

Risk measures based on Gram-Charlier expansions: theory and application.

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Abstract

The reliability of risk measures for financial portfolios crucially rests on the availability of sound representations of the involved random variables. The trade-off between adherence to reality and specification parsimony can find a fitting balance in a technique that "adjust" the moments of a density function by making use of its associated orthogonal polynomials. This approach rests on the Gram-Charlier expansion of a Gaussian law which, allowing for leptokurtosis to an appreciable extent, makes the resulting random variable a tail-sensitive density function.

In this paper we determine the density of sums of leptokurtic normal variables duly adjusted for excess kurtosis by means of their Gram-Charlier expansions based on Hermite polynomials. The resultant density can be effectively used to represent a portfolio return and as such proves suitable for computing some risk measures such as Value at Risk and expected short fall. An application to a portfolio of financial returns is used to provide evidence of the effectiveness of the proposed approach.

Keywords: Gram-Charlier expansion, Value at Risk, Expected shortfall.

JEL classification codes: C46, C58, C40, G10, G20

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1 Introduction

In the last decades, both the convergence of the financial and insurance markets and the evolution of financial engineering, in purely financial and financial-linked insurance contracts, have brought to the fore the importance of an accurate evaluation of financial risk. This has highlighted that the choice of the appropriate distribution function underlying the measure of financial risks is a key problem for operators and analysts.

Commonly used statistical models as well as several applications rest on the assumption that asset returns are by and large normally distributed. Empirical evidence, as highlighted by many authors like Mittnik *et al.* (2000) and Alles and Murray (2010), provides sound arguments against this hypothesis. As a matter of fact, it is well known that financial time series exhibit tails heavier than those of the normal distribution. This feature turns out to be of prominent importance in modeling volatility (Shuangzhe, 2006; Curto *et al.*, 2009) and more generally in the evaluation of the portfolio risk (Szegö, 2004). This has led, on the one hand, to the use of alternative distributions like the Student t, the Pearson type VII, inverse Gaussian and several stable distributions (see e.g., Mills and Markellos (2008); Rachev *et al.* (2010)). On the other hand approaches have been developed aiming at transforming the Gaussian law to meet the desired features (see Gallant and Tauchen (1989, 1993); Jondeau and Rockinger (2001); Zoia (2010)). This latter approach, which has the advantage of allowing for greater flexibility in fitting empirical distributions, is the one we have followed in this paper. Recently, Zoia (2010); Bagnato *et al.* (2015) have proposed a method to account for excess kurtosis of a density based on its polynomial transformation through its associated orthogonal polynomials. In the Gaussian case, these polynomials are the Hermite ones and the polynomially modified density is known as Gram-Charlier expansion. This approach is particularly interesting because it can be tailored on the specific features of the empirical distribution at hand and can be extended to other distributions besides the normal one (see Faliva *et al.* (2016)).

This paper develops the approach further so as to obtain the densities of sums of leptokurtic normal random variables with same or different kurtosis. After adjusting the parent normal laws via Hermite polynomials, the density function of the sum of the resulting Gram-Charlier expansions is obtained. The resulting density proves to be more tail-sensitive than the Gaussian and as such suitable for representing a portfolio return which can be used to measure the well known Value at Risk. Further, since information on the magnitude of high risks is extremely important, they are also applied to evaluate a coherent risk measure, namely the Expected Shortfall.

An application to a portfolio of international financial indexes with a dataset window covering the period from January 2009 to December 2014 provides evidence of the effective performance of these Gram-Charlier expansions. In accordance with the regulatory framework, the risk measures are evaluated at 97.5% and 99% levels to guarantee a prudent approach.

The structure of the paper is as follows. In section 2 we look at some standard risk-measures, typically used in financial-insurance market. Section 3 explains how to obtain distributions of sums of Gram-Charlier expansions. Section 4 provides closed-form expressions of the expected short-fall based on these distributions. Section 5 shows an application of these densities to a portfolio of

financial returns which provides evidences of the effectiveness of the proposed approach. Section 6 draws some conclusions. An Appendix completes the paper stating the essentials notions about sums of densities of normal random variables and Gram-Charlier expansions.

2 A glance at risk measures

As it is well known, different approaches are available to measure financial and/or insurance risks (see, for all, Albrecht (2004) and Dowd and Blake (2006), and the reference quoted therein). Descriptive measures based on the moments of a probability distribution give only a partial representation of the risk. To overcome this problem, a combination of these measures is often used, as happens for example with the mean and standard deviation in Markowitz portfolio theory or the skewness and kurtosis when symmetry and probability concentration in tails are of interest. Unfortunately, the estimation of the moments of a probability distribution may be quite sensitive to the sample and, when the moments are infinite, even impossible.

The standard theory for decision under risks, based on the expected utility approach, may be difficult to implement and sensitive to individual risk tolerance, due to the critical choice of the functional form of the utility function and the complex evaluation of the risk attitude parameter.

Risk measures based on quantiles became very popular at the end of the 1980s, because of their implementation in determining the regulatory capital requirements of the US commercial banks. Value at risk based models were introduced in the Basel II agreement and later used for the calibration of the Solvency Capital Requirement, in the Solvency II agreement.

The Value at Risk (VaR) represents the minimum loss within a certain period of time for a given probability. By denoting with $F_X(x)$ the distribution function of a variable X representing the loss and with $v_q = \inf\{x : F_X(x) \geq q\}$, $q \in (0, 1)$, the quantile function, then the VaR can be defined as

$$VaR_X(q) = \inf\{x : F_X(x) \geq q\} = F_X^{-1}(q)$$

Since VaR is simply the threshold at a given probability q , that is

$$VaR_X(q) = v_q \tag{1}$$

it does not provide information about the size of any losses beyond this point of the distribution, although knowledge of the default size is crucial for shareholders, management and regulators. In addition, VaR is not a coherent risk measure (see Artzner P. (1999)) because it is not subadditive. Sub-additivity is very important in several financial applications such as portfolio optimization, where VaR can discourage diversification. In addition, VaR estimates give incorrect results when losses/returns are not normally distributed and this shortcoming turns out to be very critical in the presence of fat tails. Furthermore, VaR -models based on typical scenarios for discrete data series, can exhibit multiple local extrema (see Uryasev (2000)).

The interest of financial and insurance managers in tail risks clearly justifies the introduction of risk measures offering information on the magnitude of high risks. The Tail Conditional Expectation (TCE) is defined as

$$TCE_X(q) = \mathbb{E}[X|X \geq v_q] \tag{2}$$

and provides the possible worst average loss. The *TCE* is not generally a coherent measure of risk, because it can be not sub-additive. This drawback is evident when dealing with discontinuous distributions (for example with portfolios containing derivatives) when the measure becomes very sensitive to small changes in the confidence level.

A risk measure that respects the axioms of coherence is the Expected Shortfall (*ES*)

$$ES_X(q) = \frac{1}{q} (\mathbb{E}[X \mathbf{1}_{\{X \geq v_q\}}] + v_q(\mathbb{P}[X \geq v_q]) - (1 - q)) \quad (3)$$

which is in general continuous with respect to the confidence level.

For real-valued random variables with continuous and strictly increasing distribution function and finite mean, the following proves true (see Acerbi and Tasche (2002))

$$TCE_X(q) = ES_X(q) \quad (4)$$

3 On the distribution of the sum of polynomially-modified Gaussian variables

In this section we tackle the issue of specifying the density function of the sum of polynomially-modified (namely Gram-Charlier expansions of) Gaussian variables assuming independence.

