VIX computation based on affine stochastic volatility models in discrete time

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Abstract

We propose a class of discrete-time stochastic volatility models that, in a parsimonious way, capture the time-varying higher moments observed in financial series. Three desirable results are obtained. First, we have a recursive procedure for the log-price characteristic function which allows a semi-analytical formula for option prices as in Heston and Nandi [2000]. Second, we reproduce some features of the VIX Index. Finally, we derive a simple formula for the VIX index and use it for option pricing.
Keywords: *Affine Stochastic Volatility; VIX; Implied Volatility Surface.*

1 Introduction

The Black and Scholes model [see Black and Scholes, 1973] is probably the most famous model proposed for option pricing. Despite its success, the drawbacks in representing real market stylized facts are well documented by an increasing empirical literature [see Embrechts et al., 1997, and the references therein]. Since Mandelbrot [1963], empirical results have shown that the process describing log returns is not a Brownian motion. Indeed, financial time series exhibit heavy tails, asymmetric distribution, persistence and clustering in volatility.

Several models have been proposed in continuous and discrete time. A first improvement is obtained through the introduction of the Lévy processes with jumps in finance. For instance, Merton [1976] introduced a Jump diffusion model in the evaluation of option prices. The success of these models in finance is justified, on one hand, by their analitycal tractability (the marginal distribution can be determined through the characteristic function) and, on the other hand, by the ability on reproducing asymmetry and heavy tails in financial time series [see Schoutens, 2003, Cont and Tankov, 2003, for a general survey]. A special attention deserves the process whose distribution at time one is a Normal Variance Mean Mixture. Particular cases widely applied in finance are the Variance Gamma process introduced by Madan and Seneta [1990], the Normal Inverse Gaussian [see Barndorff-Nielsen and Shephard, 2001], the Hyperbolic and the Generalized Hyperbolic [see Barndorff-Nielsen, 1977, Eberlein and Prause, 1998]. Their main drawback is related to the independence of the increments that makes them inadequate in capturing the dynamic of higher moments [see Iacus, 2011, for formulas of some Lévy processes applied in option pricing].

A way to overcome these limits is by using stochastic volatility models for describing log return dynamics. There are two sources of risk in these models: the first drives the volatility dynamics and the second directly log returns. The main problem is that the volatility process is not observable in the market.

In discrete time the most commonly used class for modeling financial time series is the family of GARCH models [Engle, 1995]. Despite the success in financial econometrics and risk management, their use for option pricing is not yet very well understood, as observed in Christoffersen et al. [2012]. Monte Carlo technique is often used to compute option prices in GARCH models [see Duan, 1995, Duan and Simonato, 1998, for the efficiency of Monte Carlo estimator]. Another approach is using approximate formulas based on Edgeworth expansion [see Duan et al., 1999, 2006]. It is well known that the Monte Carlo procedure is time consuming when calibration exercise is considered, while the Edgeworth expansion seems to be less accurate for option pricing with long or medium time to maturity.

A major breakthrough occurred with the paper of Heston and Nandi [2000] where the authors derived a recursive procedure for the characteristic function of the log price at maturity, obtaining a semi analytical formula for European call options based on Inverse Fourier Transform, as in Carr and Madan [1999]. Following the same idea a new class of GARCH models, namely affine GARCH, has been developed assuming different

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assumptions for the innovations. In particular, Christoffersen et al. [2006] considered the Inverse Gaussian innovations while Bellini and Mercuri [2007] Gamma innovations. Later Mercuri [2008] generalized the class of affine GARCH models assuming that log returns are conditionally Tempered Stable distributed [see Ornthanalai, 2008, for more details on affine GARCH models].

As observed in Christoffersen et al. [2006], the extreme asymmetry of the affine GARCH models with non-normal innovations gives an advantage for options with very short time to maturity. However, the fit is less accurate for options with medium maturity probably due to the fact that the medium time to maturity return distribution slowly converges to the Normal distribution.

To overcome this limit, starting from the affine GARCH model and assuming that the conditional distribution of log returns is a Normal Variance Mean Mixture, we construct a discrete time stochastic volatility model in a simple way. Indeed, substituting the mixing random variable with an affine GARCH, we obtain a recursive procedure for the computation of the characteristic function for the log-price at maturity. Option prices are obtained via Fourier transform. The introduction of this new class is motivated from the fact that affine models (usually in continuous time) are quite natural for option pricing but the discrete time models are easily estimated. Although the literature on affine stochastic volatility in continuous time is wide, the discrete counterpart did not receive the same attention. The substitution of the mixing r.v. with an affine GARCH process gives to our models the capability of capturing time dependence in financial times series, for instance persistence in squared returns. This affine GARCH process controls also the magnitude of the return movements and plays a similar role as the variance process in the continuous time models. Moreover, it generates time varying higher order moments. Volatility [see Chicago Board Options Exchange, 2003] and Skew [see Chicago Board Options Exchange, 2011] indexes cannot exist in a world with constant higher moments since they would be useless. Time-dependence of these moments is coherent with price movements observed in the market making our approach more realistic.

In our model, it is possible to extrapolate information from the VIX data and use it in option pricing. Indeed, we find a linear relation between the variance dynamics and the VIX^2. A similar result has been obtained in discrete time by Hao and Zhang [2013] under the GARCH assumption and, for these models, the procedure for extrapolating information from VIX in pricing Options on S&P500 has been considered recently in Kanniainen et al. [2014] while Liu et al. [2015] analyze how to assess the risk premium in GARCH(1,1), GJR, and HestonNandi models. However, our model is able to generate time-varying skewness and kurtosis that standard GARCH models can not reproduce.

The paper is organized as follows. Section 2 explains the construction of stochastic volatility models in discrete time. In Section 3 we prove that, in our setup, the VIX index is an autoregressive process with heteroskedastic innovations: we derive a linear relation between the unobservable variance and the current level of VIX index. In Section 4 we derive explicit formulas specifying the conditional distribution of log returns. In Section 5 empirical results using the implied volatility surface obtained from Bloomberg data provider are given. In Section 6 we draw some conclusions.

2 General Setup

In this section we propose a class of stochastic volatility models, in discrete time, through which we are able to price options using the information extrapolated from the VIX index.

Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, we consider a market with two assets:

- riskless with dynamics: $B_t = B_{t-1} \exp(r)$
- risky with price dynamics:

$$\begin{aligned} S_t &= S_{t-1} \exp(X_t) \\ X_t &= r + \lambda_0 h_t + \lambda_1 V_t + \sigma \sqrt{V_t} Z_t \end{aligned} \tag{1}$$

where r is the deterministic free rate observed in the market; λ_0 and λ_1 are real valued model parameters while σ must be non negative; X_t is a discrete time stochastic process describing log returns; $Z_t \sim N(0,1),\ \forall t = 1, ..., T$ and is independent from V_t.

