# A MODEL PROBLEM FOR ULTRAFUNCTIONS

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ABSTRACT. In this article. we show that non-Archimedean mathematics (NAM), namely mathematics which uses infinite and infinitesimal numbers, is useful to model some physical problems which cannot be described by the usual mathematics. The problem which we will consider here is the minimization of the functional
$$E(u, q) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx + u(q).$$
When $\Omega \subset \mathbb{R}^N$ is a bounded open set and $u \in \mathcal{C}_0^2(\overline{\Omega})$, this problem has no solution since $\inf E(u, q) = -\infty$. On the contrary, as we will show, this problem is well posed in a suitable non-Archimedean frame. More precisely, we apply the general ideas of NAM and some of the techniques of Non Standard Analysis to a new notion of generalized functions, called ultrafunctions, which are a particular class of functions based on a Non-Archimedean field. In this class of functions, the above problem is well posed and it has a solution.

## 1. INTRODUCTION

By non-Archimedean mathematics (NAM) we mean mathematics based on non-Archimedean fields, namely ordered fields which contain infinite and infinitesimal numbers. We believe that, in many circumstances, NAM allows to construct models of the physical world in a very elegant and simple way. In the years around 1900, NAM was investigated by prominent mathematicians such as Du Bois-Reymond [7], Veronese [13], David Hilbert [10] and Tullio Levi-Civita [9], but then it has been forgotten until the '60s when Abraham Robinson presented his Non Standard Analysis (NSA) [12]. We refer to Ehrlich [8] for a historical analysis of these facts and to Keisler [11] for a very clear exposition of NSA (see also [1, 4])

The purpose of this article is to show that NAM is useful to model some Physical problems which cannot be described by the usual mathematics even if they are relatively simple.

The notion of material point is a basic tool in Mathematical Physics since the times of Euler who introduced it. Even if material points do not exist, nevertheless they are very useful in the description of nature and they simplify the models so that they can be treated by mathematical tools. However, as new notions entered

in Physics (such as the notion of field), the use of material points led to situations which required new mathematics. For example, in order to describe the electric field generated by a charged point, we need the notion of Dirac measure $\delta_q$, namely this field satisfies the following equation:

$$\Delta u = \delta_q \tag{1.1}$$

where $\Delta$ is the Laplace operator.

In this article, we will describe a simple problem whose model requires NAM. Let $\Omega \subseteq \mathbb{R}^2$ be an open bounded set which represents a (ideal) membrane. Suppose that in $\Omega$ is placed a material point $P$, which is left free to move.

Suppose that the point has a unit weight and the only forces acting on it are the gravitational force and the reaction of the membrane. If $q \in \Omega$ is the position of the point and $u(x)$ represents the profile of the membrane, it follows that equation (1.1) holds in $\Omega$ with boundary condition $u = 0$ on $\partial\Omega$.

The question is: which is the point $q_0 \in \Omega$ that the particle will occupy?
The natural way to approach this problem would be the following: for every $q \in \Omega$, the energy of the system is given by the elastic energy plus the gravitational energy, namely

$$E(u, q) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx + u(q). \tag{1.2}$$

If the couple $(u_0, q_0)$ minimizes $E$, then $q_0$ is the equilibrium point. For every $q \in \Omega$, let $u_q(x)$ be the configuration when $P$ is placed in $q$, namely the solution of equation (1.1). So the equilibrium point $q_0$ is the point in which the function

$$F(q) = E(u_q, q) \tag{1.3}$$

has a minimum.

In the classical context, this "natural" approach cannot be applied; in fact $u_q(x)$ has a singularity at the point $q$ which makes $u(q)$ not well defined and the integral in (1.2) to diverge. On the contrary, this problem can be treated in NAM as we will show. In fact, since infinite numbers are allowed, we will be able to find a minimum configuration for the energy (1.2).

