SEMI-PURITY FOR CYCLES WITH MODULUS

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Abstract. In this paper, we prove a form of purity property for the $\overline{\square} = (\mathbb{P}^1, \infty)$-invariant replacement $h_0^{\overline{\square}}(\mathfrak{X})$ of the Yoneda object $\mathbb{Z}_{\mathrm{tr}}(\mathfrak{X})$ for a proper modulus pair $\mathfrak{X} = (\overline{X}, X_\infty)$ over a field k of characteristic zero, consisting of a smooth proper k-scheme and an effective Cartier divisor on it. As application, we prove the analogue in the modulus setting of Voevodsky's fundamental theorem on the homotopy invariance of the cohomology of homotopy invariant sheaves with transfers, based on a main result of [23]. This plays an essential role in the development of the theory of motives with modulus in the sense of Kahn-Saito-Yamazaki.

Contents

1. Introduction

In a series of landmark papers from the 1990's ([29], [30], [31]), Voevodsky introduced and studied the derived category of effective motives $\mathbf{DM}^{\mathrm{eff}}(k)$ over a field k. It is defined as the localization of the derived category $D(\mathbf{NST})$ of (unbounded) complexes of Nisnevich sheaves on the category of smooth correspondences (i.e. the category of smooth schemes with some extra functoriality, called *transfers structure*, for finite surjective maps) over k with respect to the projection maps $\mathbb{A}^1 \times X \to X$. This condition leads to the homotopy invariance of every homology theory representable in $\mathbf{DM}^{\mathrm{eff}}(k)$, and is a cornerstone of the construction. Explicitly, a sheaf F is called homotopy invariant if the natural map

$$F(X) \to F(X \times \mathbb{A}^1)$$

induced by the projection $X \times \mathbb{A}^1 \to X$ is an isomorphism. On this note, recall the following

Theorem 1.0.1 (5.6 [30], 24.1[21])**.** *Let F be a homotopy invariant presheaf (of abelian groups) with transfers on $\mathbf{Sm}(k)$. Then the Nisnevich sheaf with transfers F_{Nis} associated with F is homotopy invariant. If k is perfect, the presheaves $H^i_{\mathrm{Nis}}(-, F_{\mathrm{Nis}})$ have a canonical structure of homotopy invariant presheaves with transfers.*

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The above result is usually called the *homotopy invariance of the cohomology* and plays a fundamental role in the theory of motives. Theorem 1.0.1 implies that $\mathbf{DM}^{\mathrm{eff}}(k)$ is equivalent to the triangulated subcategory of $D(\mathbf{NST})$ consisting of those complexes having homotopy invariant cohomology sheaves [31, p. 205] (the perfectness assumption was removed later up to inverting p in the coefficients. See [27]). Another consequence of 1.0.1 is the fact that the standard t-structure on the derived category $D(\mathbf{NST})$ induces a t-structure on $\mathbf{DM}^{\mathrm{eff}}(k)$, whose heart is equivalent to the category of homotopy invariant sheaves. This is the so-called *homotopy t-structure*. Thus, thanks to Voevodsky's result, the category of motives, abstractly defined as a Bousfield localization of $D(\mathbf{NST})$, admits a fairly concrete description.

Imposing $\mathbb{A}^1$-invariance is enough for capturing many geometric and arithmetic properties of smooth schemes. However, it was already noticed by Voevodsky in [31, 2.2] that this is too much to ask in other interesting situations. For example, the Picard group $\mathrm{Pic}(X)$ of a smooth k-scheme X is canonically isomorphic to the motivic cohomology group $\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}(k)}(M(X), \mathbb{Z}(1)[2])$, where $M(X)$ denotes the image of the Yoneda object $\mathbb{Z}_{\mathrm{tr}}(X)$ of X in $\mathbf{DM}^{\mathrm{eff}}(k)$. On the other hand, we do not have any "motivic" description for $\mathrm{Pic}(X)$ when X is a singular variety, since the functor $X \mapsto \mathrm{Pic}(X)$ considered on $\mathbf{Sch}(k)$ (rather then on $\mathbf{Sm}(k)$) is not $\mathbb{A}^1$-invariant.

But even if one is interested in smooth schemes only, there are natural objects which do not admit a description in Voevodsky's world. Perhaps the most striking example is given by the abelianized fundamental group $\pi_1^{ab}(-)$ computed on smooth varieties over positive characteristic fields. In this case, it is well-known that $\pi_1^{ab}(\mathbb{A}^1_k)$ is far from being trivial, due to the existence of the Artin-Schrier covers $x \mapsto x - x^p$. Other examples are given by the sheaves of differential forms Ω^*_-, the de Rham-Witt complexes, $W_*\Omega^*_-$ (see e.g. [5]), the commutative algebraic groups (with unipotent part) considered as sheaves with transfers (see e.g. [26]), as well as the natural generalization of Bloch's higher Chow groups, like additive Chow groups [18], [19] and the higher Chow groups with modulus [3], [16].

With the goal of developing a theory of motives in which non $\mathbb{A}^1$-invariant objects can be represented, the second author together with Kahn and Yamazaki introduced a generalized framework in [15], [13], [14], [23], based on the principle that the category of smooth schemes over k should be replaced by the larger category of *modulus pairs*, $\underline{\mathbf{M}}\mathbf{Cor}(k)$ over a (perfect) field k. Objects are (as the name suggests) pairs $\mathfrak{X} = (\overline{X}, X_\infty)$ consisting of a separated k-scheme of finite type $\overline{X}$ and an effective (possibly empty) Cartier divisor X_∞ on it (the *modulus*) such that the complement $\overline{X} \setminus X_\infty = X$ is smooth. Morphisms are given by finite correspondences between the smooth complements, subject to certain admissibility conditions. Together, they form a symmetric monoidal category. See 2.1 for the precise definition. Write $\underline{\mathbf{M}}\mathbf{PST}$ for the category of additive presheaves of abelian groups on $\underline{\mathbf{M}}\mathbf{Cor}(k)$, and $\mathbf{MPST}$ for the subcategory of presheaves defined on the smaller category $\mathbf{MCor}$ of *proper* modulus pairs, i.e. pairs $\mathfrak{X} = (\overline{X}, X_\infty)$ such that the total space $\overline{X}$ is proper.

The idea behind this is that the pair $\mathfrak{X}$ looks like a (partial) compactification of the smooth scheme X in which the scheme structure of the boundary (the divisor X_∞) plays a non-trivial role. This approach is inspired by the theory of cycles with modulus [3], [16], a generalization in higher dimension of the classical theory of Jacobian varieties of Rosenlicht and Serre [25].

In this context, the role played by the affine line in Voevodsky's world is played by its compactified cousin

$$\overline{\square} = (\mathbb{P}^1, \infty)$$

i.e. the projective line $\mathbb{P}^1$, with reduced divisor at infinity (informally called "the cube"). The key property which replaces $\mathbb{A}^1$-invariance is the so-called *cube invariance*: a presheaf of abelian groups F on $\mathbf{MCor}(k)$ is called $\overline{\square}$-invariant if for every (proper) modulus pair $\mathfrak{X}$ the canonical map

$$F(\mathfrak{X}) \to F(\mathfrak{X} \otimes \overline{\square})$$

induced by $\mathfrak{X} \otimes \overline{\square} \to \mathfrak{X}$ is an isomorphism. We write $\mathbf{CI} \subset \mathbf{MPST}$ for the category of $\overline{\square}$-invariant presheaves with transfers on modulus pairs. Starting from this, it is possible to define a new category of effective motivic complexes, $\mathbf{MDM}^{\mathrm{eff}}(k)$, obtained as localization of the category of complexes of Nisnevich sheaves with transfers on $\mathbf{MCor}(k)$ (note that the sheaf property and the sheafification functor are rather subtle in this context, see §2.1 and [12]) with respect to the projection maps $\mathfrak{X} \otimes \overline{\square} \to \mathfrak{X}$. The canonical forgetful functor $\mathfrak{X} = (\overline{X}, X_\infty) \mapsto \omega(\mathfrak{X}) := X = \overline{X} \setminus X_\infty$ induces then an adjoint pair

$$\omega_{\mathrm{eff}} \colon \mathbf{MDM}^{\mathrm{eff}}(k) \leftrightarrows \mathbf{DM}^{\mathrm{eff}}(k) \colon \omega^{\mathrm{eff}}$$

which satisfies the property that $\omega^{\mathrm{eff}}(M(X)) = M(X, \emptyset)$ for every smooth and proper k-scheme X. See [13, 7.3]. The functor ω^{eff} is fully faithful, so that Voevodsky's category $\mathbf{DM}^{\mathrm{eff}}(k)$ can be presented as a further localization of the bigger category $\mathbf{MDM}^{\mathrm{eff}}(k)$.

We now have all the ingredients to state one of the main results of the present paper.

Theorem 1.0.2 (see Theorem 2.2.1)**.** *Assume that k is of characteristic zero. Let $F \in \mathbf{CI}^\tau$ be a $\overline{\square}$-invariant presheaf with transfers. Then for every $\mathfrak{X} \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$, the map*

$$H^i(\mathfrak{X}_{\mathrm{Nis}}, F_{\mathrm{Nis}}) \to H^i((\mathfrak{X} \otimes \overline{\square})_{\mathrm{Nis}}, F_{\mathrm{Nis}})$$

induced by $\mathfrak{X} \otimes \overline{\square} \to \mathfrak{X}$ is an isomorphism (see §2.1 for the notation F_{Nis} and $H^q(\mathfrak{X}_{\mathrm{Nis}}, F_{\mathrm{Nis}})$).

The superscript $\mathbf{CI}^\tau$ denotes the essential image of $\mathbf{CI}$ under $\tau_! \colon \mathbf{MPST} \to \underline{\mathbf{M}}\mathbf{PST}$, i.e. the left Kan extension of a presheaf defined on proper modulus pairs to a presheaf defined on every modulus pair, while the subscript $\underline{\mathbf{M}}\mathbf{Cor}_{ls}$ stands for the subcategory of *log smooth* modulus pairs, i.e. $\mathfrak{X} = (\overline{X}, X_\infty)$ such that $\overline{X}$ is smooth and $|X_\infty|$ is a strict normal crossing divisor on $\overline{X}$.

Thanks to [23], Theorem 1.0.2 was previously known to hold under the extra assumption of *semi-purity* of the sheaf F_{Nis}. In informal terms, this property can be understood as an analogue of the Gersten property in the modulus setting: let $(\mathscr{O}, (\Pi))$ be a pair consisting of a regular henselian ring $\mathscr{O}$ (essentially of finite type over k), and (Π) is the ideal generated by a non-zero element Π in the maximal ideal of $\mathscr{O}$. We think $(\mathscr{O}, (\Pi))$ as limit object of $\underline{\mathbf{M}}\mathbf{Cor}$ in the usual way. Let K be the function field of $\mathscr{O}$. Then a Nisnevich sheaf F on $\underline{\mathbf{M}}\mathbf{Cor}$ is semipure if the map

$$F(\mathscr{O}, (\Pi)) \to F(K, \emptyset)$$

is injective for every such pair (this definition is actually slightly stronger then the one used in the paper, see 2.1). Unfortunately, this property is not satisfied by every object in $\mathbf{CI}^\tau_{\mathrm{Nis}}$. It is however (somehow surprisingly) satisfied by a large class of objects: the Nisnevich sheaves of the form $\tau_! \underline{a}_{\mathrm{Nis}} h_0^{\overline{\square}}(\mathfrak{X})$ for every modulus pair $\mathfrak{X} = (\overline{X}, X_\infty)$ with $\overline{X}$ smooth and projective. Here, the functor $\underline{a}_{\mathrm{Nis}}$ (resp. the non-underlined version a_{Nis}) denotes the left adjoint to the inclusion functor $\underline{\mathbf{M}}\mathbf{NST} \to \underline{\mathbf{M}}\mathbf{PST}$ (resp. to the inclusion functor $\mathbf{MNST} \to \mathbf{MPST}$). The existence of such sheafification functor, as well as its properties, are discussed in [12].

The symbol $h_0^{\overline{\square}}(-)$ denotes the $\overline{\square}$-replacement functor on $\mathbf{MPST}$ (i.e. the left adjoint of the inclusion $\mathbf{CI} \subset \mathbf{MPST}$, see 2.1), applied then to the representable presheaf $\mathbb{Z}_{\mathrm{tr}}(\mathfrak{X})$. This is the analogue of the Suslin-Voevodsky $h_0(-)$ construction (see [28], [21]). The key result of this paper is then the following

Theorem 1.0.3 (see Theorem 2.1.3)**.** *Assume that k is of characteristic zero. For $\mathfrak{X} \in \mathbf{MCor}^{\mathrm{proj}}$, we have that $\tau_! \underline{a}_{\mathrm{Nis}} h_0^{\overline{\square}}(\mathfrak{X}) = a_{\mathrm{Nis}} h_0^{\overline{\square}}(\mathfrak{X})$ is semi-pure.*

The proof of this result, which can be reformulated into a problem about algebraic cycles, occupies Sections 3-5. From this, it is possible to prove Theorem 1.0.2 without semi-purity assumptions on F, together with all the other mentioned corollaries.

We now briefly discuss the contents of the different sections, and we give some ideas about the proofs.

Section 2 contains a recollection of definitions and results from [12], [13], [14] and [23], together with the exact statements of the main theorem and some applications. We then begin with the proof of the semipurity result for modulus pairs of dimension 1 (i.e. for curves). This is achieved by means of a direct and explicit computation in Section 3. In fact, we prove a more general result, namely Theorem 3.1.2. To pass from the case of curves to the case of surfaces, we need to develop some moving techniques for cycles with modulus, and we do so in Section 6 after some preliminaries in Section 4. The idea behind the technical arguments in Section 6 is ultimately simple, and goes back to the Bloch-Quillen's formula, relating the group of 0-cycles on a surface to the cohomology of the sheaf $\mathcal{K}_2$. Let us explain it for the reader's convenience.

Let X be a regular integral surface defined over a field F and let η be its generic point. Recall that the sheaf $\mathcal{K}_{2,X}$ has a Gersten resolution

$$0 \to \mathcal{K}_{2,X} \to \iota_{\eta,*}K_2(k(\eta)) \xrightarrow{T} \bigoplus_{y\in X^{(1)}} \iota_{y,*}K_1(k(y)) \xrightarrow{\partial} \bigoplus_{x\in X^{(2)}} \iota_{x,*}K_0(k(x)) \tag{1.1}$$

where the first map T is the tame symbol $T = (T_y)_{y\in X^{(1)}}$ and ∂ agrees with the divisor map under the identification $\bigoplus_{x\in X^{(2)}} \iota_{x,*}K_0(k(x)) = \mathcal{Z}_0(X)$, where the latter denotes the free abelian group of zero cycles on X. The cokernel of ∂ is then canonically identified with the Chow group of zero cycles $\mathrm{CH}_0(X)$. It is a well known fact (first discovered by Bloch [4]) that (1.1) is a flasque resolution of the sheaf $\mathcal{K}_{2,X}$, so that $H^2(X, \mathcal{K}_{2,X}) \cong \mathrm{CH}_0(X)$. The important remark for us, however, is simply the fact the (1.1) is a complex, so that the composition $\partial\circ T$ is zero in the free abelian group $\mathcal{Z}_0(X)$, and that the image of ∂ is the subgroup of zero cycles rationally equivalent to zero.

In particular, we can add to any class $\gamma = \sum_y (f_y) \in \bigoplus_{y\in X^{(1)}} K_1(k(y))$ for rational functions $f_y \in k(y)^\times$, an element of the form $T\{a, b\}$ without altering the image under ∂, i.e. without altering the cycle $\partial(\gamma)$, rationally equivalent to zero. We call any such class $T\{a, b\}$ a *moving symbol*. A careful choice of $\{a, b\} \in K_2(k(\eta))$ allows us to rearrange in a convenient way any relation in the Chow group.

In geometric terms, this can be summarized as follows. Assume that X is quasi-projective. Suppose for simplicity that $\partial(\gamma) = \mathrm{div}_Z(f)$, for an integral curve $Z \subset X$, and that Z is the zero locus of a global section s of a very ample line bundle on X. Choose another section t of the same line bundle so that $s/t \in k(\eta)^\times$ is a rational function on X. Next, choose another rational function a/b, ratio of two global sections of a (possibly different) line bundle on X which gives a global lift of the rational function f on Z. We now have

$$\partial T(\{s/t, a/b\}) = \mathrm{div}_Z(f) - \mathrm{div}_{(t)}(a/b) + \mathrm{div}_{(a)}(s/t) - \mathrm{div}_{(b)}(s/t) = 0 \in \mathcal{Z}_0(X)$$

so that $\mathrm{div}_Z(f)$ is now the sum of 3 new cycles. The key observation is that the global sections a, b and t can be chosen in a very controlled way by means of suitable applications of Bertini-type theorems, giving a good control on the newly found cycles.

In our application, this simple picture becomes substantially more intricate. The regular surface X is replaced by the smooth generic fiber $X = \mathscr{X}_K$ of an integral relative surface $\mathscr{X}$ over the spectrum of a henselian local ring $\mathscr{O}$, which comes equipped with an effective Cartier divisor $\mathscr{D}$, surjective over $S = \operatorname{Spec} \mathscr{O}$. We briefly mention the three main points that we have to address.

First, we need to impose some extra conditions on the rational functions on the curves on X, the modulus condition, i.e. we have to replace $K_1(k(y))$ with suitable relative K_1 groups with respect to the divisor $\mathscr{D} \times_S K$ on X. Second, we need to work with a subgroup of the group $\mathcal{Z}_0(X)$, generated by closed points in X whose closure in $\mathscr{X}$ has a special behavior with respect to the special fiber $\mathscr{D}_s$ of $\mathscr{D}$. We call it the *modulus condition over* $\mathscr{O}$ (see Definition 3.1.1). Finally, the global sections like a, b and t above need to have a model over S satisfying a (rather weak) good intersection property with respect to $\mathscr{D}_s$. This is achieved by means of some new Bertini-type theorems over a local base, which we develop in Section 4.

With a suitable combination of these moving techniques, the global injectivity result, Theorem 2.1.3, can be then reduced from the case of surfaces to the case of curves.

The case in arbitrary dimension is essentially achieved by reducing to the case of surfaces, using again the Bertini theorems of Section 4 and some injectivity results for blow-ups, discussed at the beginning of Section 5. This latter reduction step requires the assumption that k has resolution of singularities in the strong form, so it forces us to assume that our ground field has characteristic zero.

Notations and conventions. Throughout this paper, k will denote a fixed ground field and $\mathscr{O}$ a Noetherian local domain, whose residue field is a finite extension of k (most of the time, $\mathscr{O}$ will be essentially of finite type over k). We write K for the function field of $\mathscr{O}$. The field k will be assumed to be of characteristic zero in Section 5 in order to apply resolution of singularities [9] and resolution of marked ideals in the form of [1], .

We will use Roman capital letters to denote schemes over k or over K, and we follow the convention that Script letters (like $\mathscr{X}$) will denote schemes over $\mathscr{O}$. Our main object of interest is the category of *modulus pairs* $\mathbf{MCor}$ over k (see Section 2.1), and its variants. Objects of $\mathbf{MCor}$ will be denote by Gothic letters (like $\mathfrak{X}$).

2. Main theorem and applications to $\overline{\square}$-invariant sheaves

2.1. **Review on basic definitions and statement of the main theorem.** We collect some basic definitions and results from [12] and [13]. We fix a base field k which is assumed to be perfect. Let $\mathbf{Sch}$ be the category of k-schemes separated and of finite type over k and $\mathbf{Sm} \subset \mathbf{Sch}$ be the full subcategory of smooth schemes. Let $\underline{\mathbf{M}}\mathbf{Cor}$ be the category of *modulus pairs*: The objects are pairs $\mathfrak{X} = (\overline{X}, X_\infty)$ with $\overline{X} \in \mathbf{Sch}$ and a (possibly empty) effective Cartier divisor $X_\infty \subset \overline{X}$ such that $\overline{X} - |X_\infty| \in \mathbf{Sm}$. A modulus pair $\mathfrak{X} = (\overline{X}, X_\infty)$ is *proper* (resp. *projective*) if $\overline{X}$ is proper (resp. *projective*) over k. Let $\mathbf{MCor} \subset \underline{\mathbf{M}}\mathbf{Cor}$ be the full subcategory of proper modulus pairs and write $\mathbf{MCor}^{\mathrm{proj}}$ for the subcategory of projective modulus pair. A basic example of an object of $\mathbf{MCor}$ is the cube

$$\overline{\square} := (\mathbb{P}^1_k, \infty).$$

An *admissible prime correspondence from* $\mathfrak{X} = (\overline{X}, X_\infty)$ *to* $\mathfrak{Y} = (\overline{Y}, Y_\infty)$ is a prime correspondence $V \in \mathbf{Cor}(X, Y)$ with $X = \overline{X} - |X_\infty|$ and $Y = \overline{Y} - |Y_\infty|$ satisfying the following condition

$$X_{\infty|\overline{V}^N} \geq Y_{\infty|\overline{V}^N},$$

where $\overline{V}^N \to \overline{V} \subset \overline{X} \times \overline{Y}$ is the normalization of the closure of V. It is called *left proper* if $\overline{V}$ is proper over $\overline{X}$. We denote by $\underline{\mathbf{M}}\mathbf{Cor}(\mathfrak{X}, \mathfrak{Y}) \subset \mathbf{Cor}(X, Y)$ the subgroup generated by left proper admissible prime correspondences. By [13, Prop 1.2.3] $\underline{\mathbf{M}}\mathbf{Cor}(\mathfrak{X}, \mathfrak{Y})$ is preserved by the composition of finite correspondences so that we can define the category $\underline{\mathbf{M}}\mathbf{Cor}$ with objects the modulus pairs and morphisms given by admissible left proper correspondences. We denote by $\mathbf{MCor}$ the full subcategory with objects the proper modulus pairs.

Let $\underline{\mathbf{M}}\mathbf{PST}$ (resp. $\mathbf{MPST}$) be the category of additive presheaves of abelian groups on $\underline{\mathbf{M}}\mathbf{Cor}$ (resp. $\mathbf{MCor}$). For $F \in \underline{\mathbf{M}}\mathbf{PST}$ and $\mathfrak{X} = (\overline{X}, X_\infty) \in \underline{\mathbf{M}}\mathbf{Cor}$ write $F_{\mathfrak{X}}$ for the presheaf on the étale site $\overline{X}_{\text{ét}}$ over $\overline{X}$ given by $U \to F(\mathfrak{X}_U)$ for $U \to \overline{X}$ étale, where $\mathfrak{X}_U = (U, X_\infty \times_{\overline{X}} U) \in \underline{\mathbf{M}}\mathbf{Cor}$. We say F is a Nisnevich sheaf if so is $F_{\mathfrak{X}}$ for all $\mathfrak{X} \in \underline{\mathbf{M}}\mathbf{Cor}$ (see [12, Section 4]). For $F \in \underline{\mathbf{M}}\mathbf{PST}$ and $\mathfrak{X} = (\overline{X}, X_\infty)$, we write $(F_{\mathfrak{X}})_{\mathrm{Nis}}$ for the Nisnevich sheafification of the presheaf $F_{\mathfrak{X}}$ on $\overline{X}_{\text{ét}}$. For $F \in \underline{\mathbf{M}}\mathbf{NST}$, we write

$$H^i(\mathfrak{X}_{\mathrm{Nis}}, F) = H^i(\overline{X}_{\mathrm{Nis}}, F_{\mathfrak{X}}) \text{ for } F \in \underline{\mathbf{M}}\mathbf{NST}. \tag{2.1}$$

In order to explain the sheafification functor in the modulus context, we need to introduce more notation. Let Σ^{fin} be the subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ having the same objects as $\underline{\mathbf{M}}\mathbf{Cor}$ and such that the morphisms $f\colon \mathfrak{X} \to \mathfrak{Y}$ with $\mathfrak{X} = (\overline{X}, X_\infty)$ and $\mathfrak{Y} = (\overline{Y}, Y_\infty)$ are the graphs of morphisms $f\colon \overline{X} \to \overline{Y}$ which induce isomorphisms $X \cong Y$ between the smooth interiors, and satisfying $f^*Y_\infty = X_\infty$.

We write $\underline{\mathbf{M}}\mathbf{NST} \subset \underline{\mathbf{M}}\mathbf{PST}$ for the full subcategory of Nisnevich sheaves. By [12, Theorem 2, (1)] the inclusion $\underline{\mathbf{M}}\mathbf{NST} \to \underline{\mathbf{M}}\mathbf{PST}$ has an exact left adjoint $\underline{a}_{\mathrm{Nis}}$, satisfying

$$(\underline{a}_{\mathrm{Nis}}F)(\mathfrak{X}) = \varinjlim_{\mathfrak{Y} \in \Sigma^{\mathrm{fin}} \downarrow \mathfrak{X}} (F_{\mathfrak{Y}})_{\mathrm{Nis}}(\mathfrak{Y})$$

for every $F \in \underline{\mathbf{M}}\mathbf{PST}$ and $\mathfrak{X} \in \underline{\mathbf{M}}\mathbf{Cor}$.

Similarly, let $\mathbf{MNST} \subset \mathbf{MPST}$ be the full subcategory of those F such that $\tau_! F \in \underline{\mathbf{M}}\mathbf{NST}$. By [12, Theorem 3,(1)] the inclusion $\mathbf{MNST} \to \mathbf{MPST}$ has an exact left adjoint a_{Nis} such that $\underline{a}_{\mathrm{Nis}}\tau_! = \tau_! a_{\mathrm{Nis}}$.

