Generalized solutions in PDEs and the Burgers' equation

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Abstract

In many situations, the notion of function is not sufficient and it needs to be extended. A classical way to do this is to introduce the notion of weak solution; another approach is to use generalized functions. Ultrafunctions are a particular class of generalized functions that has been previously introduced and used to define generalized solutions of stationary problems in [4, 7, 9, 11, 12]. In this paper we generalize this notion in order to study also evolution problems. In particular, we introduce the notion of Generalized Ultrafunction Solution (GUS) for a large family of PDEs, and we confront it with classical strong and weak solutions. Moreover, we prove an existence and uniqueness result of GUS's for a large family of PDEs, including the nonlinear Schroedinger equation and the nonlinear wave equation. Finally, we study in detail GUS's of Burgers' equation, proving that (in a precise sense) the GUS's of this equation provide a description of the phenomenon at microscopic level.

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1. Introduction

In order to solve many problems of mathematical physics, the notion of function is not sufficient and it is necessary to extend it. Among people working in partial differential equations, the theory of distributions of Schwartz and the notion of weak solution are the main tools to be used when equations do not have classical solutions. Usually, these equations do not have classical solutions since they develop singularities. The notion of weak solution allows to obtain existence results, but uniqueness may be lost; also, these solutions might violate

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the conservation laws. As an example let us consider the Burgers' equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0. \tag{BE}$$

A local classical solution $u(t,x)$ is unique and, if it has compact support, it preserves the momentum $P=\int u\ dx$ and the energy $E=\frac{1}{2}\int u^2\ dx$ as well as other quantites. However, at some time a singularity appears and the solution can be no longer described by a smooth function. The notion of weak solution is necessary, but the problem of uniqueness becomes a central issue. Moreover, in general, E is not preserved.

An approach that can be used to try to overcome these difficulties is the use of generalized functions (see e.g. [15, 16, 25], where such an approach is developed using ideas in common with Colombeau theory). In this paper we use a similar approach by means of non-Archimedean analysis, and we introduce the notion of ultrafunction solution for a large family of PDEs using some of the tools of Nonstandard Analysis (NSA). Ultrafunctions are a family of generalized functions defined on the field of hyperreals, which are a well known extension of the reals. They have been introduced in [4], and also studied in [7, 8, 10, 11, 12, 13]. The non-Archimedean setting in which we will work (which is a reformulation, in a topological language, of the ultrapower approach to NSA of Keisler) is introduced in Section 2. In Section 3 we introduce the spaces of ultrafunctions, and we show their relationships with distributions. In Section 4 we introduce the notion of generalized ultrafunction solutions (GUS). We prove an existence and uniqueness theorem for these generalized solutions, and we confront them with strong and weak solutions of evolution problems. In particular, we show the existence of a GUS even in the presence of blow ups (as e.g. in the case of the nonlinear Schroedinger equation), and we show the uniqueness of GUS for the nonlinear wave equation. Finally, in Section 5 we study in detail Burgers' equation and, in a sense precised in Section 5.4, we show that in this case the unique GUS of this equation provides a description of the phenomenon at microscopic level.

1.1. Notations

Let Ω be a subset of $\mathbb{R}^N$: then

- $\mathcal{C}\left(\Omega\right)$ denotes the set of continuous functions defined on $\Omega\subset\mathbb{R}^N$;
- $\mathcal{C}_c\left(\Omega\right)$ denotes the set of continuous functions in $\mathcal{C}\left(\Omega\right)$ having compact support in Ω;
- $\mathcal{C}_0\left(\overline{\Omega}\right)$ denotes the set of continuous functions in $\mathcal{C}\left(\Omega\right)$ which vanish on $\partial\Omega$;
- $\mathcal{C}^k\left(\Omega\right)$ denotes the set of functions defined on $\Omega\subset\mathbb{R}^N$ which have continuous derivatives up to the order k;
- $\mathcal{C}_c^k\left(\Omega\right)$ denotes the set of functions in $\mathcal{C}^k\left(\Omega\right)$ having compact support;

- $\mathscr{D}(\Omega)$ denotes the set of the infinitely differentiable functions with compact support defined on $\Omega \subset \mathbb{R}^N$; $\mathscr{D}'(\Omega)$ denotes the topological dual of $\mathscr{D}(\Omega)$, namely the set of distributions on Ω;
- for any set X, $\mathcal{P}_{fin}(X)$ denotes the set of finite subsets of X;
- if W is a generic function space, its topological dual will be denated by W' and the paring by $\langle \cdot, \cdot \rangle_W$, or simply by $\langle \cdot, \cdot \rangle$.

2. Λ-theory

2.1. Non-Archimedean Fields

In this section we recall the basic definitions and facts regarding non-Archimedean fields, following an approach that has been introduced in [13] (see also [4, 6, 7, 8, 9, 10, 11, 12]). In the following, $\mathbb{K}$ will denote an ordered field. We recall that such a field contains (a copy of) the rational numbers. Its elements will be called numbers.

Definition 1. Let $\mathbb{K}$ be an infinite ordered field. Let $\xi \in \mathbb{K}$. We say that:

- ξ is infinitesimal if, for all positive $n \in \mathbb{N}$, $|\xi| < \frac{1}{n}$;
- ξ is finite if there exists $n \in \mathbb{N}$ such that $|\xi| < n$;
- ξ is infinite if, for all $n \in \mathbb{N}$, $|\xi| > n$ (equivalently, if ξ is not finite).

An ordered field $\mathbb{K}$ is called non-Archimedean if it contains an infinitesimal $\xi \neq 0$.

It's easily seen that all infinitesimal are finite, that the inverse of an infinite number is a nonzero infinitesimal number, and that the inverse of a nonzero infinitesimal number is infinite.

Definition 2. A superreal field is an ordered field $\mathbb{K}$ that properly extends $\mathbb{R}$.

It is easy to show, due to the completeness of $\mathbb{R}$, that there are nonzero infinitesimal numbers and infinite numbers in any superreal field. Infinitesimal numbers can be used to formalize a notion of "closeness":

Definition 3. We say that two numbers $\xi, \zeta \in \mathbb{K}$ are infinitely close if $\xi - \zeta$ is infinitesimal. In this case we write $\xi \sim \zeta$.

Clearly, the relation "$\sim$" of infinite closeness is an equivalence relation.

Theorem 4. *If $\mathbb{K}$ is a superreal field, every finite number $\xi \in \mathbb{K}$ is infinitely close to a unique real number $r \sim \xi$, called the **shadow** or the **standard part** of ξ.*

Given a finite number ξ, we denote its shadow as $sh(\xi)$.

2.2. The Λ-limit

In this section we introduce a particular non-Archimedean field by means of $\Lambda-$theory[2], in particular by means of the notion of $\Lambda-$limit (for complete proofs and for further informations the reader is referred to [2], [4], [7] and [13]). To recall the basics of $\Lambda-$theory we have to recall the notion of superstructure on a set (see also [22]):

Definition 5. Let E be an infinite set. The superstructure on E is the set

$$V_\infty(E) = \bigcup_{n\in\mathbb{N}} V_n(E),$$

where the sets $V_n(E)$ are defined by induction by setting

$$V_0(E) = E$$

and, for every $n \in \mathbb{N}$,

$$V_{n+1}(E) = V_n(E) \cup \mathcal{P}\left(V_n(E)\right).$$

Here $\mathcal{P}\left(E\right)$ denotes the power set of E. Identifying the couples with the Kuratowski pairs and the functions and the relations with their graphs, it follows that $V_\infty(E)$ contains almost every usual mathematical object that can be constructed starting with E; in particular, $V_\infty(\mathbb{R})$, which is the superstructure that we will consider in the following, contains almost every usual mathematical object of analysis.

Throughout this paper we let

$$\mathfrak{L} = \mathcal{P}_{fin}(V_\infty(\mathbb{R}))$$

and we order $\mathfrak{L}$ via inclusion. Notice that $(\mathfrak{L}, \subseteq)$ is a directed set. We add to $\mathfrak{L}$ a "point at infinity" $\Lambda \notin \mathfrak{L}$, and we define the following family of neighborhoods of Λ :

$$\{\{\Lambda\} \cup Q \mid Q \in \mathcal{U}\},$$

where $\mathcal{U}$ is a fine ultrafilter on $\mathfrak{L}$, namely a filter such that

- for every $A, B \subseteq \mathfrak{L}$, if $A \cup B = \mathfrak{L}$ then $A \in \mathcal{U}$ or $B \in \mathcal{U}$;
- for every $\lambda \in \mathfrak{L}$ the set $I_\lambda = \{\mu \in \mathfrak{L} \mid \lambda \subseteq \mu\} \in \mathcal{U}$.

In particular, we will refer to the elements of $\mathcal{U}$ as qualified sets and we will write $\Lambda = \Lambda(\mathcal{U})$ when we want to highlight the choice of the ultrafilter. We are interested in considering real nets with indices in $\mathfrak{L}$, namely functions

$$\varphi : \mathfrak{L} \to \mathbb{R}.$$

[2] Readers expert in nonstandard analysis will recognize that Λ-theory is equivalent to the superstructure constructions of Keisler (see [22] for a presentation of the original constructions of Keisler, and [13] for a comparison between these two approaches to nonstandard analysis).

In particular, we are interested in $\Lambda-$limits of these nets, namely in

$$\lim_{\lambda\to\Lambda}\varphi(\lambda).$$

The following has been proved in [13].

Theorem 6. *There exists a non-Archimedean superreal field* $(\mathbb{K},+,\cdot,<)$ *and an Hausdorff topology* τ *on the space* $(\mathfrak{L}\times\mathbb{R})\cup\mathbb{K}$ *such that*

1. $(\mathfrak{L}\times\mathbb{R})\cup\mathbb{K}=cl_{\tau}\left(\mathfrak{L}\times\mathbb{R}\right);$
2. *for every net* $\varphi:\mathfrak{L}\to\mathbb{R}$ *the limit*

$$L=\lim_{\lambda\to\Lambda}\left(\lambda,\varphi(\lambda)\right)$$

exists, it is in $\mathbb{K}$ *and it is unique; moreover for every* $\xi\in\mathbb{K}$ *there is a net* $\varphi:\mathfrak{L}\to\mathbb{R}$ *such that*

$$\xi=\lim_{\lambda\to\Lambda}\left(\lambda,\varphi(\lambda)\right);$$

3. $\forall\ c\in\mathbb{R}$ *we have that*

$$\lim_{\lambda\to\Lambda}\left(\lambda,c\right)=c;$$

4. *for every* $\varphi,\psi:\mathfrak{L}\to\mathbb{R}$ *we have that*

$$\begin{aligned}\lim_{\lambda\to\Lambda}\left(\lambda,\varphi(\lambda)\right)+\lim_{\lambda\to\Lambda}\left(\lambda,\psi(\lambda)\right) &= \lim_{\lambda\to\Lambda}\left(\lambda,(\varphi+\psi)(\lambda)\right);\\ \lim_{\lambda\to\Lambda}\left(\lambda,\varphi(\lambda)\right)\cdot\lim_{\lambda\to\Lambda}\left(\lambda,\varphi(\lambda)\right) &= \lim_{\lambda\to\Lambda}\left(\lambda,(\varphi\cdot\psi)(\lambda)\right).\end{aligned}$$

Proof. For a complete proof of Theorem 6 we refer to [13]. The idea[3] is to set

$$I=\left\{\varphi\in\mathfrak{F}\left(\mathfrak{L},\mathbb{R}\right)\ \mid\varphi(x)=0\text{ in a qualified set}\right\};$$

it is not difficult to prove that I is a maximal ideal in $\mathfrak{F}\left(\mathfrak{L},\mathbb{R}\right)$, and hence

$$\mathbb{K}:=\frac{\mathfrak{F}\left(\mathfrak{L},\mathbb{R}\right)}{I}$$

is a field. Now the claims of Theorem 6 follows by identifying every real number $c\in\mathbb{R}$ with the equivalence class of the constant net $[c]_I$ and by taking the topology τ generated by the basis of open sets

$$b(\tau)=\left\{N_{\varphi,Q}\mid\varphi\in\mathfrak{F}\left(\mathfrak{L},\mathbb{R}\right),Q\in\mathcal{U}\right\}\cup\mathcal{P}(\mathfrak{L}\times\mathbb{R}),$$

where

$$N_{\varphi,Q}:=\left\{\left(\lambda,\varphi(\lambda)\right)\mid\lambda\in Q\right\}\cup\left\{[\varphi]_I\right\}$$

is a neighborhood of $[\varphi]_I$. $\square$

[3] To work, this idea needs some additional requirement on the ultrafilter $\mathcal{U}$, see e.g. [5], [13].

