}_0=u_0$.

For any $v\in H^1(\Omega)$, we set $|||v|||=\|v\|_{L^2(\Omega)}+\|v\|_{L^{s'}(\partial\Omega\cap\partial\Omega_1)}$, where as usual $s'=s/(s-1)$. We recall that $w\in W^{1,\infty}(\Omega)$ if and only if $w\in L^\infty(\Omega)$ and $\nabla w\in L^\infty(\Omega,\mathbb{R}^N)$ and $\|w\|_{W^{1,\infty}(\Omega)}=\|w\|_{L^\infty(\Omega)}+\|\nabla w\|_{L^\infty(\Omega)}$.

For any $0<\varepsilon\leqslant 1$, let us define $\mathcal{F}_\varepsilon$ as the following functional on $L^1(\Omega)$. For any $u\in L^1(\Omega)$, we set:

$$\begin{aligned}\mathcal{F}_\varepsilon(u)={}&\frac{1}{\varepsilon}\sup_{\substack{v\in H^1(\Omega)\\ |||v|||\leqslant 1}}\bigg(\int_\Omega\nabla u\cdot\nabla v-\int_{\partial\Omega\cap\partial\Omega_1}f_\varepsilon v\bigg)^2+\frac{1}{\varepsilon}\sup_{\substack{w\in W^{1,\infty}(\Omega)\\ \|w\|_{W^{1,\infty}(\Omega)}\leqslant 1}}\bigg(\int_\Omega\nabla u\cdot\nabla(uw)-\int_{\partial\Omega\cap\partial\Omega_1}f_\varepsilon uw\bigg)^2\\ &+\frac{1}{\varepsilon}\int_\gamma|u-g_\varepsilon|^2+\int_\Omega|\nabla u|^q+\big[S(u)\big]\quad\text{if } u\in H \text{ and } \|u\|_{L^\infty(\Omega)}\leqslant E,\end{aligned}\tag{5.3}$$

whereas $\mathcal{F}_\varepsilon(u)=+\infty$ otherwise.

On the other hand, let $\mathcal{F}_0$ be the following functional on $L^1(\Omega)$. We call H_0 the following subset of H.

Definition 5.1. We say that $u\in H_0$ if $u\in H$, $\int_\Omega\nabla u\cdot\nabla v=\int_{\partial\Omega}f_0v$ for any $v\in H^1(\Omega)$, $\int_\Omega\nabla u\cdot\nabla(uw)=\int_{\partial\Omega}f_0uw$ for any $w\in W^{1,\infty}(\Omega)$, and $u=g_0$ on γ.

We remark that, since u is only an $SBV(\Omega)$ function, if $w\in W^{1,\infty}(\Omega)$, in general uw does not belong to $H^1(\Omega)$. However, the jump set of uw is essentially contained in the jump set of u.

Then, for any $u\in L^1(\Omega)$ we set:

$$\mathcal{F}_0(u)=\int_\Omega|\nabla u|^q+\big[S(u)\big]\quad\text{if } u\in H_0 \text{ and } \|u\|_{L^\infty(\Omega)}\leqslant E,\tag{5.4}$$

whereas $\mathcal{F}_0(u)=+\infty$ otherwise.

We begin with the following important proposition.

Proposition 5.2. *For any ε, $0<\varepsilon\leqslant 1$, we have that $\mathcal{F}_\varepsilon$ admits a minimum on $L^1(\Omega)$. Furthermore, there exists a compact subset S of $L^1(\Omega)$ such that* $\min_S\mathcal{F}_\varepsilon=\min_{L^1(\Omega)}\mathcal{F}_\varepsilon$ *for any ε, $0<\varepsilon\leqslant 1$.*

Proof. We have that there exists a constant C, such that for any $0<\varepsilon\leqslant 1$ we have $\mathcal{F}_\varepsilon(u_0)\leqslant C$, and $\mathcal{F}_\varepsilon(\tilde{u}_\varepsilon)\leqslant C$, where $\tilde{u}_\varepsilon=u(f_\varepsilon,K_0)$. This implies, in particular, that $\mathcal{F}_\varepsilon$ is not identically equal to $+\infty$, for any $0<\varepsilon\leqslant 1$. Furthermore, for any constant C, we have that $S_\varepsilon=\{u\in L^1(\Omega)\colon\ \mathcal{F}_\varepsilon(u)\leqslant C\}$ is contained in $S=\{u\in SBV(\Omega)\colon\ \|u\|_{L^\infty(\Omega)}\leqslant E$ and $\int_\Omega|\nabla u|^q+[S(u)]\leqslant C\}$ which is compact in $L^1(\Omega)$ by the SBV compactness theorem (Theorem 2.1).

We need to prove existence of a minimum of the functional $\mathcal{F}_\varepsilon$. Let us fix ε, $0<\varepsilon\leqslant 1$, and let us consider a minimizing sequence $\{u_h\}_{h=1}^\infty$ of the functional $\mathcal{F}_\varepsilon$. By the previous reasoning we may assume, without loss of generality, that u_h converges, as $h\to\infty$, to a function u in the following sense. We have that $u_h\to u$ in $L^p(\Omega)$ for any p, $1\leqslant p<+\infty$, and almost everywhere (therefore $\|u\|_{L^\infty(\Omega)}\leqslant E$), $u_h\overset{*}{\rightharpoonup}u$ weakly* in $BV(\Omega)$, $\nabla u_h\rightharpoonup\nabla u$ weakly in $L^q(\Omega,\mathbb{R}^N)$, $u_h\rightharpoonup u$ weakly in $H^1(\widetilde{\Omega}_1)$ (therefore, by the compactness of the trace operator, u_h converges to u strongly in $L^2(\gamma)$ and strongly in $L^{s'}(\partial\Omega\cap\partial\Omega_1)$). By Lemma 4.3 we conclude that $u\in H$. We need to prove the semicontinuity of the functional $\mathcal{F}_\varepsilon$. We need the following property of u_h. There exist $a_h\in L^2(\Omega)$ and $\eta_h\in L^s(\partial\Omega\cap\partial\Omega_1)$ such that

$$\int_\Omega\nabla u_h\cdot\nabla v=\int_\Omega a_hv+\int_{\partial\Omega\cap\partial\Omega_1}(f_\varepsilon+\eta_h)v\quad\text{for any } v\in H^1(\Omega).$$

