Backward stochastic differential equations in infinite dimensions with continuous driver and applications

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Abstract

In this paper we prove the existence of solution to backward stochastic differential equations (BSDEs) in infinite dimensions with continuous driver under various assumptions. We apply our results to a stochastic game problem with infinitely many players.

1 Introduction

In this paper we consider the following backward stochastic differential equation (BSDE), in the sense of [18], on a finite time interval $[0, T]$, in an infinite dimensional setting:

$$dY_t = -BY_t\, dt - \psi(t, X_t, Y_t, Z_t)\, dt + Z_t\, dW_t, \qquad Y_T = \phi(X_T). \tag{1.1}$$

In the above, W is a cylindrical Wiener process in a Hilbert space Ξ, B is the infinitesimal generator of a strongly continuous dissipative compact semigroup (e^{tB}) in a Hilbert space K, X is a Markov process with respect to the filtration generated by W, ψ and ϕ are deterministic functions with values in K. The solution (Y, Z) takes values in $K \times L_2(\Xi, K)$, where $L_2(\Xi, K)$ denotes the space of Hilbert-Schmidt operators from Ξ to K. The solution is understood in an appropriate sense, see below.

BSDEs in infinite dimensions were first studied in [17]. In this paper the authors proved existence and uniqueness of the solution to BSDE (1.1) assuming that the driver ψ is uniformly Lipschitz with respect to (y, z).

BSDEs in infinite dimensions were also studied in [1], [2], [3], [10], [14], [19], [21], in the more general case when the driver ψ can be random. In [9], [10], [11], [12], equation (1.1) was considered when the process X takes values in a Hilbert space H and is defined as the solution to a stochastic evolution equation of the form

$$dX_t = AX_t\, dt + F(t, X_t)\, dt + G(t, X_t)\, dW_t, \qquad X_0 = x \in H. \tag{1.2}$$

Here A is the infinitesimal generator of a strongly continuous semigroup (e^{tA}) in H, F and G are appropriate functions with values in H and in the space of bounded linear operators from Ξ to H, respectively. Various problems were considered in these papers, including applications to nonlinear partial differential equations for functions defined on $[0,T]\times H$ and optimal stochastic control. In [13] the fully coupled case is addressed, i.e. when F and G may depend on the unknown processes Y and Z.

In this paper we prove existence of a solution to BSDE (1.1) assuming that ψ is only continuous with respect to (y,z).

Our starting point is the result in [15], where all the processes W, X, Y, Z take values in finite-dimensional vector spaces. In that paper ψ is assumed to have linear growth with respect to (y,z); this allows to prove the existence result for the BSDE and to prove existence of a Nash equilibrium in an N-player stochastic differential game. A crucial assumption in that paper is a condition on the densities of transition probabilities of the process X with respect to the Lebesgue measure. This condition is fulfilled in the case when G is uniformly non degenerate. The result of [15] was generalized in [16] to the case of discrete-functional-type drivers.

In our paper we also impose conditions on the transition probabilities of the process X. However, due to the infinite dimensional nature of the state space H, we need completely different assumptions.

In section 3 we consider the case when X is an Ornstein-Uhlenbeck process, i.e. it solves (1.2) with $F=0$ and G constant. In this case explicit conditions are known to ensure equivalence of transition probabilities. We prove a formula for mutual densities, generalizing a result in [4], and use it to prove the existence of a solution to (1.1) assuming that ψ has linear growth with respect to (y,z). Generalizations of this result to more general processes X seem to be possible, for instance using the formulae for transition densities introduced in [22], [23], [24]. The present result is however sufficient for the applications to stochastic games that we present.

In section 6 we apply the existence result for the BSDE to prove existence of a Nash equilibrium in a stochastic game. The underlying controlled process has a nonlinear drift and constant diffusion coefficient: see equation (6.1). This time, using the infinite-dimensionality of the process Y, we are able to study a stochastic game with infinitely many players. Stochastic games with an infinite number of players are a mathematical model used to describe a variety of economical and financial markets, but so far a dynamical setting with continuous time was not considered to our knowledge, perhaps due to the complexity of the techniques involved.

In sections 4 and 5 we only assume that X is a Markov process with values in a metric space, and we prove the existence of solution to the BSDE assuming that ψ is bounded. We impose two kinds of conditions. First, in section 4, we require the transition probabilities of X to be equivalent to each other (but no condition is imposed on the corresponding densities). An application is given in example 4.1, again in the case of a process solution of an evolution equation of the form (1.2). In section 5 we address a case where transition probabilities can be even singular, and we require a continuity condition with respect to the variation norm: see (5.2). This kind of property is customary in the theory of stochastic evolution equations in infinite-dimensional spaces: it has been deeply investigated in connection with the so-called strong Feller property and several conditions are known which guarantee that it is verified: see [7]. One example is given below, see example 5.1, to show applicability of the general result.

In section 2 we introduce notation, we state a general approximation lemma and recall some facts about the Ornstein-Uhlenbeck process in a Hilbert space.

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2 Preliminaries

In this section we collect material that will be used in the sequel. First we recall some notation, then we define the Ornstein-Uhlenbeck semigroup that is used in sections 3 and 6, finally we state and prove an approximation lemma that is frequently used afterwards.

2.1 Notation

In this paper the letters H, K, Ξ denote Hilbert spaces. All Hilbert spaces are assumed to be real and separable. The norm is denoted $|\cdot|$ and the scalar product $\langle\cdot,\cdot\rangle$, with a subscript to indicate the space, if necessary. $L(H,K)$ denotes the space of linear bounded operators from H to K, with its usual norm. We shorten $L(H,H)$ to $L(H)$. $L_2(H,K)$ denotes the space of Hilbert-Schmidt operators from H to K, with the Hilbert-Schmidt norm. Operator norms are also denoted by $|\cdot|$, with a subscript if necessary.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A cylindrical Wiener process $\{W_t,\ t \geq 0\}$ in a Hilbert space Ξ is a family of linear mappings $\xi \to W_t^\xi$, defined for $\xi \in \Xi$ with values in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, such that $\{W_t^\xi,\ t \geq 0\}$ is a real Wiener process and $\mathbb{E}\,[W_t^\xi W_s^\eta] = (t \wedge s)\langle \xi, \eta\rangle$ for $\xi, \eta \in \Xi$ and $t, s \geq 0$. By $\mathcal{F}_t$ we denote the σ-algebra generated by the random variables $\{W_s^\xi,\ s \in [0,t], \xi \in \Xi\}$ and by the $\mathbb{P}$-null sets of $\mathcal{F}$. We call $(\mathcal{F}_t)_{t\geq 0}$ the Brownian filtration of W.

Stochastic integration theory can be defined with respect to W: we refer to [6] for details. If $\{\Psi_t,\ t \in [0,T]\}$ is an $(\mathcal{F}_t)$-predictable process with values in $L_2(\Xi, H)$, satisfying $\mathbb{P}$-a.s. $\int_0^T |\Psi_t|^2_{L_2(\Xi,H)} < \infty$ then the stochastic integral $\{\int_0^t \Psi_s\, dW_s,\ t \in [0,T]\}$ is an $(\mathcal{F}_t)$-local martingale with values in H admitting a continuous version.

2.2 The Ornstein-Uhlenbeck process

Let H, Ξ be Hilbert spaces. We are given two linear operators $A : D(A) \subset H \to H$ and $G \in L(\Xi, H)$ such that

Hypothesis 1 *(i) The operator $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{tA}, t \geq 0\}$ of bounded linear operators in H.*

(ii) $G : \Xi \to H$ is a bounded linear operator.

(iii) The operators

$$Q_t x = \int_0^t e^{sA} G G^* e^{sA^*} x ds, x \in H,$$

are of trace class for all $t \geq 0$.

(iv) $e^{tA}(H) \subset Q_t^{\frac{1}{2}}(H)$, for all $t > 0$.

We define the Ornstein-Uhlenbeck process as the solution of the following stochastic equation:

$$dX_t = AX_t\, dt + G\, dW_t, \qquad X_0 = x \tag{2.1}$$

where $x \in H$ is given and W is a cylindrical Wiener process in Ξ. Equation (2.1) is understood in the so-called mild sense: the solution is by definition the process

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A} G\, dW_s, \qquad t \geq 0. \tag{2.2}$$

It is well known (see e.g. [6]) that under the assumptions $(i) - (iii)$ in Hypothesis 1 the Ito integral is well defined and X_t is a random variable with values in H with law $\mathcal{N}(e^{tA}x, Q_t)$, i.e.

the Gaussian measure with mean $e^{tA}x$ and covariance operator Q_t. Moreover, condition (iv) ensures that the measures $\{\mathcal{N}(e^{tA}x, Q_t),\ t > 0, x \in H\}$ are all equivalent. In the following we fix $0 < t \le T$, $x \in H$ and we denote by $k_{tT}(x, \cdot)$ the density of $\mathcal{N}(e^{tA}x, Q_t)$ with respect to $\mathcal{N}(0, Q_T)$.

Lemma 2 *Assume that Hypothesis 1 holds, and let $0 < t \le T$ and $x \in H$ be given. Define*

$$\Theta_{tT} = Q_T^{-\frac{1}{2}} e^{tA} Q_{T-t} (Q_T^{-\frac{1}{2}} e^{tA})^*. \tag{2.3}$$

Then $1 - \Theta_{tT}$ is a positive operator with bounded inverse and we have, for $\mathcal{N}(0, Q_T)$-almost every $y \in H$,

$$\begin{aligned} k_{tT}(x, y) &= \det(1 - \Theta_{tT})^{-\frac{1}{2}} \exp\{ -\frac{1}{2}\langle (1 - \Theta_{tT})^{-1} Q_T^{-\frac{1}{2}} e^{tA}x, Q_T^{-\frac{1}{2}} e^{tA}x\rangle \\ &\quad + \langle (1 - \Theta_{tT})^{-1} Q_T^{-\frac{1}{2}} e^{tA}x, Q_T^{-\frac{1}{2}} y\rangle - \frac{1}{2}\langle \Theta_{tT}(1 - \Theta_{tT})^{-1} Q_T^{-\frac{1}{2}} y, Q_T^{-\frac{1}{2}} y\rangle\}. \end{aligned} \tag{2.4}$$

We also have the following estimates:

$$|(1 - \Theta_{tT})^{-1}| \le 1 + |Q_{T-t}||Q_t^{-1/2} e^{tA}|^2 \tag{2.5}$$

and

$$\det(1 - \Theta_{tT})^{-1} \le \exp\{(1 + |Q_{T-t}||Q_t^{-1/2} e^{tA}|^2)\ |Q_t^{-1/2} e^{tA}|^2\ Trace\, Q_{T-t}\}. \tag{2.6}$$

By 1 we also denote the identity operator. These formulae need some explanations. First we note that, as a consequence of Hypothesis 1, one can prove that the operators $Q_T^{-1/2} e^{tA}$ and $Q_t^{-1/2} e^{tA}$ are everywhere defined and bounded and that Θ_{tT} is a symmetric trace class operator satisfying $0 \le \Theta_{tT} < 1$. Next, the determinant occurring in (2.4) and (2.6) is understood as the infinite product of eigenvalues. It is well defined, since Θ_{tT} is trace class. Finally, for arbitrary $b \in H$ and trace class symmetric operator M the functions $\langle b, Q_T^{-1/2} y\rangle$ and $\langle M Q_T^{-1/2} y, Q_T^{-1/2} y\rangle$, $y \in H$, are defined by the formulae

$$\langle b, Q_T^{-1/2} y\rangle = \sum_{j=1}^{\infty} \lambda_j^{-1/2} \langle b, e_j\rangle \langle y, e_j\rangle, \tag{2.7}$$

and

$$\langle M Q_T^{-1/2} y, Q_T^{-1/2} y\rangle = \sum_{j,k=1}^{\infty} \lambda_j^{-1/2} \lambda_k^{-1/2} \langle M e_k, e_j\rangle \langle y, e_j\rangle \langle y, e_k\rangle,$$

where (e_k), (λ_k) are the eigenvectors and eigenvalues of Q_T, the eigenvalues are strictly positive. The series converge in $L^2(H, \mathcal{N}(0, Q_T))$ so that the formula (2.4) defines a function $k_{tT}(x, \cdot)$ up to a set of $\mathcal{N}(0, Q_T)$ measure 0. In particular, the function $y \to \langle b, Q_T^{-1/2} y\rangle$ defined in (2.7) has centered gaussian law with covariance $|b|^2$ on the probability space $(H, \mathcal{N}(0, Q_T))$ and it follows that

$$\int_H \exp\{\langle b, Q_T^{-1/2} y\rangle\}\, \mathcal{N}(0, Q_T)(dy) = \exp\{\frac{1}{2}|b|^2\}. \tag{2.8}$$

Lemma 2 is similar to Proposition 4.2 in [4], where densities with respect to invariant measure of the process X were considered instead of densities with respect to $\mathcal{N}(0, Q_T)$. Here we do not assume that X has an invariant measure. The proof of Lemma 2 is postponed to the appendix.

2.3 An approximation procedure

Lemma 3 *Let M be a metric space, H and K Hilbert spaces and $\psi : M \times H \to K$ a Borel measurable function satisfying*

$$|\psi(m,h)| \le C(|h| + g(m)), \qquad m \in M, h \in H$$

for some constant $C > 0$ and some function $g : M \to [0,\infty)$. Let $\psi(m,\cdot) : H \to K$ be a continuous function for every $m \in M$.

Then there exists a sequence of Borel measurable functions $\psi_n : M \times H \to K$ satisfying the following conditions.

(i) There exists a constant $C' > 0$ such that for every n

$$|\psi_n(m,h)| \le C'(|h| + g(m) + 1), \qquad m \in M, h \in H.$$

(ii) For every $m \in M$, $\psi_n(m,\cdot) : H \to K$ is infinitely Fréchet differentiable.

(iii) There exist constants $C_n > 0$ such that for every n

$$|\psi_n(m,h) - \psi_n(m,k)| \le C_n|h-k|, \qquad m \in M; h,k \in H.$$

(iv) If $h_n \to h$ in H then $\psi_n(m,h_n) \to \psi(m,h)$ in K, for every $m \in M$.