Theorem 1. *Consider n identically and independently distributed random variables X_1, \dots, X_n having as common density a Gram-Charlier expansion defined as follows(see Definition 2 in Appendix)*

$$f_X(x; \beta) = \left(1 + \frac{\beta}{4!} p_4(x)\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (5)$$

where β is a positive parameter subject to $f_X(x; \beta)$ being non-negative definite, and

$$p_{4j}(x) = \sum_{i=0}^{\frac{4j}{2}} (-1)^i \binom{4j}{2i} x^{4j-2i}. \quad (6)$$

is the $4j$ -th degree Hermite polynomial.

Then, the density function of the sum $Y = X_1 + \dots + X_n$ is given by

$$f_Y(x_1 + \dots + x_n; \beta) = \sum_{j=0}^n \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \frac{1}{\sqrt{2n\pi}} \left(\frac{1}{\sqrt{n}}\right)^{4j} e^{-\frac{y^2}{2n}} p_{4j}\left(\frac{y}{\sqrt{n}}\right). \quad (7)$$

Proof. Bearing in mind the following property of Fourier transforms,

$$\frac{d^n f(x)}{dx^n} \leftrightarrow (i\omega)^n F(\omega) \quad (8)$$

together with the noteworthy property of the Gaussian law,

$$\frac{d^n \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}{dx^n} = (-1)^n \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} p_n(x) \quad (9)$$

the following proves true (see formula (46) in Appendix)

$$(-1)^n \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} p_n(x) \leftrightarrow (i\omega)^n e^{-\frac{\omega^2}{2}} \quad (10)$$

This entails that the characteristic function associated to (5) is

$$F_X(\omega; \beta) = \left(1 + \frac{\beta}{4!} \omega^4\right) e^{-\frac{\omega^2}{2}}. \quad (11)$$

By following the same argument, put forward in Lemma 1 in Appendix, the characteristic function of the sum of n Gram-Charlier expansions turns out to be

$$F_Y(\omega; \beta) = \left(1 + \frac{\beta}{4!} \omega^4\right)^n e^{-\frac{n\omega^2}{2}} = \sum_{j=0}^n \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \omega^{4j} e^{-\frac{n\omega^2}{2}} \quad (12)$$

Now, thanks to the following property of Fourier transforms

$$|a|f(ay) \leftrightarrow F\left(\frac{\omega}{a}\right), \quad (13)$$

formula (8) can be conveniently generalized as follows

$$\frac{d^n |a| f_X(ax)}{dx^n} \leftrightarrow \left(i\frac{\omega}{a}\right)^n F\left(\frac{\omega}{a}\right) \quad (14)$$

and this, in light of (10), entails the following

$$(-1)^n \frac{|a|}{\sqrt{2\pi}} e^{-\frac{(ax)^2}{2}} p_n(ax) \leftrightarrow \left(i\frac{\omega}{a}\right)^n e^{-\frac{1}{2}\left(\frac{\omega}{a}\right)^2}. \quad (15)$$

Then, setting $a = \frac{1}{\sqrt{n}}$ and $n = 4j$ in formula (15), yields

$$\left(\frac{1}{\sqrt{n}}\right)^{4j} \frac{1}{\sqrt{2n\pi}} e^{-\frac{y^2}{2n}} p_{4j}\left(\frac{y}{\sqrt{n}}\right) \leftrightarrow \omega^{4j} e^{-\frac{n\omega^2}{2}} \quad (16)$$

which, taking into account (12), eventually leads to the density in formula (7). \square

The density of the sum variable $Y = X_1 + \dots + X_n$ given in (7) depends on the parameter β which plays the role of common excess kurtosis (with respect to the standard Gaussian law) of each variable X_i . In Zoia (2010) it is shown that the Gram-Charlier expansion (5) has positive density if $0 \leq \beta \leq 4$ and is unimodal if $0 \leq \beta \leq 2, 4$. These constraints also hold in the case of the sum of n i.i.d variables, according to the Theorem 1.6 in Dharmadhikari (1988).

The graphs in Figure 1 depict the density functions of the sums of Gram-Charlier expansions for different values of n ($n = 1$, $n = 2$ and $n = 3$) and β . In each graph β has been set equal to 0 (its minimum value), equal to 2,4 (the maximum value which guarantees the unimodality of the Gram-Charlier density), and equal to 1 (an intermediate value in its range of variation).

As a further extension of the Theorem 1, we prove the following corollary which covers the case of Gram-Charlier expansions of sums of variables characterized by different excess kurtosis β' s.

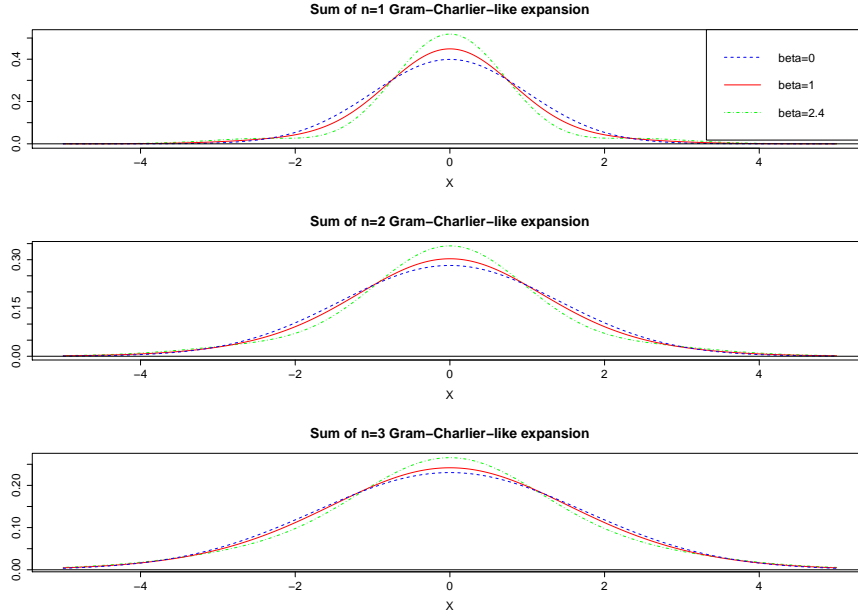


Figure 1: Densities of sums of Gram-Charlier expansions for $n = 1, 2, 3$, and $\beta = 0, 1, 2.4$.

Corollary 1. *Let us consider n independent Gram-Charlier expansions of the random variables X_1, \dots, X_n , characterized by excess kurtosis β_1, \dots, β_n , respectively. Then, the density function of the sum $Y = X_1 + \dots + X_n$, denoted with $f_Y(x_1 + \dots + x_n; \beta_1, \dots, \beta_n)$, is*

$$f_Y(x_1 + \dots + x_n; \beta_1, \dots, \beta_n) = \sum_{j=0}^n \left(\frac{b_{n,j}}{(4!)^j} \right) \frac{1}{\sqrt{2n\pi}} \left(\frac{1}{\sqrt{n}} \right)^{4j} e^{-\frac{y^2}{2n}} p_{4j} \left(\frac{y}{\sqrt{n}} \right) \quad (17)$$

where $b_{n,j}$ is the sum of the combinations of the n parameters β_j taken j at a time without repetition.