We require V_t to be an adapted positive process such that the conditional moment generating function (m.g.f. hereafter) of V_t given the information available at time $t-1$ is:

$$E[\exp(cV_t)|\mathcal{F}_{t-1}] = \exp(h_t f(c, \theta)) \tag{2}$$

and $\forall$ fixed vector θ, $\exists\ \delta > 0$ such that $\forall c \in (-\delta, \delta)$ the function $f(c, \theta) \in C^\infty$ and $f(0, \theta) = 0$. The vector θ contains the parameters of distribution V_t given information at time $t-1$. We assume h_t to be a predictable process defined as:

$$h_t = \alpha_0 + \alpha_1 V_{t-1} + \beta h_{t-1}. \tag{3}$$

The process h_t is positive if the parameters α_0, α_1 and β are non negative.
It is worth noting that if V_t is constant (i.e. $V_t = \bar{V},\ t = 1, 2, \ldots$ and consequently $h_t = \bar{h}$), the sequence $\{X_t\}_{t=1,2,\ldots}$ is composed by i.i.d gaussian r.v.' s and a general sample path is centered in $r + \lambda_0\bar{h} + \lambda_1\bar{V}$. The magnitude of movements depends on the value of $\bar{V}$. In this case, the oscillating behaviour of returns in quiet and in turbulent markets can not be reproduced. The same observation holds if we assume the sequence $\{V_t\}_{t=1,2,\ldots}$ to be composed by i.i.d. random variables.
From (2) we have:

$$E[V_t|\mathcal{F}_{t-1}] = \left.\frac{\partial E[\exp(cV_t)|\mathcal{F}_{t-1}]}{\partial c}\right|_{c=0}.$$

Let $g(\theta)$ be defined as:

$$g(\theta) := \left.\frac{\partial f(c,\theta)}{\partial c}\right|_{c=0}, \tag{4}$$

the analytical expression for conditional mean of V_t becomes:

$$E[V_t|\mathcal{F}_{t-1}] = h_t g(\theta). \tag{5}$$

Adding and subtracting the quantity $\alpha_1 g(\theta)h_{t-1}$ in ((3)) we obtain for h_t a new representation:

$$h_t = \alpha_0 + (\alpha_1 g(\theta) + \beta)h_{t-1} + \alpha_1(V_{t-1} - g(\theta)h_{t-1}). \tag{6}$$

Observe that h_t is an AR(1) with heteroskedastic error $V_{t-1} - g(\theta)h_{t-1}$. Therefore, if we extrapolate from the market the realizations of h_t, the generalized least square technique gives us estimates for the quantities α_0, α_1, and $\alpha_1 g(\theta) + \beta$. In our model, the conditional variance evolves according to the stochastic process h_t:

$$Var\left[V_t|\,\mathcal{F}_{t-1}\right] = h_t \left.\frac{\partial^2 f(c,\theta)}{(\partial c)^2}\right|_{c=0}.$$

An essential requirement, based on empirical evidence, is the negative correlation between returns and volatility which implies:

$$Cov\left(V_t,X_t|\,\mathcal{F}_{t-1}\right) = \lambda_1 Var(V_t|\mathcal{F}_{t-1}) < 0, \tag{7}$$

meaning that λ_1 must be negative.
If we compute the conditional expectation of X_t we have:

$$E\left(X_t\,|\mathcal{F}_{t-1}\right) = r + \left(\lambda_0 + \lambda_1 g\left(\theta\right)\right)h_t. \tag{8}$$

Looking to relation in (8) is natural for a financial interpretation to require $\lambda_0 + \lambda_1 g\left(\theta\right) > 0$ since it implies a positive risk premium for the asset.
In the special case when $\sigma = 0$ the process describing X_t is an affine GARCH as in Christoffersen et al. [2006], Bellini and Mercuri [2007] and Mercuri [2008].

Our approach tries to generalize the Lévy processes built on the Normal Variance Mean Mixtures since we introduce a dependence structure. Indeed the conditional distribution evolves through time due to the predictable process h_t.

Both h_t and σ are crucial for the variability of the process X_t but σ does not introduce any heteroskedasticity in the model and for obtaining time dependent higher moments we need h_t to be defined as in (6). Through the predictable process h_t, we are able to generalize the Lévy process built on the Normal Variance Mean Mixture obtaining a distribuition of increments that evolves in time.

The next step is to show how to price a European call option with maturity T where the dynamics of the log returns for the risky asset is defined in (1). Here, we provide a simple recursive procedure through which we obtain the conditional m.g.f. of $\ln\left(S_T\right)$ using a similar approach as that introduced in Heston and Nandi [2000].

Proposition 1 *Under condition* (2)*, the m.g.f. of the random variable* $\ln S_T$ *given the information at time* t *exists and is given by:*

$$E[\exp(c\ln\left(S_T\right))|\mathcal{F}_t] = S_t^c \exp[A(t;T,c) + B(t;T,c)h_{t+1}].$$

The time-dependent coefficients $A(t;T,c)$ *and* $B(t;T,c)$ *are:*

$$\begin{cases} A(t;T,c) & = cr + A(t+1;T,c) + \alpha_0 B(t+1;T,c) \\ B(t;T,c) & = c\lambda_0 + \beta B(t+1;T,c) + \\ & \quad f(c\lambda_1 + \alpha_1 B(t+1;T,c) + \frac{c^2\sigma^2}{2}, \theta) \end{cases} \tag{9}$$

with the following final conditions:

$$\begin{aligned} A(T;T,c) &= 0 \\ B(T;T,c) &= 0. \end{aligned}$$

(see Appendix 7.1)

The existence of m.g.f. allows us to obtain the characteristic function and the distribution function is achieved by the inverse Fourier transform.

Our aim is to price options and compute implied volatility indexes. In order to ensure the martingale condition under Q measure, we use the following proposition.

Proposition 2 *Under the assumptions $E(S_t) < +\infty$ and $\lambda_0 = -f(\lambda_1 + \frac{\sigma^2}{2}, \theta)$, the discounted price is a martingale.*
(see Appendix 7.2)

We have obtained in Proposition 1 the m.g.f. for the underlying of a call option. The next step is the evaluation of a European call option as in Heston [1993]

$$\begin{aligned} C(K,T) &= S_0\Pi_1 - Ke^{-rT}\Pi_2 \\ \Pi_1 &= \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \Re\left(\frac{K^{-iu}E_0^Q[S_T^{i(u-i)}]}{iuE_0^Q[S_T]}\right)du \\ \Pi_2 &= \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \Re\left(\frac{K^{-iu}E_0^Q[S_T^{iu}]}{iu}\right)du \end{aligned}$$

The exercise probabilities Π_1 and Π_2 can be computed following Feller [1968].