Now we want to define the Λ-limit of nets $(\lambda, \varphi(\lambda))_{\lambda\in\mathfrak{L}}$, where $\varphi(\lambda)$ is any bounded net of mathematical objects in $V_\infty(\mathbb{R})$ (a net $\varphi : \mathfrak{L} \to V_\infty(\mathbb{R})$ is called bounded if there exists n such that $\forall\lambda \in \mathfrak{L},\ \varphi(\lambda) \in V_n(\mathbb{R})$). To this aim, let us consider a net

$$\varphi : \mathfrak{L} \to V_n(\mathbb{R}). \tag{2.1}$$

We will define $\lim_{\lambda\to\Lambda}\ (\lambda, \varphi(\lambda))$ by induction on n.

Definition 7. For $n = 0$, $\lim\limits_{\lambda\to\Lambda}(\lambda, \varphi(\lambda))$ exists by Theorem (6); so by induction we may assume that the limit is defined for $n-1$ and we define it for the net (2.1) as follows:

$$\lim_{\lambda\to\Lambda}\ (\lambda, \varphi(\lambda)) = \left\{\lim_{\lambda\to\Lambda}(\lambda, \psi(\lambda)) \mid \psi : \mathfrak{L} \to V_{n-1}(\mathbb{R}) \text{ and } \forall\lambda \in \mathfrak{L},\ \psi(\lambda) \in \varphi(\lambda)\right\}.$$

From now on, we set

$$\lim_{\lambda\uparrow\Lambda}\varphi(\lambda) := \lim_{\lambda\to\Lambda}\ (\lambda, \varphi(\lambda))\,.$$

Notice that it follows from Definition 7 that $\lim\limits_{\lambda\uparrow\Lambda}\varphi(\lambda)$ is a well defined object in $V_\infty(\mathbb{R}^*)$ for every bounded net $\varphi : \mathfrak{L} \to V_\infty(\mathbb{R})$.

2.3. Natural extension of sets and functions

In this section we want to show how to extend subsets and functions defined on $\mathbb{R}$ to subsets and functions defined on $\mathbb{K}$.

Definition 8. Given a set $E \subseteq \mathbb{R}$, we set

$$E^* := \left\{\lim_{\lambda\uparrow\Lambda}\psi(\lambda) \mid \forall\lambda \in \mathfrak{L}\,\psi(\lambda) \in E\right\}.$$

E^* is called the **natural extension** of E.

Thus E^* is the set of all the limits of nets with values in E. Following the notation introduced in Def. 8, from now on we will denote $\mathbb{K}$ by $\mathbb{R}^*$. Similarly, it is possible to extend functions.

Definition 9. Given a function

$$f : A \to B$$

we call natural extension of f the function

$$f^* : A^* \to B^*$$

such that

$$f^*\left(\lim_{\lambda\to\Lambda}\ (\lambda, \varphi(\lambda))\right) := \lim_{\lambda\to\Lambda}\ (\lambda, f\,(\varphi(\lambda)))$$

for every $\varphi : \mathfrak{L} \to A$.

That Definition 9 is well posed has been proved in [13]. Let us observe that, in particular, $f^*(a) = f(a)$ for every $a \in A$ (which is why f^* is called the extension of f).

3. Ultrafunctions

3.1. Definition of Ultrafunctions

We follow the construction of ultrafunctions that we introduced in [12]. Let N be a natural number, let Ω be a subset of $\mathbb{R}^N$ and let $V \subset \mathfrak{F}(\Omega, \mathbb{R})$ be a function vector space such that $\mathscr{D}(\Omega) \subseteq V(\Omega) \subseteq L^2(\Omega)$.

Definition 10. We say that $(V_\lambda(\Omega)_{\lambda\in\mathfrak{L}})$ is an **approximating net for** $V(\Omega)$ if

1. $V_\lambda(\Omega)$ is a finite dimensional vector subspace of $V(\Omega)$ for every $\lambda \in \mathfrak{L}$;
2. if $\lambda_1 \subseteq \lambda_2$ then $V_{\lambda_1}(\Omega) \subseteq V_{\lambda_2}(\Omega)$;
3. if $W(\Omega) \subset V(\Omega)$ is a finite dimensional vector space then there exists $\lambda \in \mathfrak{L}$ such that $W(\Omega) \subseteq V_\lambda(\Omega)$ (i.e., $V(\Omega) = \bigcup\limits_{\lambda\in\mathfrak{L}} V_\lambda(\Omega)$).

Let us show two examples.

Example 11. Let $V(\mathbb{R}) \subseteq L^2(\mathbb{R})$. We set, for every $\lambda \in \mathfrak{L}$,

$$V_\lambda(\Omega) := Span(V(\Omega) \cap \lambda).$$

Then $(V_\lambda(\Omega))_{\lambda\in\mathfrak{L}}$ is an approximating net for $V(\Omega)$.

Example 12. Let

$$\{e_a\}_{a\in\mathbb{R}}$$

be a Hamel basis[4] of $V(\Omega) \subseteq L^2$. For every $\lambda \in \mathfrak{L}$ let

$$V_\lambda(\Omega) = Span\left\{e_a \mid a \in \lambda\right\}.$$

Then $(V_\lambda(\Omega))_{\lambda\in\mathfrak{L}}$ is an approximating net for $V(\Omega)$.

Definition 13. Let $\mathcal{U}$ be a fine ultrafilter on $\mathfrak{L}$, let $\Lambda = \Lambda(\mathcal{U})$ and let $(V_\lambda(\Omega))_{\lambda\in\mathfrak{L}}$ be an approximating net for $V(\Omega)$. We call **space of ultrafunctions generated by** $(V_\lambda(\Omega))$ the Λ-limit

$$V_\Lambda(\Omega) := \lim_{\lambda\uparrow\Lambda} V_\lambda(\Omega) = \left\{\lim_{\lambda\uparrow\Lambda} f_\lambda \mid \forall\lambda \in \mathfrak{L}\ f_\lambda \in V_\lambda(\Omega)\right\}.$$

[4] We recall that $\{e_a\}_{a\in\mathbb{R}}$ is a Hamel basis for W if $\{e_a\}_{a\in\mathbb{R}}$ is a set of linearly indipendent elements of W and every element of W can be (uniquely) written has a finite sum (with coefficients in $\mathbb{R}$) of elements of $\{e_a\}_{a\in\mathbb{R}}$. Since a Hamel basis of W has the continuum cardinality we can use the points of $\mathbb{R}$ as indices for this basis.

In this case we will also say that the space $V_\Lambda(\Omega)$ is based on the space $V(\Omega)$. When $V_\lambda(\Omega) := Span(V(\Omega) \cap \lambda)$ for every $\lambda \in \mathfrak{L}$, we will say that $V_\Lambda(\Omega)$ is a **canonical space of ultrafunctions**.

Using the above definition, if $V(\Omega)$, $\Omega \subset \mathbb{R}^N$, is a real function space and $(V_\lambda(\Omega))$ is an approximating net for $V(\Omega)$ then we can associate to $V(\Omega)$ the following three hyperreal functions spaces:

$$V(\Omega)^\sigma = \{f^* \mid f \in V(\Omega)\};\tag{3.1}$$

$$V_\Lambda(\Omega) = \left\{\lim_{\lambda\uparrow\Lambda} f_\lambda \mid \forall\lambda \in \mathfrak{L}\ f_\lambda \in V_\lambda(\Omega)\right\};\tag{3.2}$$

$$V(\Omega)^* = \left\{\lim_{\lambda\uparrow\Lambda} f_\lambda \mid \forall\lambda \in \mathfrak{L}\ f_\lambda \in V(\Omega)\right\}.\tag{3.3}$$

Clearly we have

$$V(\Omega)^\sigma \subset V_\Lambda(\Omega) \subset V(\Omega)^*.$$

So, given any vector space of functions $V(\Omega)$, the space of ultrafunctions generated by $V(\Omega)$ is a vector space of hyperfinite dimension that includes $V(\Omega)^\sigma$, and the ultrafunctions are Λ-limits of functions in $V_\lambda(\Omega)$. Hence the ultrafunctions are particular internal functions

$$u : (\mathbb{R}^*)^N \to \mathbb{C}^*.$$

Since $V_\Lambda(\Omega) \subset \left[L^2(\mathbb{R})\right]^*$, we can equip $V_\Lambda(\Omega)$ with the following scalar product:

$$(u, v) = \int^* u(x)v(x)\ dx,\tag{3.4}$$

where $\int^*$ is the natural extension of the Lebesgue integral considered as a functional

$$\int : L^1(\Omega) \to \mathbb{R}.$$

Therefore, the norm of an ultrafunction will be given by

$$\|u\| = \left(\int^* |u(x)|^2\ dx\right)^{\frac{1}{2}}.$$

Sometimes, when no ambiguity is possible, in order to make the notation simpler we will write $\int$ istead of $\int^*$.

Remark 14. Notice that the natural extension f^* of a function f is an ultrafunction if and only if $f \in V(\Omega)$.

Proof. Let $f \in V(\Omega)$ and let $(V_\lambda(\Omega))$ be an approximating net for $V(\Omega)$. Then, eventually, $f \in V_\lambda(\Omega)$ and hence

$$f^* = \lim_{\lambda\uparrow\Lambda} f \in \lim_{\lambda\uparrow\Lambda}\ V_\lambda(\Omega) = V_\Lambda(\Omega).$$

Conversely, if $f \notin V(\Omega)$ then $f^* \notin V^*(\Omega)$ and, since $V_\Lambda(\Omega) \subset V^*(\Omega)$, this entails the thesis. $\square$

3.2. Canonical extension of functions, functionals and operators

Let $V_\Lambda(\Omega)$ be a space of ultrafunctions based on $V(\Omega) \subseteq L^2(\Omega)$. We have seen that given a function $f \in V(\Omega)$, its natural extension

$$f^* : \Omega^* \to \mathbb{R}^*$$

is an ultrafunction in $V_\Lambda(\Omega)$. In this section we investigate the possibility to associate an ultrafunction $\widetilde{f}$ to any function $f \in L^1_{loc}(\Omega)$ in a consistent way. Since $L^2(\Omega) \subseteq V'(\Omega)$, this association can be done by means of a duality method.

Definition 15. Given $T \in \left[L^2(\Omega)\right]^*$, we denote by $\widetilde{T}$ the unique ultrafunction such that $\forall v \in V_\Lambda(\Omega)$,

$$\int_{\Omega^*} \widetilde{T}(x)v(x)\,dx = \int_{\Omega^*} T(x)v(x)\,dx.$$

The map

$$P_\Lambda : \left[L^2(\Omega)\right]^* \to V_\Lambda(\Omega)$$

defined by $P_\Lambda T = \widetilde{T}$ will be called the **canonical projection.**

The above definition makes sense, as T is a linear functional on $V(\Omega)^*$, and hence on $V_\Lambda(\Omega) \subset V(\Omega)^*$.

Since $V(\Omega) \subset L^2(\Omega)$, using the inner product (3.4) we can identify $L^2(\Omega)$ with a subset of $V'(\Omega)$, and hence $\left[L^2(\Omega)\right]^*$ with a subset of $[V'(\Omega)]^*$; in this case, $\forall f \in \left[L^2(\Omega)\right]^*$, $\forall v \in V_\Lambda(\Omega)$,

$$\int \widetilde{f}(x)v(x)\ dx = \int f(x)v(x)\ dx,$$

namely the map $P_\Lambda f = \widetilde{f}$ restricted to $\left[L^2(\Omega)\right]^*$ reduces to the orthogonal projection

$$P_\Lambda : \left[L^2(\Omega)\right]^* \to V_\Lambda(\Omega).$$

If we take any function $f \in L^1_{loc}(\Omega) \cap L^2(\Omega)$, then $f^* \in \left[L^1_{loc}(\Omega) \cap L^2(\Omega)\right]^* \subset \left[L^2(\Omega)\right]^*$ and hence $\widetilde{f^*}$ is well defined by Def. 15. In order to simplify the notation we will simply write $\widetilde{f}$. This discussion suggests the following definition:

Definition 16. Given a function $f \in L^1_{loc}(\Omega) \cap L^2(\Omega)$, we denote by $\widetilde{f}$ the unique ultrafunction in $V_\Lambda(\Omega)$ such that $\forall v \in V_\Lambda(\Omega)$,

$$\int \widetilde{f}(x)v(x)\ dx = \int f^*(x)v(x)\ dx.$$

$\widetilde{f}$ is called the canonical extension of f.