Proof. We use the construction in [20]. Let (e_i) denote a basis of H and define the projection $P_n : H \to \mathbb{R}^n$ setting $P_n h = (\langle e_i, h\rangle)_{i=1}^n$, $h \in H$. Then for $y = (y_i)_{i=1}^n \in \mathbb{R}^n$ we have $P_n^* y = \sum_{i=1}^n y_i e_i$. Let $\rho_n : \mathbb{R}^n \to [0,\infty)$ be infinitely differentiable functions such that $\int_{\mathbb{R}^n} \rho_n(y)dy = 1$ with support contained in $\{y \in \mathbb{R}^n : |y|_{\mathbb{R}^n} \le 1/n\}$. Define

$$\overline{\psi}_n(m,h) = \int_{\mathbb{R}^n} \psi(m, P_n^*(P_n h + y))\rho_n(y)dy, \qquad h \in H, m \in M.$$

It is easy to show that $\overline{\psi}_n(m,\cdot) : H \to K$ is infinitely Fréchet differentiable, that $\overline{\psi}_n(m,h_n) \to \psi(m,h)$ whenever $h_n \to h$ in H, and to prove the estimate $|\overline{\psi}_n(m,h)| \le C'(|h| + g(m) + 1)$, for some constant C'. Next we take $\eta_n \in C^\infty(\mathbb{R})$ such that $\eta_n(x) = 1$ for $x \le n$, $\eta_n(x) = 0$ for $x \ge n+1$, $|\eta_n(x)| + |\eta_n'(x)| \le c$ for some constant c. Then setting

$$\psi_n(m,h) = \eta_n(\sqrt{1+|h|^2} - 1 + g(m))\, \overline{\psi}_n(m,h), \qquad h \in H, m \in M,$$

it is easy to show that the gradient of ψ_n is bounded by some constant (depending on n) and that all the conclusions of the Lemma are satisfied. □

3 BSDE with linear growth continuous driver

In this section we consider a BSDE of the form:

$$dY_t = -BY_t\, dt - \psi(t, X_t, Y_t, Z_t)\, dt + Z_t\, dW_t, \qquad Y_T = \phi(X_T), \tag{3.1}$$

for t varying on a bounded time interval $[0,T]$. W is a cylindrical Wiener process in a Hilbert space Ξ and we denote by $(\mathcal{F}_t)$ its Brownian filtration. The unknown processes Y and Z take values in a Hilbert space K and in the Hilbert space $L_2(\Xi, K)$ respectively. X is a given $(\mathcal{F}_t)$-predictable process in another Hilbert space H. On the drivers B and ψ and the final datum ϕ we assume the following.

Hypothesis 4 *(i) The operator $B : D(B) \subset K \to K$ is the infinitesimal generator of a strongly continuous dissipative semigroup $\{e^{tB}, t \geq 0\}$ of linear bounded operators on K.*

(ii) $\phi : H \to K$ and $\psi : [0,T] \times H \times K \times L_2(\Xi, K) \to K$ are Borel measurable functions, and there exist two constants $C > 0$ and $p \geq 1$ such that

$$|\phi(x)| \leq C(1 + |x|^p), \qquad x \in H,$$

$$|\psi(t, x, y, z)| \leq C(1 + |x|^p + |y| + |z|), \qquad t \in [0,T], x \in H, y \in K, z \in L_2(\Xi, K).$$

(iii) For every $t \in [0,T]$ and $x \in H$, the function $\psi(t, x, \cdot, \cdot) : K \times L_2(\Xi, K) \to K$ is continuous.

Let us suppose that $\sup_{t \in [0,T]} \mathbb{E}|X_t|^{2p} < \infty$. We say that an $(\mathcal{F}_t)$-predictable process (Y, Z) with values in $K \times L_2(\Xi, K)$ is a mild solution of (3.1) if

$$\sup_{t \in [0,T]} \mathbb{E}\,|Y_t|^2 + \mathbb{E} \int_0^T |Z_t|^2 dt < \infty \tag{3.2}$$

and for every $t \in [0,T]$ the following equality holds:

$$Y_t + \int_t^T e^{(s-t)B} Z_s\, dW_s = e^{(T-t)B}\phi(X_T) + \int_t^T e^{(s-t)B}\psi(s, X_s, Y_s, Z_s)\, ds, \qquad \mathbb{P} - a.s. \tag{3.3}$$

The result of [17] states that there exists a unique mild solution if, in addition to the previous assumptions, one supposes that the function $\psi(t, x, \cdot, \cdot)$ is Lipschitz continuous. In the following we will drop the Lipschitz condition and prove some existence results. We first need some preliminary estimates.

Lemma 5 *Assume that Hypothesis 4 holds and let X be an $(\mathcal{F}_t)$-predictable process satisfying $\sup_{t \in [0,T]} \mathbb{E}|X_t|^{2p} < \infty$. Let (Y, Z) be a mild solution to (3.1). Then*

$$\sup_{t \in [0,T]} \mathbb{E}\,|Y_t|^2 + \mathbb{E} \int_0^T |Z_t|^2 dt \leq C \sup_{t \in [0,T]} \mathbb{E}(1 + |X_t|^{2p}). \tag{3.4}$$

If ψ', ϕ' are functions satisfying Hypothesis 4 and (Y', Z') is a corresponding mild solution then

$$\mathbb{E} \int_0^T |Z_t - Z'_t|^2 dt \leq \mathbb{E}|\phi(X_T) - \phi'(X_T)|^2 + C \left(\sup_{t \in [0,T]} \mathbb{E}(1 + |X_t|^{2p}) \right)^{1/2} \left(\mathbb{E} \int_0^T |Y_t - Y'_t|^2 dt \right)^{1/2}. \tag{3.5}$$

In (3.4) and (3.5) the constant C depends only on T and on the constants C, p in Hypothesis 4.

Proof. Let us introduce the operators $J_k = k(k \cdot 1 - B)^{-1}$, $k > 0$. A direct computation shows that $BJ_k = k^2(k \cdot 1 - B)^{-1} - k \cdot 1$, so in particular the operators BJ_k are bounded (they are called the Yosida approximations of B). We set $Y_t^k = J_k Y_t$, $Z_t^k = J_k Z_t$. We now verify that Y^k admits the Itô differential

$$dY_t^k = -BY_t^k\, dt - J_k\psi(t, X_t, Y_t, Z_t)\, dt + Z_t^k\, dW_t. \tag{3.6}$$

In fact applying J_k to both sides of (3.3) we have

$$Y_t^k + \int_t^T e^{(s-t)B} Z_s^k\, dW_s = e^{(T-t)B} J_k\phi(X_T) + \int_t^T e^{(s-t)B} J_k\psi(s, X_s, Y_s, Z_s)\, ds. \tag{3.7}$$

Applying B to both sides and integrating we obtain, for every $r \in [0,T]$,

$$\begin{aligned} &\int_r^T BY_t^k\, dt + \int_r^T \int_t^T e^{(s-t)B} BZ_s^k\, dW_s\, dt \\ &\qquad = \int_r^T e^{(T-t)B} BJ_k\phi(X_T)\, dt + \int_r^T \int_t^T e^{(s-t)B} BJ_k\psi(s,X_s,Y_s,Z_s)\, ds\, dt. \end{aligned} \tag{3.8}$$

We have

$$\int_r^T e^{(T-t)B} BJ_k\phi(X_T)\, dt = e^{(T-r)B} J_k\phi(X_T) - J_k\phi(X_T)$$

and, applying the stochastic Fubini theorem (see e.g. [6])

$$\int_r^T \int_t^T e^{(s-t)B} BZ_s^k\, dW_s\, dt = \int_r^T \int_r^s e^{(s-t)B} BZ_s^k\, dt\, dW_s = \int_r^T (e^{(s-r)B} Z_s^k - Z_s^k)\, dW_s.$$

Substituting in (3.8) and comparing with (3.7) gives

$$\int_r^T BY_t^k\, dt = Y_r^k + \int_r^T Z_s^k\, dW_s - J_k\phi(X_T) - \int_r^T J_k\psi(s,X_s,Y_s,Z_s)\, ds,$$

which proves (3.6).

Applying the Itô formula to $|Y_t^k|^2$ we obtain

$$\begin{aligned} &|Y_t^k|^2 + \int_t^T |Z_s^k|^2 ds \\ &= |J_k\phi(X_T)|^2 + 2\int_t^T (\langle Y_s^k, BY_s^k\rangle + \langle Y_s^k, J_k\psi(s,X_s,Y_s,Z_s)\rangle)\, ds - 2\int_t^T \langle Y_s^k, Z_s^k dW_s\rangle. \end{aligned}$$

We have

$$\mathbb{E}\left(\int_0^T |(Z_s^k)^* Y_s^k|^2 ds\right)^{1/2} \le \mathbb{E}\left[\sup_{s\in[0,T]} |Y_s^k| \left(\int_0^T |Z_s^k|^2 ds\right)^{1/2}\right] < \infty, \tag{3.9}$$

since it follows from (3.6) and Burkholder's inequality that $\mathbb{E}\sup_{t\in[0,T]} |Y_t^k|^2 < \infty$. (3.9) ensures that we can take expectation in the previous equality and obtain

$$\mathbb{E}|Y_t^k|^2 + \mathbb{E}\int_t^T |Z_s^k|^2 ds = \mathbb{E}|J_k\phi(X_T)|^2 + 2\mathbb{E}\int_t^T (\langle Y_s^k, BY_s^k\rangle + \langle Y_s^k, J_k\psi(s,X_s,Y_s,Z_s)\rangle)\, ds.$$

Now we use the dissipativity of B and we obtain

$$\mathbb{E}|Y_t^k|^2 + \mathbb{E}\int_t^T |Z_s^k|^2 ds \le \mathbb{E}|J_k\phi(X_T)|^2 + 2\mathbb{E}\int_t^T \langle Y_s^k, J_k\psi(s,X_s,Y_s,Z_s)\rangle\, ds.$$

It is well known that $|J_k|_{L(K)} \le 1$ and $J_k h \to h$ for every $h \in K$. By the growth condition on ψ, the hypothesis $\sup_{t\in[0,T]} \mathbb{E}|X_t|^{2p} < \infty$ and by (3.2) we can apply the dominated convergence theorem and we arrive at

$$\mathbb{E}|Y_t|^2 + \mathbb{E}\int_t^T |Z_s|^2 ds \le \mathbb{E}|\phi(X_T)|^2 + 2\mathbb{E}\int_t^T \langle Y_s, \psi(s,X_s,Y_s,Z_s)\rangle\, ds. \tag{3.10}$$

Next we have, for every $\epsilon > 0$ and for some constant C_ϵ,

$$\langle Y_s, \psi(s,X_s,Y_s,Z_s)\rangle \le C|Y_s|(1 + |X_s|^p + |Y_s| + |Z_s|) \le \epsilon|Z_s|^2 + C_\epsilon(1 + |X_s|^{2p} + |Y_s|^2).$$

Choosing ϵ sufficiently small we obtain, for some $C, c > 0$,

$$\begin{aligned} \mathbb{E}|Y_t|^2 + c\mathbb{E}\int_t^T |Z_s|^2 ds &\leq \mathbb{E}|\phi(X_T)|^2 + 2\mathbb{E}\int_t^T (1 + |X_s|^{2p} + |Y_s|^2)\, ds \\ &\leq C \sup_{t\in[0,T]} (1 + \mathbb{E}\,|X_t|^{2p}) + C\mathbb{E}\int_t^T |Y_s|^2\, ds, \end{aligned}$$

and (3.4) follows from Gronwall's lemma.

In order to prove (3.5) we write the equation satisfied by $(Y - Y', Z - Z')$ and, introducing the operators J_k and proceeding as before, instead of (3.10) we arrive at

$$\begin{aligned} &\mathbb{E}|Y_t - Y_t'|^2 + \mathbb{E}\int_t^T |Z_s - Z_s'|^2 ds \\ &\quad\leq \mathbb{E}|\phi(X_T) - \phi'(X_T)|^2 + 2\mathbb{E}\int_t^T \langle Y_s - Y_s', \psi(s, X_s, Y_s, Z_s) - \psi'(s, X_s, Y_s', Z_s')\rangle\, ds. \end{aligned}$$

We set $f_s = \psi(s, X_s, Y_s, Z_s) - \psi'(s, X_s, Y_s', Z_s')$ and note that

$$|f_s| \leq C(1 + |X_s|^p + |Y_s| + |Z_s| + |Y_s'| + |Z_s'|).$$

From estimate (3.4) we deduce

$$\mathbb{E}\int_0^T |f_s|^2 ds \leq C \sup_{t\in[0,T]} \mathbb{E}\,(1 + |X_t|^{2p})$$

and we obtain

$$\begin{aligned} &\mathbb{E}|Y_t - Y_t'|^2 + \mathbb{E}\int_t^T |Z_s - Z_s'|^2 ds \\ &\quad\leq \mathbb{E}|\phi(X_T) - \phi'(X_T)|^2 + 2\left(\mathbb{E}\int_t^T |Y_s - Y_s'|^2\, ds\right)^{1/2} \left(\mathbb{E}\int_t^T |f_s|^2\, ds\right)^{1/2} \\ &\quad\leq \mathbb{E}|\phi(X_T) - \phi'(X_T)|^2 + C\left(\sup_{t\in[0,T]} \mathbb{E}\,(1 + |X_t|^{2p})\right)^{1/2} \left(\mathbb{E}\int_t^T |Y_s - Y_s'|^2\, ds\right)^{1/2}. \end{aligned}$$

(3.5) follows immediately. □

We are now able to state and prove the main result of this section, where for the process X we take the Ornstein-Uhlenbeck process introduced in section 2.2: Given $x_0 \in H$ we define

$$X_t = e^{tA}x_0 + \int_0^t e^{(t-s)A}G\, dW_s. \tag{3.11}$$

Theorem 6 *Assume that Hypotheses 1 and 4 hold and suppose that the operators e^{tB} are compact for $t > 0$. Let X be the Ornstein-Uhlenbeck process defined by (3.11).*

Then there exists a mild solution (Y, Z) to equation (3.1).

Moreover there exist Borel measurable functions $u : [0, T]\times H \to K$, $v : [0, T]\times H \to L_2(\Xi, K)$ such that, $\mathbb{P}$-a.s.,

$$Y_t = u(t, X_t),\ \textit{for all } t \in [0, T]; \qquad Z_t = v(t, X_t),\ \textit{for almost all } t \in [0, T].$$

Proof - First Step. Approximation. We apply Lemma 3 to the metric space $[0,T] \times H$ and the Hilbert space $K \times L_2(\Xi, K)$ and obtain a sequence of functions $\psi^n : [0,T] \times H \times K \times L_2(\Xi, K) \to K$ such that, for any $n \geq 1$,

$$|\psi^n(t,x,y,z)| \leq C(1 + |x|^p + |y| + |z|), \tag{3.12}$$

and for fixed n, ψ^n is Lipschitz with respect to (y,z) uniformly with respect to (t,x).

Let $(Y^{n,t,x}, Z^{n,t,x})$ be the unique mild solution of

$$dY_s^{n,t,x} = -BY_s^{n,t,x}\, ds - \psi^n(s, X_s^{t,x}, Y_s^{n,t,x}, Z_s^{n,t,x})\, ds + Z_s^{n,t,x}\, dW_s, \quad Y_T^{n,t,x} = \phi(X_T^{t,x}), \tag{3.13}$$

where $X_s^{t,x}$ is the Ornstein-Uhlenbeck process starting from x at time t:

$$X_s^{t,x} = e^{(s-t)A}x + \int_t^s e^{(s-r)A} G\, dW_r, \qquad 0 \leq t \leq s \leq T,$$

(we define $X_s^{t,x} = x$ for $s < t$). It is easy to prove that $\sup_{s\in[0,T]} \mathbb{E}|X_s^{t,x}|^{2p} \leq C(1+|x|^{2p})$ and (3.4) implies

$$\sup_{s\in[t,T]} \mathbb{E}|Y_s^{n,t,x}|^2 + \mathbb{E}\int_0^T |Z_s^{n,t,x}|^2 ds \leq C(1+|x|^{2p}). \tag{3.14}$$

Moreover there exist Borel measurable functions $u^n : [0,T] \times H \to K$ and $v^n : [0,T] \times H \to L_2(\Xi, K)$, such that

$$Y_s^{n,t,x} = u^n(s, X_s^{t,x}), \qquad Z_s^{n,t,x} = v^n(s, X_s^{t,x}). \tag{3.15}$$

The proof of (3.15) can be found in [8] (see also [9], Proposition 3.2, for a direct proof in the infinite dimensional case).