3 VIX Index

In this Section we show how to derive the Volatility Implied Index (VIX) in our model. In particular, the linear relation between VIX and the process h_t is derived. From a theoretical point of view, this relation implies that the VIX is a mean-reverting autoregressive process with heteroskedastic errors. A similar result has been proposed in Zhang and Zhu [2006] under the assumption that the SPX dynamics is described by Heston [1993]. The methodology for computing the VIX index is based on the replication of a variance swap [see Demeterfi et al., 1999] and the current level of VIX is related to the value of the portfolio composed by out-of-the money call/put options on the S&P500. Assuming that the strike prices vary continuously from 0 to $+\infty$, the VIX squared formula is the following:

$$\begin{aligned} \left(\frac{VIX_t}{100}\right)^2 &= \frac{2e^{r(T-t)}}{T-t}\left[\int_0^{S^*}\frac{1}{K^2}P(S_t,K)dK + \int_{S^*}^{+\infty}\frac{1}{K^2}C(S_t,K)dK\right] \\ &= \frac{2e^{r(T-t)}}{T-t}\left[E_t^Q\left(\frac{S_T - S^*}{S^*} - \ln\left(\frac{S_T}{S^*}\right)\right)\right]. \end{aligned} \tag{10}$$

$C(S_t,K)$ and $P(S_t,K)$ are out-of-the money call and put option prices. S^* is the forward price of the SPX index.
The main result of our model is reported in the following proposition.

Proposition 3 *Under the conditions:*

$$\begin{aligned} &\alpha_1 g(\theta) + \beta < 1 \\ &\lambda_1 g(\theta) - f\left(\lambda_1 + \tfrac{\sigma^2}{2}, \theta\right) \leq 0 \\ &h_{t+1} > 0 \end{aligned} \tag{11}$$

the VIX squared is an affine linear function of the predictable process h_t:

$$\left(\frac{VIX_t}{100}\right)^2 = -\frac{2e^{r(T-t)}}{T-t}\left[C(t;T) + D(t;T)h_{t+1}\right] \tag{12}$$

where $C(t;T)$ and $D(t;T)$ are functions of the model parameters, given by:

$$\begin{cases} C(t;T) = \alpha_0\left[\lambda_1 g(\theta) + \lambda_0\right]\left\{\frac{T-t-1-[\alpha_1 g(\theta)+\beta]\frac{1-[\alpha_1 g(\theta)+\beta]^{(T-t)-1}}{1-[\alpha_1 g(\theta)+\beta]}}{1-[\alpha_1 g(\theta)+\beta]}\right\} \\ D(t;T) = \left[\lambda_1 g(\theta) + \lambda_0\right]\frac{1-[\alpha_1 g(\theta)+\beta]^{T-t}}{1-[\alpha_1 g(\theta)+\beta]} \end{cases} \tag{13}$$

with $T - t = 30$ days.
(See Appendix 7.3)

Considering the fact that VIX is a measure of the 30 days implied volatility on S&P500, equation (12) becomes:

$$\left(\frac{VIX_t}{100}\right)^2 = -\frac{2e^{r30}}{30}\left[C_{30} + D_{30}h_{t+1}\right]$$

where r is the one month Libor rate on daily basis.
We define the adjusted VIX as:

$$VIX_t^{adj} = -\frac{30}{2e^{r30}}\frac{VIX_t^2}{10^4}.$$

Notice that $VIX_t^{adj} < 0\ \forall t$ since it is a decreasing linear transformation of the VIX squared.
Using Proposition 3 we have:

$$VIX_t^{adj} = C_{30} + D_{30}h_{t+1} \Rightarrow h_{t+1} = \frac{VIX_t^{adj} - C_{30}}{D_{30}}. \tag{14}$$

The requirement $h_{t+1} > 0$ implies that $0 > VIX_t^{adj} > C_{30}\ \forall t$.
Using the definition (6) of h_t, we have following proposition:

Proposition 4 *Under the same conditions of Proposition 3, defining the heteroskedastic error term* $\tau_t := \alpha_1(V_t - g(\theta)h_t)D_{30}$, *the* VIX_t^{adj} *is an AR(1) defined as:*

$$VIX_t^{adj} = int + slope VIX_{t-1}^{adj} + \tau_t$$

where

$$\begin{cases} int = 30\alpha_0\left(\lambda_1 g(\theta) + \lambda_0\right) \\ slope = \alpha_1 g(\theta) + \beta \end{cases}$$

(see Appendix 7.4)

Given the model parameters, the current and the one-day-ahead VIX level we have the heteroskedastic error term defined as:

$$\tau_{t+1} = VIX_{t+1} - int - slope VIX_t.$$

From equation (14) we extract h_{t+1} and obtain the value of the main "unobservable" variable of our model, i.e V_{t+1}:

$$V_{t+1} = g(\theta) + \frac{\tau_{t+1}}{\alpha_1 D_{30}}.$$

Once estimated *int* and *slope* we can redefine D_{30} and C_{30} in order to extrapolate a multiple of h_{t+1} from the quoted VIX_t. In particular we get:

$$D_{30} = \frac{D_{30}^*}{\alpha_0} = \frac{int\left(1 - slope^{30}\right)}{30 * (1 - slope)}\frac{1}{\alpha_0}$$

$$C_{30} = \left[\frac{29 - slope\frac{1-slope^{29}}{1-slope}}{1 - slope}\right]\frac{int}{30}$$

$$\frac{VIX_t^{adj} - C_{30}}{D_{30}^*} = \frac{h_{t+1}}{\alpha_0} > 0$$

The quantity $\frac{h_{t+1}}{\alpha_0}$ can be used to compute the m.g.f. of $\ln(S_T)|\mathcal{F}_t$ needed in option pricing.
If $slope < 1$, VIX_t^{adj} is mean reverting. The long term mean and the reverting speed are respectively:

$$\frac{int}{1 - slope}, \quad 1 - slope.$$

The conditional mean of the error term is zero but we are in presence of heteroskedasticity:

$$E\left[\tau_t|\,\mathcal{F}_{t-1}\right] = 0, \quad Var\left[\tau_t|\,\mathcal{F}_{t-1}\right] = \alpha_1^2 D_{30}^2 Var\left[V_t|\,\mathcal{F}_{t-1}\right].$$

Although $Cov\left[\tau_{t+1}, \tau_t \middle| \mathcal{F}_{t-1}\right] = 0$ and $Cov\left[\tau_{t+1}, \tau_t^2 \middle| \mathcal{F}_{t-1}\right] = 0$, the error time-dependence structure is more complex than a linear one. The following quantities are different from zero and time dependent:

$$Cov\left[\tau_{t+1}^2, \tau_t \middle| \mathcal{F}_{t-1}\right] = \alpha_1^3 D_{30}^3 \left.\frac{\partial^2 f}{(\partial c)^2}\right|_{c=0} \left[\alpha_0 + \alpha_1 \left.\frac{\partial^2 f}{(\partial c)^2}\right|_{c=0}\right] h_t$$

$$Cov\left[\tau_{t+1}^2, \tau_t^2 \middle| \mathcal{F}_{t-1}\right] = \alpha_1^4 D_{30} \left.\frac{\partial^2 f(c,\theta)}{(\partial c)^2}\right|_{c=0} \left[\alpha_0 + (\alpha_1 g(\theta) + \beta) \left.\frac{\partial^2 f(c,\theta)}{(\partial c)^2}\right|_{c=0} h_t^2 + \alpha_1^2 \mu_3\right]$$

where $\mu_3 = E\left[(V_t - g(\theta))^3 \middle| \mathcal{F}_{t-1}\right]$.

