EXPLICIT SHORT INTERVALS FOR PRIMES IN ARITHMETIC PROGRESSIONS ON GRH

ADRIAN W. DUDEK, LOÏC GRENIÉ, AND GIUSEPPE MOLTENI

Abstract. We prove explicit versions of Cramér's theorem for primes in arithmetic progressions, on the assumption of the generalised Riemann hypothesis.

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1. Motivations and results

The purpose of this article is to combine techniques from analytic number theory with computation to furnish explicit short interval results for primes in arithmetic progressions. This is done on the assumption of the generalised Riemann hypothesis (GRH), and builds on the earlier work of the authors [1], where the problem was considered without reference to residue classes.

Throughout this paper, unless it is mentioned, we will be assuming GRH to be true. Let $q \in \mathbb{N}$ and $a \in \mathbb{Z}$ with $(a, q) = 1$. Unconditionally, both McCurley [9] and later Kadiri [7] proved that, for every positive ϵ and q_0, there exists $\alpha = \alpha(\epsilon, q_0)$ such that if $\log x \geq \alpha_\epsilon \log^2 q$ and $q \geq q_0$, then $[x, e^\epsilon x]$ contains a prime p congruent to a modulo q. They provide pairs of explicit values for α and q_0 dependent on the choice of ϵ; Kadiri's work improves on that of McCurley by providing smaller values of α.

Clearly, on the assumption of GRH, the result should improve significantly; Dusart proved in his Ph.D. Thesis [2, Th. 3.7, p. 114] that when $x \geq \max(\exp(\frac{5}{4}q), 10^{10})$ one has

$$\Big|\psi(x; q, a) - \frac{x}{\varphi(q)}\Big| \leq \frac{1}{4\pi}\sqrt{x}\log^2 x.$$

This implies that there is a prime in $[x-h, x+h]$ which is congruent to a modulo q provided that $h > (\frac{1}{4\pi}+\epsilon)\varphi(q)\sqrt{x}\log^2 x$ and $x \geq x_0(q, \epsilon)$ for every $\epsilon > 0$.
Recently, in joint work with the second and third authors, Perelli [5, Th. 1] proved that there exist absolute (i.e., independent of x and q) positive constants x_0, c_1 and c_2 such that for $x \geq x_0$ and $c_1\varphi(q)\sqrt{x}\log x \leq h \leq x$ one has

$$\pi(x+h; q, a) - \pi(x; q, a) \geq c_2 \frac{h}{\varphi(q)\log x}. \tag{1.1}$$

This is in some sense the best result we can hope to prove, but the constants are not explicit.
In the present paper we prove the following result.

Theorem 1.1. *Assume GRH, $q \geq 3$ and $(a, q) = 1$. Let α, δ, ρ, m and m' be as in Table 1 and assume*

$$h \geq \varphi(q)(\alpha \log x + \delta \log q + \rho)\sqrt{x}$$

and $x \geq (m\varphi(q)\log q)^2$. Then there is a prime p which is congruent to a modulo q with $|p-x| < h$. Furthermore, if we assume

$$h \geq \varphi(q)((\alpha+1)\log x + \delta \log q + \rho)\sqrt{x}$$

and $x \geq (m'\varphi(q)\log q)^2$, then there are at least $\sqrt{x}$ such primes.

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Table 1. Parameters for Theorem 1.1

α	δ	ρ	m	m'	α	δ	ρ	m	m'
1/2	1	12	23	46	1.253/2	0.1	7	500	1500
1/2	1/2	9	86	188	1	0	8	23	46
1/2	1/3	9	1500	3500	0.9	0	7	31	66
1.253/2	1	14	18	34	0.8	0	6	52	120
1.253/2	1/2	9	34	74	0.7	0	5	200	500
1.253/2	0.2	7	110	260					

The claim of this theorem has the same qualitative behavior in its dependencies on q as what is predicted in (1.1), but the constants m and m' ruling the minimum x are quite large. This is the effect of the fact that under the hypotheses for the theorem the quotient h/x is for sure small in the long run for x, but this happens uniformly in q only for very large values of q. In fact, when x has its lowest value we have

$$\frac{h}{x_{\min}} = \frac{\varphi(q)(\alpha\log x_{\min}+\delta\log q+\rho)}{\sqrt{x_{\min}}} = \frac{2\alpha+\delta+o(1)}{m+o(1)}.$$

This considerably affects the computations, because they are more effective for smaller h/x, and so we are forced to choose larger values of m and m'. This also means that for small q and some limited range of x, extensive numerical tests have to be performed to complete the proof.

From the same general formulas we also deduce the following result.

Theorem 1.2. *Assume GRH, $q \geq 3$, $(a,q) = 1$,*

$$h \geq \varphi(q)\Big(\frac{1}{2}\log(q^2x)+15\Big)\sqrt{x}$$

and $x \geq (8\varphi(q)\log q\log\log q)^2$. Then there is a prime p which is congruent to a modulo q with $|p-x| < h$. Furthermore, if we assume

$$h \geq \varphi(q)\Big(\frac{1}{2}\log(q^2x^3)+15\Big)\sqrt{x}$$

and $x \geq (15\varphi(q)\log q\log\log q)^2$, then there are at least $\sqrt{x}$ such primes.

Theorem 1.2 is worse in its dependency of the minimum x on q, but the constants are better. As a consequence, its claims improve on the case $\alpha = 1/2$, $\delta = 1$ in Theorem 1.1 for all $q \leq \exp(\exp(23/8)) \simeq 5{\cdot}10^7$ (resp. $q \leq \exp(\exp(46/15)) \simeq 2{\cdot}10^9$).

Both theorems could be adapted to include the cases $q = 1$ and $q = 2$, but for them we have already proved a better result in [1] where the conclusions are proved with $h = \frac{1}{2}\log x+2$ for any $x \geq 2$.

The conclusions improve significantly if, following Dusart, we select a lower bound for x of exponential type in terms of q. In fact, the same formulas producing Theorems 1.1 and 1.2 allow us to prove the following result.

Theorem 1.3. *Assume GRH, $(a,q) = 1$ and $x \geq \exp(q)$. Let*

$$h \geq \frac{\varphi(q)}{2}\log(q^2x)\sqrt{x}.$$

Then for each $q \geq 35$ there is a prime p which is congruent to a modulo q with $|p-x| < h$. Furthermore, assuming

$$h \geq \frac{\varphi(q)}{2}\log(q^2x^3)\sqrt{x}$$

and $q \geq 67$, there are at least $\sqrt{x}$ such primes.

This claim is always stronger than what we deduce from Dusart's result, apart from the larger minimum value for q.

Note that Theorem 1.1 (case $\alpha = 1/2$, $\delta = 1$) shows that the least prime congruent to a modulo q is lower than

$$(24^2{+}o(1))(\varphi(q)\log q)^2$$

where $o(1)$ is explicit. According to computations in Section 5 (see Table 3), the constant reduces to $21^2{+}2{\cdot}21$ for extremely large values of q but this is notably weaker than the bound $(\varphi(q)\log q)^2$ which has been proved by Lamzouri, Li and Soundararajan [8, Cor. 1.2] for all $q \geq 4$.

Also, from Theorem 1.1, one deduces the following explicit version of a quasi-Dirichlet's conjecture for primes close to squares of integers.

Corollary 1.4. *Assume GRH and let $q \geq 1$ and $n \geq 8\varphi(q)\log q$. Then the interval*

$$\big(n^2, (n{+}\varphi(q)(12{+}2\log(qn)))^2\big)$$

contains a prime which is congruent to a modulo q, for every a coprime to q.

Similar corollaries may be deduced from Theorems 1.2 and 1.3.

2. Functional equation and integral representation

Let χ be a character modulo q; let χ^* be the primitive character inducing χ and let q_χ be its conductor. Let $a_\chi := (1{-}\chi(-1))/2$ denote the parity of χ, so that

$$L(s,\chi) = L(s,\chi^*)\prod_{p|q}\big(1{-}\chi^*(p)p^{-s}\big)$$

and

$$\xi(1{-}s,\chi^*) = \frac{\tau(\chi^*)}{i^{a_\chi}\sqrt{q_\chi}}\xi(s,\overline{\chi^*})$$

where we have that

$$\xi(s,\chi^*) := s(s{-}1)\Big(\frac{q_\chi}{\pi}\Big)^{\frac{s+a_\chi}{2}}\Gamma\Big(\frac{s{+}a_\chi}{2}\Big)L(s,\chi^*).$$

We also let

$$\psi_\chi^{(1)}(x) := \int_0^x \psi_\chi(u)\,\mathrm{d}u = \sum_{n\leq x}\chi(n)\Lambda(n)(x{-}n) \tag{2.1}$$

and recall the integral representation

$$\psi_\chi^{(1)}(x) = -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{L'}{L}(s,\chi)\frac{x^{s+1}}{s(s{+}1)}\,\mathrm{d}s \tag{2.2}$$

which holds for all $x \geq 1$. The next lemma gives an alternative formula for $\psi_{\chi^*}^{(1)}(x)$ based on the representation (2.2) applied to the character χ^*.

Lemma 2.1. *We have that*

$$\psi_{\chi^*}^{(1)}(x) = \frac{x^2}{2}\delta_{\chi^*=\mathbf{1}} - \sum_{\rho\in Z_{\chi^*}}\frac{x^{\rho+1}}{\rho(\rho{+}1)} - xr_{\chi^*} + r'_{\chi^*} + R_{\chi^*}^{(1)}(x) \tag{2.3}$$

where Z_{χ^*} is the set of nontrivial zeros of $L(s,\chi^*)$, and r_{χ^*}, r'_{χ^*} are the constants

$$\begin{aligned} r_{\chi^*} &= \frac{L'}{L}(0,\chi^*)(a_\chi\delta_{\chi^*\neq\mathbf{1}}{+}\delta_{\chi^*=\mathbf{1}}) - (1{+}\beta)(1{-}a_\chi)\delta_{\chi^*\neq\mathbf{1}},\\ r'_{\chi^*} &= (1{-}\alpha)a_\chi + \frac{L'}{L}(-1,\chi^*)(1{-}a_\chi), \end{aligned}$$

with α and $\beta \in \mathbb{C}$ defined in the proof,

$$R_{\chi^*}^{(1)}(x) = -\sum_{n=1}^{\infty}\frac{x^{1-2n-a_\chi}}{(2n{+}a_\chi)(2n{+}a_\chi{-}1)} + a_\chi\log x - (1{-}a_\chi)\delta_{\chi^*\neq\mathbf{1}}\,x\log x,$$

$\delta_{\chi^*=\mathbf{1}}$ *is* 1 *when* $\chi^*=\mathbf{1}$ *and* 0 *otherwise, and* $\delta_{\chi^*\neq\mathbf{1}}=1-\delta_{\chi^*=\mathbf{1}}$.

Proof. The poles of $-\frac{L'}{L}(s,\chi)\frac{x^{s+1}}{s(s+1)}$ at the trivial zeros $s=-2n-a_\chi$ are simple for every integer $n\geq 1$. Moreover, when $a_\chi=1$, the pole at $s=0$ is simple and its contribution to $R^{(1)}_{\chi^*}(x)$ is $-\frac{L'}{L}(0,\chi^*)x$, while the one in $s=-a_\chi=-1$ (i.e. $n=0$) is a double pole with contribution

$$1-\alpha+\log x,$$

where $-\frac{L'}{L}(-1+\epsilon,\chi^*)=:-\frac{1}{\epsilon}+\alpha+O(\epsilon)$.
Lastly, when $a_\chi=0$, the pole at $s=-1$ is simple and its contribution to $R^{(1)}_{\chi^*}(x)$ is $\frac{L'}{L}(-1,\chi^*)$, while the one in $s=-a_\chi=0$ (i.e. $n=0$) is double and its contribution is

$$x(1+\beta-\log x),$$

where $-\frac{L'}{L}(\epsilon,\chi^*)=:-\frac{1}{\epsilon}+\beta+O(\epsilon)$ when χ^* is not trivial, and is simple with contribution equal to $-\frac{L'}{L}(0,\chi^*)x$ when χ^* is trivial. □

3. General setting and partial results

Let $q\in\mathbb{N}$ and $a\in\mathbb{Z}$ with $(a,q)=1$. For any sequence $f=\{f_\chi\}$ of objects depending on the character χ modulo q let

$$M_{a,q}f_\chi := \frac{1}{\varphi(q)}\sum_{\chi\in\widehat{(\mathbb{Z}/q\mathbb{Z})^*}}\overline{\chi}(a)f_\chi.$$

The operator $M_{a,q}$ selects for the integers which are congruent to a modulo q. Notice that $M_{a,q}$ is akin to the mean value, since if $|f_\chi|\leq M$ for every character, then $|M_{a,q}f_\chi|\leq M$.
Moreover, for any function $f\colon\mathbb{R}\to\mathbb{C}$ we let

$$\Delta_{2,h}f := f(x+h)-2f(x)+f(x-h).$$

The operator $\Delta_{2,h}$ will select the integers which are in the interval $(x-h,x+h)$.
Notably, the operators $M_{a,q}$ and $\Delta_{2,h}$ commute and

$$M_{a,q}\Delta_{2,h}\psi^{(1)}_\chi(x)=\sum_{n=a\,[q]}\Lambda(n)K(x-n;h)$$

where $K(u;h):=\max\{h-|u|,0\}$, so that it is supported in $|u|\leq h$, is positive in the open set, and has a unique maximum at $u=0$ with $K(0;h)=h$. The theorem follows from this basic equality by estimating, in the standard way, the function appearing on the left hand side. Since Lemma 2.1 is valid only for χ^*, we firstly need to connect $\psi^{(1)}_\chi$ with $\psi^{(1)}_{\chi^*}$. To this end, we let

$$B(\chi,x):=\psi^{(1)}_\chi(x)-\psi^{(1)}_{\chi^*}(x)$$

and prove the following lemma.