Second Step. In this step we prove that there exists a subsequence of $u^n(t,x)$ which is convergent in K for every t, x. This is obvious for $t = T$, since $u^n(T,x) = \phi(x)$, so we can assume $t < T$.

We denote by $\mu_t(x, dy)$ the gaussian measure $\mathcal{N}(e^{tA}x, Q_t)(dy)$ and by $\mu_T(dy)$ the measure $\mathcal{N}(0, Q_T)(dy)$, and we note that the law of $X_s^{t,x}$ is $\mu_{s-t}(x, dy)$, $0 \leq t \leq s \leq T$. Noting that $u^n(t,x) = Y_t^{n,t,x}$, taking expectation in the BSDE we have

$$\begin{aligned} u^n(t,x) &= \mathbb{E}\, e^{(T-t)B}\phi(X_T^{t,x}) + \mathbb{E}\int_t^T e^{(s-t)B}\psi^n(s, X_s^{t,x}, Y_s^{n,t,x}, Z_s^{n,t,x})\, ds \\ &= \mathbb{E}\, e^{(T-t)B}\phi(X_T^{t,x}) + \mathbb{E}\int_t^T e^{(s-t)B}\psi^n(s, X_s^{t,x}, u^n(s, X_s^{t,x}), v^n(s, X_s^{t,x}))\, ds \\ &= \mathbb{E}\, e^{(T-t)B}\phi(X_T^{t,x}) + \int_t^T e^{(s-t)B}\int_H \Psi^n(s,y)\, \mu_{s-t}(x,dy)\, ds, \end{aligned} \tag{3.16}$$

where $\Psi^n(s,y) = \psi^n(s, y, u^n(s,y), v^n(s,y))$. For $t < T$ and $\delta > 0$ so small that $t + \delta \leq T$ we decompose $u^n(t,x)$ as follows:

$$u^n(t,x) = q(t,x) + a_\delta^n(t,x) + b_\delta^n(t,x), \tag{3.17}$$

where $q(t,x) = \mathbb{E}\, e^{(T-t)B}\phi(X_T^{t,x})$,

$$a_\delta^n(t,x) = \int_t^{t+\delta} e^{(s-t)B}\int_H \Psi^n(s,y)\, \mu_{s-t}(x,dy)\, ds,$$

$$b_\delta^n(t,x) = \int_{t+\delta}^T e^{(s-t)B}\int_H \Psi^n(s,y)\, \mu_{s-t}(x,dy)\, ds.$$

We note that the inequality

$$\left|\int_H \Psi^n(s,y)\;\mu_{s-t}(x,dy)\right| = \left|\mathbb{E}\,\psi^n(s,X_s^{t,x},Y_s^{n,t,x},Z_s^{n,t,x})\right| \\ \le C\,\mathbb{E}\,(1+|X_s^{t,x}|^p + |Y_s^{n,t,x}| + |Z_s^{n,t,x}|) \tag{3.18}$$

implies

$$\begin{aligned} |a_\delta^n(t,x)| &\le C\,\mathbb{E}\int_t^{t+\delta}(1+|X_s^{t,x}|^p + |Y_s^{n,t,x}| + |Z_s^{n,t,x}|)\,ds \\ &\le C\,\delta^{1/2}\left(\mathbb{E}\int_t^{t+\delta}(1+|X_s^{t,x}|^{2p} + |Y_s^{n,t,x}|^2 + |Z_s^{n,t,x}|^2)\,ds\right)^{1/2} \\ &\le C_x\,\delta^{1/2}, \end{aligned} \tag{3.19}$$

by (3.14). Next we consider $b_\delta^n(t,x)$ that we rewrite

$$b_\delta^n(t,x) = \int_{t+\delta}^T e^{(s-t)B}\int_H \Psi^n(s,y)d^{s,t}(x,y)\;\mu_s(0,dy)\,ds,$$

where we have denoted $d^{s,t}(x,y)$ the density of $\mu_{s-t}(x,\cdot)$ with respect to $\mu_s(0,\cdot)$. Let us consider the Hilbert space of Borel measurable functions $[0,T]\times H\to K$, square summable with respect to the measure $\mu_s(0,dy)ds$, equipped with the usual inner product. It will be denoted $L^2([0,T]\times H;\mu_s(0,dy)ds;K)$. Let us check that (Ψ^n) is a bounded set in this space: Indeed we have

$$\begin{aligned} \int_0^T\int_H |\Psi^n(s,y)|^2\;\mu_s(0,dy)\,ds &= \mathbb{E}\int_0^T |\psi^n(s,X_s^{0,0},Y_s^{n,0,0},Z_s^{n,0,0})|^2 ds \\ &\le C\,\mathbb{E}\int_0^T (1+|X_s^{0,0}|^{2p} + |Y_s^{n,0,0}|^2 + |Z_s^{n,0,0}|^2)\,ds \\ &\le C, \end{aligned}$$

by (3.14). The sequence (Ψ^n) is therefore weakly compact and there exists a subsequence (still denoted (Ψ^n)) which is weakly convergent in $L^2([0,T]\times H;\mu_s(0,dy)ds;K)$.

For fixed $k\in K$ define

$$\varphi(s,y) = 1_{[t+\delta,T]}(s)d^{s,t}(x,y)\;e^{(s-t)B^*}k$$

and assume for a moment that φ (which of course depends also on t,x,δ,k) belongs to $L^2([0,T]\times H;\mu_s(0,dy)ds;K)$. The function φ is chosen so that

$$\langle b_\delta^n(t,x),k\rangle = \int_{t+\delta}^T\int_H \langle e^{(s-t)B}\Psi^n(s,y),k\rangle d^{s,t}(x,y)\;\mu_s(0,dy)\,ds = \langle \Psi^n,\varphi\rangle_{L^2([0,T]\times H;\mu_s(0,dy)ds;K)}.$$

It follows that for integers $n,m\ge 1$,

$$\begin{aligned} \langle u^n(t,x)-u^m(t,x),k\rangle &= \langle a_\delta^n(t,x)-a_\delta^m(t,x),k\rangle + \langle b_\delta^n(t,x)-b_\delta^m(t,x),k\rangle \\ &= \langle a_\delta^n(t,x)-a_\delta^m(t,x),k\rangle + \langle \Psi^n-\Psi^m,\varphi\rangle_{L^2([0,T]\times H;\mu_s(0,dy)ds;K)}. \end{aligned}$$

From (3.19) it follows that

$$|\langle u^n(t,x)-u^m(t,x),k\rangle| \le C\,\delta^{1/2}|k| + |\langle \Psi^n-\Psi^m,\varphi\rangle_{L^2([0,T]\times H;\mu_s(0,dy)ds;K)}|,$$

and since (Ψ^n) is weakly convergent we conclude that $(\langle u^n(t,x),k\rangle)_n$ is a Cauchy sequence for every $k\in K$, so that, for all t,x, $(u^n(t,x))_n$ is a weakly convergent sequence in K.

It remains to check that $\varphi \in L^2([0,T] \times H; \mu_s(0,dy)ds; K)$. From Lemma 2, the density $d^{s,t}(x,y)$ has the form

$$d^{s,t}(x,y) = \det(1-\Theta^{s,t})^{-\frac{1}{2}} \exp\{-\frac{1}{2}\langle (1-\Theta^{s,t})^{-1} Q_s^{-\frac{1}{2}} e^{(s-t)A}x, Q_s^{-\frac{1}{2}} e^{(s-t)A}x\rangle$$
$$+\langle (1-\Theta^{s,t})^{-1} Q_s^{-\frac{1}{2}} e^{(s-t)A}x, Q_s^{-\frac{1}{2}} y\rangle - \frac{1}{2}\langle \Theta^{s,t}(1-\Theta^{s,t})^{-1} Q_s^{-\frac{1}{2}} y, Q_s^{-\frac{1}{2}} y\rangle\},$$

where $\Theta^{s,t} = Q_s^{-1/2} e^{(s-t)A} Q_t (Q_s^{-1/2} e^{(s-t)A})^*$. So setting $h_{s,t,x} = (1-\Theta^{s,t})^{-1} Q_s^{-\frac{1}{2}} e^{(s-t)A}x$, we obtain $0 \le d^{s,t}(x,y) \le \det(1-\Theta^{s,t})^{-1/2} \exp(\langle h_{s,t,x}, Q_s^{-\frac{1}{2}} y\rangle)$ and recalling formula (2.8) we find

$$\begin{aligned}\int_0^T \int_H |\varphi(s,y)|^2 \mu_s(0,dy)ds &\le C \int_{t+\delta}^T \int_H |d^{s,t}(x,y)|^2 \mu_s(0,dy)ds \\ &\le C \int_{t+\delta}^T \det(1-\Theta^{s,t})^{-1} \exp(2|h_{s,t,x}|^2) ds.\end{aligned} \tag{3.20}$$

By (2.6) we have

$$\det(1-\Theta^{s,t})^{-1} \le \exp\left[(1+|Q_t||Q_{s-t}^{-\frac{1}{2}} e^{(s-t)A}|^2)\, |Q_{s-t}^{-\frac{1}{2}} e^{(s-t)A}|^2 \, Trace\, Q_t\right]$$

and, taking into account (2.5),

$$\begin{aligned}|h_{s,t,x}| &\le |(1-\Theta^{s,t})^{-1}||Q_s^{-\frac{1}{2}} Q_{s-t}^{\frac{1}{2}}||Q_{s-t}^{-\frac{1}{2}} e^{(s-t)A}||x| \\ &\le (1+|Q_t||Q_{s-t}^{-\frac{1}{2}} e^{(s-t)A}|^2)|Q_s^{-\frac{1}{2}} Q_{s-t}^{\frac{1}{2}}||Q_{s-t}^{-\frac{1}{2}} e^{(s-t)A}||x|.\end{aligned}$$

Since $Q_s \ge Q_{s-t}$ it follows that $|Q_s^{-\frac{1}{2}} Q_{s-t}^{\frac{1}{2}}| \le 1$. Using the inequality (7.6) and noting that $s-t \ge \delta$ we obtain $|Q_{s-t}^{-\frac{1}{2}} e^{(s-t)A}| \le |Q_\delta^{-\frac{1}{2}} e^{\delta A}|$. It follows that

$$\det(1-\Theta^{s,t})^{-1} \le \exp\left[(1+|Q_t||Q_\delta^{-\frac{1}{2}} e^{\delta A}|^2)\, |Q_\delta^{-\frac{1}{2}} e^{\delta A}|^2 \, Trace\, Q_t\right],$$

$$|h_{s,t,x}| \le (1+|Q_t||Q_\delta^{-\frac{1}{2}} e^{\delta A}|^2)|Q_\delta^{-\frac{1}{2}} e^{\delta A}||x|.$$

This shows that the right-hand side of (3.20) is finite and therefore φ belongs to $L^2([0,T] \times H; \mu_s(0,dy)ds; K)$.

So far in Step 2 we have proved that for all t,x, the sequence $(u^n(t,x))_n$ is weakly convergent in K. We will now prove that the convergence takes place in the norm of K. To this purpose it is enough to show that, for fixed t,x, the sequence $(u^n(t,x))_n$ is relatively compact in K or, equivalently, that it is totally bounded.

Let us fix (t,x) and let $\epsilon > 0$ be arbitrary. Let us consider again the decomposition (3.17). By (3.19) we can choose δ such that $|a_\delta^n(t,x)| < \epsilon/2$ for every n. Next note that

$$b_\delta^n(t,x) = e^{\delta B} \int_{t+\delta}^T e^{(s-t-\delta)B} \int_H \Psi^n(s,y)\, \mu_{s-t}(x,dy)\, ds,$$

and from (3.18) it follows that

$$\left|\int_{t+\delta}^T e^{(s-t-\delta)B} \int_H \Psi^n(s,y)\, \mu_{s-t}(x,dy) ds\right| \le C\, \mathbb{E} \int_0^T (1+|X_s^{t,x}|^p + |Y_s^{n,t,x}| + |Z_s^{n,t,x}|) ds \le C(t,x,\delta)$$

by (3.14). Since $e^{\delta B}$ is compact by our assumptions, the sequence $(b^n_\delta(t,x))_n$ is relatively compact, hence totally bounded. So there exists a finite set $A \subset K$ such that for any n there exists $a \in A$ satisfying $|b^n_\delta(t,x) - a| < \epsilon/2$. So for every n there exists $a \in A$ such that $|u^n(t,x) - q(t,x) - a| < \epsilon$. This proves that $(u^n(t,x))_n$ is totally bounded. We have now proved that $(u^n(t,x))_n$ is a convergent sequence in K for every (t,x).

Third Step. Convergence of Y^n and Z^n.

Let us consider again the the Ornstein-Uhlenbeck process $X_s = X^{0,x_0}_s$ defined in (3.11) and let us denote $Y^n_s = Y^{n,0,x_0}_s$, $Z^n_s = Z^{n,0,x_0}_s$. Denoting by $u(t,x)$ the limit of $u^n(t,x)$ then obviously $Y^n_s = u^n(s,X_s)$ converges to $u(s,X_s)$, which we denote by Y_s. Setting $s=t$ in (3.14) we have $|u^n(t,x)| = \mathbb{E}|Y^{n,t,x}_t| \leq C(1+|x|^p)$ and consequently

$$|Y^n_s|^2 = |u^n(s,X_s)|^2 \leq C(1+|X_s|^{2p});$$

since $\mathbb{E}\int_0^T |X_t|^{2p} dt < \infty$ we conclude that Y^n converges to Y in $L^2(\Omega\times[0,T];K)$. ¿From inequality (3.5) of Lemma 5 it follows that

$$\begin{aligned}\mathbb{E}\int_0^T |Z^n_t - Z^m_t|^2 dt &\leq C\left(\sup_{t\in[0,T]} \mathbb{E}\,(1+|X_t|^{2p})\right)^{1/2}\left(\mathbb{E}\int_0^T |Y^n_t - Y^m_t|^2 dt\right)^{1/2}\\ &\leq C_{x_0}\left(\mathbb{E}\int_0^T |Y^n_t - Y^m_t|^2 dt\right)^{1/2}\end{aligned} \tag{3.21}$$

from which we conclude that (Z^n) is a Cauchy sequence in $L^2(\Omega\times[0,T];L_2(\Xi,K))$. Let us denote by Z its limit. Passing to a subsequence, we can assume that $|Z^n_t - Z_t| \to 0$, $\mathbb{P}$-a.s. for almost every t. Let us define a function $v : [0,T]\times H \to L_2(\Xi,K)$ by setting $v(t,x) = \lim_{n\to\infty} v^n(t,x)$ for all (t,x) for which the limit exists, $v(t,x) = 0$ elsewhere. Then v is Borel measurable and we have $Z_t = v(t,X_t)$, $\mathbb{P}$-a.s. for almost every t.