To pursue this program, we apply the general ideas of NAM and some of the techniques of NSA to a new notion of generalized functions which we have called **ultrafunctions** (see [2]). Ultrafunctions are a particular class of functions based on a superreal field $\mathbb{R}^* \supset \mathbb{R}$. More exactly, to any continuous function $f : \mathbb{R}^N \to \mathbb{R}$ we associate in a canonical way an ultrafunction $f_\Phi : (\mathbb{R}^*)^N \to \mathbb{R}^*$ which extends $f$; but the ultrafunctions are much more than the functions and among them we can find solutions of functional equations such as equation (1.1) which are defined in every point of $\Omega^* \subset (\mathbb{R}^*)^N$. Thus, the energy ((1.2) is well defined for every ultrafunction even if it might assume infinite values.

Now we list some of the peculiar properties of the ultrafunctions:

- the space of ultrafunctions is larger than the space of distributions, namely, to every distribution $T$, we can associate in a canonical way an ultrafunction $T_\Phi$ (for details see [2]); in particular the Dirac measure can be represented by an ultrafunction $\delta_q(x)$ and, for every ultrafunction $u$, we have that

$$\int u(x)\delta_q(x)dx = u(q);$$

- similarly to the distributions, the ultrafunctions are motivated by the need of having generalized solutions and also by the need to model extreme physical situations which cannot be described by functions defined in $\mathbb{R}^N$; however, while the distributions are no longer functions, the ultrafunctions are still functions even if they have larger domain and range;
- unlike the distributions, the space of ultrafunctions is suitable for non linear problems such as the one described above;
- if a problem has a unique classical solution $u$, then $u_\Phi$ is the only solution in the space of ultrafunctions;
- the main strategy to prove the existence of generalized solutions in the space of ultrafunction is relatively simple; it is just a variant of the Faedo-Galerkin method.

Before concluding the introduction, we refer to [3] and to [5] where other situations which require NAM are presented.

1.1. **Notation.** Let $\Omega$ be a subset of $\mathbb{R}^N$: then

- $\mathcal{C}(\Omega)$ denotes the set of real continuous functions defined on $\Omega$;
- $\mathcal{C}_0(\overline{\Omega})$ denotes the set of real continuous functions on $\overline{\Omega}$ which vanish on $\partial\Omega$;
- $\mathcal{C}^k(\Omega)$ denotes the set of functions defined on $\Omega \subset \mathbb{R}^N$ which have continuous derivatives up to the order $k$;
- $\mathcal{C}_0^k(\overline{\Omega}) = \mathcal{C}^k(\overline{\Omega}) \cap \mathcal{C}_0\left(\overline{\Omega}\right)$;
- $\mathcal{D}(\Omega)$ denotes the set of the infinitely differentiable functions with compact support defined on $\Omega \subset \mathbb{R}^N$;
- $H^1(\Omega)$ is the usual Sobolev space defined as the set of functions $u \in L^2(\Omega)$ such that $\nabla u \in L^2(\Omega)$;
- $H_0^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$;
- $H^{-1}(\Omega)$ is the topological dual of $H_0^1(\Omega)$.

## 2. Ultrafunctions

In this section we briefly recall the notion of $\Lambda$-limit and of ultrafunction which have been introduced in [2].

2.1. **The $\Lambda$-limit.** The idea behind the concept of $\Lambda$-limit is the following: let $\mathbb{U}$ denote a "mathematical universe" (which will be precisely introduced in definition (2.4)), and $\mathcal{F}$ the set of finite subsets of $\mathbb{U}$, ordered by inclusion. The $\Lambda$-limit can be thought as a way to associate to every net $\varphi : \mathcal{F} \to \mathbb{R}$ a limit $\lim_{\lambda\uparrow\mathbb{U}} \varphi(\lambda)$ that satisfies a few properties of coherence.

These limits will be elements of a Non-Archimedean field $\mathbb{K}$; since this leads to work in such fields, we recall a few basic facts and definitions:

**Definition 2.1.** Let $\mathbb{K}$ be an ordered field. Let $\xi \in \mathbb{K}$. We say that:

- $\xi$ is infinitesimal if for all $n \in \mathbb{N}$ $|\xi| < \frac{1}{n}$;
- $\xi$ is finite if there exists $n \in \mathbb{N}$ such as $|\xi| < n$;
- $\xi$ is infinite if for all $n \in \mathbb{N}$ $|\xi| > n$.