Lemma 3.1. *Assume* $x\geq 1$ *and* $0<h\leq x$. *Then*

$$|M_{a,q}\Delta_{2,h}B(\chi,x)|\leq\omega(q)h\log(2x).$$

Proof. We will prove that

$$|\Delta_{2,h}B(\chi,x)|\leq\omega(q)h\log(2x), \tag{3.1}$$

and the claim will immediately follow by the mean value property of $M_{a,q}$. By (2.1) we have that

$$B(\chi,x)=\sum_{n\leq x}(\chi(n)-\chi^*(n))\Lambda(n)(x-n).$$

Thus, only those integers that are coprime to q_χ and not q will be counted, giving

$$B(\chi,x)=-\sum_{\substack{n\leq x\\(n,q)>1\\(n,q_\chi)=1}}\chi^*(n)\Lambda(n)(x-n).$$

It follows that

$$\Delta_{2,h}B(\chi,x) = -\sum_{\substack{(n,q)>1\\(n,q_\chi)=1}} \chi^*(n)\Lambda(n)K(x-n;h),$$

and therefore

$$|\Delta_{2,h}B(\chi,x)| \leq \sum_{\substack{(n,q)>1\\(n,q_\chi)=1}} \Lambda(n)K(x-n;h).$$

Recalling the definition of Λ and removing the restriction $(n,q_\chi)=1$, we get

$$|\Delta_{2,h}B(\chi,x)| \leq \sum_{p|q}\log p\sum_{k\geq 1}K(x-p^k;h). \tag{3.2}$$

The inner sum is trivially bounded by

$$h\sum_{\substack{p^k<x+h\\1\leq k}} 1 \leq h\left\lfloor\frac{\log(x+h)}{\log p}\right\rfloor.$$

Finally, (3.2) gives

$$|\Delta_{2,h}B(\chi,x)| \leq h\sum_{p|q}\log(x+h) = \omega(q)h\log(x+h)$$

which is (3.1) under the restriction $0<h\leq x$. □

Lemma 3.2. *Assume $q\geq 3$, $x\geq 100$ and $0<h\leq\frac{5}{6}x$. Then*

$$M_{a,q}\Delta_{2,h}\psi^{(1)}_{\chi^*}(x) = \frac{h^2}{\varphi(q)} +\theta\Big[\Big|M_{a,q}\Delta_{2,h}\sum_{\rho\in Z_{\chi^*}}\frac{x^{\rho+1}}{\rho(\rho+1)}\Big|+1.7\Big|\delta_{\pm1[q]}(a)-\frac{2}{\varphi(q)}\Big|\frac{h^2}{x}+\delta_{\pm1[q]}(a)\frac{6h^2}{x(x-1)}\Big]$$

for some $\theta=\theta(a,q,x,h)\in[-1,1]$, where $\delta_{\pm1[q]}(a)=1$ when $a=\pm1 \pmod q$ and 0 otherwise.

The value $\frac{5}{6}$ in the upper bound $h\leq\frac{5}{6}x$ could be changed in a quite large interval without affecting the final result. However, in order to bound the secondary terms as h^2/x and h^2/x^2 respectively, it is essential to have an upper bound for h/x strictly smaller than 1.

Proof. We apply the operator $M_{a,q}\Delta_{2,h}$ to (2.3). We notice that $\Delta_{2,h}x^j=0$ for $j=0$, 1, and in general

$$\Delta_{2,h}f(x) = \int_0^h (h-u)(f''(x+u)+f''(x-u))\,\mathrm{d}u$$

for every C^2 function. Thus

$$M_{a,q}\Delta_{2,h}\Big(\frac{x^2}{2}\delta_{\chi^*=\mathbf{1}}\Big) = M_{a,q}(\delta_{\chi^*=\mathbf{1}})\cdot\Delta_{2,h}\Big(\frac{x^2}{2}\Big) = \frac{h^2}{\varphi(q)}, \tag{3.3}$$

$$M_{a,q}\Delta_{2,h}(-xr_{\chi^*}+r'_{\chi^*}) = 0, \tag{3.4}$$

and we have still to bound $M_{a,q}\Delta_{2,h}R^{(1)}_{\chi^*}(x)$. This is the sum of three terms:

$$M_{a,q}\Delta_{2,h}\big(a_\chi\log x\big),\qquad M_{a,q}\Delta_{2,h}\big((1-a_\chi)\delta_{\chi^*\neq\mathbf{1}}\,x\log x\big),$$

$$M_{a,q}\Delta_{2,h}\Big(\sum_{n=1}^{\infty}\frac{x^{1-2n-a_\chi}}{(2n+a_\chi)(2n+a_\chi-1)}\Big).$$

Since the set of even characters is a subgroup of $\widehat{(\mathbb{Z}/q\mathbb{Z})^*}$ of index two, we have

$$M_{a,q}(1-a_\chi) = \frac{1}{\varphi(q)}\sum_{\substack{\chi\in\widehat{(\mathbb{Z}/q\mathbb{Z})^*}\\\chi\text{ even}}}\overline{\chi(a)} = \frac{1}{2}\delta_{\pm1[q]}(a),$$

$$M_{a,q}((1-a_\chi)\delta_{\chi^*\neq\mathbf{1}}) = \frac{1}{\varphi(q)}\Big(\sum_{\substack{\chi\in\widehat{(\mathbb{Z}/q\mathbb{Z})^*}\\ \chi\,\text{even}}} \overline{\chi(a)}-1\Big) = \frac{1}{2}\delta_{\pm1[q]}(a)-\frac{1}{\varphi(q)},$$

$$M_{a,q}(a_\chi) = \frac{1}{\varphi(q)}\sum_{\substack{\chi\in\widehat{(\mathbb{Z}/q\mathbb{Z})^*}\\ \chi\,\text{odd}}} \overline{\chi(a)} = \frac{1}{2}\delta_{\pm1[q]}(a)\eta(a),$$

where η is any odd character modulo q. Moreover,

$$\begin{aligned}|\Delta_{2,h}\log x| &= \int_0^h (h-u)\Big(\frac{1}{(x+u)^2}+\frac{1}{(x-u)^2}\Big)\,\mathrm{d}u = \int_0^h \frac{2(h-u)(x^2+u^2)}{(x^2-u^2)^2}\,\mathrm{d}u\\ &\leq \frac{2}{(x^2-h^2)^2}\int_0^h (h-u)(x^2+u^2)\,\mathrm{d}u = \frac{h^2(x^2+\frac{h^2}{6})}{(x^2-h^2)^2} \leq 12\frac{h^2}{x^2},\end{aligned}$$

where for the last inequality we have used the assumption $0 < h \leq 5x/6$. Thus we have that

$$|M_{a,q}\Delta_{2,h}\big(a_\chi \log x\big)| \leq 6\frac{h^2}{x^2}\delta_{\pm1[q]}(a). \tag{3.5}$$

Similarly, for $0 \leq h \leq 5x/6$ it follows that

$$|\Delta_{2,h}x\log x| \leq \frac{xh^2}{x^2-h^2} \leq 3.4\frac{h^2}{x},$$

and so

$$|M_{a,q}\Delta_{2,h}\big((1-a_\chi)\delta_{\chi^*\neq\mathbf{1}}\, x\log x\big)| \leq 1.7\Big|\delta_{\pm1[q]}(a)-\frac{2}{\varphi(q)}\Big|\frac{h^2}{x}. \tag{3.6}$$

Lastly,

$$\begin{aligned}\Delta_{2,h}\sum_{n=1}^{\infty}\frac{x^{1-2n-a_\chi}}{(2n+a_\chi)(2n+a_\chi-1)} &= \int_0^h (h-u)\sum_{n=1}^{\infty}\Big((x+u)^{-1-2n-a_\chi}+(x-u)^{-1-2n-a_\chi}\Big)\,\mathrm{d}u\\ &= \int_0^h (h-u)\Big(\frac{(x+u)^{-1-a_\chi}}{(x+u)^2-1}+\frac{(x-u)^{-1-a_\chi}}{(x-u)^2-1}\Big)\,\mathrm{d}u.\end{aligned}$$

Using $h \leq 5x/6$ (and taking $u = vh$ with the fact that the function increases in h), we have that the above expression is bounded above by

$$\begin{aligned}&\frac{h^2}{x^{3+a_\chi}}\int_0^1 (1-v)\Big(\frac{(1+5v/6)^{-1-a_\chi}}{(1+5v/6)^2-x^{-2}}+\frac{(1-5v/6)^{-1-a_\chi}}{(1-5v/6)^2-x^{-2}}\Big)\,\mathrm{d}v\\ &= \frac{6h^2}{5x^{3+a_\chi}}\int_0^{5/6}\Big(1-\frac{6}{5}w\Big)\Big(\frac{(1+w)^{-1-a_\chi}}{(1+w)^2-x^{-2}}+\frac{(1-w)^{-1-a_\chi}}{(1-w)^2-x^{-2}}\Big)\,\mathrm{d}w.\end{aligned}$$

Since $x \geq 100$, this is bounded above by

$$\frac{6h^2}{5x^{3+a_\chi}}\int_0^{5/6}\Big(1-\frac{6}{5}w\Big)\Big(\frac{(1+w)^{-1-a_\chi}}{(1+w)^2-100^{-2}}+\frac{(1-w)^{-1-a_\chi}}{(1-w)^2-100^{-2}}\Big)\,\mathrm{d}w \leq 12\frac{h^2}{x^{3+a_\chi}}.$$

Thus, we have that

$$\Big|M_{a,q}\Delta_{2,h}\Big(\sum_{n=1}^{\infty}\frac{x^{1-2n-a_\chi}}{(2n+a_\chi)(2n+a_\chi-1)}\Big)\Big| \leq 6\frac{h^2}{x^3}\Big(1+\frac{1}{x}\Big)\delta_{\pm1[q]}(a), \tag{3.7}$$

and now the claim follows from (2.3) and (3.3–3.7). □

We split the sum on zeros as

$$\sum_{\rho\in Z_{\chi^*}}\frac{x^{\rho+1}}{\rho(\rho+1)} =: \Sigma_{\chi^*,1}+\Sigma_{\chi^*,2},$$

with $\Sigma_{\chi^*,1}$ and $\Sigma_{\chi^*,2}$ representing the sums on zeros with $|\mathrm{Im}(\rho)| \leq T$ and $|\mathrm{Im}(\rho)| > T$, respectively, for a convenient parameter $T > 0$. The next lemma provides a bound for $\Sigma_{\chi^*,2}$.

Lemma 3.3. *Assume GRH, $q \geq 3$, $0 \leq h \leq x$ and $T \geq 16$. Then*

$$\big|M_{a,q}\Delta_{2,h}\Sigma_{\chi^*,2}\big| \leq \frac{4}{\pi}\Big(x^{3/2}+\frac{h^2}{4\sqrt{x}}\Big)\Big(1+\frac{2.89}{T}\Big)\frac{\log(qT)}{T}.$$

Proof. For $z \in [0,1]$ by double squaring we get $(1+z)^{3/2}+(1-z)^{3/2} \leq 2+z^2$ which implies that $(x+h)^{3/2}+2x^{3/2}+(x-h)^{3/2} \leq 4x^{3/2}+h^2/\sqrt{x}$ for $0 \leq h \leq x$. Thus, GRH gives us that

$$\big|\Delta_{2,h}\Sigma_{\chi^*,2}\big| \leq 4\Big(x^{3/2}+\frac{h^2}{4\sqrt{x}}\Big)\sum_{\substack{\rho\in Z_{\chi^*}\\ |\mathrm{Im}(\rho)|>T}}\frac{1}{|\rho(\rho+1)|},$$

so that

$$\big|M_{a,q}\Delta_{2,h}\Sigma_{\chi^*,2}\big| \leq \frac{4}{\varphi(q)}\Big(x^{3/2}+\frac{h^2}{4\sqrt{x}}\Big)\sum_{\chi\in\widehat{(\mathbb{Z}/q\mathbb{Z})^*}}\ \sum_{\substack{\rho\in Z_{\chi^*}\\ |\mathrm{Im}(\rho)|>T}}\frac{1}{|\rho(\rho+1)|}.$$

Each inner sum on zeros could be estimated by partial summation using the known formulas for the number of zeros of each $L(s,\chi)$ (see Trudgian [14]), but we can reduce the error term by connecting the sum with a similar sum for a Dedekind zeta function. In fact one has the factorization $\zeta_{\mathbb{Q}[q]}(s) = \prod_{\chi\in\widehat{(\mathbb{Z}/q\mathbb{Z})^*}} L(s,\chi^*)$, where $\mathbb{Q}[q]$ is the cyclotomic field of q-roots of unity (see [15, Th. 4.3]), and thus

$$|M_{a,q}\Delta_{2,h}\Sigma_{\chi^*,2}| \leq \frac{4}{\varphi(q)}\Big(x^{3/2}+\frac{h^2}{4\sqrt{x}}\Big)\sum_{\substack{\rho\in Z_q\\ |\mathrm{Im}(\rho)|>T}}\frac{1}{|\rho(\rho+1)|}, \tag{3.8}$$

where Z_q is the multiset of zeros of $\zeta_{\mathbb{Q}[q]}$. This sum has already been estimated in [4, Eq. (3.7)] for a generic number field $\mathbb{K}$, the result being that

$$\sum_{|\gamma|\geq T}\frac{\pi}{|\rho|^2} \leq \Big(1+\frac{2.89}{T}\Big)\frac{W_{\mathbb{K}}(T)}{T}+\Big(1+\frac{18.61}{T}\Big)\frac{n_{\mathbb{K}}}{T}+\frac{17.31}{T^2}$$

for all $T \geq 5$ where $W_{\mathbb{K}}(T) := \log\Delta_{\mathbb{K}}+n_{\mathbb{K}}\log(T/2\pi)$, $\Delta_{\mathbb{K}}$ is the absolute value of the discriminant of $\mathbb{K}$ and $n_{\mathbb{K}}$ its degree. For $\mathbb{K} = \mathbb{Q}[q]$, one has that $\log\Delta_{\mathbb{K}} = \varphi(q)\log q-\varphi(q)\sum_{p|q}\frac{\log p}{p-1}$ (see [15, Proposition 2.7]) and $n_{\mathbb{K}} = \varphi(q)$, thus this formula becomes

$$\sum_{|\gamma|\geq T}\frac{\pi}{|\rho|^2} \leq \Big(1+\frac{2.89}{T}\Big)\frac{\varphi(q)}{T}\Big(\log q-\sum_{p|q}\frac{\log p}{p-1}+\log\Big(\frac{T}{2\pi}\Big)\Big)+\Big(1+\frac{18.61}{T}\Big)\frac{\varphi(q)}{T}+\frac{17.31}{T^2} \tag{3.9}$$

for all $T \geq 5$. We simplify this to

$$\sum_{|\gamma|\geq T}\frac{\pi}{|\rho|^2} \leq \Big(1+\frac{2.89}{T}\Big)\frac{\varphi(q)}{T}\log(qT) \tag{3.10}$$

for all $T \geq 16$. Indeed, (3.9) shows that (3.10) holds as soon as

$$1+\frac{18.61-2.89\log 2\pi}{T}+\frac{17.31}{\varphi(q)T} \leq \Big(1+\frac{2.89}{T}\Big)\sum_{p|q}\frac{\log p}{p-1}+\log 2\pi,$$

which is implied for $T \geq 16$ by

$$1+\frac{18.61-2.89\log 2\pi}{16}+\frac{17.31}{16\varphi(q)} \leq \sum_{p|q}\frac{\log p}{p-1}+\log 2\pi. \tag{3.11}$$