Fourth Step. Existence of solution. For every $t \in [0,T]$, (Y^n, Z^n) satisfies $\mathbb{P}$-a.s.:

$$Y^n_t + \int_t^T e^{(t-s)B} Z^n_s \, dW_s = e^{(T-t)B}\phi(X_T) + \int_t^T e^{(t-s)B}\psi^n(s,X_s,Y^n_s,Z^n_s)\, ds.$$

To prove that (Y,Z) is a solution to (3.3) it remains to check that

$$\mathbb{E}\int_0^T |\psi^n(s,X_s,Y^n_s,Z^n_s) - \psi(s,X_s,Y_s,Z_s)|\, ds \to 0.$$

From (iv) of Lemma 3 we obtain $\psi^n(s,x,y_n,z_n) \to \psi(s,x,y,z)$ in K, whenever $y_n \to y$ in K and $z_n \to z$ in $L_2(\Xi,K)$, for every $s\in[0,T]$, $x \in H$. Taking into account (3.12) and (3.14) we have

$$\mathbb{E}\int_0^T |\psi^n(s,X_s,Y^n_s,Z^n_s)|^2\, ds \leq C\mathbb{E}\int_0^T (1+|X_s|^{2p} + |Y^n_s|^2 + |Z^n_s|^2)\, ds \leq C$$

which shows that $(\psi^n(s,X_s,Y^n_s,Z^n_s))$ is uniformly integrable on $\Omega\times[0,T]$ and the required convergence follows immediately. □

4 BSDE with bounded continuous generator

In this section and in the following one we adopt a more general approach and we consider a process X with values in a metric space. We will assume that X is a Markov process with respect to a Brownian filtration. More precisely, in the sequel we will make the following assumptions.

(1) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and $\{W_t,\ t \in [0,T]\}$ is a cylindrical Wiener process in a Hilbert space Ξ. For an arbitrary interval $[s,t] \subset [0,T]$ we denote by $\mathcal{F}_{[s,t]}$ the σ-algebra generated by the random variables $\{W_r^\xi - W_s^\xi,\ r \in [s,t], \xi \in \Xi\}$ and by the $\mathbb{P}$-null sets of $\mathcal{F}$.

(2) $X = \{X_s^{t,x}(\omega),\ \omega \in \Omega, 0 \le t \le s \le T, x \in M\}$ is a stochastic process with values in a complete separable metric space M, measurable with respect to $\mathcal{F} \times \mathcal{B}(\Delta) \times \mathcal{B}(M)$ and $\mathcal{B}(M)$ respectively (here by Δ we denote the set $\{(t,s),\ 0 \le t \le s \le T\}$ and by $\mathcal{B}(\Lambda)$ the Borel σ-algebra of any topological space Λ).

(3) For every $t \in [0,T]$ and $x \in M$, the process $\{X_s^{t,x},\ s \in [t,T]\}$ has continuous paths and is adapted to the filtration $\{\mathcal{F}_{[t,s]},\ s \in [t,T]\}$.

(4) For $0 \le t \le s \le T$ and $x \in M$ we have, $\mathbb{P}$-a.s.,

$$X_t^{t,x} = x, \qquad X_r^{s,X_s^{t,x}} = X_r^{t,x}, \qquad \tau \in [s,T]. \tag{4.1}$$

Let us denote by

$$\mu_s^{t,x}(A) = \mathbb{P}(X_s^{t,x} \in A), \qquad 0 \le t \le s \le T,\ x \in M,\ A \in \mathcal{B}(M),$$

the transition probabilities. Standard arguments show that X is a Markov process, in the sense that for every bounded Borel function ϕ on M and for $0 \le t \le s \le r \le T$ and $x \in M$, we have

$$\mathbb{E}^{\mathcal{F}_s}\phi(X_r^{t,x}) = \int_M \phi(y)\, \mu_r^{s,X_s^{t,x}}(dy), \qquad \mathbb{P} - a.s.$$

We need the following lemma, that has been proved in [9], Proposition 3.2, in the special case when M is a Hilbert space. Exactly the same arguments carry over to the general case.

Lemma 7 *Assume the properties* **(1)** $-$ **(4)** *above. Suppose that*

(i) $z = \{z(\omega, s, t, x),\ \omega \in \Omega, 0 \le t \le s \le T, x \in M\}$ *is a stochastic process with values in a Hilbert space* V*, measurable with respect to* $\mathcal{F} \times \mathcal{B}(\Delta) \times \mathcal{B}(M)$ *and* $\mathcal{B}(V)$ *respectively.*

(ii) For every $t \in [0,T]$ *and* $x \in M$*, the process* $\{z(s,t,x),\ s \in [t,T]\}$ *is predictable with respect to the filtration* $\{\mathcal{F}_{[t,s]},\ s \in [t,T]\}$.

(iii) For $0 \le t \le s \le T$ *and* $x \in M$ *we have,* $\mathbb{P}$*-a.s.,*

$$z(r,s,X_s^{t,x}) = z(r,t,x), \qquad \textit{for almost all } r \in [s,T]. \tag{4.2}$$

Then there exists a Borel measurable function $v : [0,T] \times M \to V$ *such that, for* $t \in [0,T]$ *and* $x \in H$*, we have* $\mathbb{P}$*-a.s.*

$$z(s,t,x) = v(s, X_s^{t,x}), \qquad \text{for almost all } s \in [t,T]. \tag{4.3}$$

We fix arbitrary $x \in M$ and consider the following BSDE:

$$dY_t = -BY_t\, dt - \psi(t, X_t^{0,x}, Y_t, Z_t)\, dt + Z_t\, dW_t, \qquad Y_T = \phi(X_T^{0,x}), \tag{4.4}$$

under the following assumptions.

Hypothesis 8 *(i) The process X satisfies the properties* **(1)**-**(4)** *above.*

(ii) The operator $B : D(B) \subset K \to K$ *is the infinitesimal generator of a strongly continuous dissipative semigroup* $\{e^{tB}, t \geq 0\}$ *of bounded linear operators in* K.

(iii) $\phi : M \to K$ *and* $\psi : [0,T] \times M \times K \times L_2(\Xi, K) \to K$ *are Borel measurable functions,*

$$\mathbb{E}\, |\phi(X_T^{t,x})|^2 < \infty, \qquad t \in [0,T],\ x \in M,$$

and there exists a constant $C > 0$ *such that*

$$|\psi(t,x,y,z)| \leq C, \qquad t \in [0,T], x \in M, y \in K, z \in L_2(\Xi, K).$$

(iv) For every $t \in [0,T]$ *and* $x \in M$ *the function* $\psi(t,x,\cdot,\cdot) : K \times L_2(\Xi,K) \to K$ *is continuous.*

We say that an $(\mathcal{F}_t)$-predictable process (Y,Z) with values in $K \times L_2(\Xi, K)$ is a mild solution of (4.4) if

$$\sup_{t\in[0,T]} \mathbb{E}\,|Y_t|^2 + \mathbb{E}\int_0^T |Z_t|^2 dt < \infty \tag{4.5}$$

and for every $t \in [0,T]$ the following equality holds:

$$Y_t + \int_t^T e^{(t-s)B} Z_s\, dW_s = e^{(T-t)B}\phi(X_T^{0,x}) + \int_t^T e^{(t-s)B}\psi(s, X_s^{0,x}, Y_s, Z_s)\, ds, \qquad \mathbb{P}-a.s. \tag{4.6}$$

Lemma 9 *Assume that Hypothesis 8 holds and let* (Y,Z) *be a mild solution to (4.4). Then*

$$\sup_{t\in[0,T]} \mathbb{E}\,|Y_t|^2 + \mathbb{E}\int_0^T |Z_t|^2 dt \leq C\ (1 + \mathbb{E}|\phi(X_T^{0,x})|^2). \tag{4.7}$$

If ψ', ϕ' *are functions satisfying Hypothesis 8 and* (Y', Z') *is a corresponding mild solution then*

$$\mathbb{E}\int_0^T |Z_t - Z_t'|^2 dt \leq \mathbb{E}|\phi(X_T^{0,x}) - \phi'(X_T^{0,x})|^2 + C\,\mathbb{E}\int_0^T |Y_t - Y_t'|\, dt. \tag{4.8}$$

In (4.7) and (4.8) the constant C *depends only on* T *and on the constant* C *in Hypothesis 8.*

Proof. Proceeding as in the proof of Lemma 5 we obtain (compare (3.10))

$$\mathbb{E}|Y_t|^2 + \mathbb{E}\int_t^T |Z_s|^2 ds \leq \mathbb{E}|\phi(X_T^{0,x})|^2 + 2\mathbb{E}\int_t^T \langle Y_s, \psi(s, X_s^{0,x}, Y_s, Z_s)\rangle\, ds. \tag{4.9}$$

Since ψ is bounded we have

$$\begin{aligned}\mathbb{E}|Y_t|^2 + \mathbb{E}\int_t^T |Z_s|^2 ds &\leq \mathbb{E}|\phi(X_T^{0,x})|^2 + C\mathbb{E}\int_t^T |Y_s|\, ds\\ &\leq \mathbb{E}|\phi(X_T^{0,x})|^2 + C\mathbb{E}\int_t^T (1 + |Y_s|^2)\, ds,\end{aligned}$$

and (4.7) follows from Gronwall's lemma.

In order to prove (4.8) we write the equation satisfied by $(Y - Y', Z - Z')$ and proceeding as before we arrive at

$$\begin{aligned}&\mathbb{E}|Y_t - Y_t'|^2 + \mathbb{E}\int_t^T |Z_s - Z_s'|^2 ds\\ &\quad\leq \mathbb{E}|\phi(X_T^{0,x}) - \phi'(X_T^{0,x})|^2 + 2\mathbb{E}\int_t^T \langle Y_s - Y_s', \psi(s, X_s^{0,x}, Y_s, Z_s) - \psi'(s, X_s^{0,x}, Y_s', Z_s')\rangle\, ds.\end{aligned}$$

By the boundedness assumptions on ψ, ψ' we obtain

$$\mathbb{E}|Y_t - Y'_t|^2 + \mathbb{E}\int_t^T |Z_s - Z'_s|^2 ds \le \mathbb{E}|\phi(X_T^{0,x}) - \phi'(X_T^{0,x})|^2 + C\mathbb{E}\int_t^T |Y_s - Y'_s|\, ds.$$

(4.8) follows immediately. □

Theorem 10 *Assume that Hypothesis 8 holds, that the operators e^{tB} are compact for $t > 0$, and that the transition probabilities of the process X:*

$$\mu_s^{t,x}, \qquad 0 \le t < s \le T,\ x \in M$$

are all equivalent measures on M.

Then there exists a mild solution to equation (4.4).

Moreover there exist Borel measurable functions $u : [0,T] \times M \to K$, $v : [0,T] \times M \to L_2(\Xi, K)$ such that, $\mathbb{P}$-a.s.,

$$Y_t = u(t, X_t),\ \textit{for all } t \in [0,T]; \qquad Z_t = v(t, X_t),\ \textit{for almost all } t \in [0,T].$$

Proof - First Step. Approximation. We apply Lemma 3 to the metric space $[0,T] \times M$ and the Hilbert space $K \times L_2(\Xi, K)$ and obtain a sequence of functions $\psi^n : [0,T] \times M \times K \times L_2(\Xi, K) \to K$ such that, for any $n \ge 1$,

$$|\psi^n(t, x, y, z)| \le C, \tag{4.10}$$

and for fixed n, ψ^n is Lipschitz with respect to (y, z) uniformly with respect to (t, x).

Let $(Y^{n,t,x}, Z^{n,t,x})$ be the unique mild solution of

$$dY_s^{n,t,x} = -BY_s^{n,t,x}\, ds - \psi^n(s, X_s^{t,x}, Y_s^{n,t,x}, Z_s^{n,t,x})\, ds + Z_s^{n,t,x}\, dW_s, \quad Y_T^{n,t,x} = \phi(X_T^{t,x}), \tag{4.11}$$

where we use the convention $X_s^{t,x} = x$ for $s < t$. By (4.7)

$$\sup_{s\in[t,T]} \mathbb{E}|Y_s^{n,t,x}|^2 + \mathbb{E}\int_0^T |Z_s^{n,t,x}|^2 ds \le C\ (1 + \mathbb{E}|\phi(X_T^{t,x})|^2) < \infty. \tag{4.12}$$

Moreover, from the uniqueness of the solution to (4.11) it is easy to deduce the following identities: for $0 \le t \le s \le T$ and $x \in M$, we have, $\mathbb{P}$-a.s.,

$$\begin{aligned} Y_r^{n,s,X_s^{t,x}} &= Y_r^{n,t,x}, \quad \text{for all } r \in [s,T], \\ Z_r^{n,s,X_s^{t,x}} &= Z_r^{n,t,x}, \quad \text{for almost all } r \in [s,T]. \end{aligned}$$

Setting $u^n(t,x) = Y_t^{n,t,x}$ it follows immediately that for every t, x, $\mathbb{P}$-a.s.,

$$Y_s^{n,t,x} = u^n(s, X_s^{t,x}), \qquad s \in [t,T].$$

Applying Lemma 7 to the process $z(s,t,x) = Z_s^{n,t,x}$ we conclude that there exist Borel measurable functions $v^n : [0,T] \times M \to L_2(\Xi, K)$, such that for every t, x, $\mathbb{P}$-a.s.,

$$Z_s^{n,t,x} = v^n(s, X_s^{t,x}), \qquad \text{for almost all } s \in [t,T].$$

Second Step. In this step we prove that there exists a subsequence of $u^n(t,x)$ which is convergent in K for every t, x. This is obvious for $t = T$, since $u^n(T,x) = \phi(x)$, so we can assume $t < T$.