Remark 17. As we observed, for every $f : \mathbb{R} \to \mathbb{R}$ we have that ${}^*f \in V_\Lambda(\Omega)$ iff $f \in V(\Omega)$. Therefore for every $f : \mathbb{R} \to \mathbb{R}$

$$\widetilde{f} = f^* \Leftrightarrow f \in V(\Omega).$$

Let us observe that we need to assume that $V(\Omega) \subset L_c^\infty(\Omega) = (L_{loc}^1(\Omega))'$ if we want $\widetilde{f}$ to be defined for every function $f \in L_{loc}^1(\Omega)$. Using a similar method, it is also possible to extend operators:

Definition 18. Given an operator

$$\mathcal{A} : V(\Omega) \to V'(\Omega)$$

we can extend it to an operator

$$\widetilde{\mathcal{A}} : V_\Lambda(\Omega) \to V_\Lambda(\Omega)$$

in the following way: given an ultrafunction u, $\widetilde{\mathcal{A}}(u)$ is the unique ultrafunction such that

$$\forall v \in V_\Lambda(\Omega),\ \int^* \widetilde{\mathcal{A}}(u)v\ dx\ = \int^* \mathcal{A}^*(u)v\, dx;$$

namely

$$\widetilde{\mathcal{A}} = P_\Lambda \circ \mathcal{A}^*,$$

where P_Λ is the canonical projection.

Sometimes, when no ambiguity is possible, in order to make the notation simpler we will write $\mathcal{A}(u)$ instead of $\widetilde{\mathcal{A}}(u)$.

Example 19. The derivative of an ultrafunction is well defined provided that the weak derivative is defined from $V(\Omega)$ to his dual $V'(\Omega)$:

$$\partial : V(\Omega) \to V'(\Omega).$$

For example you can take $V(\Omega) = \mathcal{C}^1(\Omega)$, $H^{1/2}(\Omega)$, $BV(\Omega)$ etc. Following Definition 18, we have that the ultrafunction derivative

$$D : V_\Lambda(\Omega) \to V_\Lambda(\Omega)$$

of an ultrafunction u is defined by duality as the unique ultrafunction Du such that

$$\forall v \in V_\Lambda(\Omega),\ \int Du\ v\ dx\ = \langle \partial^* u, v \rangle\,. \tag{3.5}$$

Notice that, in order to simplify the notation, we have denoted the generalized derivative by $D = \widetilde{\partial}$.

To construct the space of ultrafunctions that we need to study Burgers' Equation we will use the following theorem:

Theorem 20. *Let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ and let $V(\Omega)$ be a vector space of functions. Let $V(\Omega)^*$ be a $|\mathfrak{L}|^+$-enlarged*[5] *ultrapower of $V(\Omega)$. Then every hyperfinite dimensional vector space $W(\Omega)$ such that $V(\Omega)^\sigma \subseteq W(\Omega) \subseteq V(\Omega)^*$ contains an isomorphic copy of a canonical space of ultrafunctions on $V(\Omega)$.*

Proof. First of all, we claim that there exist a hyperfinite set $H \in (\mathcal{P}_{fin}(\mathfrak{L}))^*$ such that $\lambda \subseteq H$ for every $\lambda \in \mathfrak{L}$ and such that $B = H \cap W(\Omega)$ is a hyperfinite basis of $W(\Omega)$. To prove this claim we set, for every $\lambda \in \mathfrak{L}$,

$$H_\lambda = \left\{ H \in (\mathcal{P}_{fin}(\mathfrak{L}))^* \mid \lambda^* \subseteq H \,\text{and}\, Span(H \cap V^*(\Omega)) = W(\Omega) \right\}.$$

Clearly, if $H_\lambda \neq \emptyset$ for every $\lambda \in \mathfrak{L}$ then the family $\{H_\lambda\}_{\lambda \in \mathfrak{L}}$ has the finite intersection property (as $H_{\lambda_1} \cap \cdots \cap H_{\lambda_k} = H_{\lambda_1 \cup \cdots \cup \lambda_k}$). To prove that $H_\lambda \neq \emptyset$ for every $\lambda \in \mathfrak{L}$, let $\lambda \in \mathfrak{L}$ be given and let B be a fixed hyperfinite basis of $W(\Omega)$ with $V(\Omega)^\sigma \subseteq B$ (whose existence can be easily deduced from the enlarging property of the extension, as $V(\Omega)^\sigma \subseteq W(\Omega)$). Let $\lambda = \lambda_0 \cup \lambda_1$, where $\lambda_0 \cap \lambda_1 = \emptyset$ and $\lambda_0 = \lambda \cap V(\Omega)$, and let $H = B \cup \lambda_1^*$. It is immediate to notice that $H \in H_\lambda$. Therefore this proves that the family $\{H_\lambda\}_{\lambda \in \mathfrak{L}}$ has the finite intersection property, and so our claim can be derived as a consequence of the $|\mathfrak{L}|^+$-enlarging property of the extension. From now on, we let H be an hyperfinite set with the properties of our claim, and we let $B = H \cap W(\Omega)$. Finally, we set $\mathcal{U} = \{X \subseteq \mathfrak{L} \mid H \in X^*\}$. Clearly, $\mathcal{U}$ is an ultrafilter on $\mathfrak{L}$; moreover, our construction of H has been done to have that $\mathcal{U}$ is a fine ultrafilter. To prove this, let $\lambda_0 \in \mathfrak{L}$. Then

$$\{\lambda \in \mathfrak{L} \mid \lambda_0 \subseteq \lambda\} \in \mathcal{U} \Leftrightarrow H \in \{\lambda \in \mathfrak{L} \mid \lambda_0 \subseteq \lambda\}^* \Leftrightarrow \lambda_0 \subseteq H,$$

and $\lambda_0 \subseteq H$ by our construction of the set H.

Now we set $V_\lambda(\Omega) = Span(V(\Omega) \cap \lambda)$ for every $\lambda \in \mathfrak{L}$, we set $V_{\Lambda(\mathcal{U})} := \lim_{\lambda \uparrow \Lambda(\mathcal{U})} v_\lambda$ and we let $\Phi : V_{\Lambda(\mathcal{U})}(\Omega) \to W(\Omega)$ be defined as follows: for every $v = \lim_{\lambda \uparrow \Lambda(\mathcal{U})} v_\lambda$,

$$\Phi\left(\lim_{\lambda \uparrow \Lambda(\mathcal{U})} v_\lambda\right) := v_B,$$

where v_B is the value of the hyperextension $v^* : \mathfrak{L}^* \to V^*(\Omega)$ of the function $v : \mathfrak{L} \to V(\Omega)$ evaluated in $B \in \mathfrak{L}^*$. Let us notice that, as $v_\lambda \in Span(V(\Omega) \cap \lambda)$ for every $\lambda \in \mathfrak{L}$, by transfer we have that $v_B \in Span(V(\Omega)^* \cap B) = W(\Omega)$, namely the image of Φ is included in $W(\Omega)$.

[5] For the notion of enlarging, as well as for other important notions in nonstandard analysis such as saturation and overspill, we refer to [22, 24].

To conclude our proof, we have to show that Φ is an embedding (so that we can take $\Phi\left(V_{\Lambda(\mathcal{U})}(\Omega)\right)$ as the isomorphic copy of a canonical space of ultrafunctions contained in $W(\Omega)$). The linearity of Φ holds trivially; to prove that Φ is injective let $v = \lim_{\lambda\uparrow\Lambda(\mathcal{V})} v_\lambda$, $w = \lim_{\lambda\uparrow\Lambda(\mathcal{V})} w_\lambda$. Then

$$\Phi(v) = \Phi(w) \Leftrightarrow v_B = w_B \Leftrightarrow B \in \{\lambda \in \mathfrak{L} \mid v_\lambda = w_\lambda\}^* \Leftrightarrow$$

$$\{\lambda \in \mathfrak{L} \mid v_\lambda = w_\lambda\} \in \mathcal{V} \Leftrightarrow v = w.$$

□

Lemma 21. *Let $V(\Omega)$ be given, let $(V_\lambda(\Omega))_{\lambda\in\mathfrak{L}}$ be an approximating net for $V(\Omega)$ and let $V_\Lambda(\Omega) = \lim_{\lambda\uparrow\Lambda} V_\lambda(\Omega)$. Finally, let $u \in V(\Omega)^* \setminus V_\Lambda(\Omega)$. Then $W(\Omega) := Span\,(V_\Lambda(\Omega) \cup \{u\})$ is a space of ultrafunctions on $V(\Omega)$.*

Proof. Let $u = \lim_{\lambda\uparrow\Lambda} u_\lambda$, where $u_\lambda \notin V_\lambda(\Omega)$ for every $\lambda \in \mathfrak{L}$, and let, for every $\lambda \in \mathfrak{L}$, $W_\lambda = Span\,(V_\lambda \cup \{u_\lambda\})$. Clearly, $(W_\lambda)_{\lambda\in\mathfrak{L}}$ is an approximating net for $V(\Omega)$. We claim that $W(\Omega) = W_\Lambda(\Omega) = \lim_{\lambda\uparrow\Lambda} W_\lambda(\Omega)$. Clearly, $V_\Lambda(\Omega) \subseteq W_\Lambda(\Omega)$ and $u \in W_\Lambda(\Omega)$, and hence $W(\Omega) \subseteq W_\Lambda(\Omega)$. As for the reverse inclusion, let $w \in W_\Lambda(\Omega)$ and let $w = \lim_{\lambda\uparrow\Lambda} w_\lambda$. For every $\lambda \in \mathfrak{L}$ let $w_\lambda = v_\lambda + c_\lambda u_\lambda$, where $v_\lambda \in V_\lambda$. Then

$$w = \lim_{\lambda\uparrow\Lambda} v_\lambda + \lim_{\lambda\uparrow\Lambda} c_\lambda \cdot \lim_{\lambda\uparrow\Lambda} u_\lambda$$

so, as $\lim_{\lambda\uparrow\Lambda} v_\lambda \in V_\Lambda(\Omega)$ and $\lim_{\lambda\uparrow\Lambda} u_\lambda = u$, we have that $w \in W(\Omega)$, and hence the thesis is proved. □

Theorem 22. *There is a space of ultrafunctions $U_\Lambda(\mathbb{R})$ which satisfies the following assumptions:*

1. $H_c^1(\mathbb{R}) \subseteq U_\Lambda(\mathbb{R})$;
2. *the ultrafunction* $\widetilde{1}$ *is the identity in* $U_\Lambda(\mathbb{R})$, *namely* $\forall u \in U_\Lambda(\mathbb{R})$, $u \cdot \widetilde{1} = u$;
3. $D\widetilde{1} = 0$;
4. $\forall u, v \in U_\Lambda(\mathbb{R})$, $\int^* (Du)\, v \; dx \; = -\int^* u\, (Dv) \; dx$.

Proof. We set

$$H_\flat^1(\mathbb{R}) = Span\{u \in L^2(\mathbb{R}) \mid \exists n \in \mathbb{N} \text{ s.t. } supp(u) \subseteq [-n, n]\,,$$
$$u(n) = u(-n),\ u \in H^1([-n, n])\}.$$

Let $\beta \in \mathbb{N}^* \setminus \mathbb{N}$; we set

$$W(\mathbb{R}) := \left\{ v \in \left[H_\flat^1(\mathbb{R})\right]^* \mid supp(u) \subseteq [-\beta, \beta],\, u(-\beta) = u(\beta) \right\}$$

and we let $V_\Lambda(\mathbb{R})$ be a hyperfinite dimensional vector space that contains the characteristic function $1_{[-\beta,\beta]}(x)$ of $[-\beta, \beta]$ and such that[6]

$$\left[H_\flat^1(\mathbb{R})\right]^\sigma \subseteq V_\Lambda(\mathbb{R}) \subseteq W(\mathbb{R}).$$

[6] To have this property we need the nonstandard extension to be a $|\mathcal{P}(\mathbb{R})|^+$-enlargment.

As $W(\mathbb{R}) \subseteq \left[H^1_\flat(\mathbb{R})\right]^*$ we can apply Theorem 20 to deduce that $V_\Lambda(\mathbb{R})$ contains an isomorphic copy of a canonical space of ultrafunctions on $H^1_\flat(\mathbb{R})$. If this isomorphic copy does not contain $1_{[-\beta,\beta]}$, we can apply Lemma 21 to construct a space of ultrafunctions included in $V_\Lambda(\Omega)$ that contains $1_{[-\beta,\beta]}$. Let $U_\Lambda(\Omega)$ denote this space of ultrafunctions on $H^1_\flat(\mathbb{R})$.

