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Ciclo XXXI

Lorentz invariance violation effects on ultra high energy cosmic rays propagation, a geometrical approach

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To my family

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Introduction

Motivation

In scientific literature with the term "*Cosmic rays*" (CRs) are indicated highly energetic particles of extraterrestrial origin [1, 2, 3]. They were originally discovered at the beginning of the twentieth century and were originally thought as composed mainly by electromagnetic radiation, because of their great penetrating power. The study of CRs developed through various stages, till the confirm of the existence of the Extended Air Showers (EAS) by Pierre Auger in 1938 [4]. The EAS are cascades of secondary products generated by the interaction of charged particles, that constitutes the original CRs, with the atmosphere. In fact it is well known that cosmic rays are of different nature, made by bare atomic nuclei, principally of two types: very light ones (principally protons or alpha particles) or heavier ones (mainly atomic nuclei close to Fe element) [5, 6]. Their energy spans many decades and their energetic spectrum ranges from 10^8 eV to 10^{21} eV. Because they include the most energetic particles, accessible nowadays, it is very interesting to study this kind of radiation, that can furnish new data about the production of particles in highly energetic environments [7, 8]. For example, a cosmic proton, with an energy of 10^{20} eV, colliding with a stationary identical particle of the atmosphere, can originate a reaction with a free energy (energy in the center of momentum reference frame) of about 400 TeV. This quantity must be confronted with the 14 TeV of maximum free energy available at LHC, motivating the interest in this kind of highly energetic collisions.

Another important motivation to study cosmic rays is due to the necessity of understand their origin, intended as their creation and acceleration mechanism. Gravity is supposed to be the ultimate engine of CRs acceleration, even if till now the process is not fully understood, it is possible only modeling part of it. Therefore it is supposed that CRs are generated in gigantic gravitational collapses, such as in supernovae explosions, the accretion disks of black holes, gamma ray bursts (GBR) or active galactic nuclei (AGN). Solving the problem of CR origin and composition can provide an insight of the accelerating structures and can so be useful in conducting astrophysical observations.

To study CRs properties it is necessary to distinguish between *"primary cosmic rays"*, the ones produced by the sources, and *"secondary CRs"*, revealed by facilities at soil, produced by the interaction of primary CRs with the atmosphere. This interaction changes the properties of the primary, modifying the composition and reducing their energy. The reconstruction of the primary nature from the secondary therefore is not a simple task. For this motivation and for the low frequency of the highest energetic events, the study of CRs requires gigantic facilities like Pierre Auger Experiment, in Argentina [9], or Telescope Array, in the northern emisphere [10]. Another important separation in CRs nature is the distinction between the ones generated inside our galaxy (the Milky Way) from the extragalactic ones. This separation is very difficult and hides the low level of comprehension of CRs generation mechanisms. A deeper understanding of this process in fact can allow how to separate galactic from extra-galactic ones.

Other problems in modeling CRs occur when one tries to describe their propagation through the accelerating structures and then in free space [11, 12]. CRs transport in accelerating media is outlined using the theory of diffusion, generated by resonant scattering with plasma waves of charged particles. This description can be misleading simple, but it hides the non-linearity of this kind of picture. In this sense the plasma physics determines from small scale the macroscopic physics of CRs. Another issue is given by the description of CRs propagation in deep space, with the relative lost of energy [13, 14, 15, 16]. The diffusion of cosmic rays is influenced by their interaction with the Cosmic Microwave Background Radiation (CMBR) and other physical backgrounds. Moreover considering extragalactic CRs, it is necessary to contemplate even the adiabatic waste of energy due to the universe expansion. To describe the CRs propagation it is necessary to take into account even the interaction of this charged particles with the galactic and extragalactic magnetic fields ($3\ \mu G$ and $3\ nG$ respectively). These magnetic fields can deflect the path of charged CRs, transforming it from straight to extremely chaotic, for low energy CRs. But in case of Ultra High Energy Cosmic Rays (UHECR - those with an energy equal or superior to $5 \cdot 10^{19}\ eV$) the interaction with magnetic fields can deflect their propagation only for few degrees. This means that UHECRs can be used to conduct anisotropy researches, correlating their origin with known position astrophysical objects. These objects must be collocated inside a sphere with radius determined by the energy and the nature of the UHECR considered. In fact, due to the interaction with the CMBR, Universe is not transparent for the propagation of CRs. Protons, revealed with an energy exceeding $5 \cdot 10^{19}\ eV$, can have been originated only inside a ball of defined dimension, because of photo-production of π particles caused by the interaction with the CMBR (effect known as GZK from the names of the physicists Greizen-Zatsepin-Kuz'min). Instead for heavier nuclei the interaction with the cosmic background originates a photo-dissociation effect, even dissipating part of the original kinetic energy. In both eventuality, the original CR, after a determined propagation length, has dissipated its original energy and it can be revealed only sub energy threshold. In this way the opacity to the propagation of UHECR permits anisotropy searches. In fact Universe is presumed to be anisotropic inside a finite space region with a diameter of about $100\ MpC$ (the dimension of the GZK foreseen opacity sphere). If UHECRs sources are collocated inside this sphere and are not uniformly distributed, it is possible to expect an anisotropy in arrival direction, if energy is sufficient to prevent a significant propagation path magnetic induced deflection.