Noting that $u^n(t,x) = Y_t^{n,t,x}$, taking expectation in the BSDE we have

$$
\begin{aligned}
u^n(t,x) &= \mathbb{E}\, e^{(T-t)B}\phi(X_T^{t,x}) + \mathbb{E}\int_t^T e^{(s-t)B}\psi^n(s, X_s^{t,x}, Y_s^{n,t,x}, Z_s^{n,t,x})\, ds \\
&= \mathbb{E}\, e^{(T-t)B}\phi(X_T^{t,x}) + \mathbb{E}\int_t^T e^{(s-t)B}\psi^n(s, X_s^{t,x}, u^n(s, X_s^{t,x}), v^n(s, X_s^{t,x}))\, ds \\
&= \mathbb{E}\, e^{(T-t)B}\phi(X_T^{t,x}) + \int_t^T e^{(s-t)B}\int_M \Psi^n(s,y)\, \mu_s^{t,x}(dy)\, ds,
\end{aligned}
\tag{4.13}
$$

where $\Psi^n(s,y) = \psi^n(s, y, u^n(s,y), v^n(s,y))$. We fix an arbitrary $x_0 \in M$ and note that, from our assumptions, $\mu_s^{t,x}$ is absolutely continuous with respect to μ_s^{0,x_0} for $s > t$ and $x \in M$. Let us denote by $d^{s,t}(x,y)$ the corresponding density. Then

$$
u^n(t,x) = \mathbb{E}\ e^{(T-t)B}\phi(X_T^{t,x}) + \int_t^T \int_M e^{(s-t)B}\Psi^n(s,y) d^{s,t}(x,y)\, \mu_s^{0,x_0}(dy)\, ds.
$$

Since (Ψ^n) is uniformly bounded, this family is a bounded set in $L^\infty([0,T]\times M; \mu_s^{0,x_0}(dy)\, ds; K)$, whence relatively compact in the weak* topology. Since, in addition, the space $L^1([0,T] \times M; \mu_s^{0,x_0}(dy)\, ds; K)$ is separable, there exists a sequence (still denoted Ψ^n) and a function $\Psi^0 \in L^\infty([0,T] \times M; \mu_s^{0,x_0}(dy)\, ds; K)$ such that for any $\varphi \in L^1([0,T] \times M; \mu_s^{0,x_0}(dy)\, ds; K)$ we have

$$
\lim_{n\to\infty} \int_0^T \int_M \langle \Psi^n(s,y) - \Psi^0(s,y), \varphi(s,y)\rangle_K\, \mu_s^{0,x_0}(dy)\, ds = 0.
$$

For any fixed (t,x) and for every $k \in K$,

$$
\begin{aligned}
\int_0^T \int_M 1_{s\in[t,T]} d^{s,t}(x,y)\, |e^{(s-t)B^*}k|\, \mu_s^{0,x_0}(dy)\, ds &= \int_t^T \int_M |e^{(s-t)B^*}k|\, \mu_s^{t,x}(dy)\, ds \\
\le C \int_t^T \int_M \mu_s^{t,x}(dy)\, ds = C\cdot(T-t),
\end{aligned}
$$

which shows that $\varphi(s,y) = 1_{s\in[t,T]} d^{s,t}(x,y) e^{(s-t)B^*}k$ belongs to $L^1([0,T] \times M; \mu_s^{0,x_0}(dy)\, ds; K)$. We conclude that

$$
\begin{aligned}
\lim_{n\to\infty} \langle u^n(t,x), k\rangle &= \mathbb{E}\ \langle e^{(s-t)B}\phi(X_T^{t,x}), k\rangle + \lim_{n\to\infty} \int_0^T \int_M \langle \Psi^n(s,y), \varphi(s,y)\rangle_K\, \mu_s^{0,x_0}(dy)\, ds \\
&= \mathbb{E}\ \langle e^{(s-t)B}\phi(X_T^{t,x}), k\rangle + \int_t^T \int_M \langle \Psi^0(s,y), e^{(s-t)B^*}k\rangle d^{s,t}(x,y)\, \mu_s^{0,x_0}(dy)\, ds.
\end{aligned}
$$

and so that $(u^n(t,x))_n$ is weakly convergent in K for every t, x.

To prove that $(u^n(t,x))_n$ is convergent in the norm of K we will show that, for every (t,x), the sequence $(u^n(t,x))_n$ is totally bounded.

For $t < T$ and $\delta > 0$ so small that $t + \delta \le T$ we decompose $u^n(t,x)$ as follows (compare (4.13)):

$$
u^n(t,x) = q(t,x) + a_\delta^n(t,x) + b_\delta^n(t,x),
\tag{4.14}
$$

where $q(t,x) = \mathbb{E}\, e^{(T-t)B}\phi(X_T^{t,x})$,

$$
a_\delta^n(t,x) = \int_t^{t+\delta} e^{(s-t)B}\int_M \Psi^n(s,y)\, \mu_s^{t,x}(dy)\, ds, \quad b_\delta^n(t,x) = \int_{t+\delta}^T e^{(s-t)B}\int_M \Psi^n(s,y)\, \mu_s^{t,x}(dy)\, ds.
$$

Let us fix (t,x) and let $\epsilon > 0$ be arbitrary. Since (Ψ^n) is uniformly bounded, we have $\left|\int_M \Psi^n(s,y)\,\mu_s^{t,x}(dy)\,ds\right| \le C$, so it follows that $|a_\delta^n(t,x)| \le C\,\delta$, and we can choose δ such that $|a_\delta^n(t,x)| < \epsilon/2$ for every n. Next note that

$$b_\delta^n(t,x) = e^{\delta B} \int_{t+\delta}^T e^{(s-t-\delta)B} \int_M \Psi^n(s,y)\,\mu_s^{t,x}(dy)\,ds,$$

and

$$\left| \int_{t+\delta}^T e^{(s-t-\delta)B} \int_M \Psi^n(s,y)\,\mu_s^{t,x}(dy)\,ds \right| \le C.$$

Since $e^{\delta B}$ is compact by our assumptions, the sequence $(b_\delta^n(t,x))_n$ is relatively compact, hence totally bounded. So there exists a finite set $A \subset K$ such that for any n there exists $a \in A$ satisfying $|b_\delta^n(t,x) - a| < \epsilon/2$. So for every n there exists $a \in A$ such that $|u^n(t,x) - q(t,x) - a| < \epsilon$. This shows that $(u^n(t,x))_n$ is totally bounded and the claim is proved.

Third Step. Convergence of Y^n and Z^n.

Let us denote $Y_s^n = Y_s^{n,0,x_0}$, $Z_s^n = Z_s^{n,0,x_0}$. Denoting by $u^0(t,x)$ the limit of $u^n(t,x)$ then obviously $Y_s^n = u^n(s,X_s)$ converges to $u(s,X_s)$, which we denote by Y_s. ¿From (4.12) it follows that

$$\sup_n \mathbb{E} \int_0^T |Y_s^n|^2 ds < \infty$$

and we deduce that Y^n converges to Y in $L^1(\Omega \times [0,T]; K)$. ¿From inequality (4.8) of Lemma 9 it follows that

$$\mathbb{E} \int_0^T |Z_t^n - Z_t^m|^2 dt \le C\mathbb{E} \int_0^T |Y_t^n - Y_t^m|\,dt,$$

from which we conclude that (Z^n) is a Cauchy sequence in $L^2(\Omega \times [0,T]; L_2(\Xi, K))$. Let us denote by Z its limit. Passing to a subsequence, we can assume that $|Z_t^n - Z_t| \to 0$, $\mathbb{P}$-a.s. for almost every t. Let us define a function $v : [0,T] \times H \to L_2(\Xi, K)$ setting $v(t,x) = \lim_{n\to\infty} v^n(t,x)$ for all (t,x) for which the limit exists, $v(t,x) = 0$ elsewhere. Then v is Borel measurable and we have $Z_t = v(t,X_t)$, $\mathbb{P}$-a.s. for almost every t.

Fourth Step. Existence of solution. For every $t \in [0,T]$, (Y^n, Z^n) satisfies $\mathbb{P}$-a.s.:

$$Y_t^n + \int_t^T e^{(t-s)B} Z_s^n\,dW_s = e^{(T-t)B}\phi(X_T^{0,x_0}) + \int_t^T e^{(t-s)B}\psi^n(s, X_s^{0,x_0}, Y_s^n, Z_s^n)\,ds.$$

To prove that (Y,Z) is a solution to (3.3) it remains to check that

$$\mathbb{E} \int_0^T |\psi^n(s, X_s^{0,x_0}, Y_s^n, Z_s^n) - \psi(s, X_s^{0,x_0}, Y_s, Z_s)|\,ds \to 0.$$

From (iv) of Lemma 3 we obtain $\psi^n(s,x,y_n,z_n) \to \psi(s,x,y,z)$ in K, whenever $y_n \to y$ in K and $z_n \to z$ in $L_2(\Xi,K)$, for every $s \in [0,T]$, $x \in H$. Taking into account (4.10) the required convergence follows from the dominated convergence theorem. □

4.1 Example

Let W be a cylindrical Wiener process in a Hilbert space Ξ with Brownian filtration $(\mathcal{F}_t)$. Consider the following equation on the time interval $[0,T]$ for an unknown process X with values in a Hilbert space H:

$$dX_t = AX_t\,dt + F(t,X_t)\,dt + G\,dW_t, \qquad X_0 = x,$$

where $x \in H$, the operators A and G satisfy Hypothesis 1, $F : [0, T] \times H \to H$ is a Borel measurable mapping such that, for some constant $C \geq 0$,

$$|F(t, x) - F(t, x')| \leq C\,|x - x'|, \quad |F(t, x)| \leq C\,(1 + |x|), \qquad t \in [0, T],\ x, x' \in H,$$

and there exists $\alpha > 0$ such that

$$\text{Trace} \int_0^T s^{-\alpha}\, e^{sA} GG^* e^{sA^*} ds < \infty$$

(this is a stronger assumption than Hypothesis 1-(iii)).

It is well known (see e.g. [6]) that under these conditions there exists a unique mild solution, i.e. an $(\mathcal{F}_t)$-adapted process X, with continuous paths in H, such that, $\mathbb{P}$-a.s.,

$$X_t = e^{tA} x + \int_0^t e^{(t-s)A} F(s, X_s)\, ds + \int_0^t e^{(t-s)A} G\, dW_s, \qquad t \geq 0.$$

X is unique up to indistinguishability. Let us denote by $\mu_t^{0,x}$ the law of X_t.

We assume further that the image of F is contained in the image of G and there exists $C \geq 0$ such that

$$|G^{-1} F(t, x)| \leq C, \qquad t \in [0, T],\ x \in H,$$

where G^{-1} denotes the pseudo-inverse of G. We consider the Ornstein-Uhlenbeck process X' solution of

$$dX'_t = AX'_t\, dt + G\, dW_t, \qquad X'_0 = x.$$

By the Girsanov theorem, setting

$$\rho = \exp\left(\int_0^T \langle G^{-1} F(s, X'_s), dW_s \rangle - \frac{1}{2} \int_0^T |G^{-1} F(s, X'_s)|^2\, ds \right),$$

we have $\mathbb{E}\,\rho = 1$ and the process $W'_t = W_t - \int_0^t G^{-1} F(s, X'_s)\, ds$, $t \in [0, T]$, is a cylindrical Wiener process with respect to the probability $\mathbb{P}'$ admitting density ρ with respect to $\mathbb{P}$. Then we have

$$dX'_t = AX'_t\, dt + F(t, X'_t)\, dt + G\, dW'_t, \qquad X_0 = x,$$

and it follows that the law of X' under $\mathbb{P}'$ is the same as the law of X under $\mathbb{P}$. Since $\mathbb{P}$ and $\mathbb{P}'$ are equivalent measures, it follows in particular that the $\mu_t^{0,x}$ is equivalent to $\mathcal{N}(e^{tA}x, Q_t)$, and therefore that $\{\mu_t^{0,x},\ t \in (0, T], x \in H\}$ is a family of equivalent measures. In the same way one proves that the process $X_s^{t,x}$, solution in the mild sense to the equation

$$dX_s^{t,x} = AX_s^{t,x}\, ds + F(s, X_s^{t,x})\, dt + G\, dW_s, \qquad X_t = x,$$

on the interval $[t, T] \subset [0, T]$, satisfies all the requirements of Theorem 10. So if B, ψ, ϕ satisfy the assumptions in Hypothesis 8 and the operators e^{tB} are compact for $t > 0$, then there exists a mild solution to equation (4.4).

5 BSDE with bounded continuous generator: second case

In this section we still consider a Markov process $X = \{X_s^{t,x},\ 0 \leq t \leq s \leq T, x \in M\}$, with values in a complete separable metric space M, satisfying the properties $(\mathbf{1}) - (\mathbf{4})$ of section 4. We denote by $\mu_s^{t,x}$ the transition probabilities of X. We suppose that Hypothesis 8 holds and, in addition, that the function ϕ is bounded. In particular the conclusions of Lemma 9 still hold.

We fix arbitrary $x \in M$ and we consider the same BSDE as in formula (4.4)

$$dY_t = -BY_t\,dt - \psi(t, X_t^{0,x}, Y_t, Z_t)\,dt + Z_t\,dW_t, \qquad Y_T = \phi(X_T^{0,x}). \tag{5.1}$$

As before an $(\mathcal{F}_t)$-predictable process (Y, Z) with values in $K \times L_2(\Xi, K)$ is called a mild solution of (5.1) if it satisfies (4.5) and (4.6).

In this section we replace the requirement of mutual absolute continuity of the transition probabilities of X with a continuity assumption of the map $x \to \mu_s^{t,x}$ with respect to the variation norm.

More precisely we assume that for every sequence x_n converging to x in M and for $0 \le t < s \le T$ we have

$$Var\;(\mu_s^{t,x} - \mu_s^{t,x_n}) \to 0, \tag{5.2}$$

for $n \to \infty$, where Var denotes the total variation.

Theorem 11 *Assume that Hypothesis 8 holds, that the operators e^{tB} are compact for $t > 0$, that the transition probabilities of the process X satisfy (5.2), and that $|\phi(x)| \le C$ for some constant $C > 0$ and every $x \in M$.*

Then there exists a mild solution to equation (5.1).

Moreover there exist Borel measurable functions $u : [0, T] \times M \to K$, $v : [0, T] \times M \to L_2(\Xi, K)$ such that, $\mathbb{P}$-a.s.,

$$Y_t = u(t, X_t),\ \textit{for all } t \in [0, T]; \qquad Z_t = v(t, X_t),\ \textit{for almost all } t \in [0, T].$$

Proof - First Step. Approximation. Applying Lemma 3 we construct a sequence of functions $\psi^n : [0, T] \times M \times K \times L_2(\Xi, K) \to K$ such that, for any $n \ge 1$,

$$|\psi^n(t, x, y, z)| \le C \tag{5.3}$$

and for fixed n, ψ^n is Lipschitz with respect to (y, z) uniformly with respect to (t, x). Let $(Y^{n,t,x}, Z^{n,t,x})$ be the unique mild solution of

$$dY_s^{n,t,x} = -BY_s^{n,t,x}\,ds - \psi^n(s, X_s^{t,x}, Y_s^{n,t,x}, Z_s^{n,t,x})\,ds + Z_s^{n,t,x}\,dW_s, \quad Y_T^{n,t,x} = \phi(X_T^{t,x}), \tag{5.4}$$

where we define $X_s^{t,x} = x$ for $s < t$. By (4.7) and the boundedness of ϕ,

$$\sup_{s\in[t,T]} \mathbb{E}|Y_s^{n,t,x}|^2 + \mathbb{E}\int_0^T |Z_s^{n,t,x}|^2 ds \le C\;(1 + \mathbb{E}|\phi(X_T^{t,x})|^2) \le C. \tag{5.5}$$

Arguing as in the proof of Theorem 10 we deduce that there exist Borel measurable functions $u^n : [0, T] \times M \to K$, $v^n : [0, T] \times M \to L_2(\Xi, K)$, such that for every t, x, $\mathbb{P}$-a.s.,

$$Y_s^{n,t,x} = u^n(s, X_s^{t,x}), \qquad s \in [t, T],$$

$$Z_s^{n,t,x} = v^n(s, X_s^{t,x}), \qquad \text{for almost all } s \in [t, T].$$

Second Step. In this step we prove that there exists a subsequence of $u^n(t, x)$ which is convergent in K for every t, x.