Condition (1) holds as $H^1_c(\mathbb{R}) \subseteq H^1_\flat(\mathbb{R})$. To prove condition (2) let us show that $\widetilde{1} = 1_{[-\beta,\beta]}$: in fact, for every $u \in U_\Lambda(\mathbb{R})$ we have

$$\int \widetilde{1} \cdot u\, dx = \int 1 \cdot u\, dx = \int_{-\beta}^{\beta} u\, dx = \int 1_{[-\beta,\beta]} \cdot u\, dx.$$

Henceforth condition (2) holds as $1_{[-\beta,\beta]} \cdot u = u$ for every $u \in U_\Lambda(\mathbb{R})$. To prove condition (3) let $u \in U_\Lambda(\mathbb{R})$. Then

$$\int D\left(1_{[-\beta,\beta]}\right) \cdot u\, dx = \int \partial\left(1_{[-\beta,\beta]}\right) \cdot u\, dx = u(\beta) - u(-\beta) = 0,$$

namely $D\left(1_{[-\beta,\beta]}\right) = 0$. Finally, as $U_\Lambda(\mathbb{R}) \subseteq [BV(\mathbb{R})]^*$, by equation (3.5), we have that

$$\int Du\ v\ dx\ = \langle \partial^* u, v\rangle = -\langle u, \partial^* v\rangle = -\int u\ Dv\ dx$$

and so condition (4) holds. □

Remark 23. Let $U_\Lambda(\mathbb{R})$ be the space of ultrafunctions given by Theorem 22. Then for every ultrafunction $u \in U_\Lambda(\mathbb{R})$ we have

$$\int^* u(x)dx = \int^* u(x) \cdot 1dx = \int^* u(x) \cdot \widetilde{1}dx = \int_{-\beta}^{\beta} u(x)dx.$$

We will use this property in Section 5 when talking about Burgers' equation.

3.3. Spaces of ultrafunctions involving time

Generic problems of evolution are usually formulated by equations of the following kind:

$$\partial_t u = \mathcal{A}(u), \tag{3.6}$$

where

$$\mathcal{A} : V(\Omega) \to L^2(\Omega)$$

is a differential operator.

By definition, a **strong solution** of equation (3.6) is a function

$$\phi \in V(I \times \Omega) := \mathcal{C}^0(I, V(\Omega)) \cap \mathcal{C}^1(I, L^2(\Omega))$$

where $I := [0, T)$ is the interval of time and $\mathcal{C}^k(I, B)$, $k \in \mathbb{N}$, denotes the space of functions from I to a Banach space B which are k times differentiable with continuity.

In equation (3.6), the independent variable is $(t,x)\in I\times\Omega\subset\mathbb{R}^{N+1}$, $I=[0,T)$. A disappointing fact is that a ultrafunction space based on $V(I\times\Omega)$ is not a convenient space where to study this equation, since these ultrafunctions spaces are not homogeneous in time in the following sense: if for every $t\in I^*$ we set

$$V_{\Lambda,t}(\Omega)=\left\{v\in V(\Omega)^*\mid\exists u\in V_\Lambda(I\times\Omega):u(t,x)=v(x)\right\},$$

for $t_2\neq t_1$ we have that

$$V_{\Lambda,t_2}(\Omega)\neq V_{\Lambda,t_1}(\Omega).$$

This fact is disappointing since we would like to see $u(t,\cdot)$ as a function defined on the same space for all the times $t\in I^*$. For this reason we think that a convenient space to study equation (3.6) in the framework of ultrafunctions is

$$\mathcal{C}^1(I^*,V_\Lambda(\Omega)),$$

defined as follows:

Definition 24. For every $k\in\mathbb{N}$ we set

$$\mathcal{C}^k(I^*,V_\Lambda(\Omega))=\left\{u\in\left[\mathcal{C}^k(I,V(\Omega))\right]^*\mid\forall t\in I^*,\ \forall i\leq k,\ \partial_t^i u(t,\cdot)\in V_\Lambda(\Omega)\right\},\ k\in\mathbb{N}.$$

The advantage in using $\mathcal{C}^1(I^*,V_\Lambda(\Omega))$ rather than $V_\Lambda(I\times\Omega)$ relays in the fact that we want to consider our evolution problem as a dynamical system on $V_\Lambda(\Omega)$, and the time as a continuous and homogeneous variable. In fact, at least in the models which we will consider, we have a better description of the phenomena in $\mathcal{C}^1(I^*,V_\Lambda(\Omega))$ rather than in $V_\Lambda(I\times\Omega)$ or in the standard space $\mathcal{C}^0(I,V(\Omega))\cap\mathcal{C}^1(I,L^2(\Omega))$.

3.4. Ultrafunctions and distributions

One of the most important properties of spaces of ultrafunctions is that they can be seen (in some sense that we will make precise later) as generalizations of the space of distributions (see also [10], where we construct an algebra of ultrafunctions that extends the space of distributions). The proof of this result is the topic of this section.

Let $E\subset\mathbb{R}^N$ be a set not necessarily open. In the applications in this paper E will be $\Omega\subset\mathbb{R}^N$ or $[0,T)\times\Omega\subset\mathbb{R}^{N+1}$.

Definition 25. The space of **generalized distribution** on E is defined as follows:

$$\mathscr{D}'_G(E)=L^2(E)^*/N,$$

where

$$N=\left\{\tau\in L^2(E)^*\mid\forall\varphi\in\mathscr{D}(E)\ \int\tau\varphi\,dx\sim 0\right\}.$$

The equivalence class of u in $L^2(E)^*$, with some abuse of notation, will be denoted by

$$[u]_{\mathscr{D}}.$$

Definition 26. For every (internal or external) vector space $W(E) \subset L^2(E)^*$, we set

$$[W(E)]_B = \left\{ u \in W(E) \mid \forall \varphi \in \mathscr{D}(E) \int u\varphi \, dx \text{ is finite} \right\}.$$

Definition 27. Let $[u]_{\mathscr{D}}$ be a generalized distribution. We say that $[u]_{\mathscr{D}}$ is a bounded generalized distribution if $u \in \left[L^2(E)^*\right]_B$.

Finally, we set

$$\mathscr{D}'_{GB}(E) := [\mathscr{D}'_G(E)]_B.$$

We now want to prove that the space $\mathscr{D}'_{GB}(E)$ is isomorphic (as a vector space) to $\mathscr{D}'(E)$. To do this we will need the following lemma.

Lemma 28. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and let $l \in \mathbb{R}$. If $\lim_{n\to+\infty} a_n = l$ then $sh(\lim_{\lambda\uparrow\Lambda} a_{|\lambda|}) = l$.*

Proof. Since $\lim_{n\to+\infty} a_n = l$, for every $\varepsilon \in \mathbb{R}_{>0}$ the set

$$I_\varepsilon = \{\lambda \in \mathfrak{L} \mid |l - a_{|\lambda|}| < \varepsilon\} \in \mathcal{U}.$$

In fact, let $N \in \mathbb{N}$ be such that $|a_m - l| < \varepsilon$ for every $m \geq N$. Then for every $\lambda_0 \in \mathfrak{L}$ such that $|\lambda_0| \geq N$ we have that $I_\varepsilon \supseteq \{\lambda \in \mathfrak{L} \mid \lambda_0 \subseteq \lambda\} \in \mathcal{U}$, and this proves that $I_\varepsilon \in \mathcal{U}$. Therefore for every $\varepsilon \in \mathbb{R}_{>0}$ we have

$$|l - \lim_{\lambda\uparrow\Lambda} a_{|\lambda|}| < \varepsilon,$$

and so $sh(\lim_{\lambda\uparrow\Lambda} a_{|\lambda|}) = l$. $\square$

Theorem 29. *There is a linear isomorphism*

$$\Phi : \mathscr{D}'_{GB}(E) \to \mathscr{D}'(E)$$

defined by the following formula:

$$\forall \varphi \in \mathscr{D},\ \langle \Phi\left([u]_{\mathscr{D}}\right), \varphi \rangle_{\mathscr{D}(E)} = sh\left(\int^* u \ \varphi^* \ dx \right).$$

Proof. Clearly the map Φ is well defined (namely $u \approx_{\mathscr{D}} v \Rightarrow \Phi\left([u]_{\mathscr{D}}\right) = \Phi\left([v]_{\mathscr{D}}\right)$), it is linear and its range is in $\mathscr{D}'(E)$. It is also immediate to see that it is injective. The most delicate part is to show that it is surjective. To see this let $T \in \mathscr{D}'(E)$; we have to find an ultrafunction u_T such that

$$\Phi\left([u_T]_{\mathscr{D}}\right) = T. \tag{3.7}$$

Since $L^2(E)$ is dense in $\mathscr{D}'(E)$ with respect to the weak topology, there is a sequence $\psi_n \in L^2(E)$ such that $\psi_n \to T$. We claim that

$$u_T = \lim_{\lambda \uparrow \Lambda} \ \psi_{|\lambda|}$$

satisfies (3.7) and $[u_T]_{\mathscr{D}} \in \mathscr{D}'_{GB}(E)$. Since u_T is a Λ-limit of $L^2(E)$ functions, we have that $u_T \in L^2(E)^*$, so $[u_T]_{\mathscr{D}} \in \mathscr{D}'_G(E)$. It remains to show that $[u_T]_{\mathscr{D}}$ is bounded and that $\Phi\left([u_T]_{\mathscr{D}}\right) = T$. Take $\varphi \in \mathscr{D}$; by definition,

$$\langle T, \varphi \rangle_{\mathscr{D}(E)} = \lim_{n \to +\infty} \int^* \psi_n \cdot \varphi dx = \lim_{n \to +\infty} a_n,$$

where we have set $a_n = \int \psi_n \cdot \varphi dx$. Then by Lemma 28 we have

$$\lim_{n \to +\infty} a_n = sh\left(\lim_{\lambda \uparrow \Lambda} a_{|\lambda|}\right) = sh\left(\lim_{\lambda \uparrow \Lambda} \int \psi_{|\lambda|} \cdot \varphi dx\right) =$$
$$sh\left(\int^* \left(\lim_{\lambda \uparrow \Lambda} \psi_{|\lambda|} \cdot \varphi\right) dx\right) = sh\left(\int^* u_T \cdot \varphi dx\right) = \left\langle \Phi\left([u_T]_{\mathscr{D}}\right), \varphi \right\rangle_{\mathscr{D}(E)},$$

therefore $\left\langle \Phi\left([u_T]_{\mathscr{D}}\right), \varphi \right\rangle_{\mathscr{D}(E)} = \langle T, \varphi \rangle_{\mathscr{D}(E)} \in \mathbb{R}$ and the thesis is proved. $\square$

From now on we will identify the spaces $\mathscr{D}'_{GB}(E)$ and $\mathscr{D}'(E)$; so, we will identify $[u]_{\mathscr{D}}$ with $\Phi\left([u]_{\mathscr{D}}\right)$ and we will write $[u]_{\mathscr{D}} \in \mathscr{D}'(E)$ and

$$\langle [u]_{\mathscr{D}}, \varphi \rangle_{\mathscr{D}(E)} := \langle \Phi[u]_{\mathscr{D}}, \varphi \rangle = sh\left(\int^* u \ \varphi^* \ dx\right).$$

Moreover, with some abuse of notation, we will write also that $[u]_{\mathscr{D}} \in L^2(E)$, $[u]_{\mathscr{D}} \in V(E)$, etc. meaning that the distribution $[u]_{\mathscr{D}}$ can be identified with a function f in $L^2(E)$, $V(E)$, etc. By our construction, this is equivalent to say that $f^* \in [u]_{\mathscr{D}}$. So, in this case, we have that $\forall \varphi \in \mathscr{D}(E)$

$$\langle [u]_{\mathscr{D}}, \varphi \rangle_{\mathscr{D}(E)} = sh\left(\int^* u \ \varphi^* \ dx\right) = sh\left(\int^* f^* \varphi^* dx\right) = \int f \ \varphi \ dx.$$

An immediate consequence of Theorem 29 is the following:

Proposition 30. *The space $\left[\mathcal{C}^1(I, V_\Lambda(\Omega))\right]_B$ can be mapped into a space of distributions by setting, $\forall u \in \left[\mathcal{C}^1(I, V_\Lambda(\Omega))\right]_B$,*

$$\forall \varphi \in \mathscr{D}(I \times \Omega), \left\langle [u]_{\mathscr{D}(I \times \Omega)}, \varphi \right\rangle = sh \int \int u(t,x) \varphi^*(t,x) dx dt. \tag{3.8}$$

Finally, let us also notice that the proof of Theorem 29 can be modified to prove the following result:

Proposition 31. *If $W(E)$ is an internal space such that $\mathscr{D}^*(E) \subset W(E) \subset L^2(E)^*$, then every distribution $[v]_{\mathscr{D}}$ has a representative $u \in W(E) \cap [v]_{\mathscr{D}}$.*

Namely, the map

$$\Phi : [W(E)]_B \to \mathscr{D}'(E)$$

defined by

$$\Phi(u) = [u]_{\mathscr{D}}$$

is surjective.