Some recent experimental evidences [17] seem to suggest the possibility that the predicted lack of transparency to UHECR diffusion may be modified, increasing the opacity sphere dimension. This because some observed UHECRs probably correlate with astrophysical objects, possible sources, situated farther than expected. Since the original paper of Coleman and Glashow [18], many theoretical attempts, to justify such experimental evidences, have been made, resorting to quantum gravity effects. But this presents relevant difficulties, because currently there is no a consistent formulation of a theory integrating Quantum Physics and General Relativity. Trying to enunciate a unified theory of this type, the first encountered difficult is due to the current impossibility to obtain sufficient energies needed to probe space-time at the Planck scale. In fact it is commonly supposed that the Planck length $\lambda_P = \sqrt{G\hbar c^{-3}}$ and the Planck energy $E_P = \sqrt{\hbar c^5 G^{-1}}$ represent the length and the energy scales separating the classical gravitation theory from the quantized one. Nevertheless, Planck-scale effects may manifest themselves as tiny violations of classical conservation laws at lower energies. Observation of UHECR is therefore very interesting *per se*, as just underlined, to put under experimental verification LIV theories. In fact very energetic particles are needed, that propagate for cosmological distances, so that the small violations can add together and manifest themselves. The study of the GZK phenomenon appears therefore very useful even to put constrains on QED and hadronic LIV operators. Nevertheless there are some experimental uncertainties in this sector of research, because the difficult in reconstructing the UHECR mass compositions. In fact recent hints about the mass composition increasing [19] at rising energies seem to suggest that heavier nuclei (Fe type) can constitute the UHECR majority, for energies above $10^{19}\ eV$. On the contrary Telescope Array [20] data seem to be in contrast with Auger ones. To clarify this confused situation, Auger collaboration is deploying an hardware upgrade [21]. The new detectors, plastic scintillators (SSD - solid state detectors), will integrate the Cherenkov water tanks (WTD - water tank detector) to increase the ability to discriminate the CR chemical composition. For this reason there is a renewed interest in studying UHECR as a possible scenario, where LIV effects can emerge to manifest some distinctive features. In fact LIV consequences can appear in GZK cut-off modifications, but even in hadronization, caused by these high energy particles. Therefore LIV can modify the heavy nuclei propagation, changing the photodissociation process. Moreover LIV can manifest even influencing the hadronization processes present in atmospheric showers creation, caused by cosmic rays.

Coleman and Glashow [18] and then Coen and Glashow [22] introduced the effects of LIV considering a personal maximum attainable velocity for every massive particle. In this way they predicted that for a proton maximum velocity, sufficiently different from the light speed (the invariant parameter introduced in special relativity), the GZK effect results totally suppressed. Alternatively in their theory the foreseen Universe opacity is not affected by the violation of Lorentz Invariance. Since those works, the introduction of LIV has been improved, exploiting the resulting effects due to the replacement of *ordinary special relativity* with the so called *modified special relativity* theories. These new hypotheses can account even continuous dilatations of the GZK opacity sphere as function of the LIV parameter. Moreover *modified special relativity* theories try to incorporate an invariant length parameter (the Planck lenght) to the invariant velocity (the speed of light) of standard special relativity.