By [12, Prop 2.2.1, Prop 2.3.1, Prop 2.4.1] there are three pairs of adjoint functors $(\omega_!, \omega^*)$, $(\underline{\omega}_!, \underline{\omega}^*)$ and $(\tau_!, \tau^*)$:

$$\mathbf{PST} \underset{\underline{\omega}^*}{\overset{\underline{\omega}_!}{\leftrightarrows}} \underline{\mathbf{M}}\mathbf{PST} \underset{\tau^*}{\overset{\tau_!}{\leftrightarrows}} \mathbf{MPST} \underset{\omega^*}{\overset{\omega_!}{\rightleftarrows}} \mathbf{PST}, \tag{2.2}$$

which are given by

$$\underline{\omega}^* F(\overline{X}, X_\infty) = F(\overline{X} \setminus |X_\infty|), \qquad \underline{\omega}_! H(X) = H(X, \emptyset), \tag{2.3}$$

$$\omega^* F(\overline{X}, X_\infty) = F(\overline{X} \setminus |X_\infty|), \qquad \omega_! G(X) \cong \varinjlim_{\mathfrak{X} \in \mathbf{MSm}(X)} G(\mathfrak{X}), \tag{2.4}$$

$$\tau^* F(\mathfrak{X}) = F(\mathfrak{X}), \qquad \tau_! G(\mathfrak{U}) \cong \varinjlim_{\mathscr{X} \in Comp(\mathfrak{U})} G(\mathfrak{X}), \tag{2.5}$$

where $\mathbf{MSm}(X)$ is the subcategory of $\mathbf{MCor}$ with objects the proper modulus pairs $(\overline{X}, X_\infty)$ such that $\overline{X} - |X_\infty| = X$ and the morphisms of modulus pairs which map to the identity in $\mathbf{Cor}(X, X)$, and $Comp(\mathfrak{U})$ is the category of compactifications of $\mathfrak{U} = (\overline{U}, U_\infty)$, namely objects are proper modulus pairs $\mathfrak{X} = (\overline{X}, \overline{U}_\infty + \Sigma)$, where $\overline{U}_\infty$ and Σ are effective Cartier divisors on $\overline{X}$ so that $\overline{X} \setminus |\Sigma| = \overline{U}$ and $\overline{U}_{\infty|\overline{U}} = U_\infty$, and the morphisms are those which map to the identity in $\underline{\mathbf{M}}\mathbf{Cor}(\mathfrak{U}, \mathfrak{U})$. The functors $\omega_!$, $\underline{\omega}_!$, $\tau_!$ are exact and we have

$$\omega_! = \underline{\omega}_! \tau_! \ \text{ and } \ \tau_! \omega^* = \underline{\omega}^*.$$

We now introduce two basic properties of $\underline{\mathbf{M}}\mathbf{PST}$ and $\mathbf{MPST}$. The fist property is *semi-purity*

Definition 2.1.1. We say $F \in \underline{\mathbf{M}}\mathbf{PST}$ is *semi-pure* if the unit map $F \to \underline{\omega}^* \underline{\omega}_! F$ is injective.

The second property is an analogue of homotopy invariance exploited by Voevodsky. In order to make sense of it, recall that the categories $\underline{\mathbf{M}}\mathbf{Cor}$ and $\mathbf{MCor}$ enjoy a symmetric monoidal structure defined as follows. For $\mathfrak{X} = (\overline{X}, X_\infty)$ and $\mathfrak{Y} = (\overline{Y}, Y_\infty)$, the modulus pair $\mathfrak{X} \otimes \mathfrak{Y}$ is given as

$$\mathfrak{X} \otimes \mathfrak{Y} = (\overline{X} \times \overline{Y}, X_\infty \times \overline{Y} + \overline{X} \times Y_\infty).$$

See [12, 1.4]. We can now give the following definition:

Definition 2.1.2. We say $F \in \underline{\mathbf{M}}\mathbf{PST}$ (resp. $F \in \mathbf{MPST}$) *cube invariant* if for any $\mathfrak{X} \in \underline{\mathbf{M}}\mathbf{Cor}$ (resp. $\mathfrak{X} \in \underline{\mathbf{M}}\mathbf{Cor}$), the pullback along $\mathfrak{X} \otimes \overline{\square} \to \mathfrak{X}$ induces an isomorphism

$$F(\mathfrak{X}) \cong F(\mathfrak{X} \otimes \overline{\square}).$$

We write $\mathbf{CI} \subset \mathbf{MPST}$ and $\underline{\mathbf{CI}} \subset \underline{\mathbf{M}}\mathbf{PST}$ for the full subcategories of cube invariant objects, and $\mathbf{CI}^\tau \subset \underline{\mathbf{M}}\mathbf{PST}$ for the essential image of $\mathbf{CI}$ under $\tau_! : \mathbf{MPST} \to \underline{\mathbf{M}}\mathbf{PST}$. We also write $\mathbf{CI}^\tau_{\mathrm{Nis}} = \mathbf{CI}^\tau \cap \underline{\mathbf{M}}\mathbf{NST}$.

By [23, Lem.1.16], we have

$$\mathbf{CI}^\tau \subset \underline{\mathbf{CI}} \;\text{ and }\; \mathbf{CI} = \tau_!^{-1}(\underline{\mathbf{CI}}). \tag{2.6}$$

By [14, Prop. 2.2.5] the inclusion $\mathbf{CI} \to \mathbf{MPST}$ admits a left adjoint $h_0^{\overline{\square}}$ given by

$$h_0^{\overline{\square}}(F)(\mathfrak{Y}) = \mathrm{coker}\big(F(\mathfrak{X} \otimes \overline{\square}) \xrightarrow{i_0^* - i_1^*} F(\mathfrak{Y})\big) \;\text{ for } F \in \mathbf{MPST},\ \mathfrak{Y} \in \mathbf{MCor},$$

where $i_\epsilon : (\operatorname{Spec} k, \emptyset) \to \overline{\square}$ is the morphism in $\mathbf{MCor}$ induced by a k-rational point $\epsilon \in \mathbb{P}^1 - \{\infty\}$. For $\mathscr{X} \in \mathbf{MCor}$ we write

$$h_0^{\overline{\square}}(\mathfrak{X}) = h_0^{\overline{\square}}(\mathbb{Z}_{\mathrm{tr}}(\mathfrak{X})),$$

where $\mathbb{Z}_{\mathrm{tr}}(\mathfrak{X}) = \mathbf{MCor}(-, \mathfrak{X}) \in \mathbf{MPST}$ is the Yoneda object of $\mathfrak{X}$. By abuse of notation we write $h_0^{\overline{\square}}(\mathfrak{X})$ for $\tau_! h_0^{\overline{\square}}(\mathfrak{X})$. By [13, Lem. 1.8.3] and (2.5) and the exactness of $\tau_!$, we have

$$h_0^{\overline{\square}}(\mathfrak{X})(\mathfrak{U}) = \mathrm{coker}\big(\underline{\mathbf{M}}\mathbf{Cor}(\mathfrak{U} \otimes \overline{\square}, \mathfrak{X}) \xrightarrow{i_0^* - i_1^*} \underline{\mathbf{M}}\mathbf{Cor}(\mathfrak{U}, \mathfrak{X})\big) \;\text{ for } \mathfrak{U} \in \underline{\mathbf{M}}\mathbf{Cor}.$$

We now state the main theorem of this paper.

Theorem 2.1.3. *Assume that $ch(k) = 0$. Then, for $\mathfrak{X} \in \mathbf{MCor}^{\mathrm{proj}}$, $\tau_! a_{\mathrm{Nis}} h_0^{\overline{\square}}(\mathfrak{X})$ is semi-pure.*

2.2. **Strict $\overline{\square}$-invariance of $\overline{\square}$-invariant sheaves.** Let $\underline{\mathbf{M}}\mathbf{Cor}_{ls} \subset \underline{\mathbf{M}}\mathbf{Cor}$ be the full subcategory of objects $(\overline{X}, X_\infty)$ with $\overline{X} \in \mathbf{Sm}$ and $|X_\infty|$ a simple normal crossing divisor on $\overline{X}$. As an application of Theorem 2.1.3, we prove the following theorem, which plays a fundamental role in theory of motives with modulus.

Theorem 2.2.1. *For $F \in \mathbf{CI}^\tau$ and $\mathfrak{X} \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$, we have an isomorphism*

$$\pi^* \colon H^q(\mathfrak{X}_{\mathrm{Nis}}, F_{\mathrm{Nis}}) \simeq H^q((\mathfrak{X} \otimes \overline{\square})_{\mathrm{Nis}}, F_{\mathrm{Nis}})$$

for any integer $q \geq 0$, induced by the projection $\pi \colon \mathfrak{X} \otimes \overline{\square} \to \mathfrak{X}$.

Proof. By [23, Theorem 0.6 and 0.8], the assertion holds if we assume further that F is semi-pure. In general we may write $F = \tau_! G$ with $G \in \mathbf{CI}$. We have the surjective map in $\mathbf{MPST}$

$$\bigoplus_{\alpha \in G(\mathfrak{Y})} \mathbb{Z}_{\mathrm{tr}}(\mathfrak{Y}) \to G,$$

where the direct sum ranges over all $\mathfrak{Y} \in \mathbf{MCor}$ and $\alpha \in G(\mathfrak{Y})$, which induce the Yoneda maps $\alpha\colon \mathbb{Z}_{\mathrm{tr}}(\mathfrak{Y}) \to G$ in **MPST**. By Chow's Lemma, we can in fact assume that the sum ranges over $\mathfrak{Y} \in \mathbf{MCor}^{\mathrm{proj}}$. Since $G \in \mathbf{CI}$, it factors through a surjective map

$$P := \bigoplus_{\alpha\in G(\mathfrak{Y})} h_0^{\overline{\square}}(\mathfrak{Y}) \to G.$$

Let H be its kernel so that we have an exact sequence in **CI**

$$0 \to K \to \bigoplus_{\alpha\in G(\mathfrak{Y})} h_0^{\overline{\square}}(\mathfrak{Y}) \to G \to 0.$$

By the exactness of $\tau_!$ and $\underline{a}_{\mathrm{Nis}}$ this induces an exact sequence

$$0 \to \underline{a}_{\mathrm{Nis}}\tau_! K \to H \to F \to 0 \ \text{ with } H = \underline{a}_{\mathrm{Nis}}(\tau_! P) = \bigoplus_{\alpha\in G(\mathfrak{Y})} \underline{a}_{\mathrm{Nis}}\tau_! h_0^{\overline{\square}}(\mathfrak{Y}). \tag{2.7}$$

By Theorem 2.1.3 H is semi-pure and hence so is $\underline{a}_{\mathrm{Nis}}\tau_! K$ (see [23, Lemma 1.25]). Moreover H and $\underline{a}_{\mathrm{Nis}}\tau_! K$ are in $\mathbf{CI}^{\tau}$ by [23, Theorem 0.6] and (2.6) and [13, Proposition 3.7.3]. Hence Theorem 2.2.1 holds for H and $\underline{a}_{\mathrm{Nis}}\tau_! K$. Now it holds also for F by the long exact sequence of cohomology groups arising from (2.7). $\square$

3. Semi-purity for relative curves

In this Section, we prove a variant (in fact, a generalization) of Theorem 2.1.3 for relative curves. This is generalization is necessary for the reduction of the main theorem from case of relative surfaces to the case of relative dimension 1. In this Section, the ground field k is arbitrary.

3.1. **Statement and first reductions.** Let $\mathscr{O}$ be a local normal domain and fix a non-zero element $\Pi \in \mathscr{O}$. Let $\mathscr{C}$ be a normal scheme proper surjective over $\mathscr{O}$, generically of relative dimension 1. Let $\mathscr{C}_\eta$ be the generic fiber. Note that $\mathscr{C}_\eta$ is regular. Let $D \subset \mathscr{C}$ be an effective Cartier divisor which is finite over $\mathscr{O}$. Let $Q(\mathscr{C})$ be the function field of $\mathscr{C}$.

Definition 3.1.1. We say that a closed point $x \in \mathscr{C}_\eta$ *satisfies the modulus condition over* $\mathscr{O}$ (for short, x satisfies $(\mathrm{MC})_{\mathscr{O}}$) if the following holds. Let Z be the closure of x in $\mathscr{C}$, and let $Z^N \to Z$ be its normalization with the natural map $\nu_Z : Z^N \to \mathscr{C}$. Then we have an inequality of Cartier divisors

$$\nu_Z^* D \leq \nu_Z^*(\Pi),$$

where (Π) is the Cartier divisor on $\operatorname{Spec}\mathscr{O}$ defined by Π.

Theorem 3.1.2. *Take $f \in Q(\mathscr{C})$ satisfying the following conditions:*

(i) *$f = 1 + \gamma s$ with $\gamma \in \mathscr{O}_{\mathscr{C},D}$, the semi-localization of $\mathscr{C}$ at the generic points of D, and $s \in \mathscr{O}_{\mathscr{C},D}$ a local equation of D.*
(ii) *Every prime component of $\operatorname{div}_{\mathscr{C}_\eta}(f)$ satisfies $(\mathrm{MC})_{\mathscr{O}}$.*

Let $W \subset \mathscr{C} \times_{\mathscr{O}} P^1_{\mathscr{O}}$ be the closure of the graph $\Gamma_f \subset \mathscr{C}_\eta \times_\eta \mathbb{P}^1_\eta$ of f. Then, for any irreducible component T of W with its normalization T^N and the natural maps $\nu_T : T^N \to \mathbb{P}^1_{\mathscr{O}}$ and $\mu_T : T^N \to \mathscr{C}$, we have

$$\mu_T^*(D) \leq \nu_T^*(1_{\mathscr{O}} + \mathbb{P}^1_{(\Pi)}), \tag{3.1}$$

where $1_{\mathscr{O}}$ and $\mathbb{P}^1_{(\Pi)}$ are Cartier divisor on $\mathbb{P}^1_{\mathscr{O}}$.

First we claim that it suffices to prove the theorem assuming $\mathscr{O}$ is a henselian DVR. Note $A = \bigcap_{\mathfrak{p}\subset A} A_{\mathfrak{p}}$ for a normal domain A, where $\mathfrak{p}$ range over all prime ideals of height one. Thus we may check (3.1) locally at a point t of codimension one in T^N. It suffices to consider the case where t lies on the inverse image of $T \cap (D \times_{\mathscr{O}} \mathbb{P}^1_{\mathscr{O}}) \subset \mathscr{C} \times_{\mathscr{O}} \mathbb{P}^1_{\mathscr{O}}$. Since $D \times_{\mathscr{O}} \mathbb{P}^1_{\mathscr{O}}$ is finite over $\mathbb{P}^1_{\mathscr{O}}$ by the assumption on D, the closure $\overline{\{t\}}$ of t in T^N is finite over $\mathbb{P}^1_{\mathscr{O}}$. Noting $\dim(\overline{\{t\}}) = \dim(T) - 1 = \dim(\mathbb{P}^1_{\mathscr{O}}) - 1$, this implies that t maps to a point of codimension one in $\mathbb{P}^1_{\mathscr{O}}$, and hence maps to a point of codimension≤ 1 in $\operatorname{Spec} \mathscr{O}$. Since (3.1) can be checked étale locally, we may replace $\mathscr{O}$ by its henselization at a point of codimension one and $\mathscr{C}$ by its base change. This proves the claim.

In what follows $\mathscr{O}$ is a henselian DVR with a prime element π and $\Pi = \pi^e$ for an integer $e > 0$. Let K be the quotient field and v be the normalized valuation of $\mathscr{O}$.

Since (3.1) can be checked étale locally, the above theorem follows from the following local version. Let A be the henselization of the local ring of $\mathscr{C}$ at a closed point, which is an integral normal local domain essentially of finite type over $\mathscr{O}$ with $\dim(A) = 2$. Let $s \in A$ be a local equation of D. By the assumption, $A/(s)$ is finite flat over $\mathscr{O}$ and $A/(\pi^e, s)$ is Artinian. By ZMT the natural map $\mathscr{O}[s] \to A$ induces finite map $\phi : \operatorname{Spec} A \to \operatorname{Spec} R$ with $R = \mathscr{O}\{s\}$. Let $Q(A)$ be the quotient field of A.

Theorem 3.1.3. *Let A and s be as above. Take $f \in Q(A)$ satisfying the following conditions:*

(i) *$f = 1 + \gamma s$ with γ in the semi-localization of A at the primes lying over $(s) \subset A$.*
(ii) *We have $\pi^e/s \in (A/\mathfrak{p})^N$ for all hight-one primes $\mathfrak{p}$ such that $v_{\mathfrak{p}}(f) \neq 0$ and $\mathfrak{p}$ does not divides (π).*

Let $W \subset \operatorname{Spec} A[\tau]$ be the closure of $(\tau - f = 0) \subset \operatorname{Spec} Q(A)[\tau]$. Then, for any irreducible component T of W with its normalization T^N, we have

$$(\tau - 1)\pi^e/s \in \Gamma(T^N, \mathscr{O}). \tag{3.2}$$

Note that Theorem 3.1.2 immediately implies Theorem 5.3.1 in dimension 1. Indeed, if $\mathfrak{X} = (\overline{C}, C_\infty)$ is in **MCor** with $\dim(\overline{C}) = 1$, we can apply Theorem 3.1.2 with $\mathscr{C} = \overline{C} \times_k S$, and $D = C_\infty \times_k S$. Then $\mathscr{C}_\eta = \overline{C}_K$ and in view of Proposition 5.4.1, Theorem 3.1.2 gives the injectivity of the map

$$j_{\mathscr{O},(\Pi)} \colon h_0^{\overline{\square}}(\mathfrak{X})(\mathscr{O}, (\Pi)) \to \mathrm{CH}_0(\overline{C}_K, C_{\infty,K})$$

as required.

3.2. Valuative criterion for modulus condition on $\mathscr{O}\{s\}$.

Definition 3.2.1. For $n \in \mathbb{Z}_{>0}$ and $f = \sum_{i\geq 0} a_i s^i \in R - \{0\}$, we define

$$\nu_e(f) \geq n \overset{def}{\Leftrightarrow} v(a_i) \geq n - ei \text{ for all } i \geq 0.$$

We easily check the following.

Lemma 3.2.2. *Let $f, g \in R - \{0\}$ and $n, m \in \mathbb{Z}_{>0}$.*

(1) *If $\nu_e(f) \geq n$ and $\nu_e(g) \geq n$, then $\nu_e(f \pm g) \geq n$.*
(2) *If $\nu_e(f) \geq n$ and $\nu_e(g) \geq m$, then $\nu_e(fg) \geq nm$.*
(3) *$\nu_e(\pi^m f) \geq n + m$ if and only if $\nu_e(f) \geq n$.*

Lemma 3.2.3. *Let $\mathfrak{p} \subset R$ be a height-one prime not dividing (π) such that $\pi^e/s \in (R/\mathfrak{p})^N$. Then $\mathfrak{p}$ is generated by an element of the form*

$$g = a_0 + a_1 s + \cdots a_m s^m \in \mathscr{O}[s]$$

such that $a_m \in \mathscr{O}^\times$ and $\nu_e(g) \geq v(a_0)$.

Proof. Since $s \mod \mathfrak{p} \in R/\mathfrak{p}$ is finite over $\mathscr{O}$, there is a monic irreducible polynomial

$$g = a_0 + a_1 s + \cdots a_m s^m \in \mathscr{O}[s] \ \ (a_m = 1)$$

such that $g \in \mathfrak{p}$. This implies $\mathfrak{p} = (g)$ by the irreducibility of g. Put $\theta = \pi^e/s \mod \mathfrak{p} \in Q(R/\mathfrak{p})$. From $g = 0 \in R/\mathfrak{p}$, we get

$$\theta^m + \sum_{1\leq i\leq m} \frac{a_i \pi^{ei}}{a_0} \theta^{m-i} = 0 \in Q(R/\mathfrak{p}).$$

Since $g \in \mathscr{O}[s]$ is irreducible over K, this gives a minimal equation of θ over K. Since θ is finite over $\mathscr{O}$ by the assumption, this implies $a_i\pi^{ei}/a_0 \in \mathscr{O}$ for all i, which implies $\nu_e(g) \geq v(a_0)$. □

Lemma 3.2.4. *Let $f \in R$ and $a_0 = f \mod s \in \mathscr{O}$. Assume*

$(MC)_R$ *For any height-one prime $\mathfrak{p}$ dividing f but not dividing (π), $\pi^e/s \in (R/\mathfrak{p})^N$.*

Assume further $a_0 \neq 0$. Then $\nu_e(f) \geq v(a_0)$.

Proof. Considering the prime decomposition of f in R, this follows from Lemmas 3.2.2 and 3.2.3. □

3.3. **Criterion for modulus condition on A.** Let the assumption be as in the statement of Theorem 3.1.3 Write $X = \operatorname{Spec} A$ and $D = (s) \subset X$. Let $\psi : \tilde{X} \to X$ be the blowup with center in (s, π^e) and $D' \subset \tilde{X}$ be the proper transform of D. Write

$$X_+ = \tilde{X} - D' = \operatorname{Spec} A[t]/(st - \pi^e)$$

with the induced map $\psi_+ : X_+ \to X$. For a height-one prime $\mathfrak{p} \subset A$ write $Z_\mathfrak{p} = \overline{\{\mathfrak{p}\}} \subset X$ and let $Z'_\mathfrak{p} \subset \tilde{X}$ be its proper transform.

Lemma 3.3.1. *Assume $\mathfrak{p}$ does not divide (π). Then the following conditions are equivalent.*

(i) $\pi^e/s \in (A/\mathfrak{p})^N$.
(ii) $Z'_\mathfrak{p} \subset X_+ \Leftrightarrow Z'_\mathfrak{p} \cap D' = \emptyset$.

Proof. (ii) is equivalent to that $Z'_{\mathfrak{p},+} := Z'_\mathfrak{p} \cap X_+$ is finite over $\mathscr{O}$. Note that $Z'_{\mathfrak{p},+}$ is the irreducible component of $\psi_+^{-1}(Z_\mathfrak{p}) = X_+ \times_X Z_\mathfrak{p}$ which is flat over $\mathscr{O}$. Let $\phi : Z_\mathfrak{p}^N = \operatorname{Spec}(A/\mathfrak{p})^N \to Z_\mathfrak{p}$ be the normalization and $Z^N_{\mathfrak{p},+} = Z_{\mathfrak{p},+} \times_{Z_\mathfrak{p}} Z^N_\mathfrak{p}$. Note that $Z^N_{\mathfrak{p},+}$ is the irreducible component of $X_+ \times_X Z^N_\mathfrak{p}$ which is flat over $\mathscr{O}$. The map ϕ induces a finite surjective map $Z^N_{\mathfrak{p},+} \to Z_{\mathfrak{p},+}$. Hence it suffices to show that (i) is equivalent to that $Z^N_{\mathfrak{p},+}$ is finite over $\mathscr{O}$. Note that the latter condition is equivalent to $Z^N_{\mathfrak{p},+} = Z^N_\mathfrak{p}$. Now the desired equivalence follows easily by noting

$$X_+ \times_X Z^N_\mathfrak{p} = \operatorname{Spec}(A/\mathfrak{p})^N[t]/(st - \pi^e). \tag{3.3}$$

Indeed, if (i) holds, there exist $\theta \in (A/\mathfrak{p})^N$ such that $s\theta = \pi^e$. Hence (3.3) implies $Z^N_{\mathfrak{p},+} \simeq \operatorname{Spec}(A/\mathfrak{p})^N$. To see the converse, assume $Z^N_{\mathfrak{p},+} \simeq \operatorname{Spec}(A/\mathfrak{p})^N$. Note $(A/\mathfrak{p})^N$ is a DVR and let Π be a prime element of it. Write $s = u\Pi^n, \pi = v\Pi^m$ in A with $m, n \in \mathbb{Z}_{\geq 0}$ and $u, v \in ((A/\mathfrak{p})^N)^\times$. It suffices to show $m \geq n$. Assume the contrary $m < n$. Then

$$Z^N_{\mathfrak{p},+} \simeq \operatorname{Spec}(A/\mathfrak{p})^N[t]/(u\Pi^{n-m}t - v) \simeq \operatorname{Spec} A[1/\Pi],$$

which contradicts the assumption. □

Lemma 3.3.2. *Let $g \in A$ satisfies the condition:*

$(MC)_A$ *For any height-one prime $\mathfrak{p} \subset A$ dividing (g) but not dividing (π), $\pi^e/s \in (A/\mathfrak{p})^N$.*

Then there exists $N > 0$ such that for any $\alpha \in \mathscr{O}$, $g + \alpha\pi^N \in A$ satisfies $(MC)_A$.

This follows from Lemma 3.3.1 and the following.