Proof. Following the same arguments put forward in Theorem 1, the characteristic function, $F_{Y=X_1+\dots+X_n}(\omega; \beta_1, \dots, \beta_n)$, of the sum of n Gram-Charlier expansions with different excess kurtosis $\beta_j, j = 1, \dots, n$ is

$$\begin{aligned} F_{Y=X_1+\dots+X_n}(\omega; \beta_1, \dots, \beta_n) &= e^{-\omega^2} \prod_{j=1}^n \left(1 + \frac{\beta_j}{4!} \omega^4 \right) = \\ e^{-\omega^2} \left(1 + \frac{\beta_1 + \dots + \beta_n}{4!} \omega^4 + \frac{\beta_1 \beta_2 + \dots + \beta_{n-1} \beta_n}{(4!)^2} \omega^8 + \dots + \frac{\prod_{j=1}^n \beta_j}{(4!)^n} \omega^{n4} \right) &= \quad (18) \\ \sum_{j=0}^n \frac{b_{n,j}}{(4!)^j} \omega^{4j} e^{-\omega^2}. \end{aligned}$$

Then, taking into account formulas (12) and (16) simple computations lead to (17). \square

This approach can be extended to other densities, besides the normal. However, when other distributions are considered, the density of the sum may be more conveniently obtained by making the convolution of the densities of the variables involved in the sum.

4 Expected Shortfall for sum of Gram-Charlier expansions

Gram-Charlier expansions (*GC*) are able to capture the excess of kurtosis and asymmetry of a random variable (*rv*) better than the usual normal density. This property is true also for densities which are sums of Gram-Charlier expansions, *GCS* hereafter, with respect to densities of sums of simple Gaussian laws.

Hence, the next step is to use *GCS* to measure risks related to insurance or financial assets portfolios.

In this section, following the analysis of Landsman and Valdez (2003) on *TCE* for sums of elliptic distributions and bearing in mind the studies of Acerbi and Tasche (2002), we show how to compute the expected short fall, *ES* henceforth, to evaluate the right-tail risk of a sum of *GC* expansions. First we will consider the case of **rvs** with same excess kurtosis, then with different excess kurtosis.

Assuming that the loss is likely to exceed a certain value v_q (referred to as the q -th-quantile), the *ES* is defined as follows

$$ES_Y(v_q) = E(Y|Y > v_q) = \frac{\int_{v_q}^{\infty} yf(y)dy}{\int_{v_q}^{\infty} f(y)dy} \quad (19)$$

where, for our purpose, $f(y) = f(x_1 + x_2 + \dots + x_n)$.

The following theorem shows how the integrals in (19) can be evaluated by making use of the definition and properties of the error function and of the Hermite polynomials

Theorem 2. *Let us consider the sum of n i.i.d Gram-Charlier expansions $Y = X_1 + \dots + X_n$. Then, the $ES_Y(v_q)$ has the following form*

$$ES_Y(v_q) = \frac{\sqrt{\frac{n}{2\pi}} e^{-\frac{v_q^2}{2n}} \left[1 + \sum_{j=1}^n \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \left(p_{4j} \left(\frac{v_q}{\sqrt{n}}\right) + 4j p_{4j-2} \left(\frac{v_q}{\sqrt{n}}\right) \right) \right]}{\frac{1}{2} \operatorname{erfc} \left(\frac{v_q}{\sqrt{2n}}\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{v_q^2}{2n}} \left[\sum_{j=1}^n \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j p_{4j-1} \left(\frac{v_q}{\sqrt{n}}\right) \right]} \quad (20)$$

Proof. Let us proceed by considering separately the numerator and the denominator of formula (20) which, in the following, will be denoted by A and B , respectively.

By replacing in the numerator A the density function $f(y, \beta)$ defined as in (7) we obtain

$$\begin{aligned} A &= \sum_{j=0}^n \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \frac{1}{\sqrt{2\pi}} \int_{v_q}^{\infty} \frac{y}{\sqrt{n}} e^{-\frac{y^2}{2n}} p_{4j} \left(\frac{y}{\sqrt{n}}\right) dy = \\ &= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{v_q}^{\infty} \frac{y}{\sqrt{n}} e^{-\frac{y^2}{2n}} dy}_{A_1} + \underbrace{\sum_{j=1}^n \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \frac{1}{\sqrt{2\pi}} \int_{v_q}^{\infty} \frac{y}{\sqrt{n}} e^{-\frac{y^2}{2n}} p_{4j} \left(\frac{y}{\sqrt{n}}\right) dy}_{A_2} \end{aligned} \quad (21)$$

As far as A_1 is concerned, setting $\frac{y}{\sqrt{n}} = t$ in this integral and bearing in mind that $p_1(t) = t$, we get

$$A_1 = \sqrt{\frac{n}{2\pi}} \int_{\frac{v_q}{\sqrt{n}}}^{\infty} t e^{-\frac{t^2}{2}} dt = \sqrt{\frac{n}{2\pi}} \int_{\frac{v_q}{\sqrt{n}}}^{\infty} p_1(t) e^{-\frac{t^2}{2}} dt. \quad (22)$$

Now, in light of (9), the following

$$\frac{d}{dy} \left[\frac{d^n}{dy^n} e^{-\frac{y^2}{2}} \right] = \frac{d^{n+1}}{dy^{n+1}} e^{-\frac{y^2}{2}} = (-1)^{n+1} e^{-\frac{y^2}{2}} p_{n+1}(y) \quad (23)$$

holds true.

This entails that

$$\begin{aligned} \int (-1)^{n+1} e^{-\frac{y^2}{2}} p_{n+1}(y) dy &= \int \frac{d^{n+1}}{dy^{n+1}} e^{-\frac{y^2}{2}} = \\ &= \frac{d^n}{dy^n} e^{-\frac{y^2}{2}} = \\ &= (-1)^n e^{-\frac{y^2}{2}} p_n(y) \end{aligned} \quad (24)$$

By using this result and bearing in mind that $p_0(t) = 1$, formula (22) becomes

$$\begin{aligned} A_1 &= - \left. \sqrt{\frac{n}{2\pi}} e^{-\frac{y^2}{2}} \right|_{\frac{v_q}{\sqrt{n}}}^{\infty} \\ &= \sqrt{\frac{n}{2\pi}} e^{-\frac{v_q^2}{2n}} \end{aligned} \quad (25)$$

As far as A_2 is concerned, setting $t = \frac{y}{\sqrt{n}}$ in this integral we get

$$A_2 = K \int_{\frac{v_q}{\sqrt{n}}}^{\infty} t e^{-\frac{t^2}{2}} p_{4j}(t) dt. \quad (26)$$

where $K = \sqrt{\frac{n}{2\pi}} \sum_{j=1}^n \binom{n}{j} \left(\frac{1}{\sqrt{n}}\right)^{4j} \left(\frac{\beta}{4!}\right)^j$.

Now, in light of the following property of Hermite polynomials

$$p_{n+1}(t) = t p_n(t) - n p_{n-1}(t) \quad (27)$$

the integral (26) can be rewritten as:

$$A_2 = K \int_{\frac{v_q}{\sqrt{n}}}^{\infty} \left[e^{-\frac{t^2}{2}} p_{4j+1}(t) + 4j e^{-\frac{t^2}{2}} p_{4j-1}(t) \right] dt. \quad (28)$$

which, in light of (24), becomes

$$\begin{aligned} A_2 &= K \left[4j e^{-\frac{t^2}{2}} p_{4j-2}(t) + e^{-\frac{t^2}{2}} p_{4j}(t) \right] \Big|_{\frac{v_q}{\sqrt{n}}}^{\infty} = \\ &= \sqrt{\frac{n}{2\pi}} \sum_{j=1}^n \binom{n}{j} \left(\frac{1}{\sqrt{n}}\right)^{4j} \left(\frac{\beta}{4!}\right)^j e^{-\frac{v_q^2}{2n}} \left(p_{4j} \left(\frac{v_q}{\sqrt{n}}\right) - 4j p_{4j-2} \left(\frac{v_q}{\sqrt{n}}\right) \right). \end{aligned} \quad (29)$$