Furthermore, $\|a_h\|_{L^2(\Omega)} \leqslant (\varepsilon\mathcal{F}_\varepsilon(u_h))^{1/2}$ and $\|\eta_h\|_{L^s(\partial\Omega\cap\partial\Omega_1)} \leqslant (\varepsilon\mathcal{F}_\varepsilon(u_h))^{1/2}$. In fact, for any $v \in H^1(\Omega)$ such that $v = 0$ on $\partial\Omega \cap \partial\Omega_1$, we have that $\int_\Omega \nabla u_h \cdot \nabla v \leqslant (\varepsilon\mathcal{F}_\varepsilon(u_h))^{1/2}\|v\|_{L^2(\Omega)}$. By density, we infer that $\int_\Omega \nabla u_h \cdot \nabla v = \int_\Omega a_h v$ for any $v \in H^1(\Omega)$ such that $v = 0$ on $\partial\Omega \cap \partial\Omega_1$, a_h satisfying the required properties. Then, let $v \in H^1(\Omega)$ and let $v_n \in H^1(\Omega)$, $n \in \mathbb{N}$, be such that $v_n = 0$ on $\partial\Omega \cap \partial\Omega_1$ and $v_n \to v$ in $L^2(\Omega)$ as $n \to \infty$. Then,

$$\int_\Omega \nabla u_h \cdot \nabla v - \int_\Omega a_h v - \int_{\partial\Omega\cap\partial\Omega_1} f_\varepsilon v = \int_\Omega \nabla u_h \cdot \nabla(v - v_n) - \int_\Omega a_h(v - v_n) - \int_{\partial\Omega\cap\partial\Omega_1} f_\varepsilon v$$
$$\leqslant \big(\varepsilon\mathcal{F}_\varepsilon(u_h)\big)^{1/2}\big(\|v - v_n\|_{L^2(\Omega)} + \|v\|_{L^{s'}(\partial\Omega\cap\partial\Omega_1)}\big) + \|a_h\|_{L^2(\Omega)}\|v - v_n\|_{L^2(\Omega)}.$$

Letting $n \to \infty$, we obtain that

$$\int_\Omega \nabla u_h \cdot \nabla v - \int_\Omega a_h v - \int_{\partial\Omega\cap\partial\Omega_1} f_\varepsilon v \leqslant \big(\varepsilon\mathcal{F}_\varepsilon(u_h)\big)^{1/2}\|v\|_{L^{s'}(\partial\Omega\cap\partial\Omega_1)},$$

for any $v \in H^1(\Omega)$. We have that the space of traces of $H^1(\Omega)$ functions on $\partial\Omega \cap \partial\Omega_1$ is dense in $L^{s'}(\partial\Omega \cap \partial\Omega_1)$. Therefore, there exists $\eta_h \in L^s(\partial\Omega \cap \partial\Omega_1)$ such that, for any $v \in H^1(\Omega)$, we have:

$$\int_\Omega \nabla u_h \cdot \nabla v - \int_\Omega a_h v - \int_{\partial\Omega\cap\partial\Omega_1} f_\varepsilon v = \int_{\partial\Omega\cap\partial\Omega_1} \eta_h v,$$

and clearly $\|\eta_h\|_{L^s(\partial\Omega\cap\partial\Omega_1)} \leqslant (\varepsilon\mathcal{F}_\varepsilon(u_h))^{1/2}$.

By (2.4), the weak convergence of the gradients and the strong convergence of the traces on γ and on $\partial\Omega \cap \partial\Omega_1$, we can obtain that

$$\frac{1}{\varepsilon}\sup_{\substack{v\in H^1(\Omega)\\ |||v|||\leqslant 1}}\bigg(\int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} f_\varepsilon v\bigg)^2 + \frac{1}{\varepsilon}\int_\gamma |u - g_\varepsilon|^2 + \int_\Omega |\nabla u|^q + \big[S(u)\big]$$
$$\leqslant \liminf_{h\to\infty}\frac{1}{\varepsilon}\sup_{\substack{v\in H^1(\Omega)\\ |||v|||\leqslant 1}}\bigg(\int_\Omega \nabla u_h \cdot \nabla v - \int_{\partial\Omega} f_\varepsilon v\bigg)^2 + \frac{1}{\varepsilon}\int_\gamma |u_h - g_\varepsilon|^2 + \int_\Omega |\nabla u_h|^q + \big[S(u_h)\big].$$

It remains to evaluate the most delicate term, the nonlinear one, that is

$$\frac{1}{\varepsilon}\sup_{\substack{w\in W^{1,\infty}(\Omega)\\ \|w\|_{W^{1,\infty}(\Omega)}\leqslant 1}}\bigg(\int_\Omega \nabla u \cdot \nabla(uw) - \int_{\partial\Omega} f_\varepsilon uw\bigg)^2.$$