**Definition 2.2.** An ordered field $\mathbb{K}$ is called non-Archimedean if it contains an infinitesimal $\xi \neq 0$.

We are interested in fields that extend $\mathbb{R}$:

**Definition 2.3.** A superreal field is an ordered field $\mathbb{K}$ that properly extends $\mathbb{R}$.

Since $\mathbb{R}$ is complete, it is easily seen that every superreal field contains infinitesimal and infinite numbers.

In order to precise the notion of $\Lambda$-limit, we need to define the notion of "mathematical universe". For our applications, we take as mathematical universe the superstructure on $\mathbb{R}$:

**Definition 2.4.** The **superstructure** on $\mathbb{R}$ is

$$\mathbb{U} = \cup_{n=0}^{\infty} \mathbb{U}_n$$

where $\mathbb{U}_n$ is defined by induction as follows:

$$\mathbb{U}_0 = \mathbb{R};$$

$$\mathbb{U}_{n+1} = \mathbb{U}_n \cup \mathcal{P}(\mathbb{U}_n).$$

Here $\mathcal{P}(E)$ denotes the power set of $E$. If we identify the couples with the Kuratowski pairs and the functions and the relations with their graphs, $\mathbb{U}$ formalizes the intuitive idea of mathematical universe.

We denote by $\mathcal{F}$ the set of finite subsets of $\mathbb{U}$. Ordered with the relation of inclusion, $\mathcal{F}$ becomes a direct set; following the usual nomenclature, we call *net* (with values in $E$) any function $\varphi : \mathcal{F} \to E$.

Following [2], we introduce the $\Lambda$-limit axiomatically:

(A1) **Existence Axiom.** There is a superreal field $\mathbb{K} \supset \mathbb{R}$ such that for every net $\varphi : \mathcal{F} \to \mathbb{R}$ there exists a unique element $L \in \mathbb{K}$ called the "$\Lambda$-limit" of $\varphi$. The $\Lambda$-limit will be denoted by

$$L = \lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda).$$

Moreover we assume that every $\xi \in \mathbb{K}$ is the $\Lambda$-limit of some net $\varphi : \mathcal{F} \to \mathbb{R}$.

(A2) **Real numbers axiom.** If $\varphi(\lambda)$ is eventually constant, namely $\exists \lambda_0 \in \mathcal{F} : \forall \lambda \supset \lambda_0, \ \varphi(\lambda) = r$, then

$$\lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) = r.$$

(A3) **Sum and product Axiom.** For all $\varphi, \psi : \mathcal{F} \to \mathbb{R}$:

$$\lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) + \lim_{\lambda \uparrow \mathbb{U}} \psi(\lambda) = \lim_{\lambda \uparrow \mathbb{U}} (\varphi(\lambda) + \psi(\lambda));$$

$$\lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) \cdot \lim_{\lambda \uparrow \mathbb{U}} \psi(\lambda) = \lim_{\lambda \uparrow \mathbb{U}} (\varphi(\lambda) \cdot \psi(\lambda)).$$

**Theorem 2.5** ([2, Theorem 7]). *Axioms* (A1)–(A3) *are consistent.*

We say that a net $\varphi : \mathcal{F} \to \mathbb{U}$ is **bounded** if there exists $n \in \mathbb{N}$ such that,

$$\forall \lambda \in \mathcal{F}, \ \varphi(\lambda) \in \mathbb{U}_n.$$

The notion of $\Lambda$-limit can be extended to bounded nets by induction on $n$: for $n = 0$, $\lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda)$ is defined by the axioms (A1)–(A3); so by induction we may assume that the limit is defined for $n - 1$ and we define it for a net $\varphi : \mathcal{F} \to \mathbb{U}_n$ as follows:

$$\lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) = \left\{ \lim_{\lambda \uparrow \mathbb{U}} \psi(\lambda) : \psi : \mathcal{F} \to \mathbb{U}_{n-1} \text{ and } \forall \lambda \in \Lambda, \ \psi(\lambda) \in \varphi(\lambda) \right\}.$$