Proof. We can argue as in the proof of Theorem 29, by substituting $L^2(E)$ with $\mathscr{D}(E)$. This is possible since $\mathscr{D}(E)$ is dense in $L^2(E)$ (and so, in particular, $W(E)$ is dense in $L^2(E)$), and the density property was the only condition needed to prove the surjectivity of the embedding. $\square$

In the following sections we want to study problems such as equation (3.6) in the context of ultrafunctions. To do so we will need to restrict to the following family of operators:

Definition 32. We say that an operator

$$\mathcal{A} : V(\Omega) \to V'(\Omega)$$

is weakly continuous if, $\forall u, v \in [V_\Lambda(\Omega)]_B$, $\forall \varphi \in \mathscr{D}(\Omega)$, we have that if

$$\int u\varphi^* \, dx \sim \int v\varphi^* \, dx$$

then

$$\int \mathcal{A}^*(u) \; \varphi^* \, dx \sim \int \mathcal{A}^*(v) \; \varphi^* \, dx.$$

For our purposes, the important property of weakly continuous operators is that if

$$\mathcal{A} : V(\Omega) \to V'(\Omega)$$

is weakly continuous then it can be extended to an operator

$$[\mathcal{A}]_{\mathscr{D}} : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$$

by setting

$$[\mathcal{A}]_{\mathscr{D}} \left([u]_{\mathscr{D}}\right) = [\mathcal{A}\,(w)]_{\mathscr{D}}\,,$$

where $w \in [u]_{\mathscr{D}} \cap V(\Omega)$. In the following, with some abuse of notation we will write $[\mathcal{A}\,(u)]_{\mathscr{D}}$ instead of $[\mathcal{A}]_{\mathscr{D}} \left([u]_{\mathscr{D}}\right)$.

Remark 33. Definition 32 can be reformulated in the classical language as follows: $\mathcal{A}$ is weakly continuous if for every weakly convergent sequence u_n in $\mathscr{D}'(\Omega)$ the sequence $\mathcal{A}\,(u_n)$ is weakly convergent in $\mathscr{D}'(\Omega)$.

4. Generalized Ultrafunction Solutions (GUS)

In this section we will show that an evolution equation such as equation (3.6) has Generalized Ultrafunction Solutions (GUS) under very general assumptions on $\mathcal{A}$, and we will show the relationships of GUS with strong and weak solutions. However, before doing this, we think that it is helpful to give the feeling of the notion of GUS for stationary problems. This will be done in Section 4.1 providing a simple typical example. We refer to [4], [7] and [9] for other examples.

4.1. Generalized Ultrafunction Solutions for stationary problems

A typical stationary problem in PDE can be formulated ad follows:

$$\textit{Find} \quad u \in V(\Omega) \quad \textit{such that}$$

$$\mathcal{A}(u) = f, \tag{4.1}$$

where $V(\Omega) \subseteq L^2(\Omega)$ is a vector space and $\mathcal{A} : V(\Omega) \to V'(\Omega)$ is a differential operator and $f \in L^2(\Omega)$.

The "typical" formulation of this problem in the framework of ultrafunctions is the following one:

$$\textit{Find} \quad u \in V_\Lambda(\Omega) \quad \textit{such that}$$

$$\widetilde{\mathcal{A}}(u) = \widetilde{f}. \tag{4.2}$$

In particular, if $\mathcal{A} : V(\Omega) \to L^2(\Omega)$ and $f \in L^2(\Omega)$, the above problem can be formulated in the following equivalent "weak form":

$$\textit{Find} \quad u \in V_\Lambda(\Omega) \quad \textit{such that}$$

$$\forall \varphi \in V_\Lambda(\Omega), \ \int_{\Omega^*}^* \mathcal{A}^*(u)\varphi dx = \int_{\Omega^*}^* f^* \varphi dx. \tag{4.3}$$

Such an ultrafunction u will be called a **GUS** of Problem (4.2).

Usually, it is possible to find a classical solution for problems of the type (4.1) if there are a priory bounds, but the existence of a priori bounds is not sufficient to guarantee the existence of solutions in $V(\Omega)$. On the contrary, the existence of a priori bounds is sufficient to find a GUS in $V_\Lambda(\Omega)$ (as we are going to show).

Following the general strategy to find a GUS for Problem (4.2), we start by solving the following approximate problems for every λ in a qualified set :

$$\textit{Find } u_\lambda \in V_\lambda(\Omega) \quad \textit{such that}$$

$$\forall \varphi \in V_\lambda(\Omega), \ \int_\Omega \mathcal{A}(u_\lambda)\varphi dx = \int_\Omega f \varphi dx.$$

A priori bounds in each space $V_\lambda(\Omega)$ are sufficient to guarantee the existence of solutions. The next step consists in taking the Λ-limit. Clearly, this strategy can be applied to a very large class of problems. Let us consider a typical example in details:

Theorem 34. *Let $\mathcal{A} : V(\Omega) \to V'(\Omega)$ be a hemicontinuous*[7] *operator such that for every finite dimensional space $V_\lambda \subset V(\Omega)$ there exists $R_\lambda \in \mathbb{R}$ such that*

$$\textit{if } u \in V_\lambda \textit{and } \|u\|_\sharp = R_\lambda \textit{ then } \langle \mathcal{A}(u), u \rangle > 0, \tag{4.4}$$

where $\|\cdot\|_\sharp$ is any norm in $V(\Omega)$. Then the equation (4.2) has at least one solution $u_\Lambda \in V_\Lambda(\Omega)$.

Proof. If we set

$$B_\lambda = \left\{ u \in V_\lambda | \ \|u\|_\sharp \leq R_\lambda \right\}$$

and if $\mathcal{A}_\lambda : V_\lambda \to V_\lambda$ is the operator defined by the following relation:

$$\forall v \in V_\lambda, \ \langle \mathcal{A}_\lambda(u), v \rangle = \langle \mathcal{A}(u), v \rangle$$

then it follows from the hypothesis (4.4) that $\deg(\mathcal{A}_\lambda, B_\lambda, 0) = 1$, where $\deg(\cdot, \cdot, \cdot)$ denotes the topological degree (see e.g. [1]). Hence, $\forall \lambda \in \mathfrak{L}$,

$$\exists u \in V_\lambda, \forall v \in V_\lambda, \ \langle \mathcal{A}_\lambda(u), v \rangle = 0.$$

Taking the Λ-limit of the net (u_λ) we get a GUS $u_\Lambda \in V_\Lambda(\Omega)$ of equation (4.2). □

Example 35. Let Ω be an open bounded set in $\mathbb{R}^N$ and let

$$a(\cdot, \cdot, \cdot) : \mathbb{R}^N \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}^N, \ b(\cdot, \cdot, \cdot) : \mathbb{R}^N \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$$

be continuous functions such that $\forall \xi \in \mathbb{R}^N, \forall s \in \mathbb{R}, \forall x \in \overline{\Omega}$ we have

$$a(\xi, s, x) \cdot \xi + b(\xi, s, x)s \geq \nu\left(|\xi|\right), \tag{4.5}$$

where ν is a function (not necessarely negative) such that

$$\nu\left(t\right) \to +\infty \text{ for } t \to +\infty. \tag{4.6}$$

We consider the following problem:

$$\begin{aligned} &\textit{Find} \quad u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_0(\overline{\Omega}) \quad \textit{s.t.} \\ &\nabla \cdot a(\nabla u, u, x) = b(\nabla u, u, x). \end{aligned} \tag{4.7}$$

In the framework of ultrafunctions this problem becomes

$$\textit{Find} \quad u \in V(\Omega) := \left[\mathcal{C}^2(\Omega) \cap \mathcal{C}_0(\overline{\Omega})\right]_\Lambda \quad \textit{such that}$$

[7] An operator between Banach spaces is called *hemicontinuous* if its restriction to finite dimensional subspaces is continuous.

$$\forall\varphi\in V(\Omega),\ \int_\Omega \nabla\cdot a(\nabla u,u,x)\ \varphi\ dx=\int_\Omega b(\nabla u,u,x)\varphi dx.$$

If we set

$$\mathcal{A}(u)=-\nabla\cdot a(\nabla u,u,x)+b(\nabla u,u,x)$$

it is not difficult to check that conditions (4.5) and (4.6) are sufficient to guarantee the assumptions of Theorem 34. Hence we have the existence of a ultrafunction solution of problem (4.7). Problem (4.7) covers well known situations such as the case in which $\mathcal{A}$ is a maximal monotone operator, but also very pathological cases. E.g., by taking

$$a(\nabla u,u,x)=(|\nabla u|^{p-2}-\nabla u);\ b(\nabla u,u,x)=f(x),$$

we get the equation

$$\Delta_p u-\Delta u=f.$$

Since

$$\int_\Omega(-\Delta_p u+\Delta u)\ \ u\ dx=\|u\|_{W_0^{1,p}}^p-\|u\|_{H_0^1}^2,$$

it is easy to check that we have a priori bounds (but not the convergence) in $W_0^{1,p}(\Omega)$. Therefore we have GUS, and it might be interesting to study the kind of regularity of these solutions.

4.2. Strong and weak solutions of evolution problems

As usual, let

$$\mathcal{A}:V(\Omega)\to V'(\Omega)$$

be a differential operator.

We are interested in the following Cauchy problem for $t\in I:=[0,T)$: find u such that

$$\begin{cases}\partial_t u=\mathcal{A}(u);\\ \\ u(0)=u_0.\end{cases}\tag{4.8}$$

A solution $u=u(t,x)$ of problem (4.8) is called a **strong solution** if

$$u\in C^0(I,V(\Omega))\cap C^1(I,V'(\Omega)).$$

It is well known that many problems of type (4.8) do not have strong solutions even if the initial data is smooth (for example Burgers' equation BE). This is the reason why the notion of weak solution becomes necessary. If $\mathcal{A}$ is a linear operator and $\mathcal{A}(\mathscr{D}(\Omega))\subset\mathscr{D}'(\Omega)$, classically a distribution $T\in V'(I\times\mathbb{R})$ is called a weak solution of problem (4.8) if

$$\forall\varphi\in\mathscr{D}(I\times\mathbb{R}),\ -\langle T,\partial_t\varphi\rangle+\int_\Omega u_0(x)\ \varphi(0,x)dx=\left\langle T,\mathcal{A}^\dagger\varphi\right\rangle,$$

where $\mathcal{A}^\dagger$ is the adjoint of $\mathcal{A}$.

If $\mathcal{A}$ is not linear there is not a general definition of weak solution. For example, if you consider Burgers' equation, a function $w \in L^1_{loc}(I \times \Omega)$ is considered a weak solution if

$$\forall \varphi \in \mathscr{D}(I\times\Omega),\ -\int\int w\partial_t\varphi\, dxdt - \int_\Omega u_0(x)\,\varphi(0,x)dx + \frac{1}{2}\int\int w^2\partial_x\varphi\, dxdt = 0.$$

However, if we use the notion of generalized distribution developed in section 3.4 we can give a definition of weak solution for problems involving weakly continuous operators that generalizes the classical one for linear operators:

Definition 36. Let $\mathcal{A} : W \to \mathscr{D}'$ be weakly continuous. We say that $u \in W$ is a weak solution of Problem (4.8) if the following condition is fulfilled: $\forall \varphi \in \mathscr{D}(I \times \Omega)$

$$\int u(t,x)\varphi_t(t,x)\, dxdt - \int u(0,x)\varphi(0,x)dx = \langle A(u), \varphi\rangle.$$

From the theory developed in Section 3.4, the notion of weak solution given by Definition 36 can be written in nonstandard terms as follows: $[w]_{\mathscr{D}}$ is a weak solution of Problem (4.8) if

$$\begin{cases} w \in \left[C^1(I, V(\Omega))^*\right]_B; \\ \forall \varphi \in \mathscr{D}(I \times \Omega),\ \int_0^T \int_\Omega \partial_t w\varphi^* dxdt + \int_0^T \mathcal{A}(w)\,\varphi^* dt \sim 0; \\ w(0,x) = u_0(x). \end{cases}$$

By the above equations, any strong solution is a weak solution, but the converse is not true. A very large class of problems (such as BE) which do not have strong solutions have weak solutions, or even only distributional solutions. Unfortunately, there are problems which do not have even weak (or distributional) solutions, and worst than that there are problems (such as BE) which have more than one weak solution, namely the uniqueness of the Cauchy problem is violated, and hence the physical meaning of the problem is lost. This is why we think that it is worthwhile to investigate these kind of problems in the framework of generalized solutions in the world of ultrafunctions.