Lemma 3.3.3. *Let the notation be as in Lemma 3.3.1. Assume that $g \in A$ satisfies $(MC)_A$. Then there exists $N > 0$ such that for any $\alpha \in \mathscr{O}$ and any irreducible component T of $Z_\alpha := (g + \alpha\pi^N) \subset X = \operatorname{Spec} A$ which is flat over $\mathscr{O}$, we have $T' \cap D' = \emptyset$, where $T' \subset \tilde{X}$ is the proper transform of T.*

Proof. Consider the finite map

$$\phi : X \to Y = \operatorname{Spec} R$$

and put $W = \phi(Z)$ and $W_\alpha = \phi(Z_\alpha)$. Then $W = (h)$ and $W_\alpha = (h_\alpha)$ with $h = N_{A/R}(g)$ and $h_\alpha = N_{A/R}(g_\alpha)$. Removing from W (resp. W_α) the component not flat over $\mathscr{O}$, we get an effective divisor $W_{fl} \subset Y$ (resp. $W_{\alpha,fl} \subset Y$). Let $\psi_Y : \tilde{Y} \to Y$ be the blowup with center in (s, π^e) with the exceptional divisor $E \subset \tilde{Y}$. We have $\tilde{X} \simeq \tilde{Y} \times_Y X$ with $\tilde{\phi} : \tilde{X} \to \tilde{Y}$ the projection. Let D'_Y (resp. W'_{fl}, resp. $W'_{\alpha,fl}$) be the proper transform in $\tilde{Y}$ of $D_Y = (s) \subset Y$ (resp. W_{fl}, resp. $W_{\alpha,fl}$). Then we have $D' = \tilde{\phi}^{-1}(D'_Y)$ and $T' \subset \tilde{\phi}^{-1}(W'_{\alpha,fl})$ for T' as in the lemma. Thus it suffices to show $W'_{\alpha,fl} \cap D'_Y = \emptyset$. The assumption implies that h satisfies $(MC)_R$ and so Lemma 3.3.1 (in case $A = R$) implies $W'_{fl} \cap D'_Y = \emptyset$. By the assumption we can write $h_\alpha = h + \lambda\pi^N$ with $\lambda \in R$. Around $S := D'_Y \cap E$, $\tilde{Y}$ is regular and (σ, π) with $\sigma = s/\pi^e$ is a system of regular parameters. The condition $W'_{fl} \cap D'_Y = \emptyset$ implies $h = \pi^m h'$ with $m \in \mathbb{Z}_{\geq 0}$ and $h' \in \mathscr{O}^\times_{\tilde{Y},S}$. Taking N so large that $N > m$, we get

$$h_\alpha = h + \lambda\pi^N = \pi^m h' + \lambda\pi^N = \pi^m(h' + \lambda\pi^{N-m})$$

and $h' + \lambda\pi^{N-m} \in \mathscr{O}^\times_{\tilde{Y},S}$, which implies $W'_{\alpha,fl} \cap D'_Y = \emptyset$ as desired. □

Let $d = [A : R]$. For $f \in A$ and $\alpha \in R$ we can express the norm of $\alpha + f$ to R as

$$N_{A/R}(\alpha + f) = \sum_{i=0}^{d} \alpha^{d-i}\sigma_i(f), \tag{3.4}$$

where $\sigma_i(f)$ are symmetric polynomials of homogeneous degree i in the conjugates $f_1, \dots, f_r$ of f over R, which are independent of α.

Lemma 3.3.4. *Assume that $g = a + \lambda s \in A$ where $a \in \mathscr{O} - \{0\}$ and $\lambda \in A$, satisfies $(MC)_A$ from Lemma 3.3.2. Then we have $\nu_e(\sigma_i(g)) \geq iv(a)$.*

Proof. We may assume $g = \pi^n + \lambda s \in A$ with $n \in \mathbb{Z}_{\geq 0}$ and $\lambda \in A$. For each $i \in \mathbb{Z}_{\geq 0}$ there exist integers $c_{i,j}$ with $0 \leq j \leq i$ such that for any $\beta \in \mathscr{O}$, we have

$$\sigma_i(g + \beta) = \sum_{j=0}^{i} c_{i,j}\sigma_j(g)\beta^{i-j}. \tag{3.5}$$

Thus it suffices to show $\nu_e(\sigma_i(\lambda s)) \geq ni$. By Lemma 3.3.3 there exists an integer $N > n$ such that for all $\alpha \in \mathscr{O}$, $g_\alpha = u_\alpha\pi^n + \lambda s \in A$ satisfies $(MC)_A$, where $u_\alpha = 1 + \alpha\pi^{N-n} \in \mathscr{O}^\times$. By [] this implies that $h_\alpha := N_{A/R}(g_\alpha) \in R$ satisfies $(MC)_R$. Noting $h_\alpha \mod s = u_\alpha^d\pi^{dn}$, this implies $\nu_e(h_\alpha) \geq dn$ by Lemma 3.2.4. By (3.4) we have

$$h_\alpha = \sum_{i=0}^{d} (u_\alpha\pi^n)^{d-i}\sigma_i(\lambda s).$$

By Lemma 3.2.2, $\nu_e(h_\alpha) \geq dn$ for various choices of α implies $\nu_e(\sigma_i(\lambda s)) \geq ni$ as desired. □

3.4. **Proof of the main theorem for curves.** We now prove Theorem 3.1.3. Let ϕ : $\operatorname{Spec} A[\tau] \to \operatorname{Spec} R[\tau]$ be the finite map induced by $\phi : \operatorname{Spec} A \to \operatorname{Spec} R$ and put $W_R = \phi(W)$. It suffices to show (3.2) for any irreducible component T of W_R. Let Σ be the set of height-one primes $\mathfrak{p} \subset A$ not dividing (π) such that $v_{\mathfrak{p}}(f) \neq 0$. For each $\mathfrak{p} \in \Sigma$ take a generator $h_{\mathfrak{p}}$ of $\mathfrak{p} \cap R$ as in Lemma 3.2.3. Let $\Sigma_- \subset \Sigma$ be the subset of $\mathfrak{p}$ such that $v_{\mathfrak{p}}(f) < 0$. By Theorem 3.1.3(i), $f \in A_{\mathfrak{q}}^\times$ for any prime $\mathfrak{q}$ dividing (s). From this we see that Σ_- coincides with the set of height-one primes $\mathfrak{p} \subset A$ such that $v_{\mathfrak{p}}(\gamma) < 0$. We can choose $l \in \mathbb{Z}_{\geq 0}$ and $e_{\mathfrak{p}} \in \mathbb{Z}_{>0}$ for $\mathfrak{p} \in \Sigma_-$ such that $h\gamma \in A$ for

$$h = \pi^l \prod_{\mathfrak{p} \in \Sigma_-} h_{\mathfrak{p}}^{e_{\mathfrak{p}}}.$$

Put $a = h \mod s \in \mathscr{O}$. Note $a \neq 0$ since $h_{\mathfrak{p}}$ for $\mathfrak{p} \in \Sigma$ is not divisible by s. We have

$$g := hf = a + \lambda s \quad \text{ with } \lambda = h\gamma + (h-a)/s \in A.$$

By the construction we have

$$W \subset (h\tau - g) \subset \operatorname{Spec} A[\tau],$$

and hence

$$W_R \subset (N_{A/R}(h\tau - g)) \subset \operatorname{Spec} R[\tau]. \tag{3.6}$$

Put $\sigma_i = \sigma_i(h-g)$ and $t = \tau - 1$. Since $h - g \in sA$, we have

$$\sigma_i \in s^i R. \tag{3.7}$$

By (3.4), we have

$$N_{A/R}(h\tau - g) = N_{A/R}(ht + (h-g)) = \sum_{i=0}^{d} t^{d-i} h^{d-i} \sigma_i.$$

By the construction, h (resp. g) satisfies the condition $(MC)_R$ (resp. $(MC)_A$). Hence Lemmas 3.2.4 and 3.3.4 imply

$$\nu_e(h) \geq v(a) \text{ and } \nu_e(\sigma_i(g)) \geq iv(a).$$

By (3.5) this implies

$$\nu_e(\sigma_i) \geq iv(a) \text{ and } \nu_e(h^{d-i}\sigma_i) \geq dv(a). \tag{3.8}$$

By (3.7) and (3.8), we can write

$$h^{d-i}\sigma_i = \sum_{j \geq i} c_{i,j} s^j \text{ with } v(c_{i,j}) \geq dv(a) - ej. \tag{3.9}$$

We then get

$$N_{A/R}(h\tau - g) = \sum_{i=0}^{d} t^{d-i} \Big(\sum_{j \geq i} c_{i,j} s^j \Big) = \sum_{j \geq 0} s^j \Big(\sum_{0 \leq i \leq \min\{j,d\}} c_{i,j} t^{d-i} \Big). \tag{3.10}$$

Write $n = v(a)$ and take an integer $N > 0$ such that $Ne \geq nd$. Let T be an irreducible component of W_R and $Q(T)$ be its function field. Multiplying (3.10) by t^{N-d}/s^N, (3.6) implies an equality in $Q(T)$:

$$0 = \sum_{0\leq j} \frac{1}{s^{N-j}} \Big(\sum_{0\leq i\leq \min\{j,d\}} c_{i,j} t^{N-i} \Big)$$
$$= \sum_{0\leq j\leq N} (t/s)^{N-j} \Big(\sum_{0\leq i\leq \min\{j,d\}} c_{i,j} t^{j-i} \Big) + \sum_{j\geq N+1} s^{j-N} \Big(\sum_{0\leq i\leq \min\{j,d\}} c_{i,j} t^{N-i} \Big).$$

Put

$$\theta = \pi^e t/s \;\text{ and }\; \rho_j = \pi^{ej-nd},\ \gamma_j = \sum_{0\leq i\leq \min\{j,d\}} c_{i,j} t^{j-i} \;\text{ for } 1 \leq j \leq N.$$

Multiplying the latter equation by ρ_N, we get an equality on $Q(T)$:

$$\sum_{0\leq j\leq N} \gamma_j \rho_j \theta^{N-j} + \rho_N \sum_{j\geq N+1} s^{j-N} \Big(\sum_{0\leq i\leq \min\{j,d\}} c_{i,j} t^{N-i} \Big) = 0.$$

Note that the second term lies in $\Gamma(T, \mathscr{O})$ and that $\gamma_j \rho_j \in \Gamma(T, \mathscr{O})$ for $0 \leq j \leq N$ in view of (3.9). By definition we have $\gamma_0 = c_{0,0} = h^d \mod s = a^d$ and $\rho_0 = \pi^{-nd}$. Hence, writing $a = u\pi^n$ with $u \in \mathscr{O}^\times$, we have $\gamma_0\rho_0 = u^d \in \mathscr{O}^\times$. Hence θ is integral over T so that $\theta \in \Gamma(T^N, \mathscr{O})$ as desired.

4. Bertini theorems over a base

In this Section, let S be the spectrum of a Noetherian local domain $\mathscr{O}$. Let $\mathfrak{m}$ be the maximal ideal of $\mathscr{O}$, and write k for the residue field $\mathscr{O}/\mathfrak{m}$. Finally, let K be the function field of $\mathscr{O}$. If $\mathscr{X}$ is any S-scheme, write $\mathscr{X}_\eta$ for its generic fiber and $\mathscr{X}_s$ for the reduced special fiber. Recall some notations and definitions from e.g. [10] or [24, Section 4].

Definition 4.0.1. A *hyperplane* $H \subset \mathbb{P}^N_S$ is a closed subscheme of the projective space $\mathbb{P}^N_S$ over S corresponding to an S-rational point of the dual $(\mathbb{P}^N_S)^\vee := \mathrm{Gr}_S(N-1, N)$.

By definition, $\mathrm{Gr}_S(N-1, N)(S)$ is the set consisting of surjective maps of $\mathscr{O}$-modules, $q\colon \mathscr{O}^{\oplus N+1} \to E$ such that E is a free $\mathscr{O}$-module, of rank N. Fixing a basis $\{e_0, \ldots, e_N\}$ of $\mathscr{O}^{\oplus N+1}$, we can write the kernel of q as $\sum_{i=0}^N \langle a_i\rangle e_i \subset \mathscr{O}^{N+1}$, for elements $a_i \in \mathscr{O}$, where $\langle a\rangle$ denotes the submodule of $\mathscr{O}$ generated by a. Fixing homogeneous coordinates $X_0, \ldots, X_n$ on $\mathbb{P}^N_S$, then the hyperplane H corresponding to q is the zero locus of the linear polynomial $q(X) = \sum_{i=1}^N a_i X_i$. Note that, since the map is surjective, at least one of the a_i's is not in the maximal ideal $\mathfrak{m}$.

The same polynomial $q(X)$ defines the hyperplane $H_\eta \subset \mathbb{P}^N_K$, corresponding to the K-point of $\mathrm{Gr}_S(N-1, N)$ (or equivalently, to a K point of $\mathrm{Gr}_K(N-1, N)$) given by $q_K\colon K^{\oplus N+1} \to E \otimes_S K$. Conversely, given a hyperplane $L \subset \mathbb{P}^N_K$, defined by a linear polynomial $p(X) = \sum_{i=1}^N \lambda_i X_i$ with $\lambda_i \in K$, its closure $\overline{L} \subset \mathbb{P}^N_S$ corresponds to the quotient $\mathscr{O}^{\oplus N+1} \to \mathscr{O}^{\oplus N+1}/(\sum_{i=1}^N \langle\lambda_i\rangle e_i \cap \mathscr{O}^{\oplus N+1})$. Explicitly, it is given by the linear polynomial $q(X)$ obtained by $p(X)$ by removing the denominators. In general, $\overline{L}$ might contain the fiber over the closed point $\mathbb{P}^N_s = \mathbb{P}^N_S \times_S k$. In this case, $\overline{L}$ does not define an S-point of $\mathrm{Gr}_S(N-1, N)$, since the $\mathscr{O}$-module $\mathscr{O}^{\oplus N+1}/(\sum_{i=1}^N \langle\lambda_i\rangle e_i \cap \mathscr{O}^{\oplus N+1})$ is not free.

4.0.1. It is convenient to have a coordinate free description. For a free $\mathscr{O}$-module of finite rank, write $\mathbb{P}_S(E)$ for the associated projective bundle over S. In this case, we will write $\mathrm{Gr}_S(E)$ for the Grassmannian of hyperplanes in $\mathbb{P}_S(E)$. An S-point of $\mathrm{Gr}_S(E)$ is a surjective map of $\mathscr{O}$-modules $q\colon E \to M$ with M free and with $N = \ker(q)$ free of rank 1. Any such q induces a closed immersion $H = \mathbb{P}_S(M) \hookrightarrow \mathbb{P}_S(E)$, and we call H the hyperplane corresponding to q (or, equivalently, to N).

4.0.2. Let E be again a free $\mathscr{O}$-module of finite rank, and let F be a submodule of E such that the quotient E/F is also free. The inclusion of F into E determines a closed immersion $\mathrm{Gr}_S(F) \hookrightarrow \mathrm{Gr}_S(E)$, see e.g. [6, III.2.7]. On S-points, the inclusion

$$\mathrm{Gr}_S(F)(S) \hookrightarrow \mathrm{Gr}_S(E)(S)$$

is explicitly given as follows. Let N be a 1-dimensional free submodule of F such that the quotient F/N is free. Since by assumption the quotient E/F is free, the quotient E/N is again free and therefore determines a point $q\colon E \to E/N$ of $\mathrm{Gr}_S(E)$. Thus sending $F \to F/N$ to $E \to E/N$ gives a well defined map on the set of S-points of the two Grassmannians.

4.1. **Specialization.** If S is the spectrum of a DVR, we have $\mathrm{Gr}_S(K) = \mathrm{Gr}_S(S)$ by the valuative criterion of properness. This gives a well defined specialization map

$$\begin{gathered}\mathrm{Gr}_S(N-1,N)(K) \to \mathrm{Gr}_k(N-1,N)(k),\\ (q\colon \mathscr{O}^{N+1} \to E) \mapsto (q\otimes_S k\colon k^{\oplus N+1} \to E\otimes_S k)\end{gathered} \tag{4.1}$$

given in coordinates by taking a polynomial $q(X) = \sum_{i=1}^N a_i X_i$ and reducing it modulo the maximal ideal of $\mathscr{O}$. In this case, we can in fact assume that every $a_i \in \mathscr{O}$ and at least one of the does not belong to $\mathfrak{m}$ (up to dividing by a suitable power of a uniformizer of $\mathscr{O}$), so that $\dim_k(E\otimes_S k) = \mathrm{rk}_S(E) = N$. When $\dim(S) > 1$, this is in general not the case, since the closure of a hyperplane L given by a K-rational point of $\mathrm{Gr}_S(N-1,N)$ might contain the whole fiber $\mathbb{P}^N_s$, and thus it cannot be specialized to a hyperplane in $\mathbb{P}^N_s$. In other words, $\mathrm{Gr}_S(N-1,N)(K) \neq \mathrm{Gr}_S(N-1,N)(S)$.

The specialization map is however always defined when we restrict to the set of S-points

$$\mathrm{sp}\colon \mathrm{Gr}_S(N-1,N)(S) \to \mathrm{Gr}_k(N-1,N)(k)$$

or, in coordinate free terms

$$\mathrm{sp}_E\colon \mathrm{Gr}_S(E)(S) \to \mathrm{Gr}_k(E)(k), \quad (E\to M) \mapsto (E\otimes_S k \to M\otimes_S k)$$

which is always surjective.

The following Lemma is a variant of an argument due to Jannsen and the second author in [10]. If S is regular and of dimension 1, the proof is easier, and it can be extracted from the proof of [10, Theorem 0.1]. If $\dim(S) > 1$, the argument is more delicate.

Lemma 4.1.1. *Let E be a free $\mathscr{O}$ module of finite rank, and let $P = \mathrm{Gr}_S(E)$. Let V be a Zariski dense open subset of P_K and let U be a Zariski dense open subset of P_s. Suppose that k is infinite. Then the set*

$$\mathrm{sp}^{-1}(U(k)) \cap V(S) \subset P(S)$$

is not empty. Here $V(S) = V(K) \cap P(S) \subset P(K)$.

Proof. Let Z be the complement of V in P_K, and let $\overline{Z}$ be the closure of Z in P. Let $\overline{Z}(S)$ be the set of S-points of $\overline{Z}$, i.e. $\overline{Z}(S) = \overline{Z}(K) \cap P(S)$. We begin by noting that it is enough to show that $\mathrm{sp}^{-1}(U(k)\setminus \mathrm{sp}(\overline{Z}(S))) \neq \emptyset$, i.e. that

$$\mathrm{sp}^{-1}(U(k)) \cap V(S) \supset \mathrm{sp}^{-1}(U(k)\setminus \mathrm{sp}(\overline{Z}(S))) \tag{4.2}$$

and that the latter is not empty. To show the inclusion (4.2), note that

$$\mathrm{sp}^{-1}(U(k)\setminus \mathrm{sp}(\overline{Z}(S))) = \mathrm{sp}^{-1}(U(k)) \setminus \mathrm{sp}^{-1}(\mathrm{sp}(\overline{Z}(S))) = \mathrm{sp}^{-1}(U(k)) \cap \mathrm{sp}^{-1}(\mathrm{sp}(\overline{Z}(S)))^c$$

where the complement of $\mathrm{sp}^{-1}(\mathrm{sp}(\overline{Z}(S)))$ is taken in $P(S)$. It is now easy to see that

$$\mathrm{sp}^{-1}(\mathrm{sp}(\overline{Z}(S)))^c \subset V(S),$$

since $V(S) = (\overline{Z}(S))^c$ and $\overline{Z}(S) \subset \mathrm{sp}^{-1}(\mathrm{sp}(\overline{Z}(S)))$. This proves (4.2).

Since sp is surjective, the set $\mathrm{sp}^{-1}(U(k) \setminus \mathrm{sp}(\overline{Z}(S)))$ is not empty if and only if $U(k) \setminus \mathrm{sp}(\overline{Z}(S))$ is not empty. Since U is open and P_s is an irreducible rational variety over an infinite field k, the set $U(k) \setminus \mathrm{sp}(\overline{Z}(S))$ is not empty as long as $\mathrm{sp}(\overline{Z}(S))$ is nowhere dense in P_s. Up to shrinking V further and choosing coordinates on P, we can assume that $Z \subset P_K$ is a hypersurface, given by a homogeneous polynomial $\sum c_I X^I$, with the obvious multi-index convention. Cleaning the denominators, we get an integral homogeneous equation $\sum a_I X^I$, with $a_I \in \mathscr{O}$, defining the closure $\overline{Z}$ of Z in P.

We can now divide the proof in two cases. Suppose first that there exists an index I such that $a_I \notin \mathfrak{m}$. Then $\overline{Z}$ intersects the special fiber P_s properly, and $\mathrm{sp}(\overline{Z}(S)) = (\overline{Z} \cap P_s)(k)$. Since $\overline{Z} \cap P_s$ is a proper closed subset of the irreducible scheme P_s, it follows that $U \setminus (\overline{Z} \cap P_s)$ is open and dense in P_s, and therefore has a k-rational point as remarked above.

Suppose now that $a_I \in \mathfrak{m}$ for every I. This is the case when $\overline{Z} \supset P_s$. Let $P(S)^o \subset P(S)$ be the subset consisting of those points $(x_0 : \ldots : x_N) \in P(S)$ such that $x_i \notin \mathfrak{m}$ for every $i = 0, \ldots, N$. Similarly, write $V(S)^o$ for the intersection $V(S) \cap P(S)^o$ and $Z(S)^o = \overline{Z}(S) \cap P(S)^o$. We have an inclusion

$$V(S) \cap \mathrm{sp}^{-1}(U(k)) \supset V(S)^o \cap \mathrm{sp}^{-1}(U(k)^o) \supset \mathrm{sp}^{-1}(U(F)^o \setminus \mathrm{sp}(\overline{Z}(S)))$$

where $U(F)^o$ is the set of F-points of the open dense subset of P_s given by $U \setminus \bigcup_{i=0}^N (X_i = 0)$. It is clear that the restriction of sp to $P(S)^o$ is also surjective, so that in order to show our claim it is enough to prove that $\mathrm{sp}(Z(S)^o)$ is nowhere dense in P_s. Let

$$n = \min\{m \in \mathbb{Z}_{\geq 0} \,|\, a_I \in \mathfrak{m}^m \text{ for every } I, \text{ but } a_J \notin \mathfrak{m}^{m+1} \text{ for some } J\} \geq 1$$

and write $A = \{I \,|a_I \notin \mathfrak{m}^{n+1}\}$. Let L be the subspace of the finite dimensional k-vector space $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ generated by the set $\{a_I\}_{I \in A}$, and choose a basis $\{b_\lambda\}_{\lambda \in \Lambda}$ for L. Note that $V \neq 0$.

For every $(x_0 : \ldots : x_N) \in Z(S)^o$ (which we can assume to be non empty, since otherwise there is nothing to prove) we have then a non trivial linear relation $\sum_{J \in \Sigma} a_J x^J = 0$ in $L \subseteq \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Spelling this out using the basis $\{b_\lambda\}_{\lambda \in \Lambda}$, we get

$$\sum_{J \in \Sigma} a_J x^J = \sum_{\lambda \in \Lambda} (\sum_{J \in \Sigma} \nu_{J,\lambda} x^J) b_\lambda = 0$$

where $a_J = \sum_{\lambda \in \Lambda} \nu_{J,\lambda} b_\lambda \in V$, for $\nu_{J,\lambda}$ in k. Since the b_λ are a basis, we have $(\sum_{J \in \Sigma} \nu_{J,\lambda} x^J) = 0$. Write W_λ for the proper closed subscheme of P_s given by $(\sum_{J \in \Sigma} \nu_{J,\lambda} X^J = 0)$. It follows from the above discussion that

$$\mathrm{sp}(Z(S)^o) \subset \bigcap_{\lambda \in \Lambda} W_\lambda(F),$$

and since $\bigcap_{\lambda \in \Lambda} W_\lambda$ is a proper closed subset of the irreducible scheme P_s, it is nowhere dense as required. □

Remark 4.1.2. The conclusions of Lemma 4.1.1 hold if we replace $\mathbb{P}^N_S$ or $\mathrm{Gr}_S(N-1, N)$ with any projective S-scheme P such that P has irreducible fibers, $P_s = P \otimes_S k$ is a rational variety, the specialization map $\mathrm{sp}\colon P(S) \to P_s(k)$ is surjective and U and V are open subsets (dense) in their fibers.

Thanks to the previous Lemma we can parametrize good hyperplanes $H \subset \mathbb{P}^N_S$ over S using subsets of the form $V(S) \cap \mathrm{sp}^{-1}(U(k))$, for an open subset V of $\mathrm{Gr}_K(N-1, N)$, giving the prescribed behavior on the generic fiber H_η, and an open subset U of $\mathrm{Gr}_k(N-1, N)$, imposing conditions on the special fiber H_s. We will call a hyperplane $H \subset \mathbb{P}^N_S$ general if it corresponds to an S-rational point of a set of the form $V(S) \cap \mathrm{sp}^{-1}(U(k)) \subset \mathrm{Gr}_S(N-1, N)(S)$. See [10, Remark 0.2.(i)]

4.2. **Constructing good sections.** We now explain how to apply the previous construction. As in the previous Section, S will denote the spectrum of a local domain $\mathscr{O}$, with function field K and residue field k. Let X be a smooth projective geometrically integral variety over K and let $\mathscr{X}$ be a model of X over S, i.e. an integral projective S-scheme which is surjective over S and such that $\mathscr{X}_\eta = X$. Let $\mathscr{D}$ be an effective Cartier divisor in $\mathscr{X}$, and suppose that $\mathscr{D}$ restricts to an effective Cartier divisor on the special fiber $\mathscr{X}_s$. Assume in this section that $\dim(X) \geq 2$.

Theorem 4.2.1. *Let $(\mathscr{X}, \mathscr{D})$ be as above, and fix a projective embedding $\mathscr{X} \subset \mathbb{P}^N_S$. If k is infinite, there exists a general (in the sense specified above) hyperplane $H \subset \mathbb{P}^N_S$ such that the intersection $H \cdot \mathscr{X} = H \times_{\mathbb{P}^N_S} \mathscr{X}$ is surjective over S, has smooth geometrically integral generic fiber $(H \cdot \mathscr{X})_\eta$ and such that $\mathscr{D} \cdot H$ is an effective Cartier divisor on $H \cdot \mathscr{X}$ which restricts to an effective Cartier divisor on the special fiber $(H \cdot \mathscr{X})_s$.*

Proof. By the classical Theorem of Bertini, [11, Theorem 6.3], there exists a dense open subset $V \subset \mathrm{Gr}_K(N-1, N)$ such that for every $H \in V(K)$, the intersection $H \cap X$ is smooth, geometrically integral, and intersects properly $D = \mathscr{D}_\eta$, i.e. $H \cap D$ is a Cartier divisor in $H \cap X$. Similarly, there exists a dense open subset U of $\mathrm{Gr}_k(N-1, N)$ such that no hyperplane L corresponding to a k-rational point of U satisfies $\mathrm{Ass}(L \cap \mathscr{X}_s) \cap |\mathscr{D}_s| \neq \emptyset$, since $\mathscr{D}_s$ is a Cartier divisor on $\mathscr{X}_s$, and therefore its support $|\mathscr{D}_s|$ does not contain any associated point of $\mathscr{X}_s$. Note that $L \cap \mathscr{X}_s \neq \emptyset$ for every hyperplane L over k, since $\dim(\mathscr{X}_s) \geq 2$.