In this work first of all a convincing modification of the special relativity is introduced, which can take into account the changed kinematics of massive particles in a Lorentz violating scenario. The interaction with the quantum background of space-time is described modifying the geometry, underlying this phenomenon. It is well known that, in this kind of framework, it is necessary to introduce a geometric structure, that depends not only on the local coordinates, as in Riemaniann geometry, but even on the energy of the particles, probing the structure of space. It appears obvious to resort to a kind of geometry that can take into account this dependence on frequencies, and it is the *Finsler* geometry [23, 24, 25]. Using these theories, it is demonstrated that the propagation of CRs happens in a FWR (Friedmann-Robertson-Walker) asymptotically flat space-time. Moreover it is founded a correspondence with the minimal Standard Model extension introduced by Kostelecky [26]. This results permits to deduce that the cross section of the interaction with the CMBR is not dramatically changed in a LIV scenario with strongly constrained violating parameters. It is even demonstrated that the more substantial consequences are all concentrated in the kinematical aspects of the interaction of UHECRs with the CMBR. This changes have the effect of reducing the phase space available for the products of the reaction, so diminishing the probability of a photo-pion production [27, 28]. All these hypotheses permit to obtain some interesting results on the UHECR propagations, determining a LIV parameter dependent prediction for the dimension of the opacity sphere. This approach permits to describe a continuous transition from the GZK phenomenon to the Coleman and Glashow foreseen total suppression. As direct consequence, this theory permits to justify the most recent experimental evidences. It can give a prediction to use UHECR as probes of the structure of space-time, using them to put under experimental verification the presence of LIV.

Part I

Introduction to UHECR GZK puzzle and LIV theories

CHAPTER 1

Cosmic Rays: basic properties

To introduce the topics relative to *cosmic rays* (CR), first of all it is fundamental to consider how they are generated and accelerated [29, 30, 31]. The current physical theories on their creation divide in two principal classes: the bottom-up and the top-down scenario [32, 33]. In the bottom-up framework charged particles are accelerated by astrophysical structures from lower to higher energies. These models describe the acceleration as a result of the interaction of charged particles with shock-waves, generated, for example, by supernovae explosions or AGNs. Moreover they can explain even the shape of the energy spectrum, with the suppression founded for the highest values. All the models presented here share the property that the acceleration regions have to act as "*particle accelerators*". This means that a magnetic field must confine a particle inside the region, where other mechanisms increase its energy. The maximum attainable energy is therefore proportional to:

$$E_{max} \div Ze\, R\, B \tag{1.1}$$

where Ze represents the electric charge of the CR particle, B the intensity of the magnetic field and R the size of the accelerating region of space.
In the top-down scenario the CRs are generated as decay products of what are called WIMP (weakly interacting massive particles), the ones candidate to be most of *dark matter*. In this case the acceleration is caused by the fact that the decay products are lighter than the original particles, so part of their mass is transformed in kinetic energy. Recent observational data, from experiments like Auger [34], for example, hint the exclusion of the second type of framework, because the lack of the required produced secondary photons. For this reason, in this work only the bottom-up scenario is illustrated.

1.1 Cosmic Rays acceleration mechanisms

1.1.1 Fermi II° order

The first acceleration model [35], originally proposed by Fermi in 1949 [36], suggests that particles increase their energy by stochastic collisions inside inhomogeneous magnetic regions (clouds of matter). This can explain how a particle is accelerated by a shock wave. In fact the inhomogeneity may arise as remnant of catastrophic gravitational collapses.
Suppose a charged particle, with given energy E_1 and velocity v in an arbitrary "fixed" reference frame. This particle moves toward a shock wave, that is the boundary of a region with different density. This dust cloud propagates with velocity w in the same coordinates system, along the "x" axis. In a reference frame (the primed one) attached to the cloud, the particle energy and momentum are:

$$\begin{aligned} E_1' &= \gamma(E_1 - p_1\beta c\cos\theta_1) \\ p_1' &= \gamma\left(p_1 - E_1\frac{\beta}{c}\cos\theta_1\right) \end{aligned} \tag{1.2}$$

where θ_1 is the incident angle of the particle, p_1 is the particle momentum before the interaction, $\beta = \frac{w}{c}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ is the relativistic factor.