4.3. Generalized Ultrafunction Solutions and their first properties

In Section 4.1 we gave the definition of GUS for stationary problems. The definition of GUS for evolution problems is analogous:

Definition 37. An ultrafunction $u \in \mathcal{C}^1(I^*, V_\Lambda(\Omega))$, is called a Generalized Ultrafunction Solution (GUS) of problem (4.8) if $\forall v \in V_\Lambda(\Omega)$,

$$\begin{cases} \int \partial_t uv\ dx = \int \mathcal{A}^*(u)v\ dx; \\ u(0,x) = u_0(x). \end{cases} \tag{4.9}$$

Problem (4.9) can be rewritten as follows:

$$\begin{cases} u \in \mathcal{C}^1(I^*, V_\Lambda); \\ \partial_t u = P_\Lambda \mathcal{A}^*(u); \\ u(0,x) = u_0(x), \end{cases}$$

where P_Λ is the orthogonal projection. The main Theorem of this section states that problem (4.8) locally has a GUS. As for the ordinary differential equations in finite dimensional spaces, this solution is defined for an interval of time which depends on the initial data.

Theorem 38. *Let $\mathcal{A}|_{V_\lambda(\Omega)}$ be locally Lischitz continuous $\forall \lambda \in \mathfrak{L}$; then there exists a number $T_\Lambda(u_0) \in (0,T]_{\mathbb{R}^*}$ such that problem (4.8) has a unique GUS u_Λ in $[0, T_\Lambda(u_0))_{\mathbb{R}^*}$.*

Proof. For every $\lambda \in \mathfrak{L}$ let us consider the approximate problem

$$\begin{cases} u \in C^1(I, V_\lambda(\Omega)) \ \ and \ \ \forall v \in V_\lambda(\Omega); \\ \int_\Omega \partial_t u(t,x) \ v(x) dx = \int_\Omega \mathcal{A}(u(t,x)) \ v(x) dx; \\ u_\lambda(0) = \int_\Omega u_0(x) \ v(x) dx. \end{cases} \tag{4.10}$$

It is immediate to check that this problem is equivalent to the following one

$$\begin{cases} u \in C^1(I, V_\lambda(\Omega)); \\ \partial_t u(t,x) = P_\lambda \mathcal{A}(u(t,x)); \\ u_\lambda(0) = P_\lambda u_0, \end{cases} \tag{4.11}$$

where the "projection" $P_\lambda : L^2(\Omega) \to V_\lambda(\Omega)$ is defined by

$$\int_\Omega P_\lambda w(x) v(x) dx = \langle w, v \rangle, \ \forall v \in V_\lambda(\Omega). \tag{4.12}$$

The Cauchy problem (4.11) is well posed since $V_\lambda(\Omega)$ is a finite dimensional vector space and $P_\lambda \circ \mathcal{A}$ is locally Lipschitz continuous on V_λ. Then there exists a number $T_\lambda(u_0) \in (0,T]_{\mathbb{R}}$ such that problem (4.11) has a unique solution in $[0, T_\lambda(u_0))_{\mathbb{R}}$. Taking the Λ-limit, we get the conclusion. □

Definition 39. We will refer to a solution u_Λ given as in Theorem 38 as to a local GUS.

Clearly the GUS is a global solution (namely a function defined for every $t \in [0,T)$) if $T_\lambda(u_0)$ is equal to T. In concrete applications, the existence of a global solution usually is a consequence of the existence of a coercive integral of motion. In fact, we have the following corollary:

Corollary 40. *Let the assumptions of Theorem 38 hold. Moreover, let us assume that there exists a function $I : V(\Omega) \to \mathbb{R}$ such that if $u(t)$ is a local GUS in $[0, T_\lambda)$, then*

$$\partial_t I^* (u(t)) \leq 0 \tag{4.13}$$

(or, more in general, that $I^* (u(t))$ is not increasing) and such that $\forall \lambda \in \mathfrak{L}$, $I|_{V_\lambda(\Omega)}$ is coercive (namely if $u_n \in V_\lambda(\Omega)$ and $\|u_n\| \to \infty$ then $I(u_n) \to \infty$). Then $u(t)$ can be extended to the full interval $[0, T)$.

Proof. By our assumptions, there is a qualified set Q such that $\forall \lambda \in Q$, if $u_\lambda(t)$ is defined in $[0, T_\lambda)$, then

$$\partial_t I (u_\lambda(t)) \leq 0 \tag{4.14}$$

since otherwise the inequality (4.13) would be violated. By (4.14) and the coercivity of $I|_{V_\lambda(\Omega)}$ we have that $T_\lambda(u_0) = T$. Hence also $u(t)$ is defined in the full interval $[0, T)$. □

4.4. GUS, weak and strong solutions

We now investigate the relations between GUS, weak solutions and strong solutions.

Theorem 41. *Let $u \in C^1(I^*, V_\Lambda(\Omega))$ be a GUS of Problem (4.8), and let us assume that $\mathcal{A}$ is weakly continuous. Then*

1. *if*
$$u \in \left[\mathcal{C}^1(I^*, V_\Lambda(\Omega))\right]_B$$
then the distribution $[u]_{\mathscr{D}}$ is a weak solution of Problem (4.8);
2. *moreover, if*
$$w \in [u]_{\mathscr{D}} \cap \mathcal{C}^1(I, V(\Omega))$$
then w is a strong solution of Problem (4.8).

Proof. (1) In order to simplify the notations, in this proof we will write $\int$ instead of $\int^*$. Since u is a GUS, then for any $\varphi \in \mathscr{D}(I \times \Omega) \subset \mathcal{C}_B^\infty(I^*, V_\Lambda(\Omega))$ (we identify φ and φ^*) we have that

$$\int_0^T \int_{\Omega^*} \partial_t u\varphi \; dx \; dt = \int_0^T \int_{\Omega^*} \mathcal{A}^*(u)\varphi dx \; dt.$$

Integrating in t, we get

$$\int_0^T \int_{\Omega^*} u(t,x) \; \partial_t\varphi dx \; dt - \int_\Omega u_0(x)\varphi(0,x)dx + \int_0^T \int_{\Omega^*} \mathcal{A}^*(u(t,x))\varphi dx \; dt = 0.$$

By the definition of $[u]_{\mathscr{D}}$, and as $\mathcal{A}$ is weakly continuous, we have that

$$\int_0^T \int_{\Omega^*} u(t,x) \; \partial_t\varphi dx \; dt \sim \int_0^T \int_\Omega [u]_{\mathscr{D}}(t,x) \; \partial_t\varphi dx \; dt,$$

$$\int_{\Omega^*} u_0(x)\varphi(0,x)dx \sim \int_{\Omega} \left([u]_{\mathscr{D}}\right)_0(x)\varphi(0,x)dx,$$

$$\int_0^T \int_{\Omega^*} \mathcal{A}^*(u(t,x))\varphi dx\ dt \sim \int_0^T \int_{\Omega} \mathcal{A}([u]_{\mathscr{D}}(t,x))\varphi dx\ dt.$$

Henceforth

$$\int_0^T \int_{\Omega} [u]_{\mathscr{D}}(t,x)\ \partial_t\varphi dx\ dt - \int_{\Omega} \left([u]_{\mathscr{D}}\right)_0(x)\varphi(0,x)dx + \int_0^T \int_{\Omega} \mathcal{A}([u]_{\mathscr{D}}(t,x))\varphi dx\ dt \sim 0.$$

Since all three terms in the left hand side of the above equation are real numbers, we have that their sum is a real number, and so

$$\int_0^T \int_{\Omega} [u]_{\mathscr{D}}(t,x)\ \partial_t\varphi dx\ dt - \int_{\Omega} \left([u]_{\mathscr{D}}\right)_0(x)\varphi(0,x)dx + \int_0^T \int_{\Omega} \mathcal{A}([u]_{\mathscr{D}}(t,x))\varphi dx\ dt = 0,$$

namely $[u]_{\mathscr{D}}$ is a weak solution of Problem (4.8).

(2) If there exists $w \in [u]_{\mathscr{D}} \cap \mathcal{C}^1(I, V(\Omega))$ then $u \in \left[\mathcal{C}^1(I^*, V_\Lambda(\Omega))\right]_B$, so from (1) we get that w is a weak solution of Problem (4.8). Moreover, $w \in \mathcal{C}^1(I, V(\Omega)) \subseteq \mathcal{C}^0(I, V(\Omega)) \cap \mathcal{C}^1(I, V'(\Omega))$, and hence w is a strong solution. □

Usually, if problem (4.8) has a strong solution w, it is unique and it coincides with the GUS u in the sense that $[w^*]_{\mathscr{D}} = [u]_{\mathscr{D}}$ and in many cases we have also that

$$\|u - w^*\| \sim 0. \tag{4.15}$$

If problem (4.8) does not have a strong solution but only weak solutions, often they are not unique. Thus the GUS selects one weak solution among them.

Now suppose that $w \in L^1_{loc}$ is a weak solution such that $[u]_{\mathscr{D}} = [w^*]_{\mathscr{D}}$ but (4.15) does not hold. If we set

$$\psi = u - w^*$$

then $\|\psi\|$ is not an infinitesimal and ψ carries some information which is not contained in w. Since u and w define the same distribution, $[\psi]_{\mathscr{D}} = 0$, i.e.

$$\forall \varphi \in \mathscr{D},\ \int \psi\varphi^*\ dx = 0.$$

So the information contained in ψ cannot be contained in a distribution. Nevertheless this information might be physically relevant. In Section 5.4, we will see one example of this fact.

4.5. First example: the nonlinear Schroedinger equation

Let us consider the following nonlinear Schroedinger equation in $\mathbb{R}^N$:

$$i\partial_t u = -\frac{1}{2}\Delta u + V(x)u - |u|^{p-2}u;\ p > 2, \tag{4.16}$$

where, for simplicity, we suppose that $V(x) \in \mathcal{C}^1(\mathbb{R}^N)$ is a smooth bounded potential. A suitable space for this problem is

$$V(\mathbb{R}^N) = H^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N).$$

In fact, if $u \in V(\mathbb{R}^N)$, then the energy

$$E(u) = \int \left[\frac{1}{2} |\nabla u|^2 + V(x) |u|^2 + \frac{2}{p} |u|^p \right] dx \tag{4.17}$$

is well defined; moreover, if $u \in V(\mathbb{R}^N)$ we have that

$$-\frac{1}{2} \Delta u + V(x) u - |u|^{p-2} u \in V'(\mathbb{R}^N),$$

so the problem is well-posed in the sense of ultrafunctions (see Def. 37). It is well known, (see e.g. [18]) that if $p < 2 + \frac{4}{N}$ then the Cauchy problem (4.16) (with initial data in $V(\mathbb{R}^N)$) is well posed, and there exists a strong solution

$$u \in C^0(I, V(\mathbb{R}^N)) \cap C^1(I, V'(\mathbb{R}^N)).$$

On the contrary, if $p \geq 2 + \frac{4}{N}$, the solutions, for suitable initial data, blows up in a finite time. So in this case weak solutions do not exist. Nevertheless, we have GUS:

Theorem 42. *The Cauchy problem relative to equation (4.16) with initial data $u_0 \in V_\Lambda(\mathbb{R}^N)$ has a unique GUS $u \in \mathcal{C}^1(I, V_\Lambda(\mathbb{R}^N))$; moreover, the energy (4.17) and the L^2-norm are preserved along this solution.*

Proof. Let us consider the functional

$$I(u) = \int |u|^2 dx.$$

On every approximating space $V_\lambda(\Omega)$ we have that

$$\frac{d}{dt} \int |u|^2 dx = \int \frac{d}{dt} |u|^2 dx = 2Re \int \left(u, \frac{d}{dt} u \right) = 0,$$

therefore I^* (namely, the L^2-norm) is constant on GUS. A similar direct computation can be used to prove that also the energy is constant on GUS. Moreover, it is easily seen that $\forall \lambda \in \mathfrak{L}$, $I|_{V_\lambda(\Omega)}$ is coercive. Since also the other hypotheses of Theorem 38 are verified, we can apply Corollary 40 to get the existence and uniqueness of the GUS. $\square$

Now it is interesting to know what these solutions look like, and if they have any reasonable meaning from the physical or the mathematical point of view. For example, when $p < 2 + \frac{4}{N}$ the dynamics given by equation (4.16), for suitale initial data, produces solitons (see e.g. [14] or [3]); so we conjecture that in the case $p \geq 2 + \frac{4}{N}$ solitons with infinitesimal radius will appear at the concentration points and that they will behave as pointwise particles which follow the Newtonian Dynamics.