By Lemma 4.1.1, the set $T = \mathrm{sp}^{-1}(U(k)) \cap V(S)$ is not empty. For $H \in T$, we now claim that all the other required properties are satisfied. Since $(H \cdot \mathscr{X})_s = H_s \cdot \mathscr{X}_s$ is in particular not empty, $H \cdot \mathscr{X}$ is automatically surjective over S. Let $(H \cdot \mathscr{X})^{ns} \subset H \cdot \mathscr{X}$ be the union of the irreducible components of $H \cdot \mathscr{X}$ which are not surjective over S. Note that $(H \cdot \mathscr{X})^{ns}_\eta = \emptyset$ so that we can then replace $H \cdot \mathscr{X}$ with the closure in $\mathscr{X}$ of $H \cdot \mathscr{X} \setminus ((H \cdot \mathscr{X})^{ns})$ without altering the generic fiber. So we can assume that every component of $H \cdot \mathscr{X}$ is surjective over S. But since the generic fiber $H_\eta \cdot X$ is smooth and geometrically integral, $H \cdot \mathscr{X}$ is now automatically geometrically irreducible, and generically geometrically reduced. Replacing $H \cdot \mathscr{X}$ with $(H \cdot \mathscr{X})_{\mathrm{red}}$, we can finally assume that $H \cdot \mathscr{X}$ is integral.

Since $H_s \in U(k)$, we have by construction that $\mathscr{D}$ restricts to an effective Cartier divisor on $(H \cdot \mathscr{X})_s$. Finally, note that no component $\mathscr{D}'$ of $\mathscr{D}$ can contain the generic point of $H \cdot \mathscr{X}$, since otherwise $\mathscr{D}'_s$ contains an irreducible component of $H_s \cdot \mathscr{X}_s$ and by assumption $\mathrm{Ass}(H_s \cdot \mathscr{X}_s) \cap |\mathscr{D}_s| = \emptyset$. Thus $\mathscr{D}$ restricts to a Cartier divisor on $H \cdot \mathscr{X}$. $\square$

We will need a finer version of the previous Theorem, namely an Altman-Kleiman type Bertini theorem for hypersurface sections containing a (closed) subscheme of the generic fiber. We first recall some notation.

Let F be a field and let Y be an F-scheme of finite type. Let y be a point of Y. The embedding dimension $e_y(Y)$ of Y at y is defined as the dimension of the $k(y)$ vector space $\dim_{k(y)}(\Omega^1_{Y/F,y} \otimes_{\mathcal{O}_{Y,y}} k(y))$. Following the convention of [17], for a positive integer $e > 0$ write Y_e for the locally closed subscheme of Y consisting of those $y \in Y$ such that $e_y(Y) = e$.

Theorem 4.2.2. *Let $(\mathscr{X}, \mathscr{D})$ be as above and suppose that $d = \dim(X) \geq 3$. Let Z be a closed subscheme of X, and suppose that the estimate*

$$\max_{e \leq d-1}\{e + \dim(Z_e)\} \leq d - 1 \tag{4.3}$$

holds. Let $\mathscr{Z}$ be the closure of Z in $\mathscr{X}$, and suppose moreover that $\mathscr{D}$ restricts to an effective Cartier divisor (possibly empty) on $\mathscr{Z}_s = \mathscr{Z} \times_S k$. If k is infinite, there exists a hypersurface section H of $\mathscr{X}$, of large degree, such that $H \supset \mathscr{Z}$, the generic fiber H_η is smooth and *geometrically irreducible, and such that $\mathscr{D}\cdot H$ is an effective Cartier divisor on $H\cdot\mathscr{X}$ which restricts to an effective Cartier divisor on the special fiber $(H\cdot\mathscr{X})_s$.*

Proof. Fix an embedding $\iota\colon \mathscr{X}\to\mathbb{P}^N_S$ and let $\mathcal{O}(n)=i^*\mathcal{O}_{\mathbb{P}^N_S}(n)$ for $n\geq 1$. Let $I_{\mathscr{Z}}$ be the ideal sheaf of $\mathscr{Z}$ in $\mathscr{X}$, and let $I_{\mathscr{D}}$ be the (locally principal) ideal of $\mathscr{D}$ in $\mathscr{X}$. Write $I_{\mathscr{Z}_s}$ (resp. $I_{\mathscr{D}_s}$) for the restriction of $I_{\mathscr{Z}}$ (resp. of $I_{\mathscr{D}}$) to the special fiber $\mathscr{X}_s$. Write $\mathscr{D}^1_s,\dots,\mathscr{D}^m_s$ for the irreducible components of $\mathscr{D}_s$. Finally, let $\mathscr{J}$ be the ideal sheaf of $\mathscr{X}_s$ in $\mathscr{X}$. By assumption, the restriction $I_{\mathscr{D}_s}\otimes\mathcal{O}_{\mathscr{Z}_s}$ as well as the restrictions $I_{\mathscr{D}^i_s}\otimes\mathcal{O}_{\mathscr{Z}_s}$ for $i=1,\dots,m$ are the ideal sheaves of a Cartier divisor on $\mathscr{Z}_s$. Choose n sufficiently large so that

$$H^1(\mathscr{X},\mathcal{O}(n))=H^1(\mathscr{X},I_{\mathscr{Z}}\otimes\mathcal{O}(n))=H^1(\mathscr{X},\mathscr{J}\otimes I_{\mathscr{Z}}\otimes\mathcal{O}(n))=0,\tag{4.4}$$

$$H^1(\mathscr{X},I_{\mathscr{D}\cap\mathscr{Z}}(n))=H^1(\mathscr{X}_s,I_{\mathscr{D}_s\cap\mathscr{Z}_s}(n))H^1(\mathscr{X}_s,I_{\mathscr{D}^i_s\cap\mathscr{Z}_s}(n))=0\tag{4.5}$$

for every $i=1,\dots,m$. Write E_n for the free $\mathscr{O}$-module of finite rank $H^0(\mathscr{X},\mathcal{O}(n))$ and $\tilde{E}_n$ for the finitely generated torsion free submodule $H^0(\mathscr{X},I_{\mathscr{Z}}\otimes\mathcal{O}(n))\subset E_n$. We have a commutative diagram for each i

$$\begin{tikzcd}
\tilde{E}_n \arrow[r] \arrow[d] & H^0(\mathscr{X},I_{\mathscr{Z}}/I_{\mathscr{Z}\cap\mathscr{D}}(n)) \arrow[d] \\
H^0(\mathscr{X},I_{\mathscr{Z}_s}(n)) \arrow[r] & H^0(\mathscr{X}_s,I_{\mathscr{Z}_s}/I_{\mathscr{Z}_s\cap\mathscr{D}^i_s}(n))
\end{tikzcd}\tag{4.6}$$

where every arrow is surjective (the left vertical one by (4.4) and the horizontal ones by (4.5)). Note that the last term $H^0(\mathscr{X}_s,I_{\mathscr{Z}_s}/I_{\mathscr{Z}_s\cap\mathscr{D}^i_s}(n))$ is non zero thanks to the assumption on $\mathscr{Z}_s$ and $\mathscr{D}_s$. Choose a section $s_0\in\tilde{E}_n$ such that the restrictions $s_0\mapsto\overline{s}^i_0\in H^0(\mathscr{X}_s,I_{\mathscr{Z}_s}/I_{\mathscr{Z}_s\cap\mathscr{D}^i_s}(n))$ are non zero for all i. Let F_n be a maximal free submodule of $\tilde{E}_n$ containing s_0 (i.e. F_n is a free submodule of E_n containing s_0, such that its rank $\mathrm{rk}_S(F_n)$ is maximal i.e. $\mathrm{rk}_S(F_n)=\dim_K\tilde{E}_n\otimes_S K$). Note that this is possible since $\tilde{E}_n$ is torsion free. We are going to consider S-points of the Grassmannian $\mathrm{Gr}_S(F_n)$ to parametrize our good sections. We have

$$\mathrm{Gr}_S(F_n)(S)\subset\mathrm{Gr}_S(F_n\otimes_S K)(K)\xrightarrow{\iota_K}\mathrm{Gr}_K(E_n\otimes_S K)(K)$$

and since $F_n\otimes_S K=\tilde{E}_n\otimes_S K=H^0(X,I_Z\otimes\mathcal{O}(n))$, we can identify the image of ι_K with the set

$$\mathrm{Gr}_S(F_n\otimes_S K)(K)=\{H_\eta\subset\mathbb{P}(E_n\otimes_S K)\,|\,H_\eta\supset Z\}$$

i.e. with the set of degree n hypersurface sections of X containing Z. By the Bertini theorem of Altman and Kleiman [17, Theorem 7], the estimate (4.3) implies that a general element of $\mathrm{Gr}_K(F_n\otimes_S K)(K)$ is smooth and geometrically irreducible. Let U be such open set of $\mathrm{Gr}_K(F_n\otimes_S K)$. Let

$$\mathrm{sp}\colon\mathrm{Gr}_S(F_n)(S)\to\mathrm{Gr}_k(F_n)(k)$$

be the specialization map. Let M_n be the image of $F_n\otimes_S k$ in $H^0(\mathscr{X}_s,I_{\mathscr{Z}_s}(n))$ and let $K_n=\ker F_n\otimes_S k\to H^0(\mathscr{X}_s,I_{\mathscr{Z}_s}(n))$. Then $\mathrm{Gr}_k(K_n)$ is a proper closed subspace of $\mathrm{Gr}_k(F_n\otimes_S k)$ (see 4.0.2), and the specialization map sp restricts to a surjective specialization map

$$\mathrm{sp}\colon\mathrm{Gr}_S(F_n)(S)\setminus\mathrm{sp}^{-1}(\mathrm{Gr}_k(K_n)(k))\to\mathrm{Gr}_k(M_n)(k)$$

Now, the short exact sequence

$$0\to H^0(\mathscr{X}_s,I_{\mathscr{Z}_s\cap\mathscr{D}^i_s}(n))\to H^0(\mathscr{X}_s,I_{\mathscr{Z}_s}(n))\to H^0(\mathscr{X}_s,I_{\mathscr{Z}_s}/I_{\mathscr{Z}_s\cap\mathscr{D}^i_s}(n))\to 0.$$

pulls back to a short exact sequence

$$0\to V^i_n\cap M_n\to M_n\to M_n/(V^i_n\cap M_n)\to 0$$

where V_n^i is the subspace $H^0(\mathscr{X}_s, I_{\mathscr{Z}_s\cap\mathscr{D}_s^i}(n))$ and the last term $M_n/(V_n^i\cap M_n)$ is non zero, since $s_0\in F_n$ restricts by construction to a non zero element of $H^0(\mathscr{X}_s, I_{\mathscr{Z}_s}/I_{\mathscr{Z}_s\cap\mathscr{D}_s^i}(n))$. In particular, $\mathrm{Gr}_k(V_n\cap M_n)$ defines a proper closed subspace of $\mathrm{Gr}_k(M_n)$, so that the set

$$\Phi = \mathrm{Gr}_S(F_n)(S)\setminus(\mathrm{sp}^{-1}(\mathrm{Gr}_k(K_n)(k))\cup\bigcup_{i=1}^m \mathrm{sp}^{-1}(\mathrm{Gr}_k(V_n^i\cap M_n)(k)))$$

is not empty. An element of Φ corresponds to a hypersurface section H of $\mathscr{X}$ containing $\mathscr{Z}$, and such that no component of $\mathscr{D}$ can contain its generic point, since its defining equation is non zero in

$$H^0(\mathscr{X}_s, I_{\mathscr{Z}_s}/I_{\mathscr{Z}_s\cap\mathscr{D}_s^i}(n))\subset H^0(\mathscr{X}_s, \mathcal{O}_{\mathscr{X}_s}/I_{\mathscr{D}_s^i}(n)).$$

Thus, a general (in the sense specified above) hypersurface section of $\mathscr{X}$ containing $\mathscr{Z}$ satisfies the property that $\mathscr{D}\cdot H$ is an effective Cartier divisor which restricts to a Cartier divisor on the special fiber, as in the proof of Theorem 4.2.1. Now any element in $U(S)\cap\Phi$ satisfies all the required properties. □

We now discuss another version of a Bertini-type theorem concerning sections of relative surfaces, imposing very mild conditions on the special fiber. Before that, we introduce the following Definition.

Definition 4.2.3. Let $\mathscr{C}$ be an integral scheme, proper surjective and of relative dimension 1 over S. Let D be an effective Cartier divisor on $\mathscr{C}$, and suppose that D is finite over S. Let $\mathscr{C}_\eta$ be the generic fiber of $\mathscr{C}$. We say that a closed point $x\in\mathscr{C}_\eta\setminus|D|$ satisfies the strong modulus condition over $\mathscr{O}$ (for short, x satisfies $(\mathrm{SMC})_{\mathscr{O}}$) if the closure $Z=\overline{\{x\}}$ of x in $\mathscr{C}$ satisfies $Z_s\cap|D_s|=\emptyset$.

We remark that if a closed point x satisfies $(\mathrm{SMC})_{\mathscr{O}}$, then in particular its closure Z satisfies the weaker modulus condition $(\mathrm{MC})_{\mathscr{O}}$ in the sense of Definition 3.1.1. Explicitly, for every non-zero element $\Pi\in\mathscr{O}$ we have

$$\nu_Z^*(D)\le\nu_Z^*(\Pi)$$

where $\nu_Z\colon Z^N\to Z\to\mathscr{C}$ is the composition of the normalization morphism with the inclusion $Z\subset\mathscr{C}$ and (Π) is the Cartier divisor on S defined by Π, as well as its pullback to $\mathscr{C}$. This is clear, since if x satisfies $(\mathrm{SMC})_{\mathscr{O}}$, $\nu_Z^*(D)=0$ as Weil divisor on Z^N, and $\nu_Z^*(\Pi)$ is always effective.

Proposition 4.2.4. *Let $(\mathscr{X},\mathscr{D})$ be as in 4.2, suppose moreover that $\mathscr{X}$ has dimension 2 over S and that k is infinite. Fix an embedding $\mathscr{X}\hookrightarrow\mathbb{P}_S(E)$ for a free $\mathscr{O}$-module E. Let Z be a purely 1 dimensional closed subscheme of X, let $\mathscr{Z}$ be its closure in $\mathscr{X}$ and suppose that $\mathscr{D}_s\cap\mathscr{Z}_s$ is a finite set of closed points. Fix an open subset $V\subset\mathrm{Gr}_K(E\otimes_S K)$. Then a general section $H\in\mathrm{Gr}_S(E)(S)$ satisfies the following conditions*

- *i) The generic fiber H_η belongs to $V(K)$, when we see the hyperplane H as represented by an S-point of $\mathrm{Gr}_S(E)$.*
- *ii) The intersection $H\cdot\mathscr{X}$ is surjective over S, and $\mathscr{D}\cdot H$ is an effective Cartier divisor D on $H\cdot\mathscr{X}$ which is finite over S.*
- *iii) Every $x\in H_\eta\cap Z$ satisfy $(\mathrm{SMC})_{\mathscr{O}}$ with respect to D.*

Proof. By assumption, $\mathscr{D}$ restricts to an effective Cartier divisor on the special fiber $\mathscr{X}_s$. Let $|\mathscr{D}_s|$ be its support. It is a 1 dimensional closed subscheme of $\mathscr{X}_s$, and in particular it does not contain any associated point of $\mathscr{X}_s$. Thus, there is an open subset $U'\subset\mathrm{Gr}_k(E\otimes_S k)$ such that for every $L\in U'(k)$, we have $\mathrm{Ass}(L\cap\mathscr{X}_s)\cap|\mathscr{D}_s|=\emptyset$, as in the proof of Theorem 4.2.1.

In particular, $\mathscr{D}_s$ restricts to a Cartier divisor on $L \cap \mathscr{X}_s$ for every such L, and its support is therefore a finite set of closed points.

In a similar way, we have by assumption that $\mathscr{D}_s \cap \mathscr{Z}_s = W \subset \mathscr{Z}_s$ consists of finitely many closed points of $\mathscr{X}_s$. Let U'' be the subset of $\mathrm{Gr}_k(E \otimes_S k)$ such that for every $L \in U''(k)$, $L \cap W = \emptyset$. Since W is zero dimensional, U'' is open and dense in $\mathrm{Gr}_k(E \otimes_S k)$. Let $U = U' \cap U''$. Now, the set $T = \mathrm{sp}^{-1}(U(k)) \cap V(S)$ is not empty by Lemma 4.1.1 and we claim that every $H \in T$ satisfies all the required conditions. The item (i) is satisfied by definition. All the properties in (ii) are achieved thanks to Theorem 4.2.1, apart from the finiteness of D over S, which follows from Zariski's Main Theorem, noting that the fiber of D over the closed point of S is finite and not empty.

Finally, for every $x \in H_\eta \cap Z$, note that the closure $\overline{\{x\}}$ of x in $\mathscr{X}$ is contained in $\mathscr{Z}$, so that $\overline{\{x\}} \cap |\mathscr{D}_s| \subset W$. But $\overline{\{x\}} \subset H \cdot \mathscr{X}$ as well, and since by choice $H_s \in U''(k)$, we must have $\overline{\{x\}} \cap |\mathscr{D}_s| = \emptyset$ giving the required strong modulus condition over $\mathscr{O}$. □

Remark 4.2.5. In this Section, we have assumed in every statement that the residue field k of S is infinite to guarantee the existence of k-rational points in dense open subsets $U \subset \mathrm{Gr}_k(E \otimes_S k)$ of the restriction to k of a Grassmannian $\mathrm{Gr}_S(E)$ for some free $\mathscr{O}$-module E (of finite rank).

If k is finite, this is the case over the maximal pro ℓ-extension of k for every prime number ℓ, hence over some extension k'/k of degree a power of ℓ. If the ring $\mathscr{O}$ is moreover assumed to be Henselian (and this is the case in our applications), let $\mathscr{O}'$ be the unramified extension of $\mathscr{O}$ corresponding to k'/k and let $S' = \mathrm{Spec}\,(\mathscr{O}')$. We have then a surjective specialization map $\mathrm{sp}\colon \mathrm{Gr}_{S'}(E \otimes_S S')(S') \to \mathrm{Gr}_{k'}(E \otimes_S k')(k')$, and we can lift k'-rational points to S'-rational points.

In other words, we can find good hyperlane sections (in the sense specified above) for $\mathscr{X}$ over S'. Note anyway that the results in this Section will be applied in Section 5 below only for fields of characteristic 0, which are of course infinite.

5. Reductions of the proof

This section contains the proof of the main Theorem, up to a Claim (namely Claim 5.4.5), whose proof is given in Section 6. We begin with some injectivity results under blow-ups.

5.1. Injectivity result for blowups.

Proposition 5.1.1. *Assume $ch(k) = 0$. Take $F \in \mathbf{CI}^\tau$ and $\mathscr{S} = (\overline{S}, S_\infty) \in \underline{\mathbf{M}}\mathbf{Cor}$ with $S = \overline{S} - |S_\infty|$. Let $\phi : \overline{S}' \to \overline{S}$ be a proper birational morphism such that $S' = \phi^{-1}(S) \in \mathbf{Sm}$ so that $\mathscr{S}' = (\overline{S}', S'_\infty) \in \underline{\mathbf{M}}\mathbf{Cor}$ with $S'_\infty = S_\infty \times_{\overline{S}} \overline{S}'$. Then $\phi^* : F(\mathscr{S}) \to F(\mathscr{S}')$ is injective.*

We need some preliminaries.

Lemma 5.1.2. *Take $F \in \mathbf{CI}^\tau \cap \underline{\mathbf{M}}\mathbf{NST}$ and $\mathscr{S} = (\overline{S}, S_\infty) \in \underline{\mathbf{M}}\mathbf{Cor}$ such that $\overline{S} \in \mathbf{Sm}$. Let $X \in \mathbf{Sm}$ and $U \subset X$ be a dense open subset. Then $F(X \times_k \mathscr{S}) \to F(U \times_k \mathscr{S})$ is injective.*

Proof. This is shown by a slight modification of the proof of [23, Theorem 3.1]. The question is Zariski local and hence we may assume that $\overline{S}$ is affine and that $U = X - Z$ for an effective Cartier divisor Z on X such that (X, Z) is a V-pair over some $B \in \mathbf{Sm}$ in the sense of [23, Def.2.1]. Then $(X \times_k \overline{S}, Z \times_k \overline{S})$ is a V-pair over $B \times_k \overline{S}$. Then the desired injectivity follows from [23, Theorem 2.10(1)]. □

Lemma 5.1.3. *Take $F \in \mathbf{CI}^\tau \cap \underline{\mathbf{M}}\mathbf{NST}$ and $\mathscr{S} = (\overline{S}, S_\infty) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. Let $\overline{Z} \subset \overline{S}$ be a regular closed subscheme which intersects transversally with $|S_\infty|$. Let $\phi : \overline{S}' \to \overline{S}$ be the blowup of $\overline{S}$ along $\overline{Z}$ and $S'_\infty = S_\infty \times_{\overline{S}} \overline{S}'$ and $\mathscr{S}' = (\overline{S}', S'_\infty) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. Then $\phi^* : F(\mathscr{S}) \to F(\mathscr{S}')$ is injective.*

Proof. Note that the question is Nisnevich local. Take a point $x \in \overline{Z}$. By the perfectness of k, after replacing $\overline{Z}$ by its étale neighborhood of x, $\overline{Z} \to \operatorname{Spec} k$ factors through a map $\overline{Z} \to \operatorname{Spec} k(x)$ which splits $x \to \overline{Z}$ (see [23, Lemma 7.21]). Moreover the assumption implies that there exists a smooth scheme W over $k(x)$ such that Nisnevich locally at x, we have isomorphisms

$$(\overline{S}, S_\infty) \simeq (\overline{Z}, Z_\infty) \times_{k(x)} W \ \text{ and } \ (\overline{S}', S'_\infty) \simeq (\overline{Z}, Z_\infty) \times_{k(x)} W',$$

where W' is the blowup of W at the closed point w such that x corresponds to (x, w) in the first isomorphism. Thus it suffices to show the injectivity of

$$F(\mathfrak{Z} \times_{k(x)} W) \to F(\mathfrak{Z} \times_{k(x)} W'), \tag{5.1}$$

where $\mathfrak{Z} = (\overline{Z}, Z_\infty)$ with $Z_\infty = S_\infty \times_{\overline{S}} \overline{Z}$. Letting $E \subset W'$ be the exceptional divisor, the composite of (5.1) and

$$F(\mathfrak{Z} \times_{k(x)} W') \to F(\mathfrak{Z} \times_{k(x)} (W' - E)) = F(\mathfrak{Z} \times_{k(x)} (W - w))$$

is injective by Lemma 5.1.2. This completes the proof. □

Proof of Proposition 5.1.1: Let $\phi : \overline{S}' \to \overline{S}$ be as in the proposition. By [9] there exists a proper birational morphism $\psi : \overline{S}'' \to \overline{S}$ such that $\overline{S}''$ is smooth and $S''_\infty := S_\infty \times_{\overline{S}} \overline{S}''$ is a SNCD on $\overline{S}''$. Then the induced map $(\overline{S}'', S''_\infty) \to (\overline{S}, S_\infty)$ is an isomorphism in $\underline{\mathbf{M}}\mathbf{Cor}$. Hence we may assume $\mathscr{S} \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. By resolution of marked ideals ([1, the case $d = 1$ of Th.1.3]), there exists a proper birational morphism $\psi : \overline{S}'' \to \overline{S}$ which factors through ϕ and is the composite of morphisms

$$\overline{S}'' = \overline{S}_n \to \cdots \to \overline{S}_2 \to \overline{S}_1 \to \overline{S}_0 = \overline{S},$$

where $\overline{S}_{i+1} \to \overline{S}_i$ is a blowup in a smooth center $\overline{Z}_i \subset \overline{S}_i$ which is normal crossing to the total transforms $S_{\infty,i} \subset \overline{S}_i$ of S_∞. Here we say that a smooth subscheme $\overline{Z}_i \subset \overline{S}_i$ is normal crossing with $S_{\infty,i}$ (which is a simple normal crossing divisor on $\overline{S}_i$ by the construction) if for any point x of $\overline{S}_i$, there exists a system $z_1, \cdots, z_d$ of regular parameters of $\overline{S}_i$ at x satisfying the following conditions:

- Locally at x, $Z_i = \{z_1 = \cdots z_r = 0\}$ for some $1 \le r \le d$.
- Locally at x, $S_{\infty,i} = \{z_j = 0,\ j \in J\}$ for some $J \subset \{1, \cdots, d\}$.

(Note the intersection of J and $\{1, ..., r\}$ may be non-empty. For example Z_i may be contained in $S_{\infty,i}$). Then $\psi^* : F(\mathscr{S}) \to F(\mathscr{S}'')$ is injective by Lemma 5.1.3, where $\mathscr{S}'' = (\overline{S}'', S''_\infty)$ with the total transform S''_∞ of S_∞. This implies the desired injectivity of $\phi^* : F(\mathscr{S}) \to F(\mathscr{S}')$.