We observe that, for this term, weak convergence of the gradients is not enough to guarantee lower semicontinuity. However, if we had that ∇u_h converges to ∇u strongly in $L^2(\Omega, \mathbb{R}^N)$, then semicontinuity would follow. In fact, it is easy to see that $\int_\Omega \nabla u_h \cdot \nabla(u_h w) = \int_\Omega w|\nabla u_h|^2 + \int_\Omega u_h \nabla u_h \cdot \nabla w$ would converge, as $h \to \infty$, to $\int_\Omega \nabla u \cdot \nabla(uw)$ for any $w \in W^{1,\infty}(\Omega)$. Therefore our aim is to prove that ∇u_h converges to ∇u strongly in $L^2(\Omega, \mathbb{R}^N)$. We proceed as follows. Let $A_h \in \mathcal{B}$ be such that $[S(u_h)\backslash A_h] = 0$ and let us assume, without loss of generality, that $A_h \to A \in \mathcal{B}$ in the Hausdorff distance as $h \to \infty$. We recall that $[S(u)\backslash A] = 0$. Let us consider a sequence D_n of open subsets of Ω such that D_n is compactly contained in $\Omega\backslash A$ and satisfies $|\Omega\backslash D_n| \leqslant 1/n$. Such a sequence D_n exists since $A \in \mathcal{B}$, hence $|A| = 0$. Let $d_n = \operatorname{dist}(\overline{D}_n, \partial\Omega \cup A)$. For any n, we have that there exists $h_0 \in \mathbb{N}$, depending on n, such that $\operatorname{dist}(\overline{D}_n, \partial\Omega \cup A_h) > d_n/2$ for any $h \geqslant h_0$. For any $h \geqslant h_0$, we have that on $B_{d_n/2}(D_n)$ the function u_h satisfies $\Delta u_h = a_h$, where $a_h \in L^2(\Omega)$ is the function previously defined. We can assume, without loss of generality, that, as $h \to \infty$, a_h converges to a function $a \in L^2(\Omega)$ weakly in $L^2(\Omega)$. Since $u_h \rightharpoonup u$ weakly in $H^1(B_{d_n/2}(D_n))$, we conclude that $\Delta u = a$ in $B_{d_n/2}(D_n)$ (actually in $\Omega\backslash A$). Then, a Caccioppoli's type inequality guarantees that $\nabla u_h \to \nabla u$ strongly in $L^2(D_n, \mathbb{R}^N)$. Let $\int_\Omega \|\nabla u_h - \nabla u\|^2 = \int_{D_n} \|\nabla u_h - \nabla u\|^2 + \int_{\Omega\backslash D_n} \|\nabla u_h - \nabla u\|^2$. As $h \to \infty$, the first term goes to zero by the previous reasoning, whereas the second can be made arbitrarily small since $|\Omega\backslash D_n| \leqslant 1/n$ and we have a uniform bound on the $L^q(\Omega, \mathbb{R}^N)$ norm, with $q > 2$, of ∇u_h and ∇u. □

Theorem 5.3. *As $\varepsilon \to 0^+$, $\mathcal{F}_\varepsilon$ Γ-converges to $\mathcal{F}_0$ with respect to the $L^1(\Omega)$ norm. Furthermore, $\mathcal{F}_0$ admits a minimum over $L^1(\Omega)$ and we have*:

$$\min_{L^1(\Omega)} \mathcal{F}_0 = \lim_{\varepsilon\to 0^+} \min_{L^1(\Omega)} \mathcal{F}_\varepsilon.$$

Finally, if $u_\varepsilon \in L^1(\Omega)$ converges in $L^1(\Omega)$, as $\varepsilon \to 0^+$, to a function u and satisfies $\lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0^+} \min_{L^1(\Omega)} \mathcal{F}_\varepsilon$, then u is a minimum point for $\mathcal{F}_0$.

We divide the proof of Theorem 5.3 in several steps. In the next lemmas we prove the Γ-lim inf and Γ-lim sup inequalities, respectively.

Lemma 5.4. *Let $u_\varepsilon \in L^1(\Omega)$, for any ε, $0 < \varepsilon \leqslant 1$, be such that $u_\varepsilon \to u$ in $L^1(\Omega)$ as $\varepsilon \to 0^+$ and such that, for some constant $C > 0$, we have $\mathcal{F}_\varepsilon(u_\varepsilon) \leqslant C$ for any ε, $0 < \varepsilon \leqslant 1$. Then $u \in H_0$ and $\mathcal{F}_0(u) \leqslant \liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon)$.*

Proof. Without loss of generality, let, for any $h \in \mathbb{N}$, $u_h = u_{\varepsilon_h}$ be a sequence such that $0 < \varepsilon_{h+1} < \varepsilon_h$, for any h, $\lim_{h\to\infty} \varepsilon_h = 0$, $\lim_{h\to\infty} \mathcal{F}_{\varepsilon_h}(u_{\varepsilon_h}) = \liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon)$, and u_h converges, as $h \to \infty$, to u in the following sense. We have that $u_h \to u$ in $L^p(\Omega)$ for any p, $1 \leqslant p < +\infty$, and almost everywhere (therefore $\|u\|_{L^\infty(\Omega)} \leqslant E$), $u_h \overset{*}{\rightharpoonup} v$ weakly* in $BV(\Omega)$, $\nabla u_h \rightharpoonup \nabla u$ weakly in $L^q(\Omega, \mathbb{R}^N)$, $u_h \rightharpoonup u$ weakly in $H^1(\widetilde{\Omega}_1)$ (therefore, by the compactness of the trace operator, u_h converges to u strongly in $L^2(\gamma)$ and strongly in $L^{s'}(\partial\Omega \cap \partial\Omega_1)$). Reasoning as in the proof of Proposition 5.2, we immediately obtain that $u \in H$, $\int_\Omega \nabla u \cdot \nabla v = \int_{\partial\Omega\cap\partial\Omega_1} f_0 v$ for any $v \in H^1(\Omega)$ and that $u|_\gamma = g_0$. Furthermore, with the same arguments, we obtain that $\nabla u_h \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^N)$, thus $\int_\Omega \nabla u \cdot \nabla(uw) = \int_{\partial\Omega\cap\partial\Omega_1} f_0 uw$ for any $w \in W^{1,\infty}(\Omega)$. Hence $u \in H_0$ and $\mathcal{F}_0(u) = \int_\Omega |\nabla u|^q + [S(u)] \leqslant \lim_{h\to\infty} \mathcal{F}_{\varepsilon_h}(u_{\varepsilon_h})$ by (2.4). □

