

**SOLVING A LONGSTANDING CONUNDRUM: THE UNEASY
COEXISTENCE OF ALTERNATIVE EQUILIBRIUM CONCEPTIONS IN
WALRAS'S THEORETICAL SYSTEM**

FRANCO DONZELLI

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DIPARTIMENTO DI ECONOMIA, MANAGEMENT E METODI QUANTITATIVI

Via Conservatorio 7
20122 Milano

tel. ++39 02 503 21501 (21522) - fax ++39 02 503 21450 (21505)

<http://www.economia.unimi.it>

E Mail: dipeco@unimi.it

Franco Donzelli
Department of Economics, Management, and Quantitative Methods
Università degli Studi di Milano

**Solving a Longstanding Conundrum: The Uneasy Coexistence of Alternative Equilibrium
Conceptions in Walras's Theoretical System**

Abstract

The evolution of Walras's theoretical ideas over the last thirty years of the 19th century is highly tormented: from the early outline of Walras's theoretical system, as sketched in his surviving Geneva lecture notes (1871-72), through the various editions of the *Éléments* (1st: 1874-77; 2nd: 1889; 3rd: 1896; 4th: 1900), Walras's exposition of General Equilibrium Theory undergoes a process of ceaseless revisions, which often do not blend with each other, occasionally giving rise to self-contradictory statements. This paper purports to show that such a tortured development can ultimately be traced to the uneasy coexistence in Walras's approach of two different conceptions of economic equilibrium: a more traditional one, where an equilibrium of the economy is viewed as the outcome of a balancing of forces, themselves unrelated or only barely connected with individual optimizing behavior; and a more innovative conception, resting on the then revolutionary idea that an equilibrium ought to be viewed as a consistent array of optimally chosen individual plans of action. Starting from the first conception, which pre-dates Walras's discovery of the link between individual demand and utility maximization (fall 1872), Walras strives to progressively enhance the role played by the second, without ever fully succeeding in his attempt. This is the root cause of most of the inconsistencies marring Walras's *Éléments*, as well as indirectly affecting the subsequent developments of General Equilibrium Theory.

JEL Codes: B13, B21, B31, D50

Keywords: Walras, general equilibrium, *tâtonnement*, balance of forces, compatibility of plans

1. Introduction

Léon Walras (1834-1910) is universally regarded as the founder of the theoretical approach currently known as General Equilibrium Theory (GET). Such approach, which experienced an extraordinary blossoming in the second half of the 20th century, still represents nowadays one of the main cornerstones of contemporary economic theory. In view of this, not only those historians of economics who are conversant with economic analysis, but also those theorists who do not disdain reconstructing the process of filiation of the ideas that they currently employ in their own research, have found and still find it quite natural to go back to Walras's writings to better understand the intricacies of a research program which, after one century and a half of incessant advances, has by now reached a remarkable degree of complexity.

Walras developed his own version of GET over the last three decades of the 19th century. Even if it is always somewhat arbitrary to fix starting and ending dates for complicated mental processes spanning a long period of time, in this case one can quite confidently single out a beginning and an end in the process through which Walras arrived at fleshing out his theoretical system. For it is only in the fall of 1871 that, after more than a decade of almost inconclusive efforts, he eventually succeeded in sketching a fairly detailed outline of his work plan (1871-72¹), which, though still embryonic and incomplete in a number of respects, nonetheless covered a significant portion of what would become in a very short time his fully-fledged theoretical system; as to the terminal date, let it suffice to recall that the 4th edition of Walras's *Éléments*² finally saw the light in 1900, definitely bringing to a close the author's creative phase: in fact, nothing significant would have been written by Walras after the turn of the century, with perhaps the only exception of the short article "Économique et mécanique" (1909).

¹ In this paper all references to Walras's works other than the *Éléments d'économie politique pure*, or, in short, the *Éléments*, are simply given by mentioning, between parentheses, the year of first publication or, in case of unpublished material, as it is the case here, by mentioning the year or years when such material was presumably written, without specifying the name of the author; page indications always refer to the reprint of Walras's work under question in the relevant volume of the *Œuvres économiques complètes* of Auguste et Léon Walras, quoted in full in the References at the end of the paper. In the various versions of Walras's works reprinted in the Volumes of the *Œuvres économiques complètes* there occasionally occur minor changes in either the titles of the writings or the wording of some sentences with respect to the original publications. In any case, all such changes are signaled in the critical edition we refer to. Occasionally we shall refer to sentences or documents due to the editor(s) of one or the other of the volumes of the *Œuvres économiques complètes* of Auguste et Léon Walras. In such a case, our quotation will be (Walras, year of publication of the volume concerned, page(s) of the sentence or document we are referring to). As to the *Éléments*, we direct the reader to the following footnote.

² During Walras's lifetime there appeared four editions of the *Éléments*: the 1st was published in two instalments in 1874 and 1877, respectively; the 2nd in 1889; the 3rd in 1896; the 4th in 1900. A 5th edition, containing a few minor changes and additions to the 4th edition, written by Walras himself at the beginning of the 20th century in view of a planned reprint of the *Éléments*, was posthumously published in 1926. When quoting from the *Éléments*, we shall invariably refer to the comparative edition, published in 1988 as vol. 8 of the *Œuvres économiques complètes* of Auguste et Léon Walras. Each reference in the text will have the following structure: (1988, **n** or **n**₁-**n**₂, p or p₁-p₂), where **n**, **n**₁, and **n**₂ are edition numbers, with **n** running from 1 to 5, **n**₁ from 1 to 4, and **n**₂ from 2 to 5, while p, p₁, p₂ are page numbers, always referring to the 1988 comparative edition. When no edition number is specified, it is implied that no change occurred in the quoted passage or expression over all the five editions; on the contrary, when just one edition number is specified (resp., when two edition numbers joined by a hyphen are specified), it is implied that the quoted passage or expression only occurs in the specified edition (resp., in all the editions between the specified two, themselves included). Jaffé's English translation of the *Éléments* will be referred to as (Walras 1954).

Over this thirty-year time period Walras's exposition of his ideas underwent a process of ceaseless revisions, often not blending with each other and occasionally giving rise to inconsistent statements. Confronted with such a distressing conundrum, most readers and interpreters of Walras's theoretical writings have tried to find out some simplifying accommodation, helping to make sense of a stuff which might often appear to be unmanageable at first sight.

The more theoretically oriented scholars have generally tended to play down the most obvious obscurities and inconsistencies, blaming such deficiencies on the pioneering character of Walras's undertaking and confining themselves to pick from Walras's plentiful menu those ideas that best fit their own idiosyncratic readings of the Walrasian theoretical system. This was the attitude, e.g., of such early interpreters of Walras's thought as Edgeworth (1881) and (1889), or Pareto (1896-97), (1906), and (1909), who also significantly changed his mind in this respect from the first to the last two of the works mentioned, or, a few decades later, Hicks (1934) and (1939). But it was also the stance taken in the three decades immediately following World War II by the founders of the so-called 'neo-Walrasian research program', such as Arrow and Debreu (1954), Debreu (1959), Arrow and Hahn (1971), and Arrow (1974), or by a few theorists related to that 'program' and interested in specific aspects of Walrasian economics, such as Diewert (1977).

Quite recently, some more historically oriented scholars, among whom Walker (2006) and Walker and van Daal (2014) deserve particular mention, have tried to account for the tormented evolution of Walras's theoretical ideas over his active scientific life by resorting to questionable neuro-psychological explanations: namely, such authors have evoked Walras's ailing health and declining mental powers as the only possible justification for the changes he decided to bring about in his theoretical system over the last few years of the 19th century. According to the above-mentioned scholars, Walras's change of mind intervened after the publication of the 3rd edition of the *Éléments* (1896) had the consequence of marring the 4th edition of the same work (1900), the last during the author's lifetime; this, in turn, prejudiced the subsequent developments of GET, since such developments have been mostly inspired by the contents of such edition of Walras's masterpiece.

On this issue, the present paper takes a view which is quite different from those prevailing in the literature. In our opinion, the labored evolution of Walras's theoretical ideas over the period 1871 to 1900 should neither be shelved as a minor episode, scarcely interesting from a theoretical viewpoint and at most deserving the attention of historians of ideas with a strictly archeological inclination; nor regarded as the inevitable confusion associated with any path-breaking enterprise, what would authorize all the subsequent generations of general equilibrium theorists to select from Walras's extensive legacy whatever ideas they deem apposite, while ignoring without any qualms or misgivings all that they might consider as inappropriate or outdated; nor, finally, set aside as a pathological phenomenon due to Walras's psychic disorder in the last stage of his active life. On the contrary, the painstaking efforts made by Walras over thirty years to clarify, amend, and improve upon all the successive versions of his own theory ought to be read as a lifelong attempt to solve a deep-seated dilemma connoting Walras's theoretical system from the start, a dilemma that, in spite of all his endeavors, he was unable to settle to his complete satisfaction and was consequently forced to pass on to his successors as an open question.

In Walras's original approach such dilemma can ultimately be traced to the uneasy coexistence of two alternative broad conceptions of economic equilibrium. On the one hand, in Walras's writings one can easily detect the widespread existence and persistence of a traditional conception of economic equilibrium, adapted from the equilibrium notions prevailing in the natural sciences (especially classical mechanics), where an equilibrium of either the entire economy or one of its subsystems ought to be viewed as the final outcome of a process of 'balancing of forces', themselves unrelated or only barely related with the optimizing behavior of the individual participants in the economy or subsystem concerned; on the other hand, side by side with such a traditional conception, one can also find a more innovative understanding of the equilibrium concept, resting on the idea, quite common nowadays, but utterly revolutionary in Walras's own times, that an equilibrium of either the economy or one of its subsystems ought to be viewed as an array of mutually compatible plans of action, which are also optimally chosen by the individual participants in the economy or subsystem concerned.

As will be shown in the sequel, the traditional conception is deeply rooted in Walras's juvenile system of thought: not only does it pervade his pre-analytic vision of the functioning of the economy, but it also affects his originally suggested formalization of such functioning. As a matter of fact, it was only in the fall of 1872 that Walras, luckily stumbling, with the essential help of an academic colleague, on the analytical link connecting individual demand with utility maximization, became aware that, with reference to one particular subsystem of the economy, namely, the pure-exchange subsystem, an equilibrium state can be viewed as the result of the optimizing choices of the individuals taking part in the subsystem concerned, rather than as the outcome of the 'balancing' of anonymous 'forces', whose relations with individual behavior need not be formally specified.

Starting from that flash of inspiration, over the following decades Walras strived to progressively enhance the role played by the newly discovered equilibrium conception to the detriment of the more traditional one. As a matter of fact, he quickly succeeded in reformulating the part of his pure-exchange model dealing with the determination of equilibrium in accordance with the new findings, so that he was able to fit such new version of the model in his 1874 writings – i.e., in both his first theoretical *mémoire* on the theory of exchange (Walras 1874) and the first instalment of the 1st edition of the *Éléments*, published in 1874. Yet, the part of this model dealing with the equilibration mechanism supposedly at work in the subsystem concerned, i.e., in Walras's own words, the *tâtonnement* part of the model, was not initially affected by Walras's 1872 discovery of the new equilibrium conception. Similarly, such discovery had no apparent effect on either the equilibrium determination or the *tâtonnement* parts of the other three models populating Walras's theoretical system, i.e., the two formalized models dealing with the phenomena of production and capital formation, respectively, and the relatively informal treatment of a number of monetary issues, whose formalization by means of a true and proper, albeit incomplete, model will have to wait up until the very end of the 19th century (1899)³.

³ In the following, for the sake of brevity, we shall refer to the two components of each of Walras's four equilibrium models as the 'equilibrium determination' and the 'equilibration' part of the model, respectively. Walras's own word for 'equilibration' is *tâtonnement*, so that the two terms will be used interchangeably in the following. At the beginning of section 3 below we shall explain why the two components should not be referred to, in this context, as the 'static' and the 'dynamic' part of each model, respectively. Moreover, and always for the sake of brevity, we shall denote as

Over the years, Walras tried hard to reformulate the equilibrium determination parts of the three models left originally unaffected by his 1872 findings, as well as the *tâtonnement* analysis of the equilibration processes assumedly at work in all of his four subsystems, in such a way as to make the overall treatment of all the issues involved more and more consistent with the newly discovered equilibrium conception. Yet, as will be shown in the sequel, Walras was never able to fully attain the results he was expecting from the planned changes and revisions: as regards the equilibrium determination part of his theory, Walras's incomplete success was due, at least to some extent, to his own analytical insufficiencies, as such amendable by his followers and successors; yet, as far as the *tâtonnement* part is concerned, Walras's partial failure appears to be intimately related to the underlying conflict between the two equilibrium conceptions uneasily coexisting in Walras's mind and writings.

The paper is structured as follows. In Section 2 we set the stage for the following discussion, by reviewing those of Walras's theoretical writings of the period 1871-1900 that we shall have to take into account in order to prove our main contention. Section 3 will explain how the two alternative equilibrium conceptions shape the analytical structure of the equilibrium determination part of the pure-exchange equilibrium model. Section 4 will similarly examine the equilibrium determination parts of the other three equilibrium models extensively discussed by Walras in his writings. Section 5 will explain how the conflict between the two alternative equilibrium conceptions coexisting in Walras's theoretical system affects the *tâtonnement* analysis of the equilibration processes supposedly at work in Walras's four equilibrium models. Section 6 concludes.

2. Setting the stage

The earliest reasonably detailed statement of the work plan that in a short time will mature into Walras's full-grown theoretical system can be found in a series of undated Notes, organized in ten "Leçons", presumably written in the fall 1871 and early winter 1871-72 in view of the projected delivery of a series of public lectures to be held in Geneva at the beginning of 1872. Such Notes have been recently published in Volume XI of the *Œuvres économiques complètes* of Auguste et Léon Walras under the title "Système des phénomènes économiques. Nature de la richesse" (1871-72, 417-73)⁴.

In Walras's 1871-72 Notes, on top of the outline of the intended work, there is also room for relatively extended discussions of a few specific topics, some passages of which will reappear *verbatim* in all of Walras's later writings. This particularly applies to the first four Leçons and the

'models' all of Walras's four theoretical constructs, even if, as mentioned in the text, only the first three would really deserve to be called 'models' for most of Walras's scientific life.

⁴ In the same Volume one can also find the edited versions of a number of manuscripts written by Walras in the period 1869 to 1871, published under the respective titles: "Application des mathématiques à l'économie politique (2^e tentative 1869-1870)" (1869-70a), "Capitaux et revenus" (1869-70b), "Le rétablissement de la théorie de la valeur" (1869-70c), "Application des mathématiques à l'économie politique (1871) 3^e tentative (La bonne, v.p. 15)" (1871). Though such writings attest the progressive evolution of Walras's ideas in the fields of exchange and production, anticipating some of the notions and constructs that will be fully developed in the immediately following years, they are still far away from the degree of coordination and complexity characterizing the 1871-72 Notes.

first half of the 5^e (1871-72, 418-48), devoted to the exposition of the theory of exchange: there one finds both a comprehensive analysis of the equilibrium of the exchange between two commodities, lacking however any derivation of the individual demand and supply functions from the solution of the traders' constrained utility maximization problems (2^e Leçon), and a fairly extensive analysis of the equilibrium of the exchange between several commodities (4^e Leçon), where there also appears the expression “équilibre général du marché” in the special meaning attached to it by Walras (1871-72, 438-41)⁵. At the end of the 1^{re} Leçon (1871-72, 420-23) one can also find a summary discussion of the equilibration process supposedly leading to the establishment of the equilibrium (money) price in the market for one specific commodity (the “rente française 3 %”) exchanged against money. The example used by Walras to illustrate the working of the equilibration process will be taken up again, with only minor modifications (with one exception), in all of his subsequent writings on this subject. The process itself is synthesized by Walras by means of a ‘law’, the so-called “«loi de l’offre effective et de la demande effective» [qui] est la base de toute la théorie de la valeur d’échange”; according to this ‘law of supply and demand’, the (money) price of the commodity concerned changes in the same direction as the excess demand for that commodity.