Accordingly the integral A turns out to be

$$A = \sqrt{\frac{n}{2\pi}} e^{-\frac{v_q^2}{2n}} + \sqrt{\frac{n}{2\pi}} e^{-\frac{v_q^2}{2n}} \sum_{j=1}^n \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \left(p_{4j}\left(\frac{v_q}{\sqrt{n}}\right) + 4j p_{4j-2}\left(\frac{v_q}{\sqrt{n}}\right)\right) \quad (30)$$

Similarly, after replacing $f(y, \beta)$, defined as in (7), in the denominator of (19), we get

$$\begin{aligned} B &= \sum_{j=0}^n \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \frac{1}{\sqrt{2\pi}} \int_{v_q}^{\infty} \frac{1}{\sqrt{n}} e^{-\frac{y^2}{2n}} p_{4j}\left(\frac{y}{\sqrt{n}}\right) dy = \\ &= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{v_q}^{\infty} \frac{1}{\sqrt{n}} e^{-\frac{y^2}{2n}} dy}_{B_1} + \underbrace{\sum_{j=1}^n \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \frac{1}{\sqrt{2\pi}} \int_{v_q}^{\infty} \frac{1}{\sqrt{n}} e^{-\frac{y^2}{2n}} p_{4j}\left(\frac{y}{\sqrt{n}}\right) dy}_{B_2}. \end{aligned} \quad (31)$$

As far as B_1 is concerned, setting $t = \frac{y}{\sqrt{2n}}$ in the integral we get

$$B_1 = \frac{1}{\sqrt{\pi}} \int_{\frac{v_q}{\sqrt{2n}}}^{\infty} e^{-t^2} dt = \frac{1}{2} \operatorname{erfc}\left(\frac{v_q}{\sqrt{2n}}\right). \quad (32)$$

where erfc is the complementary error function (see formula 7.1.2 in Abramowitz and Stegun (1964)).

Similarly, setting $t = \frac{y}{\sqrt{n}}$ in the integral B_2 yields

$$B_2 = \tilde{K} \int_{\frac{v_q}{\sqrt{n}}}^{\infty} e^{-\frac{t^2}{2}} p_{4j}(t) dt. \quad (33)$$

where $\tilde{K} = \sum_{j=1}^n \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \frac{1}{\sqrt{2\pi}}$.

Then, by using result (24), B_2 becomes

$$\begin{aligned} B_2 &= \tilde{K} \left[e^{-\frac{t^2}{2}} p_{4j-1}(t) \right] \Big|_{\frac{v_q}{\sqrt{n}}}^{\infty} \\ &= \sum_{j=1}^n \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j p_{4j-1}\left(\frac{v_q}{\sqrt{n}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{v_q^2}{2n}}. \end{aligned} \quad (34)$$

Accordingly, the integral B turns out to be

$$B = \frac{1}{2} \operatorname{erfc}\left(\frac{v_q}{\sqrt{2n}}\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{v_q^2}{2n}} \left[\sum_{j=1}^n \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j p_{4j-1}\left(\frac{v_q}{\sqrt{n}}\right) \right] \quad (35)$$

Finally, formula (20) is obtained by substituting the numerator and the denominator of formula (19) with A and B given in (30) and (35), respectively. \square

The same procedure can be simply generalized to the case of n random variables with different extra-kurtosis parameters β_i .

Corollary 2. *Let us consider n independent Gram-Charlier expansions of the random variables X_1, \dots, X_n , characterized by extra-kurtosis $\beta_1 \dots, \beta_n$, respectively. Then, the expected shortfall of the sum $Y = X_1 + \dots + X_n$, $ES_Y(v_q)$, has the following form*

$$ES_Y(v_q) = \frac{\sqrt{\frac{n}{2\pi}} e^{-\frac{v_q^2}{2n}} \left[1 + \sum_{j=1}^n \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \binom{b_{n,j}}{(4!)^j} \left(p_{4j} \left(\frac{v_q}{\sqrt{n}}\right) + 4j p_{4j-2} \left(\frac{v_q}{\sqrt{n}}\right) \right) \right]}{\frac{1}{2} \operatorname{erfc} \left(\frac{v_q}{\sqrt{2n}}\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{v_q^2}{2n}} \left[\sum_{j=1}^n \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{b_{n,j}}{(4!)^j} p_{4j-1} \left(\frac{v_q}{\sqrt{n}}\right) \right]}$$
(36)

Proof. Observe that the density of the sum of n Gram-Charlier expansions with different parameters, given by (17), differs from that of n Gram-Charlier expansions with equal parameters, given by (7), only for the coefficients of the Hermite polynomials $p_{4j} \left(\frac{y}{\sqrt{n}}\right)$. Hence, replacing in (20) the coefficients of the density (17) with those of the density (7), yields (36). \square

5 An application to financial asset indexes

In this section the effective performance of GC expansions of sums of r.v. in dealing with financial asset indexes is proved. To this end, we have considered a set of 4 european (UK, Germany, Italy, France) and 2 asian (China, Japan) stock exchange indexes and 2 arbitrary indexes of the pharmaceutical and alimentary industries. The preliminary statistics for these data are reported in Tables 1 and 2.

Table 1 shows the mean (μ), the standard deviation (sd), the skewness (sk), the kurtosis index (k). Since the analysis carried out in the previous section is valid for independent rvs, we use seven couples of indexes characterized by low correlation as reported in Table 2.

Table 1: Summary statistics of losses

	$\hat{\text{FTSE}}$	$\hat{\text{GDAXI}}$	FTSEMIB.MI	$\hat{\text{FCHI}}$	$\hat{\text{HSI}}$	$\hat{\text{N225}}$	SXDP.Z	KO
μ	-0,0260	-0,0493	0,0046	-0,0170	-0,0300	-0,0478	-0,0537	-0,0590
sd	1,1252	1,4278	1,8222	1,4813	1,4350	1,5138	0,8937	1,1148
sk	0,0781	0,0476	0,2106	-0,0242	-0,1662	0,5063	0,3305	-0,1975
k	6,4554	5,7822	5,7534	6,3576	7,0437	6,8020	4,9275	8,1970

The table reports for each loss the mean (μ), the standard deviation (sd), the skewness index (sk), the kurtosis index (k).

Table 2: Correlation coefficient of the losses

ρ	$\hat{\text{FTSE}}$	$\hat{\text{GDAXI}}$	FTSEMIB.MI	$\hat{\text{FCHI}}$	$\hat{\text{HSI}}$	SXDP.Z	KO
$\hat{\text{N225}}$	0.3036	0.2921	0.2490	0.2917	0.5729	0.1776	0.0954

Since we are interested in measuring losses, the returns from data have been computed as minus the logarithm of the ratio between the prices at time t and $t - 1$.

The sample size has been divided into two periods. The data of the first period (from 01/01/2009 to 09/17/2013) have been used to estimate the Gram-Charlier (GC) densities and compute the corresponding risk functions. The data of the second period (from 09/18/2013 to 12/31/2014) have been used to evaluate the goodness of the risk measure forecasts.

The GC densities of the sum of each couple of indexes are given by

$$f_Y(x_1 + x_2; \beta_1, \beta_2) = \left(1 + \frac{1}{4} \left(\frac{\beta_1 + \beta_2}{4!} \right) p_4 \left(\frac{y}{\sqrt{2}} \right) + \frac{1}{16} \frac{\beta_1 \beta_2}{(4!)^2} p_8 \left(\frac{y}{\sqrt{2}} \right) \right) \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}}. \quad (37)$$

where $p_4(x)$ and $p_8(x)$ are defined as follows

$$p_4(x) = x^4 - \binom{4}{2} x^{4-2} + 3 \binom{4}{4} x^{4-4} = x^4 - 6x^2 + 3 \quad (38)$$

$$p_8(x) = x^8 - \binom{8}{2} x^6 + 3 \binom{8}{4} x^4 - 15 \binom{8}{6} x^2 + 105 \binom{8}{8} x. \quad (39)$$

Table 3 reports the **estimates** of the extra-kurtosis $\hat{\beta}$ for each couple of series under consideration.