The collisions inside the inhomogeneity are supposed totally elastic. Because the mass of the cloud is much larger than the particle one, only the CR energy and momentum are affected by the interaction. After one collision they become, in the reference frame attached to the inhomogeneity:

$$\begin{aligned} E_2' &= E_1' \\ p_2' &= -p_1' \end{aligned} \tag{1.3}$$

The final energy of the particle expressed in the fixed frame is then:

$$\begin{aligned} E_2 &= \gamma(E_2' - p_2'\beta c\cos\theta_2) = \gamma(E_1' + p_1'\beta c\cos\theta_2) = \\ &= \gamma^2\left(E_1 - p_1\beta c\mu_1 + p_1\beta c\mu_2 - E_1\beta\mu_1\mu_2\right) \end{aligned} \tag{1.4}$$

where $\mu_1 = \cos\theta_1$, $\mu_2 = \cos\theta_2$ and θ_2 is the outgoing angle.
Using the fact that:

$$\frac{p}{E_1} = \frac{m\gamma v}{m\gamma c^2} = \frac{v}{c^2} \simeq c \tag{1.5}$$

the final energy can be written as:

$$E_2 = \gamma^2 E_1\left(1 - \beta\mu_1 + \beta\mu_2 - \beta^2\mu_1\mu_2\right) \tag{1.6}$$

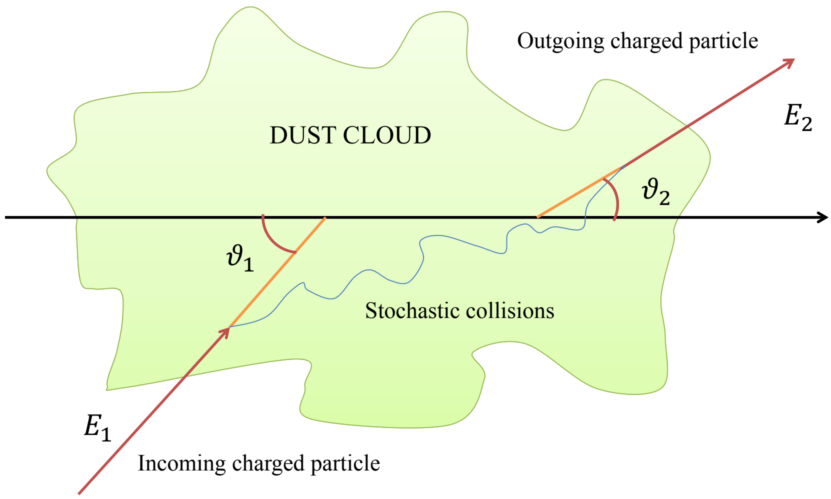


Figure 1.1: Charged particle interaction with a dust cloud in II° order Fermi mechanism.

The relative increment of energy can be expressed as:

$$\frac{\Delta E}{E_1} = \frac{E_2 - E_1}{E_1} = \gamma^2(1 - \beta\mu_1 + \beta\mu_2 - \beta^2\mu_1\mu_2) - 1 \tag{1.7}$$

Taking into account that the impacts number per time unit must be proportional to the relative velocity of the particle respect to the dust cloud, the probability of a collision must be proportional to:

$$P(\mu_1) \div (v - \beta\mu_1 c) \;\Rightarrow\; P(\mu) = A(1 - \beta\cos\theta_1) \tag{1.8}$$

where A is the normalization constant, given by:

$$\int_{-1}^{1} P(\mu)d\mu = A\int_{-1}^{1}(1 - \beta\mu_1)d\mu = 1 \;\Rightarrow 2A = 1 \;\Rightarrow\; A = \frac{1}{2} \tag{1.9}$$