Lemma 5.5. *Let $u \in H_0$ be such that $\mathcal{F}_0(u) < +\infty$. Then $\mathcal{F}_\varepsilon(u) \to \mathcal{F}_0(u)$ as $\varepsilon \to 0^+$.*

Proof. The first term of $\mathcal{F}_\varepsilon(u)$ is bounded by $\frac{1}{\varepsilon}\|f_0 - f_\varepsilon\|^2_{L^s(\partial\Omega\cap\partial\Omega_1)}$, whereas the second is bounded by $\frac{1}{\varepsilon}\|u\|^2_{L^{s'}(\partial\Omega\cap\partial\Omega_1)}\|f_0 - f_\varepsilon\|^2_{L^s(\partial\Omega\cap\partial\Omega_1)}$. The third term is bounded by $\frac{1}{\varepsilon}\|g_0 - g_\varepsilon\|^2_{L^2(\gamma)}$, finally the last two terms are constant and equal to $\mathcal{F}_0(u)$. By (5.1), the conclusion immediately follows. □

We are now able to conclude the proof of Theorem 5.3.

Proof of Theorem 5.3. Given Lemmas 5.4 and 5.5, and Proposition 5.2, the proof is an immediate application of the definition of Γ-convergence and of the Fundamental Theorem of Γ-convergence (Theorem 2.3). □

Our aim is to prove that the minimum of $\mathcal{F}_0(u)$ is strictly related to the looked for potential $u_0 = u(f_0, K_0)$. In order to guarantee that this is the case, we need to restrict our attention only to some of the classes of Definition 4.1, namely we have to assume that $k = 1$, that is we are dealing with the $C^{1,\alpha}$ case, for some α, $0 \leqslant \alpha \leqslant 1$. It would be very interesting to extend the result also to the Lipschitz, that is $C^{0,1}$, case.

Proposition 5.6. *Under the previous assumptions, let us further assume that $k = 1$. Then, under the corresponding hypothesis on K_0, $u_1 \in H_0$ if and only if $u_1 = u_0$ almost everywhere in G_{K_0} and $\nabla u_1 = \nabla u_0$ almost everywhere in Ω.*

Proof. We essentially follow the proof of Theorem 3.3 in [31]. Let A_1 in the appropriate class $\mathcal{B}$ be such that $[S(u_1)\backslash A_1] = 0$. We call K_1 the minimal closed set such that $[S(u_1)\backslash K_1] = 0$. We observe that $K_1 \cap \widetilde{\Omega}_1$ is empty, that K_1 is contained in A_1 and that $S(u_1) \cap K_1$ is dense in K_1. We call G_{K_1} the connected component of $\Omega\backslash K_1$ containing $\widetilde{\Omega}_1$ and G the connected component of $\Omega\backslash(K_0 \cup K_1)$ containing $\widetilde{\Omega}_1$. Since u_1 satisfies $\Delta u_1 = 0$ in $\Omega\backslash K_1$ and u_1 has the same Cauchy data of u_0 on γ, by unique continuation we infer that $u_1 = u_0$ on G. We notice that

$$\int_\Omega |\nabla u_0|^2 = \int_{\partial\Omega\cap\partial\Omega_1} f_0 u_0 = \int_{\partial\Omega\cap\partial\Omega_1} f_0 u_1 = \int_\Omega |\nabla u_1|^2.$$

We observe that if we prove

$$\int_G |\nabla u_0|^2 = \int_{\partial\Omega\cap\partial\Omega_1} f_0 u_0, \tag{5.5}$$

then the proof is concluded. In fact, in such a case, we would have that $\nabla u_0 = 0$ almost everywhere in $(\Omega\backslash K_0)\backslash G$. Since u_0 is a nonconstant harmonic function in G_{K_0}, its critical set in G_{K_0} cannot have positive measure, therefore $|G_{K_0}\backslash G| = 0$. The same reasoning applied to u_1 would lead to the fact that $|G_{K_1}\backslash G| = 0$. Since $u_1 = u_0$ almost everywhere in G, and $\nabla u_i = 0$ almost everywhere on $\Omega\backslash G_{K_i}$, for any $i = 0, 1$, the conclusion would follow.

In order to prove (5.5), we follow the proof of Theorem 3.3 in [31]. The only step we need to modify is the proof of Lemma 4.2 in [31]. Let us recall the statement of that lemma. Let D be an open set compactly contained in G_{K_0} such that $[\partial D]$ is finite. Let $\widetilde{D}$ be the intersection of D with G. Then, for any $v \in C^\infty(G_{K_0})$, we have:

$$\int_{\partial^*\widetilde{D}\backslash\partial D} (\nabla u_0 \cdot \nu) v \, \mathrm{d}\mathcal{H}^{N-1} = 0. \tag{5.6}$$