By inspection we test that this inequality holds for each $q = 3,\ldots,1000$. On the other hand, if $q \geq 1000$, then using the multiplicativity of $\varphi(q)/q^{3/4}$ one can prove easily that $\varphi(q) \geq q^{3/4} > 170$. Thus (3.11) still holds because $1+\frac{18.61-2.89\log 2\pi}{16}+\frac{17.31}{16\cdot 170} \leq \log 2\pi$.
The proof concludes combining (3.8) and (3.10). □

Collecting the results in Lemmas 2.1, 3.1, 3.2 and 3.3 we get

$$\sum_{n=a\,[q]} \Lambda(n)K(x-n;h) \geq \frac{h^2}{\varphi(q)} - |M_{a,q}\Delta_{2,h}\Sigma_{\chi^*,1}| - \frac{4}{\pi}\Big(x^{3/2}+\frac{h^2}{4\sqrt{x}}\Big)\Big(1+\frac{2.89}{T}\Big)\frac{\log(qT)}{T}$$
$$-\omega(q)h\log(2x) - \Big[1.7\Big|\delta_{\pm1[q]}(a)-\frac{2}{\varphi(q)}\Big|\frac{h^2}{x}+\delta_{\pm1[q]}(a)\frac{6h^2}{x(x-1)}\Big].$$

We simplify it by noticing that

$$1.7\Big|\delta_{\pm1[q]}(a)-\frac{2}{\varphi(q)}\Big|\frac{h^2}{x}+\delta_{\pm1[q]}(a)\frac{6h^2}{x(x-1)} \leq 1.7\frac{h^2}{x}$$

when $q \geq 3$ (thus $\varphi(q) \geq 2$) and $x \geq 2\varphi(q)+1$. In this way we deduce that

$$\sum_{n=a\,[q]} \Lambda(n)K(x-n;h) \geq \frac{h^2}{\varphi(q)} - |M_{a,q}\Delta_{2,h}\Sigma_{\chi^*,1}| - 4\Big(x^{3/2}+\frac{h^2}{4\sqrt{x}}\Big)\Big(1+\frac{2.89}{T}\Big)\frac{\log(qT)}{\pi T}$$
$$-\omega(q)h\log(2x) - 1.7\frac{h^2}{x}.$$

Now we remove the contribution of prime powers. We get

$$\sum_{n=a\,[q]} \Lambda(n)K(x-n;h) \leq h \sum_{\substack{|n-x|<h\\ n=a\,[q]}} \Lambda(n) = h\Big(\sum_{\substack{|p-x|<h\\ p=a\,[q]}} \log p + \sum_{2\leq k}\sum_{\substack{|p^k-x|<h\\ p^k=a\,[q]}} \log p\Big)$$

and removing the arithmetical condition one gets

$$\sum_{2\leq k}\sum_{\substack{|p^k-x|<h\\ p^k=a\,[q]}} \log p \leq \sum_{2\leq k}\sum_{|p^k-x|<h} \log p \leq [\psi(x+h)-\vartheta(x+h)]-[\psi(x-h)-\vartheta(x-h)]$$
$$\leq (1+10^{-6})\sqrt{x+h}+3\sqrt[3]{x+h}-0.998684\sqrt{x-h}$$

for every $x \geq 121$ (see [12, Cor. 2] and [13, Th. 6]). Assuming that $h \leq 5x/6$ (as we have done for Lemma 3.2) we have

$$\sum_{2\leq k}\sum_{\substack{|p^k-x|<h\\ p^k=a\,[q]}} \log p \leq 0.95\sqrt{x}+3.7\sqrt[3]{x}. \tag{3.12}$$

Note that the Brun–Titchmarsh theorem for primes in arithmetic progressions (eventually in intervals – see [10]) produces a much better bound, but only when x and q are much larger than what we need to prove our theorem. As a consequence we have decided not to use this tool.
To summarise so far, we have proved that for $q \geq 3$, $x \geq \max(121, 2\varphi(q)+1)$ and $h \leq \frac{5}{6}x$ one has

$$\begin{aligned}\sum_{\substack{|p-x|<h\\ p=a\,[q]}} \log p \geq{}& \frac{h}{\varphi(q)} - \frac{1}{h}|M_{a,q}\Delta_{2,h}\Sigma_{\chi^*,1}| - 4\Big(x^{3/2}+\frac{h^2}{4\sqrt{x}}\Big)\Big(1+\frac{2.89}{T}\Big)\frac{\log(qT)}{\pi hT}\\ &-(0.95\sqrt{x}+3.7\sqrt[3]{x})-\omega(q)\log(2x)-1.7\frac{h}{x}.\end{aligned} \tag{3.13}$$

In Section 4 we provide an upper bound for $M_{a,q}\Delta_{2,h}\Sigma_{\chi^*,1}$. In this way we will be able to prove the theorems in Sections 5 and 6.

4. Bound for $M_{a,q}\Delta_{2,h}\Sigma_{\chi^*,1}$

Lemma 4.1. *Let $y > 0$. Then for some $\theta \in [-1,1]$ we have that*

$$\int_0^y \frac{\sin^2 t}{t^2}\,\mathrm{d}t = \frac{\pi}{2}-\frac{1}{2y}+\frac{\theta}{4y^2}.$$

The claim with $\theta \in [-2,2]$ has a very simple proof. We optimize the result by proving the stronger bound $\theta \in [-1,1]$.

Proof. We note that

$$\int_0^y \frac{\sin^2 t}{t^2}\,\mathrm{d}t = \frac{\pi}{2} - \int_y^\infty \frac{\sin^2 t}{t^2}\,\mathrm{d}t = \frac{\pi}{2} - \int_y^\infty \frac{1-\cos(2t)}{2t^2}\,\mathrm{d}t = \frac{\pi}{2} - \frac{1}{2y} + \int_y^\infty \frac{\cos(2t)}{2t^2}\,\mathrm{d}t.$$

Therefore the claim states that $|f(x)| \le 1$ when $x > 0$, where $f(x) := x^2 \int_x^{+\infty} \frac{\cos v}{v^2}\,\mathrm{d}v$. We prove this statement in two steps.

Step 1) The claim holds in $[0,6]$.
We notice that f is the unique bounded solution in $(0,+\infty)$ of the ODE $y' = \frac{2}{x}(y - \frac{x}{2}\cos x)$. We can use this equation to trace the graph of $f(x)$ in $[0,6]$. The extremal points of f solve $f(x) = \frac{x}{2}\cos x =: g(x)$, so $|f(x)| < 1$ when $0 < x \le 2.4$, since $\frac{x}{2}|\cos x| < 1$ here. Moreover, $f(5) = 0.896\ldots$ so that $f'(5) > \frac{2}{5}(0.896 - \frac{5}{2}\cos(5)) > 0$, and $f(5.1) = 0.899\ldots$ so that $f'(5) < \frac{2}{5.1}(0.9 - \frac{5.1}{2}\cos(5.1)) < 0$, thus f is increasing for $x \in [2.4, 5]$ (by the differential equation) and smaller than 1 here (because $f(5) < 1$). Moreover there is a maximum for f in $[5, 5.1]$, and since $g(x) < 0.97$ here, we conclude that $|f(x)| < 1$ in $[0, 5.1]$. Moreover, $g(x)$ increases for $x \in [5.1, 6]$ and $f(6) = 0.50\ldots < g(6)$, thus $f(x)$ decreases here, and the value of $f(6)$ completes the proof of this step.

Step 2) The claim holds for $x \ge 6$.
Four integrations by parts give

$$f(x) = -\sin x + 2\frac{\cos x}{x} + 6\frac{\sin x}{x^2} - 24\frac{\cos x}{x^3} + 120x^2 \int_x^{+\infty} \frac{\cos v}{v^6}\,\mathrm{d}v$$

so that

$$|f(x)| \le \Big|\Big(1 - \frac{6}{x^2}\Big)\sin x - \frac{2}{x}\Big(1 - \frac{12}{x^2}\Big)\cos x\Big| + \frac{24}{x^3}.$$

We prove that this function is lower than 1 for $x \ge 6$. Multiplying by x^2, we have to prove that

$$-x^2 + \frac{24}{x} < (x^2-6)\sin x - \frac{2}{x}(x^2-12)\cos x < x^2 - \frac{24}{x}.$$

The first inequality is evident when $\cos x \le 0$, and the second when $\cos x \ge 0$, respectively (because we are assuming $x \ge 6$). Assuming $\cos x > 0$ for the first one, and $\cos x < 0$ for the second one, both remaining inequalities are implied by the stronger bound:

$$(x^2-6)\Big|\sin x - \frac{2}{x}\cos x\Big| < x^2 - \frac{24}{x}.$$

Since $|\sin x - \alpha\cos x| \le \sqrt{1+\alpha^2} \le 1 + \frac{\alpha^2}{2}$ (the first inequality by elementary trigonometry, the second by convexity), it is sufficient to prove that

$$(x^2-6)\Big(1 + \frac{2}{x^2}\Big) < x^2 - \frac{24}{x},$$

which in fact holds for $x \ge 6$.

□

Lemma 4.2. *Let $0 \le h < x$. Then for every $\gamma \in \mathbb{R}$ there exists $\theta \in \mathbb{C}$ with $|\theta| \le 1$ such that*

$$\Big(1+\frac{h}{x}\Big)^{\frac{3}{2}+i\gamma} - 2 + \Big(1-\frac{h}{x}\Big)^{\frac{3}{2}+i\gamma} = -4\sin^2\Big(\frac{\gamma h}{2x}\Big) + \theta(2|\gamma|+1)\frac{h^2}{x^2}.$$

Proof. The proof is straightforward and follows from the Taylor expansion of $\log(1+u)$ and some elementary inequalities. □

The definitions of $\Delta_{2,h}$ and $\Sigma_{\chi^*,1}$ show that

$$\Delta_{2,h}\Sigma_{\chi^*,1} = \sum_{\substack{\rho\in Z_{\chi^*} \\ |\mathrm{Im}(\rho)|\le T}} \frac{(x+h)^{\rho+1} - 2x^{\rho+1} + (x-h)^{\rho+1}}{\rho(\rho+1)},$$

so that by Lemma 4.2 we deduce that

$$|\Delta_{2,h}\Sigma_{\chi^*,1}| \leq 4x^{3/2} \sum_{\substack{\frac{1}{2}+i\gamma\in Z_{\chi^*}\\ |\gamma|\leq T}} \frac{\sin^2\left(\frac{\gamma h}{2x}\right)}{|(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)|} + \frac{h^2}{\sqrt{x}} \sum_{\substack{\frac{1}{2}+i\gamma\in Z_{\chi^*}\\ |\gamma|\leq T}} \frac{2|\gamma|+1}{|(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)|}.$$

As we have done for $\Sigma_{\chi^*,2}$ we use the factorization of the Dedekind zeta function $\zeta_{\mathbb{K}}$ of the cyclotomic field $\mathbb{K} := \mathbb{Q}[q]$ of q-th roots of unity as products of $L(s,\chi^*)$; in this way we deal with all zeros in $\cup_{\chi\in\widehat{(\mathbb{Z}/q\mathbb{Z})^*}} Z_{\chi^*}$ as a unique step. This does not affect the main part of the theorem, but reduces the size of the secondary terms, and makes the ranges for q and x wider in the theorem.

$$\begin{aligned}\varphi(q)|M_{a,q}\Delta_{2,h}\Sigma_{\chi^*,1}| &\leq \sum_{\chi\in\widehat{(\mathbb{Z}/q\mathbb{Z})^*}} \Bigg[4x^{3/2} \sum_{\substack{\frac{1}{2}+i\gamma\in Z_{\chi^*}\\ |\gamma|\leq T}} \frac{\sin^2\left(\frac{\gamma h}{2x}\right)}{|(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)|} + \frac{h^2}{\sqrt{x}} \sum_{\substack{\frac{1}{2}+i\gamma\in Z_{\chi^*}\\ |\gamma|\leq T}} \frac{2|\gamma|+1}{|(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)|}\Bigg] \\ &= 4x^{3/2} \sum_{\substack{\frac{1}{2}+i\gamma\in Z_q\\ |\gamma|\leq T}} \frac{\sin^2\left(\frac{\gamma h}{2x}\right)}{|(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)|} + \frac{h^2}{\sqrt{x}} \sum_{\substack{\frac{1}{2}+i\gamma\in Z_q\\ |\gamma|\leq T}} \frac{2|\gamma|+1}{|(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)|}. \qquad (4.1)\end{aligned}$$

We deduce a bound for the second sum from two computations already made by the second and third author for Dedekind zeta functions.