Mediating on all the allowed angle values, it is possible to evaluate the average relative energy increment rate per every cycle of interactions. Every particle inside the dust cloud suffers a great number of stochastic collisions, so the cosmic rays outgoing directions must be randomly distributed. This implies that the averaged value of terms proportional to $\cos\theta_2$ are equal to 0 (that is $\langle\cos\theta_2\rangle = 0$)

and can be neglected in the following computation. In this way the mean increment becomes:

$$\left\langle \frac{\Delta E}{E_1} \right\rangle = \int_{-1}^{1} P(\mu_1)\left(\gamma^2(1-\beta\mu_1)-1\right)d\mu_1 = \frac{1}{2}\int_{-1}^{1}(1-\beta\mu_1)\left(\gamma^2(1-\beta\mu_1)-1\right)$$
$$d\mu_1 = \frac{1}{2}\int_{-1}^{1}\gamma^2(1-2\beta\mu_1+\beta^2\mu_1^2)d\mu_1 - 1 = \gamma^2\left(1+\frac{\beta^2}{3}\right)-1 \simeq \frac{4}{3}\beta^2 \tag{1.10}$$

The energy gain, in this scenario, is due to the fact that a particle, which encounters a dust cloud tends to thermalize with the medium and acquires part of its energy. It is clear now why this process is indicated as *secon order Fermi mechanism*, in fact the energy gain for every process cycle is proportional to β^2. But its efficiency is not enough high to explain the energy spectrum of cosmic rays, in fact typically $\beta \simeq 10^{-4}$. So it is necessary to introduce a linear process in the parameter β, the *first order Fermi mechanism*.

1.1.2 Fermi I$^\circ$ order

To find a theory linear in the ratio $\beta = \frac{v}{c}$ it is necessary to reconsider the interaction of the charged particle with a shock wave [37], generated by a gravitational collapse. In second order Fermi mechanism the particle is accelerated by the casual interactions inside a dust cloud and diffused by the chaotic magnetic field of the dust. In this case the particle is supposed to be inside a fluid where a wave propagates. The wave constitutes the separation surface between two space regions occupied by fluids, with different physical properties, and through the surface there is a flux of matter. In fact the particle passes through the discontinuity and is accelerated by the head-on collisions with the gas encountered, inverting its direction and crossing many times the shock wave front. Taking into account a little space region, the separation surface results planar and so instead of evaluating randomly distributed shock waves it is possible to use the planar wave approximation. In this way it is introduced the Diffusive Shock Acceleration (DSA) mechanism, which will result linear in the β ratio.
In a reference frame attached to the discontinuity surface, the fluid of the unperturbed region (upstream region) is moving towards the separation, with velocity v_1, density ρ_1 and pression P_1, while the perturbed one (downstream region), is running away with velocity v_2, density ρ_2 and pression P_2. A turbulent magnetic field is present inside both the perturbed and unperturbed regions. From mass,

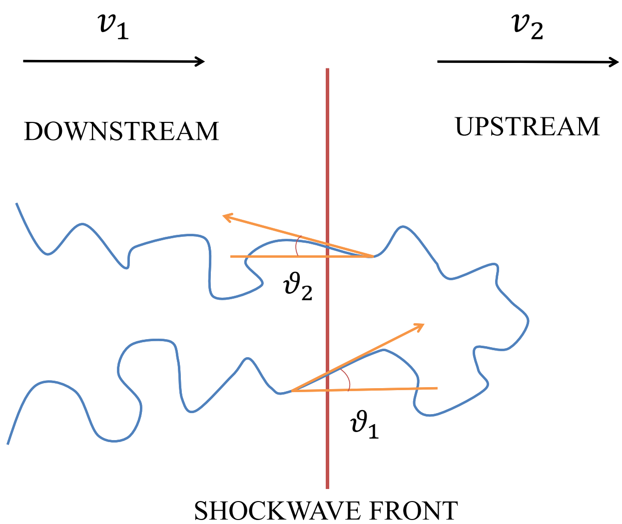


Figure 1.2: Charged particle interaction with a shock wave front in I° order Fermi mechanism.

