Alternative count regression models for modeling football outcomes

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Abstract

In this work, we propose the use of discrete counterparts of the Weibull distribution along with a copula function for modeling football results, as an alternative to existing bivariate Poisson regression models and extensions thereof. We expect that the choice of the marginal distribution and dependence structure, which try to capture known features of the data, can be beneficial in terms of fitting of the developed models; to check this conjecture, an application to the Italian Serie A championship is provided.

1 Introduction

Football is by far the most popular participant and spectator sport in the world. In many countries, especially in Europe, television and internet companies compete strongly to win the rights to broadcast games. Huge sums of money are involved, from players wages to transfer fees and sports betting. The simplicity of football's objectives and rules along with the uncertainty of games are probably responsible for such an inexhaustible attractiveness. The latter feature has captured the attentions of statisticians, who have proposed a multitude of stochastic models for analyzing (and predicting) several events associated with a football game: the first half or final result (expressed as number of goals scored by the two teams or simply as win-draw-loss), the number of shots-for and shots-against, the time to the first goal, the number of yellow or red cards, etc.

In this work, we propose the use of discrete counterparts of the Weibull distribution for modeling football results, as an alternative to existing bivariate Poisson regression models and modifications/extensions thereof, such as diagonally inflated or generalized Poisson models.

The simple bivariate Poisson model, with independent components, was the first used in football data analysis for modeling the outcome of a game (number of goals scored by the two competing teams) due to its ease of use and interpretation. Later, more complex models allowing for non-null correlation were explored, since real data often show a slight but non-negligible positive correlation between the numbers of goals scored by the two teams; or allowing for overdispersion and excess in draws, which usually characterize football outcomes.

The discrete Weibull distributions derived as analogues of the homonym continuous distribution seem to be more flexible than Poisson, since adjusting their two parameters can model a variety of different features. The numbers of goals scored by the two teams can be regarded as a joint observation from a bivariate random vector with discrete Weibull margins, linked through a copula function that accommodates dependence. The parameters of the distribution are assumed to depend on covariates such as the attack and defense abilities of the two teams and the "home effect". Several discrete Weibull regression models are proposed, by varying the type of discretization, the copula function, the choice of covariates, and are then applied to the Italian Serie A championship.

Even if the interpretation of parameters is less immediate than in Poisson models, yet they represent a suitable alternative, as the application demonstrates, and can be employed as a statistical tool for better understanding the performance of teams in order to improve predictions, from a betting perspective, or to deploy corrective actions, from a managerial point of view.

The next Section briefly recaps the basic ideas underlying bivariate count regression models usually employed when analysing football results. In Section 3 we will draw our attention on alternative marginal distributions derived as discrete counterparts from the continuous Weibull distribution; in Section 4, we will focus on the choice of the copula function; in Section 5, we will discuss an application to the Italian Serie A championship.

2 Modelling the Numbers of Goal in a Football Game through a Count Regression

Focusing on the final result of a football game, many bivariate models have been discussed in statistical literature. Most of them are an extension of the simple bivariate Poisson model with independent components. These proposals, taking the cue from the bivariate Poisson model by Holgate [10] with correlated components, take into account the specific features these data usually exhibit, namely non-negligible correlation, overdispersion and bivariate zero-inflation, and propose count regression models where the two count variables are regressed towards covariates such as team attack and defence potential, home effect, etc. [16, 15, 6, 7, 11, 12, 1, 13]. More recently, some contributions suggested the use of alternative discrete probability distributions, related to the continuous Weibull random variable [5, 3], and dependence structures, by naturally considering copula functions.

In very general terms, the stochastic model can be structured as follows. Let Y_{1i} be the number of goals scored by the home team in game i, and Y_{2i} the number of goals scored by the away team in game i; $p_1(y;\boldsymbol{\theta}_{1i})$ and $p_2(y;\boldsymbol{\theta}_{2i})$ are the discrete probability distributions modelling Y_{1i} and Y_{2i}, belonging to the same parametric family, with $\boldsymbol{\theta}_{1i}$ and $\boldsymbol{\theta}_{2i}$ being the distribution parameters (scalars or, more generally, vectors). These latter, or a transformation thereof, are expressed as a linear model, for example