The 6^e and 7^e Leçons, focusing on the theory of circulation and money, anticipate some of the fundamental themes of Walras’s subsequent analysis on this subject-matter, in particular the distinction between what he calls the *numéraire*, playing the role of unit of account, and money proper (“*monnaie*”), playing the role of medium of exchange (1871-72, 451-6). Therein one can also find a rudimentary theory of the “value of money”, exploiting the idea that the demand for money, essentially stemming from transaction purposes, is proportional to the values of the various commodities entering the circulation process according to commodity specific “coefficients de circulation”. From this Walras deduces a simple version of the quantity theory of money, admittedly taken over from the monetary debates of the past (1871-72, 457-64).

Finally, the last three Leçons, together with the second half of the 5^e, are chiefly devoted to the theory of production and capital formation, also viewed from a historical perspective. In the 8^e Leçon, Walras introduces his fundamental distinction, inherited from his father Auguste, between the two categories respectively designed as “capitaux” and “revenus”, a distinction to which he will firmly stick throughout all of his writings: by “capitaux” (capitals) he means all the objects of wealth that survive their first use, i.e., all durables, whereas by “revenus” (incomes) he means all the objects of wealth that do not survive their first use, i.e., consumer goods⁶ and raw materials, provided that they are non-durables, as well as the services rendered by the various types of “capitaux” (1871-72, 465)⁷. The “capitaux”, in turn, are grouped in the three categories of the

⁵ We shall come back to this point in section 3 below.

⁶ For Walras all consumer goods are nondurables, since all durables are grouped under the general heading of “capitaux”, irrespective of whether they are used in consumption, in which case they render consumable services (“services consommables”), or in production, in which case they render productive services (“services productives”).

⁷ In principle, the categories of “capitaux” and “revenus” are so defined by Walras as to appear mutually exclusive and exhaustive. Yet, in the very same page quoted in the text, there pops up a third category, that of circulating capitals (“capitaux circulants”), which apparently lies in between the other two, ambiguously sharing some properties of both: as a matter of fact, such category is first introduced into the picture by the equivocal expression “revenus (ou capitaux circulants)”, which might appear as a contradiction in terms. This category, while part of Walras’s conceptual system from the start, will be set aside and left practically unused for theoretical purposes up until 1899, when it will be retrieved from oblivion and extensively employed, together with an associated novel category of services, the ‘services of availability’ or ‘services of storage’ (“services d’approvisionnement”), in the *mémoire* “Équations de la circulation”

“capitaux fonciers” or “terres” (unperishable, unproduced, marketable), “capitaux personnels” or “personnes” (perishable, unproduced, unmarketable, at least in a non-slave society), and “capitaux mobiliers” or “capitaux proprement dits” (perishable, produced, marketable). In the 8^e, 9^e, and 10^e Leçons (1871-72, 465-73), one also finds a thorough discussion of the role of the entrepreneur, a statement of the competitive equilibrium conditions in the product market (equality of “prix courant” and “prix de revient”), as well as a discussion of the quantity adjustment rule governing the equilibration process in the same market, already examined in the second part of Leçon 5^e (1871-2, 449-51). Finally, in the 8^e Leçon one can find a brief, yet relatively complete, analysis of the relation between the net income of a perishable capital good and its price, together with a statement of the equilibrium condition prescribing the equality of the net rates of return on all capital goods (1871-72, 466-7); in the 9^e Leçon, instead, one can find a concise description of the situation that will be described in the *Éléments* with the expression “marché permanent” (continuous market), i.e., a situation where the data of the economy, hence the equilibrium, change from instant to instant (1871-72, 470).

In view of the above remarks, Walras’s 1871-72 Notes can be confidently regarded as the true starting point of Walras’s theoretical undertaking. As a matter of fact, less than two years later, in August 1873, Walras delivers a conference at the Académie des sciences morales et politiques in Paris, whose text will be published in January of the following year under the title “Principe d’une théorie mathématique de l’échange” (1874, 27-46). This *mémoire*, representing Walras’s first published theoretical work, provides a complete analysis of the problem of the exchange of two commodities (consumer goods) among an arbitrary finite number, greater than or equal to two, of traders (consumers) under competitive conditions. The corresponding model is formally described by a system of ordinary algebraic equations, including the equations expressing the traders’ demands for, and supplies of, both commodities as functions of the relative price of one commodity in terms of the other, as well as the market-clearing equations for both commodities. While in the 1871-72 Notes the individual demand and supply functions had been taken as empirically given, in this *mémoire* they are obtained as the solution of the traders’ constrained utility maximization problems. By solving the specified equation system, one can determine, according to Walras, the equilibrium relative price and the traders’ equilibrium demands for, and supplies of, both commodities. Finally, in briefly discussing the equilibration process supposedly leading to the establishment of the equilibrium (money) price in the market for one specific commodity exchanged against money (“blé”, in this case), Walras essentially follows the same lines as those adopted in this regard in his 1871-72 Notes (1874, 31-2).

The first *mémoire* is followed in the next few years by other three *mémoires*, respectively titled “Équations de l’échange” (1876a, 53-72), “Équations de la production” (1876b, 73-99), and “Équations de la capitalization et du credit” (originally, “Équations de la capitalization”) (1877a,

(1899, 563-82), written in view of the preparation of the 4th edition of the *Éléments*. ‘Services of availability’ is Jaffé’s preferred English rendering of the French original expression; ‘services of storage’ is Jaffé’s literal translation of the same expression (Walras 1954, 214 and 526, translator’s note [5]). In the 4th edition of the *Éléments*, when the “services d’approvisionnement” rendered by the circulating capitals are first introduced, they are contrasted with the “services d’usage”, both “consommables” and “productives”, rendered by durable capitals; the latter are qualified as “capitaux fixes” in that context (1988, 4-5, 265-7). As will be seen in the following, the theory put forward in the 1899 *mémoire* will be reproduced, with only slight modifications and additions, in the 29^e and 30^e Leçons of the *Éléments* (1988, 4-5, 439-84), representing the bulk of the wholly revised theory of circulation and money expounded therein.

101-133). With the publication of these *mémoires* Walras comes close to completing the fundamental structure of his theoretical system, the only significant exception being represented by the theory of money which is still largely lacking.

The second *mémoire*, “Équations de l’échange”, extends the analysis of the problem of exchange, already tackled by the first under the assumption that there exist only two commodities in the economy, to the pure-exchange subsystem of a competitive market economy where an arbitrary finite number, greater than or equal to three, of commodities (consumer goods) are traded by an arbitrary finite number of traders (consumers). The corresponding model, that will be referred to as the pure-exchange model in what follows, is formally described by a system of ordinary algebraic equations, comprising the equations expressing the traders’ demands for, and supplies of, all commodities as functions of the commodity prices expressed in terms of one commodity singled out as the *numeraire* of the economy, as well as the market-clearing equations for all commodities. The individual demand and supply functions of all commodities, where the arguments of the functions are the prices expressed in terms of the chosen *numeraire*, are obtained through a lengthy process in two steps: at first, by building on the results already reached in the first *mémoire*, for each pair of commodities the individual demand and supply functions are obtained by solving the individuals’ constrained utility maximization problems as functions of the relative price of one commodity in terms of the other (1876a, 56-60); then, by exploiting the “arbitrage” conditions already discussed by Walras in his 1871-72 Notes, the individual demand and supply functions are re-formulated as functions of the commodity prices expressed in terms of the chosen *numeraire* (1876a, 60-4).

By solving the specified equation system, one can determine, according to Walras, the equilibrium prices and the traders’ equilibrium demands for, and supplies of, all commodities. Finally, after showing how the problem of the exchange of several commodities among themselves can be “theoretically solved”, Walras discusses how the very same “theoretical solution” is found out “in practice” or “empirically” by the market through the “mechanism of competition”. Here one can find a first instance of that kind of analysis of the equilibration process, designed as the “*tâtonnement*” process by Walras, that will become an essential component of his theoretical system, recurring in all his formal models (1876a, 64-9). The rule driving the “*tâtonnement*” process in the pure-exchange model is a suitably adapted extension of the so-called ‘law of supply and demand’ already encountered in Walras’s 1871-72 Notes and in his first *mémoire*: namely, the ‘law’ becomes here a price adjustment rule prescribing that the *numéraire* price of those commodities whose excess demand is positive (resp., negative) should increase (resp., decrease).

Similarly, the third *mémoire*, “Équations de la production”, provides a formalized analysis of the problem of production and pricing of an arbitrary finite number of consumer goods by means of an arbitrary finite number of services of the various types of factors of production (in Walras’s terminology, “capitiaux”), under the assumption of a fixed-coefficient, single-output technology. The corresponding model, that will be referred to as the production model in the following, can be viewed as a suitably modified extension of the model analyzing the problem of the exchange of several commodities (consumer goods) among themselves, where the original assumption that the quantities of the consumer goods to be traded are exogenously given is replaced by the assumption

that such quantities are obtained as the result of a production process employing the services of exogenously given quantities of the various types of factors of production, hence of their respective services. The model is formally described by a system of ordinary algebraic equations, comprising: the equations expressing the consumers' demands for all commodities (consumer goods and consumable services) as functions of the commodity prices expressed in terms of one commodity singled out as the *numeraire* of the economy, from which the supply functions of the various types of services can be deduced, given the available quantities of services and their distribution among the consumers; the equations expressing the demands for all the productive services as functions of the consumers' demands for all the consumer goods, hence indirectly of all commodity prices, given the fixed-coefficient, single-output technology governing the production process; the equations expressing the equality between the price of each consumer good and its average cost of production; and, finally, the market-clearing equations for all the services.

By solving the specified equation system, one can determine, according to Walras, the equilibrium prices of all commodities (consumer goods and services), the consumers' equilibrium demands for all consumer goods, coinciding with the quantities produced of such goods, and for all consumable services, as well as the owners' equilibrium supplies of all the services, coinciding with the quantities demanded of such services for consumption and production purposes (1876a, 73-84). In this case, too, after showing how the problem of the exchange and production of consumer goods can be "theoretically solved", Walras discusses how the very same "theoretical solution" is found out "in practice" by the market through the "mechanism of competition". In the production model the equilibration process, once again called "*tâtonnement*" process by Walras, is driven by a twofold rule: the so-called 'law of supply and demand', already encountered in the pure-exchange model, provides the price adjustment rule according to which the price of all commodities (consumer goods and services) whose excess demand is positive (resp., negative) should increase (resp., decrease); the second rule, which is instead specific to the production model, is a quantity adjustment rule prescribing that the entrepreneurs engaged in a fixed-coefficient, single-output process of production whose output is any given consumer good should increase (resp., decrease) their scale of production whenever the price of the good concerned exceeds (resp., falls short of) its unit cost of production (1876b, 85-95).

Finally, the fourth *mémoire*, "Équations de la capitalization et du credit", provides a formalized analysis of the problem of production and pricing of an arbitrary finite number of new capital goods proper by means of an arbitrary finite number of services of the various types of factors of production, under the assumption of a fixed-coefficient, single-output technology. The corresponding model, that will be referred to as the capital formation model in the following, can be viewed as a suitably modified extension of the production model, where not only the consumer goods, but also the capital goods proper can be obtained as outputs of production processes employing as inputs the services of exogenously given quantities of the various types of factors of production⁸. The formal construction of the model requires the preliminary introduction of some new concepts: the net income ("revenu net") of a unit of a capital good proper is defined as the

⁸ It is assumed that each unit of capital renders one unit of service per period. Hence, from a numerical point of view, the quantities of capitals always coincide with the quantities of the services they render per period. Yet, from a dimensional point of view, the two magnitudes are obviously different.

difference between its gross income (“revenu brut”), coinciding with the price of the service rendered by one unit of that capital good, and the sum of the depreciation charge and the insurance premium levied on one unit of the same capital good, themselves assumed to be proportional to its price⁹; for each capital good proper, the rate of net income specific to that capital good is given by the ratio between the net income of a unit of that capital good and its price; the rate of net income (“taux de revenu net”), without any qualification, is the value of an abstract ratio between the net income of a unit of an unspecified capital good proper and its price: at a general equilibrium state, it must coincide with the common value, denoted by i , of all the rates of net income of all the capital goods existing in the economy; finally, aggregate savings (“épargne”) are defined as the positive difference between the excess of income (itself the value of the services supplied by the owners of the corresponding capitals) over consumption (itself the value of the consumer goods and consumable services demanded by the consumers) and the amount necessary to cover the depreciation and insurance of the capital goods proper (1876b, 101-105, 107-8).

These clarifications made, the formal model can now be described by a system of ordinary algebraic equations, including not only all the equations of the production model, duly modified, whenever necessary, in order to take into account that the capital formation model comprises new variables which were inexistent in the production model, but also a new set of equations, corresponding in number to the newly introduced variables. Precisely, the equations already present in the production model must be modified to take into account that the consumers’ demands for consumer goods and consumable services now depend not only on the prices of all consumer goods and services, but also on the rate of net income i , and that the demands for productive services now depend not only on the demands for all consumer goods, but also on the demands for all capital goods proper, hence indirectly not only on the prices of all consumer goods and services, but also on the rate of net income i . Such changes indirectly affect the market-clearing equations for all the services, since they modify the list of the arguments of the demand and supply functions entering such equations, while they leave formally unaffected the equations expressing the equality between the price of each consumer good and its average cost of production. As to the equations to be newly introduced, one has the following five sets of equations: the equations expressing the price of each capital good proper as the discounted value of the infinite series of net incomes, specific to that capital good and tacitly assumed to be stationary over the infinite sequence of time periods supposedly spanned by the economy, under the assumption that the discount factor is similarly stationary and set equal to $1/(1+i)$; the equations expressing the equality between the price of each capital good proper and its average cost of production, given the fixed-coefficient, single-output technology; the equations expressing the equality between the price of each already existing capital good and the ratio between its net income and the rate of net income i ; the equation expressing the aggregate savings of the economy (assumed “progressive”, i.e., characterized by positive aggregate savings) as a function of the prices of all consumer goods and services and of the rate of net income i ; and, finally, the equation expressing the equality between such aggregate savings and the value of the quantities produced of all new capital goods proper.

⁹ It should be noted that here the terms “revenu net” and “revenu brut” are employed by Walras to denote a value, rather than a quantity.

By solving the specified equation system, one can determine, according to Walras, the equilibrium prices of all commodities, i.e., in this model, the equilibrium prices of all consumer goods, services, and marketable capital goods, the equilibrium rate of net income, the consumers' equilibrium demands for all consumer goods and newly produced capital goods proper, coinciding with the quantities produced of all such goods, as well as the consumers' equilibrium demands for consumable services and the owners' equilibrium supplies of all services, coinciding with the quantities demanded of such services for consumption and production purposes; moreover, the equilibrium value of the aggregate savings coincides with the equilibrium value of the quantities produced of all the new capital goods proper (1876b, 106-107, 109-115). In this case, too, after showing how the problem of the exchange and production of consumer goods, services and new capital goods proper can be "theoretically solved", Walras discusses how the very same "theoretical solution" is found out "in practice" by the market through the "mechanism of competition". In the capital formation model the equilibration process, once again called "*tâtonnement*" process by Walras, is driven by the twofold rule already encountered in the production model: on the one hand, there is at work a price adjustment rule, provided by the so-called 'law of supply and demand', which is supposed here to govern not only the price changes of all consumer goods and services, as before, but also the changes in the rate of net income (actually a price ratio), which is presumed to increase or decrease according to whether the value of the quantities produced of all new capital goods proper exceeds or falls short of the aggregate savings of the economy; on the other hand, there is at work a quantity adjustment rule, which is supposed here to govern the changes in the scale of production not only of all consumer goods, as before, but also of the newly produced capital goods proper (1876b, 115-23).