Lemma 4.3. *Assume GRH and let $T \geq 20$. Then*

$$\frac{1}{\varphi(q)} \sum_{\substack{\frac{1}{2}+i\gamma\in Z_q\\ |\gamma|\leq T}} \frac{2|\gamma|+1}{|(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)|} \leq \frac{1}{\pi}\log(q^2T)\log T + 1.93\log q - 4.35 + \frac{21.67}{\varphi(q)}.$$

Proof. In [4, Eq. (3.8)] it is proved that

$$\sum_{\frac{1}{2}+i\gamma\in Z_q} \frac{\pi}{|\frac{1}{2}+i\gamma|} \leq \left(\log\left(\frac{T}{2\pi}\right)+4.01\right)\log\Delta_q + \left(\frac{1}{2}\log^2\left(\frac{T}{2\pi}\right)-1.41\right)\varphi(q)+25.57;$$

and in [3, Lemma 4.1] that

$$\sum_{\frac{1}{2}+i\gamma\in Z_q} \frac{1}{|(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)|} \leq 0.54\log\Delta_q - 1.03\varphi(q) + 5.39$$

(both for $T \geq 5$). Thus,

$$\begin{aligned}\sum_{\substack{\frac{1}{2}+i\gamma\in Z_q\\ |\gamma|\leq T}} \frac{2|\gamma|+1}{|(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)|} &\leq \sum_{\substack{\frac{1}{2}+i\gamma\in Z_q\\ |\gamma|\leq T}} \frac{2}{|\frac{1}{2}+i\gamma|} + \sum_{\frac{1}{2}+i\gamma\in Z_q} \frac{1}{|(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)|} \\ &\leq \frac{2}{\pi}\Bigg[\left(\log\left(\frac{T}{2\pi}\right)+4.01\right)\log\Delta_q + \left(\frac{1}{2}\log^2\left(\frac{T}{2\pi}\right)-1.41\right)\varphi(q)+25.57\Bigg] + 0.54\log\Delta_q - 1.03\varphi(q) + 5.39\end{aligned}$$

and recalling that $\log\Delta_q = \varphi(q)\log q - \varphi(q)\sum_{p|q}\frac{\log p}{p-1} \leq \varphi(q)\log q$, we get

$$\begin{aligned}\frac{1}{\varphi(q)} \sum_{\substack{\frac{1}{2}+i\gamma\in Z_q\\ |\gamma|\leq T}} \frac{2|\gamma|+1}{|(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)|} &\leq \left(\frac{2}{\pi}\log\left(\frac{T}{2\pi}\right)+3.1\right)\log q + \left(\frac{1}{\pi}\log^2\left(\frac{T}{2\pi}\right)-1.927\right) + \frac{21.67}{\varphi(q)} \\ &= \frac{1}{\pi}\log(q^2T)\log T + \left(3.1-\frac{2}{\pi}\log(2\pi)\right)\log q - \frac{2}{\pi}\log T\log(2\pi) + \frac{1}{\pi}\log^2(2\pi) - 1.927 + \frac{21.67}{\varphi(q)}.\end{aligned}$$

The claim follows by recalling that we are assuming $T \geq 20$ so that the contribution of all secondary terms is -4.35, at most. □

Lemma 4.4. *Assume GRH and Let $\mathbb{K}$ be any number field. Then*

$$\sum_{|\gamma|\leq 5} \frac{\gamma^2}{|(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)|} \leq 1.5\log\Delta_{\mathbb{K}}+1.651n_{\mathbb{K}}-1.577.$$

Proof. We apply the same technique we have already used for Lemmas 3.1 and 4.1 in [3] and for Lemma 3.1 in [4], stemming from the remark that the function $f_{\mathbb{K}}(s) := \sum_\rho \mathrm{Re}\big(\frac{2}{s-\rho}\big)$ can be exactly computed via the alternative representation

$$f_{\mathbb{K}}(s) = 2\mathrm{Re}\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s)+\log\frac{\Delta_{\mathbb{K}}}{\pi^{n_{\mathbb{K}}}}+\mathrm{Re}\Big(\frac{2}{s}+\frac{2}{s-1}\Big)+(r_1+r_2)\mathrm{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big)+r_2\mathrm{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{s+1}{2}\Big). \tag{4.2}$$

Let

$$f(s,\gamma) := \frac{4(2s-1)}{(2s-1)^2+4\gamma^2},$$

so that $f_{\mathbb{K}}(s) = \sum_\gamma f(s,\gamma)$, and let

$$g(\gamma) := \begin{cases} \frac{\gamma^2}{((\frac{1}{4}+\gamma^2)(\frac{9}{4}+\gamma^2))^{1/2}} & \text{if } |\gamma|\leq 5 \\ 0 & \text{otherwise.} \end{cases}$$

so that $\sum_{|\gamma|\leq 5} \frac{\gamma^2}{|(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)|} = \sum_\gamma g(\gamma)$. We look for a finite linear combination of $f(s,\gamma)$ at suitable points s_j such that

$$g(\gamma) \leq F(\gamma) := \sum_j a_j f(s_j,\gamma) \tag{4.3}$$

for all $\gamma\in\mathbb{R}$ so that

$$\sum_{|\gamma|\leq 5} \frac{\gamma^2}{|(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)|} \leq \sum_j a_j f_{\mathbb{K}}(s_j). \tag{4.4}$$

Once (4.4) is proved, we recover a bound for the sum on zeros by recalling the identity (4.2). According to this approach, the final coefficient of $\log\Delta_{\mathbb{K}}$ will be the sum of all a_j, and thus we are interested in the linear combinations for which this sum is as small as possible. We set $s_j = 3/4+j/2$ with $j = 1, \dots, 2\kappa+3$ for a suitable integer κ. Let $\Upsilon \subset (0,+\infty)$ be a set with κ numbers. We require:

(1) $F(\gamma) = g(\gamma)$ for all $\gamma\in\Upsilon\cup\{0,5\}$,
(2) $F'(\gamma) = g'(\gamma)$ for all $\gamma\in\Upsilon$,
(3) $\lim_{\gamma\to\infty}\gamma^2F(\gamma) = \lim_{\gamma\to\infty}\gamma^2 g(\gamma) = 0$.

This produces a set of $2\kappa+3$ linear equations for the $2\kappa+3$ constants a_j, and we hope that these satisfy (4.3) for every γ. We choose $\kappa := 10$ and $\Upsilon := \{0.5, 1.5, 2, 2.4, 2.8, 7.9, 18, 10^2, 10^3, 10^5\}$. Finally, with an abuse of notation we take for a_j the solution of the system, rounded above to 10^{-7}: this produces the numbers in Table 2.

Table 2. Values of the coefficients.

j	$a_j{\cdot}10^7$	j	$a_j{\cdot}10^7$
1	−10417203	13	−18920268046344982450
2	1056404889	14	29659178484686316889
3	−65191418930	15	−37103060687919097856
4	2306235683461	16	36963001195180424340
5	−50953892956052	17	−29124459758424138052
6	745294415104297	18	17917680016161661642
7	−7554469767270438	19	−8424311293805783518
8	55069155554895360	20	2923218093750242944
9	−297487524612176257	21	−705518033170496127
10	1219731091815491142	22	105765338120745449
11	−3866974934911032963	23	−7417073631321810
12	9612711864719121022		

Then, using Sturm's algorithm, we prove that the values found actually give an upper bound for g, so that (4.4) holds with such a_j's. These constants verify

$$\sum_j a_j = 1.4999\dots, \qquad \sum_j a_j\Big(\frac{2}{s_j}+\frac{2}{s_j-1}\Big) \leq -1.577,$$
$$\sum_j a_j \frac{\Gamma'}{\Gamma}\Big(\frac{s_j}{2}\Big) \leq 0.6552, \qquad \sum_j a_j \frac{\Gamma'}{\Gamma}\Big(\frac{s_j+1}{2}\Big) \leq 0.7314. \tag{4.5}$$

We write $\sum_j a_j \frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s_j)$ as

$$-\sum_n \tilde{\Lambda}_{\mathbb{K}}(n) S(n) \quad \text{with} \quad S(n) := \sum_j \frac{a_j}{n^{s_j}}.$$

We check numerically that $S(n) < 0$ for $n \leq 10284$ with the exception of $S(4)$, which is in any case ≤ 0.0237. Then, since the sign of a_j alternates, we can easily prove that each pair $\frac{a_1}{n^{s_1}}+\frac{a_2}{n^{s_2}}$, $\dots$, $\frac{a_{2q+1}}{n^{s_{2q+1}}}+\frac{a_{2q+2}}{n^{s_{2q+2}}}$ and the last term $\frac{a_{2q+3}}{n^{s_{2q+3}}}$ are negative for every $n \geq 10284$, thus

$$\begin{aligned}\sum_j a_j \frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s_j) &= -\sum_n \tilde{\Lambda}_{\mathbb{K}}(n) S(n) \leq -n_{\mathbb{K}} \sum_{n\neq 4} \Lambda(n) S(n)\\ &= -n_{\mathbb{K}}\Big[\sum_{n=1}^{\infty} \Lambda(n)S(n) - \Lambda(4)S(4)\Big]\\ &= n_{\mathbb{K}}\Big[\sum_j a_j \frac{\zeta'}{\zeta}(s_j) + \Lambda(4)S(4)\Big] \leq 1.3372 n_{\mathbb{K}}.\end{aligned} \tag{4.6}$$

The result now follows from (4.2), and (4.4–4.6). □

Thus, by (4.1) and Lemmas 4.3 and 4.4 we get

$$\begin{aligned}|M_{a,q}\Delta_{2,h}\Sigma_{\chi^*,1}| \leq& \frac{4x^{3/2}}{\varphi(q)} \sum_{\substack{\frac12+i\gamma\in Z_q\\ 5\leq|\gamma|\leq T}} \frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{|(\frac12+i\gamma)(\frac32+i\gamma)|}\\ &+\frac{h^2}{\sqrt{x}}\Big(\frac{1}{\pi}\log(q^2T)\log T + 3.43\log q - 2.699 + \frac{20.1}{\varphi(q)}\Big).\end{aligned} \tag{4.7}$$

To bound the sum by partial summation we need a formula for $N_q(T)$, the number of zeros ρ of $\zeta_{\mathbb{K}}$ with $\mathrm{Re}(\rho) \in (0,1)$ and $|\mathrm{Im}(\rho)| \leq T$. Let $W_q(T) := \frac{T}{\pi}\log\big((\frac{T}{2\pi e})^{\varphi(q)}\Delta_q\big)$, and let $U_q(T) :=$

$N_q(T)-W_q(T)$. Then

$$\Big|\frac{1}{\varphi(q)}N_q(T)-\frac{T}{\pi}\log\Big(\frac{T}{2\pi e}(\Delta_q)^{1/\varphi(q)}\Big)\Big| = \frac{|U_q(T)|}{\varphi(q)}$$
$$\leq d_1\log\Big(\frac{T}{2\pi}(\Delta_q)^{1/\varphi(q)}\Big)+d_2+\frac{d_3}{\varphi(q)} =: \frac{R_q(T)}{\varphi(q)} \qquad T\geq 5$$

with $d_1 = 0.395$, $d_2 = 3.459$ and $d_3 = 2.559$ (this particular set of values is computed using the algorithm of Trudgian [14] with $\eta = 0.36$, $p = -\eta$ and $r = \frac{1+\eta-p}{1/2+\eta} = 2$.) Thus, by partial summation we get

$$\sum_{\substack{\frac{1}{2}+i\gamma\in Z_q\\ 5<|\gamma|\leq T}}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2} = \int_{5^+}^{T^+}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\,\mathrm{d}N_q(\gamma)$$
$$=\frac{\sin^2\big(\frac{hT}{2x}\big)}{T^2}N_q(T)-\frac{\sin^2\big(\frac{5h}{2x}\big)}{5^2}N_q(5^+)-\int_5^T\Big[\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\Big]'N_q(\gamma)\,\mathrm{d}\gamma$$
$$=\frac{\sin^2\big(\frac{hT}{2x}\big)}{T^2}U_q(T)-\frac{\sin^2\big(\frac{5h}{2x}\big)}{5^2}U_q(5^+)$$
$$+\int_5^T\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\log\Big(\frac{\gamma^{\varphi(q)}\Delta_q}{2\pi}\Big)\frac{\mathrm{d}\gamma}{\pi}-\int_5^T\Big[\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\Big]'U_q(\gamma)\,\mathrm{d}\gamma.$$

Recalling the upper bound $|U_q(T)|\leq R_q(T)$ we get

$$\frac{1}{\varphi(q)}\sum_{5<|\gamma|\leq T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\leq\frac{R_q(T)}{\varphi(q)T^2}+\Big(\frac{h}{2x}\Big)^2\frac{R_q(5)}{\varphi(q)}$$
$$+\int_5^T\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\log\Big(\frac{\gamma(\Delta_q)^{1/\varphi(q)}}{2\pi}\Big)\frac{\mathrm{d}\gamma}{\pi}+\frac{1}{\varphi(q)}\int_5^T\Big|\frac{h}{2x}\frac{\sin\big(\frac{\gamma h}{x}\big)}{\gamma^2}-2\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^3}\Big|R_q(\gamma)\,\mathrm{d}\gamma.$$

Using the inequality $|\frac{\sin(2v)}{2}-\frac{\sin^2 v}{v}|\leq\frac{4}{5}$, we simplify to get

$$\frac{1}{\varphi(q)}\sum_{5<|\gamma|\leq T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\leq\frac{R_q(T)}{\varphi(q)T^2}+\Big(\frac{h}{2x}\Big)^2\frac{R_q(5)}{\varphi(q)}$$
$$+\int_5^T\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\log\Big(\frac{\gamma(\Delta_q)^{1/\varphi(q)}}{2\pi}\Big)\frac{\mathrm{d}\gamma}{\pi}+\frac{4h}{5\varphi(q)x}\int_5^T\frac{R_q(\gamma)}{\gamma^2}\,\mathrm{d}\gamma,$$

and since $\int_5^{+\infty}\frac{R_q(\gamma)}{\gamma^2}\,\mathrm{d}\gamma\leq 0.079\log\Delta_q+0.7528\varphi(q)+0.5118$, we get

$$\frac{1}{\varphi(q)}\sum_{5<|\gamma|\leq T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\leq\frac{h}{2\pi x}\int_0^{\frac{hT}{2x}}\frac{\sin^2 t}{t^2}\,\mathrm{d}t\log\Big(\frac{T(\Delta_q)^{1/\varphi(q)}}{2\pi}\Big)$$
$$+\frac{(0.253\log\Delta_q+2.409\varphi(q)+1.638)h}{4\varphi(q)x}+\frac{R_q(T)}{\varphi(q)T^2}+\Big(\frac{h}{2x}\Big)^2\frac{R_q(5)}{\varphi(q)}.$$