5.2. **Relative correspondences.** In order to prove our main result, we need to slightly generalize the notion of admissible correspondence to schemes that are not necessarily smooth and of finite typer over a field.

We fix $\mathscr{S} = (S, E) \in \underline{\mathbf{M}}\mathbf{Cor}$ such that S is integral and K be the field of functions of S. *An S-modulus pair* is a pair $\mathfrak{X} = (\mathscr{X}, \mathscr{D})$ consisting of an S-scheme $f\colon \mathscr{X} \to S$, separated and of finite type over S, and an effective Cartier divisor $\mathscr{D}$ on it such that the complement $\mathscr{X}^\circ = \mathscr{X} \setminus \mathscr{D}$ is generically regular i.e. $\mathscr{X}^\circ_K = \mathscr{X}^o \times_S K$ is a regular K-scheme. Let $\mu\colon \mathscr{X}^N \to \mathscr{X}$ be the normalization morphism, and let $\nu\colon \mathscr{X}^N \to S$ be the composition $f \circ \mu$.

We say that $\mathfrak{X}$ is *admissible for $\mathscr{S}$* if there is an inequality of Cartier divisors

$$\mu^*(\mathscr{D}) \ge \nu^*(E).$$

A basic example of S-modulus pair which is admissible for $\mathscr{S}$ is a map $\mathfrak{X} \to \mathscr{S}$ in $\underline{\mathbf{M}}\mathbf{Cor}$, in particular the relative cube

$$\overline{\square}_{\mathscr{S}} = (\mathbb{P}^1_S, 1_S + \mathbb{P}^1_E).$$

Let $\mathfrak{X}_i = (\mathscr{X}_i, \mathscr{D}_i)$ with $i = 1, 2$ be S-modulus pairs. Assume that $\mathfrak{X}_1$ is $\mathscr{S}$-admissible and that $\mathfrak{X}_2$ is proper (i.e. that $\mathscr{X}_2$ is proper over S). We define the group of *admissible $\mathscr{S}$-correspondences* as the subgroup

$$\underline{\mathbf{M}}\mathbf{Cor}_{\mathscr{S}}(\mathfrak{X}, \mathfrak{X}) \subset \mathbf{Cor}_K(\mathscr{X}_1^\circ \times_S K, \mathscr{X}_2^\circ \times_S K) \tag{5.2}$$

generated by the finite prime correspondences $V \in \mathbf{Cor}_K(\mathscr{X}_1^\circ \times_S K, \mathscr{X}_2^\circ \times_S K)$ satisfying the condition

$$\nu_V^*(\mathscr{D}_2) \leq \nu_V^*(\mathscr{D}_1)$$

where $\overline{V}$ is the closure of V in $\mathscr{X}_1 \times_S \mathscr{X}_2$ and $\nu_V \colon \overline{V}^N \to \mathscr{X}_1 \times_S \mathscr{X}_2$ is the composition of the (finite) normalization map with the inclusion.

Example 5.2.1. Let $\mathfrak{X} = (\mathscr{X}, \mathscr{D})$ be a proper S-modulus pair, and let $X^\circ = \mathscr{X}^\circ \times_S K$ be the generic fiber of the complement of $\mathscr{D}$. By the definition of (5.2), we get that

$$\underline{\mathbf{M}}\mathbf{Cor}_{\mathscr{S}}(\mathscr{S}, \mathfrak{X}) \subset \mathscr{Z}_0(X^\circ)$$

is the subgroup of the group $\mathscr{Z}_0(X^\circ) = \mathbf{Cor}_K(\mathrm{Spec}\,(K), X^\circ)$ of zero cycles on X generated by closed points $x \in X \subset \mathscr{X}$ satisfying

$$\nu_V^*(\mathscr{D}) \leq \nu_V^*(E), \tag{5.3}$$

where $\overline{V} \subset \mathscr{X}$ is the closure of x and $\nu_V \colon \overline{V}^N \to \mathscr{X}$ is the composition of the (finite) normalization map with the inclusion. This is a generalization of the modulus condition $(\mathrm{MC})_{\mathscr{O}}$ introduced in Definition 3.1.1. In what follows we let $(\mathrm{MC})_{\mathscr{O}}$ mean (5.3) also in the generalized context.

Let $\mathfrak{X}$ be a proper S-modulus pair and $\mathscr{S}' = (S', E') \to \mathscr{S}$ be a map in $\underline{\mathbf{M}}\mathbf{Cor}$ induced by a birational morphism $S' \to S$, we define $h_0^{\overline{\square}}(\mathfrak{X})(\mathscr{S}')$ as the cokernel

$$\underline{\mathbf{M}}\mathbf{Cor}_{\mathscr{S}}(\mathscr{S}' \otimes \overline{\square}, \mathfrak{X}) \xrightarrow{i_0^* - i_\infty^*} \underline{\mathbf{M}}\mathbf{Cor}_{\mathscr{S}}(\mathscr{S}', \mathfrak{X}) \tag{5.4}$$

where $\mathscr{S} \otimes \overline{\square} = \overline{\square}_{\mathscr{S}}$ and the map $i_0^* - i_\infty^*$ is induced by

$$\mathbf{Cor}_K(\mathbb{P}^1_K - \{1\}, \mathscr{X}_K^\circ) \xrightarrow{i_0^* - i_\infty^*} \mathbf{Cor}_K(\mathrm{Spec}\,(K), \mathscr{X}_K^\circ) = \mathscr{Z}_0(X^\circ)$$

Note that the modulus condition is preserved under i_0^* and i_∞^*, thanks to the containment Lemma (see e.g. [19, 2.2]).

If $\mathfrak{X} = (\overline{X} \times_k S, X_\infty \times_k S)$ for a proper modulus pair $(\overline{X}, X_\infty) \in \mathbf{MCor}$, this recovers precisely the definition in Section 2.

Remark 5.2.2. The notation $\underline{\mathbf{M}}\mathbf{Cor}_{\mathscr{S}}$ is suggesting that the group of admissible S-correspondences can be taken as group of morphisms in an additive category of S-modulus pairs (possibly after further restrictions). We do not need to investigate this point further in this paper. In particular, we do not claim that our definition is closed under composition.

5.3. **A reformulation in terms of algebraic cycles.** Assume now that k is again a field of characteristic 0 (this is to apply Proposition 5.1.1). Let $(X, D) \in \mathbf{MCor}$ a proper modulus pair over k. Assume that the total space X is projective. Consider $a_{\mathrm{Nis}} h_0^{\overline{\square}}(\mathfrak{X})$, where a_{Nis} : $\mathbf{MPST} \to \underline{\mathbf{M}}\mathbf{NST}$ is the left adjoint of the inclusion functor $\mathbf{MNST} \to \mathbf{MPST}$ from [12] (see section 2.1 for details). Recall the statement of Theorem 2.1.3.

Theorem 5.3.1. *For $\mathfrak{X}$ as above, the sheaf $a_{\mathrm{Nis}} h_0^{\overline{\square}}(\mathfrak{X})$ is semi pure, i.e. that the natural map*

$$a_{\mathrm{Nis}} h_0^{\overline{\square}}(\mathfrak{X}) \to \omega^* \omega_! a_{\mathrm{Nis}} h_0^{\overline{\square}}(\mathfrak{X})$$

is an injective morphism in $\mathbf{MPST}$.

The assertion follows from the injectivity of the map

$$j_{\mathscr{S}}\colon h_0^{\overline{\square}}(\mathfrak{X})(\mathscr{S}) \to h_0^{\overline{\square}}(\mathfrak{X})(K,\emptyset) \text{ for } \mathscr{S} = (S,E),$$

where S is the spectrum of a local integral domain with the function field K and E is an effective Cartier divisor on S.

Now we introduce the following terminology. Let $\phi : \tilde{S} \to S$ be a proper birational morphism such that $\phi^{-1}(S-E)$ is smooth and let S' be the product of the all localizations of $\tilde{S}$. Then $(\mathscr{X} \times_S S', \mathscr{D} \times_S S', S')$ is called *an admissible replacement of* $(\mathscr{X}, \mathscr{D}, S)$. By Proposition 5.1.1 (this is where the assumption on the characteristic is used) it suffices to show that for $\alpha \in \mathrm{Ker}(j_{\mathscr{S}})$, there exists $(\mathscr{X}', \mathscr{D}', S')$ obtained by a finite number of admissible replacements from $(\mathscr{X}, \mathscr{D}, S)$ such that the image of α in $h_0^{\overline{\square}}(\mathfrak{X}')(\mathscr{S}')$ vanishes, where $\mathscr{S}' = (S', E \times_S S')$ and $\mathfrak{X}' = (\mathscr{X}', \mathscr{D}')$. Here we extend the statement of the proposition to the case of modulus pairs over an (infinite) products of local domains in an obvious way. We prove it in the following generalized form: Let $\mathscr{S} = (S,E)$ be as above. Let $\mathfrak{X} = (\mathscr{X}, \mathscr{D})$ be a proper modulus pair over S satisfying the following conditions:

(a) $\mathscr{X}_K$ is smooth over K and $\mathscr{X}$ is projective over S.
(b) $\mathscr{X}$ and $\mathscr{D}$ are equidimensional over S.

Note that these condition are preserved by admissible replacements. Consider the natural map

$$j_{\mathscr{S}}\colon h_0^{\overline{\square}}(\mathfrak{X})(\mathscr{S}) \to h_0^{\overline{\square}}(\mathfrak{X})(K,\emptyset). \tag{5.5}$$

where $h_0^{\overline{\square}}(\mathfrak{X})(\mathscr{S})$ and $h_0^{\overline{\square}}(\mathfrak{X})(K,\emptyset)$ are defined as (5.4).

Claim 5.3.2. *Let $\mathfrak{X} = (\mathscr{X}, \mathscr{D})$ be as above. For $\alpha \in \mathrm{Ker}(j_{\mathscr{S}})$ there exists $(\mathscr{X}', \mathscr{D}', S')$ obtained by a finite number of admissible replacements from $(\mathscr{X}, \mathscr{D}, S)$ such that the image of α in $h_0^{\overline{\square}}(\mathfrak{X}')(\mathscr{S}')$ vanishes.*

5.4. **Proof of Claim 5.3.2.** First we note the following.

Proposition 5.4.1. *Let $\mathfrak{X}$ be as in Claim 5.3.2. Then*

$$h_0^{\overline{\square}}(\mathfrak{X})(K,\emptyset) = \mathrm{CH}_0(\mathscr{X}_K|\mathscr{D}_K).$$

Proof. See [3, Section 3] for a comparison between the groups $\mathrm{CH}^r(\mathscr{X}_K|\mathscr{D}_K, 0)$, defined by means of the cubical cycle complex and the relative Chow groups $\mathrm{CH}^r(\mathscr{X}_K|\mathscr{D}_K)$ defined in terms of divisors of rational functions. This, together with the definition of $h_0^{\overline{\square}}(\mathfrak{X})(K,\emptyset)$ immediately gives the proof. $\square$

Let the assumption be as in Claim 5.3.2. Take $\alpha \in \ker(j_{\mathscr{S}})$. By definition α is the class of a zero-cycle

$$\alpha = \sum_{j=1}^{N} m_j [x_j]$$

for closed points

$$x_j \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathscr{S}}(\mathscr{S}, \mathfrak{X}) \subset \mathcal{Z}_0(\mathscr{X}_K^\circ) \ \ (\mathscr{X}^\circ = \mathscr{X} - |\mathscr{D}|),$$

i.e. satisfying the condition $(\mathrm{MC})_{\mathscr{O}}$ (cf. (5.3)). According to the Definition (see [16]), $j_{\mathscr{S}}(\alpha) = 0$ in $\mathrm{CH}_0(\mathscr{X}_K, \mathscr{D}_K)$ means that there exist integral curves $C_1, \ldots, C_n$ contained in $\mathscr{X}_K$, and rational functions f_i on C_i for $1 \le i \le n$, which *satisfy the modulus condition with respect to* $\mathscr{D}_K$, i.e.

$$f_i \in G(C_i^N, C_{i,\infty}) := \bigcap_{x \in C_{i,\infty}} \ker(\mathcal{O}^\times_{C_i^N,,x} \to \mathcal{O}^\times_{C_{i,\infty},,x}), \tag{5.6}$$

where C_i^N is the normalization of C_i and $C_{i,\infty} = C_i^N \times_{\mathscr{X}_K} \mathscr{D}_K$, such that

$$\alpha = \sum_{j=1}^{N} m_j[x_j] = \sum_{i=1}^{n} \mathrm{div}_{C_i}(f_i) = \sum_{i=1}^{n} \nu_{i,*}\mathrm{div}_{C_i^N}(f_i),$$

where $\nu_i \colon C_i^N \to C_i \subset \mathscr{X}_K$ is the composition of the normalization map with the inclusion.

Write C for the union of the C_i's and $\alpha = \mathrm{div}_C(f)$, for $f \in K(C)^\times = \prod_{i=1}^n K(C_i)^\times$ a meromorphic function restricting to f_i on the integral component Z_i.

Lemma 5.4.2. *Assume S is local and* $\dim(\mathscr{X}_K) = 1$. *Then the map* (5.5) *is injective.*

Proof. By the above description of $\alpha \in \ker(j_{\mathscr{S}})$, the lemma follows from Theorem 3.1.2. □

Remark 5.4.3. To apply Theorem 3.1.2 to the proof of the above lemma, we may weaken the condition (b) only assuming that $\mathscr{D}$ is finite over S (without assuming that $\mathscr{X}$ is equidimensional over S).

5.4.1. *Step 1.* Assume $\dim(\mathscr{X}_K) \geq 2$. Let C_S be the closure of C in $\mathscr{X}$ and C'_S be its normalization. Since $C'_S \to C_S$ is finite (so in particular projective), there is a closed embedding $C'_S \hookrightarrow \mathscr{X}' = \mathscr{X} \times_S \mathbb{P}^N_S$ for $N > 0$. Put $C' = C'_S \times_S K$. By definition C' is a regular curve over K. Put $\mathscr{D}' = \mathscr{D} \times_S \mathbb{P}^N_S \subset \mathscr{X}'$ and write $\mathfrak{X}'$ for the modulus pair $(\mathscr{X}', \mathscr{D}')$ over S. We have an induced morphism

$$\rho \colon \mathfrak{X}' \to \mathfrak{X},$$

which yields a commutative diagram

$$\begin{array}{ccc}
h_0^{\overline{\square}}(\mathfrak{X}')(\mathscr{S}) & \xrightarrow{j'_{\mathscr{S}}} & \mathrm{CH}_0(\mathscr{X}'_K|\mathscr{D}'_K) \\
\downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_{K,*}} \\
h_0^{\overline{\square}}(\mathfrak{X})(\mathscr{S}) & \xrightarrow{j_{\mathscr{S}}} & \mathrm{CH}_0(\mathscr{X}_K|\mathscr{D}_K).
\end{array}$$

Claim 5.4.4. *There exists $\alpha' \in \mathrm{Ker}(j'_{\mathscr{S}})$ such that $\rho_*(\alpha') = \alpha$.*

Proof. We have a commutative diagram

$$\begin{array}{ccc}
\underline{\mathbf{M}}\mathbf{Cor}_{\mathscr{S}}(\mathscr{S}, \mathfrak{X}') & \xrightarrow{\iota'} & \mathcal{Z}_0(\mathscr{X}'^{\circ}_K) \\
\downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_{K,*}} \\
\underline{\mathbf{M}}\mathbf{Cor}_{\mathscr{S}}(\mathscr{S}, \mathfrak{X}) & \xrightarrow{\iota} & \mathcal{Z}_0(\mathscr{X}^{\circ}_K)
\end{array}$$

where $\mathscr{X}'^{\circ} = \mathscr{X}' - |\mathscr{D}'|$. By our assumption, $\alpha = \rho_{K,*}\mathrm{div}_{C'}(f') \in \mathrm{Im}(\iota)$, where f' is the rational function on C' induced by f on C noting that $K(C)^\times = K(C')^\times$. Notice that each rational function f_i on C_i induces one on a component C'_i of C' lying over C_i and it satisfies the modulus condition with respect to $\mathscr{D}'_K$. Indeed, we have a commutative diagram

$$\begin{tikzcd}
C_i^N \arrow[r, "\cong"] \arrow[dr, "\nu_i"'] & C'_i \arrow[d] \\
& C_i
\end{tikzcd}$$

where the vertical map is induced by the normalization map, and the horizontal map is an isomorphism by definition. Moreover we have $C_i^N \times_{\mathscr{X}'_K} \mathscr{D}'_K = C_i^N \times_{\mathscr{X}_K} \mathscr{D}_K$.

A priori, $\mathrm{div}_{C'}(f')$ does not belong to the image of ι', since there may be new cancellations. Namely, there may be distinct closed points y and y' in support of $\mathrm{div}_{C'}(f')$ such that $x =$

$\rho_K(y) = \rho_K(y')$ and $x \notin |\alpha|$. To remedy this, let $x \in C \setminus \mathscr{D}_K$, and write $y_1, \dots, y_r$ for the set $C' \cap \rho_K^{-1}(x)$. Let γ_x be the 0-cycle on $\mathscr{X}'_K$ given by

$$\gamma_x = \sum_{j=1}^{r} v^j_{C'_K}(f')[y_j]$$

where $v^j_{C'_K}(f')$ is the order of f' at the point y_j. Note that we can assume that this number is well defined, since each y_j lies exactly in one component of C' since C' is regular.

We can now choose a chain of rational lines $L_\nu \cong \mathbb{P}^1_{K'}$ and rational functions $g_\nu \in K(L_\nu)^\times = K'(t)$ (for some finite field extension K'/K) such that $\gamma_x = \sum \mathrm{div}_{L_\nu}(g_\nu)$. Note that this is possible since each L_ν lives inside $\mathbb{P}^N_x$, a projective space lying over x. By construction, each function g_ν automatically satisfies the modulus condition with respect to $\mathscr{D}'_K$, since $x \notin \mathscr{D}_K$ so that $L_\nu \cap \mathscr{D}'_K = \emptyset$, and we have

$$\rho_{K,*}(\gamma_x) = \rho_{K,*}(\sum \mathrm{div}_{L_\nu}(g_\nu)) = 0.$$

Repeat the argument for every such x. Consider $C'' = C' \cup \bigcup_\nu L_\nu$ and rational functions $(f_1, \dots, f_n, (g_\nu)_\nu)$ on C''. We have $e_x(C'') \le 2$ for each closed point $x \in C''$. By the construction the functions f_i on C'_i as well as g_ν on L_ν satisfy the modulus condition with respect to $\mathscr{D}'_K$. Then the cycle

$$\alpha' = \sum_{1 \le i \le n} \mathrm{div}_{C'_i}(f_i) - \sum_x \gamma_x \in \mathscr{Z}_0(\mathscr{X}'^\circ),$$

lies in the kernel of $\mathscr{Z}_0(\mathscr{X}'^\circ) \to \mathrm{CH}_0(\mathscr{X}'|\mathscr{D}'_K)$ and its image in $\mathscr{Z}_0(\mathscr{X}^\circ_K)$ via $\rho_{K,*}$ agrees with α. We also have $\alpha' \in \mathrm{Im}(\iota')$. Indeed, take $x \in |\alpha'|$ and $W \subset \mathscr{X}'$ be its closure. By the above construction $y = \rho_K(x) \in |\alpha|$ and $V = \rho(W)$ is the closure of y. Since $W \to V$ is proper, the condition (5.3) for V implies that for W. This completes the proof of the claim. □

By Claim 5.4.4, for the proof of Claim 5.3.2, we may replace $\mathfrak{X}, \alpha, C, (f_1, \dots, f_n)$ with $\mathfrak{X}', \alpha', C'', (f_1, \dots, f_n, (g_\nu)_\nu)$ respectively, and assume the following

(c) The embedding dimension $e_x(C) \le 2$ for all $x \in C$.

5.4.2. *Step 2.* In this step we assume the condition (c). Write f for the rational function on C defined by the functions $f_1, \dots, f_n$ on each component C_i of C. We introduce the following two sets

$$T_0 = \{x \in \mathscr{X}_K | f \in \mathfrak{m}_x \mathcal{O}_{C,x}\}, \quad \widetilde{T_\infty} = \{x \in \mathscr{X}_K | f \notin \mathcal{O}^\times_{C,x}\} - T_0$$

Let C_{sing} be the set of singular points of C, and let $T_\infty = C_{sing} \cup \widetilde{T_\infty}$. Let $T_{\infty,S} \subset \mathscr{X}$ be its closure. By [22, Cor. 5.7.10] we can find a proper birational morphism $S' \to S$ such that $C_S \times_S S'$ (resp. $T_{\infty,S} \times_S S'$) is equi-dimensional of relative dimension one (resp. finite) over S. For the proof of Claim 5.3.2, we may replace S by S'_x for a point $x \in S'$ and $\mathscr{X}$ and $\mathscr{D}$ by $\mathscr{X} \times_S S'_x$ and $\mathscr{D} \times_S S'_x$, and assume the following

(d) S is local and C_S and the closure $T_{\infty,S}$ of T_∞ are equi-dimensional over S.

(Note that the equi-dimensionality of $\mathscr{X}$ and $\mathscr{D}$ is preserved by the base change $S' \to S$.)

By an iterated application of Theorem 4.2.2, we can find a relative surface $\mathscr{H}$ over S (i.e. $\dim_S(\mathscr{H}_K) = 2$), containing C_S and satisfying the following properties

(3) $\mathscr{H}$ is integral and equi-dimensional over S, and the generic fiber H_K is a smooth projective geometrically integral K-surface.
(4) $\mathscr{D}_H = \mathscr{D} \cap \mathscr{H}$ is an effective Cartier divisor on $\mathscr{H}$, which is equi-dimensional over S.

Now we can consider the modulus pair $\mathfrak{H} = (\mathscr{H}, \mathscr{D}_H)$ over S. We have an induced morphism $i\colon \mathfrak{H} \to \mathfrak{X}$, which yields a commutative diagram

$$\begin{tikzcd} h_0^{\overline{\square}}(\mathfrak{H})(\mathscr{S}) \arrow[r, "j^H_{\mathscr{S}}"] \arrow[d, "i_*"] & \mathrm{CH}_0(H|D_H) \arrow[d, "i_{K,*}"] \\ h_0^{\overline{\square}}(\mathfrak{X})(\mathscr{S}) \arrow[r, "j_{\mathscr{S}}"] & \mathrm{CH}_0(\mathscr{X}_K|\mathscr{D}_K). \end{tikzcd}$$

where and $\mathscr{D}_H \times_S K = D_H$. By the construction $\alpha \in \mathrm{Ker}(j_{\mathscr{S}})$ comes via i_* from an element of $\alpha^H \in \mathrm{Ker}(j^H_{\mathscr{S}})$. Hence, for the proof of Claim 5.3.2, we may replace $\mathfrak{X}, \alpha, C$ with $\mathfrak{H}, \alpha^H, C$ respectively.

5.4.3. By summarizing what we have done in the above reduction steps, we are now given a proper modulus pair $\mathfrak{X} = (\mathscr{X}, \mathscr{D})$ over S which is local and a pair (Z, f) of a curve Z on $X = \mathscr{X}_K$ satisfying the following conditions:

- (♠1) $\mathscr{X}$ is equi-dimensional of relative dimension 2 over S and $\mathscr{D} \subset \mathscr{X}$ is an effective Cartier divisor equi-dimensional of relative dimension 1 over S.
- (♠2) Z is a SNCD on $X = \mathscr{X}_K$. The closure $\mathscr{Z}$ of Z in $\mathscr{X}$ is equi-dimensional over S and the closure of the set T_∞ of non-zero and non-regular points of f is finite over S.

Let $Z_1, \dots, Z_n$ be the irreducible components of Z. We are then given rational functions f_i on Z_i such that each f_i on Z_i satisfies the modulus condition with respect to $D = \mathscr{D}_K$ and that writing

$$\alpha = \sum_{i=1}^{n} \mathrm{div}_{Z_i}(f_i) = \sum_{j=1}^{N} m_j[x_j] \ \in \mathscr{Z}_0(X^\circ) \ \ (X^\circ = X - |D|),$$

we have $x_j \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathscr{S}}(\mathscr{S}, \mathfrak{X})$, i.e. x_j satisfies the modulus condition (5.3).

The following claim will be proved in the next section.