4 Special cases

The conditional distribution of log returns belongs to the family of Normal Variance Mean Mixture since Z_t in (1) is normally distributed. A univariate Normal Variance Mean Mixture [see Barndorff-Nielsen et al., 1982] is a random variable defined as:

$$X \stackrel{d}{=} \mu + \lambda V + \sigma \sqrt{V} Z$$

where Z and V are independent univariate random variables, $Z \sim N(0,1)$, and V is defined on the positive real line. Below we introduce three special cases of our approach where the conditional distribution of log returns is respectively Variance Gamma [see Madan and Seneta, 1990], Normal Inverse Gaussian [see Barndorff-Nielsen and Shephard, 2001] and Normal Tempered Stable [see Barndorff-Nielsen and Shephard, 2001].

4.1 Dynamic Variance Gamma

Assuming that the affine GARCH process V_t is conditionally Gamma distributed [see Bellini and Mercuri, 2007] then X_t in (1) follows a Dynamic Variance Gamma model introduced by Bellini and Mercuri [2011].
The conditional m.g.f. of the V_t is:

$$\begin{array}{ll} E\left[e^{cV_t} \middle| \mathcal{F}_{t-1}\right] & = \exp\left[-h_t \ln(1-c)\right] \\ f(c,\theta) & = -\ln(1-c) \\ g(\theta) & = 1. \end{array}$$

System (9) becomes:

$$\begin{cases} A(t;T,c) & = cr + A(t+1;T,c) + \alpha_0 B(t+1;T,c) \\ B(t;T,c) & = c\lambda_0 + \beta B(t+1;T,c) + \\ & - \ln\left(1 - c\lambda_1 - \alpha_1 B(t+1;T,c) - \frac{c^2\sigma^2}{2}\right). \end{cases} \tag{15}$$

System (13) becomes:

$$\begin{cases} C(t;T,c) & = \alpha_0 (\lambda_1 + \lambda_0) \left\{ \frac{(T-t) - (\alpha_1+\beta)\frac{1-(\alpha_1+\beta)^{T-t-1}}{1-(\alpha_1+\beta)}}{1-(\alpha_1+\beta)} \right\} \\ D(t;T,c) & = (\lambda_1 + \lambda_0) \frac{1-(\alpha_1+\beta)^{T-t}}{1-(\alpha_1+\beta)}. \end{cases} \tag{16}$$

with final conditions $C(T;T,c) = 0$ and $D(T;T,c) = 0$. We have the following restrictions on the parameters:

$$\begin{cases} \lambda_1 \leq 0 \\ \lambda_0 = \ln\left(1 - \lambda_1 - \frac{\sigma^2}{2}\right) \\ \alpha_1 + \beta \leq 1 \\ \lambda_1 + \ln\left(1 - \lambda_1 - \frac{\sigma^2}{2}\right) \leq 0. \end{cases} \tag{17}$$

4.2 Dynamic Normal Inverse Gaussian

If the affine GARCH process V_t is conditionally Inverse Gaussian distributed [see Christoffersen et al., 2006] than log-returns X_t, given the information at time $t-1$, have a Normal Inverse Gaussian distribution [see Barndorff-Nielsen, 1997].
The density of a Inverse Gaussian distribution is:

$$f_V(v) = \frac{h_t}{\sqrt{2\pi v^3}} \exp\left[-\frac{1}{2}\left(\sqrt{v} - \frac{h_t}{\sqrt{x}}\right)^2\right].$$

The conditional m.g.f. of the V_t is:

$$\begin{array}{ll} E\left[e^{cV_t}\middle|\mathcal{F}_{t-1}\right] & = \exp\left[h_t\left(1-\sqrt{1-2c}\right)\right] \\ f(c,\theta) & = \left(1-\sqrt{1-2c}\right) \\ g(\theta) & = 1. \end{array}$$

System (9) becomes:

$$\left\{\begin{array}{ll} A(t;T,c) & = xr + A(t+1;T,c) + \alpha_0 B(t+1;T,c) \\ B(t;T,c) & = c\lambda_0 + \beta B(t+1;T,c) + \\ & \sqrt{1-2\left(c\lambda_1 + \alpha_1 B(t+1;T,c) + \frac{c^2\sigma^2}{2}\right)}. \end{array}\right. \tag{18}$$

System (13) becomes

$$\left\{\begin{array}{ll} C(t;T,c) & = \alpha_0\left(\lambda_1+\lambda_0\right)\left\{\frac{(T-t)-(\alpha_1+\beta)\frac{1-(\alpha_1+\beta)^{T-t-1}}{1-(\alpha_1+\beta)}}{1-(\alpha_1+\beta)}\right\} \\ D(t;T,c) & = (\lambda_1+\lambda_0)\frac{1-(\alpha_1+\beta)^{T-t}}{1-(\alpha_1+\beta)} \end{array}\right. \tag{19}$$

with final conditions $C(T;T,c)=0$ and $D(T;T,c)=0$. We have the following restrictions on the parameters:

$$\left\{\begin{array}{l} \lambda_1 \leq 0 \\ \lambda_0 = -\left(1-\sqrt{1-2\left(\lambda_1+\frac{\sigma^2}{2}\right)}\right) \\ \lambda_1 - 1 + \sqrt{1-2\left(\lambda_1+\frac{\sigma^2}{2}\right)} < 0 \\ \alpha_1+\beta < 0 \end{array}\right. \tag{20}$$

4.3 Dynamic Normal Tempered Stable

Consider the affine process V_t proposed in Mercuri [2008] then log returns follow a conditional Normal Tempered Stable as introduced in Barndorff-Nielsen and Shephard [2001]. We recall that the Normal Tempered Stable is obtained as a Normal Variance Mean Mixture where the mixing density is a Tempered Stable [see Tweedie, 1984] that is obtained by tempering the tail of a positively skewed $\alpha-$ Stable distribution. The Normal Tempered Stable has as special cases the Variance Gamma and the Normal Inverse Gaussian.
The conditional m.g.f. of $V_t|\mathcal{F}_{t-1}$ is:

$$E\left[e^{cV_t}\middle|\mathcal{F}_{t-1}\right] = \exp\left[h_t b\left(1-(1-2cb^{-1/\alpha})^{\alpha}\right)\right] \tag{21}$$

where $\alpha \in (0,1)$ and $b>0$.
Comparing (21) with (2), we have:

$$f(c,\theta) = b\left(1-(1-2cb^{-1/\alpha})^{\alpha}\right)$$

and

$$g(\theta) = 2\alpha b^{(\alpha-1)/\alpha}.$$

Applying Proposition 1, we obtain the recursive system of equations for time dependent coefficients:

$$\left\{\begin{array}{ll} A(t;T,c) & = cr + A(t+1;T,c) + \alpha_0 B(t+1;T,c) \\ B(t;T,c) & = c\lambda_0 + \beta B(t+1;T,c) + \\ & b\left\{1-\left[1-2b^{-\frac{1}{\alpha}}\left(c\lambda_1+\alpha B(t+1;T,c)+\frac{c^2\sigma^2}{2}\right)\right]^{\alpha}\right\}. \end{array}\right. \tag{22}$$