A set that is a $\Lambda$-limit of sets is called **internal**. The $\Lambda$-limit provides a way to extend subset of $\mathbb{R}$ and functions defined on (subsets of) $\mathbb{R}$ to $\mathbb{K}$:

**Definition 2.6.** Given a set $E \subset \mathbb{R}$, let $c_E : \mathcal{F} \to \mathbb{U}$ be the net such that for all $\lambda \in \mathcal{F}$ $c_E(\lambda) = E$. Then

$$E^* := \lim_{\lambda \uparrow \mathbb{U}} c_E(\lambda) = \left\{ \lim_{\lambda \uparrow \mathbb{U}} \psi(\lambda) : \psi(\lambda) \in E \right\}$$

is called **natural extension** of $E$.

Using the above definition we have that

$$\mathbb{K} = \mathbb{R}^*.$$

A function $f$ can be extended by identifying $f$ and its graph, and this extension satisfies the following properties:

**Theorem 2.7.** *For every sets* $A, B \in \mathbb{U}$, *the* **natural extension** *of a function* $f : A \to B$ *is a function*

$$f^* : A^* \to B^*;$$

*moreover for every* $\varphi : \Lambda \cap \mathcal{P}(A) \to A$, *we have that*

$$\lim_{\lambda \uparrow \mathbb{U}} f(\varphi(\lambda)) = f^* \left( \lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) \right).$$

A property that is natural to ask for the $\Lambda$-limit of a net $\varphi$ is that some properties of the limit can be deduced from the properties of $\varphi$. This is ensured by the following important theorem:

**Theorem 2.8** (Leibnitz Principle). *Let* $\mathcal{R}$ *be a relation in* $\mathbb{U}_n$ *for some* $n \geq 0$ *and let* $\varphi, \psi : \mathcal{F} \to \mathbb{U}_n$. *If for all* $\lambda \in \mathcal{F}$, $\varphi(\lambda) \mathcal{R} \psi(\lambda)$, *then*

$$\left( \lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) \right) \mathcal{R}^* \left( \lim_{\lambda \uparrow \mathbb{U}} \psi(\lambda) \right).$$

The last key concept that we need is that of hyperfinite set.

**Definition 2.9.** An internal set is called **hyperfinite** if it is the $\Lambda$-limit of finite sets.

All the internal finite sets are hyperfinite, but there are hyperfinite sets which are not finite, e.g. the set

$$\mathbb{R}^\circ := \lim_{\lambda \uparrow \mathbb{U}} (\mathbb{R} \cap \lambda)$$

is not finite. The hyperfinite sets are very important since, by Leibnitz Principle, they inherit many properties of finite sets; e.g., $\mathbb{R}^\circ$ has a maximum and a minimum element, and every internal function (i.e. a function such that its graph is an internal set)

$$f : \mathbb{R}^\circ \to \mathbb{R}^*$$

has a maximum and a minimum as well. Intuitively, hyperfinite sets can be thought as having an hyperfinite number $\beta$ of elements, where $\beta$ is an element of $\mathbb{N}^*$. Given a set $A \in \mathbb{U}$ we denote by $A^\circ$ its hyperfinite extension:

$$A^\circ = \lim_{\lambda \uparrow \mathbb{U}} (\lambda \cap A).$$

By this construction, if a hyperfinite set consists of numbers, or vectors, it is possible to add all its elements. Let

$$A := \lim_{\lambda \uparrow \mathbb{U}} A_\lambda$$

be a hyperfinite set; the hyperfinite sum of the elements of $A$ is defined as follows:

$$\sum_{a \in A} a = \lim_{\lambda \uparrow \mathbb{U}} \sum_{a \in A_\lambda} a.$$