The writing and publication of Walras's first four *mémoires* almost exactly overlaps with the writing and publication of the two instalments of the 1st edition of the *Éléments*; therefore it is by no means surprising that the contents of the four *mémoires* should largely correspond to the subject matter of the 1st edition of Walras's masterpiece. Precisely, the theory of exchange, developed in the first two *mémoires*, is the heart of the first part of the first instalment of the 1st edition, the second part dealing essentially with monetary matters; the theory of production and the theory of capital formation, expounded in the third and fourth *mémoire*, respectively, represent instead the bulk of the second instalment¹⁰.

As to the first *mémoire*, its substance, duly revised and largely expanded in view of the different publishing location, is reproduced, often with the same wording, in the first half of Section II of the 1st edition, dealing with the theory of the exchange of two commodities for one another. The only notable difference consists in the example chosen to illustrate the working of the "mechanism of competition" in a market where a specific commodity is traded for money: for the commodity used as an instance here is the same as that employed in Walras's 1871-72 Notes ("rente française 3 %"), instead of that appearing in the 1st *mémoire* ("blé"); such turnaround, which is not accidental, will be confirmed in all the subsequent editions of the *Éléments* (1988, 71). The text of the 2nd *mémoire*,

¹⁰ In 1877, the year when the second instalment of the 1st edition of the *Éléments* is eventually published, the four *mémoires* are collected in a brochure, titled *Théorie mathématique de la richesse sociale* (1877b, 18-133). A few years later (1883) an enlarged collection of papers, bearing the same title as the 1877 collection, but containing three more *mémoires*, will be published.

instead, is somewhat different from the text that can be found in the second half of Section II of the 1st edition, dealing with the theory of the exchange of several commodities among themselves. The 2nd *mémoire*, in fact, is partly written and revised after the publication of the first instalment of the 1st edition of the *Éléments*; as such, it reflects a few afterthoughts, chiefly of an expository nature, that Walras will later incorporate in the 2nd and subsequent editions of the *Éléments*: in particular, Walras makes explicit use of the *numéraire* in deriving the general equilibrium conditions concerning the relations among commodity price and discusses at much greater length the arbitrage operations supposedly bringing about such result (1876a, 60-9).

The second part of the first instalment of the 1st edition of the *Éléments*, coinciding with Section III in that edition, deals with monetary matters, both theoretical and applied; it immediately follows, in that edition only, the Section coping with the theory of exchange. As far as the theory of money is concerned, Walras takes the same approach as he had already adopted in the 6^e and 7^e Leçons of his 1871-72 Notes: the theory, based on the consideration of the so-called “circulation à desservir”¹¹, assumes fixed “circulation coefficients” in order to determine what might be regarded as a transaction demand for money; once again, exploiting this assumption, Walras derives a version of the quantity theory of money (1988, 1, 460-80).

As to the second instalment of the 1st edition of the *Éléments*, it should be recognized that most of its contents consist in an enlarged version, basically faithful to the original, of the theories of production and capital formation developed in the third and fourth *mémoires*, respectively. Apart from a few additions, the only significant changes in passing from the *mémoires* to the text of the 1st edition concern some particular aspects of the theory of capital formation, of which the most momentous are probably the partial revisions of the tormented exposition of the theory of the *tâtonnement* as applied to the capital formation model¹² and the suppression of a few statements concerning the marketability of the already existing capitals¹³.

The elaboration of the 2nd enlarged and revised edition of the *Éléments* takes an exceedingly long time: the new edition appears in fact only in 1889, twelve years after the completion of the 1st. Walras himself provides an explanation for such delay. In fact, in the preface to the second edition (1988, 2-3, 4), he writes: “[L]a publication de la seconde [édition] à été retardée uniquement par les études qui j’avais entreprises sur la question de la monnaie”, of which no less than eight are quoted in an attached footnote. As a matter of fact, at the moment of the publication of the fourth *mémoire* (1876) and, slightly later, of the second instalment of the 1st edition of the *Éléments* (1877), Walras regarded as essentially accomplished the task of solving the first three “great problems of theoretical economics”, namely, the “problems of exchange, production, and capital formation and credit” (1988, 2-3, 6); only the fourth “great problem”, “that of money”, still remained to be scientifically approached and solved.

¹¹ This expression is translated by Jaffé as “circulation to be cleared” (Walras 1954, 38). On the intricate evolution of Walras’s monetary theory it may be useful to consult Marget (1931) and (1935), Patinkin (1965, Note C. Walras’ Theory of Money, 541-72), and Jaffé’s Translator’s Notes to the Lessons 29 and 30 in Jaffé’s English translation of the 4th edition of the *Eléments* (Walras 1954, 542-8). Yet, it should be pointed out that the interpretations of Walras’s monetary theory advanced by these economists, as well as their comments and remarks, cannot always be endorsed.

¹² Compare the text on this topic in the 1877 *mémoire* (1877a, 115-123) with the corresponding text in the 1st edition of the *Eléments* (1988, 375-404).

¹³ On this issue we shall come back in section 4 below.

In 1889, in prefacing the 2nd edition of the *Éléments*, Walras proclaims that, through the relentless efforts spent in the twelve years intervened between the finishing of the 1st and the publication of the 2nd edition, he has eventually succeeded in solving the fourth and last “great problem” that still remained to be confronted. In fact, in considering the chief issue arising in the theory of money, that “of the value of money”, he writes (1988, 2-3, 4-6): [J]’ai substitué, en la complétant convenablement, [...] à la démonstration fondée sur la considération de la « circulation à desservir » que j’avais empruntée aux économistes dans la première édition de ces *Éléments d’économie politique pure*, la démonstration fondée sur la considération de l’« encaisse désirée »¹⁴ dont je me suis servi dans la *Théorie de la monnaie* [(1886)], so that « on a, dans le quatre leçons de théorie [monétaire] pure, la solution du quatrième grand problème qui se présente en économie politique pure [...] : celui de la monnaie”.

In effect, the theory of money developed in the 2nd edition is based on premises quite different from those underlying the 1st: first of all, contrary to what he had previously supposed, Walras now regards as “evidently advantageous”, for the purposes of building the “pure theory of money”, to start from the assumption that the “thing” serving as money be “useless” in itself, and consequently unidentifiable with a true and proper commodity¹⁵ (1988, 2-3, 446-52); in the second place, instead of supposing, as before, that the demand for money exclusively depends on the “natural circumstances” characterizing the transaction technology, Walras now assumes it to depend also on human “desires” and “will”, in the same way as the demand for consumer goods and services is assumed to depend on the individuals’ marginal utility curves (“courbes de besoin”), (1988, 2-3, 452-4). Yet, in spite of this explicitly established analogy with marginal utility analysis, no attempt is made to develop a theory of demand for money (*qua* money proper, as distinguished from any commodity-money) resting on the same principles as those employed in developing the theory of demand for consumer goods and services. So that, in the end, the formal theory of money put forward in the 2nd edition does not significantly differ from that of the 1st edition, even if the “encaisse désirée” of the 2nd edition is in principle quite different from the “circulation a desservir” of the 1st (1988, 454-60).

The second most important change that, according to Walras, differentiates the 2nd edition from the 1st has to do with the introduction of a new theorem, establishing that “la condition d’égalité du taux de revue net [est] aussi la condition d’utilité maxima pour les capitaux neufs” (1988, 2-3, 6). Walras attaches an extraordinary relevance to this theorem, to the point of writing to one of his correspondents: “[Le] *Théorème de l’utilité maxima de capitaux neufs* [...] forme vraiment le couronnement de tout l’édifice de l’économie mathématique, comme le *Théorème de la satisfaction maxima des besoins* en forme la base”¹⁶. It should be noted that also in this case, as in the case of

¹⁴ This expression is translated by Jaffé as “desired cash balance” (Walras 1954, 38).

¹⁵ At the time of the publication of the 2nd edition of the *Éléments*, Walras is still convinced that in the real world the money function must ultimately be performed by a commodity-money. Yet, for the theory’s sake, he now deems that the best starting point is to assume that the object acting as money lacks any intrinsic utility. The problems arising when the money function is performed by a commodity-money are better postponed to a later stage of the analysis.

¹⁶ Letter 859 of December 6, 1888 to H. S. Foxwell (Jaffé 1965, vol. II, 277; italics in the original). The publication of this theorem gives rise to a longish, and occasionally bitter, controversy with Edgeworth, in which also von Bortkiewicz takes part. On this it may be useful to consult Jaffé (1953), Jaffé’s Translator Notes to Lessons 26 and 27 in (Walras 1954, 533-42), and the Editor’s Introduction to a preliminary version of the *Théorème* in (Walras 1993, 511-22).

the theory of money, Walras is trying to reconnect one specific element of his theory (the equality of the rate of net income on all the newly produced capital goods), already introduced in both the fourth *mémoire* and the 1st edition of the *Éléments* without adequate explanations, to what he regards as the true foundation of all his theoretical system, i.e., that *Théorème de la satisfaction maxima des besoins* which, in the theory of exchange, establishes the proportionality of prices and individual *raretés*, or the equality of the weighted marginal utilities of all commodities for all consumers. On this we shall come back in section 4 of this paper.

A third set of changes differentiating the 2nd edition from the 1st has to do with the revision of either the interpretation or the analytic representation of the equilibration or *tâtonnement* processes supposedly at work in the three economic subsystems (exchange, production, and capital formation) for which formalized models are available. Such changes, the first step in a long sequence that will terminate only in 1900, are crucial in a number of respects. Yet, Walras tends to play them down, for reasons that will become apparent later on: “[J]’ai fait peu de changements importants dans les sections consacrées aux trois autres problèmes. [...] J’ai amélioré sur plusieurs points de détail, en la laissant telle quelle dans son ensemble, la résolution des équations de l’échange¹⁷, de la production, de la capitalisation et du crédit.”

As regards the equilibration process assumedly at work in the pure-exchange subsystem, Walras adds a few words to the already mentioned illustration of the working of the “mechanism of competition” in the market where a specific commodity (“rente française 3 %”) is traded for money. The added words, affecting just three lines in the passage concerned, amount to sterilizing any actual trade at out-of-equilibrium prices, while leaving intact in all other respects the previously given description of the equilibration process where, in case of excess demand (resp., supply) on the market, unsatisfied buyers (resp., sellers) are supposed to make bids with a view to increasing (resp., decreasing) the current market price (1988, 2-5, 71-2). Such additions, anticipated by a long footnote quite incongruously inserted at the beginning of a paper devoted to the presentation and discussion of the economic contributions of Walras’s (and Jevons’s) forerunner Hermann-Henri Gossen (1885, 312, footnote 1), represent an attempt to answer the objections raised a few years before by the mathematician J. Bertrand who, in reviewing Walras’s *Théorie mathématique de la richesse sociale* (1883), had maintained that the latter’s theory of exchange is indeterminate whenever it is admitted, as apparently Walras had been ready to do in the original version of the illustration mentioned above¹⁸, that actual trades are allowed to take place at out-of-equilibrium prices, that is, when there is excess demand or excess supply in the market concerned.

As regards the equilibration processes supposedly at work in the production and in the capital formation subsystems, Walras still keeps to the assumption, already made before, that a genuine production activity, where actual amounts of services are transformed into actual amounts of products, does take place at out-of-equilibrium prices and quantities, while the equilibration process is going on: for, according to Walras, under the assumed single-output, fixed-coefficients technology, changing the scale of operation of the various lines of production is the only way by

¹⁷ By “résolution des équations de l’échange”, Walras means to refer to the analysis of the *tâtonnement* process concerning the pure-exchange subsystem. Of course the same also applies to the other two subsystems.

¹⁸ As has been shown, such version can be traced back to a very early stage of development of Walras’s ideas, namely, to the drafting of his 1871-72 Notes.

means of which the entrepreneurs engaged in any specified activity can react to the existing inequality between price (“prix de vente”) and unit cost of production (“prix de revient”) of the product concerned, be it a consumer good or a newly produced capital good, thereby priming the equilibration process and driving the production subsystem towards the realization of the price-cost equilibrium condition in all lines of production (1988, 307-12). Yet, while in the 1st edition Walras had provisionally imagined the existence of a foreign market where the domestic entrepreneurs could purchase the amounts of services required to carry out their production activities, in the 2nd one he drops such assumption, probably dictated by the need to avoid possible jams in the regular flow of the production process, but wholly extraneous to the logic of an otherwise closed general equilibrium model, replacing it with the assumption that, at each reprise of the *tâtonnement* process, there exist domestic owners of the various factors of production ready to provide the entrepreneurs with the required amounts of productive services, under unchanged technological and endowment conditions (1988, 1, 312, and 2-3, 308).

The 3rd edition of the *Éléments*, appearing in 1896, does not display any significant textual difference with respect to the 2nd, with the only exception of a few additions consisting in three Appendices, the first two of which will survive also in the following two editions, whereas the third one will be discarded, being replaced in the 4th and 5th editions by some additions to the main body of the text. Of these three Appendices, the most interesting for our present purposes are the first and the third.

Appendix I, titled “Théorie géométrique de la détermination des prix”, puts together two *mémoires* on a similar subject which had appeared at the beginning of the 1890s (1891, 523-38, and 1892b, 539-50). The greatest interest of this Appendix resides not so much in the geometrical method discovered by Walras, allowing him to employ geometrical, instead of algebraic, techniques to formally analyze problems involving more than two commodities at a time, a task that had been regarded as beyond reach beforehand; but rather in the fact that, in less than twenty pages (1988, 3-5, 692-710), Walras is able to summarize his entire theoretical system, not only as regards the determination of the equilibrium conditions in the three models available at the time (exchange, production, and capital formation), but also as regards the establishment of such equilibrium conditions by means of the *tâtonnement* processes supposedly at work in the three subsystems concerned. Such processes, which in the body of the *Éléments* are separately analyzed in distinct Sections of the book, each referring to a specific model or subsystem and each physically distant from the other two, here for the first time are jointly discussed, in a compact sequence of carefully specified steps. Yet this unified presentation cannot but make apparent the inconsistencies marring the entire investigation, particularly if one considers that, in Walras’s overall analysis of the *tâtonnement* construct, there still coexist, in the first half of the 1890s, two alternative interpretations of the nature of the process itself: a purely virtual interpretation, where no observable action is allowed to take place until the process has come to an end, which exclusively concerns the equilibration process in the pure-exchange model; and an actual one, where observable actions are supposed to take place all along the course of the process, an interpretation which concerns instead the equilibration processes in the production and capital formation models. On this we shall come back in section 5 below.