From Proposition 2 we have the following constraint:

$$\lambda_0 = -b\left[1-\left(1-2\left(\lambda_1+\frac{\sigma^2}{2}\right)b^{1/\alpha}\right)^{\alpha}\right]$$

and, implementing the Fast Fourier Transform, we price the European call option.
Using Proposition 3, we obtain the following time varying coefficients that allow us to extrapolate h_t from current level of VIX:

$$\left\{\begin{array}{ll} C(t;T) & = \alpha_0\left(2\alpha b^{(\alpha-1)/\alpha}\lambda_1+\lambda_0\right) * \\ & * \left\{\frac{(T-t)-(2\alpha b^{(\alpha-1)/\alpha}\alpha_1+\beta)\frac{1-\left(2\alpha b^{(\alpha-1)/\alpha}\alpha_1+\beta\right)^{T-t-1}}{1-\left(2\alpha b^{(\alpha-1)/\alpha}\alpha_1+\beta\right)}}{1-\left(2\alpha b^{(\alpha-1)/\alpha}\alpha_1+\beta\right)}\right\} \\ D(t;T) & = (2\alpha b^{(\alpha-1)/\alpha}\lambda_1+\lambda_0)\frac{1-\left(2\alpha b^{(\alpha-1)/\alpha}\alpha_1+\beta\right)^{T-t}}{1-\left(2\alpha b^{(\alpha-1)/\alpha}\alpha_1+\beta\right)} \end{array}\right. . \tag{23}$$

In this case the condition (11) becomes:

$$\begin{cases} 2\alpha b^{(\alpha-1)/\alpha}\lambda_1 - b\left[1-\left(1-2\left(\lambda_1+\frac{\sigma^2}{2}\right)b^{-1/\alpha}\right)^{\alpha}\right] \leq 0. \\ \alpha_1+\beta<1 \end{cases} \tag{24}$$

5 Empirical Analysis

We investigate in details the ability of our models to reproduce the behavior of European option prices on SPX index. We have two main objectives: replicate the market option volatilities and compare the theoretical VIX derived in our models with the observed one. The dataset is composed by the implied volatility surfaces observed each Wednesday going from May 2011 to April 2012, moneyness ranging from 0.9 to 1.1 and time to maturity 30, 60 and 90 days (the total number of observations is 1008). We choose Wednesday's observations to remove possible weekend effects as those discussed in French [1980]. From equation 1 we see that we need the term structure of the risk-free rate in order to compute the m.g.f of the variable $\ln S_T$. The Libor curve can be a possible choice though we know it is not the only one. We downloaded the needed curve from Bloomberg.

The first Wednesdays of each month are the in-sample data (231 observations). The remaining dataset (777 observations) is used for the out-of-sample analysis. We calibrate the model in each in-sample period. The values obtained for the parameters are used as input for the out-of-sample analysis. The error measure considered is:

$$\sqrt{percMSE} = \sqrt{\frac{\sum_{k=1}^{K}\sum_{t=1}^{T}\left[\frac{\sigma^{mkt}(k,t)-\sigma^{theo}(k,t)}{\sigma^{mkt}(k,t)}\right]^2}{N_T * N_K}}$$

where $\sigma^{mkt}(k,t)$, $\sigma^{theo}(k,t)$ are respectively the implied volatilities observed in the market and those obtained by the models. N_T, N_K represent respectively the number of the available maturities and strikes.

Tables 1, 2 and 3 report the values of the calibrated parameters and the corresponding in-sample errors.

Insert here Tab. 1, 2 and 3.

Our calibration exercise takes into account the possibility of extrapolating the latent process h_t directly from the VIX index. We find that for the DNTS model the in-sample errors are the lowest except only in one case where the DNIG model has the best performance. This result strongly supports our initial guess that two additional parameters would allow to better capture the market dynamics. Observe that if $b = 2^a$ and $\alpha = \frac{1}{a}$ for $a \to 0$ we obtain the DVG model, while for $b = 1$ and $\alpha = \frac{1}{2}$ the model is the DNIG.

The out-of-sample results suggest the use of the DNTS model in the considered dataset. Indeed, computing the $\sqrt{percMSE}$ on the entire out-of-sample data, we find that the DNTS reaches an error level of 5.05% which is a reduction error of 21.10% with respect to DNIG (the second best model). To deeply analyse the out of sample error, Figure 1 reports the results obtained in 36 out-of-sample Wednesdays. In 72% of the cases the DNTS shows a lower error level than the other two while the DNIG has the lowest error level only in 14% of the cases.

Insert here Fig. 1.

We remark that in our model the square of the VIX is an autoregressive process. The conditional expected value of the VIX is not available in a closed form formula. However, using Jensen's inequality, we easily derive the following upper bound that we use in our analysis:

$$E\left[VIX_{t+1}|\,\mathcal{F}_t\right] = E\left[\sqrt{VIX^2_{t+1}}\bigg|\,\mathcal{F}_t\right] \leq \sqrt{E\left[VIX^2_{t+1}|\,\mathcal{F}_t\right]} = VIX^{ub}_{t+1}.$$

Using Proposition 4 and equation (14), our upper bound becomes:

$$VIX^{ub}_{t+1} = \sqrt{-\frac{2e^{30r}10^4}{30}int + slopeVIX^2_t}$$

where all quantities are on daily basis and the year conversion is necessary for comparison with its observed level.

We calibrate the model on the first Wednesday of each month (in total there are 12 calibration periods). The resulting parameters are maintained fixed until the next in-sample day. From Figure 4 and Table 5 we observe that the DVG model displays the worst performance.

Insert here Fig. 2.

Insert here Tab. 4.

Instead of having fixed parameters for the entire month we can decide to make the recalibration period dynamic. Intuitively, if the market conditions change a lot (i.e. we observe a jump of the implied volatility), it is reasonable to think that in order to have a better prediction for the VIX level we must update the model parameters. This update for us means to recalibrate the model using the option volatilities observed after the jump has occurred.
We face the problem of defining the jump in terms of relative daily variation of the VIX Index level. If the observed VIX level is lower than 30 percent we recalibrate if the next day relative variation is higher than 30%. For example if the current level of VIX is 15% we recalibrate the model if the next day value is higher than 20% or lower than 10%. For higher levels of the VIX index (more than 30%) the required daily relative variation is fixed at 25%. This decision comes from the fact that VIX levels higher than 39% are rarely observed. In Figure 3 we report a comparison between the VIX and S&P500 for the considered dates.

Insert here Fig. 3.

The number of calibrations is reduced from 12 (when the parameters for the entire month are fixed) to 9 (when decision is dependent on the VIX level). Comparing the results reported in Table 4, and Table 5, the error term, defined as $\sqrt{E\left[(VIX_{mkt} - VIX_{ub})^2\right]}$, is reduced when the calibration time is based on VIX index level. This is also confirmed from Figures 3 and 4.

Insert here Fig. 4.

Insert here Tab. 5.
The choice of the DNTS showed in the calibration exercise seems to be weaker when we try to forecast the VIX index level. In particular, the DNIG seems to behave better in some extreme market conditions.