4.6. *Second example: the nonlinear wave equation*

Let us consider the following Cauchy problem relative to a nonlinear wave equation in a bounded open set $\Omega \subset \mathbb{R}^N$:

$$\begin{cases} \Box\psi + |\psi|^{p-2}\psi & = & 0 \;\text{ in } I \times \Omega; \\ \\ \psi & = & 0 \;\text{ on } I \times \partial\Omega; \\ \\ \psi(0,x) & = & \psi_0(x), \partial_t\psi(0,x) = \psi_1(x), \end{cases} \tag{4.18}$$

where $\Box = \partial_t^2 - \Delta,\ p > 2\,, I = [0,T)$. In order to formulate this problem in the form (4.8), we reduce it to a system of first order equations (Hamiltonian formulation):

$$\begin{cases} \partial_t\psi = \phi; \\ \\ \partial_t\phi = \Delta\psi - |\psi|^{p-2}\psi. \end{cases}$$

If we set

$$u = \begin{bmatrix} \psi \\ \phi \end{bmatrix};\ \mathcal{A}(u) = \begin{bmatrix} \phi \\ \Delta\psi - |\psi|^{p-2}\psi \end{bmatrix},$$

then problem (4.18) reduces to a particular case of problem (4.8).

A suitable space for this problem is

$$V(\Omega) = \left[\mathcal{C}^2(\Omega) \cap \mathcal{C}_0(\overline{\Omega})\right] \times \mathcal{C}(\Omega).$$

If $u \in V(\Omega)$, the energy

$$E(u) = \int_\Omega \left[\frac{1}{2}\left|\phi\right|^2 + \frac{1}{2}\left|\nabla\psi\right|^2 + \frac{1}{p}|\psi|^p\right] dx \tag{4.19}$$

is well defined.

It is well known, (see e.g. [23]) that problem (4.18) has a weak solution; however, it is possible to prove the global uniqueness of such a solution only if $p < \frac{N}{N-2}$ (any p if $N = 1, 2$).

On the contrary, in the framework of ultrafunctions we have the following result:

Theorem 43. *The Cauchy problem relative to equation (4.18) with initial data $u_0 \in V_\Lambda(\Omega)$ has a unique solution $u \in \mathcal{C}^1(I^*, V_\Lambda(\Omega))$; moreover, the energy (4.19) is preserved along this solution.*

Proof. We have only to apply Theorem 38 and Corollary 40, where we set

$$I(u) := E(u) = \int_\Omega \left[\frac{1}{2}\left|\phi\right|^2 + \frac{1}{2}\left|\nabla\psi\right|^2 + \frac{1}{p}|\psi|^p\right] dx. \qquad \square$$

5. The Burgers' equation

5.1. Preliminary remarks

In section 4.5 we have shown two examples which show that:

- equations which do not have weak solutions usually have a unique GUS;
- equations which have more than a weak solution have a unique GUS.

So ultrafunctions seem to be a good tool to study the phenomena modelled by these equations. At this point we think that the main question is to know what the GUS look like and if they are suitable to represent properly the phenomena described by such equations from the point of view of Physics. Of course this question might not have a unique answer: probably there are phenomena which are well represented by GUS and others which are not. In any case, it is worthwhile to investigate this issue relatively to the main equations of Mathematical Physics such as (4.16), (4.18), Euler equations, Navier-Stokes equations and so on.

We have decided to start this program with the (nonviscous) Burgers' equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0,$$

since it presents the following peculiarities:

- it is one of the (formally) simplest nonlinear PDE;
- it does not have a unique weak solution, but there is a physical criterium to determine the solution which has physical meaning (namely the entropy solution);
- many solutions can be written explicitly, and this helps to confront classical and ultrafunction solutions.

We recall that an other interesting approach to Burgers' equation by means of generalized functions (in the Colombeau sense) has been devoloped by Biagioni and Oberguggenberger in [16].

5.2. Properties of the GUS of Burgers' equations

The first property of Burgers' equation (BE) that we prove is that its smooth solutions with compact support have infinitely many integrals of motion:

Proposition 44. *Let $G(u)$ be a differentiable function, $G\in\mathcal{C}^1(\mathbb{R})$, $G(0)=0$, and let $u(t,x)$ be a smooth solution of (BE) with compact support. Then*

$$I(u)=\int G(u(t,x))dx$$

is a constant of motion of (BE) (provided that the integral converges).

Proof. The proof of this fact is known, we include it here only for the sake of completeness. Multiplying both sides of equation (BE) by $G'(u)$, we get the equation

$$G'(u)\partial_t u + G'(u)u\partial_x u = 0,$$

which gives

$$\partial_t G(u) + \partial_x H(u) = 0,$$

where

$$H(u) = \int_0^u sG'(s)ds. \tag{5.1}$$

Since u has compact support, we have that $-\int \partial_x H(u)dx = 0$, and hence

$$\partial_t \int G(u)dx = -\int \partial_x H(u)dx = 0. \qquad \square$$

Let us notice that Proposition 44 would hold also if we do not assume that u has a compact support, provided that it decays sufficiently fast.

In the literature, any function G as in the above theorem is called entropy and H is called entropy flux (see e.g. [17, 19]), since in some interpretation of this equation G corresponds (up to a sign) the the physical entropy. But this is not the only possible interpretation.

If we interpret (BE) as a simplification of the Euler equation, the unknown u is the velocity; then, for $G(u) = u$ and $G(u) = \frac{1}{2}u^2$, we have the following constants of motion: the **momentum**

$$P(u) = \int u dx$$

and the **energy**

$$E(u) = \frac{1}{2}\int u^2 dx.$$

However, in general the solutions of Burgers' equation are not smooth; in fact, if the initial data $u_0(x)$ is a smooth function with compact support, the solution develops singularities. Hence we must consider weak solutions which, in this case, are solutions of the following equation in weak form: $w \in L^1_{loc}(I \times \Omega)$, and $\forall \varphi \in \mathscr{D}(I \times \Omega)$

$$\int_0^T \int_\Omega w(t,x)\partial_t\varphi(t,x)\ dxdt - \int_\Omega u_0(x)\varphi(0,x)dx + \frac{1}{2}\int_0^T \int_\Omega w(t,x)^2\partial_x\varphi(t,x)\ dxdt = 0. \tag{5.2}$$

Nevertheless, the momentum and the energy of the GUS of Burgers' equation are constants of motion as we will show in Theorem 46. This result holds if we work in $\mathcal{C}^1(I^*, U_\Lambda(\mathbb{R}))$, where $U_\Lambda(\mathbb{R})$ is the space of ultrafunctions described in

Theorem 22.

With this choice of the space of ultrafunctions, a GUS of the Burgers' equation, by definition, is a solution of the following problem:

$$\left\{ \begin{array}{l} u \in \mathcal{C}^1(I^*, U_\Lambda(\mathbb{R})) \quad and \quad \forall v \in U_\Lambda(\mathbb{R}) \\ \\ \quad \int (\partial_t u)\, v dx = - \int (u \partial_x u)\, v dx; \\ \\ \qquad \quad u\,(0,x) = u_0\,(x)\,, \end{array} \right. \tag{5.3}$$

were $u_0 \in U_\Lambda(\mathbb{R})$ (mostly, we will consider the case where $u_0 \in \left(H^1_c(\mathbb{R})\right)^\sigma$). Let us recall that, by Definition 24, for every $u \in \mathcal{C}^1(I^*, U_\Lambda(\mathbb{R}))$, we have $\partial_t u(t, \cdot) \in U_\Lambda(\mathbb{R})$.

We have the following result:

Theorem 45. *For every initial data $u_0 \in U_\Lambda(\mathbb{R})$ the problem (5.3) has a GUS.*

Proof. It is sufficient to apply Theorem 38 to obtain the local existence of a GUS u, and then Corollary 40 with

$$I(w) = E(w) = \frac{1}{2} \int w^2 dx$$

to deduce that the local GUS is, actually, global. In fact if we take $v(t,x) = u(t,x)$ in the weak equation that defines Problem (5.3), we get

$$\begin{array}{rcl} \displaystyle\int (\partial_t u)\, u dx & = & \displaystyle -\int [u\partial_x u]\, u dx = \\ \displaystyle -\int [u\partial_x u]\, u \cdot \widetilde{1} dx & = & \displaystyle -\int_{-\beta}^{\beta} [u\partial_x u]\, u dx = \\ \displaystyle -\frac{1}{3}\int_{-\beta}^{\beta} \partial_x u^3 dx & = & 0, \end{array}$$

as $u(\beta) = u(-\beta)$. Then

$$\partial_t E(u) = 0$$

and hence Corollary 40 can be applied. □

Theorem 46. *Problem (5.3) has two constants of motion: the energy*

$$E = \frac{1}{2} \int u^2 \ dx$$

and the momentum

$$P = \int u \ dx.$$

Proof. We already proved that the energy is constant in the proof of Theorem 45. In order to prove that also P is constant take $v = \widetilde{1} \in U_\Lambda(\mathbb{R})$ in equation

(5.3). Then we get

$$\partial_t P = \partial_t \int u dx = \int \partial_t u \widetilde{1} dx = -\int u \partial_x u \widetilde{1} dx = -\frac{1}{2} \int_{-\beta}^{+\beta} \partial_x u^2 dx = 0,$$

as $u(-\beta) = u(\beta)$. $\square$

Let us notice that Theorems 45 and 46 hold even if u_0 is a very singular object, e.g. a delta-like ultrafunction.

Remark 47. Proposition 44 shows that the strong solutions of (BE) have infinitely many constants of motion; is this fact true for the GUS? Let us try to prove that

$$\int G(u(t,x))dx$$

is constant following the same proof used in Theorem 45 and 46. We set

$$v(t,x) = P_\Lambda G'(u) \in C(I^*, U_\Lambda)$$

and we replace it in eq. (5.3), so that

$$\begin{aligned} \partial_t \int G(u(t,x))dx &= \int \partial_t u G'(u) dx \\ &= \int \partial_t u P_\Lambda G'(u) dx \quad (\text{since } \partial_t u(t,\cdot) \in U_\Lambda) \\ &= -\int u \partial_x u P_\Lambda G'(u) dx. \end{aligned}$$

Now, if we assume that $G'(u(t,.)) \in U_\Lambda(\mathbb{R})$, we have that $P_\Lambda G'(u) = G'(u)$ and hence

$$\begin{aligned} \partial_t \int G(u(t,x))dx &= -\int u \partial_x u G'(u) dx \\ &= -\int \partial_x H(u) dx = 0 \end{aligned}$$

where $H(u)$ is defined by (5.1). Thus $\int G(u(t,x))dx$ is a constant of motion provided that

$$G'(u) \in C(I, U_\Lambda). \tag{5.4}$$

However, this is only a sufficient condition. Clearly, in general the analogous of condition (5.4) will depend on the choice of the space of ultrafunctions $V_\Lambda(\mathbb{R})$: different choices of this space will give different constants of motion. Our choice $V_\Lambda(\mathbb{R}) = U_\Lambda(\Omega)$ was motivated by the fact that GUS of equation (5.3) in $U_\Lambda(\Omega)$ preserves both the energy and the momentum.

5.3. GUS and weak solutions of BE

In this section we consider equation (5.3) with $u_0 \in \left(H^1_c(\mathbb{R})\right)^\sigma$. Our first result is the following:

Theorem 48. *Let u be the GUS of problem (5.3) with initial data $u_0 \in \left(H^1_c(\mathbb{R})\right)^\sigma$. Then $[u]_{\mathscr{D}(I\times\Omega)}$ is a weak solution of problem BE.*

Proof. From Theorem 45 we know that the problem admits a GUS u, and from Theorem 46 we deduce that $[u]_{\mathscr{D}}$ is a bounded generalized distribution: in fact, for every $\varphi \in \mathscr{D}(I\times\mathbb{R})$ we have

$$\left|\int \overline{u}\varphi dx\right| \leq \left(\int \overline{u}^2 dx\right)^{\frac{1}{2}} \left(\int \varphi^2 dx\right)^{\frac{1}{2}} < +\infty$$

as $\int \overline{u}^2 dx = \int u_0^2 dx < +\infty$ by the conservation of energy on GUS. Therefore, from Theorem 41 we deduce that $w := [\bar{u}]_{\mathscr{D}(I\times\Omega)}$ is a weak solution of problem BE. $\square$

Thus the GUS of problem 5.3 is unique and it is associated with a weak solution of problem BE. It is well known (see e.g. [17] and references therein) that weak solutions of (BE) are not unique: hence, in a certain sense, the ultrafunctions give a way to choose a particular weak solution among the (usually infinite) weak solutions of problem BE.