$$g_j(\theta_{1ji}) = \boldsymbol{\beta}'_{1j}\boldsymbol{x}_{1ji}, \quad g_j(\theta_{2ji}) = \boldsymbol{\beta}'_{2j}\boldsymbol{x}_{2ji}$$

with $j = 1, \ldots, p$, where p is the dimension of the parameter vectors $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$; $\boldsymbol{x}_{1ji}$ and $\boldsymbol{x}_{1ji}$ are the two corresponding vectors of covariates, not necessarily the same; $\boldsymbol{\beta}_{1j}$ and $\boldsymbol{\beta}_{2j}$ the vector of regression parameters; $i = 1, \ldots, n$, being n the sample size. For example, if we consider the Poisson distribution with parameter λ, being $p = 1$, the model can be written as

$$\begin{cases} Y_{1i} \sim \text{Pois}(\lambda_{1i}), & \log(\lambda_{1i}) = \boldsymbol{\beta}'_1\boldsymbol{x}_{1i} \\ Y_{2i} \sim \text{Pois}(\lambda_{2i}), & \log(\lambda_{2i}) = \boldsymbol{\beta}'_2\boldsymbol{x}_{2i} \end{cases}$$

In order to accommodate possible association between the two count variables, we resort to copulas. The cumulative distribution functions of the two count variables Y_{1i} and Y_{2i}, say F_{1i} and F_{2i}, are linked through a parametric bivariate copula function $C(u_1, u_2; \theta)$:

$$F(y_{1i}, y_{2i}) = C(F_{1i}(y_{1i}), F_{2i}(y_{2i}); \theta),$$

so that the joint probability mass function is derived as

$$P(Y_{1i} = y_{1i}, Y_{2i} = y_{2i}) = F(y_{1i}, y_{2i}) - F(y_{1i} - 1, y_{2i}) - F(y_{1i}, y_{2i} - 1) + F(y_{1i} - 1, y_{2i} - 1).$$

3 Marginal Distribution: Discrete Analogue of the Continuous Weibull Distribution

At least three probability distributions have been derived so far as a discrete counterpart of the continuous Weibull model.

A first discrete Weibull distribution was introduced by [18] and is usually referred to as 'type I discrete Weibull distribution', in order to distinguish it from two other models proposed later by [23] (type II discrete Weibull) and [21] (type III discrete Weibull). A continuous Weibull random variable (rv) T has probability density function given by

$$f_t(t;\lambda,\beta) = \lambda\beta t^{\beta-1}\mathrm{e}^{-\lambda t^{\beta}} \quad t>0, \tag{1}$$

with $\lambda, \beta > 0$, and cumulative distribution function (cdf)

$$F_t(t;\lambda,\beta) = 1-\mathrm{e}^{-\lambda t^{\beta}}. \tag{2}$$

If we consider the rv $Y = \lfloor T \rfloor$, where $\lfloor T \rfloor$ denotes the largest integer equal to or smaller than T, it can be easily shown that its probability mass function (pmf), defined on the non-negative integers only, is given by

$$p(y;q,\beta) = F_t(y+1) - F_t(y) = \mathrm{e}^{-\lambda y^{\beta}} - \mathrm{e}^{-\lambda (y+1)^{\beta}} = q^{y^{\beta}} - q^{(y+1)^{\beta}} \quad y \in \mathbb{N} = \{0,1,2,\dots\}, \tag{3}$$

with $q = \mathrm{e}^{-\lambda}$, and then $0 < q < 1$. The corresponding cdf is

$$F(y;q,\beta) = 1 - q^{(y+1)^{\beta}} \quad y \in \mathbb{N}. \tag{4}$$

This distribution retains the expression of the cumulative distribution function of the continuous Weibull model – just compare Eq.(2) to Eq.(4). The first parameter q has a nice interpretation: since $P(X=0) = 1-q$, it represents the probability of a positive value. As to the second parameter β, it does not possess an equally immediate meaning. However, if we define the hazard rate function of Y as $r(y) = p(y)/P(Y \geq y)$, it has been shown [18] that $r(y)$ is a constant function if $\beta = 1$ (in this case, (3) reduces to the geometric pmf), an increasing function if $\beta > 1$, a decreasing function if $\beta < 1$.