Appendix III, titled “Note sur la réfutation de la théorie anglaise du fermage de M. Wicksteed” (1988, 3, 715-22), had been preceded by an almost identical text published in a *Receuil* of the Université de Lausanne (1896). It will be suppressed in the subsequent editions of the *Éléments*, the 4th and the 5th, where a relatively small part of its content, coinciding with the final *Post Scriptum*, will be summarized, with revisions and amendments, in the newly added paragraph 326 of Lesson 36. The chief interest of this Appendix lies precisely in its *Post Scriptum*, where Walras eventually comes back, albeit in the context of the extraneous controversy with Wicksteed and some of his British supporters (namely, Edgeworth and Marshall) on the so-called English theory of rent and the marginal productivity theory of distribution, to his old idea, dating back to the 1st edition of the *Éléments*, that “it would be easy” (“il serait facile”) to drop the assumption of fixed production coefficients, only provisionally accepted in view of its simplicity, in favor of the more realistic assumption of variable coefficients, to be determined by solving a cost minimization problem (1988, 1-3, 305-6). Here Walras, also availing himself of the hints coming from the published material by Pareto and the contemporaneous correspondence on this issue with Pareto, von Bortkiewicz, and the Italian economist Barone, summarizes his view on the cost minimization issue as of the mid-1890s¹⁹: starting from a single-output, several-input technology, as summarized by a production function tacitly assumed to be homogeneous of the first degree, he arrives at solving the cost minimization problem, thereby identifying the conditional demand functions for all the services entering as inputs into the production function and expressing the cost-minimizing marginal cost function as the common value of the ratios of the prices of the various services and their respective marginal products. Yet, he is still unable (or unwilling) to take the further fundamental step, which would consist in moving from cost minimization to profit maximization. The reasons for this will be explained in the section 4 below.

Finally, the 4th edition of the *Éléments* is characterized by three major, and a few minor, changes with respect to the third. The three major changes are summarized by Walras himself in the Preface to the 4th edition, where however Walras freely cannibalizes the Preface to the second edition, thereby mixing changes which authentically occur in the passage from the 3rd to the 4th edition, i.e., in the years between 1896 and 1900, with changes which had already taken place much earlier, in the passage from the 1st to the 2nd or from the 2nd to the 3rd edition, i.e., in the period 1876 to 1889 or 1889 to 1896.

The three major changes concern respectively: first, the amendments brought to the theory of capital formation, especially as regards the nature of the savings function; secondly, the comprehensive transformation of the theory of money; thirdly, the overall revision of the interpretation of the *tâtonnement* construct, hence also of the equilibrium concept, in Walras’s theoretical system. The last two changes are anticipated by the publication in 1899 of the *mémoire* “Équations de la circulation”, which will be reproduced almost *verbatim*, with only a few changes in the distribution of the various parts of the document, in the new Lessons 29 and 30 of the 4th edition of the *Éléments*; the first is instead an unanticipated amendment directly brought about in the 4th edition.

¹⁹ On this intricate historiographical and theoretical issue it is very useful to consult Jaffé (1964), where one can also find Walras’s French translation of a fundamental paper by Barone, reviewing Wicksteed’s *Essay* due to Barone, whose original unfortunately is lost.

As regards the changes affecting the theory of capital formation, they essentially concern the savings function. Instead of assuming that function as empirically given, as he had done in all his previous writings, Walras tries to deduce it rationally from the same principles as those employed in deriving the demand and supply functions of consumer goods and services since the publication of the first *mémoire*. To this end he introduces an ideal commodity, denoted by (E) and called “revenu net perpétuel” (perpetual net income) and, whose unit consists in the promise of receiving one unit of *numéraire* in perpetuity. The price in *numéraire* of commodity (E), p_e , is equal to the reciprocal of the “taux de revenu net perpétuel” (rate of perpetual net income), i , that is, $p_e = 1/i$. The quantities of such commodity enter the consumers’ utilities functions exactly in the same way as the quantities of the other consumer goods and consumable services; hence, for each consumer, there is supposed to exist a marginal utility function of commodity (E). The individual demands for such commodity represent the way in which individual consumers can save. In view of the above definitions and assumptions, the individual demands for commodity (E), hence individual savings, can be determined as functions of the prices of all commodities, including the price of commodity (E) itself, $p_e = 1/i$, by solving standard constrained utility maximization problems.

As regards the theory of money, one should recall that in 1889, in prefacing the 2nd edition of the *Éléments*, Walras had declared that, with the changes and the improvements brought about in the twelve years intervened between the completion of the 2nd and the publication of the 3rd edition, he had been able to solve the fourth “great problem” of theoretical economics, namely, the problem of money. Yet, ten years later, he is still patently dissatisfied with the solution adopted, to the point of writing the above mentioned *mémoire* “Équations de la circulation” (1899), which radically changes the foundations of Walras’s theory of money, though, somewhat paradoxically, leaving unchanged its conclusions. So that it is not surprising that, in the preface to the 4th edition, Walras should write again:

Mais c’est surtout la théorie de la monnaie qui a été sensiblement²⁰ modifiée par suite des études que j’ai poursuivies de 1876 à 1899 sur cette question.

And, similarly, it is not surprising, though a bit disconcerting, that he should once again conclude the part of the Preface dealing with the changes in the theory of money and circulation, now jointly considered, by saying that:

les six leçons qui la composent fournissent la solution du quatrième grand problème de l’économie pure : celui de la circulation. (1988, 4-5, 7, 9).

As can be grasped from the preceding quotations, the change in the theory of money is brought about through the development of a previously inexistent theory of circulation. In this regard, Walras retrieves the general category of circulating capital (“capital circulant”), dating back to Walras’s 1871-72 Notes (1871-72, 465), as well as its sub-categories, as detailed in the third *mémoire* on the “Équations de la production” (1876b, 76), among which one finds the “stocks of

²⁰ In the 5th edition this adverb is curiously changed into “entièrement” (1988, 5, 7, footnote ee).

income goods consisting of consumers' goods" and "raw materials"²¹ ("approvisionnement de revenus consistant en objets de consommation" and "matières premières"). All such categories had been set aside and left nearly unused for almost thirty years, being only mentioned as "phénomènes accessoires" (accessory phenomena) in the framework of a sort of simplified national accounting table for the entire economy (1988, 1-3, 574, 576). On the contrary, in Walras's 1899 *mémoire*, and then in the 4th and 5th editions of the *Éléments*, all the categories of circulating capital take on a position which, from an analytical point of view, turns out to be basically the same as the position already occupied by the categories of "fixed" capital²². In particular, as regards the specific services rendered by the various types of circulating capitals, Walras writes (1988, 4-5, 267):

Nous devons constater, dans la théorie de la circulation, que les approvisionnements de revenus, en attendant de donner leur unique *service d'usage*, donnent, eux aussi, un *service d'approvisionnement* qui peut être soit consommable soit producteur.

The demand for the consumable services rendered by the various types of circulating capital goods is determined in the same way as the demand for all consumer goods and the consumable services rendered by the various type of fixed capitals. The demand for the productive services rendered by the various types of circulating capital goods is determined in the same way as the demand for the productive services rendered by all the various types of fixed capitals, according to a given fixed-coefficient, single-output technology. As usual, the prices of the services rendered by the various types of circulating capital goods are determined by the market-clearing conditions.

Money, being a special type of circulating capital, renders a corresponding "service d'approvisionnement". The demand for such service is proportional to the value of the "services d'approvisionnement" rendered by the various types of circulating capital goods, in accordance with fixed circulation coefficients: for money is supposed to partly replace physical stocks of circulating capital goods in their role as providers of storage services. Taking this relationship into account, the price of the "service d'approvisionnement" rendered by money can be indirectly determined, once the price of the "services d'approvisionnement" rendered by the various types of circulating capital goods are known.

Finally, as regards the prices of the circulating capital goods themselves, Walras proceeds as follows. Since, for the sake of simplicity, he decides to ignore the depreciation and insurance of circulating capital goods (1988, 4-5, 449, footnote 1), all circulating capital, though consisting in nondurable "revenus", paradoxically turns out to be as perfectly durable as all landed capital. Therefore, the price of every circulating capital good is equal to the ratio between the price of the

²¹ These are Jaffé's English translations of the French expressions in parentheses mentioned in the text (Walras 1954, 219).

²² The pre-existing distinction between "capitaux" (durables) and "revenus" (nondurables) must now be further qualified, as far as it is possible, due to the theoretical role now played by the various categories of circulating capital. In fact, the "capitaux circulants" are "revenus", hence in principle nondurables (as already mentioned in footnote 7 above, this definition is dangerously close to being self-contradictory, but this is another matter). As such, they should be distinguished by the "capitaux", which instead continue to be interpreted as durables. Therefore, when in the 4th and 5th editions the two categories of "capitaux" appear side by side, in the same sentence or paragraph, Walras generally tries to distinguish between them, by characterizing the previously unqualified term "capitaux" with the qualifier "fixes". See, e.g., (1988, 4-5, 265).

corresponding “service d’approvisionnement” and the rate of perpetual net income, i . The rate of perpetual net income, in turn, is determined by the condition that the aggregate savings, itself coinciding with the value of the quantity demanded of commodity (E), be equal to the aggregate value of the quantities produced of all the new capital goods, both fixed and circulating. Since money is a special type of circulating capital, its price is equal to the ratio between the price of its “service d’approvisionnement” and the rate of perpetual net income.

Proceeding in the way illustrated above, the specific variables concerning circulating capital and money can be integrated into a comprehensive general equilibrium model, described by a system of equations whose number is equal to the number of the unknowns to be determined. By solving the specified equation system, one can determine, according to Walras, the equilibrium prices and quantities of all commodities, including the prices and quantities demanded and supplied of the circulating capital goods and their “services d’approvisionnement”, as well as the price of money and its “service d’approvisionnement”, and the equilibrium price and quantity demanded and supplied of the commodity perpetual net income.

In this case, too, after showing how the problem of the exchange and production of consumer goods, services and new capital goods proper can be “theoretically solved”, Walras discusses how the very same “theoretical solution” is found out “in practice” by the market through the “mechanism of competition”. In the model with circulating capital and money Walras’s discussion of the equilibration process, once again called “*tâtonnement*” process, is very concise: taking for granted that in the markets for all commodities the equilibration process works as already explained following the usual rules, Walras focuses his attention on the market for the services of money, where the equilibrium is reached “almost without any *tâtonnement*”, by means of simple changes of the price of the service of money in the same direction as the excess demand prevailing in that market (1988, 4-5, 465). This being said, Walras uses the “equation of monetary circulation”, establishing the equality between demand for and supply of the services of money (or money itself), to derive once again a version of the quantity theory of money.

As to the general meaning of the *tâtonnement* construct, we have already hinted at the confused situation still dominating Walras’s reasoning in the mid-1890s. As a matter of fact, in the 3rd edition of the *Éléments* there still coexist alternative interpretations of the nature of the *tâtonnement* processes supposedly at work in the various subsystems identified by Walras: for, while in the pure-exchange model the equilibration process is viewed as a purely virtual process, giving rise to no observable action out of equilibrium, in the models with production and capital formation, instead, it is viewed as a real process, giving rise to observable activities all along the equilibration path. In 1899, in developing in a specific *mémoire* his new theory of circulating capital and money, Walras appends to his *mémoire* a final Note where, after explicitly recognizing that in the previous editions of the *Éléments* the operations contemplated by the *tâtonnement* construct in both the production and the capital formation models had been supposed to be “effectively” carried out, he introduces an assumption, called “hypothèse des bons”, according to which no observable productive activity is actually carried out during the equilibration processes, all contracts being only provisionally stipulated under the condition that they are null and void if the markets for services and products are not in equilibrium (1899, 581-2). This assumption therefore generalizes to all the *tâtonnement*

processes supposedly at work in all of Walras's equilibrium models the assumption already made with reference to the pure-exchange model alone, where all the out-of-equilibrium transactions had been forbidden since the 2nd edition of the *Éléments*.

As a result of such "hypothèse", the time structure of Walras's analysis changes significantly, as specified in the following passage of the 4th and 5th edition of the *Éléments* (1988, 4-5, 447, italics in the original), replicating the text of the 1899 Note with slight modifications:

Au moyen de l'hypothèse des *bons*, on peut distinguer nettement, surtout si on les suppose successives, les trois phases suivantes:

1° La phase des *tâtonnements préliminaires* en vue de l'établissement de l'équilibre en principe ;

2° La phase *statique* de l'établissement effective *ab ovo* de l'équilibre relatif à la livraison des services producteurs et des produits pendant la période de temps considérée, aux conditions convenues, sans changements dans les données du problème ;

3° Une phase *dynamique* de trouble continu de l'équilibre par des changements dans ces données et de rétablissement continu de l'équilibre ainsi troublé.

In conformity with the new assumptions, Walras changes a few passages concerning the description of the *tâtonnement* process in both the production and the capital formation model (1988, 4-5, 309, 377) and a few further sentences here and there in the book.

3. Alternative equilibrium conceptions in Walras's pure-exchange model

As has been shown above, with reference to the four "great problems" arising in the field of economics, to wit, the problems of exchange, production, capital formation, and money, Walras invariably proceeds in a standardized way: first, for each "great problem", he puts forward a system of ordinary algebraic equations, whose "theoretical solution", corresponding to any given set of data, would allow the theorist, at least in principle, to determine the equilibrium values of all the unknown variables (essentially, prices and quantities of all the commodities traded in the economic subsystem corresponding to the "problem" at hand²³); secondly, once again for each "great problem", he less formally introduces a system of functional equations (typically, difference equations), allegedly describing the "empirical" or "practical" working of the "mechanism of competition" in the relevant economic subsystem, whose evolution over time supposedly describes the equilibration process driving the economic subsystem concerned towards the very same "theoretical solution" as the one in principle determined by solving the system of ordinary equations specified under the first heading.

For Walras the two above-mentioned aspects of equilibrium analysis are inseparable from one another. To give an idea of Walras's stance on this issue, we quote in detail the passage from

²³ In the most comprehensive model, the one also dealing with the fourth "great problem", that of circulation and money, among the unknowns one must also consider the price of money (in terms of *numéraire*), even if money is not a commodity proper.

Walras's *mémoire* on the "Equations de l'échange" (1876a, 64), where for the first time he makes his position explicit in this regard:

Voilà comment, les équations de demande étant données, les prix en résulte mathématiquement. Reste seulement à montrer, et c'est là le point essentiel, que ce même problème de l'échange dont nous venons de fournir la solution théorique et aussi celui que se résout pratiquement sur le marché par le mécanisme de la libre concurrence.

While the above passage clearly concerns the first systematic exposition of the pure-exchange model with several commodities, almost identical sentences are repeated over and over again with regard to all the four models put forward by Walras, in all of his subsequent writings. Two questions arise in this connection.

The first one has to do with the terms that might be employed to designate the two components of equilibrium analysis that are clearly identified by Walras with reference to all his models. It might be tempting to use the terms 'statics' and 'dynamics' to qualify the first and the second component of Walras's equilibrium analysis, respectively. But this would not be a good idea. In the first place, only late in his career Walras eventually accepts to parsimoniously use the adjectives "statique" and "dynamique"²⁴, as well as their corresponding nouns, and he does so only reluctantly, because he deems such words to be "dangerous"²⁵. In the second place, when Walras does actually employ the terms "statique" and "dynamique" in the 4th and 5th editions of the *Éléments*, he typically uses them to qualify a "phase" or a "point de vue", almost never an "equilibrium". In the third place, while the term "statics" might perhaps be forced, without excessive strain, to designate the first branch of Walras's general equilibrium analysis, namely, that dealing with the characterization of the equilibrium conditions prevailing in a given subsystem for a given set of data, as well as with the computation of the "theoretical solution" associated with such conditions; the term "dynamics", instead, ought never to be employed to designate the second branch of Walras's analysis, namely, that dealing with the explanation of the equilibration process supposedly at work in the subsystem concerned, aimed at finding out "empirically", or "in practice", the very same "solution" already "theoretically" obtained by solving the system of ordinary algebraic equations describing the model concerned, under the assumption of unchanging data. As we have seen, in fact, for Walras "dynamic" has the special meaning of characterizing a kind of event, process, or analysis that is inextricably associated with a change in the data of the economic system or subsystem concerned, so that that term cannot possibly be employed to designate Walras's theory of the *tâtonnement*, a construct or process which is assumed to operate or to take place under the assumption of no change

²⁴ This only occurs after the publication, in the same year 1889, of two papers by Edgeworth directly or incidentally concerning Walras's system of thought, the first being Edgeworth's critical review of the 2nd edition of the *Éléments* (Edgeworth 1889a), and the second his "Presidential Address to Section F of the British Association" (Edgeworth 1889b). In these two writings the author, in levelling his criticism at Walras's theory of the *tâtonnement* and related topics, make extensive use of the terms 'statics' and 'dynamics' thereby inducing Walras to employ the same or similar terms, at first in his correspondence and private notes, and later also in his publications, in order to favor reciprocal communication and understanding.