Claim 5.4.5. *Under the above assumption, there exists integral curves $\tilde{Z}_1, \dots, \tilde{Z}_m$ on X and rational functions g_ν on $\tilde{Z}_\nu$ for $1 \le \nu \le m$ satisfying the conditions:*

- (♣1) $\alpha = \sum_{\nu=1}^{m} \mathrm{div}_{\tilde{Z}_\nu}(g_\nu)$,
- (♣2) *for each ν, g_ν as a rational function on the normalization $\tilde{Z}_\nu^N$ of $\tilde{Z}_\nu$, satisfies the modulus condition with respect to $D \times_X \tilde{Z}_\nu^N$.*
- (♣3) *After a finite number of admissible replacements, the following property hold: Let $\tilde{\mathscr{Z}}_\nu$ be the closure of $\tilde{Z}_\nu$ in $\mathscr{X}$ for each ν. Then $\tilde{\mathscr{Z}}_\nu \times_{\mathscr{X}} \mathscr{D}$ is finite over S. Moreover all points in the support of $\mathrm{div}_{\tilde{Z}_\nu}(g_\nu)$ satisfy the modulus condition (5.3) for $(\mathscr{X}, \mathscr{D})$.*

5.4.4. *Step 3.* In this step we finish the proof of Claim 5.3.2 (and hence that of Theorem 5.3.2) admitting Claim 5.4.5. Let the notation be as in the claim. Let $\tilde{\mathscr{Z}}_\nu^N$ be the normalization of $\tilde{\mathscr{Z}}_\nu$ and $\tilde{\mathscr{Z}}_{\nu,\infty}^N = \tilde{\mathscr{Z}}_\nu^N \times_{\mathscr{X}'} \mathscr{D}$. Then $\mathfrak{Z}_\nu = (\tilde{\mathscr{Z}}_\nu^N, \tilde{\mathscr{Z}}_{\nu,\infty}^N)$ is an $\mathscr{S}$-modulus pair for each ν. The condition (♣3) implies that all points in $|\mathrm{div}_{\tilde{Z}_\nu^N}(g_\nu)|$ satisfy the modulus condition (5.3) for $\mathfrak{Z}_\nu$. The condition (♣2) implies that the class of $\mathrm{div}_{\tilde{Z}_\nu^N}(g_\nu)$ in $h_0^{\overline{\square}}(\mathfrak{Z}_\nu)(\mathscr{S})$ lies in the kernel of

$$j_{\mathscr{S}'_x}\colon h_0^{\overline{\square}}(\mathfrak{Z}_\nu)(\mathscr{S}) \to h_0^{\overline{\square}}(\mathfrak{Z}_\nu)(K, \emptyset).$$

By Lemma 5.4.2 and Remark 5.4.3, we get $\alpha_\nu = 0 \in h_0^{\overline{\square}}(\mathfrak{Z}_\nu)(\mathscr{S})$, which implies $\alpha = 0 \in h_0^{\overline{\square}}(\mathfrak{X}')(\mathscr{S})$ by (♣1). This completes the proof of Claim 5.3.2.

6. Moving by tame symbols

This Section contains the final ingredient of the proof of the main theorem, i.e. the proof of Claim 5.4.5. The main idea is to replace an arbitrary finite set of relations in the Chow group of zero cycles with modulus on a surface by a more controlled one, in which "cancellations", in the appropriate sense, no longer occur. This is achieved by an iterated applications of a moving argument by tame symbols, i.e. by adding cycles which are obtained as boundaries of classes in the K_2 of the function field of the surface. In this form, our argument is inspired by a moving lemma in the context of cycles on a singular variety due to Marc Levine, see [20], [7]. A similar moving technique has been used in [2]. A supplementary difficulty in our approach is the need to take care of the models over the base S of every constructed object.

6.1. **Setting.** We now fix the setting. Let S be again the spectrum of a local domain $\mathscr{O}$, with fraction field K and infinite residue field k. The arguments in this section do not require any assumption on the characteristic of k, but the restriction to characteristic zero is necessary to apply these results to the proof of the main theorem.

Let $\mathfrak{X} = (\mathscr{X}, \mathscr{D})$ be a proper modulus pair over S. Let $X = \mathscr{X}_K$ be its smooth generic fiber, and write $D = \mathscr{D}_K$. Let (Z, f) be a pair, consisting of a reduced, purely 1-dimensional closed subscheme of X, with irreducible components $Z_1, \ldots, Z_n$, and a rational function $f = (f_i) \in K(Z)^\times = \prod K(Z_i)^\times$ satisfying the modulus condition with respect to D (which means, by definition, that each f_i satisfies the modulus condition (5.6) on Z_i with respect to D).

We assume that $\mathfrak{X}$ and (Z, f) satisfy the conditions ($\spadesuit$1) and ($\spadesuit$2) of 5.4.3. In order to prove Claim 5.4.5, we need to replace the curves Z_i with new curves, $\tilde{Z}_\nu$ such that conditions ($\clubsuit$1), ($\clubsuit$2) and ($\clubsuit$3) are satisfied. In particular, we need to ensure that each closed point in the support of the new rational function g_ν on $\tilde{Z}_\nu$ satisfy the modulus condition $(\mathrm{MC})_{\mathscr{O}}$ (cf. (5.3)). This is the most delicate condition, since (for the given set (Z, f)), there are a priori points in the support of $\mathrm{div}_{Z_i}(f_i)$ which are not in the support of $\gamma = \mathrm{div}_Z(f) = \sum_{i=1}^n \mathrm{div}_{Z_i}(f_i)$. In other words, cancellation occur.

We write Σ for the set of such cancellation points in the expression of γ. Explicitly, let $\gamma_i = \mathrm{div}_{Z_i}(f_i)$. Then

$$\Sigma = \{x \in X_{(0)} | \, x \notin |\gamma|, \text{ but } x \in |\gamma_i| \text{ for some } i = 1, \ldots, n\}.$$

Note that $\Sigma \cap D = \emptyset$.

Our goal in this Section is to rewrite the cycle γ as a sum of divisors of functions on carefully chosen curves so that cancellations in the above sense no longer occur. This will allow us to apply directly the results of Section 3 to each individual term. We summarize the precise statement in the point 6.3.1 below.

Recall that a *model* of a scheme W defined over K is an integral proper S-scheme $\mathscr{W}$, surjective over S and such that $\mathscr{W} \times_S K \cong W$.

6.2. **Reduction I.** We keep the notation of the previous section. Apoint $x \in X$ is a cancellation point for $\mathrm{div}_Z(f)$ if $x \notin |\mathrm{div}_Z(f)|$ (i.e. x does not appear in the support of the divisor) but there exists $i \in \{1, \ldots, n\}$ such that $f_i \in \mathfrak{m}_x \mathcal{O}_{Z_i, x}$, i.e. x appears as a zero of the restriction of f to one of the components of Z. Note that if this happens, there must exist another component $j \neq i$ such that f_j has a pole at x, with matching multiplicity.

We fix the following two sets, as in the previous section:

$$T_0 = \{x \in X | f \in \mathfrak{m}_x \mathcal{O}_{Z,x}\}, \quad T_\infty = (\{x \in X | f \notin \mathcal{O}_{Z,x}^\times\} - T_0) \cup Z_{\mathrm{sing}}$$

where Z_{sing} is the set of singular points of the SNCD curve Z. Informally, we will refer to T_0 as the set of zeros of f (note that these are regular points of f), and to T_∞ as the set of "poles" of f. In particular, T_∞ contains every point x of intersection of different irreducible components of Z where f does not extend to a regular function in the local ring of Z at x. Note that by condition the modulus condition of f, we have $T_\infty \cap D = \emptyset$. By construction, the set T_∞ contains also every cancellation point $x \in \Sigma$. As specified above, thanks to condition ($\spadesuit$2), we have that the closure $\overline{T_\infty}$ of T_∞ is finite over S.

We write Λ for the set

$$\Lambda = (T_0 \cup T_\infty) \amalg (D \cap Z).$$

The argument will involve the choice of several auxiliary objects. We begin by choosing a free S-module E of finite rank, corresponding to a projective embedding

$$\iota_E \colon \mathscr{X} \hookrightarrow \mathbb{P}_S(E). \tag{6.1}$$

Write $\iota_E^* \mathcal{O}_{\mathbb{P}_S(E)}(1) = \mathcal{O}(\Gamma)$, where $\Gamma \subset \mathscr{X}$ is an ample hypersurface section of $\mathscr{X}$, and $H^0(X, \mathcal{O}_X(\Gamma_K)) = E \otimes_S K$. By the Bertini Theorem 4.2.1, we can assume that Γ is surjective over S, that $\Gamma_K \cap \Lambda = \emptyset$, that $\Gamma_s \cap \mathscr{D}_s$ is a finite set and finally that $\Gamma \cap \overline{T_\infty}$ is empty. Note that the last condition can be achieved since $\overline{T_\infty}_s$ is a finite set, so that we can impose that $\Gamma_s \cap \overline{T_\infty}_s = \emptyset$ and apply the same token of Theorem 4.2.1 (so that both $\Gamma_K \cap T_\infty = \emptyset$ and $\Gamma_s \cap \overline{T_\infty}_s = \emptyset$).

Choose a global section $\beta \in H^0(\mathscr{X}, \mathcal{O}_{\mathscr{X}}(\Gamma))$ such that its divisor of zeros, $\mathscr{B} = (\beta)$ satisfies the following conditions. Let $B = \mathscr{B}_K$. Then B is integral, satisfies $B \cap \Lambda = \emptyset$ and intersects $|D|$ transversally (which means that B intersects $|D|$ in the regular locus, and transversally). This can be achieved by the classical theorem of Bertini over K, and we will use our refinement, Theorem 4.2.1 to take care of the model over $\mathscr{O}$ of the section. In particular, $\mathscr{B}$ is surjective over S, and $\mathscr{D}$ restricts to an effective Cartier divisor on it, finite over S. We may also assume that $\mathscr{B}_s \cap \overline{T_\infty}_s = \emptyset$. This last condition can be achieved since $\overline{T_\infty}_s$ is a finite set of points. Note that $\mathscr{B}$ is equidimensional over S.

The following lemma is another application of Bertini's theorem 4.2.1.

Lemma 6.2.1. *There exists a hypersurface section $\mathcal{L} \subset \mathscr{X}$, of sufficiently large degree, such that the following conditions are satisfied.*

a) $\mathcal{L}$ is surjective over S, and satisfies the property that $\mathscr{D} \cap \mathcal{L}$ is an effective Cartier divisor, finite over S. In particular, $\mathcal{L}_s \cap \mathscr{D}_s$ is finite.
b) Let $L = \mathcal{L}_\eta$. Then $L \cap \Lambda = \emptyset$ and L intersects D properly.
c) $H^1(X, \mathcal{O}_X(L) \otimes I_{Z \cup D \cup B}) = H^1(\mathscr{X}, \mathcal{O}_{\mathscr{X}}(\mathcal{L}) \otimes I_{\mathscr{Z} \cup \mathscr{D} \cup \mathscr{B}}) = 0$
d) There exists a section $t_0 \in H^0(\mathscr{X}, \mathcal{O}_{\mathscr{X}}(\mathcal{L}) \otimes I_{\mathscr{Z}})$ such that the divisor of zeros of t_0 satisfies $(t_0) = \mathscr{Z} \cup \mathscr{Z}''$. Here $\mathscr{Z}'' \subset \mathscr{X}$, and if $Z'' = \mathscr{Z}''_K$, then we have

$$Z'' \cap \Lambda = Z'' \cap Z \cap (B \cup D) = Z'' \cap B \cap D = \emptyset, \tag{6.2}$$

and Z'' intersects $|D| + B$ transversally.
e) The sets $\mathscr{Z}_s \cap \mathscr{Z}''_s$ and $\mathscr{Z}''_s \cap \mathscr{B}_s$ are finite, and $\mathscr{Z}_s \cap \mathscr{B}_s \cap \mathscr{Z}''_s = \emptyset$.

Proof. Everything follows from Bertini's theorem and its variants. For condition e), note that by construction no component of $\mathscr{B}$ is contained in the special fiber $\mathscr{X}_s$ (in fact, $\mathscr{B}$ is equidimensional over S). The same holds for $\mathscr{D}$, since it is equidimensional over S. In particular, X contains every generic point of $\mathscr{D}$ and of $\mathscr{B}$. Thus condition (6.2) on the generic fiber, together with the fact that Z and D intersect properly, is enough to guarantee that t_0 does not vanish identically on $\mathscr{B} \cup \mathscr{D}$. In particular, the restriction morphism

$$H^0(\mathscr{X}, \mathcal{O}_{\mathscr{X}}(\mathcal{L}) \otimes I_{\mathscr{Z}}) \to H^0(\mathscr{B} \cup \mathscr{D}, \mathcal{O}_{\mathscr{X}}(\mathcal{L}))$$

sends t_0 to a section whose zero locus is nowhere dense in $\mathscr{B}\cup\mathscr{D}$. Note that it can still happen that (t_0) contains a component of $\mathscr{D}_s$ (this is the case if the original model $\mathscr{X}$ of Z contained a component of $\mathscr{D}_s$). $\square$

As before, we may assume that there exists a free S-module M, corresponding to another projective embedding

$$\iota_M\colon \mathscr{X} \hookrightarrow \mathbb{P}_S(M), \quad \text{such that } \mathscr{O}_{\mathscr{X}}(\mathcal{L}) = \iota_M^*(\mathscr{O}_{\mathbb{P}_S(M)}(1)) \tag{6.3}$$

and $M\otimes_S K = H^0(X, \mathcal{O}_X(L))$.

Write $(t_0) = \mathscr{Z}_0$ and Z_0 for its generic fiber $(\mathscr{Z}_0)_K$. Let f_0 be the rational function on Z_0 induced by f on Z and by the constant function 1 on Z''. Let T'_∞ be the set of "poles" of f_0 in the above sense, $T'_\infty = (\{x\in X | f_0 \notin \mathcal{O}^\times_{Z_0,x}\} - T_0)\cup Z_{0,\mathrm{sing}}$. Then $T'_\infty \supset T_\infty$, and by construction it is disjoint from $\Lambda - T_\infty$. Moreover, $T'_\infty\cap D = \emptyset$ (this follows from the construction of Z in the previous section), and $f_0 \in 1 + I_D\mathcal{O}_{Z_0,x}$ for every $x\in D\cap Z_0$, i.e. the function f_0 satisfies the modulus condition on Z_0 with respect to D.

Note also that the closure of T'_∞ in $\mathscr{X}$ is also finite over S. This is automatic for the points in T_∞, thanks to (♠2). As for the points in $T'_\infty\setminus T_\infty$, they are all contained in $Z\cap Z''$. Thus finiteness of their closure over S is guaranteed by e) above.

We summarize the situation so far. Up to adding extra components on the generic fiber, we can assume that the curve Z_0 is given by the generic fiber of the divisor of zeros of a global section of a very ample line bundle on $\mathscr{X}$. Such divisor gives an equidimensional model for Z_0 over S. On Z_0 we are given a rational function f_0 which satisfies the modulus condition with respect to D. The set T'_∞ of points where f_0 is not defined or where f_0 has a pole (a finite set of closed points disjoint from D) satisfy the property that the closure in the model is finite over S. Finally, we have chosen an auxiliary flat divisor $\mathscr{B}$ on the model, missing the zeros of f_0 as well as T'_∞ (and their closure over S), with the additional property that the model $\mathscr{D}$ of D restricts to an effective divisor on $\mathscr{B}$, finite over S.

The next step is to choose yet another section, lifting the restriction of t_0 to $\mathscr{D}\cup\mathscr{B}$.

Lemma 6.2.2. *There exists a lift $t_\infty \in H^0(\mathscr{X}, \mathcal{O}_{\mathscr{X}}(\mathcal{L}))$ of the restriction of t_0 to $\mathscr{D}\cup\mathscr{B}$ such that the divisor of zeros $(t_\infty) = \mathscr{Z}_\infty$ of t_∞ satisfies the following properties. The generic fiber $Z_\infty = (\mathscr{Z}_\infty)_K$ is regular, intersects $|D|$ transversally and $Z_\infty\cap(T_0\cup T'_\infty) = \emptyset$. Moreover, there exists a proper birational morphism $\phi\colon S'\to S$ such that the following condition is satisfied. For every $x\in S'$, let S'_x be the localization at x. Write $\mathscr{X}'$ (resp. $\mathscr{D}'$, resp $\mathscr{Z}'_\infty$, resp. $\mathscr{Z}'_0$) for the base change of $\mathscr{X}$ to S'_x (resp. the base change of $\mathscr{D}$, resp. the base change of $\mathscr{Z}_\infty$, resp. the base change of $\mathscr{Z}_0$). Let V be the set of points in $Z_\infty\cap Z_0\setminus D$ and let $\mathscr{V}'$ be the closure of V in $\mathscr{X}'$. Then $\mathscr{V}'$ is finite over S'_x.*

Proof. Consider the torsion-free submodule

$$\widetilde{M} = H^0(\mathscr{X}, \mathcal{O}_{\mathscr{X}}(\mathcal{L})\otimes I_{\mathscr{D}\cup\mathscr{B}}) \subset H^0(\mathscr{X}, \mathcal{O}_{\mathscr{X}}(\mathcal{L}))$$

of the free S-module M given in (6.3). Choose a basis $t_1,\ldots t_m$ of the maximal free submodule M' of $\widetilde{M}$, and denote by the same letters the corresponding sections of $H^0(X, \mathcal{O}_X(L)\otimes I_D\otimes I_B)$. Adding t_0 to the set $(t_1,\ldots,t_m)$ defines another free submodue $M'' = M'\oplus\langle t_0\rangle$ of M, of rank $m+1$ since $t_0\notin\widetilde{M}$ thanks to conditions d) and e) of Lemma 6.2.1.

The t_i's define then a morphism $\mathscr{X}\setminus((\mathscr{D}\cup\mathscr{B})\cap\mathscr{Z}_0)\to\mathbb{P}_S(M'')$. On the generic fiber X, it corresponds to the rational map $\psi\colon X \dashrightarrow \mathbb{P}^m_K$ given by the sections $(t_0, t_1,\ldots,t_m)$ of $H^0(X, \mathcal{O}_X(L))$. It is a locally closed immersion on $X\setminus|D\cup B|$, and since the base locus of the linear system associated to $(t_0,t_1,\ldots,t_m)$ is

$$(B\cup D)\cap Z_0 = (D\cap Z_0)\amalg B\cap Z_0,$$

it is in fact a morphism away from $(B \cup D) \cap Z_0$ (which is a zero dimensional set). Thus ψ is birational, hence separable, and has image of dimension equal to two. By the classical theorem of Bertini, a general divisor in the linear system $V(t_0, t_1, \dots, t_m)$ is irreducible and generically reduced, and regular away from $(B \cup D) \cap Z_0$. Since $(T_0 \cup T'_\infty) \cap D = \emptyset$, and also $(T_0 \cup T'_\infty) \cap B = \emptyset$, the set $(T_0 \cup T'_\infty)$ (which is zero dimensional) is away from the base locus, hence we can assume that the intersection of a general divisor with $(T_0 \cup T'_\infty)$ is empty.

Now, the linear system $V(t_0, t_1, \dots, t_m)$ (on the generic fiber X) is the linear system associated to the sub-vector space $M'' \otimes_S K$ of $M \otimes_S K$ or, equivalently, to the K-points of the Grassmannian $\mathrm{Gr}_S(M'' \otimes_S K)(K)$. Let $U \subset \mathrm{Gr}_S(M'' \otimes_S K)$ be the Zariski open subset such that for every point in $t_\infty \in U(K)$, the corresponding divisor $Z_\infty = (t_\infty)$ satisfies the above-mentioned properties.

As in the proof of Lemma 4.1.1, we can assume that $t_\infty \in U(S)^o \subset U(K)$ (see the notation in loc. cit.). In particular, up to an element in $\mathscr{O}^\times$, the section t_∞ is of the form $t_\infty = t_0 + \alpha$, with $\alpha \in H^0(\mathscr{X}, \mathcal{O}_{\mathscr{X}}(\mathcal{L}) \otimes I_{\mathscr{D} \cup \mathscr{B}})$.

Note that by construction,

$$\mathscr{Z}_0 \cap \mathscr{D} = \mathscr{Z}_\infty \cap \mathscr{D}, \quad \mathscr{Z}_0 \cap \mathscr{B} = \mathscr{Z}_\infty \cap \mathscr{B}.$$

By condition c) of the previous Lemma, together with the fact that Z is SNCD, we can assume that at every point $y \in Z_0 \cap D$, the functions (t_0) is part of a system of parameters of $\mathcal{O}_{X,y}$, which is a regular, 2-dimensional local ring. Let now π be a local equation for $|D|$ (note that $I_{B,y} \cong \mathcal{O}_{X,y}$ since $y \notin B$ by choice). If $I_{D,y} = (\pi)$, we can then choose α satisfying

$$\alpha_y = a_y \pi \in I_{D,y} \subset \mathcal{O}_{X,y}$$

with $a_y \in \mathcal{O}_{X,y}^\times$, so that $t_{\infty,y} = t_0 + a_y \pi$ defines a regular divisor at y, tangent to Z_0. This can be checked as follows. The statement is equivalent to say that $t_{\infty,y} \in \mathfrak{m}_y \setminus \mathfrak{m}_y^2$. This is clear if $\pi \in \mathfrak{m}_y^2$. Otherwise, choose $t'_0 \in \mathcal{O}_{X,y}$ such that (t_0, t'_0) forms a regular system of parameters of $\mathcal{O}_{X,y}$. Write $\pi = \lambda t_0 + \lambda' t'_0$ in $\mathfrak{m}_y / \mathfrak{m}_y^2$. Then $t_{\infty,y} \in \mathfrak{m}^2$ if and only if $\lambda' = 0$ and $a_y \lambda = -1$. We can assume that for general α this is not the case.

The same argument works if we replace D by B, noting that $(D \cap Z_0) \cap B \cap Z_0 = \emptyset$ and that Z_0 intersects B transversally. Thus we can assume that Z_∞ is regular at every point of $Z_\infty \cap B$ as well. Explicitly, let b be a local equation for B in a neighborhood of $y \in Z_0 \cap B$. As before, note that $I_{D,y} \cong \mathcal{O}_{X,y}$ since this time $y \notin D$. If $I_{B,y} = (b)$, we can choose α satisfying the additional property

$$\alpha_y = c_y b \in I_{B,y}$$

with $c_y \in \mathcal{O}_{X,y}^\times$, so that $t_{\infty,y} = t_0 + c_y b$ defines a regular divisor at y (note that, in this case, B itself is regular at y).

Finally, let $V = (Z_\infty \cap Z_0) \setminus D$. We can apply [22, Cor. 5.7.10] to achieve finiteness of the closure $\mathscr{V}'$ of V after a base change along a proper birational map $\phi\colon S' \to S$ (followed by a localization $S'_x \to S'$ at a point $x \in S'$). As before, the map induces an isomorphism on the generic fiber, so that all the remaining properties of $\mathscr{Z}_\infty$ are left untouched. □

Remark 6.2.3. All the constructions performed so far (i.e. the properties of the chosen sections t_0, t_∞ and β) are stable under base change $\phi\colon S' \to S$ along proper birational maps followed by localizations, and so is the statement of Claim 5.4.5. In particular, we can replace S with S'_x in the previous Lemma and assume that the finiteness of the closure of V is already achieved over S.

We now extend the function f_0 to a function h on $Z_0 \cup Z_\infty$ by setting $h = (f_0, 1) \in K(Z_0)^\times \times K(Z_\infty)^\times$. The next claim proves that h is in fact a regular function on the union $Z_0 \cup Z_\infty$ at every point of intersection with D.

Claim 6.2.4. *For every $x \in Z_0 \cap D = Z_\infty \cap D$, we have $h \in \mathcal{O}^\times_{Z_0\cup Z_\infty, x}$.*

Proof. In a neighborhood of $x \in Z_0 \cap D$, the scheme $Z_0 \cup Z_\infty$ is defined by the principal ideal

$$I_{Z_0} I_{Z_\infty} = (t_0(t_0 + a_x\pi)),$$

where $(\pi) = I_{D,x}$ and $a_x \in \mathcal{O}^\times_{X,x}$ as in the previous lemma. We have then the following exact sequence of $\mathcal{O}_{X,x}$-modules

$$0 \to \mathcal{O}_{X,x}/(t_0(t_0 + a_x\pi)) \to \mathcal{O}_{X,x}/(t_0) \times \mathcal{O}_{X,x}/(t_0 + a_x\pi) \to \mathcal{O}_{X,x}/(t_0, a_x\pi) \to 0$$

By assumption, the function f_0 satisfies the modulus condition, i.e. $f \in 1 + I_D\mathcal{O}_{X,x}$. Thus $f_0 - 1 = 0 \bmod (t_0, a_x\pi)$. In other words, the pair $(f_0, 1)$ determines a regular function

$$h \in \mathcal{O}_{X,x}/(t_0(t_0 + a_x\pi)) = \mathcal{O}_{Z_0\cup Z_\infty, x},$$

as required. Note that h is automatically invertible at x. □

Write $\hat{T}_\infty = \{x \in X \,|\, h \notin \mathcal{O}^\times_{Z_0\cup Z_\infty,x}\} \setminus T_0$, and write $\hat{T}_0 = T_0$ coherently (note that $\hat{T}_0$ is precisely the set of zeros of h, in the sense discussed above). By Claim 6.2.4, we have that $\hat{T}_\infty \cap D = \emptyset$. Note also that the closure of $\hat{T}_\infty$ in $\mathscr{X}$ is finite over S. Indeed, the set $\hat{T_\infty}$ is, by construction, given by the original set of poles of f on Z, i.e. T_∞, together with the intersections

$$Z \cap Z'', \quad (Z_0 \cap Z_\infty) \setminus D.$$

For the points in the first set, the finiteness of their closure is guaranteed by e) of Lemma 6.2.1. For the points in $(Z_0 \cap Z_\infty) \setminus D$, finiteness of their closure follows from the last property of Lemma 6.2.2.

We now choose yet another section of $\mathcal{O}(\mathcal{L})$, as in the following lemma.

Lemma 6.2.5. *Let $\hat{\Sigma} \subset \hat{T}_\infty$ be the cancellation set for h (or, equivalently, for f_0, or equivalently for f). There exists a section $s_\infty \in H^0(\mathscr{X}, \mathcal{O}_{\mathscr{X}}(\mathcal{L}))$ such that its divisor of zeros $\mathscr{H} = (s_\infty)$ satisfies the following properties. Write $H = \mathscr{H}_\eta$ for the generic fiber.*

i) H is integral, intersects $|D|$ transversally (which means that H intersects $|D|$ only in the regular locus, and transversally).

ii) H contains $\hat{T}_\infty$, and is regular along each point of $\hat{\Sigma}$ (note that $\hat{T}_\infty \supset \hat{\Sigma}$).

iii) The restriction of s_∞ to $Z_0 \cup Z_\infty$ determines a global section

$$hs_\infty \in H^0(Z_0 \cup Z_\infty, \mathcal{O}_X(L))$$

iv) $H \cap (\Lambda - \hat{T}_\infty) = H \cap Z_0 \cap D = H \cap D \cap L = H \cap D \cap B = H \cap B \cap Z_0 = H \cap B \cap Z_\infty = \emptyset$.