In particular, if $A = \{a_1, \ldots, a_\beta\}$ consists of $\beta$ elements, with $\beta \in \mathbb{N}^*$, we use the notation

$$\sum_{a \in A} a = \sum_{j=1}^{\beta} a_j.$$

2.2. **Definition of the ultrafunctions.** Let $\Omega$ be a subset of $\mathbb{R}^N$, and let $V_G(\Omega)$ be a vector space such that $\mathcal{D}(\overline{\Omega}) \subseteq V_G(\Omega) \subseteq \mathcal{C}(\overline{\Omega}) \cap L^2(\Omega)$. Let $\varphi_{V_G(\Omega)}$ be the net such that, for every $\lambda \in \mathcal{F}$, $\varphi_{V_G(\Omega)}(\lambda) = V_\lambda(\Omega)$, where

$$V_\lambda(\Omega) = \mathrm{span}(V_G(\Omega) \cap \lambda).$$

**Definition 2.10.** The **set of ultrafunctions generated by** $V_G(\Omega)$ is

$$V(\Omega) = \lim_{\lambda \uparrow \mathbb{U}} V_\lambda(\Omega) = \mathrm{span}(V_G(\Omega)^\circ);$$

any element $u(x)$ of $V(\Omega)$ is called **ultrafunction** and $V_G(\Omega)$ is called the generating space.

Observe that, being the $\Lambda$-limit of a net of vector spaces of finite dimensions, $V(\Omega)$ is a vector space of hyperfinite dimension. Its dimension, that we denote by $\beta$, is

$$\beta = \lim_{\lambda \uparrow \mathbb{U}} \dim(V_\lambda(\Omega)).$$

The ultrafunctions are $\Lambda$-limits of continuous functions in $V_\lambda(\Omega)$, so they are internal functions

$$u : \Omega^* \to \mathbb{C}^*.$$

(we recall that a function is called "internal" if it is a $\Lambda$-limit of functions).

Notice that $V(\Omega)$ inherits an Euclidean structure that is the $\Lambda$-limit of the Euclidean structure of every space $V_\lambda(\Omega)$ given by the usual $L^2(\Omega)$ scalar product; also, since $V(\Omega)$ is a subset of $L^2(\Omega)^*$, it can be equipped with the following scalar product

$$(u, v) = \int_\Omega^* u(x)\overline{v(x)}\, dx.$$

where $\int_\Omega^*$ is the natural extension of the Lebesgue integral considered as a functional. Being a vector space of hyperfinite dimension, $V(\Omega)$ admits an hyperfinite orthonormal basis $\{e_i(x) : i \leq \beta\}$. Having fixed a basis, we can make two important constructions in an explicit form. The first is the extension to $V(\Omega)$ of continuous functions $f(x)$ such that

$$\forall v(x) \in V(\Omega), \ -\infty < \int_\Omega^* f^*(x)v(x)dx < +\infty.$$

Let $f(x)$ be such a function, and let $\Phi$ denote the orthogonal projection

$$\Phi : \mathcal{C}(\Omega)^* \to V(\Omega).$$

We call **canonical extension** of $f(x)$ the ultrafunction

$$f_\Phi(x) = \Phi(f^*(x)).$$

Observe that $f_\Phi = f^* \Leftrightarrow f(x) \in V_G(\Omega)$ as expected, and that for every function $f(x)$ the following holds:

$$\forall v(x) \in V(\Omega), \ \int^* f^*(x)v(x)dx = \int^* f_\Phi(x)v(x)dx.$$

In terms of the basis $\{e_i(x) \mid i \leq \beta\}$, the operator $\Phi$ has the expression

$$\Phi(f(x)) = f_\Phi(x) = \sum_{i=1}^\beta \Big( \int^* f^*(\xi)e_i(\xi)d\xi \Big)e_i(x). \tag{2.1}$$

The second important construction regards the Dirac delta functions.