Figure 1 displays the pmf of the type I discrete Weibull rv for several value combinations of q and β. From here the role of β, for a fixed value of q, clearly emerges: larger values of β lead to less dispersed distributions, with most of the probability mass concentrated on the first integer values; smaller values of β lead to more dispersed distributions. The expected value of the type I discrete Weibull rv cannot be generally computed in a closed form; it is equal to the infinite sum $\mathbb{E}(Y) = \sum_{y=1}^{\infty} q^{y^{\beta}}$, which leads to a closed-form expression if and only if $\beta = 1$: $\mathbb{E}(Y) = q/(1-q)$. It is clear $\mathbb{E}(Y)$, fixed q, is a decreasing function of β. Its value can be approximated recalling the result in [14], involving the expected value $\mathbb{E}(T)$ of the corresponding continuous distribution, which ensures that the value $\mathbb{E}(Y)$ falls between $\mathbb{E}(T)-1$ and $\mathbb{E}(T)$.

The first parameter of the type I discrete Weibull model can be related to explanatory variables $\boldsymbol{x}_i$ through a complementary log-log link function: $\log(-\log(q_i)) = \boldsymbol{\alpha}'\boldsymbol{x}_i$. Additionally, even the second parameter β can be related to explanatory variables $\boldsymbol{z}_i$, not necessarily the same as for q, through the following natural link function (remember that β takes only positive values): $\log(\beta_i) = \boldsymbol{\gamma}'\boldsymbol{z}_i$.

Contrary to the Poisson rv, which cannot adequately model count data whose variance differs from the mean, which is a circumstance often occurring in practice, the type I discrete

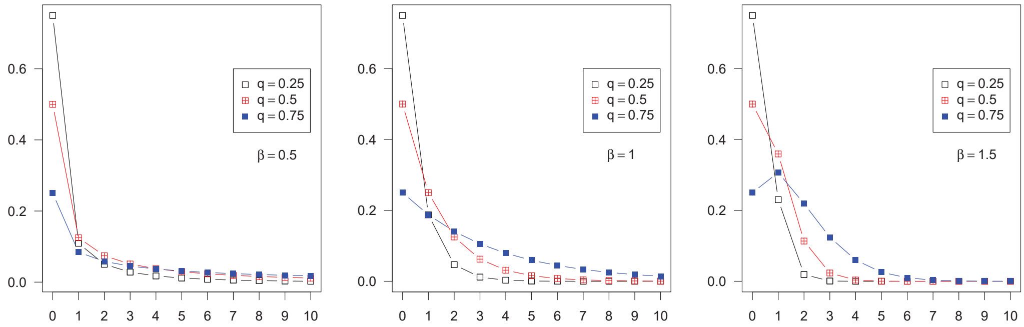

Figure 1: Graphs of the probability mass function of the type I discrete Weibull distribution for some combinations of its parameters q and β

Weibull rv can model both under-dispersed and over-dispersed data [8]. This distribution can also handle count data presenting an excess of zeros, arising in many physical situations (see again [8]); just remember that the probability of 0 is controlled by the q parameter only.

In regard to point and interval estimation of the parameters of the type I discrete Weibull distribution, one can refer to [2] and references therein, where several inferential procedures are considered and discussed and applicability issues are raised. The type I discrete Weibull model is implemented in the `R` environment [24] through the packages `DiscreteWeibull` [4] and `DWreg` [25].

As for the type II discrete Weibull rv, its distribution is derived by imposing that its hazard function has the same expression as the hazard function of the continuous Weibull rv. The resulting discrete distribution may have a finite or infinite support according to the value taken by the second parameter β of the continuous distribution. Such an odd feature depends on the fact that the hazard rate for a discrete model is bounded between 0 and 1, whereas this restriction is not needed for the hazard rate of a continuous distribution. For more details, we address the reader to the original paper [21].