²⁵ This is what Walras himself asserts in a Note for private use probably written in 1891, discovered by Jaffé among Walras's papers deposited at the University of Lausanne, and published in Vol. II of Walras's Correspondence as Part II of an attachment to Walras' letter to Vilfredo Pareto dated January 9, 1895 (Jaffé, 1965, vol. II, letter 1200, 630, fn. (*)).

in the data. To sum up, therefore, leaving aside the terms ‘statics’ and ‘dynamics’, one should better call the two components of Walras’s equilibrium analysis with their original names, that is: the first should be qualified as the determination of the equilibrium conditions relative to the model of a given economic subsystem, with a view to finding out the “theoretical solution” associated with the equation system describing the model itself; the second as the examination of the equilibration or *tâtonnement* process relative to a specified economic subsystem or the corresponding model.

This being said, it remains to decide how to develop our discussion of the effects of the conjectured coexistence of two alternative equilibrium conceptions on Walras’s equilibrium analysis. In principle, the two components of the Walrasian theoretical system are so strictly intertwined as to suggest that they should be jointly taken into account in the discussing our problem. Yet, for expository purposes, Walras himself, in all of his models, invariably brings to an end the examination of the first component before embarking upon any investigation of the second. We have given above an example of the typical sentence used by Walras at the moment of passing from the first to the second part of the analysis in all of his models. Therefore, for the same expository reasons as Walras’s, we shall proceed in the same way here: at first, we shall focus on the equilibrium conditions of the four equilibrium models and their associated “theoretical solutions”, examining the pure-exchange model in this section and the other three models (production, capital formation, and money) in the next one; then, in section 5, we shall discuss the equilibration processes coupled with the four equilibrium models.

The pure-exchange model, as embryonically developed by Walras in his 1871-72 Notes, deals at first with an arbitrary finite number of traders exchanging two commodities (consumer goods) for one another, and is subsequently extended to an arbitrary finite number of traders exchanging several commodities (three or more) among themselves. In the pure-exchange model with only two commodities, given the individual and aggregate demand functions, where the quantities demanded and supplied are expressed as functions of the relative price of one commodity in terms of the other, and assuming that such functions satisfy suitable properties, which are supposed to reflect alleged empirical characteristics of the underlying relations, only vaguely perceived and sketched out, the equilibrium determination boils down to finding the relative price in correspondence to which aggregate demand and supply are equal to one another for both commodities (1871-72, 424-37). In the case of several commodities, the condition that has just been mentioned for each pair of commodities must be supplemented by an additional clause, concerning every triple of commodity prices, a clause that Walras is able to obtain by exploiting Cournot’s no-arbitrage conditions as spelled out in Chapter III of the latter’s *Recherches* (Cournot 1838, 28-45).

Such additional condition is specified as follows: when the number n of commodities is greater than or equal to three, for any three commodities a , b , and c , the relative price of commodity c in terms of commodity b , p_{cb} , must be equal to the ratio between the relative price of commodity c in terms of commodity a , p_{ca} , and the relative price of commodity b in terms of commodity a , p_{ba} , i.e., $p_{cb} = p_{ca} / p_{ba}$. According to Walras (1871-2, 438-41), an equilibrium of a multi-market economy, where the number n of commodities is no less than three, can be qualified as “général” if and only if, for any triple of commodities, the clause specifying the no-arbitrage relations among their relative prices holds. In this definition of “équilibre général”, the aspect that is put to the forefront is obviously that of the general interdependence among all the markets and, more generally, among all

the economic phenomena characterizing a multi-commodity economy. Moreover, if and only if the above condition is met, the $n(n-1)$ relative prices for all the possible ordered pairs of commodities can be simplified into a list of n prices expressed in terms of one commodity singled out as the *numeraire* of the economy, whose price is set identically equal to 1. Behind the realization of such a “general equilibrium of the market”, with its associated list of n prices, one for each commodity, there is assumed to be at work an equilibration process, called “arbitrage” by Walras in Cournot’s wake, which is driven by the individuals’ attempts to exploit every opportunity to make a gain from trade whenever the above condition is violated (1871-2, 438-46).

As can be gathered from the preceding paragraph, Walras arrives very early at both characterizing his notion of “équilibre général” and analytically determining such equilibrium along the lines broadly illustrated above, so that the relevant concepts and the essence, though not all the details, of the proofs can already be found in his 1871-72 Notes (Walras 1993, 413-5). Yet, at the moment of writing such Notes, Walras, though devoting a lot of remarks to the notion of utility and its possible applications to the theory of exchange (1871-2, 3^e and 5^e Leçons, 430-5, 447-51), is still far away from discovering the existence of what will soon after appear as the fundamental relation between marginal utility and price, and consequently also from succeeding in deriving the individual demand and supply functions of consumer goods from the solution of a constrained utility maximization problem. As is well-known, such discovery and derivation will only occur in the fall 1872, and only thanks to the essential help of Paul Piccard, a professor of mechanics and one of Walras’s colleagues at the Academy (later University) of Lausanne²⁶.

Once this step is taken, the individual demand and supply functions, instead of being regarded as empirically given, appear as the result of the traders’ optimizing choices. Similarly, the equilibrium of the pure-exchange model, instead looking as the outcome of the balancing of objective ‘forces’, grasped on the basis of the investigator’s empirical experience and no further probed or explained from a theoretical point of view, is interpreted as a state where the individual plans of action (trade plans, in the case under discussion), optimally chosen by agents (traders) pursuing their subjective interests in a competitive setting, what implies a price-taking behavior, are mutually compatible and therefore, by virtue of such compatibility, objectively feasible. Of course, these results cannot be found in Walras’s 1871-72 Notes, where the theoretical potentialities offered by the utility maximization approach still are altogether unknown; but they lie at the very center of all of Walras’s later writings on the theory of exchange, starting from his first two *mémoires* (1874 and 1876a) and proceeding with the Sections on this topic in all the editions of the *Éléments* (Section II in the first three editions; Sections II and III in the last two, where the two-commodity case is separated from the several-commodity case) (1988, 1, 162, 168, 170, 172; 1988, 2-5, 161, 163, 167, 169, 171). Moreover, from the publication of the second *mémoire* onwards (1876a, 60-9), by

²⁶ Paul Piccard’s contribution to the solution of Walras’s problem of deriving individual demand functions from utility and quantity considerations was first disclosed by Jaffé, in his monumental edition of Léon Walras’s correspondence (Jaffé 1965, 309-11), where Piccard’s manuscript containing his answer to Walras’s request for help is described and the text published in full. Jaffé’s remarks on this issue in his 1972 paper are very useful, too (Jaffé 1972, 303-5). The text of Piccard’s manuscript is also reproduced, together with some glosses in the margins by Walras and a few comments by the Editor, in (Walras 1993, 693-5). Walras thanks Piccard for his help in a letter addressed to the latter on October 25, 1873. However, such letter, which will be published in November of the same year in the *Bulletin of the Société vaudoise des sciences naturelles* (1873, 479-81), does not appear to do full justice to Piccard, in the light of the latter’s fundamental contribution to the building of Walras’s theoretical system.

supposing the “arbitrage” operations to be so efficient as to carry their effects through before anything else has time to occur, and consequently exploiting from the start the possibility of expressing the prices of the n commodities in terms of one of them chosen as the *numéraire*, Walras is able to redefine his “general equilibrium” condition by means of a system of n market-clearing equations, one for each commodity, each stating the equality of the demand for, and the supply of, the corresponding commodity, where all the demands and supplies are functions of the n prices expressed in terms of the chosen *numéraire*.

And yet, something of the original non-optimizing approach, characteristic of Walras’s 1871-72 Notes, persists in his later writings, too. For, somewhat incongruously, in the 1874 *mémoire* on the theory of exchange between two commodities, as in all the subsequent writings on this specific topic, the determination of the competitive equilibrium solution on the basis of “empirically given” demand functions precedes the derivation of the individual demand and supply functions from the solution of the corresponding utility maximization problems. The order in which the two problems (namely, equilibrium determination and demand functions derivation) are tackled is then reverted in the 1876 *mémoire* on the theory of exchange of several commodities among themselves, where a sort of methodological justification for such reversal is provided (1876a, 54), and in all the subsequent writings on this issue (1988, 1 and 2-5, 153-254²⁷). Yet, the expository choice made by Walras in his first theoretical publication reveals that, as regards the pure-exchange model, the equilibrium conception he originally had in mind is independent of any optimizing presupposition; as a matter of fact, such a presupposition powerfully enters into the picture only at a later stage, after Paul Piccard’s disclosure of its theoretical potential, fostering the emergence of an alternative equilibrium conception, whose first application in Walras’s economics is precisely represented by the theory of exchange.

The idea that in Walras’s equilibrium analysis and, more generally, in the whole of GET, there coexist two alternative equilibrium conceptions is adumbrated, among others, by Arrow in his Nobel Prize Lecture, devoted to an examination of GET. In fact, referring precisely to Walras, Arrow (1974, 254, italics in the original) identifies in the “balancing of supply and demand” the founding element of the very notion of Walrasian general equilibrium, a notion that from this point of view conforms to the interpretation of the term ‘equilibrium’ that characterizes the natural and formal sciences:

[T]he balancing of supply and demand under these conditions may be referred to as *equilibrium* in accordance with the usual use of that term in science and mathematics.

Yet, in the following page, in formally expounding the general equilibrium model, as originally put forward by Walras and subsequently developed by his successors in the first half of the twentieth century, among whom John Hicks (1939) and Paul Samuelson (1947) are specifically mentioned, Arrow (1974, 255) identifies the compatibility among optimally chosen plans of action as the

²⁷ In the specified page interval of the comparative edition of the *Éléments*, pages reproducing the text of the 1st edition alternate with pages reproducing the text of all the following editions. These textual differences, however, have nothing to do with the issue discussed in the text of this paper.

distinctive feature of GET, where the plans are supposed to be chosen by independent individuals interacting through the competitive price mechanism:

Competitive analysis is founded on two basic principles: optimizing behavior on the part of individual agents in the presence of prices taken as given by them and the setting of prices so that, given this individual behavior, supply equals demand on each market.

With a view to facilitating the exposition, in the following we shall use two synthetic expressions – equilibrium as a ‘balance of forces’ *versus* equilibrium as an ‘array of mutually compatible optimizing plans’ – to identify the two alternative conceptions of economic equilibrium coexisting in Walras’s work, not only, as has been seen, in his theory of exchange, but also, as will be seen, in all the other parts of his theoretical system. Even if such expressions are never directly employed by Walras, they seem to effectively and concisely communicate the two contrasting viewpoints about the meaning of equilibrium in Walras’s theoretical system.

Coming back to the pure-exchange model, it should be stressed that Walras, while proving ready to take advantage of Paul Piccard’s suggestion, by making of the constrained utility maximization principle (“principe de la satisfaction maxima des besoins”) the cornerstone of his entire analysis, yet was strongly affected by the particular shape and intrinsic limitations displayed by that suggestion. In fact, when Paul Piccard put forward his joint solution of the utility maximization and exchange equilibrium problems, he was certainly conditioned by both the kind of question he had been asked to answer and the degree of mathematical knowledge he might have expected from his interlocutor. Even if we do not know the exact terms of Walras’s original question, we can infer from Piccard’s answer that Walras had asked him to solve a problem concerning only two commodities, namely, two consumer goods, where the utility functions to be maximized were, as always in Walras, additively separable function and utility was interpreted as a cardinal magnitude. Moreover, Piccard certainly knew that Walras’s mathematical skills were limited. Hence the proposed solution was very elementary in character: it employed just the indispensable minimum amount of calculus, being chiefly couched in geometrical terms. Moreover, given the additive separability of the traders’ utility functions, their marginal utility functions (Walras’s *raretés*) turned out to be functions of one variable only. Hence, no need to employ any calculus of functions of several variables, or to use more advanced constrained maximization methods, such as Lagrange’s, could possibly arise; so that, not surprisingly, no such kind of calculus or method was employed.

For his entire scientific life Walras will stick to the mathematical and theoretical approach implicit in Piccard’s solution summarized above, an approach whose intrinsic weaknesses will prejudice not only any further improvement of Walras’s pure-exchange model, but also, and foremost, the development of the with models of production, capital formation, and money. Deferring to the next section all considerations concerning the last three models, let us now complete our analysis of the shortcomings ensuing from Walras’s attempts to apply to his pure-exchange model the new conception of equilibrium as an “array of mutually compatible optimizing plans”.

Walras will never give up his cardinal interpretation of individual utilities and marginal utilities, nor his assumption that the utility functions be additively separable. As to the cardinality assumption, it must be recognized that, up to the end of the nineteenth century, most economists were ready to accept it without any qualms; moreover, for the purposes of the analysis that Walras was interested in developing, such assumption was not so dramatically dangerous. Wholly different remarks must be made about the additive separability assumption. First of all, since the early 1880s a number of well-known economists, starting from Edgeworth (1881), had grasped the serious limitations intrinsic in such assumption, dropping it in favor of more general hypotheses; so that, in this case, Walras's attachment to the assumption concerned quickly turned into a signal of backwardness²⁸. In the second place, and most important, adhering to that assumption allowed Walras to neglect almost all applications of calculus to functions of several variables, so that he found himself unprepared to deal with optimization problems involving several choice variables in an essential way. This is one of the reasons why Walras never directly tackled the problem of utility maximization in the pure-exchange model with several commodities, relying instead on an indirect two-step procedure allowing him to by-pass the intractable (for him) direct procedure. In this case the indirect procedure consists in taking sequentially the following two steps: in the first place, for all pairs of commodities, the solution of the corresponding partial utility maximization problem can be obtained by exploiting Piccard's suggestion for the two-commodity case; in the second place, the partial solutions can be made mutually consistent by exploiting Cournot's no-arbitrage conditions (1988, 1 and 2-5, 152-214²⁹).

4. Alternative equilibrium conceptions in Walras's production, capital formation, and money models

Let us now turn to the other three equilibrium models put forward by Walras, with a view to ascertaining which is the role played by the two alternative equilibrium conceptions in each case. As in the previous section, here we shall confine ourselves to considering only the first component of Walras's equilibrium analysis, that concerning the determination of the equilibrium conditions and the associated "theoretical solution". The consideration of the second component, that regarding the analysis of the equilibration or *tâtonnement* processes is deferred to the next section.