Let us begin by investigating the properties of the set A_1. Let us call κ_1 the set which is the union of all the endpoints of the arcs forming A_1, if $N = 2$, or of all the sides of the generalized triangles forming A_1, if $N = 3$. We notice that $\mathcal{H}^{N-2}(\kappa_1) < +\infty$. Let us consider a point x such that $x \in \partial G \cap G_{K_0}$. Obviously, we have that x must belong to K_1. Let us assume that x does not belong to κ_1. We can find positive constants r and t, depending on x and on L, such that, up to a rigid transformation, $Q_{r,t}(x) \Subset G_{K_0}$, $\gamma_1 = A_1 \cap Q_{r,t}(x) = \{y \in Q_{r,t}(x)\colon\ y_N = \varphi(y')\}$ and $Q_{r,t}(x)\backslash A_1$ has exactly two connected components, divided by the graph of φ, $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ being a $C^{1,\alpha}$ function with $C^{1,\alpha}$ norm bounded by L. These two connected components are $D^+ = \{y \in Q_{r,t}(x)\colon\ y_N > \varphi(y')\}$ and $D^- = \{y \in Q_{r,t}(x)\colon\ y_N < \varphi(y')\}$. Without loss of generality, we may assume that D^+ and D^- are domains with Lipschitz boundaries.

We observe that, since $x \in \partial G$, at least one of the two domains D^- and D^+ must be a subset of G. Without loss of generality, we assume that $D^- \subset G$. We notice that if D^+ is also contained in G, then $u_1 = u_0$ on $Q_{r,t}(x)\backslash A_1$, thus, since u_0 is continuous across A_1, we obtain that $[S(u_1) \cap Q_{r,t}(x)] = 0$, therefore $K_1 \cap Q_{r,t}(x) = \emptyset$, and this is a contradiction to the chosen properties of x. We may conclude that D^+ is not contained in G, therefore we have that $K_1 \cap Q_{r,t}(x) = \gamma_1$. We recall that $C_0^1(\mathbb{R}^N)$ is dense in $H^1(D)$ provided D is a bounded domain with Lipschitz boundary. For any $v \in C_0^1(Q_{r,t}(x))$, we have that $\int_{Q_{r,t}(x)} \nabla u_1 \cdot \nabla v = \int_{D^-} \nabla u_1 \cdot \nabla v + \int_{D^+} \nabla u_1 \cdot \nabla v = 0$. By the Gauss–Green formula for sets of finite perimeter, Theorem 2.2, and since $u_1 = u_0$ on D^- and $u_0 \in C^1(\overline{Q}_{r,t}(x))$, we have that $\int_{D^-} \nabla u_1 \cdot \nabla v = \int_{\gamma_1} (\nabla u_0 \cdot \nu) v$ for any $v \in C_0^1(Q_{r,t}(x))$. Here and in the sequel ν shall denote the exterior normal to D^-. Therefore, $\int_{D^+} \nabla u_1 \cdot \nabla v = -\int_{\gamma_1} (\nabla u_0 \cdot \nu) v$ for any $v \in C_0^1(Q_{r,t}(x))$ and, by the density of C_0^1 already recalled, we infer that $\int_{D^+} \nabla u_1 \cdot \nabla(u_1 v) = -\int_{\gamma_1} (\nabla u_0 \cdot \nu) u_1^+ v$. Here u_1^+ and u_1^- denote the traces of u_1 on the two sides of γ_1. However, we have that for any $v \in C_0^1(Q_{r,t}(x))$, $\int_{Q_{r,t}(x)} \nabla u_1 \cdot \nabla(u_1 v) = \int_{D^-} \nabla u_1 \cdot \nabla(u_1 v) + \int_{D^+} \nabla u_1 \cdot \nabla(u_1 v) = 0$. Again by the Gauss–Green formula, $\int_{D^-} \nabla u_1 \cdot \nabla(u_1 v) = \int_{\gamma_1} (\nabla u_0 \cdot \nu) u_0 v = \int_{\gamma_1} (\nabla u_0 \cdot \nu) u_1^- v$. We conclude that $\int_{\gamma_1} (\nabla u_0 \cdot \nu)(u_1^+ - u_1^-) v = 0$ for any $v \in C_0^1(Q_{r,t}(x))$. Therefore, $(\nabla u_0 \cdot \nu)(u_1^+ - u_1^-) = 0$ $\mathcal{H}^{N-1}$-almost everywhere on γ_1. By Lemma 2.5, we have that u_1^+ and u_1^- are continuous on γ_1. Let us consider the set $\tilde{\gamma}_1 = \{y \in \gamma_1\colon\ u_1^+(y) \neq u_1^-(y)\}$ which is an open subset of γ_1. Furthermore, $\tilde{\gamma}_1 \subset S(u_1)$ and consequently $\gamma_1 \subset \overline{\tilde{\gamma}}_1$. On $\tilde{\gamma}_1$, we have that $\nabla u_0 \cdot \nu = 0$ $\mathcal{H}^{N-1}$-almost everywhere.

Here it is the only step where we need to use the further $C^{1,\alpha}$ regularity we have imposed in our hypotheses. Otherwise, Lipschitz regularity would have been enough. In fact, if γ_1 is the graph of a $C^{1,\alpha}$ function, then ν is also continuous, hence $\nabla u_0 \cdot \nu$ is a continuous function on γ_1 which is zero on $\tilde{\gamma}_1$, a dense subset of γ_1, so it is identically equal to zero.