As for the type III discrete Weibull rv, its pmf can be expressed as

$$P(Y = y; c, \beta) = e^{-c\sum_{j=1}^{y} j^{\beta}}[1 - e^{-c(y+1)^{\beta}}], \quad y \in \mathbb{N}, \tag{5}$$

letting by convention $\sum_{j=1}^{y} j^{\beta} = 0$ if $y = 0$; with $c > 0$ and $\beta \geq -1$. Note that $P(Y = 0) = 1 - e^{-c}$. Despite its unequivocal name, the type III discrete Weibull rv is not similar in functional form to any of the functions describing a continuous Weibull distribution, although the negative exponential terms in (5) reminds us of an analogous term in (1).

These latter two discrete models have not attracted much attention so far, due to the complex expression of their pmf, which makes parameter estimation not straightforward. However, their use in a count regression model can be still feasible, although some care has to be devoted to the choice of the link functions for their parameters.

4 Dependence Structure: the Clayton Copula

Lack of independence/incorrelation between the number of goals scored by the two teams in a football match was first claimed by [6]; in [17] the use of copulas for modeling two correlated count distributions related to football games was suggested perhaps for the first time. As

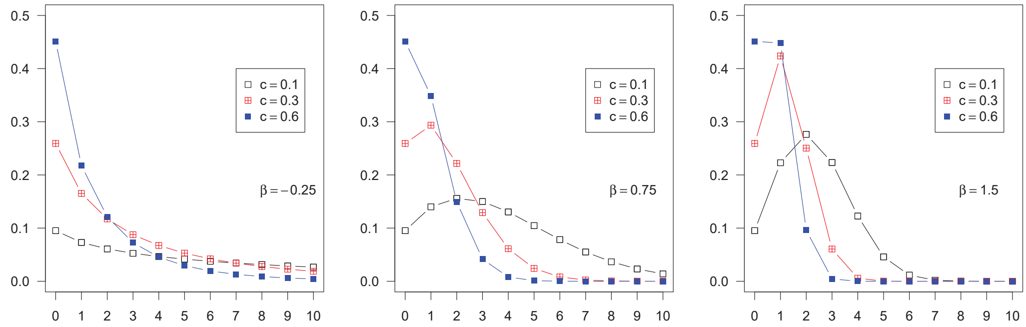

Figure 2: Graphs of the probability mass function of the type III discrete Weibull distribution for some combinations of its parameters c and β

anticipated in Section 2, we assume that the random variables modeling the number of goals scored by home and away teams, Y_{1i} and Y_{2i}, are no longer statistically independent, given the covariates; we model their dependence structure through a specific copula family.

Copulas represent a very flexible tool for modeling dependence among rvs. A bivariate copula is a joint cumulative distribution function in $[0,1]^2$ with standard uniform margins U_1 and U_2:

$$C(u_1, u_2) := P(U_1 \leq u_1, U_2 \leq u_2). \tag{6}$$

Sklar's theorem [22] states that if F is a joint distribution function with margins F_1 and F_2, then there exists a copula $C : [0,1]^2 \to [0,1]$ such that, for all x_1, x_2 in $\bar{\mathbb{R}} = [-\infty, +\infty]$,

$$F(x_1, x_2) = C(F_1(x_1), F_d(x_2)).$$

If the margins are continuous, then C is unique, otherwise C is uniquely determined on $\mathrm{Ran}(F_1) \times \mathrm{Ran}(F_2)$, with $\mathrm{Ran}(F_j)$ denoting the range of F_j. Conversely, if C is a copula and F_1, F_2 are univariate cdfs, then the function F defined in (6) is a joint distribution function with margins F_1, F_2. If the margins are continuous, the unique copula C is given by

$$C(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2)),$$

where F_j^{-1} denotes the generalized inverse of the marginal cdf F_j, i.e., $F_j^{-1}(t) = \inf\{x \in \mathbb{R} : F_j(x) \geq t\}$.

We recall that for any copula C the following constraint holds for any $(u_1, u_2) \in [0,1]^2$:

$$\max(0, u_1 + u_2 - 1) \leq C(u_1, u_2) \leq \min(u_1, u_2); \tag{7}$$

the left and right members of the inequality are called Fréchet lower bound and Fréchet upper bound, respectively [9]. $M(u_1, u_2) = \min(u_1, u_2)$ is itself a copula, named "comonotonicity copula", as well as $W(u_1, u_2) = \max(0, u_1 + u_2 - 1)$, the bivariate "countermonotonicity copula".