Moreover, $\mathscr{H}$ is surjective over S and satisfies the property that $\mathscr{D} \cap \mathscr{H}$ is an effective Cartier divisor on $\mathscr{H}$, finite over S.

Proof. We first take care of the generic fiber H. Its model over S, with the required property, will be obtained using Theorem 4.2.2 applied to the set $\hat{T}_\infty$ and with respect to the embedding ι_M in $\mathbb{P}_S(M)$ (given in (6.3)). Finiteness over S of the restriction of the divisor $\mathscr{D}$ is achieved by the same token of Proposition 4.2.4. Note that this crucially uses the fact that the closure $\mathscr{T}_\infty$ of $\hat{T}_\infty$ in $\mathscr{X}$ is finite over S (in particular, $\mathscr{T}_{\infty,s}$ is a finite set, disjoint from $\mathscr{D}_s$).

Write $\hat{\Sigma} = \{\xi_1, \ldots, \xi_r\}$, and for $i = 1, \ldots, r$. Recall that, by construction, we have

$$(Z_0 \cup Z_\infty) \cap \hat{\Sigma} = Z_0 \cap \hat{\Sigma} = Z \cap \hat{\Sigma},$$

and that at each point $\xi_i \in \hat{\Sigma}$, there are exactly two regular components of Z passing through ξ, and intersecting transversally there (this is again guaranteed by condition (♠2)). We can then choose a regular system of parameters $(u_{1,i}, u_{2,i}) \subset \mathcal{O}_{X,\xi_i}$ generating the maximal ideal of the local ring of X at ξ_i such that $I_{Z_0,\xi_i} = (u_{1,i}, u_{2,i})$, so that $u_{1,i}$ and $u_{2,i}$ are local parameters for the two components $Z_{1,i}$ and $Z_{2,i}$ of Z passing through ξ_i. Since ξ_i is a cancellation point, we have

$$\mathrm{ord}_{Z_{1,i},\xi_i}(f) + \mathrm{ord}_{Z_{2,i},\xi_i}(f) = 0.$$

Write $\lambda_i = \mathrm{ord}_{Z_{2,i},\xi_i}(f)$, and suppose (up to exchanging the components) that $\lambda_i > 0$. We can then write $f = \frac{a_i}{u_1^{\lambda_i}} \mod u_2$, for $a_i \in \mathcal{O}^\times_{Z_2,\xi_i}$ and $f = b_i u_2^{\lambda_i} \mod u_1$, for $b_i \in \mathcal{O}^\times_{Z_1,\xi_i}$, using that $\mathcal{O}_{Z_1,\xi_i}$ and $\mathcal{O}_{Z_2,\xi_i}$ are DVRs.

Let $I_{\xi_i} \subset \mathcal{O}_X$ be the ideal sheaf of the point ξ_i, and define $J_{\xi_i} \subset I_{\xi_i}$ to be the subsheaf of I_{ξ_i} locally generated by $(J_{\xi_i})_{\xi_i} = (u_{1,i}^{\lambda_i+1}, u_{2,i}) \subset \mathcal{O}_{X,\xi_i}$. Let J be the product $\prod_{i=1}^r J_{\xi_i}$ and let $\hat{J} = JJ_\infty$, where $J_\infty \subset \mathcal{O}_X$ is the ideal sheaf of the points in $\hat{T}_\infty \setminus \hat{\Sigma}$.

We now choose $s_\infty \in H^0(X, \mathcal{O}_X(L) \otimes \hat{J})$ such that $s_{\infty,\xi_i} \neq 0$ in $(J_{\xi_i}/J_{\xi_i} \cap I^2_{\xi_i})_{\xi_i}$. Note that, if necessary, we could have replaced $\mathcal{L}$ with another hypersurface section of $\mathscr{X}$ so to have enough global sections of $\mathcal{O}_X(L) \otimes J$ from the beginning (we remark that J does not depend on the later modifications, such as the choices of t_0 and t_∞).

Write H for the divisor of zeros of s_∞. As remarked above, the classical theorem of Bertini implies that H is integral, that it intersects transversally $|D|$ (since $|D| \cap \hat{T}_\infty = \emptyset$) and that satisfies condition iv) on the avoidance of the specified set of closed points. In particular, each point of $H \cap D$ is a regular point of H. In a neighborhood of $\xi \in \hat{\Sigma}$, we can write $s_\infty = u_1^{\lambda+1} + \varepsilon u_2$, with $\varepsilon \in \mathcal{O}^\times_{X,\xi}$ by assumption, so that H is regular at ξ. Condition ii) is then achieved.

The function h is a rational function on $Z_0 \cup Z_\infty$ (i.e. an element of the ring of total quotients of $Z_0 \cup Z_\infty$), and it's not defined precisely at the finite set of points $\hat{T}_\infty$. Consider now the function $s_\infty h$. We claim that s_∞ can be chosen in the previous step so that $s_\infty h \in \mathcal{O}_{Z_0 \cup Z_\infty, x}$ for every $x \in \hat{T}_\infty$. In particular, the function $s_\infty h$ gives rise to an element of $H^0(Z_0 \cup Z_\infty, \mathcal{O}_X(L))$.

For $x = \xi \in \hat{\Sigma}$, this is a direct consequence of the construction. Indeed, $\mathcal{O}_{Z_0 \cup Z_\infty, x} = \mathcal{O}_{Z_1 \cup Z_2, x}$, where Z_1 and Z_2 are the two regular components of $Z_0 \cup Z_\infty$ passing through x. We have then a short exact sequence, as in the proof of Claim 6.2.4

$$0 \to \mathcal{O}_{Z_1 \cup Z_2, x} \to \mathcal{O}_{X,x}/(u_2) \times \mathcal{O}_{X,x}/(u_1) \to \mathcal{O}_{X,x}/(u_1, u_2) \to 0$$

and an equality

$$hs_\infty = (\frac{a}{u_1^\lambda}, bu_2^\lambda)(u_1^{\lambda+1} + \varepsilon u_2) = (au_1, \varepsilon b u_2^{\lambda+1}) \in \mathcal{O}_{X,x}/(u_2) \times \mathcal{O}_{X,x}/(u_1), \tag{6.4}$$

with a and εb units, from which it follows that $(hs_\infty)_{Z_1' \cap Z_2'} = 0$ in $\mathcal{O}_{X,x}/(u_1, u_2) = k(x)$, so that hs_∞ gives rise to an element of $\mathcal{O}_{Z_1 \cup Z_2, x}$. Note in particular that the required lifting property can be achieved together with the regularity of H. For $x \in \hat{T}_\infty \setminus \hat{\Sigma}$, where the regularity of H is not required, it is enough to choose s_∞ such that $s_{\infty,x} \in \hat{J}_x^N = \mathfrak{m}_x^N \subset \mathcal{O}_{X,x}$ for a sufficiently large N (depending on x) so that $hs_\infty = 0 \mod \hat{J}_x$. This forces hs_∞ to be regular on $Z_0 \cup Z_\infty$ at x. Note that H will be, in general, highly singular there.

Finally, since the base locus of the linear system $|H^0(X, \mathcal{O}_X(L) \otimes \hat{J})|$ is disjoint from the zero dimensional sets $(\Lambda - \hat{T}_\infty)$, $Z_0 \cap D$, $D \cap L$, $D \cap B$, $B \cap Z_0 = B \cap Z_\infty$, thanks to the condition $\hat{T}_\infty \cap D = \emptyset$, we can assume that H satisfies condition iv). Here we are using in particular the fact that $\hat{T}_\infty \cap D \cap (Z_\infty \cup L) = \hat{T}_\infty \cap D \cap (Z_\infty \cup L) = \emptyset$, which is a consequence of Lemma 6.2.2 and the fact that h is a regular invertible function on $Z_0 \cup Z_\infty$ at every $x \in D \cap Z_0 = D \cap Z_\infty$, which is ensured by Claim 6.2.4.

Similarly, since $\hat{T}_\infty \cap |D| = \emptyset$, condition i) (i.e. the transversality of the intersection) can be achieved as well. $\square$

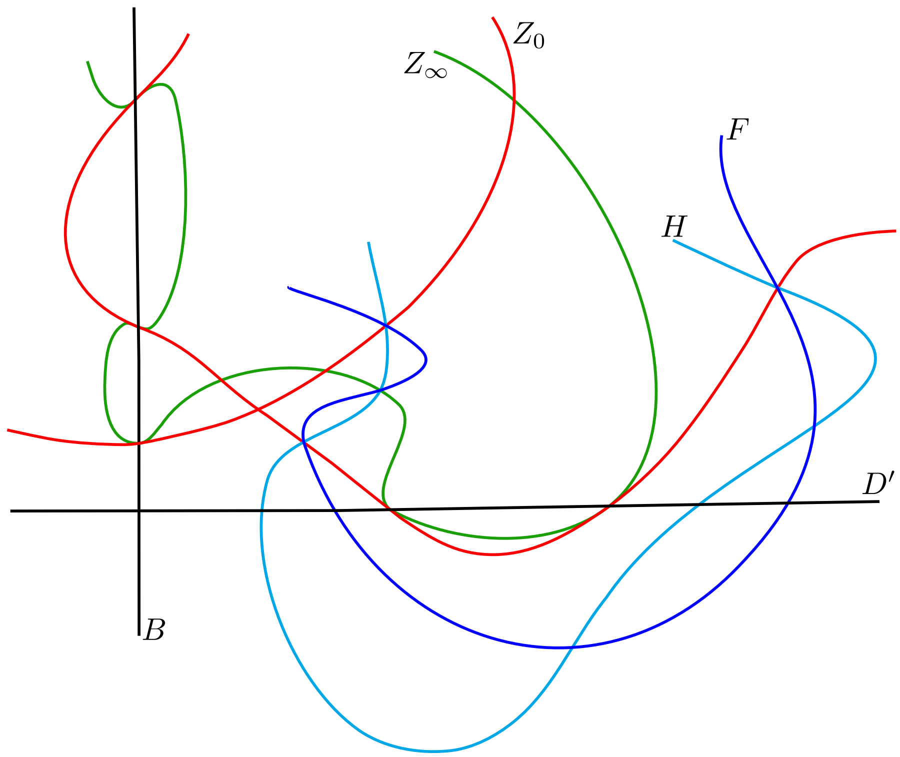

FIGURE 1. The configuration of curves on the generic fiber after Lemma 6.2.6

Recall that we have fixed two free S-modules E and M in (6.1) and (6.3) respectively, corresponding to two projective embeddings ι_E and ι_M. Now, for every $N > 0$, write E_N for the free S-module

$$E_N = H^0(\mathscr{X}, \iota_E^* \mathcal{O}_{\mathbb{P}_S(E)}(N)).$$

Combining the closed embedding ι_M with the N-fold embedding ι_{E_N} and following the result with the Segre embedding (see [8, 4.3.3]), we get a composite embedding

$$\iota\colon \mathscr{X} \hookrightarrow \mathbb{P}_S(M \otimes E_N)$$

For $N > 0$ sufficiently large, we have that

$$H^1(X, \mathcal{O}_X(L + N\Gamma_K) \otimes I_{Z_0 \cup Z_\infty}) = H^1(\mathscr{X}, \iota_M^* \mathcal{O}_{\mathbb{P}_S(M)}(1) \otimes \iota_E^* \mathcal{O}_{\mathbb{P}_S(E)}(N) \otimes I_{\mathscr{Z}_0 \cup \mathscr{Z}_\infty}) = 0$$

by Serre's vanishing theorem. This gives in particular two surjections

$$H^0(X, \mathcal{O}_X(L + N\Gamma_K)) \xrightarrow{\phi_\eta} H^0(Z_0 \cup Z_\infty, \mathcal{O}_X(L + N\Gamma_K)) \to 0$$

$$H^0(\mathscr{X}, \mathscr{O}_{\hat{\mathscr{X}}}(\mathcal{L} + N\Gamma)) \xrightarrow{\phi} H^0(\mathscr{Z}_0 \cup \mathscr{Z}_\infty, \mathscr{O}_{\mathscr{X}}(\mathcal{L} + N\Gamma)) \to 0.$$

Thanks to the previous Lemma, we are given the section $s_\infty h \in H^0(Z_0 \cup Z_\infty, \mathcal{O}_X(L))$. If we multiply it with the restriction of $\beta^N \in H^0(X, \mathcal{O}_X(N\Gamma_K))$ to $Z_0 \cup Z_\infty$, we get a section

$$\beta^N s_\infty h \in H^0(Z_0 \cup Z_\infty, \mathcal{O}_X(L + N\Gamma_K)),$$

which we are going to lift in an appropriate way thanks to the following result.

Lemma 6.2.6. *There exists a section $s_0 \in H^0(X, \mathcal{O}_X(L+N\Gamma_K))$ such that $\phi_\eta(s_0) = \beta^N s_\infty h$, and such that its divisor of zeros $F = (s_0)$ satisfies the following properties.*

v) F is integral, regular along each point of $\hat{\Sigma}$.

vi) $F \cap D \cap L = F \cap D \cap Z_0 = F \cap H \cap D = \emptyset$, *and* F *intersect* $|D|$ *transversally.*

vii) F *intersects* $H \setminus (Z_0 \cup Z_\infty)$ *transversally, which means that every intersection point of* F *and* H *away from* $Z_0 \cup Z_\infty$ *is a regular point of both* F *and* H*, and that they intersect transversally there.*

Moreover, let $\mathscr{F}$ *be the closure of* F *in* $\mathscr{X}$*. Then* $\mathscr{F}$ *is surjective over* S *and satisfies the property that* $\mathscr{D} \cap \mathscr{F}$ *is an effective Cartier divisor on* $\mathscr{F}$*, finite over* S*.*

Proof. We keep the notations of Lemma 6.2.5. Consider the torsion-free submodule

$$\widetilde{M \otimes_S E_N} = H^0(\mathscr{X}, \iota_M^* \mathcal{O}_{\mathbb{P}_S(M)}(1) \otimes \iota_E^* \mathcal{O}_{\mathbb{P}_S(E)}(N) \otimes I_{\mathscr{Z}_0 \cup \mathscr{Z}_\infty})$$

of the free S-module $M \otimes_S E_N$.

Let $s_1, \dots, s_m$ be a basis for a maximal free submodule E' of $\widetilde{M \otimes_S E_N}$, and denote by the same letters the corresponding sections of $H^0(X, \mathcal{O}(L + N\Gamma_K) \otimes I_{Z_0 \cup Z_\infty})$. As in the proof of Lemma 6.2.2, the s_i's define a rational map

$$X \dashrightarrow \mathbb{P}^{m-1}_K$$

that is locally a closed immersion on $X \setminus (Z_0 \cup Z_\infty)$. Let $\tilde{s_0}$ be any lift of $\beta^N s_\infty h$ to a global section in $H^0(X, \mathcal{O}_X(L + N\Gamma_K))$, and let $\lambda \in \mathscr{O} - \{0\}$ be a non-zero element such that $\lambda \tilde{s_0} \in M \otimes_S E_N$. Adding $\lambda \tilde{s_0}$ to the set $(s_1, \dots, s_m)$ defines another free submodule $E'' = E' \oplus \langle \lambda \tilde{s_0} \rangle$ of $M \otimes_S E_N$, of rank $m+1$, since $\tilde{s_0} \notin H^0(X, \mathcal{O}(L + N\Gamma_K) \otimes I_{Z_0 \cup Z_\infty})$ (so that, a fortiori, λs_0 is S-linearly independent of the s_i's). By construction, we have that the linear system $V(\tilde{s}_0, s_1, \dots, s_m)$ is nothing but the linear system associated to the sub-vector space $E'' \otimes_S K$ of the vector space $(M \otimes_S E_N) \otimes_S K$ or, equivalently to the K-points of the Grassmannian $\mathrm{Gr}_S(E'' \otimes_S K)(K)$.

The meromorphic section $\tilde{s_0}$ has poles along the divisor of λ in $\mathscr{X}$ by construction, and its divisor of zeros $(\tilde{s_0})$ a priori has the following undesired property. Let $\mathscr{D}_s^1, \dots, \mathscr{D}_s^r$ be the irreducible components of $\mathscr{D}_s$. Then we may have that $(\tilde{s_0}) \cap \mathscr{D} \supset \mathscr{D}_s^i$ for some i. In order to improve the situation, we make the following

Claim 6.2.7. *There exists* $\gamma \in \mathscr{O} = \Gamma(S, \mathcal{O}) \setminus \{0\}$ *and* $e > 0$ *such that, if* Θ *denotes the divisor of zeros of* γ *in* $\hat{\mathscr{X}}$*, for any* $\alpha \in \widetilde{M \otimes_S E_N}$*, the section* $s(e, \alpha) = \tilde{s_0}/\gamma^e + \alpha \in H^0(\mathscr{X}, \mathcal{O}_{\mathscr{X}}(\mathcal{L} + N\Gamma + e\Theta))$ *satisfies the following property: any component of* $(s(e, \alpha)) \cap \mathscr{D}$ *is dominant over* S*.*

Proof. Let $\Pi_1, \dots, \Pi_n$ be the irreducible components of $(\tilde{s_0}) \cap \mathscr{D}$, and let $f \colon \mathscr{D} \to S$ be the restriction to $\mathscr{D}$ of the structure map. Note that every Π_i is of codimension 1 in $\mathscr{D}$ (since it is the restriction of a divisor). Let $f(\Pi_i)$ be the image of Π_i in S. It is closed in S, since f is proper, and its codimension is at most 1 since f is equidimensional and $\Pi_i \subset f^{-1}(f(\Pi_i)) \subset \mathscr{D}$. Choose $0 \neq \gamma \in \Gamma(S, \mathcal{O})$ such that $\Pi_i \subset f^{-1}((\gamma)) =: \Theta$ for every Π_i which is not dominant over S. Here (γ) denotes as customary the zero locus of γ in S.

Next, let $\mathfrak{p} \in S^{(1)}$ be a codimension 1 point of S such that $\mathfrak{p} \in (\gamma)$. Let $\mathscr{D}_\mathfrak{p}$ be the fiber of $\mathscr{D}$ over $\mathfrak{p}$ and let $\eta_\mathfrak{p}$ be a generic point of $\mathscr{D}_\mathfrak{p}$. Finally, let $R_\mathfrak{p} = \mathcal{O}^N_{\mathscr{D}_{\mathrm{red}}, \eta_\mathfrak{p}}$ be the normalization of the local ring of $\mathscr{D}_{\mathrm{red}}$ at $\eta_\mathfrak{p}$ (note that we can replace $\mathscr{D}$ with $\mathscr{D}_{\mathrm{red}}$ to prove the claim, since the property is topological). It is a discrete valuation ring, so we can choose $\varepsilon_\mathfrak{p}$ a prime element. Thus we may write

$$\gamma = \varepsilon_\mathfrak{p}^l w, \quad \tilde{s}_{0|R_\mathfrak{p}} = \varepsilon_\mathfrak{p}^n u$$

for $w, u \in R_\mathfrak{p}^\times$ and $l, n \in \mathbb{Z}$. For $e > 0$ and $\alpha \in \widetilde{M \otimes_S E_N}$ write $s(e, \alpha) = \tilde{s_0}/\gamma^e + \alpha$ and write $s(e, \alpha)_{|R_\mathfrak{p}}$ for its restriction to $Q(R_\mathfrak{p})$. For the restriction of α to $R_\mathfrak{p}$, we can write

$\alpha_{\mathfrak{p}} = b(t_0 t_\infty)_{|R_{\mathfrak{p}}}$ for $b \in R_{\mathfrak{p}}$, since the ideal sheaf $I_{\mathscr{Z}_0 \cup \mathscr{Z}_\infty}$ is locally principal generated by the product the images of t_0 and t_∞. So write $(t_0 t_\infty)_{|R_{\mathfrak{p}}} = \varepsilon_{\mathfrak{p}}^m v$ for $m \in \mathbb{Z}$ and $v \in R_{\mathfrak{p}}^\times$. Then

$$s(e, \alpha)_{|R_{\mathfrak{p}}} = \varepsilon_{\mathfrak{p}}^{n-el}(uw^{-1} + b\varepsilon_{\mathfrak{p}}^{el+m-n})$$

For $e > 0$ sufficiently large, we have $n - el < 0$ and $el + m - n > 0$. This means exactly that the divisor of zeros of the modified section does not contain the component of $\mathscr{D}_{\mathfrak{p}}$ corresponding to $\eta_{\mathfrak{p}}$. We repeat the procedure for every generic point $\eta_{\mathfrak{p}}$ and for every codimension 1 point of S contained in (γ) to find an integer e big enough so that the restriction to $\mathscr{D}$ of the divisor of zeros of $(s(e, \alpha))$, independently of α, does not have any component whose image in S has positive codimension. □

Thanks to the claim, we can replace $\tilde{s_0}$ with any lift of the form $\tilde{s_0}/\gamma^e + \alpha$. Since $\gamma \in \mathscr{O} \setminus \{0\}$, we clearly have that the divisor of zeros of the generic fiber of $\tilde{s_0}/\gamma^e + \alpha$ agrees with $\tilde{s_0} + \gamma^e \alpha$, and since α is a section of $\widetilde{M \otimes_S E_N}$, we have $\phi_\eta(\tilde{s_0}/\gamma^e + \alpha) = \beta^N s_\infty h$ for every α and for every $e > 0$. Fix such a lift and call it s_0'. For every $\alpha \in \widetilde{M \otimes_S E_N}$, write F_α for the zero locus of $s_0' + \alpha$ (the α is clearly different from the previous one). Let $\mathscr{F}_\alpha$ be the closure of F_α in $\mathscr{X}$. Then every component of $\mathscr{F}_\alpha \cap \mathscr{D}$ is dominant over S.

We now proceed to further modify the lift in order to achieve the properties v), vi) and vii) on the generic fiber: this corresponds to the choice of a suitable α.

Notice that the base locus of $V(s_0', s_1, \ldots, s_m)$ is zero dimensional, given by $(s_0') \cap (Z_0 \cup Z_\infty)$, and disjoint from the sets $D \cap L$, $D \cap Z_0 = D \cap Z_\infty$, and $D \cap H$. Indeed, $D \cap H \cap (Z_0 \cup Z_\infty) = \emptyset$ by condition iv) of Lemma 6.2.5, while $D \cap L \cap (Z_0 \cup Z_\infty) = \emptyset$ by the choice of L (missing Λ) and the choice of t_∞ in Lemma 6.2.1. As for the set $D \cap (Z_0 \cup Z_\infty)$, note that by choice s_0' restricts to $\beta^N s_\infty h$ (up to a unit in $K^\times$) on $Z_0 \cup Z_\infty$, and we have that $B \cap Z_0 \cap D = \emptyset$ by the choice of B (which forces $B \cap Z_\infty \cap D = \emptyset$, since $Z_0 \cap D = Z_\infty \cap D$). In particular, β^N is a unit at each point $x \in Z_0 \cup Z_\infty \cap D$.

We are then left to consider the term $s_\infty h$ in a neighborhood of any $x \in D \cap Z_0 = D \cap Z_\infty$ in $Z_0 \cup Z_\infty$ in order to prove that the base locus of $V(s_0', s_1, \ldots, s_m)$ is disjoint from $D \cap Z_0$. Restricting ${s_0}'/\beta^N$ to the components Z_0 and Z_∞ we get (up to a common unit in $K^\times$)

$$s_0'/\beta^N_{|Z_0} = f_0 \cdot s_{\infty|Z_0}, \quad s_0'/\beta^N_{|Z_\infty} = 1 \cdot s_{\infty|Z_\infty}. \tag{6.5}$$

The function f_0 satisfies the modulus condition on Z_0 with respect to D, so f_0 is a unit at every $x \in D \cap Z_0$. By Lemma 6.2.5, $H = (s_\infty)$ is disjoint from $Z_0 \cap D$, thus s_∞ is a unit at every $x \in D \cap Z_0$. The same analysis works for the restriction to Z_∞. Thus, independently of the choice of the lift s_0, the base locus of $V(\tilde{s}_0, s_1, \ldots, s_m)$ is disjoint from the sets in vi).

By the classical theorem of Bertini, there exists an open subset U of $\mathrm{Gr}_S(E'' \otimes_S K)$ such that every hypersurface section F corresponding to a K-point of U is irreducible and generically reduced, and satisfies both vi) and vii). Chose a section $s_0 \in E''$ such that $F = (s_0)$ belongs to U. Up to a unit, such section is of the form $s_0' + \alpha$ for general $\alpha \in \widetilde{M \otimes_S E_N}$.

We now turn to the regularity (over K) in a neighborhood of each cancellation point $\xi \in \hat{\Sigma}$. Thanks to the fact that β is a unit at ξ (since $B \cap \hat{\Sigma} = \emptyset$) and the fact that $s_\infty h$ is given by the expression in (6.4), in a neighborhood of ξ any lifting of $\beta^N h s_\infty$ is of the form

$$s_0 = \mu_1 u_1 + \mu_2 u_2^{\lambda+1} + \mu_{12} u_1 u_2, \quad \mu_{12} \in \mathcal{O}_{X,\xi}, \mu_1, \mu_2 \in \mathcal{O}_{X,\xi}^\times$$

in particular, $F = (s_0)$ is automatically regular at ξ as required.