6 Conclusions

In this paper we proposed a class of discrete time stochastic volatility models. We started from the affine GARCH model and assumed that the conditional distribution of log returns is a Normal Variance Mean Mixture with support the entire real line. We obtained a recursive procedure for the computation of the characteristic function for the log-price at maturity. Option prices were than obtained via Fourier transform.
In our model, it is possible to extrapolate information from the VIX data. The VIX^2 index resulted to be an autoregressive process and the information extracted was used for pricing options on S&P500.
We specified some special cases for our general model. The Dynamic Normal Inverse Gaussian based model resulted to be more flexible in capturing market dynamics especially in turbulent periods.

7 Appendix

7.1 Conditional Moment Generating Function

Following the approach proposed in Heston and Nandi [2000] we derive a system of recursive equations for the time dependent coefficients of the conditional m.g.f. of the random variable $\ln(S_T)$ given the available information at time t. We want to prove that the conditional m.g.f. is given by the following formula:

$$E_t\left[\exp\left(c\ln\left(S_T\right)\right)\middle|\,\mathcal{F}_t\right] = S_t^c \exp\left[A\left(t;T,c\right) + B\left(t;T,c\right)h_{t+1}\right]. \tag{25}$$

We use the mathematical induction method.

1. We observe that relation (25) holds at time T since $A(T;T,c) = 0$ and $B(T;T,c) = 0$.

2. We suppose the relation (25) holds at time $t+1$ and, by the law of iterated expectations, we prove it at time t.

$$
\begin{array}{l}
E\left[E\left[S_T^c\right|\mathcal{F}_{t+1}]\right|\mathcal{F}_t] = E\left[\exp\left[A\left(t+1;T,c\right)+B\left(t+1;T,c\right)h_{t+2}\right]\right|\mathcal{F}_t] \\
= E\left[\exp\left[c\ln\left(S_T\right)+cr+A(t+1;T,c)\right.\right. \\
+\ c\lambda_0 h_{t+1}+c\lambda_1 V_{t+1}+c\sigma\sqrt{V_{t+1}}Z_{t+1}+ \\
\left.\left.+\alpha_0 B\left(t+1;T,c\right)+\alpha_1 B\left(t+1;T,c\right)V_{t+1}+\beta B\left(t+1;T,c\right)h_{t+1}\ \right]\,\right|\mathcal{F}_t] \\
= S_t^c\exp\left[cr+A\left(t+1;T,c\right)+\alpha_0 B\left(t+1;T,c\right)+\left(c\lambda_0+\beta B\left(t+1;T,c\right)\right)h_{t+1}\right]* \\
*E\left[\exp\left[\left(c\lambda_1+\alpha_1 B\left(t+1;T,c\right)+\frac{c^2\sigma^2}{2}\right)V_{t+1}\right]\middle|\,\mathcal{F}_t\right].
\end{array} \tag{26}
$$

Using the conditional m.g.f. of the r.v. V_{t+1}, equation (26) becomes:

$$
\begin{aligned}
E\left[E\left[S_T^c\right|\mathcal{F}_{t+1}]\right|\mathcal{F}_t] \quad &= \quad S_t^c\exp\left[cr+A\left(t+1;T,c\right)+\alpha_0 B\left(t+1;T,c\right)+\right. \\
&\left.+\left(c\lambda_0+\beta B\left(t+1;T,c\right)+f\left(c\lambda_1+\alpha_1 B\left(t+1;T,c\right)+\frac{c^2\sigma^2}{2},\theta\right)\right)h_{t+1}\right]
\end{aligned} \tag{27}
$$

By comparing the expression obtained in equation (27) with (25) we obtain the following recursive system:

$$
\left\{\begin{array}{ll}
A(t;T,c) & = cr+A(t+1;T,c)+\alpha_0 B(t+1;T,c) \\
B(t;T,c) & = c\lambda_0+\beta B(t+1;T,c)+ \\
 & \quad f(c\lambda_1+\alpha_1 B(t+1;T,c)+\frac{c^2\sigma^2}{2},\theta)
\end{array}\right. \tag{28}
$$

with $A(T;T,c)=0$ and $B(T;T,c)=0$.

7.2 Martingale condition

We want to prove that $\forall s\leq t$:

$$
\lambda_0=-f(\lambda_1+\frac{\sigma^2}{2};\theta)\ \overset{(1)}{\Longrightarrow}\ E\left[\frac{S_t}{e^r}\middle|\mathcal{F}_{t-1}\right]=S_{t-1}\overset{(2)}{\Longrightarrow}E\left[\frac{S_t}{e^{r(t-s)}}\middle|\mathcal{F}_s\right]=S_s. \tag{29}
$$

$(\overset{(1)}{\Longrightarrow})$

We assume r to be constant but the proof holds even assuming r to be a predictable process. By simple calculus, we obtain:

$$
E\left[\frac{S_t}{e^r}\middle|\mathcal{F}_{t-1}\right]=S_{t-1}\exp\left[\left(\lambda_0+f\left(\lambda_1+\frac{\sigma^2}{2};\theta\right)\right)h_{t-1}\right] \tag{30}
$$

substituting $\lambda_0=-f(\lambda_1+\frac{\sigma^2}{2};\theta)$ in (30) we obtain the result.

$(\overset{(2)}{\Longrightarrow})$

By the iterated law of conditional expectation we have:

$$
\begin{aligned}
E\left[\frac{S_t}{e^{r(t-s)}}\middle|\mathcal{F}_s\right] &= E\left[E\left[\frac{S_t}{e^{r(t-s)}}\middle|\mathcal{F}_{t-1}\right]\middle|\mathcal{F}_s\right] \\
&= E\left[\frac{1}{e^{r(t-s-1)}}\underbrace{E\left[\frac{S_t}{e^r}\middle|\mathcal{F}_{t-1}\right]}_{S_{t-1}}\middle|\mathcal{F}_s\right] \\
&= \ ...\ = E\left[\frac{S_{s+1}}{e^r}\middle|\mathcal{F}_s\right]=S_s.
\end{aligned}
$$

7.3 VIX Index: derivation formula

We derive an analytical formula for the VIX index when the dynamics of S&P 500 belongs to our class. Defined S^* as the forward price of S_t with maturity $T-t$, we start from the VIX definition:

$$
\left(\frac{VIX_t}{100}\right)^2=\frac{2e^{r(T-t)}}{T-t}\left[\underbrace{E^Q\left[\frac{S_T-S^*}{S^*}\middle|\mathcal{F}_t\right]}_{(*)}-\underbrace{E^Q\left[\ln\left(\frac{S_T}{S^*}\right)\middle|\mathcal{F}_t\right]}_{(**)}\right].
$$

The quantity in $(*)$ is 0 since:

$$
E^Q\left[\frac{S_T-S^*}{S^*}\middle|\mathcal{F}_t\right]=\frac{1}{S_te^{r(T-t)}}E^Q\left[S_T\right|\mathcal{F}_t]-1=0.
$$

Given the spot price S_t, we have $S_T = S_t \exp\left(\sum_{d=t+1}^{T} X_d\right)$ and by substituting in $(**)$ we get the following expression for VIX squared:

$$\left(\frac{VIX_t}{100}\right)^2 = -\frac{2e^{r(T-t)}}{T-t} \underbrace{E\left[\sum_{d=t+1}^{T} \lambda_1 V_d + \lambda_0 h_d \middle| \mathcal{F}_t\right]}_{(\Delta)}. \tag{31}$$

In order to compute the quantity (Δ) in (31) we use the mathematical induction method. $\forall\, l = t, \ldots, T$ we assume that:

$$E\left[\sum_{d=t+1}^{T} \lambda_1 V_d + \lambda_0 h_d \middle| \mathcal{F}_l\right] = C(l;T) + D(l;T)h_{l+1} + \sum_{d=t+1}^{l} \lambda_1 V_d + \lambda_0 h_d \tag{32}$$

with $C(T;T) = 0$ and $D(T;T) = 0$. First, we notice that all the quantities on the right side of (32) are known given the information at time l.