From among the multitude of parametric bivariate copulas, we pick Clayton's copula, belonging to the so-called Archimedean family. The expression of the one-parameter Clayton copula is

$$C(u_1, u_2) = \max\left\{(u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}, 0\right\}, \quad \theta \in (-1, +\infty) \setminus \{0\}. \tag{8}$$

The Clayton copula is interesting as it can model various kinds of dependence, ranging from comonotonicity in the limit as $\theta \to +\infty$, independence if $\theta \to 0$, and countermonotonicity if $\theta \to -1$.

The values of the θ parameter can be better interpreted resorting to the expression of Kendall's correlation ρ_τ for the Clayton copula (valid however for continuous margins only; see [19]):

$$\rho_\tau = \frac{\theta}{2+\theta}.$$

Moreover, the Clayton copula is also able to capture lower tail dependence. For a bivariate absolutely continuous rv (X_1, X_2), with marginal cdfs F_1 and F_2, and generalized inverse functions $F_1^{\leftarrow}$ and $F_2^{\leftarrow}$, respectively, the coefficient of lower tail dependence is defined as

$$\lambda_L = \lim_{u\to 0^+} P(X_2 \le F_2^{-1}(u) | X_1 \le F_1^{-1}(u)) = \lim_{u\to 0^+} C(u,u)/u,$$

and for the Clayton copula with $\theta > 0$, we have that

$$\lambda_L = 2^{-1/\theta} > 0.$$

Other well-known one-parameter bivariate copulas, such as the Gauss, the Plackett, and the Frank, do not meet this feature, being all asymptotically lower and upper tail independent. In Figure 3, the bivariate density plot of the Clayton copula is displayed for $\theta = 2$, along with the scatter plot of a bivariate random sample generated from the same copula (size $n = 5,000$). Thus, the Clayton copula may be a suitable candidate for modelling dependence between the numbers of goals scored in football games in a football championship, usually presenting a frequency of $0-0$ draws higher than that which is caught by standard stochastic models.

[20] considered the Clayton-copula model with negative binomial marginals for modelling simultaneous spike-counts of neural populations, whereas, for computational reason, they are typically modeled by a Gaussian distribution. In [17], the Clayton copula is cited as a possible dependence structure for modelling the numbers of shots-for and shots-against a team in a football game.

5 Empirical Analysis: Italian Serie A Championship

We focus on the main Italian football championship, called "Serie A", a professional league competition for football clubs located at the top of the Italian football league system. Since 2004-05, there have been 20 clubs playing in Serie A and as in most of the European countries a true round-robin format is used. During the season, each club plays each of the other teams twice; once at home and once away, eventually totaling 38 games. In the first half of the season, called the "andata", each team plays once against each league opponent, for a total of 19 games. In the second half, called the "ritorno", the teams play in the same exact order that they did in the first half of the season, the only difference being that home and away situations are switched. Since the 1994-95 season, teams earn three points for a win, one point for a draw and no points for a loss.

Here we are interested in analysing and modeling the final result for all the 380 games played throughout the season. For game i, $1 \le i \le 380$, we denote with y_{1i} the number of goals scored by the home team, h_i, and with y_{2i} the number of goals scored by the away team, a_i. Based on these data, one can estimate all the parameters involved in the regression model of Section 2, by using the maximum likelihood method, and for each game construct a theoretical