In order to assess the goodness of fit of GCS to data, the Hellinger's entropy distance (Granger *et al.*, 2004; Maasoumi and Racine, 2002) between the empirical and the estimated distributions have been computed. Low values of this index denote a good fit of GCS to data. The last column of Table 3 shows the values of this index for the GCS densities.

Table 3: Parameter estimates of the GCS distribution on the first 1000 days with the relative Hellinger's entropy distance S_ρ .

Index 1	Index 2	$\hat{\beta}_1$	$\hat{\beta}_2$	S_ρ
^N225	^FTSE	3.9666	2.9189	0.0203
^N225	^GDAXI	3.9666	2.9189	0.0213
^N225	FTSEMIB.MI	3.9666	2.4250	0.0200
^N225	^FCHI	3.9666	2.8433	0.0185
^N225	^HSI	3.9666	3.5847	0.0215
^N225	SXDP.Z	3.9666	1.6780	0.0232
^N225	KO	3.9666	4.000	0.0173

Figure 2 shows the tails of the estimated GCS densities superimposed on those of the corresponding empirical distributions. Both the values of the Hellinger's entropy index and the graphs highlight the good fit of GCS to data, especially in the tail areas which are the loci involved in the risk measure estimates.

Table 4 and Figure 3 compare the *VaR* estimated via GCS in the first period of the sample at the 97, 5% and 99% levels with the corresponding empirical quantile. As all the *VaR* estimates exceed the corresponding empirical values, it can easily be concluded that the GCS provide precautionary *VaR* estimates against potential losses. In the same table, a comparison between the empirical quantiles and the *VaR* estimates obtained by using the normal density without

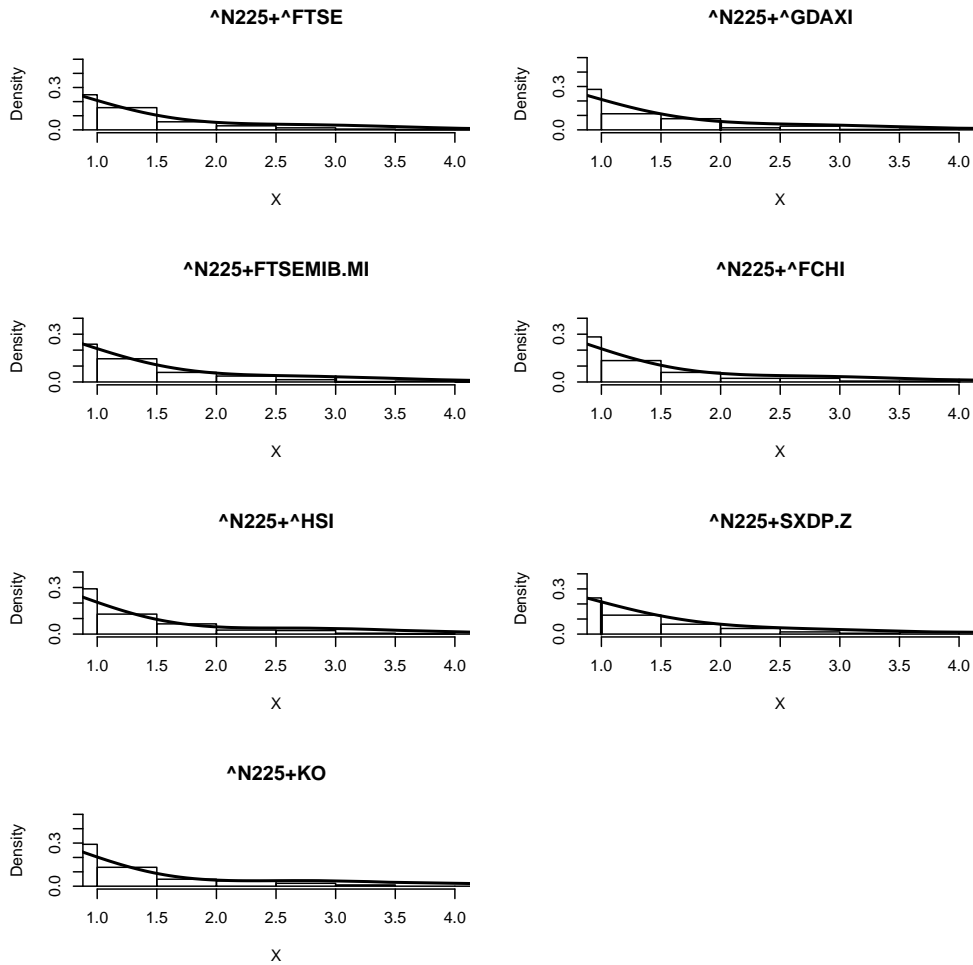


Figure 2: Histograms of the portfolio losses with the estimate GCS densities.

polynomial expansion is made. As in this case, Var estimates at both $\alpha = 0.025$ and $\alpha = 0.001$ underestimate the risk measures, we conclude that results provided by this latter density are dangerous for risk management and in stark contrast to the regulatory philosophy.

To evaluate the out-of-sample performance of the GCS densities, we have computed the VaR for $\alpha = 0.025$ and $\alpha = 0.01$ on the second part of sample (the last 374 days) which has not been used in the estimation process of the GCS densities.

Further, some punctual measures of losses in this period have been computed. These are the ABLF (average binary loss function), the AQLF (average quadratic loss function) and the UL (unexpected loss).

The values of these indexes as well as VaR values are displayed in Table 4. As happens in the sample (first 1000 days), the VaR values for GC distributions

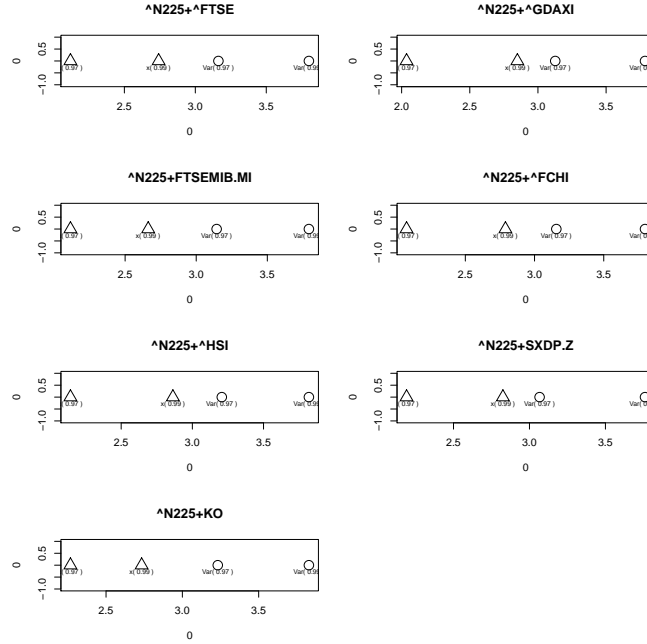


Figure 3: Empirical vs theoretical VaR of the portfolio losses. Triangles denote empirical VaR at $1 - \alpha = 0,975$, $1 - \alpha = 0,99$ while circles denote estimated VaR with GCS at the same levels.

give quite precautionary results with respect to the simple Normal. In fact all the proposed loss measures confirm that the GC distributions offer the best out of sample performance.