1. Since V_t and h_t are respectively adapted and predictable process our assumption is true for $l = T$ if $C(T;T) = 0$ and $D(T;T) = 0$.
2. We suppose the relation holds at time $l + 1$ and we prove it for time l using the property of iterated expectations.

$$E\left[\sum_{d=t+1}^{T} \lambda_1 V_d + \lambda_0 h_d \middle| \mathcal{F}_l\right] = E\left[E\left[\sum_{d=t+1}^{T} \lambda_1 V_d + \lambda_0 h_d \middle| \mathcal{F}_{l+1}\right] \middle| \mathcal{F}_l\right]. \tag{33}$$

The quantity on the right hand side of equation (33) is equal to:

$$E\left[C(l+1;T) + D(l+1;T)h_{l+2} + \sum_{d=t+1}^{l+1} \lambda_1 V_d + \lambda_0 h_d \middle| \mathcal{F}_l\right]. \tag{34}$$

Substitute in (34) the definition of h_{l+2} and get:

$$\begin{aligned} &C(l+1;T) + \alpha_0 D(l+1;T) + (\beta D(l+1;T) + \lambda_0)h_{t+1} + \textstyle\sum_{d=t+1}^{l}(\lambda_1 V_d + \lambda_0 h_d) \\ &+E\left[(\alpha_1 D(l+1;T) + \lambda_1)\, V_{l+1} | \mathcal{F}_l\right]. \end{aligned}$$

From (5) we get:

$$\begin{aligned} &C(l+1;T) + \alpha_0 D(l+1;T) + \left[(\lambda_0 + \lambda_1 g(\theta)) + (\beta + \alpha_1 g(\theta))\, D(l+1;T)\right] h_{t+1} \\ &+ \textstyle\sum_{d=t+1}^{l} \lambda_1 V_d + \lambda_0 h_d \end{aligned}$$

and, by comparison with (32), we get the following system:

$$\begin{cases} C(l;T) &= C(l+1;T) + D(l+1;T)\alpha_0 \\ D(l;T) &= [\lambda_1 g(\theta) + \lambda_0] + (\alpha_1 g(\theta) + \beta)\, D(l+1;T) \end{cases} \tag{35}$$

with final conditions $C(T;T) = 0$ and $D(T;T) = 0$.
We show that if the following two conditions are satisfied:

- $\alpha_1 g(\theta) + \beta < 1$
- $\lambda_1 g(\theta) + \lambda_0 \leq 0$

the right hand of the equation (12) is positive, coherently with the fact of being equal to the squared VIX value. We notice that $D(l;T)$ is a linear difference equation whose solution at time $l = t$, $\forall t \leq T$ is given by:

$$D(t;T) = \underbrace{[\lambda_1 g(\theta) + \lambda_0]}_{\leq 0} \underbrace{\frac{1 - [\alpha_1 g(\theta) + \beta]^{T-t}}{1 - [\alpha_1 g(\theta) + \beta]}}_{>0}. \tag{36}$$

The solution (36) and the positivity of α_0 imply negative values for $C(t;T)$:

$$\begin{aligned} C(t;T) &= \underbrace{C(T;T)}_{=0} + \underbrace{D(T;T)}_{=0} + \alpha_0 \sum_{l=t+1}^{T-1} \underbrace{D(l;T)}_{<0} \\ &= \alpha_0 \left[\lambda_1 g(\theta) + \lambda_0\right] \left\{ \frac{T - t - 1 - [\alpha_1 g(\theta) + \beta] \frac{1 - [\alpha_1 g(\theta) + \beta]^{(T-t)-1}}{1 - [\alpha_1 g(\theta) + \beta]}}{1 - [\alpha_1 g(\theta) + \beta]} \right\}. \end{aligned}$$

7.4 VIX Index: autoregressive model

In equation (6), we substitute the expression for h_{t+1} and h_t using the VIX adjusted as in (14). We obtain

$$\frac{VIX_t^{adj} - C_{30}}{D_{30}} = \alpha_0 + (\alpha_1 g(\theta) + \beta)\frac{VIX_{t-1}^{adj} - C_{30}}{D_{30}} + \alpha_1(V_t - g(\theta)h_t) \;\Rightarrow$$

$$VIX_t^{adj} = \alpha_0 D_{30} + C_{30}\left[1 - (\alpha_1 g(\theta) + \beta)\right] + (\alpha_1 g(\theta) + \beta)\, VIX_{t-1}^{adj} + \alpha_1 D_{30}(V_t - g(\theta)h_t).$$

We can easily observe that VIX_t^{adj} is an $AR(1)$ and it can be written as:

$$VIX_t^{adj} = int + slope VIX_{t-1}^{adj} + \tau_t.$$

Trivially we have:

$$\begin{aligned} int &= \alpha_0 D_{30} + C_{30}\left[1 - (\alpha_1 g(\theta) + \beta)\right] \\ slope &= (\alpha_1 g(\theta) + \beta) \\ \tau_t &= \alpha_1 D_{30}(V_t - g(\theta)h_t). \end{aligned}$$

Using the explicit solution (13) for C_{30} and D_{30} and by rearranging, we get a simple expression for int:

$$\begin{aligned} int &= \alpha_0 \left(\lambda_1 g(\theta) + \lambda_0\right) \tfrac{1-slope^{30}}{1-slope} + \alpha_0 \left(\lambda_1 g(\theta) + \lambda_0\right) \left(29 - slope \tfrac{1-slope^{29}}{1-slope}\right) \\ &= 30\alpha_0 \left(\lambda_1 g\left(\theta\right) + \lambda_0\right). \end{aligned}$$

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In sample calibration for DVG							
date	λ_0	λ_1	σ	α_0	α_1	β	Perc. error
04-May-2011	0.012	-0.012	0.014	0.033	0.493	0.379	0.048
01-Jun-2011	0.036	-0.039	0.069	0.009	0.274	0.148	0.084
06-Jul-2011	0.005	-0.005	0.006	0.033	0.344	0.633	0.027
03-Aug-2011	0.034	-0.035	0.001	0.060	0.317	0.000	0.037
07-Sep-2011	0.008	-0.008	0.011	0.032	0.538	0.444	0.015
05-Oct-2011	0.051	-0.053	0.029	0.028	0.155	0.484	0.024
02-Nov-2011	0.095	-0.100	0.007	0.018	0.057	0.085	0.039
07-Dec-2011	0.060	-0.062	0.007	0.024	0.008	0.454	0.052
04-Jan-2012	0.019	-0.019	0.020	0.023	0.207	0.644	0.048
01-Feb-2012	0.036	-0.038	0.056	0.017	0.014	0.157	0.048
07-Mar-2012	0.042	-0.043	0.029	0.000	0.000	1.000	0.088