By Lemma 4.1 and the bound $R_q(T)\leq 0.395\log(T^{\varphi(q)}\Delta_q)+2.74\varphi(q)+2.559$ for every $T\geq 5$, the bound becomes

$$\frac{1}{\varphi(q)}\sum_{5<|\gamma|\leq T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}$$
$$\leq\frac{h}{4x}\Big(1-\frac{2}{\pi}\frac{x}{hT}+\frac{2}{\pi}\frac{x^2}{h^2T^2}\Big)\log\Big(\frac{T(\Delta_q)^{1/\varphi(q)}}{2\pi}\Big)+\frac{(0.253\log\Delta_q+2.409\varphi(q)+1.638)h}{4\varphi(q)x}$$
$$+\frac{0.395\log(T^{\varphi(q)}\Delta_q)+2.74\varphi(q)+2.559}{\varphi(q)T^2}+\Big(\frac{h}{2x}\Big)^2\frac{0.395\log(5^{\varphi(q)}\Delta_q)+2.74\varphi(q)+2.559}{\varphi(q)}.$$

We substitute $\log\Delta_q=\varphi(q)\log q-\varphi(q)\sum_{p|q}\frac{\log p}{p-1}$ in the first two terms, while for the last two we simply use the bound $\log\Delta_q\le\varphi(q)\log q$. Moreover, since for $q\ge 3$ we have $\varphi(q)\ge 2$, so we use this hypothesis to simplify the terms decaying as $1/T^2$. We get

$$\le\frac{h}{4x}\Big(1-\frac{2}{\pi}\frac{x}{hT}+\frac{2}{\pi}\frac{x^2}{h^2T^2}\Big)\log\Big(\frac{qT}{2\pi}\Big)+\Big(0.253\log q-\sum_{p|q}\frac{\log p}{p-1}+2.409+\frac{1.638}{\varphi(q)}\Big)\frac{h}{4x}$$
$$+\frac{0.395\log(qT)+4.02}{T^2}+\Big(0.395\log q+3.376+\frac{2.559}{\varphi(q)}\Big)\frac{h^2}{4x^2},$$

where we used that $1.253-\frac{2}{\pi}Y+\frac{2}{\pi}Y^2\ge 1$ to simplify the coefficient of $\sum_{p|q}$. Thus (4.7) becomes

$$\begin{aligned}|M_{a,q}\Delta_{2,h}\Sigma_{\chi^*,1}|\le& h\sqrt{x}\log\Big(\frac{qT}{2\pi}\Big)-\frac{2}{\pi}h\sqrt{x}\frac{x}{hT}\log\Big(\frac{qT}{2\pi}\Big)+\frac{2}{\pi}h\sqrt{x}\frac{x^2}{h^2T^2}\log\Big(\frac{qT}{2\pi}\Big)\\&+\Big(0.253\log q-\sum_{p|q}\frac{\log p}{p-1}+2.409+\frac{1.638}{\varphi(q)}\Big)h\sqrt{x}+(1.58\log(qT)+16.08)\frac{x^{3/2}}{T^2}\\&+\Big(\frac{1}{\pi}\log(q^2T)\log T+3.9\log q+0.7+\frac{22.7}{\varphi(q)}\Big)\frac{h^2}{\sqrt{x}}.\end{aligned}\tag{4.8}$$

5. Proof of Theorem 1.1

Substituting (4.8) into (3.13) and by (3.12) we get

$$\begin{aligned}\sum_{\substack{|p-x|<h\\p=a\,[q]}}\log p\ge&\frac{h}{\varphi(q)}-\Big[\sqrt{x}\log\Big(\frac{qT}{2\pi}\Big)-\frac{2}{\pi}\sqrt{x}\frac{x}{hT}\log\Big(\frac{qT}{2\pi}\Big)+\frac{2}{\pi}\sqrt{x}\frac{x^2}{h^2T^2}\log\Big(\frac{qT}{2\pi}\Big)\\&+\Big(0.253\log q-\sum_{p|q}\frac{\log p}{p-1}+2.409+\frac{1.638}{\varphi(q)}\Big)\sqrt{x}+(1.58\log(qT)+16.08)\frac{x^{3/2}}{hT^2}\\&+\Big(\frac{1}{\pi}\log(q^2T)\log T+3.9\log q+0.7+\frac{22.7}{\varphi(q)}\Big)\frac{h}{\sqrt{x}}\Big]\\&-4\Big(x^{3/2}+\frac{h^2}{4\sqrt{x}}\Big)\Big(1+\frac{2.89}{T}\Big)\frac{\log(qT)}{\pi hT}\\&-0.95\sqrt{x}-3.7\sqrt[3]{x}-\omega(q)\log(2x)-1.7\frac{h}{x}.\end{aligned}$$

We introduce a new parameter β defined as $hT=:\beta x$. Thus, the previous inequality becomes

$$\begin{aligned}\frac{1}{\sqrt{x}}\sum_{\substack{|p-x|<h\\p=a\,[q]}}\log p\ge&\frac{h/\sqrt{x}}{\varphi(q)}-\Big(1+\frac{2}{\pi\beta}+\frac{2}{\pi\beta^2}+\frac{4\cdot 2.89}{\pi\beta T}\Big)\log(qT)-0.253\log q\\&-\Big(\frac{1}{\pi}\log(q^2T)\log T+3.9\log q+0.7+\frac{22.7}{\varphi(q)}+\frac{1.58\log(qT)+16.08}{\beta^2}+\Big(1+\frac{2.89}{T}\Big)\frac{\log(qT)}{\pi T}\Big)\frac{h}{x}\\&-1.53-\sum_{p|q}\frac{\log p}{p-1}-\frac{1.638}{\varphi(q)}-\frac{2\log(2\pi)}{\pi\beta}\Big(1-\frac{1}{\beta}\Big)-3.7x^{-1/6}-\omega(q)\frac{\log(2x)}{\sqrt{x}}-1.7\frac{h}{x^{3/2}}.\end{aligned}\tag{5.1}$$

We simplify this formula by noticing that for $\beta\ge 20$ and $x\ge(10\varphi(q)\log q)^2$ (unfortunately we cannot hope to prove anything as strong as this one, so that these assumptions will be satisfied), the function appearing in the last line is larger than -2 for $q\ge 18$ (we use the assumption $\frac{h}{x}\le\frac{5}{6}$ to bound $\frac{h}{x^{3/2}}$ with $\frac{5/6}{\sqrt{x}}$, and when $q\ge 800$ we apply the bounds $\omega(q)\le\log q$ and $\varphi(q)\ge\sqrt{q}$). Thus we have

$$\begin{aligned}\frac{1}{\sqrt{x}}\sum_{\substack{|p-x|<h\\p=a\,[q]}}\log p\ge&\frac{h/\sqrt{x}}{\varphi(q)}-\Big(1+\frac{2}{\pi\beta}+\frac{2}{\pi\beta^2}+\frac{4\cdot 2.89}{\pi\beta T}\Big)\log(qT)-0.253\log q-2\\&-\Big(\frac{1}{\pi}\log(q^2T)\log T+3.9\log q+0.7+\frac{22.7}{\varphi(q)}+\frac{1.58\log(qT)+16.08}{\beta^2}+\Big(1+\frac{2.89}{T}\Big)\frac{\log(qT)}{\pi T}\Big)\frac{h}{x}.\end{aligned}\tag{5.2}$$

We introduce three nonnegative parameters α, δ and ρ, and we further set

$$h = \varphi(q)(\alpha\log x+\delta\log q+\rho)\sqrt{x}, \qquad T = \frac{\beta}{\varphi(q)}\frac{\sqrt{x}}{\alpha\log x+\delta\log q+\rho}.$$

For the first part of the theorem, that is, the existence of a prime $p = a \pmod q$ with $|p-x| \le h$, it is sufficient to prove that the function appearing on the right hand side of (5.2) is positive. This happens when

$$(1-F)(\alpha\log x+\delta\log q+\rho) > G \tag{5.3}$$

where

$$F(q,x) := \Big(\frac{\log(q^2T)\log T}{\pi}+3.9\log q+0.7+\frac{22.7}{\varphi(q)}+\frac{1.58\log(qT)+16.08}{\beta^2}+\Big(1+\frac{2.89}{T}\Big)\frac{\log(qT)}{\pi T}\Big)\frac{\varphi(q)}{\sqrt{x}},$$

$$G(q,x) := \Big(1+\frac{2}{\pi\beta}+\frac{2}{\pi\beta^2}+\frac{4\cdot 2.89}{\pi\beta T}\Big)\log(qT)+0.253\log q+2.$$

We still have to make a choice for β, for which we have two different requirements.
CASE 1. Consider $x\to\infty$, for a fixed q. Then $\log T\sim\frac{1}{2}\log x$, as soon as $\log\beta = o(\log x)$. Thus $F \ll \frac{\log^2 x}{\sqrt{x}}$, and to prove (5.3) we need

$$\alpha\log x+\rho > \Big(\frac{1}{2}+\frac{1}{\pi\beta}+\frac{1}{\pi\beta^2}\Big)\log x+O(1),$$

not uniformly in q and in the other parameters. Thus we need

$$\alpha > \frac{1}{2}+\frac{1}{\pi\beta}+\frac{1}{\pi\beta^2},$$

and we can improve this bound to $\alpha\ge\frac{1}{2}$ if we assume that $\beta\asymp\log x$, at the cost of increasing ρ.
CASE 2. Consider $q\to\infty$, and $x = x_0(q) = (m\varphi(q)\log q)^2$ for some constant m. Then

$$T = \frac{m\beta}{2\alpha+\delta}+O\Big(\frac{\log\log q}{\log q}\Big),$$

not uniformly in α, δ, ρ and m. In particular, it stays bounded if we assume that β is bounded, and

$$F = \frac{1}{m}\Big(\frac{2}{\pi}\log T+3.9+\frac{1.58}{\beta^2}+\Big(1+\frac{2.89}{T}\Big)\frac{1}{\pi T}+O\Big(\frac{1}{\log q}\Big)\Big).$$

Thus F is small if m is large enough, and (5.3) is implied by

$$(2\alpha+\delta)\log q+\rho > (1-F)^{-1}\Big(1.253+\frac{2}{\pi\beta}+\frac{2}{\pi\beta^2}+\frac{4\cdot 2.89}{\pi\beta T}\Big)\log q+O(1),$$

because $\varphi(q)\log q\gg q$. Thus it is sufficient to have

$$2\alpha+\delta\ge\Big(1-\frac{1}{m}\Big(\frac{2}{\pi}\log T+3.9+\frac{1.58}{\beta^2}+\Big(1+\frac{2.89}{T}\Big)\frac{1}{\pi T}\Big)\Big)^{-1}\Big(1.253+\frac{2}{\pi\beta}+\frac{2}{\pi\beta^2}+\frac{4\cdot 2.89}{\pi\beta T}\Big),$$

at the cost of increasing ρ.

In order to meet both requirements for β we set

$$\beta = \ell\log\Big(\frac{\sqrt{x}}{\varphi(q)\log q}\Big), \tag{5.4}$$

for a suitable constant $\ell>0$ that we will fix later. In this way we can set $\alpha = 1/2$, and δ will be close to 0.253, specifically: $|\delta-0.253|\ll\frac{\log(\ell m)}{m}+\frac{1}{\ell\log m}$. Obviously we are interested in producing small values for m. Thus, for a fixed value of α and δ we select the value of ℓ producing the minimum m such that (δ,ℓ,m) satisfies the requirements.
If one is interested mainly in the q aspect, then one can select $\alpha = 1.253/2$; in this way δ can be chosen arbitrarily small if m and ℓ are large enough, and the value $\delta = 0$ is possible for every $\alpha > 1.253/2$. Possible choices are in Table 3.

The previous argument has showed how we have to set β, and what we can expect to be able to prove. However, in order to get a true proof we need to convert (5.3) into something decreasing in x when all other parameters are fixed, because only in this way can we prove the claim for all

$x \geq x_0$ by testing it only in x_0.
We notice that according to our definitions both β and T increase as functions of x, at least for $x \geq 10$. Moreover, setting $u := \sqrt{x}$, one sees that $\frac{1}{u}\log^2 T$ decreases if and only if

$$\frac{2\log T}{u}\frac{d\log T}{du} \leq \frac{\log^2 T}{u^2} \iff 2u\frac{d\log T}{du} \leq \log T \iff 2[\frac{1}{\log u}+1-\frac{2\alpha}{(2\alpha\log u+\delta\log q+\rho)}] \leq \log T$$
$$\Leftarrow 2+\frac{4}{\log x} \leq \log T.$$

For $x \geq 100$, this is true whenever $T \geq 20$. This suffices to prove that in this range $F(q,x)$ decreases as a function of x. Unfortunately this is false for G, thus we have to modify it into a new $\overline{G}$ having a better behavior in x and such that $\overline{G} \geq G$ so that

$$(1-F)(\alpha\log x+\delta\log q+\rho) > \overline{G} \tag{5.5}$$

implies (5.3).
Firstly, we notice that for $x =: u^2$ moderately larger than 100, the function $\frac{1}{u}\log^2 T\log u$ decreases as well. In fact, this happens if and only if

$$\frac{2\log T\log u}{u}\frac{d\log T}{du}+\frac{\log^2 T}{u^2} \leq \frac{\log^2 T\log u}{u^2} \iff 2u\frac{d\log T}{du}+\frac{\log T}{\log u} \leq \log T \tag{5.6}$$
$$\iff 2\Big[\frac{1}{\log u}+1-\frac{2\alpha}{(2\alpha\log u+\delta\log q+\rho)}\Big] \leq \Big[1-\frac{1}{\log u}\Big]\log T \Leftarrow 2\frac{\log x+2}{\log x-2} \leq \log T$$

and for $x \geq 23000$ this is true whenever $T \geq 20$, once again. This proves that in this range also $F(q,x)\log x$ decreases as a function of x. Secondly, recalling our setting for T and β, we see that

$$qT = \frac{q\ell\sqrt{x}}{\varphi(q)}\frac{\log\big(\frac{\sqrt{x}}{\varphi(q)\log q}\big)}{\alpha\log x+\delta\log q+\rho} \leq \frac{q\ell\sqrt{x}}{2\alpha\varphi(q)}.$$

We use this bound to substitute $\log(qT)$ in G, producing

$$\overline{G}(q,x) := \Big(1+\frac{2}{\pi\beta}+\frac{2}{\pi\beta^2}+\frac{4\cdot 2.89}{\pi\beta T}\Big)\log\Big(\frac{q\ell\sqrt{x}}{2\alpha\varphi(q)}\Big)+0.253\log q+2.$$

With this $\overline{G}$, Inequality (5.5) may be written as

$$\Big(\alpha-\frac{1}{2}\Big)\log x+(1-F)(\delta\log q+\rho) > \alpha F\log x+\Big(\frac{1}{\pi\beta}+\frac{1}{\pi\beta^2}+\frac{2\cdot 2.89}{\pi\beta T}\Big)\log x \tag{5.7}$$
$$+\Big(1+\frac{2}{\pi\beta}+\frac{2}{\pi\beta^2}+\frac{4\cdot 2.89}{\pi\beta T}\Big)\log\Big(\frac{q\ell}{2\alpha\varphi(q)}\Big)+0.253\log q+2.$$

When $\alpha \geq 1/2$, the function appearing on the left hand side increases in x (whenever $x \geq 100$, $T \geq 20$), while the function on the right hand side decreases in x (whenever $x \geq 23000$, $T \geq 20$). This shows that if $x \geq 23000$ and $\alpha \geq 1/2$, we can check (5.7) (and hence (5.5), since they are equivalent) for $x \geq x_0$ by testing it for x_0.
We also have to satisfy the assumption

$$\frac{1}{T} = \frac{\varphi(q)}{\sqrt{x}}\frac{\alpha\log x+\delta\log q+\rho}{\ell\log(\frac{\sqrt{x}}{\varphi(q)\log q})} \leq \frac{1}{20}, \tag{5.8}$$

and, since we have assumed $h \leq 5x/6$ in several places, we need also

$$\frac{h}{x} = \frac{\varphi(q)}{\sqrt{x}}(\alpha\log x+\delta\log q+\rho) \leq \frac{5}{6}, \tag{5.9}$$

where again the functions appearing on the left hand sides decrease in x (for $x \geq e^2$).