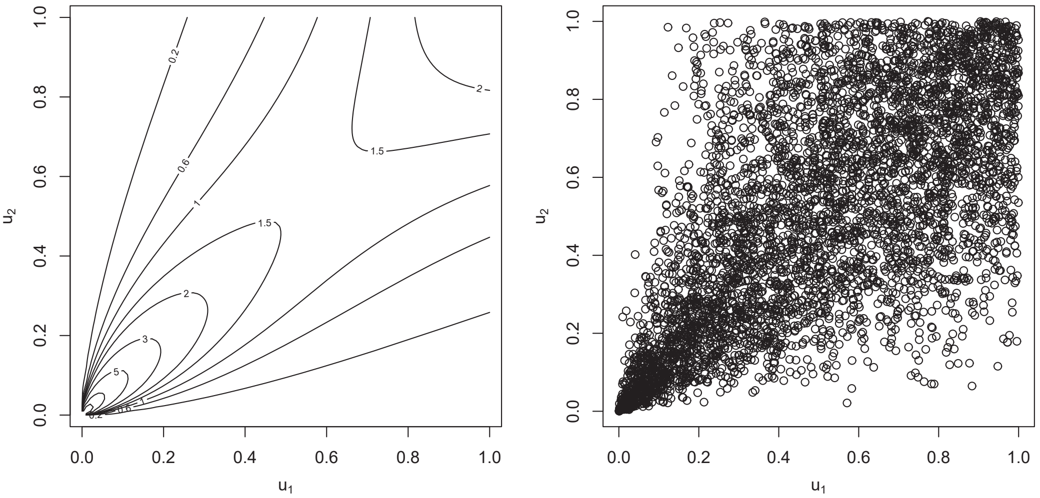

Figure 3: Clayton copula with parameter $\theta = 2$: contour density plot (on the left) and scatter plot of a random sample of size $5,000$ (on the right)

joint probability table providing the probability of any possible outcome. As an overall result, by aggregating all the single theoretical outcomes, one can reconstruct the theoretical final scoreboard and compare it with its real counterpart.

We will start from a basic copula-based model, where the margins are assumed to follow the type I discrete Weibull distribution (3) and the dependence structure is induced by the Clayton copula (8). The two q parameters of the Weibull distribution are related to covariates as follows (see [3]):

$$\begin{cases} \log[-\log(q_{1i})] &= \mu^{(q)} + \text{home}^{(q)} + \text{att}^{(q)}_{h_i} + \text{def}^{(q)}_{a_i} \\ \log[-\log(q_{2i})] &= \mu^{(q)} + \text{att}^{(q)}_{a_i} + \text{def}^{(q)}_{h_i} \end{cases}$$

where $\mu^{(q)}$ is a constant term, $\text{home}^{(q)}$ is the "home effect", $\text{att}^{(q)}_k$ and $\text{def}^{(q)}_k$ are the "attack" and "defence" parameters associated to q for team k. Note that apart from the constant term, the covariates for q are all dummy variables. The parameter β for the marginal distributions and the parameter θ of Clayton copula are assumed to be constant. Estimates for all parameters can be numerically obtained by maximizing the joint log-likelihood function. For the Italian Serie A championship, season 2015/16, the parameter estimates of the model above and their significance are reported in Table 1. Note the value of the estimate of β ($1.866 > 1$), which highlights how the distribution of scored goals is quite concentrated on the first integers; and the value of the estimate of θ (0.142), denoting a very slight correlation between the numbers of scored and conceded goals.

Additional models can be constructed by considering the other two discrete Weibull distribution, alternative copula functions, and different sets of covariates for the distribution parameters.

team	$\text{att}^{(q)}$	$\text{def}^{(q)}$
Atalanta	0.230	0.089
Bologna	0.322.	0.087
Carpi	0.314.	−0.250
Chievo	0.121	0.056
Empoli	0.184	0.030
Fiorentina	−0.334∗	0.217
Frosinone	0.287.	−0.622∗∗∗
Genoa	0.122	−0.058
Inter	0.072	0.243
Juventus	−0.557∗∗∗	0.975∗∗∗
Lazio	−0.138	−0.094
Milan	0.000	0.097
Napoli	−0.712∗∗∗	0.400∗
Palermo	0.334∗	−0.378∗
Roma	−0.714∗∗∗	0.153
Sampdoria	−0.022	−0.285.
Sassuolo	−0.020	0.226
Torino	−0.117	−0.134
Udinese	0.283.	−0.387∗
other parameters		
$\mu^{(q)}$	−1.037∗∗∗	
$\text{home}^{(q)}$	−0.385∗∗∗	
β	1.866∗∗∗	
θ	0.142.	

Table 1: Parameter estimates for the model applied to Italian Serie A championship 2015/2016. Attack and defense parameters satisfy the sum-to-zero constraint; so, for the last team in alphabetical order, Verona, we have $\text{att}^{(q)} = 0.346$ and $\text{def}^{(q)} = -0.364$.
Significance codes for p-values: 0 "***" 0.001 "**" 0.01 "*" 0.05 "." 0.1 "" 1

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