We first claim that for fixed (t, x) there exists a subsequence (n_k) (depending on (t, x)) such that $(u^{n_k}(t, x))_k$ is convergent in K. This is obvious for $t = T$, since $u^n(T, x) = \phi(x)$, so we can assume $t < T$. It is enough to show that, for fixed t, x, the sequence $(u^n(t, x))_n$ is relatively compact in K or, equivalently, that it is totally bounded.

From the definition of mild solution to (5.4) we obtain, taking expectation,

$$u^n(t,x) = Y_t^{n,t,x} = \mathbb{E}\, e^{(T-t)B}\phi(X_T^{t,x}) + \mathbb{E}\int_t^T e^{(s-t)B}\psi^n(s, X_s^{t,x}, Y_s^{n,t,x}, Z_s^{n,t,x})\, ds$$
$$= \mathbb{E}\, e^{(T-t)B}\phi(X_T^{t,x}) + \int_t^T e^{(s-t)B} g^{n,t,x}(s)\, ds,$$

where $g^{n,t,x}(s) = \mathbb{E}\,\psi^n(s, X_s^{t,x}, Y_s^{n,t,x}, Z_s^{n,t,x})$ satisfies $|g^{n,t,x}(s)| \le C$. It follows that $|u^n(t,x)| \le C$, i.e. the sequence $(u^n(t,x))_n$ is uniformly bounded. For $\delta > 0$ so small that $t+\delta \le T$ we decompose $u^n(t,x)$ as follows:

$$u^n(t,x) = q(t,x) + a_\delta^n(t,x) + b_\delta^n(t,x), \tag{5.6}$$

where $q(t,x) = \mathbb{E}\, e^{(T-t)B}\phi(X_T^{t,x})$,

$$a_\delta^n(t,x) = \int_t^{t+\delta} e^{(s-t)B} g^{n,t,x}(s)\, ds, \quad b_\delta^n(t,x) = \int_{t+\delta}^T e^{(s-t)B} g^{n,t,x}(s)\, ds.$$

Let us fix (t,x) and let $\epsilon > 0$ be arbitrary. We have $|a_\delta^n(t,x)| \le C\,\delta$, so that we can choose δ such that $|a_\delta^n(t,x)| < \epsilon/2$ for every n. Next note that

$$b_\delta^n(t,x) = e^{\delta B}\int_{t+\delta}^T e^{(s-t-\delta)B} g^{n,t,x}(s)\, ds,$$

and

$$\left|\int_{t+\delta}^T e^{(s-t-\delta)B} g^{n,t,x}(s)\, ds\right| \le C.$$

Since $e^{\delta B}$ is compact by our assumptions, the sequence $(b_\delta^n(t,x))_n$ is relatively compact, hence totally bounded. So there exists a finite set $A \subset K$ such that for any n there exists $a \in A$ satisfying $|b_\delta^n(t,x) - a| < \epsilon/2$. So for every n there exists $a \in A$ such that $|u^n(t,x) - q(t,x) - a| < \epsilon$. This shows that $(u^n(t,x))_n$ is totally bounded and the claim is proved.

Next note that

$$\begin{aligned} u^n(t,x) &= \mathbb{E}\, e^{(T-t)B}\phi(X_T^{t,x}) + \mathbb{E}\int_t^T e^{(s-t)B}\psi^n(s, X_s^{t,x}, Y_s^{n,t,x}, Z_s^{n,t,x})\, ds \\ &= \mathbb{E}\, e^{(T-t)B}\phi(X_T^{t,x}) + \mathbb{E}\int_t^T e^{(s-t)B}\psi^n(s, X_s^{t,x}, u^n(s, X_s^{t,x}), v^n(s, X_s^{t,x}))\, ds \\ &= \int_M e^{(T-t)B}\phi(y)\, \mu_T^{t,x}(dy) + \int_t^T e^{(s-t)B}\int_M \Psi^n(s,y)\, \mu_s^{t,x}(dy)\, ds, \end{aligned}$$

where $\Psi^n(s,y) = \psi^n(s, y, u^n(s,y), v^n(s,y))$.

Let us fix a dense sequence (t_j) in $[0,T]$ and a dense sequence (x_i) in M. By the previous claim and a diagonal procedure we can find a subsequence (n_k) such that $(u^{n_k}(t_j, x_i))_k$ converges for every i, j. By a change of notation we can assume that the original sequence $(u^n(t_j, x_i))_n$ is convergent for every i, j.

Next we fix j and we prove that $(u^n(t_j, x))_n$ is convergent for every $x \in M$. The assertion is trivial if $t_j = T$, so we assume $t_j < T$. We start from the inequality

$$\begin{aligned} |u^n(t_j,x) - u^m(t_j,x)| \le\ & |u^n(t_j,x) - u^n(t_j,x_i)| + |u^m(t_j,x) - u^m(t_j,x_i)| \\ & + |u^n(t_j,x_i) - u^m(t_j,x_i)|. \end{aligned} \tag{5.7}$$

We have

$$u^n(t_j,x)-u^n(t_j,x_i) = e^{(T-t)B}\int_M \phi(y)\,[\mu_T^{t_j,x}(dy)-\mu_T^{t_j,x_i}(dy)]+ + \int_{t_j}^T e^{(s-t)B}\int_M \Psi^n(s,y)\,[\mu_s^{t_j,x}(dy)-\mu_s^{t_j,x_i}(dy)]\,ds,$$

and since ϕ is bounded and Ψ^n is uniformly bounded we obtain

$$\left|\int_{t_j}^T e^{(s-t)B}\int_M \Psi^n(s,y)\,[\mu_s^{t_j,x}(dy)-\mu_s^{t_j,x_i}(dy)]\,ds\right| \le C\int_{t_j}^T Var(\mu_s^{t_j,x}-\mu_s^{t_j,x_i})\,ds,$$

and

$$\left|e^{(T-t)B}\int_M \phi(y)\,[\mu_T^{t_j,x}(dy)-\mu_T^{t_j,x_i}(dy)]\,ds\right| \le C\,Var(\mu_T^{t_j,x}-\mu_T^{t_j,x_i}).$$

We note that by (5.2) for every sequence $x_n\to x$ we have $Var(\mu_s^{t_j,x}-\mu_s^{t_j,x_n})\to 0$ for $s>t_j$. Since $Var(\mu_s^{t_j,x}-\mu_s^{t_j,x_n})\le 2$, by the dominated convergence theorem we obtain

$$\int_{t_j}^T Var(\mu_s^{t_j,x}-\mu_s^{t_j,x_n})\,ds\to 0.$$

Given $\epsilon>0$, from the previous inequalities it follows that we can choose x_i so close to x that

$$|u^n(t_j,x)-u^n(t_j,x_i)|\le\epsilon,$$

for every n. In a similar way one proves that x_i can be chosen such that in addition $|u^m(t_j,x)-u^m(t_j,x_i)|\le\epsilon$ for every m, and since $(u^n(t_j,x_i))_n$ is convergent we conclude from (5.7) that $(u^n(t_j,x))_n$ is a Cauchy sequence for every $x\in M$.

Next we prove that $(u^n(t,x))_n$ is convergent for every $t\in[0,T]$ and $x\in M$. We can assume $t<T$, otherwise the assertion is trivial. We first claim that for $t<r$ we have

$$\left|u^n(t,x)-\int_M u^n(r,y)\,\mu_r^{t,x}(dy)\right|\le C\cdot(r-t). \tag{5.8}$$

From (5.4) we obtain

$$Y_t^{n,t,x}-Y_r^{n,t,x}=\int_t^r e^{(s-t)B}\psi^n(s,X_s^{t,x},Y_s^{n,t,x},Z_s^{n,t,x})\,ds-\int_t^r e^{(s-t)B}Z_s^{n,t,x}\,dW_s.$$

Taking expectation we obtain

$$\begin{aligned}\mathbb{E}\int_t^r e^{(s-t)B}\psi^n(s,X_s^{t,x},Y_s^{n,t,x},Z_s^{n,t,x})\,ds &= \mathbb{E}\,[Y_t^{n,t,x}-Y_r^{n,t,x}]\\ &=\mathbb{E}\,[u^n(t,x)-u^n(r,X_r^{t,x})]\\ &=u^n(t,x)-\int_M u^n(r,y)\,\mu_r^{t,x}(dy),\end{aligned}$$

and since ψ^n is uniformly bounded, (5.8) follows immediately.

Then we have, for $t_j>t$,

$$\begin{aligned}|u^n(t,x)-u^m(t,x)| \le& \left|u^n(t,x)-\int_M u^n(t_j,y)\,\mu_{t_j}^{t,x}(dy)\right|\\ &+\left|u^m(t,x)-\int_M u^m(t_j,y)\,\mu_{t_j}^{t,x}(dy)\right|\\ &+\left|\int_M u^n(t_j,y)\,\mu_{t_j}^{t,x}(dy)-\int_M u^m(t_j,y)\,\mu_{t_j}^{t,x}(dy)\right|.\end{aligned}$$

Given $\epsilon > 0$, we choose j such that $t_j - t < \epsilon$. For $n, m \geq N$ we have

$$|u^n(t,x) - u^m(t,x)| \leq C \cdot \epsilon + \int_M \sup_{n,m \geq N} |u^n(t_j, y) - u^m(t_j, y)| \, \mu_{t_j}^{t,x}(dy).$$

Since the sequence $(u^n(t_j, y))_n$ is convergent for every y and it is uniformly bounded, the last integral tends to 0 for $N \to \infty$. The proof of step 2 is finished.

The third and fourth step are the same as in Theorem 10 and this concludes the proof. □

5.1 Example

Let W be a cylindrical Wiener process in a Hilbert space Ξ with Brownian filtration $(\mathcal{F}_t)$. We take $H = \Xi$ and consider the following equation on the time interval $[t, T] \subset [0, T]$ for an unknown process X with values in H:

$$dX_s = AX_s \, ds + F(X_s) \, ds + G(X_s) \, dW_s, \qquad X_t = x,$$

where $x \in H$, the operator $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{tA}, t \geq 0\}$ of bounded linear operators in H, $F : H \to H$ and $G : H \to L(H)$ are Borel measurable mappings such that, for some constant $C \geq 0$,

$$|F(x) - F(x')| \leq C\,|x - x'|, \quad |G(x) - G(x')|_{L(H)} \leq C\,|x - x'|, \qquad x, x' \in H.$$

We also assume that $e^{tA} \in L_2(H, H)$ for $t > 0$ and that $\int_0^T e^{-\alpha t} |e^{tA}|^2_{L_2(H,H)} dt < \infty$ for some $\alpha > 0$. It is well known (see e.g. [6]) that under these conditions there exists a mild solution i.e. an $(\mathcal{F}_t)$-adapted process, with continuous paths in H, such that, $\mathbb{P}$-a.s.,

$$X_s = e^{(s-t)A}x + \int_t^s e^{(s-r)A} F(X_r) \, dr + \int_t^s e^{(s-r)A} G(X_r) \, dW_r, \qquad s \in [t, T].$$

X is unique up to indistinguishability. The solution will be denoted $X_s^{t,x}$, to stress the dependence on x and t. The process X constructed in this way satisfies the conditions (**1**) – (**4**) of section 4. We denote by $\mu_s^{t,x}$ the law of $X_s^{t,x}$.

Assume now in addition that $G(x)$ is invertible for every $x \in H$ and there exists $C \geq 0$ such that $|G(x)^{-1}|_{L(H)} \leq C$ for all $x \in H$. Then the following inequality has been proved in [20] (see also [7], Theorem 7.1.1 and Lemma 7.1.5):

$$Var\,(\mu_s^{t,x} - \mu_s^{t,x'}) \leq \frac{C}{\sqrt{s-t}}\,|x - x'|, \quad 0 \leq t < s \leq T, \; x, x' \in H.$$

So under the previous assumptions condition (5.2) clearly holds, and so if B, ψ, ϕ satisfy the other requirements in Theorem 11 then there exists a mild solution to equation (5.1).

6 A stochastic game with infinitely many players

Let W be a cylindrical Wiener process in a Hilbert space Ξ, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $(\mathcal{F}_t)$ be its Brownian filtration.

We consider the Ornstein-Uhlenbeck process in a Hilbert space H defined by the equation $dX_t = AX_t \, dt + G \, dW_t$, more precisely

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A} G \, dW_s, \qquad t \in [0, T],$$

with A and G satisfying Hypothesis 1, and $x \in H$.

Hypothesis 12 *i) Let I be a finite or countable set.*

ii) For every $i \in I$, a metric space U_i is given. We denote $U = \times_{i\in I} U_i$ the product space.

iii) We assume that Borel measurable functions are given

$$R : [0,T] \times H \times U \to \Xi, \quad l^i : [0,T] \times H \times U \to \mathbb{R}, \quad \phi^i : H \to \mathbb{R},$$

for every $i \in I$. Moreover there exist constants $c_R \geq 0$, $c_i \geq 0$ such that

$$|R(t,x,v)| \leq c_R, \quad |l^i(t,x,v)| + |\phi^i(x)| \leq c_i(1 + |x|^p), \qquad t \in [0,T], x \in H, v \in U, i \in I.$$

Finally we assume that for every $t \in [0,T]$, $x \in H$ and $i \in I$ the functions

$$R(t,x,\cdot) : U \to \Xi, \qquad l^i(t,x,\cdot) : U \to \mathbb{R},$$

are continuous.

iv) For every $i \in I$ a number $\lambda_i \geq 0$ is given. If I is infinite, identifying I with the natural numbers, we assume that $\lambda_i \to +\infty$ as $i \to \infty$.

Each element $i \in I$ represents a player. U_i represents the set of actions that player i can take at any time. Coordinates of an element $v \in U$ are denoted v^i and we use the notation $v = (v^i)_i$. λ_i is a discount factor in the cost of player i, as defined below.

An $(\mathcal{F}_t)$-adapted process $u = \{u_t,\ t \in [0,T]\}$, with values in U, is called admissible decision process. Each component $u^i = \{u^i_t,\ t \in [0,T]\}$, $i \in I$, is then a process with values in U_i; u^i_t represents the action taken by player i at time t.

For every admissible decision process u, a cost $J^i(u)$ for the player $i \in I$ is defined as follows. By the Girsanov theorem the process

$$W^u_t = W_t - \int_0^t R(s, X_s, u_s)\, ds, \qquad t \in [0,T],$$

is a Wiener process under the probability measure $\mathbb{P}^u$ admitting the density ρ^u with respect to $\mathbb{P}$ given by

$$\rho^u = \exp\left(\int_0^T \langle R(s,X_s,u_s), dW_s\rangle - \frac{1}{2}\int_0^T |R(s,X_s,u_s)|^2\, ds \right).$$

We define

$$J^i(u) = \mathbb{E}^u \left[\int_0^T e^{-\lambda_i t}\, l^i(t, X_t, u_t)\, dt + e^{-\lambda_i T} \phi^i(X_T) \right], \qquad i \in I.$$

Since R is bounded, the application of the Girsanov theorem is justified and we also have $\mathbb{E}|\rho^u|^p < \infty$ for every $p \in [1,\infty)$. We note that X satisfies

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A} G R(s, X_s, u_s)\, ds + \int_0^t e^{(t-s)A} G\, dW^u_s, \qquad t \in [0,T]. \tag{6.1}$$

Therefore, under $\mathbb{P}^u$, X is the solution of a controlled stochastic equation with nonlinear drift.