The forecast performance of VaR values estimates via GCS , at a chosen significance level, has been evaluated by implementing two tests: the likelihood-ratio test and the binomial two-sided test, whose results are shown in Table 5. The null hypothesis of both tests assumes that the percentage of forecast losses is coherent with the effective one against the bi-lateral alternative which assumes that the VaR values overestimate or underestimate this percentage. A p -value lower or equal to 0.01 can be interpreted as evidence against the null (for more details see (Kupiec, 1995; Christoffersen *et al.*, 1998)).

According to the likelihood ratio test 11 out of 14 GCS engender forecasts which are coherent at the chosen α level. Looking at Figure 3, we see that rejection happens for those GCS densities whose VaR estimates are most distant from the corresponding empirical quantiles. These results are in accordance with the results of the binomial tests.

Furthermore, a reading of the likelihood-ratio test of the $VaR_{\alpha=0.01}$, inspired by the "traffic light" approach suggested by the Basel Committee, seems to place the GCS results in the "green zone".

Also the less debatable expected shortfall (ES) has been computed as risk

Table 4: Descriptive analysis of VaR .

Index 1	Index 2	$1 - \alpha$	VaR_{emp}	VaR	GCS			Normal			
					ABLF	AQLF	UL	Var	ABLF	AQLF	UL
^N225	^FTSE	0,975	2,1199	3,1628	0,0053	0,0093	0,0042	1,96	0,0213	0,05	0,019
^N225	^FTSE	0,990	2,7415	3,8004	0,0027	0,0033	0,0014	2,3263	0,016	0,0331	0,0127
^N225	^GDAXI	0,975	2,0362	3,1269	0,0080	0,0130	0,0059	1,96	0,0186	0,0487	0,017
^N225	^GDAXI	0,990	2,8491	3,7856	0,0053	0,0057	0,0014	2,3263	0,008	0,0276	0,0123
^N225	FTSEMIB.MI	0,975	2,1155	3,1437	0,0080	0,0120	0,0044	1,96	0,0186	0,0443	0,0148
^N225	FTSEMIB.MI	0,990	2,6624	3,7924	0,0027	0,0033	0,0013	2,3263	0,008	0,0245	0,0109
^N225	^FCHI	0,975	2,0755	3,1575	0,0053	0,0076	0,0032	1,96	0,0213	0,0424	0,0147
^N225	^FCHI	0,990	2,7897	3,7982	0,0027	0,0028	0,0006	2,3263	0,0106	0,023	0,0095
^N225	^HSI	0,975	2,1437	3,2068	0,0080	0,0101	0,0038	1,96	0,016	0,0401	0,0149
^N225	^HSI	0,990	2,8635	3,8186	0,0027	0,0027	0,0001	2,3263	0,008	0,0229	0,0108
^N225	SXDP.Z	0,975	2,1910	3,0666	0,0053	0,0141	0,0066	1,96	0,0426	0,083	0,0289
^N225	SXDP.Z	0,990	2,8253	3,7606	0,0053	0,0075	0,0029	2,3263	0,0293	0,0532	0,0167
^N225	KO	0,975	2,2667	3,2319	0,0106	0,0206	0,0088	1,96	0,0319	0,0908	0,0327
^N225	KO	0,990	2,7333	3,8290	0,0080	0,0107	0,0037	2,3263	0,0213	0,0597	0,0237

For each couple of indexes (first two columns) at level α (third column) the table displays the empirical VaR evaluated on the first part of the sample (fourth column), the VaR estimated via GCS (fifth column) and via Normal distribution (ninth column), as well as the indexes ABLF, AQLF and UL for both the GCS (sixth-eighth columns) and the Normal distribution (tenth-twelfth columns).

measure. The formula of the ES of the GCS is

$$\begin{aligned}
 ES_Y(v_q) &= \\
 &= \frac{1}{\sqrt{\pi}} e^{-\frac{v_q^2}{4}} \left[1 + \frac{1}{4} \left(\frac{\beta_1 + \beta_2}{4!} \right) \left(p_4 \left(\frac{v_q}{\sqrt{2}} \right) + 4p_2 \left(\frac{v_q}{\sqrt{2}} \right) \right) + \frac{1}{16} \left(\frac{\beta_1 \beta_2}{(4!)^2} \right) \left(p_8 \left(\frac{v_q}{\sqrt{2}} \right) + 8p_6 \left(\frac{v_q}{\sqrt{2}} \right) \right) \right] \\
 &\quad \frac{1}{2} \operatorname{erfc} \left(\frac{v_q}{2} \right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{v_q^2}{4}} \left[\frac{1}{4} \left(\frac{\beta_1 + \beta_2}{4!} \right) p_3 \left(\frac{v_q}{\sqrt{2}} \right) + \frac{1}{16} \left(\frac{\beta_1 \beta_2}{(4!)^2} \right) p_7 \left(\frac{v_q}{\sqrt{2}} \right) \right]
 \end{aligned} \quad (40)$$

The ES has been computed in the first period of the sample (the first 1000 days) by using the VaR estimated via GCS as quantile. This procedure has been carried out for different α levels and more precisely for $\alpha = 0.05$, $\alpha = 0.025$ and $\alpha = 0.01$. The estimates of the expected shortfall for these α values, ES_α from now on, are shown in Table 6.

In order to evaluate the out-of sample performance of this risk measure, the ES have been computed also in the second part of the sample (last 374 days). These values, denoted with ES_{emp} and reported in Table 6, have been obtained by using VaR computed from GCS estimated in the first sample period for different α values ($\alpha = 0.05$, $\alpha = 0.025$ and $\alpha = 0.01$) as quantiles.

The goodness of ES_α estimates has been evaluated by implementing two tests based on bootstrap procedure. Both of them consider the performance of the GCS density under examination inadequate if the ES_α systematically underestimates the effective losses mean (ES_{emp}), thus implying great damage. The first test proposed by McNeil and Frey (2000) is based on the following statistic

$$Z_1 = \frac{1}{N} \sum_{t=1}^N \left(\frac{X_t I_{X_t > VaR_\alpha}}{ES_\alpha} - 1 \right) \quad (41)$$

where N is the number of losses X_t in the second part of the sample (the last 274 days) lying over the VaR_α , $I_{X_t > VaR_\alpha}$ is an indicative variable which assumes values equal to 1 if $X_t > VaR_\alpha$ and 0 otherwise. ES_α is the expected shortfall estimated by using the GCS density. Under the null hypothesis, assuming the correctness of the GCS densities or equivalently the goodness of the ES_α estimates, Z_1 takes low values.

Table 5: Analysis of VaR: test.

Index 1	Index 2	$1 - \alpha$	p-val(LUrc)	p-val(VaR)
$\hat{N}225$	\hat{FTSE}	0,975	0,0031	0,0076
$\hat{N}225$	\hat{FTSE}	0,99	0,0890	0,1958
$\hat{N}225$	\hat{GDAXI}	0,975	0,0138	0,0301
$\hat{N}225$	\hat{GDAXI}	0,99	0,3165	0,5981
$\hat{N}225$	FTSEMIB.MI	0,975	0,0138	0,0301
$\hat{N}225$	FTSEMIB.MI	0,99	0,0890	0,1958
$\hat{N}225$	\hat{FCHI}	0,975	0,0031	0,0076
$\hat{N}225$	\hat{FCHI}	0,99	0,0890	0,1958
$\hat{N}225$	\hat{HSI}	0,975	0,0138	0,0301
$\hat{N}225$	\hat{HSI}	0,99	0,0890	0,1958
$\hat{N}225$	SXDP.Z	0,975	0,0031	0,0076
$\hat{N}225$	SXDP.Z	0,99	0,3165	0,5981
$\hat{N}225$	KO	0,975	0,0443	0,0947
$\hat{N}225$	KO	0,99	0,6831	1,0000

For each couple of indexes (first two columns) at level α (third column) the table displays the p-values of both the likelihood ratio test (fourth column) and the binomial test (fifth column). p-values leading to acceptance of the null hypothesis at 1% are highlighted in bold.