**Theorem 2.11** ([2, Theorem 23]). *Given a point $q \in \Omega$, there exists a unique function $\delta_q(x)$ in $V(\Omega)$ such that*

$$\forall v \in V(\Omega), \ \int^* \delta_q(x)v(x)\,dx = v(q). \tag{2.2}$$

The ultrafunction $\delta_q(x)$ is called the **Dirac ultrafunction** in $V(\Omega)$ concentrated in $q$. In terms of the basis $\{e_i(x) : i \leq \beta\}$, the $\delta_q$ has the expression

$$\delta_q(x) = \sum_{i=1}^\beta e_i(q)e_i(x), \tag{2.3}$$

which validity can be checked by a direct calculation.

**Remark 2.12.** We observe that, in the context of ultrafunctions, the Dirac ultrafunctions are actual functions, while in the classical theory of functions they are distributions. For example, in the ultrafunction context it makes perfect sense to consider objects like $\delta_q(x)^2$, $\delta_q(x) - 1$, $\delta_q(x) \cdot \delta_{q'}(x)$ and so on.

## 3. Model problem

In this section we want to solve the problem described in the introduction via a "natural" approach that can not be applied in the classical framework, while it can be applied in the ultrafunction setting. We begin by describing the Dirichlet problems in the framework of ultrafunctions.

3.1. **Dirichlet problem.** Let $\Omega$ be a bounded open set in $\mathbb{R}^N$, and consider the Dirichlet problem

$$\begin{aligned} u &\in \mathcal{C}_0^2(\Omega), \\ -\Delta u &= f(x) \quad \text{for } x \in \Omega. \end{aligned} \tag{3.1}$$

When $\partial\Omega$ and $f(x)$ are smooth, problem (3.1) has a unique solution. Otherwise, in the classical Sobolev approach, problem (3.1) is transformed into the problem

$$\begin{aligned} u &\in H_0^1(\Omega); \\ -\Delta u &= f(x). \end{aligned} \tag{3.2}$$

Problem (3.2) has a unique solution whenever $\Omega$ is a bounded open set and $f(x)$ is in $H^{-1}(\Omega)$; in this case the equation $-\Delta u = f$ is required to be satisfied in a weak sense:

$$-\int_\Omega u\Delta\varphi\,dx = \int_\Omega f\varphi\,dx \quad \forall\varphi \in \mathcal{D}(\Omega).$$

Also, the solution $u(x)$ given by this process is not a function but an equivalence class of functions defined a.e. in $\Omega$. In the approach with ultrafunctions let $V_0^2(\Omega)$ be the space of ultrafunctions generated by $C_0^2(\Omega)$. Problem (3.1) can be rewritten as follows:

$$u \in V_0^2(\Omega);$$
$$-\Delta_\Phi u = f(x) \quad \text{for } x \in \Omega^*. \tag{3.3}$$

where $\Delta_\Phi = \Phi \circ \Delta^* : V_0^2(\Omega) \to V_0^2(\Omega)$.

Observe that now we are solving the problem in an hyperfinite space, and by Leibnitz Principle it follows that there is an unique solution for every $f(x) \in V_0^2(\Omega)$ (for the details see [2, Theorem 27]). The idea of the proof is the following. The solution can be constructed by first finding a solution $u_\lambda(x)$ in each finite dimensional space $(V_0^2(\Omega))_\lambda = \mathrm{span}(C_0^2(\Omega) \cap \lambda)$, and then taking the $\Lambda$-limit

$$\overline{u}(x) = \lim_{\lambda \uparrow \mathbb{U}} u_\lambda(x).$$

The solution $\overline{u}(x)$ is an ultrafunction defined for every $x \in \Omega^*$ and, since

$$\forall x \in \partial\Omega, \ \forall \lambda \in \mathcal{F} \cap C_0^2(\Omega), \ u_\lambda(x) = 0,$$

it follows by Leibniz principle that for all $x \in \partial(\Omega^*)$, $\overline{u}(x) = 0$.