An admissible decision process $\widehat{u}$ is called a Nash equilibrium if, for each $i \in I$, the equality

$$J^i(\widehat{u}) \leq J^i(u),$$

takes place for arbitrary decision process u satisfying, for all $j \neq i$,

$$u^j_t = \widehat{u}^j_t, \qquad \mathbb{P}\text{−a.s. for almost every } t \in [0,T].$$

The aim of this section is to show that a Nash equilibrium exists under appropriate conditions. Our main assumption is Hypothesis 13 below. Before its statement we introduce some notation.

Let us fix numbers $\rho_i > 0$ such that $\sum_{i\in I} c_i^2 \rho_i < \infty$, where c_i are the constants introduced in Hypothesis 12. In the sequel we will consider backward equations for processes with values in the Hilbert space $\ell^2_\rho(I)$, the space of real sequences $(y^i)_i$ satisfying $\sum_{i\in I} |y^i|^2 \rho_i < \infty$, endowed with the inner product

$$\langle y, v\rangle_{\ell^2_\rho(I)} = \sum_{i\in I} y^i v^i \rho_i, \qquad y = (y^i)_i \in \ell^2_\rho(I),\ v = (v^i)_i \in \ell^2_\rho(I).$$

For $i \in I$ we denote g_i the element of $\ell^2_\rho(I)$ defined by $g_i^j = 0$ if $i \neq j$, $g_i^i = 1/\rho_i$. We note that $\langle y, g_i\rangle_{\ell^2_\rho(I)} = y^i$ for every $y = (y^i)_i \in \ell^2_\rho(I)$ and that the family $\{g_i\sqrt{\rho_i},\ i \in I\}$ is a complete orthonormal basis of $\ell^2_\rho(I)$. For every $z \in L_2(\Xi, \ell^2_\rho(I))$ we can define elements $z^i \in \Xi^*$ by the formula

$$z^i\xi = \langle z\xi, g_i\rangle_{\ell^2_\rho(I)}, \qquad \xi \in \Xi, i \in I.$$

Since z is a Hilbert-Schmidt operator we have

$$\sum_{i\in I} |z^i|^2_{\Xi^*}\, \rho_i < \infty, \tag{6.2}$$

so that the sequence $(z^i)_i$ belongs to the Hilbert space $\ell^2_\rho(I, \Xi^*)$ consisting of Ξ^*-valued sequences satisfying (6.2), endowed with the natural inner product. It is easy to check that the mapping $z \to (z^i)_i$ is a Hilbert space isomorphism between $L_2(\Xi, \ell^2_\rho(I))$ and $\ell^2_\rho(I, \Xi^*)$. In the sequel we will make the identification $z = (z^i)_i$.

Hypothesis 13 *There exists a Borel measurable function* $\underline{u} : [0, T] \times H \times L_2(\Xi, \ell^2_\rho(I)) \to U$ *such that for every* $t \in [0, T]$, $x \in H$, $z = (z^i)_i \in L_2(\Xi, \ell^2_\rho(I))$, $i \in I$ *the inequality*

$$z^i R(t, x, \underline{u}(t, x, z)) + l^i(t, x, \underline{u}(t, x, z)) \leq z^i R(t, x, v) + l^i(t, x, v),$$

holds for every $v \in U$ *satisfying* $v^j = \underline{u}^j(t, x, z)$ *for all* $j \neq i$. *Moreover for every* $t \in [0, T]$, $x \in H$ *and* $i \in I$ *the function* $\underline{u}^i(t, x, \cdot) : L_2(\Xi, \ell^2_\rho(I)) \to U$ *is continuous.*

Remark 14 Hypotheses 12 and 13 are easier to check in the special case

$$R(t, x, v) = \sum_{j\in I} R_j(t, x, v^j), \quad l^i(t, x, v) = \sum_{j\in I} l^i_j(t, x, v^j), \qquad t \in [0, T], x \in H, v \in U, \tag{6.3}$$

i.e. when R and each l^i are sums of functions depending only on one coordinate $v^j \in U_j$ of $v \in U$. More precisely suppose that I, U_i, ϕ^i satisfy the assumptions of Hypothesis 12 (in particular, $|\phi^i(x)| \leq \widehat{c}_i(1 + |x|^p)$ for every x, i and for some constants $\widehat{c}_i \geq 0$) and that for every $i, j \in I$ there exist Borel measurable functions

$$R_j : [0, T] \times H \times U_j \to \Xi, \quad l^i_j : [0, T] \times H \times U_j \to \mathbb{R},$$

and constants c_{Rj}, c_{ij} such that

$$|R_j(t, x, a)| \leq c_{Rj}, \quad |l^i_j(t, x, a)| \leq c_{ij}(1 + |x|^p), \qquad t \in [0, T], x \in H, a \in U_j,$$

and $\sum_j c_{Rj} < \infty$, $\sum_j c_{ij} < \infty$ for every $i \in I$. We also assume that for every $t \in [0, T]$, $x \in H$ and $i, j \in I$ the functions

$$R_j(t, x, \cdot) : U_j \to \Xi, \qquad l^i_j(t, x, \cdot) : U_j \to \mathbb{R},$$

are continuous. If R and l^i are defined by (6.3) then Hypothesis 12 is satisfied with $c_i = \sum_j c_{ij} + \widehat{c}_i$. Suppose now that there exist Borel measurable functions $\underline{u}^i : [0,T] \times H \times \Xi^* \to U_i$, $i \in I$, such that

$$\eta R_i(t,x,\underline{u}^i(t,x,\eta)) + l_i^i(t,x,\underline{u}^i(t,x,\eta)) \leq \eta R_i(t,x,a) + l_i^i(t,x,a), \tag{6.4}$$

for every $i \in I$, $t \in [0,T]$, $x \in H$, $\eta \in \Xi^*$, $a \in U_i$. Moreover assume that for every $t \in [0,T]$, $x \in H$ and $i \in I$ the function $\underline{u}^i(t,x,\cdot) : \Xi^* \to U_i$ is continuous. Then setting

$$\underline{u}(t,x,z) = (\underline{u}^i(t,x,z^i))_i$$

it is easy to verify that Hypothesis 13 is satisfied.

Note that (6.4) can be expressed as

$$\underline{u}^i(t,x,\eta) \in \operatorname*{argmin}_{a \in U_i} [\eta R_i(t,x,a) + l_i^i(t,x,a)].$$

The existence of a function $\underline{u}^i$ satisfying (6.4) and such that $\underline{u}^i(t,x,\cdot)$ is continuous can be effectively checked in particular cases. For instance, in addition to the previous assumptions, suppose that all the metric spaces U_i coincide with the ball $B(0,r)$ of radius $r > 0$ centered at the origin of another Hilbert space $\mathcal{A}$. Furthermore assume that R_j are defined by

$$R_j(t,x,a) = \overline{R}_j(t,x)a, \qquad t \in [0,T], x \in H, a \in \mathcal{A},$$

where each $\overline{R}_j(t,x)$ is a linear bounded operator from $\mathcal{A}$ to Ξ, $\overline{R}_j(\cdot,\cdot)a : [0,T] \times H \to \Xi$ is Borel measurable for every $a \in \mathcal{A}$, and $|\overline{R}_j(t,x)| \leq \overline{c}_{Rj}$, $t \in [0,T], x \in H$, for some constants $\overline{c}_{Rj} \geq 0$ satisfying $\sum_j \overline{c}_{Rj} < \infty$. Suppose finally that l_i^i have the special form $l_i^i(t,x,a) = |a|^2$, $a \in B(0,r)$. Then a minimizer of $a \to \eta R_i(t,x,a) + l_i^i(t,x,a) = \eta \overline{R}_i(t,x)a + |a|^2$ over $B(0,r)$ can be easily computed, and the required function $\underline{u}^i$ can be defined by

$$\underline{u}^i(t,x,\eta) = \begin{cases} -\dfrac{1}{2}(\eta \overline{R}_i(t,x))^* & \text{if} \quad |\eta \overline{R}_i(t,x)| \leq 2r, \\ \\ -r\, \dfrac{(\eta \overline{R}_i(t,x))^*}{|\eta \overline{R}_i(t,x)|} & \text{if} \quad |\eta \overline{R}_i(t,x)| > 2r, \end{cases}$$

for $t \in [0,T]$, $x \in H$, $\eta \in \Xi^*$, where by $(\eta \overline{R}_i(t,x))^* \in \mathcal{A}$ we denote the image of $\eta \overline{R}_i(t,x) \in \mathcal{A}^*$ under the Riesz isometry $\mathcal{A}^* \to \mathcal{A}$. □

Theorem 15 *Under Hypotheses 1, 12 and 13 there exists a Nash equilibrium $\widehat{u}$. Moreover there exists a Borel measurable function $v : [0,T] \times H \to L_2(\Xi, \ell_\rho^2(I))$ such that*

$$\widehat{u}_t = \underline{u}(t, X_t, v(t, X_t)), \qquad \mathbb{P}\text{−a.s. for almost every } t \in [0,T]. \tag{6.5}$$

Remark 16 By equality (6.5), $\widehat{u}$ is called a closed-loop Nash equilibrium. □

Proof. Let us define an operator B in $\ell_\rho^2(I)$ setting $(By)_i = -\lambda_i y^i$ for $y \in D(B) = \{(y^i)_i : \sum_{i \in I} \lambda_i^2 |y^i|^2 \rho_i < \infty\}$. B is a self-adjoint operator with eigenvectors g_i and eigenvalues $-\lambda_i$. It is the infinitesimal generator of the dissipative semigroup given by the formula $(e^{tB}y)_i = e^{-\lambda_i t} y^i$. The condition $\lambda_i \to \infty$ ensures that e^{tB} is compact for every $t > 0$.

Let us define $\phi(x) = (\phi^i(x))_i$ and $f(t,x,z) = (f^i(t,x,z))_i$, where

$$f^i(t,x,z) = z^i R(t,x,\underline{u}(t,x,z)) + l^i(t,x,\underline{u}(t,x,z)), \qquad t \in [0,T], x \in H, z \in L_2(\Xi, \ell_\rho^2(I)), \tag{6.6}$$

and let us consider the backward equation

$$dY_t = -BY_t\,dt - f(t, X_t, Z_t)\,dt + Z_t\,dW_t, \qquad Y_T = \phi(X_T), \tag{6.7}$$

where the unknown processes Y and Z take values in $\ell^2_\rho(I)$ and $L_2(\Xi, \ell^2_\rho(I))$ respectively.

Next we verify that the functions

$$f : [0, T] \times H \times L_2(\Xi, \ell^2_\rho(I)) \to \ell^2_\rho(I), \qquad \phi : H \to \ell^2_\rho(I)$$

satisfy the assumptions of Theorem 6. By Hypothesis 12,

$$\begin{aligned}
|f(t,x,z)|_{\ell^2_\rho(I)} &\le \Big(\sum_i |z^i R(t,x,\underline{u}(t,x,z))|^2\rho_i\Big)^{1/2} + \Big(\sum_i |l^i(t,x,\underline{u}(t,x,z))|^2\rho_i\Big)^{1/2} \\
&\le c_R\Big(\sum_i |z^i|^2_{\Xi^*}\rho_i\Big)^{1/2} + \Big(\sum_i c_i^2\rho_i\Big)^{1/2}(1+|x|^p) \\
&= c_R|z|_{L_2(\Xi,\ell^2_\rho(I))} + \Big(\sum_i c_i^2\rho_i\Big)^{1/2}(1+|x|^p),
\end{aligned}$$

$$|\phi(x)|_{\ell^2_\rho(I)} = \Big(\sum_i |\phi^i(x)|^2\rho_i\Big)^{1/2} \le \Big(\sum_i c_i^2\rho_i\Big)^{1/2}(1+|x|^p).$$

The functions $f^i(t,x,\cdot)$ are continuous since they are defined in terms of the continuous mappings $R(t,x,\cdot)$, $l^i(t,x,\cdot)$ and $\underline{u}(t,x,\cdot)$. To check continuity of $f(t,x,\cdot)$, let us consider a sequence z_n converging to z in $L_2(\Xi, \ell^2_\rho(I))$ and note that

$$\begin{aligned}
|f^i(t,x,z_n) - f^i(t,x,z)| &\le |z_n^i R(t,x,\underline{u}(t,x,z_n)) - z^i R(t,x,\underline{u}(t,x,z))| \\
&\quad + |l^i(t,x,\underline{u}(t,x,z_n)) - l^i(t,x,\underline{u}(t,x,z))| \\
&\le c_R|z_n^i - z^i| + |z^i|\,|R(t,x,\underline{u}(t,x,z_n)) - R(t,x,\underline{u}(t,x,z))| \\
&\quad + |l^i(t,x,\underline{u}(t,x,z_n)) - l^i(t,x,\underline{u}(t,x,z))|.
\end{aligned}$$

It follows that

$$\begin{aligned}
|f(t,x,z_n) - f(t,x,z)|_{\ell^2_\rho(I)} &= \Big(\sum_i |f^i(t,x,z_n) - f^i(t,x,z)|^2\rho_i\Big)^{1/2} \\
&\le c_R|z_n - z|_{L_2(\Xi,\ell^2_\rho(I))} \\
&\quad + \Big(\sum_i |z^i|^2\,|R(t,x,\underline{u}(t,x,z_n)) - R(t,x,\underline{u}(t,x,z))|^2\rho_i\Big)^{1/2} \\
&\quad + \Big(\sum_i |l^i(t,x,\underline{u}(t,x,z_n)) - l^i(t,x,\underline{u}(t,x,z))|^2\rho_i\Big)^{1/2}.
\end{aligned}$$

Since R is bounded, $\sum_i |z^i|^2\rho_i < \infty$, $|l^i(t,x,\underline{u}(t,x,z_n))| \le c_i(1+|x|^p)$ and $\sum_i c_i^2\rho_i < \infty$ we conclude that $|f(t,x,z_n) - f(t,x,z)|_{\ell^2_\rho(I)} \to 0$.

Theorem 6 shows that (6.7) has a solution satisfying, in particular, $\mathbb{E}\int_0^T |Z_s^i|^2\,ds < \infty$. Moreover, there exists a Borel measurable function $v : [0,T] \times H \to L_2(\Xi, \ell^2_\rho(I))$ such that $Z_t = v(t, X_t)$, $\mathbb{P}$-a.s. for almost every $t \in [0,T]$.