The second test, proposed by Acerbi and Szekely (2014), is quite similar to the previous one. The statistic test is

$$Z_2 = \frac{1}{T} \sum_{t=1}^T \frac{X_t I_{X_t > VaR_\alpha}}{\alpha ES_\alpha} - 1 \quad (42)$$

where T denotes the sample size (274 in the case under exam). The null hypothesis of this test is the same as Z_1 test and, similarly to this latter, the Z_2 statistic assumes low values under the null hypothesis.

Both tests have been performed by implementing a bootstrap simulation. In both cases, 999 bootstrap samples have been selected from the out-of-sample data-set without making any assumption on the underlying data distribution and the statistics Z_1 and Z_2 have been computed by using these 999 bootstrap samples. The p-values of both tests have been computed as percentages of the Z_1 and Z_2 statistics obtained from bootstrap samples exceeding the corresponding statistics Z_1 and Z_2 , respectively, computed on the second part of the data (last 274 days). Looking at these p-values, reported in Table 6, we can conclude that the out of the sample performance of the GCS densities is quite good in most of the cases.

All the analyses have been carried out by using software R (R Core Team, 2015). In particular, basic financial operations have been worked out by using *tseries* (Trapletti and Hornik, 2015) package, computations involving Hermite polynomials with *EQL* (Thorn Thaler, 2009) package and tests for the evaluation of

goodness of fitting have been implemented by using *np* (Hayfield and Racine, 2008) package.

Table 6: Out-of-sample *ES* performance

Index 1	Index 2	α	VaR	ES_{emp}	ES_{α}	Z1	$pval(Z_1)$	Z2	$pval(Z_2)$
^N225	^FTSE	0,950	2,4725	3,2659	3,3266	-0,0182	0,5165	-0,9863	0,4675
^N225	^FTSE	0,975	3,1628	3,9579	3,8500	0,0280	0,3443	-0,9944	0,5395
^N225	^FTSE	0,990	3,8004	4,3089	4,4539	-0,0326	0,3744	-0,9974	0,6647
^N225	^GDAXI	0,950	2,4342	3,8722	3,2944	0,1754	0,4494	-0,9901	0,4975
^N225	^GDAXI	0,975	3,1269	3,8722	3,8269	0,0118	0,4675	-0,9917	0,4535
^N225	^GDAXI	0,990	3,7856	4,0568	4,4359	-0,0855	0,4935	-0,9951	0,5355
^N225	FTSEMIB.MI	0,950	2,4513	3,6975	3,3092	0,1173	0,4184	-0,9906	0,4905
^N225	FTSEMIB.MI	0,975	3,1437	3,6975	3,8376	-0,0365	0,5005	-0,9921	0,4745
^N225	FTSEMIB.MI	0,990	3,7924	4,2733	4,4442	-0,0384	0,3744	-0,9974	0,6386
^N225	^FCHI	0,950	2,4664	3,4896	3,3217	0,0505	0,4695	-0,9912	0,5125
^N225	^FCHI	0,975	3,1575	3,7596	3,8466	-0,0226	0,5005	-0,9947	0,5355
^N225	^FCHI	0,990	3,7982	4,0289	4,4512	-0,0949	0,3764	-0,9976	0,6276
^N225	^HSI	0,950	2,5277	3,6804	3,3680	0,0928	0,4825	-0,9908	0,4935
^N225	^HSI	0,975	3,2068	3,6804	3,8788	-0,0512	0,5275	-0,9922	0,5115
^N225	^HSI	0,990	3,8186	3,8710	4,4776	-0,1355	0,3534	-0,9977	0,6547
^N225	SXDP.Z	0,950	2,3817	2,9504	3,2435	-0,0904	0,4695	-0,9745	0,4835
^N225	SXDP.Z	0,975	3,0666	4,3077	3,7887	0,1370	0,3413	-0,9938	0,5065
^N225	SXDP.Z	0,990	3,7606	4,3077	4,4080	-0,0228	0,4675	-0,9947	0,5375
^N225	KO	0,950	2,5635	3,5989	3,3926	0,0608	0,5035	-0,9792	0,4765
^N225	KO	0,975	3,2319	4,0578	3,8957	0,0416	0,4855	-0,9886	0,5105
^N225	KO	0,990	3,8290	4,2954	4,4921	-0,0438	0,4785	-0,9923	0,4885

For each couple of indexes (first two columns) at each level α (third column) there are displayed the theoretical VaR for GC distributions (fourth column), the empirical ES evaluated on the first sample (fifth column), the theoretical ES for GC distributions (sixth column), the statistic tests Z_1 and Z_2 (seventh and ninth columns) and the associated *p*-values for the GC distributions (eighth and tenth columns). The significance level is fixed at 1%.

6 Conclusion

In this paper, we devise a method to specify the distribution of the sums of leptokurtic Gaussian variables. The approach we have adopted rests on the polynomial transformation of Gaussian variables by means of their associated Hermite polynomials resulting in Gram-Charlier expansions. The sum of Gram Charlier expansions (GCS) turns out to be a tail sensitive density and as such can be effectively used to represent a portfolio return. Thus, it can be conveniently used to compute risk measures such as the Value at Risk and the expected shortfall. Its application to a portfolio of a set of financial asset indexes provides evidence of the effectiveness of the proposed technique. This is confirmed by the GCS effective performance in both VaR and expected shortfall estimation in and out of the sample period.

Appendix

In the following we run through the classic procedure to obtain the density of sum of independent standard-normal random variables. The same procedure applies to sums of Gram-Charlier expansions with due computations as shown

in Section 3, (see also e.g. Johnson and Kotz (1970); Stuart and Ord (2004)). In this connection let us first state the following

Lemma 1. *Let $Y = X_1 + X_2$, be the sum of two i.i.d. normal random variables. Then the density of Y is*

$$f_y(y) = (4\pi)^{-1/2} e^{-\frac{y^2}{4}} \quad (43)$$

Proof. As it is well known, the density of Y is

$$f_Y(y) = f_X(x_1) * f_X(x_2) \quad (44)$$

where the symbol $*$ denotes convolution. Further, the characteristic function $F_Y(\omega)$ of Y is the product of the characteristic functions of the X_1 and X_2 , that is

$$F_Y(\omega) = F_{X_1}(\omega)F_{X_2}(\omega) = F_X^2(\omega) \quad (45)$$

Now, bearing in mind the Fourier-transform pair

$$\sqrt{\frac{a}{\pi}} e^{-at^2} \leftrightarrow e^{-\frac{\omega^2}{4a}} \quad (46)$$

and setting $a = \frac{1}{2}$, yields

$$F_X(\omega) = e^{-\frac{\omega^2}{2}}. \quad (47)$$

which is the characteristic function of the standard normal variable. According to (45), the characteristic function of the sum of two i.i.d. standard normal is

$$F_Y(\omega) = e^{-\omega^2}. \quad (48)$$

In turn, by setting $a = 1/4$ in (46), the density function of the sum $f_Y(y)$ proves to be as in (43) \square

The same procedure applies to obtain the density function of the sum of two Gram-Charlier expansions as in Theorem 1 of Section 3. In this connection let us introduce the following.