We conclude that $\nabla u_0 \cdot \nu = 0$ $\mathcal{H}^{N-1}$-almost everywhere on $(\partial G \cap G_{K_0})\backslash\kappa_1$. Since $[\kappa_1] = 0$, we have that $\nabla u_0 \cdot \nu = 0$ $\mathcal{H}^{N-1}$-almost everywhere on $\partial G \cap G_{K_0}$. Since $\partial^*\widetilde{D} \subset \partial\widetilde{D}$ and $\partial\widetilde{D} \subset \partial D \cup (\partial G \cap G_{K_0})$, (5.6) immediately follows. □

Remark 5.7. In this remark, we investigate the following property. Let us consider the set $\partial G_{K_0} \cap \Omega$. By the definition of $\mathcal{B}'$, let κ_0 be the set with $\mathcal{H}^{N-2}(\kappa_0) < +\infty$ such that for any point $x \in (\partial G_{K_0} \cap \Omega)\backslash\kappa_0$ we can find positive constants r and t, depending on x, such that, up a rigid transformation, $Q_{r,t}(x) \Subset \Omega$, $\gamma_1 = \partial G_{K_0} \cap Q_{r,t}(x) = \{y \in Q_{r,t}(x)\colon$

$y_N = \varphi(y')\}$ and $Q_{r,t}(x)\backslash\partial G_{K_0}$ has exactly two connected components, divided by the graph of φ, $\varphi:\mathbb{R}^{N-1}\to\mathbb{R}$ being a $C^{k,\alpha}$ function. These two connected components are $D^+ = \{y \in Q_{r,t}(x)\colon\ y_N > \varphi(y')\}$ and $D^- = \{y \in Q_{r,t}(x)\colon\ y_N < \varphi(y')\}$, which are assumed to have Lipschitz boundaries. Let u_0^- and u_0^+ be the traces of u_0 on the two sides of γ_1, respectively. By Lemma 2.5, we recall that u_0^- and u_0^+ are continuous on γ_1. We have three cases. First, both D^+ and D^- are not contained in G_{K_0}. Actually, this case cannot occur, otherwise it contradicts the fact that x belongs to ∂G_{K_0}. In any case, u_0 would be identically equal to zero on D^+ and D^-, therefore $\gamma_1 \cap S(u_0) = \emptyset$. In the second case, both D^+ and D^- are contained in G_{K_0}. We are interested in the third case, when one of the two sets, say D^-, is contained in G_{K_0}, whereas the other, D^+, is not. We investigate the following question. We ask whether there exists a constant c such that $[\{y \in \gamma_1\colon\ u_0^-(y) = c\}] > 0$. We recall that $\Delta u_0 = 0$ in D^- and $\nabla u_0 \cdot \nu = 0$ on γ_1 in a weak sense. A rather deep result of unique continuation, see [2], says that if $k = 1$ and $\alpha = 1$, that is we are in the $C^{1,1}$ case, then $[\{y \in \gamma_1\colon\ u_0^-(y) = c\}] > 0$ implies that u_1 is identically equal to c on D^-. We conclude that, if $k = 1$ and $\alpha = 1$, then we have $[\gamma_1\backslash S(u_0)] = 0$. Let us observe that if $N = 2$ the regularity condition on γ_1 may be relaxed.

In the next theorem we illustrate the most important consequences of Theorem 5.3 and Proposition 5.6.

Theorem 5.8. *Under the hypotheses of Proposition* 5.6*, we have that for any family* u_ε*,* $0 < \varepsilon \leqslant 1$*, such that* $u_\varepsilon \to u$ *in* $L^1(\Omega)$ *as* $\varepsilon \to 0^+$ *and such that, for some constant* $C > 0$*, we have*:

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leqslant C \quad \text{for any } \varepsilon,\ 0 < \varepsilon \leqslant 1,$$

then u_ε *converges, as* $\varepsilon \to 0^+$*, to* u *weakly* in* $BV(\Omega)$*, strongly in* $L^p(\Omega)$*, for any* p*,* $1 \leqslant p < +\infty$*, and* ∇u_ε *converges to* $\nabla u = \nabla u_0$ *weakly in* $L^q(\Omega, \mathbb{R}^N)$ *and strongly in* $L^2(\Omega, \mathbb{R}^N)$*. Furthermore,* $u = u_0$ *almost everywhere in* G_{K_0}*.*

If we further assume that $\alpha = 1$ *and* u_ε *further satisfies* $\lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0^+} \min_{L^1(\Omega)} \mathcal{F}_\varepsilon$*, then* $[S(u)\backslash\partial G_{K_0}] = 0$ *and* u *is constant on any connected component of* $\Omega\backslash\partial G_{K_0}$ *different from* G_{K_0}*.*

Proof. The first part is an immediate consequence of Lemma 5.4 and its proof and of Proposition 5.6.

For what concerns the second part, we observe that, on $\Omega\backslash\overline{G}_{K_0}$, u attains at most a countable number of values c_i, $i \in \mathbb{N}$. Then we argue in the following way. With the notation of Remark 5.7, let us consider $x \in (\partial G_{K_0} \cap \Omega)\backslash\kappa_0$ and its corresponding γ_1. The first case is not interesting. In the second case, $u_0 = u$ almost everywhere in $Q_{r,t}(x)$, hence $S(u_0) = S(u)$ $\mathcal{H}^{N-1}$-almost everywhere on γ_1. In the third case, $[\gamma_1\backslash S(u)] = 0$, since this set is included in the union of $\{y \in \gamma_1\colon\ u^-(y) = c_i\}$, $i \in \mathbb{N}$. Therefore, we immediately conclude that if $k = 1$ and $\alpha = 1$ we have that $[S(u_0)\backslash S(u)] = 0$. Furthermore, under our additional assumptions, u is a minimum point of $\mathcal{F}_0$, hence $[S(u)] \leqslant [S(u_0)]$. We immediately conclude that also $[S(u)\backslash S(u_0)] = 0$ and consequently $[S(u)\backslash\partial G_{K_0}] = 0$. Thus u is constant on any connected component of $\Omega\backslash\partial G_{K_0}$ which is different from G_{K_0}. $\square$

We conclude the section with the following remark.

Remark 5.9. We can consider the following variants in the definition of the functionals $\mathcal{F}_\varepsilon$.