momentum and energy conservation principles, one obtains the equations:

$$\begin{cases} \rho_1 v_1 = \rho_2 v_2 \\ \rho_1 v_1^2 + P_1 = \rho_2 v_2^2 + P_2 \\ \frac{1}{2}\rho_1 v_1^2 + \frac{\chi}{\chi - 1} v_1 P_1 = \frac{1}{2}\rho_2 v_2^2 + \frac{\chi}{\chi - 1} v_2 P_2 \end{cases} \tag{1.11}$$

where $\chi = \frac{C_P}{C_V}$, C_P and C_V are respectively the specific heat at constant pressure and volume. Assuming an ideal fluid, it is possible to obtain the following relation:

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\chi + 1)\mathcal{M}_1^2}{(\chi - 1)\mathcal{M}_1^2 + 2} \tag{1.12}$$

where $\mathcal{M}_i$ represents the *Mach* number, that is the ratio of the velocity of the shock and the sound speed inside the matter where the shock itself propagates:

$$\mathcal{M}_i = \frac{v_i}{c_i} \tag{1.13}$$

If the shock speed is assumed much larger than the sound one in the fluid, it is reasonable to consider for the *Mach* number $\mathcal{M} \to \infty$ and the relation (1.12) becomes:

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{\chi + 1}{\chi - 1} \tag{1.14}$$

From this relation follows that in the down-stream region the gas density is increasing respect to the one of the gas in the upstream region and the velocity of the perturbed medium is decreasing.
If the fluid is considered made of a mono-atomic perfectly ionized gas, the ratio χ assumes the value:

$$\chi = \frac{C_P}{C_V} = \frac{5}{3} \tag{1.15}$$

and consequently, for the velocity:

$$v_2 = \frac{1}{4} v_1 \tag{1.16}$$

In the fixed reference frame the unperturbed medium is moving toward the shock wave front with a velocity $V = v_2 - v_1$. The energy of a particle inside the downstream region is (fixed coordinate system):

$$E_i' = \Gamma E_i \left(1 + \beta \cos \theta_i\right) \tag{1.17}$$

where $\beta = \frac{V}{c}$ and $\Gamma = \frac{1}{\sqrt{1-\beta^2}}$.
After some random interactions the particle inverts its direction, it crosses again the shock wave and its energy is now E_f' in the downstream region (reference frame attached to the discontinuity):

$$E_f = \Gamma E_f' \left(1 + \beta \cos \theta_f'\right) \tag{1.18}$$

Even in this case the energy is conserved after every cycle of interactions, so $E_f' = E_i'$ and replacing what obtained in (1.17) in (1.18):

$$E_f = \Gamma^2 E_i \left(1 + \beta \cos \theta_f'\right) \left(1 + \beta \cos \theta_i\right) \tag{1.19}$$

Therefore the relative energy increment is given by:

$$\frac{\Delta E}{E_i} = \frac{E_f - E_i}{E_i} = \Gamma^2 (1 + \beta \cos \theta_i + \beta \cos \theta_f' + \beta^2 \cos \theta_i \cos \theta_f') - 1 \tag{1.20}$$

It is necessary now to evaluate the probability distribution of the angles θ_i and θ_f' respect to the shock wave front. The probability is proportional to the solid angle ($2 \sin \theta$) and to the propagation direction ($\cos \theta$), so:

$$P(\theta) = 2 \sin \theta \cos \theta \tag{1.21}$$

and finally:

$$dn = 2\cos\theta \, d\cos\theta \tag{1.22}$$

Now, it is possible to obtain the value of the relative energy increment averaged on the angles:

$$\begin{aligned}\left\langle \frac{\Delta E}{E_i} \right\rangle &\simeq \int_0^1 (\Gamma^2(1+2\beta\cos\theta)-1)\cos\theta \, d\cos\theta + O(\beta^2) = \\ &= \int_0^1 2\beta\cos^2\theta \, d\cos\theta + O(\beta^2) = \frac{4}{3}\beta\end{aligned} \tag{1.23}$$

where the terms with the angles θ_i and θ'_f have been collected because the integration extremes are equal in both cases.
It is important to underline that, for this theoretical model, the energy gain is due to the fact that head on collisions, with the shock wave front, are more likely to occur than the following ones. The first mechanism determines an energy gain, instead the second one is a dissipative process. Therefore in this situation, the energy gain results linear in β. This effect is due to the fact that the value of the cosine of the angle defined by the propagation direction and the normal to the wave front is included between 0 and 1. Instead in second order mechanism the cosine is included between -1 and 1, so there can be even energy loss for the particle in every interaction cycle. For this reason the energy gain in second order mechanism results not enough efficient to justify the observed shape of the energy spectrum and it is necessary to introduce a more efficient process.