The combinations of values for the parameters α, δ, and m in Table 3 are in some sense unrealistic: they can be satisfied only for extremely large q. In order to have a claim which could be proved for every q we have to increase m and choose ρ accordingly. Our choices are in Table 4, and for every choice of the parameters appearing there we verify by direct computation that all requirements are satisfied by $x = x_0(q) := (m\varphi(q)\log q)^2$ when $3 \le q \le q_0$, with just a few exceptions which are in Table 5 and for which we have to test the claim directly for $x \in [x_0(q), x(q)]$.

To deal with larger q's, we set $x = x_0(q) = (m\varphi(q)\log q)^2$ in (5.5), but, again, we have to modify $F_0(q) = F(q, x_0(q))$ and $\overline{G}_0(q) = \overline{G}(q, x_0(q))$ in order to produce an inequality which will hold for every $q \ge q_0$ when verified for q_0. For this purpose we introduce

$$\tilde{F}_0(q) := \frac{1}{m}\Big(\frac{\log(q^2T_+)\log T_+}{\pi\log q}+3.9+\frac{0.7}{\log q}+\frac{22.7}{q}+\frac{1.58\log(qT_+)+16.08}{\beta_0^2\log q}+\Big(1+\frac{2.89}{T_-}\Big)\frac{\log(qT_+)}{\pi T_-\log q}\Big)$$

and

$$\tilde{G}_0(q) := \Big(1+\frac{2}{\pi\beta_0}+\frac{2}{\pi\beta_0^2}+\frac{4{\cdot}2.89}{\pi\beta_0T_-}\Big)\log\Big(\frac{\ell mq\log q}{2\alpha}\Big)+0.253\log q+2$$

with

$$\beta_0 := \ell\log m, \qquad T_- := \frac{\beta_0 m\log q}{2\alpha\log(mq\log q)+\delta\log q+\rho}, \qquad T_+ := \frac{\beta_0 m}{2\alpha+\delta}.$$

Then for $x = x_0(q)$ one has $\beta = \beta_0$, $T_- \le T \le T_+$, $\tilde{F}_0(q) \ge F_0(q)$ and $\tilde{G}_0(q) \ge \overline{G}_0(q)$, so that (5.5) for $x = x_0(q)$ holds for sure if

$$(1-\tilde{F}_0(q))(2\alpha\log(mq)+\delta\log q+\rho) \ge \tilde{G}_0(q). \tag{5.10}$$

We notice that $\tilde{F}_0(q)$ and $1/T_-$ decrease in q, thus (5.10) may be written as

$$\Big((1-\tilde{F}_0(q))(2\alpha+\delta)-\Big(1.253+\frac{2}{\pi\beta_0}+\frac{2}{\pi\beta_0^2}+\frac{4{\cdot}2.89}{\pi\beta_0T_-}\Big)\Big)\log q \ge \Big(1+\frac{2}{\pi\beta_0}+\frac{2}{\pi\beta_0^2}+\frac{4{\cdot}2.89}{\pi\beta_0T_-}\Big)\log\log q$$
$$+(\tilde{F}_0(q)-1)(2\alpha\log m+\rho)+\Big(1+\frac{2}{\pi\beta_0}+\frac{2}{\pi\beta_0^2}+\frac{4{\cdot}2.89}{\pi\beta_0T_-}\Big)\log\Big(\frac{\ell m}{2\alpha}\Big)+2$$

i.e., as

$$A\log q-B\log\log q-C \ge 0 \tag{5.11}$$

where A increases in q and B and C decrease. The function on the left hand side is increasing in q when

$$A'\log q-B'\log\log q+\frac{A}{q}-\frac{B}{q\log q}-C' > 0,$$

and for this it is sufficient to have

$$-\tilde{F}_0'(q)(2\alpha+\delta)\log q+\frac{A}{q}-\frac{B}{q\log q} > 0.$$

Since

$$-\tilde{F}_0'(q) \ge \frac{S}{q\log^2 q} \quad\text{with}\quad S := \frac{1}{m}\Big(\frac{\log^2 T_+}{\pi}+0.7+\frac{1.58\log T_++16.08}{\beta_0^2}\Big), \tag{5.12}$$

in order to have a monotonous behavior of (5.11) it is sufficient to have

$$A\log q \ge B-S(2\alpha+\delta). \tag{5.13}$$

In this way we see that if (5.11) holds for a certain $q = q_0$ large enough to satisfy (5.13), then it is proved for every $q \ge q_0$. Moreover, we notice that inequalities (5.8) and (5.9) in $x_0(q) = (m\varphi(q)\log q)^2$ are satisfied as soon as

$$\frac{1}{T_-} = \frac{2\alpha\log(mq\log q)+\delta\log q+\rho}{\ell m\log m\log q} \le \frac{1}{20}, \tag{5.14}$$

and

$$\frac{2\alpha\log(mq\log q)+\delta\log q+\rho}{m\log q} \le \frac{5}{6}. \tag{5.15}$$

Thus (finally!) we have produced the test we were looking for: we search for a q_0 satisfying (5.10), (5.13), (5.14) and (5.15). Then everything is proved for $q \ge q_0$. Our computations show that the values of q_0 appearing in Table 4 pass this test.

For $q \le q_0$ and $x \in [x_0(q), x(q)]$ we use the mighty computer procedure `Check1` described below so that now the proof of the first claim of the theorem is complete.

For the second part of the theorem, i.e. the claim ensuring that if we increase α by one then there are at least $\sqrt{x}$ primes $p = a \pmod q$ in $|p-x| \le h$, we proceed in similar way. Indeed, the inequality

$$\log(x+h) \sum_{\substack{|p-x|<h\\ p=a\,[q]}} 1 \ge \sum_{\substack{|p-x|<h\\ p=a\,[q]}} \log p,$$

allows to prove the claim by proving that the function appearing on the right hand side of (5.1) is larger than $\log(x+h)$. This amounts to modifying (5.3) into

$$(1-F)((\alpha+1)\log x+\delta\log q+\rho) > G+\log(x+h),$$

i.e. into

$$(1-F)(\alpha\log x+\delta\log q+\rho) > G+F\log x+\log(1+h/x),$$

where F and G are defined as before (but with $\alpha+1$ instead of α in the definition of T). We simplify the inequality recalling that we are assuming that $h/x \le 5/6$. Moreover, we once again use $\overline{G}$ instead of G in order to get an inequality which is proved for all x larger than x_0 when it is proved for x_0: by (5.6) this happens at least whenever $x \ge 23000$. Thus it is sufficient to prove that

$$(1-F)(\alpha\log x+\delta\log q+\rho) > \overline{G}+F\log x+\log(11/6). \tag{5.16}$$

Setting $x = x_0'(q) = (m'\varphi(q)\log q)^2$, for a diverging q the inequality becomes

$$2(\alpha-(\alpha+1)F)\log(m'\varphi(q)\log q)+(1-F)\delta\log q+O(1) > \Big(1.253+\frac{2}{\pi\beta}+\frac{2}{\pi\beta^2}+\frac{4\cdot 2.89}{\pi\beta T}\Big)\log q+O(1).$$

If we assume that $F \le \alpha/(\alpha+1)$, then the lower bound $\varphi(q)\log q \ge q$ shows that this is

$$(2\alpha+\delta-(2\alpha+\delta+2)F)\log q+O(1) > \Big(1.253+\frac{2}{\pi\beta}+\frac{2}{\pi\beta^2}+\frac{4\cdot 2.89}{\pi\beta T}\Big)\log q+O(1),$$

which forces us to select α, δ, l and m' in such a way that

$$2\alpha+\delta-(2\alpha+\delta+2)F \ge 1.253+\frac{2}{\pi\beta_0}+\frac{2}{\pi\beta_0^2}+\frac{4\cdot 2.89}{\pi\beta_0 T}$$

with $\beta_0 = \ell'\log m'$, $T = \frac{\beta m'}{2\alpha+\delta}$ and

$$F = \frac{1}{m'}\Big(\frac{2}{\pi}\log T+3.9+\frac{1.58}{\beta_0^2}+\Big(1+\frac{2.89}{T}\Big)\frac{1}{\pi T}\Big).$$

This implies that for the combinations of α and δ we have already considered before we have to select for ℓ' and m' the values in Table 3. As before, in order to get a statement provable for all q we have to further increase m', for which we select the values in Table 4. Now, for every choice of the parameters in Table 4 we verify by direct computation that (5.8), (5.9) (substituting α, m and ℓ with $\alpha+1$, m' and ℓ') and (5.16) are satisfied by $x = x_0'(q) := (m'\varphi(q)\log q)^2$ when $3 \le q \le q_0'$, with just a few exceptions which are in Table 6 and for which we test the claim directly for $x \in [x_0'(q), x'(q)]$. This proves this part of the theorem for $q \le q_0'$.

To deal with larger q's, we set $x = x_0'(q) = (m'\varphi(q)\log q)^2$ in (5.16), but, again, we substitute F and $\overline{G}$ with $\tilde{F}_0(q)$ and $\tilde{G}_0(q)$, getting

$$(1-\tilde{F}_0(q))(2\alpha\log(m'\varphi(q)\log q)+\delta\log q+\rho) > \tilde{G}_0(q)+2\tilde{F}_0(q)\log(m'\varphi(q)\log q)+\log(11/6).$$

Assuming

$$\tilde{F}_0(q) \le \alpha/(\alpha+1), \tag{5.17}$$

the inequality is implied by

$$(1-\tilde{F}_0(q))(2\alpha\log(m'q)+\delta\log q+\rho) \ge \tilde{G}_0(q)+2\tilde{F}_0(q)\log(m'q)+\log(11/6), \tag{5.18}$$

which is what we get substituting $\varphi(q)\log q$ with its upper bound q. We write this inequality as

$$\Big(2\alpha+\delta-(2\alpha+\delta+2)\tilde F_0(q)-\Big(1.253+\frac{2}{\pi\beta_0}+\frac{2}{\pi\beta_0^2}+\frac{4\cdot 2.89}{\pi\beta_0 T_-}\Big)\Big)\log q$$
$$\geq\Big(1+\frac{2}{\pi\beta_0}+\frac{2}{\pi\beta_0^2}+\frac{4\cdot 2.89}{\pi\beta_0 T_-}\Big)\log\log q+(\tilde F_0(q)-1)(2\alpha\log m'+\rho)$$
$$+\Big(1+\frac{2}{\pi\beta_0}+\frac{2}{\pi\beta_0^2}+\frac{4\cdot 2.89}{\pi\beta_0 T_-}\Big)\log\Big(\frac{\ell m'}{2\alpha}\Big)+2\tilde F_0(q)\log m'+2+\log(11/6),$$

i.e. as

$$\mathcal{A}\log q-\mathcal{B}\log\log q-\mathcal{C}\geq 0 \tag{5.19}$$

where $\mathcal{A}$ increases in q and $\mathcal{B}$ and $\mathcal{C}$ decrease. It is monotonous in q when

$$\mathcal{A}'\log q-\mathcal{B}'\log\log q+\frac{\mathcal{A}}{q}-\frac{\mathcal{B}}{q\log q}-\mathcal{C}'>0,$$

and for this it is sufficient to have

$$-\tilde F_0'(q)(2\alpha+\delta+2)\log q+\frac{\mathcal{A}}{q}-\frac{\mathcal{B}}{q\log q}>0.$$

By (5.12), in order to have a monotonous behavior of (5.19) it is sufficient to have

$$\mathcal{A}\log q\geq\mathcal{B}-S(2\alpha+\delta+2). \tag{5.20}$$

In this way we see that if (5.19) holds for a certain $q=q_0'$ large enough to satisfy (5.20), then it is proved for every $q\geq q_0'$. Thus we have produced the test we were looking for: we search for the q_0' satisfying (5.14), (5.15) (substituting α, m and ℓ with $\alpha+1$, m' and ℓ') (5.17), (5.18) and (5.20). Then everything is proved for $q\geq q_0'$. Our computations show that each q_0' appearing in Table 4 pass this test, so that also the proof of the second claim of the theorem is completed.

For $q\leq q_0'$ and $x\in[x_0'(q),x'(q)]$ we use the mighty computer procedure `CheckSqrt` described below so that now the proof of the theorem is complete.

Remark. The procedures `Check1` and `CheckSqrt` check more than what is needed: they detect the existence of prime numbers in $[x-h,x]$ except for the initial x's.