So $\overline{u}(x)$ satisfies the pointwise boundary condition, a result that is not true in the Sobolev approach. Finally, when problem (3.1) has a solution $s(x) \in \mathcal{C}^2(\overline{\Omega})$, then

$$\overline{u}(x) = s^*(x)$$

and, when problem (3.2) has a solution $g(x) \in H_0^1(\Omega)$, then we have that

$$\int_\Omega g(x)v(x)\,dx \sim \int_\Omega^* \overline{u}(x)v(x)\,dx \quad \forall v(x) \in \mathcal{C}_0^2(\overline{\Omega}).$$

3.2. **A solution by means of ultrafunctions.** Now let us consider a minimization problem inspired by the one which we have discussed in the introduction. Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set; we want to find a function $u$ defined in $\Omega$ (with $u = 0$ on $\partial\Omega$) and a point $q \in \overline{\Omega}$ which minimize the functional

$$E(u, q) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx + u(q).$$

It is well known that this problem has no solution in $\mathcal{C}_0^2(\overline{\Omega})$ and it makes no sense in the space of distributions. On the contrary it is well defined and it has a solution in $V_0^2(\Omega)$. More exactly, we have the following result.

**Theorem 3.1.** *For every point $q \in \overline{\Omega}^*$, the Dirichlet problem*

$$\Delta_\Phi u = \delta_q \quad \text{for } x \in \Omega^*;$$
$$u(x) = 0 \quad \text{for } x \in \partial\Omega^*.$$

*has a unique solution $u_q \in V_0^2(\Omega)$ whose energy $E(u_q, q) \in \mathbb{R}^*$ is an infinite number; moreover there exists $q_0 \in \Omega^*$ such that*

$$E(u_{q_0}, q_0) = \min_{q \in \overline{\Omega}^*} E(u_q, q) = \min_{q \in \overline{\Omega}^*, \ u \in V_0^2(\Omega)} E(u, q).$$

*Proof.* First of all we observe that

$$\min_{q \in \overline{\Omega}^*} E(u_q, q) = \min_{q \in \overline{\Omega}^*, \, u \in V_0^2(\Omega)} E(u, q)$$

since every stationary point $(\overline{u}, \overline{q})$ of $E(u, q)$ satisfies $-\Delta_\Phi(\overline{u}) = \delta_{\overline{q}}$.

To minimize $E(u, q)$ we use the Feado-Galerkin method, namely the finite dimensional reduction. First of all, for every $\lambda \in \mathcal{F}$, we solve the following problem in $(V_0^2(\Omega))_\lambda = \mathrm{span}(C_0^2(\Omega) \cap \lambda)$:

$$u \in (V_0^2(\Omega))_\lambda;$$
$$\int \Delta u \, v \, dx = \int \delta_q v \, dx \quad \text{for every } v \in (V_0^2(\Omega))_\lambda. \tag{3.4}$$

This problem has a unique solution $u_{q,\lambda}(x)$ for every $\lambda \in \mathcal{F} \cap \mathcal{C}_0^2(\Omega)$, since $(V_0^2(\Omega))_\lambda$ is a nonempty finite-dimensional vector space. We show that this solution depends continuously on $q$. Consider the linear operator

$$-\Delta_\lambda : (V_0^2(\Omega))_\lambda \to (V_0^2(\Omega))_\lambda$$

that associate to every $u$ of $(V_0^2(\Omega))_\lambda$ the unique element $-\Delta_\lambda(u)$ such that

$$\forall v \in (V_0^2(\Omega))_\lambda, \ \int_\Omega -\Delta_\lambda u \, v \, dx = \int_\Omega -\Delta u \, v \, dx. \tag{3.5}$$

So, $-\Delta_\lambda u$ is the orthogonal projection of $-\Delta u$ on $(V_0^2(\Omega))_\lambda$. Observe that $-\Delta_\lambda$ is a linear operator that acts on a finite dimensional vector space with $\ker(-\Delta_\lambda) = \{0\}$, so it is invertible. Now, let $e_1(x), \ldots, e_n(x)$ be an orthogonal base of $(V_0^2(\Omega))_\lambda$, and consider the function $k : \overline{\Omega} \to (V_0^2(\Omega))_\lambda$ that associates to every point $q \in \overline{\Omega}$ the unique function $\delta_{q,\lambda} \in (V_0^2(\Omega))_\lambda$ defined as follows:

$$\delta_{q,\lambda}(x) = \sum_{i=1}^n e_i(q) e_i(x).$$

Observe that, by definition, $\forall v \in (V_0^2(\Omega))_\lambda$ we have

$$\int_\Omega \delta_{q,\lambda} v \, dx = \int_\Omega \sum_{i=1}^n e_i(q) e_i(x) v(x) = \sum_{i=1}^n e_i(q) \int_\Omega e_i(x) v(x) = v(q),$$

and, since $v(q) = \int_\Omega \delta_q(x) v(x) \, dx$, we have

$$\forall v \in (V_0^2(\Omega))_\lambda, \ \int_\Omega \delta_{q,\lambda}(x) v(x) \, dx = \int_\Omega^* \delta_q(x) v^*(x). \tag{3.6}$$

Let $u_{q,\lambda}(x)$ be a solution to 3.4. Then, since $-\Delta_\lambda$ is invertible and (3.5) and (3.6) hold, we have

$$u_{q,\lambda}(x) = \Delta_\lambda^{-1} \circ k(q).$$

Since, as observed, $k$ and $(-\Delta_\lambda)^{-1}$ are continuous functions, it follows that $u_{q,\lambda}$ depends continuously on $q$. Thus also

$$F_\lambda(q) = E_\lambda(u_q, q) = \frac{1}{2} \int_\Omega |\nabla u_{q,\lambda}(x)|^2 dx + u(q)$$

is continuous. Since $\overline{\Omega}$ is compact, $F_\lambda(q)$ has a minimizer which we denote by $q_\lambda$. Now let

$$\overline{q} = \lim_{\lambda \uparrow \mathbb{U}} q_\lambda,$$

$$u_{\overline{q}} = \lim_{\lambda \uparrow \mathbb{U}} u_{q_\lambda, \lambda}.$$

By Leibniz Principle, $(u_{\overline{q}}, \overline{q})$ is the minimizer of $E(u, q)$ in $\overline{\Omega}^*$. Let us see that $\overline{q} \in \Omega^*$. By definition of Dirac ultrafunction we have that, for all $q \in \partial\Omega$, $\delta_q = 0$, so $u_q(x) = 0$ and $E(u_q, q) = 0$, while $E(u_q, q) < 0$ for every $q \in \Omega^*$. So $\overline{q} \in \Omega^*$. $\square$

**Remark 3.2.** A similar problem that can be studied with the same technique is the problem of a electrically charged pointwise free particle in a box. Representing the box with an open bounded set $\Omega \subseteq \mathbb{R}^3$, denoting by $u_q$ the electrical potential generated by the particle placed in $q \in \Omega$, then $u_q$ satisfies the Dirichlet problem

$$u \in \mathcal{C}_0^2(\Omega);$$
$$\Delta_\Phi u = \delta_q \quad \text{for } x \in \Omega.$$

The equilibrium point would be the point $q_0 \in \overline{\Omega}^*$ that minimizes the electrostatic energy which is given by

$$E_{\text{el}}(q) = \frac{1}{2} \int_\Omega |\nabla u_q(x)|^2 dx.$$

Notice that

$$E_{\text{el}}(q) = \int_\Omega \delta_q(x) u_q(x) dx - \frac{1}{2} \int_\Omega |\nabla u_q(x)|^2 dx,$$

namely, on the solution, the electrostatic energy is the opposite than the energy of a membrane-like problem in $\mathbb{R}^3$. To solve this problem we notice that, by definition of Dirac ultrafunction (2.2), we have that, for all $q \in \partial\Omega$, $\delta_q = 0$. So $E_{\text{el}}(q) \geq 0$ and $E_{\text{el}}(q) = 0$ if and only if $q \in \partial\Omega$. More precisely we have:

- $E_{\text{el}}(q)$ is infinite if the distance between $q$ and $\partial\Omega$ is larger than some positive real number;
- $E_{\text{el}}(q)$ is positive but not infinite for some $q$ infinitely close to $\partial\Omega$;
- $E_{\text{el}}(q) = 0$ if and only if $q \in \partial\Omega$.

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