Our general contention is that in all the three cases Walras starts from a traditional conception of equilibrium. Each model is originally characterized by a model-specific equilibrium concept, inherited from previous approaches: from classical political economy as regards the production model; from classical political economy combined with simplified formulae of financial mathematics as regards the capital formation model; from a transaction technology version of the quantity theory of money as regards the money model. Each equilibrium concept naturally differs from the others. Yet, such differences notwithstanding, all the three equilibrium concepts, not

²⁸ When Irving Fisher (1892, 45-7) politely criticized Walras's adoption of the additive separability assumption, implying the dependence of the *rareté* of each commodity on the amount of that commodity only, Walras justified his choice in private correspondence by saying that he had studied the problems caused by the existence of substitutes and complements, but had decided to put them aside for the sake of simplicity (Jaffé 1965, vol. II, letter 1064 of July 28, 1892, 498-9). One can legitimately doubt of such justification.

²⁹ Remarks similar to those made in footnote 27 above apply in this case, too.

differently from the equilibrium concept originally characterizing the pure-exchange model, can be regarded as specific instances of a traditional conception of equilibrium as the result of a ‘balancing of forces’.

As we have seen, in 1872, when Walras came across the new conception of equilibrium as an ‘array of compatible optimizing plans’, he quickly reacted to such discovery by adapting the formal structure of the pure-exchange model to the new ideas. Then, with delays which significantly vary with the fields of application, he strived to enlarge the scope of the new approach by applying it to the models of production, capital formation, and money. But this task proved much more difficult to accomplish: for, on the one hand, the old ideas turned out to be more deeply entrenched in the fields of production, capital formation, and money than they had proven to be in the exchange field; on the other, the new ideas not surprisingly proved to be much more at home in the exchange domain, where they had been born after all, than in the other three domains, to which, at least at first, they proved to be altogether alien.

Let us turn now to the production model. The structure of the model, clearly defined from the start, will remain basically unaltered throughout Walras’s scientific life. The distinction between “capitaux” and “revenus”, that Walras inherits from his father Auguste; the distinction between a “market for products”, where consumer goods are bought and sold, and a “market for productive services”, where the services rendered by the various types of capitals are bought and sold; the role of the entrepreneur, who buys productive services from the owners of the corresponding capitals on the market for services, combines such services in the production process, and finally sells to the consumers the products coming out of the process on the market for products; all these are basic elements of the production model which, being already present in Walras’s 1871-2 Notes (1871-72, 465-70), will be kept unchanged over the years.

Let us now consider how Walras deals with the production technology issue. As early as in 1876, when the third *mémoire* on the “Équations de la production” is published (1876b), Walras makes the assumption of a single-output, fixed-coefficient technology³⁰, an assumption to which he will stick in all the editions of the *Éléments* as far as the Section on the theory of production is concerned (1988, 1, 1-3, 1-5, 2-5, 4-5, 261-342³¹). As mentioned above, it is true that, since the 1st edition of the *Éléments* (1988, 1-3, 305, footnote ff), Walras states that “it would be easy” to endogenously determine the production coefficients by solving a cost minimization problem. Yet, as has been seen, such issue will not be formally tackled until the 3rd edition of the *Éléments*, where it is cursorily dealt with in the *Post-Scriptum* attached to Appendix III (1988, 3, 720-2), later suppressed; in the 4th and 5th editions the revised contents of the eliminated *Post-Scriptum* will be moved to §326 of Lesson 36 (1988, 4, 4-5, 5, 586-91), which is one of the last Lessons of the *Éléments*, really peripheral with respect to the analytical core of the book. Putting temporarily aside the issue of the possible relaxation of the fixed-coefficient assumption, a relaxation which is suggested by Walras, but never concretely brought to bear on the analytical structure of his

³⁰ This assumption, on the other hand, recalls that of “fixed circulation coefficients” in the theory of money, dating back to the 1871-72 Notes (1871-72, 457).

³¹ Remarks similar to those made in footnote 27 above apply in this case, too.

production model, let us now focus attention on the behavior of the entrepreneurs in the production model under the standard (for Walras) assumption of a single-output, fixed-coefficient technology.

As is well-known, the entrepreneurs play a central role in Walras's theory of production. By buying services and selling products on their respective markets, they bear costs and earn revenues, hence they make profits or suffer losses. Since the very start of his research, Walras assumes that, in each line of production, the entrepreneurs engaged in producing that product react to the occurrence of profits (resp., losses) by increasing (resp., decreasing) their scale of production, that is, the quantity of output they produce. This behavioral rule is clearly stated in Walras's 1871-72 Notes (1871-72, 449); it is then restated in the third *mémoire* on the theory of production (1876b, 78), and thereafter consistently reiterated in all the editions of the *Éléments* (1988, 283-4). It must be stressed, therefore, that Walras nowhere supposes that the entrepreneurs, taking the prices of both the services they employ and the products they produce as fixed parameters, should maximize profits. On the contrary, he invariably assumes that the entrepreneurs, for given prices of inputs and outputs, react to the ensuing profits or losses with a sort of adaptive behavior which, by borrowing a term introduced and used in a different context, might be qualified as 'satisficing', rather than maximizing.

Since profit maximization, as has been shown, does not characterize the behavior of Walrasian entrepreneurs, one is led back to consider cost minimization, wondering whether this assumption, though really extraneous to the core of Walras's system of thought, might nonetheless represent the way through which some sort of optimizing behavior enters into the picture of the Walrasian production model, thereby indirectly affecting the equilibrium conception employed therein.

When Walras tackles for the first time the issue of the variability of the coefficients of production, in the second instalment of the 1st edition of the *Éléments* (1877, 1-3, 586), he does so by supposing that the coefficients of production representing the quantities of the services of specific types of landed capitals ("terres", indexed by t), personal capitals ("personnes", indexed by p), and capital goods proper ("capitaux proprement dits", indexed by k) employed in the production of one unit of commodity b , respectively denoted by b_t , b_p , and b_k , be determined by solving a unit cost minimization problem, for given prices of services and product, under a technological constraint represented by the equation $\varphi(b_t, b_p, b_k \dots) = 0$, called "équation de fabrication" by Walras. This is the equation of the unit isoquant of the production function describing the given single-output, variable-coefficient technology, a function that Walras tacitly, and probably inadvertently, assumes to exhibit constant returns to scale.

At the end of 1876, in view of the planned discussion of the issue of the variability of the coefficients of production in the second instalment of the 1st edition of the *Éléments* then under preparation, Walras writes to Hermann Amstein, a professor of mathematics at the École d'ingénieurs of Lausanne, asking for his help in solving the cost minimization problem mentioned above. Amstein's answer, contained in a letter to Walras dated January 6, 1877, provides two formally equivalent solutions to Walras's problem, one of which based on Lagrange's method for

solving constrained optimization problems³². According to Jaffé (1964, 206-7, and 1965, Vol. I, 519, note 5), Walras fails to exploit Amstein's suggestions because his poor mathematical competence at the time prevents him from fully understanding their meaning. He will take up again Amstein's notes only in the mid-1890s, when, taking advantage of his improved mathematical skills, under the stimulus of Barone's and Pareto's concomitant reflections on the theory of production and costs, and also under the spur of the mounting controversy with Wicksteed over the theory of marginal productivity, Walras goes back to the cost minimization issue.

To sum up, the assumption that the entrepreneurs engage in an optimizing conduct of any sort plays no role in the determination of the equilibrium conditions in Walras's production model, as we know it from the Section on the theory of production in all the editions of the *Éléments* (Section IV in the 1st, 4th, and 5th editions; Section III in the 2nd and 3rd). The equilibrium condition of which Walras systematically avails himself is the traditional condition of equality between price ("prix de vente" in Walras's terminology) and unit cost (prix de revient" in Walras's terminology), itself constant as output changes, for given prices of the services, in view of the assumption of fixed coefficients (1988, 283).

Of course, Walras's condition of equilibrium in production ("condition [...] relative à l'équilibre de la production") can be, and has actually been, rationalized *a posteriori* as the outcome of the optimizing choices of entrepreneurs who maximize profits (or minimize costs) under the constraint of a single-output, constant-returns-to-scale technology. Yet, this is not the route followed by Walras. As far as the theory of production is concerned, Walras seems to draw his inspiration not so much from the new approach characteristic of the 'marginal revolution' of the 1870s, as from the pre-existing approach characteristic of classical political economy, a school of thought for which the compatibility of optimally chosen production plans is certainly not at the center of the stage. We agree therefore with the judgement expressed by Schumpeter (1954, 1010, footnote 30), an author who cannot surely be suspected of any negative bias against Walras, according to whom

on an infinitely higher level of rigor, Walras really reformulated the theories of production of A. Smith, J. B. Say, and J. S. Mill.

Let us move now to the theory of capital formation. Also in this field one should recognize that Walras, though making great progress with respect to pre-existing theories and striving to improve the original formulation of his model, no doubt achieving some success over time, yet proves unable to free himself entirely of the weight of tradition.

As regards the production of capital goods proper, one can make remarks similar to the ones already made with respect to the production of consumer goods: for in both cases the assumptions about the production technology and the behavior of the entrepreneurs are the same. Hence, also in the case of

³² Letter 364 in (Jaffé 1965, Vol. I, 516-20). In the cited place, together with Amstein's letter, one can also find the transcript of the draft of the question that Walras had put to Amstein, as well as the transcripts of an undated note, written by Walras at either the end of 1894 or the beginning of 1895, and of a few penciled notations added by Walras to Amstein's autograph letter, certainly written after 1900. All these documents are also reproduced in (Walras 1993, 660-665), together with some remarks by the Editor. Jaffé's notes to Amstein's letter are very useful. One should also consult (Jaffé 1964).

capital goods proper, one can complain that no significant role is assigned to any kind of optimizing behavior on the part of the entrepreneurs, who are the agents acting on the supply side of the market for products (including newly produced capital goods proper). In this case, however, there arises a further difficulty, which was not present in the case of the production of consumer goods: for here, unlike in the case of consumer goods, the assumption of optimizing behavior rests on shaky foundations also as regards the demand side of the market.

Let us then turn to the demand for fixed capitals, of which capital goods proper are one possible type³³. Already at the end of 1871, at the moment of writing his 1871-72 Notes, Walras tries to confront the problem of explaining how demands and prices of fixed capitals are determined in the market. But in 1871, it will be remembered, no theory of demand based on utility maximization is yet available for any type of goods, be they nondurables or durables, or consumable services. Hence, Walras cannot but confine himself to recall the utility concepts (“*utilité extensive*” and “*intensive*”), then vaguely supposed to underlie the demand for consumer goods. In the case of fixed capitals, however, a further complication arises: for, in determining the conditions ruling their demands, supplies, and prices, one must also take into account the conditions at which the services they render are exchanged in the market. In view of this, Walras concludes his discussion by introducing, by means of examples, the formula expressing the price of a fixed capital as a ratio between its net income (defined as the price of the service rendered by that fixed capital, net of depreciation charge and insurance premium) and the rate of net income (or the rate of interest), which must be the same for all fixed capitals (1871-72, 466-7).

A few years later, when Walras, in preparing his fourth *mémoire* on the “*Équations de la capitalisation*” (1876b), takes up again the problem of the determination of the demands, supplies, and prices of fixed capitals, one might expect him to strengthen the scanty theory put forward in his 1871-72 Notes, by fully exploiting the new theory of demand, supply, and prices of consumer goods, based on utility maximization, which has by then been developed and is fully available for possible extensions. Yet, no attempt is made to directly apply the new theory of consumer demand to the issue of the pricing of fixed capitals. The only way in which such new theory is indirectly brought to bear upon the issue of fixed capital pricing is through the determination of the prices of the services they render: for, in so far as these services can be consumed by the individual participants in the economy, what Walras takes always for granted, they directly enter into the consumers’ utility functions, on a par with consumer goods, so that the new theory of demand can be immediately extended to the pricing of the services of fixed capitals. Then, once the prices of the services are explained in this way, they can in turn be indirectly used for determining, given the rate of interest, the prices of fixed capitals via the above mentioned formula stating that the price of a fixed capital must be equal to the ratio between its net income and the rate of interest. Of course, the use of this formula essentially rests on the assumption, only tacitly made by Walras, that both the prices of services and the rate of interest be expected with certainty by all the individual participants in the economy to be stationary over time.

³³ Here we take into consideration fixed capitals only, since a theory of demand and pricing of circulating capital goods is missing for most of Walras’s scientific life. Moreover, when such theory is eventually put forward in the 1899 *mémoire* on the theory of circulation and money, it turns out to be formally identical with the already existing theory of demand and pricing of fixed capitals.

Hence, Walras preserves the formula already employed in his 1871-72 Notes, which is a special case, under the assumption of stationary expectations, of a more general formula stating the equality between the price of any durable asset and the discounted value of the sequence of revenues it yields – a formula that is somewhat trivial since, as Walras himself will declare in the Preface to the 2nd edition of the *Éléments*, it immediately pops up whenever one randomly opens any treatise of financial mathematics (1988, 2-5, 18). But he does not try to extend the scope of the utility maximization approach so fruitfully employed in the theory of demand of consumer goods and consumable services, by applying it directly to the theory of demand and prices of fixed capitals. The reason for this lies in Walras's very restrictive interpretation of the meaning of the individuals' utility and marginal utility functions, whose arguments must be constrained to be magnitudes producing immediate satisfaction (“revenus”, in his terminology). As we shall see in a moment, Walras for a while contemplates the possibility of including among the arguments of the individuals' utility functions also objects whose utility is not (only) immediately experienced, but can (also) be enjoyed in the future. However, from the early 1870s and for a very long period of time, not exactly knowing how to deal with such phenomena, Walras simply excludes all durables from the set of possible arguments of the individuals' utility functions.

This choice leads him most naturally to keep to a minimum the role played by fixed capitals in his theoretical system. First of all, given that fixed capitals must not enter the individuals' utility functions, it seems natural to keep them out of the individuals' budget constraints as well. This is the reason why, after recognizing that, if not persons, at least lands and capital goods proper are marketable commodities (1988, 267-271), and after discussing at length the possibility for any owner of fixed capitals of “eating one's capital” (1988, 356-7), Walras ends up by assuming that in his capital formation model those fixed capitals that already exist at the start in the economy, being part of the endowments inherited from the past, cannot be marketed³⁴. Once this step is taken, however, several other steps necessarily follow. For, if existing fixed capitals cannot be marketed, the only way for making it possible to determine the prices of fixed capitals is to suppose that new fixed capital goods proper, in excess of what is required to take care of depreciation and destruction

³⁴ Walras tries to justify his assumption by invoking reasons which are obviously different from those suggested in the text. Moreover, if the reasons adduced by Walras were sound, what we have denoted as Walras's ‘assumption’ of non-marketability of fixed capitals would turn out to be a ‘conclusion’ of his argument. Yet, Walras's justifications are untenable. To clarify the matter, let us recall that Walras's conclusion comes shortly after a passage where he reproduces in the *Éléments*, with a few qualifications, the formula already discussed above linking the price of the fixed capital under question, P , with the price of its service, p , and the rate of net income, i , account being taken of the depreciation charge, μP , and the insurance premium, νP . As can be seen, the depreciation charge and the insurance premium are assumed to be proportional to the price P of the fixed capital, μ and ν being the respective fixed proportionality coefficients. Hence p can be viewed as the gross income produced by one unit of capital, while $\pi = p - (\mu + \nu)P$ is the net income. The formula can then be written as: $[p - (\mu + \nu)P]/P = \pi/P = i$, or equivalently: $P = p/(i + \mu + \nu)$. (1988, 2-5, 349); a similar passage appears in (1988, 1, 348).

Walras then goes on in this way: « Dans ces conditions, il ne pourrait y avoir vente et achat de capitaux ; car ces capitaux ne pourraient s'échanger que les uns contre les autres en proportion de leurs revenus nets, et cette opération, qui n'aurait théoriquement aucune raison d'être, ne fournirait non plus aucun prix en numéraire. » (1988, 2-5, 349, 351); a similar passage appears in (1988, 1, 350).