Table 1: Calibrated parameters for the DVG model in the in-sample period

In sample calibration for DNIG							
date	λ_0	λ_1	σ	α_0	α_1	β	Perc. error
04-May-2011	0.049	-0.052	0.062	0.006	0.012	0.572	0.039
01-Jun-2011	0.047	-0.050	0.061	0.006	0.016	0.604	0.029
06-Jul-2011	0.009	-0.009	0.011	0.009	0.168	0.816	0.024
03-Aug-2011	0.035	-0.036	0.042	0.029	0.212	0.059	0.022
07-Sep-2011	0.067	-0.072	0.075	0.017	0.120	0.113	0.022
05-Oct-2011	0.007	-0.008	0.010	0.028	0.427	0.564	0.007
02-Nov-2011	0.060	-0.064	0.066	0.007	0.081	0.674	0.019
07-Dec-2011	0.046	-0.048	0.057	0.005	0.008	0.867	0.024
04-Jan-2012	0.029	-0.030	0.019	0.019	0.065	0.733	0.057
01-Feb-2012	0.030	-0.031	0.042	0.023	0.269	0.109	0.034
07-Mar-2012	0.013	-0.014	0.015	0.010	0.211	0.760	0.026

Table 2: Calibrated parameters for the DNIG model in the in-sample period

In sample calibration for DNTS									
date	λ_0	λ_1	σ	α_0	α_1	β	b	a	Perc. error
04-May-2011	0.212	-0.107	0.013	0.002	0.363	0.276	0.814	0.990	0.009
01-Jun-2011	0.066	-0.042	0.039	0.011	0.296	0.000	0.800	0.750	0.019
06-Jul-2011	0.005	-0.005	0.006	0.039	0.380	0.596	1.000	0.500	0.025
03-Aug-2011	0.052	-0.027	0.008	0.012	0.510	0.000	0.897	0.955	0.007
07-Sep-2011	0.005	-0.006	0.009	0.051	0.658	0.409	0.962	0.413	0.015
05-Oct-2011	0.012	-0.016	0.026	0.001	0.116	0.910	0.946	0.345	0.004
02-Nov-2011	0.003	-0.003	0.006	0.133	0.806	0.233	0.863	0.351	0.011
07-Dec-2011	0.085	-0.043	0.010	0.004	0.473	0.072	0.854	0.975	0.005
04-Jan-2012	0.003	-0.005	0.007	0.105	0.772	0.449	1.000	0.341	0.018
01-Feb-2012	0.100	-0.053	0.021	0.001	0.104	0.803	0.800	0.934	0.014
07-Mar-2012	0.018	-0.011	0.007	0.032	0.539	0.090	0.965	0.803	0.024

Table 3: Calibrated parameters for the DNTS model in the in-sample period

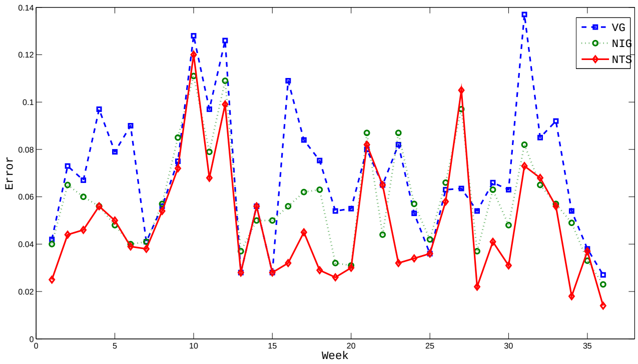

Figure 1: Out of sample weekly comparison.

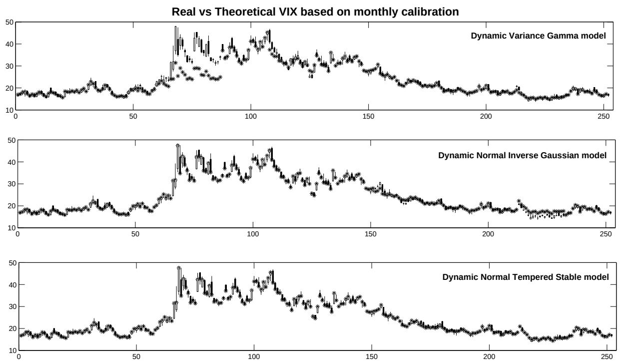

Figure 2: Comparison between the predict VIX (upper bound $*$) and next day open, closed, min, max VIX level using monthly calibration

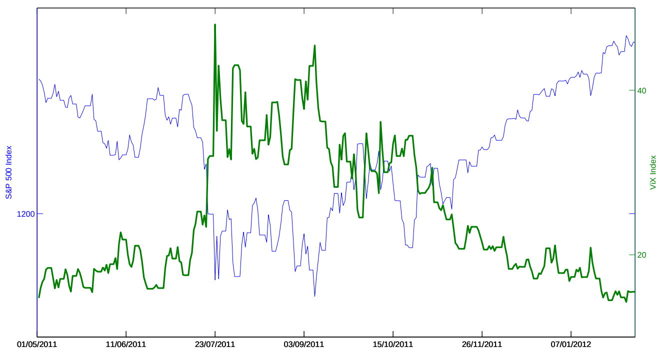

Figure 3: Comparison between VIX and S&P500.

	DVG	DNIG	DNTS
Open	1,111%	0,029%	0,140%
Closing	0,967%	0,173%	0,004%
High	2,187%	1,047%	1,216%
Low	0,028%	1,167%	0,999%

Table 4: Errors obtained when the calibration is done the first Wednesday of each month.

	DVG	DNIG	DNTS
Open	0,589%	0,005%	0,080%
Closing	0,445%	0,139%	0,064%
High	1,665%	1,081%	1,156%
Low	0,550%	1,133%	1,059%

Table 5: Errors obtained when the calibration decision depends on the VIX index level.

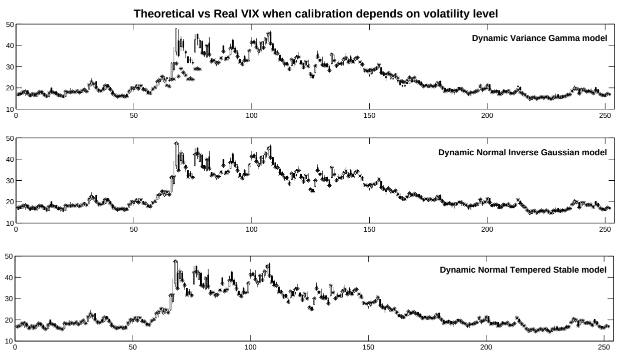

Figure 4: Comparison between the predict VIX (upper bound $*$) and next day open, closed, min, max VIX level using monthly calibration