We can finally turn to the last required condition about the model $\mathscr{F}$ of F, namely the fact that $\mathscr{D}$ restricts to an effective Cartier divisor on $\mathscr{F}$, finite over S. It is clear that $\mathscr{F}$ is surjective over S, and that $\mathscr{D}$ defines a divisor on $\mathscr{F}$. By Claim 6.2.7 and the construction, we also have that every component of $\mathscr{D} \cap \mathscr{F}$ is dominant over S and generically finite (by vi)). It can still be the case that some components are not finite over S. To remedy this, we apply [22, Corollary 5.5.2] to find a proper birational map $S' \to S$ such that after base change to S', the strict transform of every component of $\mathscr{F} \cap \mathscr{D}$ is flat over S'. Since the map $S' \to S$ induces an isomorphism on the generic fiber, the remaining properties of F are left untouched, and the same is true after localization $S'_x \to S'$ at every point $x \in S'$. As before, since the statement of Claim 5.4.5 is stable under the base change $S'_x \to S' \to S$, we can replace S with S'_x (changing the notation) and assume that F has a model $\mathscr{F}$ over S for which the restriction of $\mathscr{D}$ to it is finite over S. □

We summarize what we achieved so far in the following proposition. We keep the above notations.

Proposition 6.2.8. *Let X be as above. Then there exist integral curves H and F such that the following conditions are satisfied*

- *a') Both H and F are regular in a neighborhood of $\hat{\Sigma}$.*
- *b') Both H and F intersect transversally $|D|$ (in particular, they are regular there), and $H \cap F \cap |D| = \emptyset$.*
- *c') After base change along a proper birational map $\phi\colon S' \to S$, followed by a localization $S'_x \to S'$ at every point $x \in S'$ (see the statement of Claim 5.4.5) F and H have integral models $\mathscr{F}$ and $\mathscr{H}$ over S, which are surjective over S and satisfy the property that the restriction of $\mathscr{D}$ to them is an effective Cartier divisor, finite over S.*
- *d') Let $\tau = t_0/t_\infty \in K(X)$ as rational function on X. Then the restriction of τ to F and to H satisfy the modulus condition with respect to $D \cap F$ and to $D \cap H$ respectively.*

Moreover, we have $\gamma = \mathrm{div}_F(\tau) - \mathrm{div}_H(\tau)$ as zero cycles on X.

Proof. Properties a') and b') are direct consequences of Lemma 6.2.5 and of Lemma 6.2.6. Since moreover

$$D \cap (Z_\infty \cup L) \cap H = D \cap (Z_\infty \cup L) \cap F = D \cap (Z_0 \cup L) \cap H = D \cap (Z_0 \cup L) \cap F = \emptyset,$$

thanks to Lemma 6.2.5 part iv) and 6.2.6 part vi), the choice of t_∞ in Lemma 6.2.2 guarantees that $\tau \in 1 + I_D\mathcal{O}_{X,x}$ for $x \notin D \cap Z_\infty$, so that $\tau_{|F} \in 1 + I_D\mathcal{O}_{F,x}$ for every $x \in F \cap D$ and that $\tau_{|H} \in 1 + I_D\mathcal{O}_{H,x}$ for every $x \in H \cap D$ (note that here we are using the property that $Z_\infty \cap D \cap (F \cup H) = \emptyset$). We can now compute

$$\begin{aligned} 0 &= \mathrm{div}_{Z_\infty}(1) = F \cdot Z_\infty - H \cdot Z_\infty - N(B \cdot Z_\infty), \\ \gamma &= \mathrm{div}_Z(f) = F \cdot Z_0 - H \cdot Z_0 - N(B \cdot Z_0). \end{aligned}$$

We subtract the two equations and collect to get $\gamma = \mathrm{div}_F(\tau) - \mathrm{div}_H(\tau) - N\mathrm{div}_B(\tau)$. But $t_0/t_\infty = 1$ on B thanks to Lemma 6.2.2, thus the term $N\mathrm{div}_B(\tau)$ vanishes and we get the required expression for γ. □

6.3. **Reduction II.** We keep the notations of Section 6.2. In order to further improve the expression of γ, we introduce some auxiliary divisors.

Lemma 6.3.1. *There exists a hypersurface section $\mathscr{H}' \subset \mathscr{X}$, such that the following conditions are satisfied. Let $H' = \mathscr{H}'_\eta$ be the generic fiber.*

1. *H' is integral, regular, and $H'_{|D} \geq H_{|D} + N \cdot B_{|D}$*
2. *$H' \cap \hat{T}_0 = H' \cap \hat{T}_\infty = H' \cap (F \cap H) = H' \cap F \cap D = H' \cap F \cap B = \emptyset$.*
3. *$H^1(X, \mathcal{O}_X(H') \otimes I_D) = 0$.*
4. *$\mathscr{H}'$ is integral, surjective over S. Moreover, $\mathscr{D}$ restricts to an effective Cartier divisor on $\mathscr{H}'$ which is finite over S.*

Proof. Everything can be achieved by using the classical theorem of Bertini and its variants over S, i.e. Theorem 4.2.1.

More precisely: property (3) can be achieved by taking H' of sufficiently large degree, and property (2) is clear since all the relevant sets are zero dimensional. As for property (1), notice that H intersects $|D|$ transversally by construction (by lemma 6.2.5). Similarly, B intersects $|D|$ transversally by choice and away from H (since $H \cap D \cap B = \emptyset$ by Lemma 6.2.5). A local analysis around a neighborhood of any point in $(B \cap D) \amalg (H \cap D)$ shows that H' can be chosen to be regular there. Note that H' can be chosen to intersect D transversally at each point $x \in H \cap D$, and will be tangent to D at each point $x \in B \cap D$. We can moreover ask that H' intersects $|D|$ transversally at each point of $(H' \cap D) \setminus (D \cap (B \cup H))$.

We finally turn to property (4). Integrality of $\mathscr{H}'$, as well as the surjectivity over S are clear: the only delicate condition is the finiteness of the restriction of $\mathscr{D}$ to $\mathscr{H}'$. We first observe that condition (1) on the generic fiber imposes a closed condition on H', i.e. the containment of the points of intersection $H \cap D$ and of $B \cap D$. Both sets are finite, and the closure in the model $\mathscr{X}$ of any closed point in $(H \cap D) \cup (B \cap D)$ is finite over S thanks to the construction of H in Lemma 6.2.5 and the choice of B at the beginning of Section 6.2. We leave the details to the reader. □

Write $g = (s_0/s_\infty \beta^N)_{|D} \in H^0(D, \mathcal{O}(H + N \cdot B))$. By property (1) above, we can consider the image of g in $H^0(D, \mathcal{O}_X(H'))$, and choose a lift G along the surjection (guaranteed by property (3)).

$$H^0(X, \mathcal{O}_X(H')) \to H^0(D, \mathcal{O}_D(H')) \to 0$$

Write W_0 for the divisor of zeros of G. Note that $\mathrm{div}(G) = W_0 - H' \in \mathcal{Z}^1(X)$ by construction.

Claim 6.3.2. *W_0 is regular in a neighborhood of every point x in $W_0 \cap D$ and $W_0 \cap D \cap H' = \emptyset$.*

Proof. Again by construction, any point x in D such that either s_0 or $s_\infty \beta$ is not a unit is a regular point of $|D|$. Let then π be a local equation for $|D|$ in a neighborhood of x, so that $D = (\pi^e)$. Any lift G of g is then in a neighborhood of x in X of the form $G = (s_0 + a\pi^e s_\infty \beta^N)/(s_\infty \beta^N)$ for $a \in \mathcal{O}_{X,x}$, so that W_0 is locally given by $s_0 + a\pi^e s_\infty \beta^N = 0$.

This is enough to conclude. In fact, observe that by condition b') in Proposition 6.2.8 and condition (2) of Lemma 6.3.1, for every $x \in H' \cap |D|$, we have that $s_{0,x} \in \mathcal{O}_{X,x}^\times$. In particular, $s_0 + a\pi^e s_\infty \beta^N$ is not in the maximal ideal of $\mathcal{O}_{X,x}$, showing that $W_0 \cap D \cap H' = \emptyset$ as required. As for the regularity, note that if $x \in D \cap W_0$, we have $G_{|D} = g = (s_0/s_\infty \beta^N)_{|D}$ and this time $s_\infty \beta^N$ is a unit there. Since $F = (s_0)$ intersects $|D|$ transversally, any lift as above is automatically regular. □

Following the path of Lemmas 6.2.2 and 6.2.5, we can now alter G by any section of $H^0(X, \mathcal{O}_X(H') \otimes I_D)$ such that

1') W_0 is regular,
2') $W_0 \cap F \cap H' = W_0 \cap \hat{T}_0 = W_0 \cap \hat{T}_\infty = \emptyset$

For condition 1'), note that regularity away from D can be achieved by standard Bertini. Along D, this is guaranteed by Claim 6.3.2. As for condition 2'), it follows from the fact that $\hat{T}_0 \cap D = \hat{T}_\infty \cap D = \emptyset$ and that $F \cap H' \cap D = \emptyset$.

We now proceed as follows. Let $\hat{\Lambda}$ be the following set

$$\hat{\Lambda} = (F \cap D) \cup (H' \cap D) \cup (W_0 \cap D) \cup (\hat{T}_0 \cup \hat{T}_\infty) \cup (F \cap H') \cup (B \cap D).$$

Choose another hypersurface section $\mathcal{L}'$ of $\mathscr{X}$, of sufficiently large degree, such that, if $L' = \mathcal{L}_K$, we have $L' \cap \hat{\Lambda} = \emptyset$. As before, we may assume that $H^0(X, \mathcal{O}_X(L')) = M' \otimes_S K$ for a free S-module M', corresponding to a projective embedding $\iota_{M'}\colon \hat{\mathscr{X}} \hookrightarrow \mathbb{P}_S(M')$ so that $\mathcal{O}(\mathcal{L}') = (\iota_{M'}^*\mathcal{O}_{\mathbb{P}_S(M')}(1))$. We can choose the degree of $\mathcal{L}'$ to be sufficiently large so that

$$H^1(X, \mathcal{O}_X(L') \otimes I_{D\cup W_0}) = H^1(\mathscr{X}, \iota_{M'}^*\mathcal{O}(1) \otimes I_{\mathscr{D}\cup\mathscr{W}_0}) = 0. \tag{6.6}$$

where $\mathscr{W}_0$ is the closure of W_0 in $\hat{\mathscr{X}}$.

We now let Ξ be the set of cancellation points in the expression of γ' as $\mathrm{div}_F(\tau) - \mathrm{div}_H(\tau)$, which moreover do not satisfy $(\mathrm{MC})_{\mathcal{O}}$. Explicitly

$$\Xi = \{\xi \in F \cap H \,|\, \mathrm{ord}_{F,\xi}(\tau) = \mathrm{ord}_{H,\xi}(\tau)\} \cap \{\xi \in F \cap H \,|\, \xi \text{ does not satisfy } (\mathrm{MC})_{\mathscr{O}}\} \subseteq \hat{\Sigma}$$

Fix $\xi \in \Xi$, and let $\lambda = \lambda_\xi$ be the order of vanishing of τ at ξ, i,e.

$$\mathrm{ord}_{F,\xi}(\tau) = \mathrm{ord}_{H,\xi}(\tau) = \lambda$$

Note that the closure of ξ in $\mathscr{X}$ is finite over S, since this property is satisfied by every point in $\hat{\Sigma}$. Choose a global section $l_\infty \in H^0(X, \mathcal{O}_X(L'))$ such that, if (l_∞) denotes its divisor of zeros, we have:

i) (l_∞) is integral, and regular.
ii) $\xi \in (l_\infty)$, and $(l_\infty) \cap (\Xi \setminus \{\xi\}) = \emptyset$.
iii) (l_∞) intersects F, $|D|$, H, H', B and W_0 transversally, which means that (l_∞) intersects each divisor in the regular locus, and transversally.
iv) $(l_\infty) \cap D \cap (F \cup H' \cup B) = (l_\infty) \cap D \cap W_0 = \emptyset$, and $(l_\infty) \cap (\hat{T}_0 \cup \hat{T}_\infty) \setminus \{\xi\} = \emptyset$
v) For every closed point $y \in (l_\infty)$ in the set

$$((l_\infty) \cap H') \cup ((l_\infty) \cap F) \cup ((l_\infty) \cap H) \cup ((l_\infty) \cap B) \setminus \Xi \tag{6.7}$$

we have that y satisfies $(\mathrm{SMC})_{\mathscr{O}}$.
vi) $(l_\infty) = (\mathcal{L}_\infty)_\eta$ for a hypersurface section $\mathcal{L}_\infty$ of $\mathscr{X}$ which is surjective over S and satisfies the property that $\mathscr{D}$ restricts to an effective Cartier divisor on $\mathcal{L}_\infty$, which is finite over S.

The last two conditions follow from Theorem 4.2.2 (applied to the inclusion of the closure of the point ξ) and Proposition 4.2.4 for the embedding $\iota_{M'}$. Notice that here the condition that $\overline{\xi}$ is finite over S is crucial, as well as the fact that F, H, H' and B all have models over S such that the restriction of $\mathscr{D}$ to them is a Cartier divisor, finite over S. This is necessary to achieve condition v). A general choice of l_∞ satisfies all the other properties, noting that the only closed condition is ii), but by construction F and H are regular and tranverse to each other there, so that (l_∞) can be itself chosen to be regular at ξ, and intersecting F and H transversally.

By (6.6), we have a surjection

$$H^0(X, \mathcal{O}_X(L')) \to H^0(D \cup W_0, \mathcal{O}_X(L')_{|D\cup W_0}) \to 0$$

Claim 6.3.3. *There exists a lift l_0 of the restriction of l_∞ to $D \cup W_0$ such that*

i') $(l_0) \cap \Xi = \emptyset$, (l_0) is regular and intersect transversally $|D|$.
ii') $(l_0) \cap (\hat{T}_\infty \cup \hat{T}_0) = \emptyset$
iii') $(l_0) \cap D \cap (F \cup H) = \emptyset$
iv') For every closed point $y \in (l_0)$ in the set

$$((l_0) \cap F) \cup ((l_0) \cap H) \cup ((l_0) \cap H') \cup ((l_0) \cap B) \tag{6.8}$$

we have that y satisfies $(\mathrm{SMC})_{\mathscr{O}}$.
v') $(l_0) = (\mathcal{L}_0)_\eta$ for a hypersurface section $\mathcal{L}_0$ of $\mathscr{X}$ which is surjective over S and satisfies the property that $\mathscr{D}$ restricts to an effective Cartier divisor on $\mathcal{L}_0$, which is finite over S.

Proof. Regularity away from $D \cup W_0$ is clear, as well as the condition $(l_0) \cap \Xi = \emptyset$, since Ξ is away from D, and condition 2') guarantees that Ξ is away from W_0. Note that $D \cap W_0 \cap (l_\infty) = \emptyset$, and that (l_∞) intersects transversally both W_0 and $|D|$, so that we can choose the lift l_0 to be regular there as well.

Since $(D \cup W_0) \cap (\hat{T}_\infty \cup \hat{T}_0) = \emptyset$, condition ii') is clear. Similarly, since $(l_\infty) \cap D \cap (F \cup H) = \emptyset$ by iv), we get condition iii').

As for the last two conditions, we argue as in the proof of Lemma 6.2.6 to further refine the choice of l_0. More precisely, consider the torsion free-submodule

$$\widetilde{M}' = H^0(\mathscr{X}, \iota_{M'}^* \mathcal{O}(1) \otimes I_{\mathscr{D} \cup \mathscr{W}_0})$$

of the free module $M' = H^0(\mathscr{X}, \iota_{M'}^* \mathcal{O}(1))$.

Let $l_1, \dots, l_r$ be a basis for a maximal free submodule E'' of $\widetilde{M}'$, and denote by the same letters the corresponding sections of $H^0(X, \mathcal{O}_X(L') \otimes I_{D \cup W_0})$. Let $\tilde{l}_0$ be any lift of the restriction of l_∞ to $D \cup W_0$ to a global section in $M'_n \otimes_S K$, and let $\lambda \in \mathscr{O} - \{0\}$ be a non-zero element such that $\lambda \tilde{l}_0 \in M'_n$. We can then add $\lambda \tilde{l}_0$ to the l_i's to get a rank $r+1$ free $\mathscr{O}$-module E'''. We can now apply the theorem of Bertini over S (Proposition 4.2.4) to the corresponding locally closed immersion to get iv'). Notice that F, H, H' and B all have models over S such that the restriction of $\mathscr{D}$ to them is a Cartier divisor, finite over S. This is necessary to apply Proposition 4.2.4 4.2.4.

Finally, to get v'), we argue again as in the last part of the proof of Lemma 6.2.6. We quickly sketch the argument: a general section of $\mathrm{Gr}_k(E''' \otimes_S k)(k)$ contains the special fiber $\mathscr{V}_s$ of the closure $\mathscr{V}$ of every point in $(l_\infty) \cap (W_0 \setminus D)$. But those points satisfy $(\mathrm{SMC})_{\mathscr{O}}$ thanks to v) above, and therefore $\mathscr{V}_s$ is not contained in any component of $\mathscr{D}_s$. But then we can assume that the image of l_0 is not zero in $H^0(\mathscr{X}_s, \mathscr{O}_{X_s}/I_{\mathscr{D}_s^i})$ for each component $\mathscr{D}_s^i$ of $\mathscr{D}_s$, which means in particular that we can chose it so that $\mathcal{L}_0$ satisfies the property that $\mathscr{D}$ restricts to an effective Cartier divisor on it, finite over S as required. □

We summarize for the reader's convenience what we have achieved so far. For simplicity, we focus on the salient properties of the new objects constructed.

Proposition 6.3.4. *Let X be as above, and let ξ be a cancellation point for the cycle γ'. Then there exist integral curves, (l_0) and (l_∞), divisors of zeros of sections $l_0, l_\infty \in H^0(X, \mathcal{O}_X(L'))$ of a very ample hypersurface section L' of X such that the following conditions are satisfied.*

1. *Both (l_0) and (l_∞) are regular, and have integral models $\mathcal{L}_0$ and $\mathcal{L}_\infty$ over S, which are surjective over S and satisfy the property that the restriction of $\mathscr{D}$ to them is an effective Cartier divisor, finite over S.*
2. *Both (l_0) and (l_∞) intersect transversally $|D|$.*
3. *(l_0) is disjoint from $\hat{T}_\infty \cup \hat{T}_0 \cup \Xi$.*
4. *(l_∞) passes through ξ, and intersects both F and H and H' transversally there.*
5. *Both (l_0) and (l_∞) are chosen so that every closed points in the sets* (6.7) *and* (6.8) *satisfy the strong modulus condition* $(\mathrm{SMC})_{\mathscr{O}}$.
6. *Let $l_0/l_\infty \in K(X)$ as rational function on X. Then the restriction of l_0/l_∞ to B, F, H', H and W_0 satisfy the modulus condition with respect to D.*

The new sections (l_0) and (l_∞) will be used to remove, from the expression of γ, the specified cancellation point ξ. This is the content of the next subsection.

6.3.1. *Moving.* We can now compute. Note that by construction each zero cycles appearing is supported on $X \setminus D$. Adding to γ the boundary of $\{\frac{s_0}{s_\infty \beta^N}, \frac{l_0}{l_\infty}\} \in K_2(K(X))$ we get

$$\begin{aligned}\gamma &= \mathrm{div}_F(\tau) - \mathrm{div}_H(\tau)\\ &= \mathrm{div}_F(\tau) - \mathrm{div}_H(\tau) + \lambda\Big(\mathrm{div}_F\Big(\frac{l_0}{l_\infty}\Big) - \mathrm{div}_H\Big(\frac{l_0}{l_\infty}\Big) +\\ &+ \mathrm{div}_{(l_0)}\Big(\frac{s_0}{s_\infty\beta^N}\Big) - \mathrm{div}_{(l_\infty)}\Big(\frac{s_0}{s_\infty\beta^N}\Big) - N\mathrm{div}_B\Big(\frac{l_0}{l_\infty}\Big)\Big)\end{aligned}$$

Collecting the terms containing F and H gives then

$$\gamma = \mathrm{div}_F\Big(\frac{\tau l_0^\lambda}{l_\infty^\lambda}\Big) - \mathrm{div}_H\Big(\frac{\tau l_0^\lambda}{l_\infty^\lambda}\Big) + \lambda\mathrm{div}_{(l_0)}\Big(\frac{s_0}{s_\infty\beta^N}\Big) - \lambda\mathrm{div}_{(l_\infty)}\Big(\frac{s_0}{s_\infty\beta^N}\Big) - N\lambda\mathrm{div}_B\Big(\frac{l_0}{l_\infty}\Big), \tag{6.9}$$

and each curve appearing in the expression has an integral model over S, which is surjective and satisfies the property that $\mathscr{D}$ restricts to an effective Cartier divisor on it, finite over S.

In this newly found expression for γ, note that ξ does not appear in the support of any of the divisors involved. Indeed, ξ does not appear in the last 3 terms by construction, while the function $\frac{\tau l_0^\lambda}{l_\infty^\lambda}$ is a unit at ξ thanks to the choices of l_0 and l_∞ (note in particular that we are using the fact that $(H \cup F) \cap D \cap (l_\infty) = \emptyset$, so that the expression $\frac{\tau l_0^\lambda}{l_\infty^\lambda}$ is indeed a unit at every point of $F \cap D$ and at every point of $H \cap D$.

Since $l_0/l_\infty = 1$ along D and $(l_\infty) \cap D \cap B = \emptyset$, note that the last term $\mathrm{div}_B\Big(\frac{l_0}{l_\infty}\Big)$ satisfies the modulus condition. Finally, note that thanks to properties v) and iv') above, every divisor appearing in (6.9) except $\Xi - \{\xi\}$ satisfies the modulus condition $(\mathrm{MC})_{\mathscr{O}}$.

We are now left to correct the second and the third term of (6.9) to get the modulus condition with respect to D. Write γ'' for the cycle

$$\gamma'' := \mathrm{div}_F\Big(\frac{\tau l_0^\lambda}{l_\infty^\lambda}\Big) - \mathrm{div}_H\Big(\frac{\tau l_0^\lambda}{l_\infty^\lambda}\Big) - N\lambda\mathrm{div}_B\Big(\frac{l_0}{l_\infty}\Big).$$

Note that, by construction and what remarked above, the cycle γ'' already satisfies the modulus condition with respect to D. Arguing as above, we add to (6.9) the boundary of $\{G, \frac{l_0}{l_\infty}\} \in K_2(K(X))$ to find

$$\begin{aligned}\gamma &= \gamma'' + \lambda\Big(\mathrm{div}_{(l_0)}\Big(\frac{s_0}{s_\infty\beta^N}G^{-1}_{|(l_0)}\Big) - \mathrm{div}_{(l_\infty)}\Big(\frac{s_0}{s_\infty\beta^N}G^{-1}_{|(l_\infty)}\Big) - \mathrm{div}_{(l_0)}(G) + \mathrm{div}_{(l_\infty)}(G)\Big)\\ &= \gamma'' + \lambda\Big(\mathrm{div}_{(l_0)}\Big(\frac{s_0}{s_\infty\beta^N}G^{-1}_{|(l_0)}\Big) - \mathrm{div}_{(l_\infty)}\Big(\frac{s_0}{s_\infty\beta^N}G^{-1}_{|(l_\infty)}\Big) + \mathrm{div}_{W_0}\Big(\frac{l_0}{l_\infty}\Big) - \mathrm{div}_{H'}\Big(\frac{l_0}{l_\infty}\Big)\Big)\end{aligned}$$

We now note the following

(1) By construction, $l_0 = l_\infty \bmod I_{W_0}$. Thus the term $\mathrm{div}_{W_0}\Big(\frac{l_0}{l_\infty}\Big)$ vanishes.
(2) $\frac{l_0}{l_\infty} \in 1 + I_D\mathcal{O}_{H',x}$ for every $x \in H' \cap D$. Note that we are in particular using the fact that $(l_\infty) \cap H' \cap D = \emptyset$, which implies that l_∞ is a unit at every point $x \in H' \cap D \supset H \cap D$.
(3) The function G is constructed as global lift of the restriction of $\frac{s_0}{s_\infty\beta^N}$ to D. Moreover, for every $x \in (l_0) \cap D$ we have that $G_{|(l_0),x} \in \mathcal{O}^\times_{(l_0),x}$, since $(H' \cup W_0) \cap D \cap (l_0) = \emptyset$. In particular, we have that $G^{-1}_{|(l_0),x} \in \mathcal{O}^\times_{(l_0),x}$, so that $\frac{s_0}{s_\infty\beta^N}G^{-1}_{|(l_0)}$ is regular and invertible at every $x \in D \cap (l_0)$, and it's congruent to 1 mod I_D. In other words, it satisfies the modulus condition. The same argument applies verbatim to $\frac{s_0}{s_\infty\beta^N}G^{-1}_{|(l_\infty)}$ on (l_∞).

Thus γ simplifies as

$$\gamma = \gamma'' + \lambda\Big(\mathrm{div}_{(l_0)}\Big(\frac{s_0}{s_\infty\beta^N}G^{-1}_{|(l_0)}\Big) - \mathrm{div}_{(l_\infty)}\Big(\frac{s_0}{s_\infty\beta^N}G^{-1}_{|(l_\infty)}\Big) - \mathrm{div}_{H'}\Big(\frac{l_0}{l_\infty}\Big)\Big)$$

and every term satisfies the modulus condition with respect to D as well as every closed point appearing in the expression satisfies $(\mathrm{MC})_{\mathscr{O}}$, with the only exception of the points in $\Xi - \{\xi\}$. Repeating the argument using $\gamma'' + N\lambda\mathrm{div}_B\Big(\frac{l_0}{l_\infty}\Big)$ instead of γ, we can remove every other cancellation point $\xi' \in \Xi$ which do not satisfy $(\mathrm{MC})_{\mathscr{O}}$. This completes the proof of Claim 5.4.5. , and hence that of Theorem 5.3.2 thanks to the reduction steps of Section 5.

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