Definition 1. *Orthogonal polynomials.*

Given a density $f(x)$ with finite moments m_j , we can determine a system of polynomials $p_n(x) = \sum_j \delta_j x^j$ such that

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)f(x)dx = \begin{cases} \gamma_n & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases} \quad (49)$$

the condition (49) determines $p_n(x)$ up to a constant factor and the coefficients δ_j turn out to be algebraic function of the moments m_j

$$m_j = \int_{-\infty}^{+\infty} x^j f(x)dx \quad (50)$$

(see Faliva et al. (2016) for details).

When the density $f(x)$ is even, $p_n(x)$ is either even or odd depending on n being even or odd, respectively.

Should $f(x)$ be the Gaussian law, then $\{p_n(x)\}$ would correspond to the well known Hermite polynomials, that is

$$p_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{\partial^j}{\partial x^j} e^{-\frac{x^2}{2}}. \quad (51)$$

Orthogonal polynomials can be used to modify the moments of the parent density via Gram-Charlier expansions. In this connection we have the following

Definition 2. *Let*

$$q(x, \beta) = 1 + \frac{\beta}{\gamma_j} p_j(x) \quad (52)$$

where $p_j(x)$ is the orthogonal polynomial of degree j associated with $f(x)$, β is a positive parameter and γ_j the squared norm of $p_j(x)$. Then,

$$\varphi(x, \beta) = q(x, \beta) f(x) \quad (53)$$

subject to $q(x, \beta)$ being non-negative definite, is a density whose lower-order moments, μ_j , are related to those of $f(x)$, m_j , as follows

$$\mu_j = \begin{cases} m_i & \text{for } i = 1, 2, 3, \dots, j-1 \\ m_i + \beta & \text{for } i = j \end{cases} \quad (54)$$

Higher moments of $\varphi(x, \beta)$ turn out to be algebraic functions of the moments of $f(x)$. For the proof see Faliva et al. (2016).

When $f(x)$ is a Gaussian law, the density (53) is the classical Gram-Charlier expansion.

References

- Abramowitz, M. and Stegun, I. A. (1964). *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*, volume 55. Courier Corporation.
- Acerbi, C. and Szekely, B. (2014). Back-testing expected shortfall. *Risk*, **27**(11).
- Acerbi, C. and Tasche, D. (2002). On the coherence of expected shortfall. *Journal of Banking & Finance*, **26**(7), 1487–1503.
- Albrecht, P. (2004). Risk measures. In T. Jozef and B. Sundt, editors, *Encyclopedia of Actuarial Science*, pages 1493–1501. Wiley, N.Y.
- Alles, L. and Murray, L. (2010). Non-normality and risk in developing asian markets. *Review of Pacific Basin Financial Markets and Policies*, **13**(4), 583–605.
- Artzner P., Delbaen F., E. J. (1999). Coherent measures of risk. *Mathematical Finance*, **9**(3), 203–228.
- Bagnato, L., Potì, V., and Zoia, M. G. (2015). The role of orthogonal polynomials in adjusting hyperpolitic secant and logistic distributions to analyse financial asset returns. *Statistical Papers*, **56**(4), 1205–1234.
- Christoffersen, P., Diebold, F. X., and Schuermann, T. (1998). Horizon problems and extreme events in financial risk management. *Economic Policy Review*, **4**(3), 98–116.
- Curto, J. D., Pinto, J. C., and Tavares, G. N. (2009). Modeling stock markets' volatility using garch models with normal, student's t and stable paretian distributions. *Statistical Papers*, **50**(2), 311–321.
- Dharmadhikari, S. W. (1988). *Unimodality, convexity, and applications*, volume 27. Academic Press, N.Y.
- Dowd, K. and Blake, D. (2006). After var: the theory, estimation, and insurance applications of quantile-based risk measures. *Journal of Risk and Insurance*, **73**(2), 193–229.
- Faliva, M., Potì, V., and Zoia, M. G. (2016). Orthogonal polynomials for tailoring density functions to excess kurtosis, asymmetry, and dependence. *Communications in Statistics-Theory and Methods*, **45**(1), 49–62.
- Gallant, A. R. and Tauchen, G. (1989). Semiparametric estimation of conditionally constrained heterogeneous processes: Asset pricing applications. *Econometrica: Journal of the Econometric Society*, pages 1091–1120.
- Gallant, A. R. and Tauchen, G. (1993). A nonparametric approach to nonlinear time series analysis: estimation and simulation. in Brillinger, D., P. Caines, J. Geweke, E. Parzen, M. Rosenblatt, and M. S. Taqqu eds., *New Directions in Time Series Analysis, Part II*, pages 71–92.
- Granger, C., Maasoumi, E., and Racine, J. (2004). A dependence metric for possibly nonlinear processes. *Journal of Time Series Analysis*, **25**(5), 649–669.

- Hayfield, T. and Racine, J. S. (2008). Nonparametric econometrics: The np package. *Journal of Statistical Software*, **27**(5).
- Johnson, N. L. and Kotz, S. (1970). *Continuous Univariate Distributions*, volume 2. Wiley, N.Y., New York.
- Jondeau, E. and Rockinger, M. (2001). Gram–charlier densities. *Journal of Economic Dynamics and Control*, **25**(10), 1457–1483.
- Kupiec, P. H. (1995). Techniques for verifying the accuracy of risk measurement models. *THE J. OF DERIVATIVES*, **3**(2).
- Landsman, Z. M. and Valdez, E. A. (2003). Tail conditional expectations for elliptical distributions. *North American Actuarial Journal*, **7**(4), 55–71.
- Maasoumi, E. and Racine, J. (2002). Entropy and predictability of stock market returns. *Journal of Econometrics*, **107**(1), 291–312.
- McNeil, A. J. and Frey, R. (2000). Estimation of tail-related risk measures for heteroscedastic financial time series: an extreme value approach. *Journal of empirical finance*, **7**(3), 271–300.
- Mills, T. C. and Markellos, R. N. (2008). *The econometric modelling of financial time series*. Cambridge University Press.
- Mittnik, S., Paoletta, M., and Rachev, S. (2000). Diagnosis and treating the fat tails in financial returns data. *Journal of Empirical Finance*, **7**(3–4), 389–416.
- R Core Team (2015). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria.
- Rachev, S. T., Hoechstetter, M., Fabozzi, F. J., Focardi, S. M., *et al.* (2010). *Probability and statistics for finance*, volume 176. Wiley. com.
- Shuangzhe, Liu Chris, H. (2006). On estimation in conditional heteroskedastic time series models under non-normal distributions. *Statistical Papers*, **49**(3), 455–469.
- Stuart, A. and Ord, J. (2004). Kendall’s advanced theory of statistics (distribution theory, vol. 1). *Griffin, London*.
- Szegö, G. P. (2004). *Risk measures for the 21st century*. Wiley, N.Y.
- Thorn Thaler (2009). *Die Extended-Quasi-Likelihood-Funktion in Generalisierten Linearen Modellen*. Master’s thesis, Technische Universität Graz.
- Trapletti, A. and Hornik, K. (2015). *tseries: Time Series Analysis and Computational Finance*. R package version 0.10-34.
- Uryasev, S. (2000). Conditional value at risk: Optimization algorithms and applications. *Financial Engineering News*, (14), 1–5.
- Zoia, M. G. (2010). Tailoring the gaussian law for excess kurtosis and skewness by hermite polynomials. *Communications in Statistics-Theory and Methods*, **39**(1), 52–64.