6. Proof of Theorems 1.2 and 1.3

We keep the notations

$$h=\varphi(q)(\alpha\log x+\delta\log q+\rho)\sqrt{x},\qquad T=\frac{\beta}{\varphi(q)}\frac{\sqrt{x}}{\alpha\log x+\delta\log q+\rho},$$

but we make a different choice for β. In fact, the first two negative terms $(1+\frac{2}{\pi\beta})\log(qT)$ in (5.1), up to terms of lower order in β, are

$$\log\beta+\frac{\log(q^2x)}{\pi\beta}.$$

This expression reaches its minimum when

$$\beta=\frac{1}{\pi}\log(q^2x),$$

which is how we set β now. This choice puts restrictions on α and δ: to control the terms appearing in the equations below we need to have $\alpha\geq 1/2$, $\delta>0$ and $2\alpha+\delta\geq 2$. Since we are interested in furnishing small values for α and δ, this leaves us with the range $\alpha\in[1/2,1)$ and $\delta=2-2\alpha$. In this range we pick the case $\alpha=1/2$, $\delta=1$, which is a natural choice; the interested reader will be able to complete the similar computations needed for any other setting of α and δ. Thus, our settings are:

$$h=\varphi(q)(\tfrac12\log(q^2x)+\rho)\sqrt{x},\qquad T=\frac{\sqrt{x}}{\pi\varphi(q)}\frac{\log(q^2x)}{\frac12\log(q^2x)+\rho}.$$

As a consequence we have

$$T \le \frac{2}{\pi}\frac{\sqrt{x}}{\varphi(q)}, \tag{6.1}$$

$$\log(qT) \le \frac{1}{2}\log(q^2x)+\log\Big(\frac{2}{\pi\varphi(q)}\Big), \tag{6.2}$$

$$\frac{2\log(qT)}{\pi\beta} \le 1. \tag{6.3}$$

Moreover,

$$-\sum_{p|q}\frac{\log p}{p-1}-\log(\varphi(q)) = -\log q-\sum_{p|q}\Big[\frac{\log p}{p-1}+\log\Big(1-\frac{1}{p}\Big)\Big] \le -\log q. \tag{6.4}$$

The function appearing on the right hand side of (5.1) is surely positive when

$$\begin{aligned}\frac{1}{2}\log(q^2x)+\rho \ge{}& \log(qT)+\frac{2}{\pi\beta}\log(qT)+0.253\log q+1.53-\sum_{p|q}\frac{\log p}{p-1}+\frac{1.638}{\varphi(q)}\\ &+\frac{2\log(2\pi)}{\pi\beta}\Big(1-\frac{1}{\beta}\Big)+\frac{2}{\pi\beta^2}\log(qT)+\frac{4{\cdot}2.89}{\pi\beta}\frac{\log(qT)}{T}\\ &+\Big(\frac{1}{\pi}\log(q^2T)\log T+3.9\log q+0.7+\frac{22.7}{\varphi(q)}\Big)\frac{h}{x}+(1.58\log(qT)+16.08)\frac{h}{\beta^2x}\\ &+3.7x^{-1/6}+\omega(q)\frac{\log(2x)}{\sqrt{x}}+\frac{h}{\pi x}\Big(1+\frac{2.89}{T}\Big)\frac{\log(qT)}{T}+1.7\frac{h}{x^{3/2}}.\end{aligned}$$

Using (6.2) for the first $\log(qT)$, (6.3) for the terms $\frac{1}{\beta}\log(qT)$, (6.1) for $\log(q^2T)\log T$, and (6.4), we deduce that it is sufficient to have

$$\begin{aligned}\rho \ge{}& \log\Big(\frac{2}{\pi}\Big)+1-0.747\log q+1.53+\frac{1.638}{\varphi(q)}+\frac{2\log(2\pi)}{\pi\beta}\Big(1-\frac{1}{\beta}\Big)+\frac{1}{\beta}\\ &+\frac{2{\cdot}2.89}{T}+\Big(\frac{1}{\pi}\log\Big(\frac{2}{\pi}\frac{q^2\sqrt{x}}{\varphi(q)}\Big)\log\Big(\frac{2}{\pi}\frac{\sqrt{x}}{\varphi(q)}\Big)+3.9\log q+0.7+\frac{22.7}{\varphi(q)}\Big)\frac{h}{x}\\ &+\Big(\frac{1.58\pi}{2}+\frac{16.08}{\beta}\Big)\frac{h}{\beta x}+\frac{3.7}{x^{1/6}}+\omega(q)\frac{\log(2x)}{\sqrt{x}}+\frac{h}{\pi x}\Big(1+\frac{2.89}{T}\Big)\frac{\log(qT)}{T}+1.7\frac{h}{x^{3/2}}.\end{aligned}$$

In several places we have assumed $T \ge 20$, thus we can use this assumption to note that it implies

$$\frac{1}{\pi}\Big(1+\frac{2.89}{T}\Big)\frac{\log(qT)}{T} \le 0.02\log q+0.06.$$

We further assume $x \ge (8\varphi(q)\log q\log\log q)^2$ to bound

$$\log\Big(\frac{2}{\pi}\Big)+2.53+\frac{1.638}{\varphi(q)}+\frac{2\log(2\pi)}{\pi\beta}\Big(1-\frac{1}{\beta}\Big)+\frac{1}{\beta}+\frac{2{\cdot}2.89}{T}+\frac{3.7}{x^{1/6}}+\omega(q)\frac{\log(2x)}{\sqrt{x}}+1.7\frac{h}{x^{3/2}} \tag{6.5}$$

with $\mathcal{E}(q)$, which is 9.3 when $q \le 12$ and 4 otherwise. Hence it is sufficient to have

$$\begin{aligned}\rho \ge{}&\mathcal{E}(q)-0.747\log q\\ &+\Big(\frac{1}{\pi}\log\Big(\frac{2}{\pi}\frac{q^2\sqrt{x}}{\varphi(q)}\Big)\log\Big(\frac{2}{\pi}\frac{\sqrt{x}}{\varphi(q)}\Big)+3.92\log q+0.76+\frac{22.7}{\varphi(q)}\Big)\frac{h}{x}+\Big(\frac{1.58\pi}{2}+\frac{16.08}{\beta}\Big)\frac{h}{\beta x}.\end{aligned} \tag{6.6}$$

Recalling the definitions of h and β, (6.6) becomes:

$$(1-F(q,x))\rho \ge G(q,x) \tag{6.7}$$

with

$$\begin{aligned}F(q,x) :={}&\Big(\frac{1}{\pi}\log\Big(\frac{2}{\pi}\frac{q^2\sqrt{x}}{\varphi(q)}\Big)\log\Big(\frac{2}{\pi}\frac{\sqrt{x}}{\varphi(q)}\Big)+3.92\log q+0.76+\frac{22.7}{\varphi(q)}\Big)\frac{\varphi(q)}{\sqrt{x}}\\ &+\Big(\frac{1.58}{2}+\frac{16.08}{\log(q^2x)}\Big)\frac{\pi^2\varphi(q)}{\log(q^2x)\sqrt{x}},\end{aligned}$$

$$G(q,x):=\mathcal{E}(q)+\Big(\frac{1}{\pi}\log\Big(\frac{2}{\pi}\frac{q^2\sqrt{x}}{\varphi(q)}\Big)\log\Big(\frac{2}{\pi}\frac{\sqrt{x}}{\varphi(q)}\Big)+3.92\log q\Big)\log(q\sqrt{x})\frac{\varphi(q)}{\sqrt{x}}-0.747\log q$$
$$+\Big(0.76+\frac{22.7}{\varphi(q)}\Big)\log(q^2x)\frac{\varphi(q)}{2\sqrt{x}}+\Big(\frac{1.58}{2}+\frac{16.08}{\log(q^2x)}\Big)\frac{\pi^2\varphi(q)}{2\sqrt{x}}.$$

We notice that $F(q,x)$ and $G(q,x)$ decrease as a function of x (hence there is no need to change G, in this case), at least for $x\geq e^6=403.42\ldots$. Thus, if (6.7) holds for fixed ρ and q, for a given $x_0(q)$, then it holds for any $x\geq x_0(q)$ for the same ρ and q.
Moreover we have to satisfy the assumptions

$$\frac{1}{T}=\frac{h}{\beta x}=\frac{\pi\varphi(q)}{\sqrt{x}}\Big(\frac{1}{2}+\frac{\rho}{\log(q^2x)}\Big)\leq\frac{1}{20} \tag{6.8}$$

and

$$\frac{h}{x}=\frac{\varphi(q)}{\sqrt{x}}\Big(\frac{1}{2}\log(q^2x)+\rho\Big)\leq\frac{5}{6}, \tag{6.9}$$

where again the functions appearing on the left hand side decrease in x.
We verify by direct computation that all these requirements are satisfied for $\rho=15$ by any $x\geq x_0(q)$ with $x_0(q)$ given in Table 7, when $q\leq 660$. For this purpose, we use a variant of Procedure `Check1`.

To deal with larger q's, we choose $x_0(q):=(m\varphi(q)\ell(q))^2$, where we set $\ell(q):=\log q\log\log q$ to simplify the notation. To select a suitable value for m we note that $G_0(q):=G(q,x_0(q))$ stays bounded if and only if

$$\frac{1}{\pi}\log\Big(\frac{q^2\sqrt{x_0(q)}}{\varphi(q)}\Big)\log\Big(\frac{\sqrt{x_0(q)}}{\varphi(q)}\Big)\log(q\sqrt{x_0(q)})\frac{\varphi(q)}{\sqrt{x_0(q)}}-0.747\log q$$

is bounded, and that this happens if and only if $\frac{4}{\pi m}<0.747$. This shows that any m larger than 2, say, is allowed when $q\geq q_0(m)$ is large enough. With this choice of $x_0(q)$, inequalities (6.8) and (6.9) are satisfied as soon as

$$\frac{\pi}{m\ell(q)}\Big(\frac{1}{2}+\frac{\rho/2}{\log(mq\varphi(q)\ell(q))}\Big)\leq\frac{1}{20}, \tag{6.10}$$

and

$$\frac{1}{m\ell(q)}\big(\log(mq\varphi(q)\ell(q))+\rho\big)\leq\frac{5}{6}. \tag{6.11}$$

To deal with (6.7), (6.10) and (6.11) for arbitrary q we substitute there the arithmetical function $\varphi(q)$ with its upper bound q or its lower bound $\sqrt{q}$ in order to produce in any case upper-bounds $\tilde{F}_0(q)$ and $\tilde{G}_0(q)$ for $F_0(q):=F(q,x_0(q))$ and $G_0(q)$ respectively, and for the function to the left hand side of (6.10). In this way (6.7) changes into

$$(1-\tilde{F}_0)\rho\geq\tilde{G}_0. \tag{6.12}$$

As for Theorem 1.1, functions $\tilde{F}_0$ and those we get from (6.10) and (6.11) are decreasing in q, while this remains false for $\tilde{G}_0$. However, contrary to the situation for Theorem 1.1 the parameters α $(=\frac{1}{2})$, δ $(=1)$ and ρ $(=15)$ are now fixed, thus we can verify directly that $\rho\geq-\tilde{G}_0'/\tilde{F}_0'$ for any $q\geq 3$ and any integer $8\leq m\leq 20$. This shows that for these parameters $(1-\tilde{F}_0)\rho-\tilde{G}_0$ is increasing in the full range for q.

In this way we can conclude that when $8\leq m\leq 20$ all conditions we have to test become monotonous in their dependence of x and q, so that we can prove them for $x\geq x_0(q)$ and $q\geq q_0(m)$ by proving them for $x=x_0(q)$ and $q=q_0(m)$. We have collected some results in Table 8, for several values of m. We see that the value $m=8$ produces a small enough $q_0(m)$, hence we have selected it, as reported in Theorem 1.2. To complete the proof of Theorem 1.2 we still need to test the claim for $3\leq q<660$ and x in the interval $[(8\varphi(q)\ell(q))^2,x_0(q)]$ with $x_0(q)$ given in Table 7. For this purpose we use an analogue of Procedure `Check`.

For the second part of the theorem it is sufficient to prove that the right hand side of (5.1) is larger than $\log(x+h)$ when we increasing h to $h+\varphi(q)\sqrt{x}\log x$. This modifies (6.7) into

$$(1-F(q,x))\rho \geq G(q,x)+F(q,x)\log x+\log(1+5/6) =: G_s(q,x).$$

We proceed as before. In fact, both sides are decreasing as a function of x. Thus, we verify by direct computation that all these requirements are satisfied for $\rho = 15$ by any $x \geq x_0'(q)$ with $x_0'(q)$ given in Table 7, when $q \leq 1320$.

Again, we choose $x_0(q) := (m'\varphi(q)\ell(q))^2$, producing

$$(1-F(q,x_0'(q)))\rho \geq G(q,x_0'(q))+F(q,x_0'(q))\log(x_0'(q))+\log(1+5/6) = G_s(q,x_0'(q)). \tag{6.13}$$

In order to have $G_s(q,x_0(q))$ bounded it is necessary that $\frac{8}{\pi m'} < 0.747$, thus any $m' \geq 4$ suffices. With this choice of $x_0'(q)$, inequalities (6.8) and (6.9) are satisfied as soon as

$$\frac{\pi}{m'\ell(q)}\Big(\frac{1}{2}+\frac{\rho/2+\log(m'\varphi(q)\ell(q))}{\log(m'q\varphi(q)\ell(q))}\Big) \leq \frac{1}{20}, \tag{6.14}$$

and

$$\frac{1}{m'\ell(q)}\big(\log(m'q\varphi(q)\ell(q))+2\log(m'\varphi(q)\ell(q))+\rho\big) \leq \frac{5}{6}. \tag{6.15}$$

To deal with (6.13), (6.14) and (6.15) for arbitrary q we substitute there the arithmetical function $\varphi(q)$ with its upper bound q or its lower bound $\sqrt{q}$ in order to produce in any case upper-bounds $\tilde{F}(q,x_0(q))$ and $\tilde{G}_s(q,x_0(q))$ for $F(q,x_0(q))$ and $G_s(q,x_0(q))$ respectively, and for the function on the left hand side of (6.14). In this way (6.13) changes into

$$(1-\tilde{F})\rho \geq \tilde{G}_s.$$

Functions $\tilde{F}$, and those we get from (6.14) and (6.15) are evidently decreasing in q, but this is still false for $\tilde{G}_s$. However, $(1-\tilde{F})\rho-\tilde{G}_s$ is decreasing if and only if $\rho \geq -\tilde{G}_s'/\tilde{F}'$ and for $\rho = 15$ this holds for any $q \geq 3$ if $m' \geq 10$. In this way we can conclude that when $m' \geq 10$ all conditions we have to test become monotonous in their dependence of x and q, so that we can prove them for $x \geq x_0'(q)$ and $q \geq q_0'(m')$ by proving them for $x = x_0'(q)$ and $q = q_0'(m')$. We have collected some results in Table 8, for several values of m'. Unfortunately, the computations show that any value of m' smaller than 15 would produce an extremely large $q_0(m')$. As a consequence we have selected $m' = 15$, as reported in Theorem 1.2.