Now, Walras is of course right when he says that, under the stated conditions, there would be no reason to exchange a certain amount of a given fixed capital for an amount of another fixed capital having the same value, the two fixed capitals being perfectly equivalent from the point of view of their capacity of generating income. Yet, an individual might well be willing to sell a certain quantity of a given fixed capital that that individual owns in exchange for quantities of consumer goods or consumable services having the same value at the given prices, thereby dissaving or “eating one's capital”; or *vice versa*.

of existing fixed capital goods proper, are produced, sold and bought on the market³⁵. This in turn implies that, in order to make the theory of prices of fixed capitals possible, the economic state, which in principle may well be “rétrograde”, “stationnaire”, or “progressif”, must be assumed *a priori* to be “progressive”, that is, characterized by a positive amount of savings (which are defined as net of depreciation charges and insurance premiums)³⁶.

Once it is assumed that the existing fixed capitals cannot be marketed, the income of the consumers, who are also the owners of the unsaleable fixed capitals, consists in the revenues coming from the sales of the services rendered by the fixed capitals they own. What remains of the income of any individual consumer after deducing the amount spent for the purchase of consumer goods and consumable services represents that consumer’s savings. By aggregating individual savings over all consumers, one obtains the aggregate gross savings of the economy. What is left after deducing the depreciation charges and the insurance premiums to be paid for granting the maintenance of the existing capital goods proper represents the aggregate net savings of the economy, which, in view of the assumptions made so far, cannot but be spent for the purchase of newly produced fixed capital goods proper; hence, aggregate net savings automatically generate an equivalent demand for newly produced fixed capital goods proper, expressed in value terms. Such demand, however, is not directed towards specific capital goods proper, but towards an undifferentiated amount of value: for all newly produced fixed capital goods proper, if evaluated at equilibrium prices, are perfectly equivalent as to their income-generating capacity.

As regards the savings function obtained in the way discussed above and employed in the capital formation model presented in the first three editions of the *Éléments*, hence up until 1896, Walras explicitly recognizes that it is not theoretically founded, but only given “empiriquement” (1988, 1-3, 358). In the same passage, however, he recalls that also in the pure-exchange model the demand functions of consumer goods, initially given “empiriquement”, had been later theoretically deduced

³⁵ For Walras, any theory of commodity prices presupposes, as a necessary condition, the existence of markets on which such commodities are exchanged. At the very beginning of the Section of the *Éléments* devoted to the theory of capital formation (part of Section V in the 1st edition; part of Section IV in the 2nd and 3rd editions; Section V in the 4th and 5th editions), Walras declares: “Il ne saurait y avoir, pour nous, de prix que sur le marché.” (1988, 346).

³⁶ Walras is wavering about the assumptions discussed in text, that is about the marketability of already existing capital goods and the necessity of assuming that the economic state be “progressif”. In the fourth *mémoire*, written in 1876, one finds the following passage: “Pour avoir une offre et une demande de capitaux, il faut substituer à la conception d’un état économique stationnaire celle d’un état économique *ou rétrograde ou progressif*. Il faut supposer des entrepreneurs qui, au lieu de fabriquer des produits consommables, ont fabriqué des capitaux producteurs neufs *ou bien des propriétaires fonciers, travailleurs ou capitalistes qui, ayant acheté des produits consommables pour une somme supérieure au montant de leurs revenus producteurs, sont obligés de vendre tout ou partie de leurs capitaux producteurs*”. The italicized parts of the above passage are suppressed by Walras in 1877, when the first four *mémoires* are collected in the publication *Théorie mathématique de la richesse sociale* (1877a, 103, footnotes *d* and *e*). Such deletions are confirmed in the second instalment of the 1st edition of the *Éléments*, where one can find a passage almost identical to that of the *Théorie mathématique*, a passage which will persist with a few minor changes in the 2nd and 3rd editions (1988, 1-3, 350). It should be stressed that, in all the quoted versions of this tortured paragraph, covering a period of more than twenty years, Walras sticks to the idea that the economic state must be conceived as “stationnaire” before the issue of capital formation is introduced, that is, in the pure-exchange and production models. Every reference to the stationary state of the economy disappears in the 4th and 5th editions, where the state of the economy is explicitly assumed to be “progressif” for the purposes of the theory of capital formation. And yet, in line with the new assumptions and terminology characterizing such editions, Walras says that, independently of whether the state of the economy is “progressif” or “rétrograde”, “il peut demeurer statique si les dispositions à l’épargne, comme les dispositions à la consommation, sont supposées fixes pendant un certain temps.” (1988, 4-5, 351).

from first principles (i.e., from constrained utility maximization) in the subsequent developments of the analysis³⁷. In Walras's opinion, this precedent lets one hope that in the future

y aurait-il lieu de rechercher les éléments mathématiques constitutifs de la fonction de l'épargne, comme nous avons recherché ceux de la fonction de la demande effective. Il faudrait évidemment, pour cela, considérer l'utilité sous un aspect nouveau, la distinguer en utilité présente et utilité future.

Yet, this suggestion will not be pursued in the 4th edition of the *Éléments*, where Walras, with a view to introducing the savings function no longer only "empiriquement", but "rationnellement", follows a route altogether different from the one he had adumbrated up to the 3rd edition at the least³⁸. In fact, Walras concocts an imaginary « merchandise (E) consistant en *revenu net perpétuel* dont le prix $p_e = 1/i$ et la quantité demandée d_e s'exprimeront en unité de numéraire. i est le *taux de revenue net perpétuel* [...] » (1988, 359; italics in the original). Such commodity (E) enters as an argument in the individual utility functions; therefore one can define a marginal utility function relative to such commodity, endowed with the standard properties characterizing such functions in Walras's theory of exchange, and consequently also derive a demand function for such commodity, by solving a standard constrained utility maximization problem. According to Walras, such demand function would provide the theoretically founded savings function he had been looking for.

Apparently, therefore, at the very end of his active scientific life, Walras eventually succeeds in extending the utility maximizing approach, originally confined to the pure-exchange model, to the capital formation model, at least as far as the savings function is concerned. Yet this result is deceptive. In the first place, no explanation is provided for the sudden abandonment of a line of research that appeared to be promising, that is, the project of investigating the relationship between present and future utility. In the second place, if the artificial commodity (E), which is a perfectly durable capital, may enter the individual utility functions and be endowed with individual marginal utility functions, why should the other fixed capitals be excluded from such opportunities?

Similar remarks can be made as regards the theory of money. Also in this case Walras pursues the objective of making his theory more and more consistent with the innovative optimizing approach which underlies his theory of the demand for consumer goods. As Walras himself underlines in his Preface to the 2nd edition of the *Éléments* (1988, 2-3, 4, 6), a first step in this direction is taken when the conception of money as "circulation à desservir", a conception "empruntée aux économistes" that had been endorsed in the 1st edition of the *Éléments*, is replaced by the more satisfactory

³⁷ Here Walras is evidently referring to his theory of the exchange of two commodities for one another, as initially expounded in his first *mémoire* (1874, 27-46) and later restated in all the editions of the *Éléments* (1988, 69-117, 127-47), where at first the demand functions are assumed as "empirically" given, and the exchange equilibrium is determined on the basis of that assumption, whereas the derivation of the demand functions from the solution of the individuals' utility maximization problems is postponed to a later stage of the discussion. We have already dwelled upon this peculiar structure of Walras's exposition. Here Walras himself is implicitly suggesting what has already been hinted at above: namely, he is admitting that the adopted expository order reflects the chronological order in which the various components of the theory had come to his mind.

³⁸ As a matter of fact, a sentence very similar to the one quoted in the text appears in the *mémoire* "Equations de la circulation", published in 1899 (1899, 565). Yet, since the 1899 *mémoire* is necessarily based on the theory of capital formation of the 3rd edition of the *Éléments*, the repetition of a sentence on present and future utility might be less significant than it might appear at first sight.

conception of money as “encaisse désirée”, advocated in the 2nd and 3rd editions. However, in spite of this change, which goes in the right direction because it substitutes a more subjectively oriented view of the demand for money for the purely objective view of the 1st edition and the preceding 1871-72 Notes, much remains to be done. In fact, as Walras himself recognizes in the Preface to the 4th edition (1988, 4-5, 9),

dans [la] 2^e édition et dans la 3^e, l'équation d'égalité de l'offre et de la demande de la monnaie était toujours posé à part et empiriquement. Dans la présente édition, elle est déduite rationnellement d'équations d'échange et de satisfaction maxima en même temps que les équations d'égalité de l'offre et de la demande des capitaux circulants.

It is clear that for Walras the integration of the theory of money with that of circulating capital, at which he arrives only in 1899, represents a great progress in the direction of a unified theory of the economic phenomena, a theory based on uniform principles of optimizing behavior. However, within the Walrasian framework, it is even more difficult to deduce the theory of demand of circulating capitals from the solution of a problem of utility maximization under budget constraints than it is to deduce the theory of fixed capitals from the solution of the same constrained maximization problem. This statement depends on two facts. First, Walras's theory of circulating capitals is virtually identical, from a formal point of view, to the theory of fixed capitals; only the economic interpretation of the variables changes. Therefore, all the shortcomings that we have encountered in examining the theory of fixed capitals are inherited by the theory of circulating capitals. In the second place, the fact that circulating capitals (which belong to the category of “revenus”, or objects which do not survive their first use) are formally dealt with in the same way as fixed capitals (which belong to the category of “capitaux”, or objects which do survive their first use) gives rise to a further difficulty: for such homogenization of the two categories of objects of social wealth blurs a distinction which is fundamental for Walras's theoretical system: that between “capitaux”, which are durables, and “revenus”, which are nondurables. Hence, the theory of demand for circulating capitals, hence also the theory of demand for money, which depends on the former in Walras's integrated theory of circulation and money, rest on foundations that are unsound.

5. Equilibrium and equilibration: stationary vs. instantaneous equilibrium concepts in Walras's theoretical system

As illustrated in the two previous sections with reference to the four equilibrium models composing the Walrasian theoretical system, the two alternative equilibrium conceptions coexisting from the start in Walras's theoretical approach deeply affect the first component of the Walrasian analysis, that is, the component dealing with the “theoretical solutions” of the equation systems describing the models concerned. But they also affect the second part of that analysis, that is, the part dealing with the equilibration processes, or *tâtonnement* processes in Walras's terminology, supposedly driving the subsystems analyzed in the various models towards the concrete realization in the market of the very same “theoretical solutions” obtained in the first part by solving the equation systems describing the models themselves. Before discussing how the two equilibrium conceptions affect the *tâtonnement* process associated with each of the four equilibrium models in Walras's

theoretical system, it is convenient to examine the general properties that an equilibration process associated with an abstract model conforming to either equilibrium conception should be expected in principle to display.

Let us start from the conception according to which an equilibrium should be viewed as an ‘array of mutually compatible optimizing plans’. A plan of action, unlike an action, necessarily is a mentalistic phenomenon. An agent’s choice of a plan of action is instantaneous by its very nature. Whether the individual choices of plans of actions made by a number of agent belonging to a specified system are mutually compatible or not is a fact that can only be ascertained with reference to a specified time instant, the instant at which all the plans are chosen. Hence, when an equilibrium is viewed as an ‘array of mutually compatible optimizing plans’, it is natural, or even necessary, to speak of an ‘instantaneous equilibrium’. On the contrary, the alternative equilibrium conception, according to which an equilibrium ought to be viewed as a ‘balance’ of objective, empirically observable ‘forces’, of course so specified as to be relevant to the economic context of reference, can be easily associated with the operation of a time process which, as it occurs in several models of mechanics or other natural sciences, envisages precisely such ‘balance of forces’ as the persistent outcome of the process. In this latter case, therefore, it is natural to refer to the equilibrium state of the model concerned as the stationary state of a dynamic process; hence, the most appropriate expression which might be employed to designate such a state is that of ‘stationary equilibrium’.

Either conception naturally imposes constraints not only on the equilibrium theories associated to it, but also on the theories dealing with the equilibration processes supporting the equilibrium states associated to that conception. As regards the equilibration processes, the problems or difficulties arising in either case fall under three main headings: first, the preservation of the data; second, the out-of-equilibrium behavior of the agents; third, who or what drives the process. Let us consider each issue in turn.

As regards the idea that the equilibration process should not alter the data existing at the start of the process itself, the situation appears to differ widely, at least in principle, in the two cases under discussion. When an equilibrium is viewed as an ‘array of mutually compatible optimizing plans’, we have to do with an ‘instantaneous’ equilibrium, which is defined on the basis of the data prevailing in the economic system concerned at the instant at which the plans are chosen. It is natural in this case to require that the equilibration process associated with such an ‘instantaneous’ equilibrium should not alter the data characterizing the ‘instantaneous’ equilibrium itself. On the contrary, when an equilibrium is viewed as a ‘balance of forces’, we have to do with a ‘stationary’ equilibrium, which is defined as the persistent outcome of an equilibration process. In this case there is no *a priori* reason to require that the data prevailing at the initial instant of the process be preserved throughout the process, so that they characterize the ‘stationary’ equilibrium, if and when it is eventually reached. As a matter of fact, provided that the dynamic behavior of the economic system be clearly specified, and supposing that the equilibrium conditions identifying the ‘balance of forces’ be clearly stated as well, no risk of indetermination should arise if the initial data are not preserved without changes over the process: as in every well-defined deterministic dynamical process, once the initial conditions have been detailed and the rules governing the process have been carefully spelled out, no difficulty should arise in identifying the ‘stationary’ equilibrium

eventually reached, even if the conditions prevailing in the system at the end of the process were to differ from the initial ones. But this last remark naturally leads us to the second issue.

As regards the out-of-equilibrium behavior of the agents, once again the situation appears to differ widely in the two cases under discussion. When an equilibrium is viewed as an ‘array of mutually compatible optimizing plans’, the equilibrium theory, if it keeps firmly to its own presuppositions, is able to explain the agents’ behavior, that is, their observable actions, only at equilibrium, for only at equilibrium are the optimally chosen plans of action mutually compatible, and consequently executable. Out of equilibrium a theory conforming to this equilibrium conception can at most explain the agents’ choices of mentalistic plans, but it is wholly powerless as regards their observable actions. Moreover, in order to have a process of any kind, it is necessary that something in the agents’ situations should change. But, as we have seen, the data cannot change in this case. A possible way-out is to assume that there exist some adjustment variables that, though not being part of the data, affect the agents’ situations, changing throughout the equilibration process in such a way as to drive the economic system towards the ‘instantaneous’ equilibrium associated with the unaltered data. (In competitive equilibrium models the role of such adjustment variables is obviously played by prices.) Therefore, an equilibration process consistent with the equilibrium conception under discussion can only be a mentalistic process, describing the evolution over time of optimally chosen plans. However, since the data prevailing at the instant to which the ‘instantaneous’ equilibrium refers must not change, the agents must be prevented from carrying out observable actions capable of affecting the data throughout the process, what would almost invariably occur, barring exceptional cases. On the contrary, no such problems arise with the alternative equilibrium conception: for in this case, since the agents are not required to implement optimally chosen plans of actions, their observable behavior can be so defined as to be consistent with both equilibrium and out-of-equilibrium states.