We will show that the process $\widehat{u}_t = \underline{u}(t, X_t, Z_t) = \underline{u}(t, X_t, v(t, X_t))$, $t \in [0,T]$, is a Nash equilibrium. Writing (6.7) in the form specified by definition (3.3) and taking scalar product with g_i we obtain, for every $i \in I$,

$$Y_t^i + \int_t^T e^{-\lambda_i(s-t)} Z_s^i\,dW_s = e^{-\lambda_i(T-t)}\phi^i(X_T) + \int_t^T e^{-\lambda_i(s-t)} f^i(s, X_s, Z_s)\,ds.$$

For every admissible decision process u, by the definition of W^u we obtain

$$Y_0^i - e^{-\lambda_i T}\phi^i(X_T) = -\int_0^T e^{-\lambda_i s} Z_s^i\, dW_s^u - \int_0^T e^{-\lambda_i s} Z_s^i R_s(s, X_s, u_s)\, ds + \int_0^T e^{-\lambda_i s} f^i(s, X_s, Z_s)\, ds.$$

We recall that W^u is a Wiener process under $\mathbb{P}^u$ and we note that

$$\mathbb{E}^u \left(\int_0^T |Z_s^i|^2\, ds \right)^{1/2} = \mathbb{E}\left[\rho^u \left(\int_0^T |Z_s^i|^2\, ds \right)^{1/2} \right] \le \left(\mathbb{E}|\rho^u|^2 \right)^{1/2} \left(\mathbb{E} \int_0^T |Z_s^i|^2\, ds \right)^{1/2} < \infty.$$

It follows that $\int_0^t Z_s^i\, dW_s^u$, $t \in [0, T]$ is a $\mathbb{P}^u$-martingale. Taking expectation we obtain

$$\begin{aligned} Y_0^i &= e^{-\lambda_i T}\mathbb{E}^u \phi^i(X_T) + \mathbb{E}^u \int_0^T e^{-\lambda_i s}[f^i(s, X_s, Z_s) - Z_s^i R(s, X_s, u_s)]\, ds \\ &= J^i(u) + \mathbb{E}^u \int_0^T e^{-\lambda_i s}[f^i(s, X_s, Z_s) - Z_s^i R(s, X_s, u_s) - l^i(s, X_s, u_s)]\, ds. \end{aligned} \tag{6.8}$$

By the definition of f^i and Hypothesis 13, for every $i \in I$,

$$f^i(t, x, z) \le z^i R(t, x, v) + l^i(t, x, v), \qquad t \in [0, T], x \in H, z \in L_2(\Xi, \ell^2_\rho(I)),$$

for every $v \in U$ satisfying $v^j = \underline{u}^j(t, x, z)$ for all $j \neq i$. It follows that

$$f^i(t, X_t, Z_t) \le Z_t^i R(t, X_t, u_t) + l^i(t, X_t, u_t), \tag{6.9}$$

for every decision process such that $u_t^j = \widehat{u}_t^j = \underline{u}^j(t, X_t, Z_t)$ for all $j \neq i$.

On the other hand from (6.6) we obtain

$$f^i(t, X_t, Z_t) = Z_t^i R(t, X_t, \widehat{u}_t) + l^i(t, X_t, \widehat{u}_t). \tag{6.10}$$

From (6.8) and (6.9) it follows that $Y_0^i \le J^i(u)$; from (6.8) and (6.10) it follows that $Y_0^i = J^i(\widehat{u})$; we conclude that $J^i(\widehat{u}) \le J^i(u)$, which shows that $\widehat{u}$ is a Nash equilibrium. □

7 Appendix.

This appendix is devoted to the proof of Lemma 2. We follow closely [4], proof of Proposition 4.2. We keep the notation of section 2.2; by Im we denote the image of an operator. We first state a lemma on gaussian measures.

Lemma 17 *Suppose that Q, R are nonnegative, injective, trace class linear operators on H satisfying*

$$\operatorname{Im} Q^{1/2} = \operatorname{Im} R^{1/2}; \tag{7.1}$$

suppose moreover that the operator

$$G = (R^{-1/2}Q^{1/2})^* R^{-1/2} Q^{1/2} - 1 \tag{7.2}$$

is trace class. Then $\mathcal{N}(0, R)$ is equivalent to $\mathcal{N}(0, Q)$ and, for $\mathcal{N}(0, Q)$-a.e. $x \in H$,

$$\frac{d\mathcal{N}(0, R)}{d\mathcal{N}(0, Q)}(x) = \det(1 + G)^{1/2} \exp\left(-\frac{1}{2} \langle G Q^{-1/2} x, Q^{-1/2} x \rangle \right). \tag{7.3}$$

The determinant is understood as the infinite product of eigenvalues. It is well defined, since G is trace class. Equivalence of measures follows from the Feldman-Hajek Theorem, while the formula for the density can be found in [5], II.4.3, Remark 4.4 and formula (4.16). A simple direct proof can be found in [4].

In the rest of this appendix we assume that Hypothesis 1 holds. We state two well-known properties of the operators Q_t, whose short proofs are reported for the reader's convenience.

(i) *The operators Q_t, $t > 0$, are injective.*

Indeed, by a duality argument (see for instance [6], appendix B), Hypothesis 1-(iv) implies that for every $t > 0$ there exists $C_t > 0$ such that

$$|e^{tA^*}y| \leq C_t|Q_t^{1/2}y|, \quad y \in H.$$

So if $Q_t x = 0$ for some $t > 0$, then $Q_s x = 0,\ s \leq t$, and consequently $e^{sA^*}x = 0,\ s \leq t$; letting $s \to 0$, we obtain $x = 0$.

(ii) *For every $t > 0$, $\operatorname{Im} Q_T^{1/2} = \operatorname{Im} Q_t^{1/2}$. In particular, $Q_T^{-1/2}e^{tA}$ is a linear bounded operator on H.*

We notice the equality $Q_T = Q_t + e^{tA}Q_{T-t}e^{tA^*}$, which is a consequence of the definition of Q_t and Q_T. We obtain

$$Q_T = Q_t + e^{tA}Q_{T-t}e^{tA^*} = Q_t^{1/2}\left[1 + (Q_t^{-1/2}e^{tA})Q_{T-t}(Q_t^{-1/2}e^{tA})^*\right]Q_t^{1/2},$$

which yields, for some constant $C_{tT} > 0$,

$$|Q_T^{1/2}x|^2 = |\left[1 + (Q_t^{-1/2}e^{tA})Q_{T-t}(Q_t^{-1/2}e^{tA})^*\right]^{1/2}Q_t^{1/2}x|^2 \leq C_{tT}|Q_t^{1/2}x|^2, \quad x \in H. \tag{7.4}$$

On the other hand,

$$|Q_t^{1/2}x|^2 = \langle Q_t x, x\rangle \leq \langle Q_T x, x\rangle = |Q_T^{1/2}x|^2, \quad x \in H. \tag{7.5}$$

By a duality argument (see e.g. [6], Appendix B) we conclude that $\operatorname{Im} Q_T^{1/2} = \operatorname{Im} Q_t^{1/2}$.

(iii) *For $0 < s \leq t$ we have*

$$|Q_t^{-1/2}e^{tA}| \leq |Q_s^{-1/2}e^{sA}|. \tag{7.6}$$

We start from the easily verified identity $Q_t = Q_{t-s} + e^{(t-s)A}Q_s e^{(t-s)A^*}$, which implies $Q_t \geq e^{(t-s)A}Q_s e^{(t-s)A^*}$ and therefore $|Q_t^{1/2}x|^2 \geq |Q_s^{1/2}e^{(t-s)A^*}x|^2$, $x \in H$. By a duality argument it follows that $|Q_t^{-1/2}e^{(t-s)A}Q_s^{1/2}| \leq 1$ and consequently $|Q_t^{-1/2}e^{tA}x| = |Q_t^{-1/2}e^{(t-s)A}Q_s^{1/2}Q_s^{-1/2}e^{sA}x| \leq |Q_s^{-1/2}e^{sA}x|$, which proves the claim.

Proof of Lemma 2. The kernel k is the Radon-Nikodym density

$$k_t(x, \cdot) = \frac{d\mathcal{N}(e^{tA}x, Q_t)}{d\mathcal{N}(0, Q_T)}.$$

We will first prove the special case corresponding to $x = 0$, namely that

$$k_t(0, \cdot) = \det(1 - \Theta_{tT})^{-1/2}\exp\left\{-\frac{1}{2}\langle\Theta_{tT}(1 - \Theta_{tT})^{-1}Q_T^{-1/2}y, Q_T^{-1/2}y\rangle\right\}. \tag{7.7}$$

Since Q_{T-t} is a trace class operator and $Q_T^{-1/2}e^{tA}$ is linear bounded, the operator Θ_{tT} is trace class. Moreover, since

$$Q_t = Q_T - e^{tA}Q_{T-t}e^{tA^*} = Q_T^{1/2}\left[1-(Q_T^{-1/2}e^{tA})Q_{T-t}(Q_T^{-1/2}e^{tA})^*\right]Q_T^{1/2} = Q_T^{1/2}(1-\Theta_{tT})Q_T^{1/2}$$

we have

$$(1-\Theta_{tT})x = Q_T^{-1/2}Q_tQ_T^{-1/2}x, \quad x \in \operatorname{Im} Q_T^{1/2}. \tag{7.8}$$

Therefore, $\langle (1-\Theta_{tT})x, x\rangle \geq 0$ for $x \in \operatorname{Im} Q_T^{1/2}$, a dense subset of H; it follows that $(1-\Theta_{tT})$ is nonnegative. Equality (7.8) also implies, by standard arguments, that $(1-\Theta_{tT})$ is invertible and

$$(1-\Theta_{tT})^{-1} = (Q_t^{-1/2}Q_T^{1/2})^*Q_t^{-1/2}Q_T^{1/2}. \tag{7.9}$$

Define $G = (Q_t^{-1/2}Q_T^{1/2})^*Q_t^{-1/2}Q_T^{1/2} - 1$. Then

$$G = (1-\Theta_{tT})^{-1} - 1 = \Theta_{tT}(1-\Theta_{tT})^{-1}, \tag{7.10}$$

so G is trace class and formula (7.7) follows from Lemma 17.

To prove the general case, we use the equality

$$k_t(x,\cdot) = \frac{d\mathcal{N}(e^{tA}x, Q_t)}{d\mathcal{N}(0,Q_t)}\frac{d\mathcal{N}(0,Q_t)}{d\mathcal{N}(0,Q_T)} = \frac{d\mathcal{N}(e^{tA}x, Q_t)}{d\mathcal{N}(0,Q_t)}k_t(0,\cdot), \tag{7.11}$$

and we notice that, by the Cameron-Martin Theorem (see e.g. [6]),

$$\frac{d\mathcal{N}(e^{tA}x, Q_t)}{d\mathcal{N}(0,Q_t)}(y) = \exp\left(\langle Q_t^{-1/2}e^{tA}x, Q_t^{-1/2}y\rangle - \frac{1}{2}|Q_t^{-1/2}e^{tA}x|^2\right),$$

for $\mathcal{N}(0,Q_t)$-a.e. $y \in H$. If $m \in \operatorname{Im} Q_t$, then (7.9) implies $(1-\Theta_{tT})^{-1}Q_T^{-1/2}m = Q_T^{1/2}Q_t^{-1}m$ and we have, for $y \in H$, a.e. with respect to $\mathcal{N}(0,Q_T)$ and $\mathcal{N}(0,Q_t)$,

$$\begin{aligned}&\langle Q_t^{-1/2}m, Q_t^{-1/2}y\rangle = \langle Q_t^{-1}m, y\rangle = \langle Q_T^{1/2}Q_t^{-1}m, Q_T^{-1/2}y\rangle \\ &= \langle (1-\Theta_{tT})^{-1}Q_T^{-1/2}m, Q_T^{-1/2}y\rangle.\end{aligned} \tag{7.12}$$

(7.9) also implies

$$|Q_t^{-1/2}m|^2 = |(1-\Theta_{tT})^{-1/2}Q_T^{-1/2}m|^2. \tag{7.13}$$

The equalities (7.12) and (7.13) extend by continuity to every $m \in \operatorname{Im} Q_t^{1/2}$. So we can set $m = e^{tA}x$, and substituting into (7.11) and using (7.7), we prove the formula for k.

It remains to prove the inequalities (2.5) and (2.6).

The equality (7.9) shows that $|(1-\Theta_{tT})^{-1}| = |Q_t^{-1/2}Q_T^{1/2}|^2$. The first equality in (7.4) implies that

$$|Q_t^{-1/2}Q_T^{1/2}|^2 \leq |1 + (Q_t^{-1/2}e^{tA})Q_{T-t}(Q_t^{-1/2}e^{tA})^*|,$$

and since $(Q_t^{-1/2}e^{tA})Q_{T-t}(Q_t^{-1/2}e^{tA})^* \geq 0$, we conclude that

$$|(1-\Theta_{tT})^{-1}| \leq 1 + |(Q_t^{-1/2}e^{tA})Q_{T-t}(Q_t^{-1/2}e^{tA})^*| \leq 1 + |Q_{T-t}||Q_t^{-1/2}e^{tA}|^2, \tag{7.14}$$

which proves (2.5).

In the sequel we denote for simplicity

$$a = |Q_{T-t}||Q_t^{-1/2}e^{tA}|^2.$$

To prove (2.6) we first recall that Θ_{tT} is a trace class nonnegative operator and we denote $\lambda_0, \lambda_1, \dots$ its eigenvalues, arranged in decreasing order. Since $0 \leq \Theta_{tT} < 1$ we have $0 \leq \dots \leq \lambda_1 \leq \lambda_0 = |\Theta_{tT}| < 1$. It follows that $(1-\lambda_0)^{-1} = |(1-\Theta_{tT})^{-1}|$ and by (7.14) we have $(1-\lambda_0)^{-1} \leq 1+a$ and we first conclude that $\lambda_0 \leq a/(1+a)$.
Next we compute

$$\det(1-\Theta_{tT})^{-1} = \prod_{k=0}^{\infty}(1-\lambda_k)^{-1} = \exp\left[-\sum_{k=0}^{\infty}\log(1-\lambda_k)\right].$$

Since the function $x \to (-\log(1-x))/x$ is increasing in the interval $(0,1)$ we have in particular

$$\frac{-\log(1-\lambda_k)}{\lambda_k} \leq \frac{-\log(1-\lambda_0)}{\lambda_0} \leq \frac{-\log(1-\frac{a}{1+a})}{\frac{a}{1+a}} = \frac{\log(1+a)}{a}(1+a) \leq 1+a,$$

and we obtain

$$\det(1-\Theta_{tT})^{-1} \leq \exp\left[(1+a)\sum_{k=0}^{\infty}\lambda_k\right] = \exp\left[(1+a)\; Trace\,\Theta_{tT}\right].$$

Then we have

$$Trace\,\Theta_{tT} = Trace\,((Q_T^{-1/2}e^{tA})Q_{T-t}(Q_T^{-1/2}e^{tA})^*) \leq (Trace\,Q_{T-t})|Q_T^{-1/2}e^{tA}|^2$$

and since the inequality (7.5) implies that $|Q_T^{-1/2}Q_t^{1/2}| \leq 1$, we deduce that

$$|Q_T^{-1/2}e^{tA}| \leq |Q_T^{-1/2}Q_t^{-1/2}||Q_t^{-1/2}e^{tA}| \leq |Q_t^{-1/2}e^{tA}|.$$

Substituting, we obtain $\det(1-\Theta_{tT})^{-1} \leq \exp\left[(1+a)\;(Trace\,Q_{T-t})|Q_t^{-1/2}e^{tA}|^2\right]$ and (2.6) is proved. □

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