First of all, we observe that the results of Proposition 5.2 and of Theorem 5.3 still hold if we substitute the class $\mathcal{B}$ with any class formed by nonempty compact sets whose $\mathcal{H}^{N-1}$-measures are uniformly bounded and which is compact with respect to the Hausdorff distance.

Another possible modification is the following. In the first term, we may substitute $|||\cdot|||$ with the $L^2(\Omega)$ norm. In this case, we have that there exists a constant C such that still $\mathcal{F}_\varepsilon(\tilde{u}_\varepsilon) \leqslant C$ for any ε, $0 < \varepsilon \leqslant 1$, where $\tilde{u}_\varepsilon = u(f_\varepsilon, K_0)$. The results of Proposition 5.2, of Lemma 5.4 and of the first part of Theorem 5.8 still hold.

Another possible variant is the following. Since the $\mathcal{H}^{N-1}$-measure of any element of $\mathcal{B}$ is uniformly bounded, we can drop the term $[S(u)]$ in the definition of the functionals $\mathcal{F}_\varepsilon$ for any ε, $0 \leqslant \varepsilon \leqslant 1$. All the results in this section, except the second part of Theorem 5.8, are still valid. If we also replace the $|||\cdot|||$ with the $L^2(\Omega)$ norm, then Proposition 5.2, Lemma 5.4 and the first part of Theorem 5.8 still hold true, whereas Theorem 5.3 holds under the assumptions of Proposition 5.6. In this case, we need to modify Lemma 5.5 as follows. For any $u \in H_0$ such that $\mathcal{F}_0(u)$ is finite, we define v_ε, for any $0 < \varepsilon \leqslant 1$, in this way. We let $v_\varepsilon = \tilde{u}_\varepsilon$ in G_{K_0} and $v_\varepsilon = u$ outside G_{K_0}. We observe that, if Proposition 5.6 holds, then $u = u_0$ in G_{K_0}. By Proposition 3.1 and by the properties of K_0, we may conclude that $\tilde{u}_\varepsilon$ converges to u_0 strongly in $H^{1,q}(G_{K_0})$, therefore v_ε converges to u in $L^1(\Omega)$ and $\mathcal{F}_\varepsilon(v_\varepsilon)$ converges to $\mathcal{F}_0(u)$, where the functionals $\mathcal{F}_\varepsilon$, $0 \leqslant \varepsilon \leqslant 1$, are the modified ones.

6. Conclusions

In this paper we have shown how the electrostatic potential corresponding to suitable piecewise C^1 defects can be approximated by the solutions of free-discontinuity variational problems which depend on the available (noisy) boundary data only. We have proved a convergence result as the noise level goes to zero. We conclude the paper pointing out some of the most important features of the method.

Nonlinearity. The presence of the second term of the functional $\mathcal{F}_\varepsilon$, which is by the way the most difficult term to deal with, clearly shows the nonlinear character of the problem of recovering the electrostatic potential by Cauchy data in a partially known domain and, in turn, the nonlinearity of the inverse problem of determining defects by boundary data.

The minmax approach. In order to deal with an overdetermined problem (we have both the Cauchy data of the solution of an elliptic equation, which is however defined in a partially known domain) through a variational method, we use a minmax approach which allows us to impose one of the two data through a (somewhat relaxed) constraint. Such a minmax approach is frequently adopted in shape optimization problems.

The Mumford–Shah functional. The use of the Mumford–Shah functional, namely $\int_\Omega |\nabla u|^q + [S(u)]$, has several purposes. First, we remark that usually in the Mumford–Shah functional the exponent q is equal to 2, whereas in our case the presence of the nonlinear term imposes to us to take q strictly greater than 2. It guarantees coerciveness and consequently existence and convergence of minimizers. From a numerical point of view it should guarantee a suitable regularization, which is required by the ill-posedness of the problem.

The assumptions on H. The use of the space H instead of $SBV(\Omega)$ seems to be very restrictive and difficult to handle. For instance, it might make more difficult to approximate, in the sense of Γ-convergence, the functionals $\mathcal{F}_\varepsilon$ with functionals defined on spaces of more regular functions like Sobolev spaces. Such an approximation would be very interesting and useful for the implementation of the method. We recall that, for the Mumford–Shah functional, various approximations of this kind have been developed, see for instance [10] and its references. However, we believe that the restriction to the set H is not very severe from a practical point of view In fact, if we discretize the domain Ω through any regular mesh with reasonably good properties, then the conditions imposed on the elements of H would be automatically satisfied. Concerning the restriction to piecewise $C^{1,\alpha}$ admissible defects, instead of Lipschitz ones, again this is not an issue. In fact, it is well known that these kinds of problems are severely ill-posed, therefore, in order to recover the defects in a reasonably stable way we need to require some smoothness, particularly in dimension higher than 2, see for instance [3]. Furthermore, again for the ill-posedness, the discretization of the domain can not be too fine, and this is another motivation why working in the set H should not be so restrictive.

Variants of the method. As the analysis shows, without loss of generality, we can multiply the terms of the functionals $\mathcal{F}_\varepsilon$ by different positive constants. Furthermore, in Remark 5.9, we have illustrated some of the possible variants of the functionals. In particular, let us notice the role of the $|||\cdot|||$ norm and of the $L^2(\Omega)$ norm. If we use the $|||\cdot|||$ norm, then, roughly speaking, $\mathcal{F}(u)$ is finite provided u matches the (noisy) Neumann datum f_ε up to some error. If we replace it with the $L^2(\Omega)$ norm, then the (noisy) Neumann datum must be matched exactly, therefore it might be more restrictive than the other. On the other hand, it might have the advantage of being easier to implement.