Lastly, it is easy to prove that $F(q,e^q)$ is smaller than 1 for $q \geq 10$ and that $G(q,e^q) \leq 0$ for $q \geq 220$, and $G_s(q,e^q) \leq 0$ for $q \geq 500$ and this proves Theorem 1.3 with $q \geq 220$ for the first claim and $q \geq 500$ for the second. The first (second) claim is extended to $q \geq 35$ ($q \geq 67$, respectively) keeping the true value of (6.5) in place of $\mathcal{E}(q)$ in the definition of $G(q,x)$.

7. Proof of Corollary 1.4

We can assume $q \geq 3$, because the claim for $q = 1$ and $q = 2$ follows from the analogous (and stronger) claim proved in [1, Cor. 4.1].

By Theorem 1.1 (case $\alpha = 1/2$, $\delta = 1$) we know that there is a prime congruent to a modulo q as soon as

$$(2n+\varphi(q)A)\varphi(q)A \geq \varphi(q)\sqrt{M}(24+\log(q^2M))$$

where $A := 12+2\log(qn)$ and $M := \frac{1}{2}[n^2+(n+\varphi(q)A)^2]$. Dividing by nA and setting $B := \frac{\varphi(q)}{n}A$, the inequality becomes

$$2+B \geq \sqrt{1+B+\frac{B^2}{2}}\Big(1+\frac{12+\log(1+B+\frac{B^2}{2})}{A}\Big),$$

i.e.,

$$\sqrt{\frac{4+4B+2B^2}{4+4B+B^2}}\Big(1+\frac{12+\log(1+B+\frac{B^2}{2})}{A}\Big) \leq 2.$$

Set $H := \sqrt{\frac{4+4B+2B^2}{4+4B+B^2}}$, and notice that it is an increasing function of B, and is bounded by $\sqrt{2}$. Hence the inequality may be written as

$$1+B+\frac{B^2}{2} \leq \exp\Big(A\Big(\frac{2}{H}-1\Big)-12\Big).$$

In terms of B this is solved by

$$B \leq \Big[2\exp\Big(A\Big(\frac{2}{H}-1\Big)-12\Big)-1\Big]^{1/2}-1,$$

but needs $2\exp(A\big(\frac{2}{H}-1\big)-12) \geq 1$. Recalling the definition of B, it means that

$$\varphi(q) \leq \frac{n}{A}\Big[\Big[2\exp\Big(A\Big(\frac{2}{H}-1\Big)-12\Big)-1\Big]^{1/2}-1\Big].$$

Recalling the definition of A, we see that for every fixed value of q, the quotient n/A increases with n. Hence $B = \varphi(q)\frac{A}{n}$ decreases with n, and $1/H$ (which decreases with B) increases with n. This shows that the function appearing on the right hand side increases as a function of n, for every fixed q, if $A(2/H-1) \geq 12$. As a consequence the inequalities hold true for $n \geq n_0$ as soon they hold for $n = n_0$. It is easy to prove that for $n \geq 8\varphi(q)\log q$ they hold for all $q \geq 3$.

8. Auxiliary tables

Table 3. Parameters for $q \to \infty$.

α	δ	m	ℓ	m'	ℓ'	α	δ	m	ℓ	m'	ℓ'
1/2	1	21	7	44	6	1.253/2	0.1	142	17	373	17
1/2	1/2	56	7	139	7	1	0	21	7	44	6
1/2	1/3	179	24	475	21	0.9	0	27	7	60	5
1.253/2	1	17	8	34	6	0.8	0	40	7	97	5
1.253/2	1/2	29	6	66	5	0.7	0	95	11	245	10
1.253/2	0.2	69	9	175	8	0.627	0	21236	1652	57287	1310

Table 4. Parameters

α	δ	ρ	m	ℓ	q_0	m'	ℓ'	q_0'
1/2	1	12	23	6.4	1947657	46	5.3	1984065
1/2	1/2	9	86	14	443235	188	11	2974713
1/2	1/3	9	1500	120	2293436	3500	190	2711303
1.253/2	1	14	18	7	7991888	34	5.7	6306843
1.253/2	1/2	9	34	7	3055181	74	6	920941
1.253/2	0.2	7	110	18	3287890	260	15	3790727
1.253/2	0.1	7	500	64	2878356	1500	66	999372
1	0	8	23	6.4	1972765	46	5.3	2001416
0.9	0	7	31	6	2617343	66	5	1294983
0.8	0	6	52	9	1987447	120	8	630195
0.7	0	5	200	16	1713915	500	26	958214
0.627	0	10	10^{10}	3480	10^{438}	10^{10}	4100	10^{438}

Table 5. Exceptions: for these q's the claim has to be tested in $[x_0(q), x(q)]$

$\alpha = 1/2$, $\delta = 1$, $\rho = 12$, $m = 23$, $\ell = 6.4$

q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$
3	2553	23000	6	6793	23000	9	91940	94714
4	4066	23000	7	72111	81124	10	44875	55094
5	21924	37494	8	36598	51147	12	52263	60595

$\alpha = 1/2$, $\delta = 1/2$, $\rho = 9$, $m = 86$, $\ell = 14$

q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$
3	35706	77348	4	56854	95500	6	94976	104272

$\alpha = 1/2$, $\delta = 1/3$, $\rho = 9$, $m = 1500$, $\ell = 120$: no exceptions

$\alpha = 1.253/2$, $\delta = 1$, $\rho = 14$, $m = 18$, $\ell = 7$

q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$
3	1564	174459	6	4160	23000	9	56311	59241
4	2490	190024	7	44166	50277	10	27485	35009
5	13428	565474	8	22416	31807	12	32009	38677

$\alpha = 1.253/2$, $\delta = 1/2$, $\rho = 9$, $m = 34$, $\ell = 7$

q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$
3	5580	24333	5	47910	62458	8	79978	87897
4	8886	29766	6	14844	34684			

$\alpha = 1.253/2$, $\delta = 0.2$, $\rho = 7$, $m = 110$, $\ell = 18$

q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$
3	58416	136773	4	93015	176298	6	155383	196485

$\alpha = 1.253/2$, $\delta = 0.1$, $\rho = 7$, $m = 500$, $\ell = 64$: no exceptions

$\alpha = 1$, $\delta = 0$, $\rho = 8$, $m = 23$, $\ell = 6.4$

q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$
3	2553	23000	6	6793	23000	10	44875	52243
4	4066	23000	8	36598	47072	12	52263	58690
5	21924	32725						

$\alpha = 0.9$, $\delta = 0$, $\rho = 7$, $m = 31$, $\ell = 6$

q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$
3	4639	23000	5	39828	47524	8	66487	69419
4	7387	23032	6	12340	28176			

$\alpha = 0.8$, $\delta = 0$, $\rho = 6$, $m = 52$, $\ell = 9$

q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$
3	13054	45973	5	112066	116443	6	34723	70349
4	20786	58793						

$\alpha = 0.7$, $\delta = 0$, $\rho = 5$, $m = 200$, $\ell = 16$

q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$	q	$x_0(q)$	$x(q)$
3	193111	283439	4	307489	391345			

TABLE 6. Exceptions: for these q's the claim has to be tested in $[x_0'(q), x'(q)]$

$\alpha = 1/2,\ \delta = 1,\ \rho = 12,\ m' = 46,\ \ell' = 5.3$								
q	$x_0'(q)$	$x'(q)$	q	$x_0'(q)$	$x'(q)$	q	$x_0'(q)$	$x'(q)$
3	10215	28413	4	16266	33887	6	27172	39233
$\alpha = 1/2,\ \delta = 1/2,\ \rho = 9,\ m' = 188,\ \ell' = 11$: no exceptions								
$\alpha = 1/2,\ \delta = 1/3,\ \rho = 9,\ m' = 3500,\ \ell' = 190$: no exceptions								
$\alpha = 1.253/2,\ \delta = 1,\ \rho = 14,\ m' = 34,\ \ell' = 5.7$								
q	$x_0'(q)$	$x'(q)$	q	$x_0'(q)$	$x'(q)$	q	$x_0'(q)$	$x'(q)$
3	5580	23000	4	8886	23000	6	14844	23000
$\alpha = 1.253/2,\ \delta = 1/2,\ \rho = 9,\ m' = 74,\ \ell' = 6$								
q	$x_0'(q)$	$x'(q)$	q	$x_0'(q)$	$x'(q)$	q	$x_0'(q)$	$x'(q)$
3	26437	53359	4	42095	65485	6	70320	76541
$\alpha = 1.253/2,\ \delta = 0.2,\ \rho = 7,\ m' = 260,\ \ell' = 15$: no exceptions								
$\alpha = 1.253/2,\ \delta = 0.1,\ \rho = 7,\ m' = 1500,\ \ell' = 66$: no exceptions								
$\alpha = 1,\ \delta = 0,\ \rho = 8,\ m' = 46,\ \ell' = 5.3$								
q	$x_0'(q)$	$x'(q)$	q	$x_0'(q)$	$x'(q)$	q	$x_0'(q)$	$x'(q)$
3	10215	26091	4	16266	32379	6	27172	39992
$\alpha = 0.9,\ \delta = 0,\ \rho = 7,\ m' = 66,\ \ell' = 5$								
q	$x_0'(q)$	$x'(q)$	q	$x_0'(q)$	$x'(q)$	q	$x_0'(q)$	$x'(q)$
3	21029	40486	4	33485	50922	6	55938	62679
$\alpha = 0.8,\ \delta = 0,\ \rho = 6,\ m' = 120,\ \ell' = 8$								
q	$x_0'(q)$	$x'(q)$	q	$x_0'(q)$	$x'(q)$	q	$x_0'(q)$	$x'(q)$
3	69520	108608	4	110696	139012			
$\alpha = 0.7,\ \delta = 0,\ \rho = 5,\ m' = 500,\ \ell' = 26$: no exceptions								

TABLE 7. Constants for the proof of Theorem 1.2: small q's.

q	x_0	q	x_0	q	x_0'	q	x_0'
3	43741	9	273368	3	98197	9	826355
4	41398	10	126848	4	108188	10	419894
5	141162	11	690311	5	317506	11	2381080
6	38467	12	126684	6	122626	12	447783
7	283378	$13 \le q \le 100$	$(34\varphi(q)\ell(q))^2$	7	739830	$13 \le q \le 100$	$(52\varphi(q)\ell(q))^2$
8	131137	$100 \le q \le 660$	$(10\varphi(q)\ell(q))^2$	8	386260	$100 \le q \le 1320$	$(20\varphi(q)\ell(q))^2$

TABLE 8. Constants for the proof of Theorem 1.2: large q's.

m	q_0	m'	q_0'
8	660	14	343072
9	168	15	1320
10	111	16	330

Input: Three reals α, δ, ρ
Input: Two integers q, x
return $(\alpha \log x + \delta \log q + \rho)\varphi(q)\sqrt{x}$;

Function h1(α, δ, ρ, q, x)

```
Input: Three reals α, δ, ρ
Input: Two integers q, x
return ((α+1) log x+δ log q+ρ)φ(q)√x;
```

Function hsqrt(α, δ, ρ, q, x)

```
Input: Three reals α, δ, ρ
Input: Three integers q, x_0, x
for a ← 1 to q−1 do
    if (a, q) ≠ 1 then continue;
    M_1[a] ← x_0+h1(α, δ, ρ, q, x_0);
end
forprime p ← x_0 to x+h1(α, δ, ρ, q, x) do
    a ← p mod q;
    if (a, q) ≠ 1 then continue;
    if M_1[a] ≤ p then
        print ("Problem with class ", a, " mod ", q, " for x =", M_1[a]);
    end
    M_1[a] ← p+h1(α, δ, ρ, q, p);
endfp
for a ← 1 to q−1 do
    if (a, q) ≠ 1 then continue;
    if M_1[a] < x then
        print ("Problem with class ", a, " mod ", q, " for x =", M_1[a]);
    end
end
```

Procedure Check1(α, δ, ρ, q, x_0, x)

```
Input: Three reals α, δ, ρ
Input: Three integers q, x_0, x
for a ← 1 to q−1 do
    if (a, q) ≠ 1 then continue;
    M_s[a] ← x'_0+hsqrt(α, δ, ρ, q, x'_0);
    N[a] ← floor(M_s[a])+1;
end
forprime p ← x'_0 to x'+hsqrt(α, δ, ρ, q, x') do
    a ← p mod q;
    if (a, q) ≠ 1 then continue;
    N[a] ← N[a]−1;
    if N[a] ≠ 0 then continue;
    if M_s[a] ≤ p then
        print ("Problem for sqrt claim with class ", a, " mod ", q, " for x =", M_s[a]);
    end
    M_s[a] ← p+hsqrt(α, δ, ρ, q, p);
    N[a] ← floor(M_s[a])+1;
endfp
for a ← 1 to q−1 do
    if (a, q) ≠ 1 then continue;
    if M_s[a] < x' then
        print ("Problem for sqrt claim with class ", a, " mod ", q, " for x =", M_s[a]);
    end
end
```

Procedure CheckSqrt(α, δ, ρ, q, x'_0, x')

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`http://users.mat.unimi.it/~molteni/research/primes/progressions.gp`
the code we have used to compute the constants in this paper.

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(A. Dudek) Mathematical Sciences Institute, The Australian National University
Email address: `adrian.dudek@anu.edu.au`

(L. Grenié) Dipartimento di Ingegneria gestionale, dell'informazione e della produzione, Università di Bergamo, viale Marconi 5, 24044 Dalmine (BG) Italy
Email address: `loic.grenie@gmail.com`

(G. Molteni) Dipartimento di Matematica, Università di Milano, via Saldini 50, 20133 Milano, Italy
Email address: `giuseppe.molteni1@unimi.it`