Finally, let us turn to the third issue, which concerns the ‘force’, be it a person, a set of persons, or a mechanism, driving the equilibration process. Not surprisingly, also here a quite different situation prevails in the two cases under discussion. When an equilibrium is viewed as an ‘array of mutually compatible optimizing plans’, the driving ‘force’ behind the equilibration process is represented by the changes in the adjustment variables mentioned above. Yet such changes cannot be brought about by the agents participating in the economic system: for they are supposed to simply react to the existing data (unchanging over the process) and the specified values (changing over the process) of the adjustment variables by optimally choosing their plans of action; hence they cannot be required to change the adjustment variables themselves. The ultimate driving ‘force’ of the equilibration process, therefore, must be represented by an unexplained mechanism (Walras’s “market” or “competition mechanism”) or entity (it is here that the fictitious character of the auctioneer enters the stage in the modern versions of GET). On the contrary, no such assumption is required under the alternative equilibrium conception: for the driving ‘force’ of the equilibration process may well be represented by the very agents’ observable behavior which, as explained above, is in principle governed by the same rules at both equilibrium and out-of-equilibrium states.

To sum up, the only kind of equilibration process which is consistent with the ‘instantaneous’ equilibrium notion associated with the conception viewing an equilibrium as an ‘array of mutually

compatible optimizing plans' is a purely mentalistic process, taking place in a kind of 'logical' time consuming no instant of that 'real' time over which the economic system evolves; throughout such process no change can occur in the data characterizing the economic system at the instant to which the 'instantaneous' equilibrium refers and no observable action can take place; the only admissible changes concern on the one hand the adjustment variables, themselves governed by an unexplained mechanism or entity, and on the other the plans of action optimally chosen by the agents, who react to the given data and the changing values of the adjustment variables. On the contrary, no such requirements must be satisfied by the equilibration processes supporting the 'stationary' equilibrium notions associated with the conception viewing an equilibrium as a 'balance of forces', themselves objectively specified by means of observable variables. The 'force' driving the process in this case is represented by the behavior of the agents themselves, so that no external mechanism or entity must be brought in from outside. The initial data must not be assumed to remain unchanged throughout the process, since the possible changes in the data do not jeopardize the deterministic character of the theory, provided of course that the dynamic laws governing the equilibration process, which takes place over the same 'real' time as that over which the economic system evolves, be fully specified. Yet, since this turns out to be impossible in many cases, particularly in the economic applications of the approach, the temptation often arises to assume the initial data to be time-invariant, with a view to granting a pseudo-deterministic character to the theory, which would be lost otherwise.

In the light of the preceding observations, let us now turn to a discussion of Walras's analysis of the equilibration or *tâtonnement* processes that in his view are the indispensable companions of the "theoretical solutions" of his equilibrium models. Walras develops his analysis of the *tâtonnement* construct since 1874, the date when the first instalment of the 1st edition of the *Éléments* sees the light, taking it up again, with repeated changes and integrations, in all the subsequent editions, up until the 4th (1900). The analysis is separately applied to all the four equilibrium models, starting from the pure-exchange model with several commodities and proceeding with the production and the capital formation models, and finally, but only in the 4th edition, also with the model of circulation and money. The only place where one can find a comprehensive and compact view of the working of the *tâtonnement* processes associated with the first three models is Appendix I to the 3rd and following editions of the *Éléments*, titled "Théorie géométrique de la détermination des prix" (1988, 3-5, 692-710), where Walras specifies that in reality all the operations involved in the *tâtonnement* processes relative to the pure-exchange and production models should be thought of as taking place simultaneously, even if, for the needs of the analysis, they are separately dealt with in both the body of the book and the Appendix itself (1988, 3-5, 704). In this regard it may be interesting to observe that, referring in particular to the pure-exchange and production models, where the analysis of the *tâtonnement* process is much more protracted and detailed than in the other two models, Walras does not lay any claim to originality concerning the *tâtonnement* construct: for Walras, in fact his own analysis of the equilibration process is nothing other than a theoretically more sophisticated formulation of the traditional analysis of market equilibrium

adjustment based on the “law of supply and demand”, as it can be found expounded, e.g., in J. S. Mill’s work³⁹.

This last observation leads us to discuss Walras’s interpretation of both the equilibrium concept and the *tâtonnement* construct, as employed in his theoretical system, in the light of the remarks made above about the coexistence in his writings of the two alternative equilibrium conceptions examined in this paper. Since in Walras the traditional conception pre-exists the new one, it is only natural that he should initially believe the equilibrium concept to be of the ‘stationary’ type and that he should at first interpret an equilibrium state as the final stationary outcome of a time process, observable in ‘real’ time, driving the economic system towards such equilibrium state.

If one looks at the initial formulations of Walras’s equilibrium theory, no doubt can arise as to the ‘stationary’ character of the equilibrium eventually arrived at through the *tâtonnement* process in all the models to which that construct is originally applied (i.e., the first three models). As regards the pure-exchange model, let it suffice to recall that, already in his 1871-72 Notes (1871-72, 429), in the pure-exchange model with only two commodities, Walras designates as “prix stationnaire” the equilibrium price reached at the end of the equilibration process; this expression, together with other similar ones, will remain unchanged in all of Walras’s subsequent writings, all the editions of the *Éléments* included (1988, 93, 197; 1993, 69). Another passage, which is revealing in this respect, is the illustration of the working of the equilibration process in a market where one commodity is traded for money, the commodity in question being the “rente française 3 %” in both the 1871-72 Notes (1871-72, 420-3) and all the editions of the *Éléments* (1988, 71-72⁴⁰), curiously turning into “blé” in the first *mémoire* (1874, 32).

This illustration is particularly interesting for a number of reasons: in the first place, the expression “prix stationnaire” is employed also in this case to denote the equilibrium price reached at the end of the equilibration process; in the second place, in this relatively informal passage Walras discloses what is probably his true pre-analytic vision of the equilibration process in the pure-exchange model, a process where not only do trades take place out of equilibrium, but also the agents react to the occurrence of excess demands or supplies in the market by raising or decreasing the market price themselves. Most probably Walras did not initially perceive that out-of-equilibrium transactions necessarily (and also unpredictably, on the basis of the available theory) change the initial endowment allocation, hence the data of the problem, making the problem itself indeterminate⁴¹: in fact, since no precise theory explaining the traders’ out-of-equilibrium transactions is available, the out-of-equilibrium path cannot be traced, so that the ‘stationary’ equilibrium eventually reached, if any, cannot be identified. Similarly, he does not perceive that

³⁹ On this issue, one should consult Walras’s letters to Borkiewicz and Pareto, dated February 27, 1891, and January 9, 1895, respectively, which are particularly explicit in this respect (Jaffé, 1965, Vol. II, letter 999, 434-5, and letter 120, enclosure, 628-32).

⁴⁰ This passage will remain unchanged in all the editions of the *Éléments*, with the exception of a few words, which are inserted in the 2nd edition and then maintained in the subsequent ones. On this we shall come back in a moment.

⁴¹ In the second edition of the *Éléments* Walras compares the problems relative to the equilibration process which arise in the production model with those arising in the pure-exchange model. Among the latter, the difficulty pointed out in the text is not mentioned. This would suggest that Walras is not fully aware of its existence even at the moment of revising the text of the *Éléments* in preparation for the 2nd edition (second half of the 1880s), what would seem to imply that he had been wholly unaware of it in the mid-1870s (1988, 2-3, 308).

admitting that the traders themselves can change prices conflicts with the price-taking assumption underlying his novel theory of demand based on constrained utility maximization. Here we find a striking instance of the persistence of old ideas in a new framework with which such ideas are not compatible. And it is probably such persistence of conflicting ideas which leads Bertrand to point out, in his 1883 review of Walras's *Théorie mathématique de la richesse sociale* (Bertrand 1883), that the latter's theory of exchange is indeterminate as long as out-of-equilibrium trades are allowed to take place. As we shall see in a moment, it is precisely Bertrand's criticism which prompts Walras to react, partially changing his analysis of the *tâtonnement* processes in the mid-1880s.

Yet, before discussing such changes, it is convenient to direct our attention to the production model. Also in this case we see that Walras initially interprets the *tâtonnement* process as an observable equilibration process taking place in 'real' time, with actual transactions occurring out of equilibrium and actual production activities being carried out throughout the process⁴². Also in the production model the equilibration process is supposed to drive the economy towards a 'stationary' equilibrium. Since also here no precise theory of the consequences of the entrepreneurs' and the consumers' out-of-equilibrium behavior is provided, the problem of the unpredictability of the out-of-equilibrium path, and of the consequent indeterminacy of the equilibrium eventually arrived at, would arise in this case, too, unless an assumption of time-invariance of the data characterizing the economic system were made. And indeed Walras does make such assumption, whose logical consistency depends on whether or not the out-of-equilibrium transactions and production activities are such as to affect the data. If one were willing to accept that all existing fixed capitals cannot be bought and sold, an assumption characterizing all of Walras's theoretical system, that no capital goods proper can be produced, a standard assumption in Walras's production model, and on top of this that capital goods proper do not depreciate or disappear, a non-standard assumption that would make capital goods proper indistinguishable from landed capital, then one would be left with a pure-flow economy, of the type so much cherished by Cassel (1899, 1918), where out-of-equilibrium transactions or production activities are incapable of permanently affecting the data. Such a pure-flow economy, where only nondurables (consumer goods and services) are marketable, is the only possible instance of a production economy where the assumption of time-invariance of the data is logically tenable and the notion of a 'stationary' equilibrium can be consistently advocated, even if the market participants are allowed to carry out out-of-equilibrium observable, but unpredictable, activities.

Yet, when one considers the capital formation model, the situation changes dramatically: for the production of new capital goods proper, which is the essential novelty of the model concerned, necessarily alters the data characterizing the economic system under scrutiny, thereby making the assumption of time-invariance of the data logically untenable. Walras is ready to admit that, under the new circumstances, "il faut substituer à la conception d'un état économique stationnaire celle d'un état économique progressif" (1988, 1-3, 350). This sentence dates back to 1877. Yet, in spite of this timely understanding of the needs of the theory, for a long time (more than twenty years) Walras will not accept to give up the idea of an equilibration process in 'real' time, characterized by observable out-of-equilibrium actions, driving the economy towards a 'stationary' equilibrium state, even when the production of new capital goods is allowed for. Such protracted conservative attitude

⁴² This will be explicitly recognized by Walras himself on a number of occasions. See, e.g., (1988, 2-3, 308; 1899, 581).

is probably motivated by the strong influence that the traditional conception of equilibrium exercises on Walras's production and capital formation models from the start.

As time elapses, however, the progressive advancement of the new equilibrium conception to the detriment of the traditional one induces Walras to revise his stance as to the nature and characteristics of both the equilibrium concept and the *tâtonnement* construct, particularly as regards their time dimension. Yet, since at first the substitution of the new equilibrium conception for the old one takes place only in the field of exchange, it is only natural that Walras's change of perspective should reveal itself for the first time with reference to the pure-exchange model. In 1885, in reaction to Bertrand's criticism of the pure-exchange model with out-of-equilibrium transactions in 'real' time, Walras, though only incidentally and almost unwillingly, admits that no actual transactions must be allowed to take place out of equilibrium, explicitly acknowledging the 'instantaneous' character of the pure-exchange equilibrium notion in his model. A few years later, in the 2nd edition of the *Éléments*, by adding a few words to the above mentioned passage containing the illustration of the working of the market for a specific commodity (1988, 2-5, 71-72), he turns the equilibration process in the pure-exchange model into a purely virtual process devoid of observable effects, taking place in a 'logical' time separate from the 'real' time over which the economic system evolves.

Yet, up to the 3rd edition of the *Éléments* included (1896), Walras continues to stick to the old assumptions as far as the production and capital formation models are concerned. It is only in 1899, with the introduction of the assumption so-called "hypothèse des bons" in the *mémoire* on the "Équations de la circulation et de la monnaie" (1899, 581), an assumption reproduced one year later in Lesson 29 of the 4th edition of the *Éléments* (1988, 4-5, 447), that Walras makes the entire equilibration process, inclusive of the phenomena of production, capital formation, circulation and money, a purely virtual process, recognizing at the same time, though still incidentally, the 'instantaneous' character of equilibrium notion employed therein. However, he proves unable to translate this late recognition into a global reconsideration of his overall theoretical system, as expounded in the 4th edition of the *Éléments*. As a matter of fact, Walras confines himself to marginally changing here and there a few individual words or isolated expressions, without significantly revising the theory developed in the previous editions, a theory which still remains largely affected by the traditional view of equilibrium as a 'balance' of objective 'forces', barely related to the individuals' optimizing choices.

6. Concluding remarks

We have shown that in Walras's works there coexist two conceptions of general economic equilibrium: the first, borrowed from past economists ("empruntée aux économistes"), rests on the traditional idea of a 'balance' of objective 'forces', identifiable by means of the observation of the working of interdependent markets and the empirical experience of the occurrence of observable economic phenomena; on the contrary, the second, based on the notion of equilibrium as an 'array of mutually compatible optimizing plans', is subjective in nature, embodying the deepest spirit of the 'marginal revolution' of the early 1870s.

Walras strives to integrate the two conceptions, at first placing them side by side in his writings and then progressively substituting the second for the first in all of his models. Yet, it is only in the pure-exchange model that the two conceptions are fully amalgamated into a unitary construction, where the second conception is the prevailing one. In the other three models (production, capital formation, circulation and money) the first and more traditional conception, though continuously perfected by the insertion of elements drawn from the second, still continues to provide the background vision.

These two equilibrium conceptions, the traditional and the new one, are also associated with two alternative interpretations of the time dimension of the equilibrium state, which may conveniently be called the ‘stationary’ and the ‘instantaneous’ interpretation, respectively. Such interpretations, in turn, are related to two alternative views of the equilibration process supposedly driving the economy or one of its subsystems towards an equilibrium state: according to the first one, the equilibration process should be viewed as a process taking place in ‘real’ time, giving rise to observable actions, and driving the economy towards a state of ‘stationary’ equilibrium; according to the second one, instead, the equilibration process should be viewed as a process taking place in ‘logical’ time, giving rise to no observable actions, and driving the economy towards a state of ‘instantaneous’ equilibrium.

Given his initial endorsement of the more traditional conception of equilibrium as a ‘balance of forces’, Walras is naturally led to embrace the ‘stationary’ interpretation of the equilibrium concept and to advocate a view of the equilibration process, the *tâtonnement* process in Walras’s terminology, according to which such process takes place in ‘real’ time, giving rise to observable phenomena.

This is the sort of equilibration process initially assumed to be at work in all of Walras’s models. Yet, when Walras replaces the traditional conception of equilibrium with the new one, centered on the optimizing behavior of the economic agents, the notion of ‘stationary’ equilibrium and the associated interpretation of the *tâtonnement* process become incompatible with the new theoretical framework. This is particularly evident as far as the pure-exchange model is concerned, since this is the first model to be substantially affected by the new equilibrium conception. But also in the models that are at first less significantly affected by the new approach, such as the production and the capital formation models, the ‘stationary’ interpretation of both the equilibrium notion and the associated equilibration process cause a number of theoretical difficulties.

At first Walras reacts to such predicament by virtualizing the *tâtonnement* process associated with the pure-exchange model (1885 and 1889). Ten years later, by adopting the so-called “hypothèse des bons”, he transforms the equilibration processes of the entire economy, including those supposedly at work in the models with production, capital formation, circulation and money, into purely virtual processes, taking place in a ‘logical’ time separate from the ‘real’ time over which the economy evolves, and driving the whole economy towards a state of ‘instantaneous’ equilibrium. Yet, Walras’s change of mind is not brought to its extreme logical consequences, so that the version

of general equilibrium theory expounded in the 4th edition of the *Éléments* (1900) still displays inconsistencies due to the conflicting assumptions persisting therein.

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