However, among the weak solutions there is one that is of special interest, namely the **entropy solution**. The entropy solution is the only weak solution of (BE) satisfying particular conditions (the entropy conditions) along the curves of discontinuity of the solution (see e.g. [20], Chapter 3). For our purposes, we are interested in the equivalent characterization of the entropy solution as the limit, for[8] $\nu \to 0$, of the solutions of the following parabolic equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{5.5}$$

(see e.g. [21] for a detailed study of such equations). These equations are called the viscous Burgers' equations and they have smooth solutions in any reasonable function space. In particular, in Lemma 49, we will prove that the problem 5.5 has a unique GUS in $U_\Lambda(\mathbb{R})$ for every initial data $u_0 \in U_\Lambda(\mathbb{R})$. Now, if $\bar{u}$ is the GUS of problem 5.5 with a classical initial condition $u_0 \in L^2(\mathbb{R})$, then $[\overline{u}]_{\mathscr{D}}$ is bounded: in fact, for every $\varphi \in \mathscr{D}(I\times\mathbb{R})$ we have

$$\left|\int \overline{u}\varphi dx\right| \leq \left(\int \overline{u}^2 dx\right)^{\frac{1}{2}} \left(\int \varphi^2 dx\right)^{\frac{1}{2}} < +\infty$$

as $\int \overline{u}^2 dx \leq \int u_0^2 dx < +\infty$. Therefore, from Theorem 41 we deduce that $w := [\bar{u}]_{\mathscr{D}(I\times\Omega)}$ is a weak solution.

[8]In this approach, ν is usally called the viscosity.

We are now going to prove that it is possible to choose ν infinitesimal in such a way that w is the entropy solution. This fact is interesting since it shows that this GUS represents properly, from a Physical point of view, the phenomenon described by Burgers' equation. In order to see this let us consider the problem (5.5) with ν hyperreal.

Lemma 49. *The problem*

$$\left\{ \begin{array}{c} u \in \mathcal{C}^1(I, U_\Lambda(\Omega)) \text{ and } \forall v \in U_\Lambda(\mathbb{R}) \\ \\ \int \left(\partial_t u(t,x) + u \partial_x u(t,x)\right) v(x) dx = \int \nu \partial_x^2 u(t,x) v(x) dx, \\ \\ u\left(0\right) = u_0 \end{array} \right. \tag{5.6}$$

has a unique GUS for every $\nu \in \left(\mathbb{R}^+\right)^*$ and every $u_0 \in U_\Lambda(\mathbb{R})$.

Proof. Let $(U_\lambda(\mathbb{R}))_{\lambda \in \mathfrak{L}}$ be an approximating net of $U_\Lambda(\mathbb{R})$. Since $\nu \in \left(\mathbb{R}^+\right)^*$ and $u_0 \in U_\Lambda(\mathbb{R})$, we have that for every $\lambda \in \mathfrak{L}$ there exist $\nu_\lambda \in \mathbb{R}^+$ and $u_{0,\lambda} \in U_\lambda(\mathbb{R})$ such that

$$\nu = \lim_{\lambda \uparrow \Lambda} \nu_\lambda \text{ and } u_0 = \lim_{\lambda \uparrow \Lambda} u_{0,\lambda}.$$

Thus, we can consider the approximate problems

$$\left\{ \begin{array}{c} u \in \mathcal{C}^1(I, U_\lambda(\mathbb{R})) \text{ and } \ \forall v \in U_\lambda(\mathbb{R}) \\ \\ \int \left(\partial_t u(t,x) + u \partial_x u(t,x)\right) v(x) dx = \int \nu \partial_x^2 u(t,x) v(x) dx, \\ \\ u\left(0\right) = u_{0,\lambda}. \end{array} \right. \tag{5.7}$$

For every λ, the problem (5.7) has a unique solution u_λ. If we let $u_\Lambda = \lim_{\lambda \uparrow \Lambda} u_\lambda$ we have that u_Λ is the unique ultrafunction solution of problem (5.6). □

Let us call u_ν the GUS of Problem (5.6). A natural conjecture would be that, if u_0 is standard, then for every ν infinitesimal the distribution $[u_\nu]_{\mathscr{D}(I \times \Omega)}$ is the entropy solution of Burgers' equation. However, as we are going to show in the following Theorem, in general this property is true only "when ν is a *large* infinitesimal":

Theorem 50. *Let u_0 be standard, let z be the entropy solution of Problem BE with initial condition u_0 and, for every $\nu \in \mathbb{R}^*$, let u_ν be the solution of Problem 5.6 with initial condition u_0^*. Then there exists an infinitesimal number ν_0 such that, for every infinitesimal $\nu \geq \nu_0$, $[u_\nu]_{\mathscr{D}(I \times \Omega)} = z$; namely, the GUS of Problem 5.6, for every infinitesimal $\nu \geq \nu_0$, correspond (in the sense of Definition 25) to the entropy solution of Problem BE.*

Proof. For every real number ν we have that the standard problem

$$\begin{cases} w \in \mathcal{C}^1(I, H^1_\flat(\mathbb{R})), \\ \partial_t w(t,x) + w\partial_x w(t,x) = \nu \partial_x^2 w(t,x), \\ w\,(0) = u_0 \end{cases}$$

has a unique solution w_ν. Therefore for every real number ν we have $u_\nu = w_\nu^*$. For overspill we therefore have that there exists an infinitesimal number ν_0 such that, for every infinitesimal $\nu \geq \nu_0$, $u_\nu = w_\nu$, where w_ν is the solution of the problem

$$\begin{cases} w \in \mathcal{C}^1(I, H^1_\flat(\mathbb{R}))^*, \\ \partial_t w(t,x) + w\partial_x w(t,x) = \nu \partial_x^2 w(t,x), \\ w\,(0) = u_0^*. \end{cases}$$

But as $z = \lim_{\varepsilon \to 0^+} v_\varepsilon$, we have that for every infinitesimal number ν, for every test function φ we have that

$$\langle z^* - v_\nu, \varphi^* \rangle \sim 0.$$

In particular for every infinitesimal $\nu \geq \nu_0$,

$$\langle z^* - u_\nu, \varphi^* \rangle \sim 0,$$

and as this holds for every test function φ we have our thesis. $\square$

Theorem 50 shows that, for a standard initial value u_0, there exists a ultrafunction which corresponds to the entropy solution of Burgers' equation; moreover, this ultrafunction solves a viscous Burgers' equation for an infinitesimal viscosity (namely, it is the solution of an infinitesimal perturbation of Burgers' equation). However, within ultrafunctions theory there is another "natural" solution of Burgers' equation for a standard initial value u_0, namely the unique ultrafunction u that solves Problem 5.3. We already proved in Theorem 48 that u corresponds (in the sense of Definition 25) to a weak solution of Burgers' equation. Our conjecture is that this weak solution is precisely the entropy solution; however, we have not been able to prove this (yet!). Nevertheless, in any case it makes sense to analyse this solution: this will be done in the next section.

5.4. The microscopic part

Let $u \in C^1(I^*, U_\Lambda)$ be the GUS of (5.3) and let $w = [u]_{\mathscr{D}}$. With some abuse of notation we will identify the distribution w with a L^2 function. We want to compare u and w^* and to give a physical interpretation of their difference.

Since we have that

$$[u]_{\mathscr{D}} = [w^*]_{\mathscr{D}}$$

we can write

$$u = w^* + \psi;$$

we have that

$$\forall \varphi \in \mathscr{D}\left(I \times \Omega\right), \int \int^* u\varphi^* \ dx \ dt \sim \int \int w\varphi \ dx \ dt$$

and

$$\int \int^* \psi\varphi^* \ dx \ dt \sim 0. \tag{5.8}$$

We will call w (and w^*) the macroscopic part of u and ψ the microscopic part of u; in fact, we can interpret (5.8) by saying that ψ does not appear to a mascroscopic analysis. On the other hand, $\int^* \psi\varphi \ dx \ dt \not\sim 0$ for some $\varphi \in C^1(I^*, U_\Lambda) \backslash \mathscr{D}\left(I \times \Omega\right)$. Such a φ "is able" to detect the *infinitesimal oscillations* of ψ. This justifies the expression "macroscopic part" and "microscopic part". So, in the case of Burgers equation, the ultrafunctions do not produce a solution to a problem without solutions (as in the example of section 4.5), but they give a different description of the phenomenon, namely they provide also the information contained in the microscopic part ψ.

So let us analyze it:

Proposition 51. *The microscopic part ψ of the GUS solution of problem (5.3) satisfies the following properties:*

1. *the momentum of ψ vanishes:*
$$\int \psi \ dx = 0;$$
2. *w^* and ψ are almost orthogonal:*
$$\int \int \psi w^* \ dxdt \sim 0;$$
3. *the energy of u is the sum of the kinetic macroscopic energy, $\int |w(t,x)|^2 \ dx$, the kinetic microscopic energy (heat) $\int |\psi(t,x)|^2 \ dx$ and an infinitesimal quantity;*
4. *if w is the entropy solution then the "heat" $\int |\psi(t,x)|^2 \ dx$ increases.*

Proof. 1) $\int \psi \, dx = \int u \, dx - \int w^* \, dx$, and the conclusion follows as both u and w preserve the momentum.

2) First of all we observe that the L^2 norm of ψ is finite, as $\psi = u - w^*$ and the L^2 norms of u and w are finite. Now let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathscr{D}(I \times \Omega)$ that converges strongly to w in L^2. Let $\{\varphi_\nu\}_{\nu \in \mathbb{N}^*}$ be the extension of

this sequence. As $\varphi_n \rightharpoonup w$ in L^2, we have that for any infinite number $N \in \mathbb{N}$ $\|\varphi_N - w^*\|_{L^2} \sim 0$. For every finite number $n \in \mathbb{N}^*$ we have that

$$\int \psi\varphi_n dxdt = 0,$$

as $\varphi_n \in \mathscr{D}(I \times \Omega)$. By overspill, there exists an infinite number N such that $\int \psi\varphi_N dxdt = 0$. If we set $\eta = w^* - \varphi_N$, we have $\|\eta\|_{L^2} \sim 0$. Then

$$\begin{aligned} \left|\int \psi w^* dxdt\right| &= \left|\int \psi(\varphi_N + \eta)dxdt\right| \\ &= \left|\int \psi\varphi_N dxdt + \int \psi\eta dxdt\right| \sim 0, \end{aligned}$$

as $\int \psi\varphi_N dxdt = 0$ and $\left|\int \psi\eta dxdt\right| \leq \int |\psi|\,|\eta|\,dxdt \leq (\|\psi\|_{L^2} \cdot \|\eta\|_{L^2})^{\frac{1}{2}} \sim 0$.

3) This follows easily from (2).

4) The energy of $u = w^* + \psi$ is constant, while the energy of w^*, if w is the entropy solution, decreases. Therefore we deduce our thesis from (3). □

Now let $\Omega \subset I \times \mathbb{R}$ be the region where w is regular (say H^1) and let $\Sigma = (I \times \mathbb{R})\setminus\Omega$ be the singular region. We have the following result:

Theorem 52. *ψ satisfies the following equation in the sense of ultrafunctions:*

$$\partial_t\psi + \partial_x(\mathbf{V}\psi) = F,$$

where

$$\mathbf{V} = \mathbf{V}(w,\psi) = w(t,x) + \frac{1}{2}\psi(t,x) \tag{5.9}$$

and

$$\mathfrak{supp}\,(F(t,x)) \subset N_\varepsilon(\Sigma),$$

where $N_\varepsilon(\Sigma)$ is an infinitesimal neighborhood of Σ^*.

Proof. In Ω we have that

$$\partial_t w + w\partial_x w = 0$$

Since $u = w + \psi$ satisfies the following equation (in the sense of ultrafunctions),

$$D_t u + P(u_x\partial u) = 0$$

we have that ψ satisfies the equation,

$$D_t\psi + P\left[\partial_x\left(w\psi + \frac{1}{2}\psi^2\right)\right] = 0$$

in $\Omega^*\setminus N_\varepsilon(\Sigma)$ where $N_\varepsilon(\Sigma)$ is an infinitesimal neighborhood of Σ^*. □

As we have seen ψ^2 can be interpreted as the density of heat. Then $\mathbf{V}$ can be interpreted as the flow of ψ; it consists of two parts: w which is the macroscopic component of the flow and $\frac{1}{2}\psi(t,x)$ which is the transport due to the Brownian motion.

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