A correct tuning of the constants and the use of different variants of the method might be of help to obtain better reconstruction results in an implementation of the method.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2] V. Adolfsson, L. Escauriaza, $C^{1,\alpha}$ domains and unique continuation at the boundary, Comm. Pure Appl. Math. 50 (1997) 935–969.

[3] G. Alessandrini, E. Beretta, E. Rosset, S. Vessella, Optimal stability for inverse elliptic boundary value problems with unknown boundaries, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 29 (2000) 755–806.

[4] G. Alessandrini, E. Di Benedetto, Determining 2-dimensional cracks in 3-dimensional bodies: uniqueness and stability, Indiana Univ. Math. J. 46 (1997) 1–82.

[5] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Clarendon Press, Oxford, 2000.
[6] S. Andrieux, A. Ben Abda, Identification of planar cracks by complete overdetermined data: inversion formulae, Inverse Problems 12 (1996) 553–563.
[7] N.D. Aparicio, M.K. Pidcock, The boundary inverse problem for the Laplace equation in two dimensions, Inverse Problems 12 (1996) 565–577.
[8] N.D. Aparicio, M.K. Pidcock, On a class of free boundary problems for the Laplace equation in two dimensions, Inverse Problems 14 (1998) 9–18.
[9] T. Bannour, A. Ben Abda, M. Jaoua, A semi-explicit algorithm for the reconstruction of 3D planar cracks, Inverse Problems 13 (1997) 899–917.
[10] A. Braides, Approximation of Free-Discontinuity Problems, Springer-Verlag, Berlin/Heidelberg/New York, 1998.
[11] K. Bryan, M. Vogelius, A computational algorithm to determine crack locations from electrostatic boundary measurements. The case of multiple cracks, Int. J. Engrg. Sci. 32 (1994) 579–603.
[12] K. Bryan, M.S. Vogelius, A review of selected works on crack identification, in: C.B. Croke, I. Lasiecka, G. Uhlmann, M.S. Vogelius (Eds.), Geometric Methods in Inverse Problems and PDE Control, Springer-Verlag, New York, 2004, pp. 25–46.
[13] B. Bunck, A. Elcrat, T. Hrycak, On detecting emerging surface cracks from boundary measurements, Inverse Problems 17 (2001) 1391–1400.
[14] G. Dal Maso, An Introduction to Γ-Convergence, Birkhäuser, Boston/Basel/Berlin, 1993.
[15] J. Deny, J.L. Lions, Les espaces du type de Beppo Levi, Ann. Inst. Fourier (Grenoble) 5 (1953–54) 305–370.
[16] A.R. Elcrat, V. Isakov, O. Neculoiu, On finding a surface crack from boundary measurements, Inverse Problems 11 (1995) 343–351.
[17] L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, Boca Raton/Ann Arbor/London, 1992.
[18] A. Friedman, M. Vogelius, Determining cracks by boundary measurements, Indiana Univ. Math. J. 38 (1989) 527–556.
[19] T. Gallouet, A. Monier, On the regularity of solutions to elliptic equations, Rend. Mat. Appl. 19 (1999) 471–488.
[20] H. Kang, M. Lim, G. Nakamura, Detection of surface breaking cracks in two dimensions, Inverse Problems 19 (2003) 909–918.
[21] P.G. Kaup, F. Santosa, M. Vogelius, Method for imaging corrosion damage in thin plates from electrostatic data, Inverse Problems 12 (1996) 279–293.
[22] A. Litman, D. Lesselier, F. Santosa, Reconstruction of a two-dimensional binary obstacle by controlled evolution of a level-set, Inverse Problems 14 (1998) 685–706.
[23] V.G. Maz'ja, Sobolev Spaces, Springer-Verlag, Berlin/Heidelberg/New York, 1985.
[24] M. McIver, An inverse problem in electromagnetic crack detection, IMA J. Appl. Math. 47 (1991) 127–145.
[25] N.G. Meyers, An L^p-estimate for the gradient of solutions of second order elliptic divergence equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 17 (1963) 189–206.
[26] D. Mumford, J. Shah, Optimal approximations by piecewise smooth functions and associated variational problems, Comm. Pure Appl. Math. 42 (1989) 577–685.
[27] M.K.V. Murthy, G. Stampacchia, A variational inequality with mixed boundary conditions, Israel J. Math. 13 (1972) 188–224.
[28] C. Ramananjaona, M. Lambert, D. Lesselier, J.-P. Zolésio, Shape reconstruction of buried obstacles by controlled evolution of a level set: from a min-max formulation to numerical experimentation, Inverse Problems 17 (2001) 1087–1111.
[29] J.R. Roche, J. Sokołowski, Numerical method for shape identification problems, in: G. Da Prato, J.-P. Zolésio (Eds.), Partial Differential Equation Methods in Control and Shape Analysis, Marcel Dekker, New York/Basel/Hong Kong, 1997, pp. 275–284.
[30] L. Rondi, Uniqueness and optimal stability for the determination of multiple defects by electrostatic measurements, Ph.D. thesis, S.I.S.S.A.–I.S.A.S., Trieste, 1999 (downloadable from http://www.sissa.it/library/).
[31] L. Rondi, Unique continuation from Cauchy data in unknown non-smooth domains, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 5 (2006) 189–218.
[32] L. Rondi, F. Santosa, Enhanced electrical impedance tomography via the Mumford–Shah functional, ESAIM: COCV 6 (2001) 517–538.
[33] F. Santosa, A level-set approach for inverse problems involving obstacles, ESAIM: COCV 1 (1996) 17–33.
[34] F. Santosa, M. Vogelius, A computational algorithm to determine cracks from electrostatic boundary measurements, Int. J. Engrg. Sci. 29 (1991) 917–937.