1.2 Energy spectrum

1.2.1 Spectrum shape

Using the linear energy increment foreseen by the DSA mechanism [38, 39], it is now possible to try to understand the observed at soil cosmic rays energy spectrum. The particle energy relative increment, per every cycle of interaction, that is two crossing of the discontinuity from the downstream region to the upstream and viceversa, is:

$$\left\langle \frac{\Delta E}{E_0} \right\rangle = \left\langle \frac{E_1 - E_0}{E_0} \right\rangle = \xi E_0 \tag{1.24}$$

$\xi > 0$ is the energy gain and, from the DSA model, it follows that $\xi = \frac{4}{3}\beta$. After n different cycles, the energy will be:

$$E_n = (1+\xi)E_{n-1} = \ldots = (1+\xi)^n E_0 \tag{1.25}$$

Defining $0 < \epsilon < 1$ as the probability that a particle escapes from the accelerating space region, that is downstream one, the number $\widetilde{N}_n$ of particles after n cycles will be:

$$\widetilde{N}_n = (1-\epsilon)^n \widetilde{N}_0 \tag{1.26}$$

and the number of particles run away from the downstream region, becoming cosmic rays, is:

$$N_n = \epsilon \widetilde{N}_n = \epsilon(1-\epsilon)^n \widetilde{N}_0 = (1-\epsilon)^n N_0 \tag{1.27}$$

redefining the initial number of particles as $\epsilon \widetilde{N}_0 = N_0$.
From relations (1.25) and (1.27) it is possible to obtain:

$$\begin{cases} \ln\left(\dfrac{N_n}{N_0}\right) = n \ln(1-\epsilon) \\ \ln\left(\dfrac{E_n}{E_0}\right) = n \ln(1+\xi) \end{cases} \tag{1.28}$$

From these equations follows:

$$\ln\left(\frac{N_n}{N_0}\right) = \ln\left(\frac{E_n}{E_0}\right)\frac{\ln(1-\epsilon)}{\ln(1+\xi)} = \ln\left[\left(\frac{E_n}{E_0}\right)^{-\alpha}\right] \tag{1.29}$$

where it has been defined the ratio $\alpha = -\frac{\ln(1-\epsilon)}{\ln(1+\xi)}$.
From the previous relation it is possible to derive:

$$\left(\frac{N(E)}{N_0}\right) = \left[\left(\frac{E}{E_0}\right)^{-\alpha}\right] \tag{1.30}$$

where the final number of particles is expressed as function of the energy. Differentiating this equality to respect to the energy it is possible to obtain the differential distribution:

$$\frac{dN(E)}{dE} = N_0 E_0^{\alpha}(-\alpha)E^{-\alpha-1} \tag{1.31}$$

setting $\gamma = \alpha + 1$, the previous relation becomes the well known power law, that determines the energy spectrum shape:

$$\frac{dN(E)}{dE} \div E^{-\gamma} \tag{1.32}$$

Necessary thing left to do consists in evaluating the numeric value of the exponent γ. To obtain this result, first of all is fundamental to evaluate the probability for a particle to run away from the accelerating medium, the downstream region, per time unit. Setting the number of particles that from the upstream region enter the downstream one as $N_{up \to down}$, the number that follows the inverse path as $N_{down \to up}$ and the number that run away as $N_{escaped}$, the conservation of the total particle number gives:

$$N_{up \to down} = N_{down \to up} + N_{escaped} \tag{1.33}$$

The density per steradian of particles crossing the discontinuity, in one way or the other, per time unity, must be proportional to the total number present, multiplied by the velocity of the flux, so:

$$\begin{cases} n_{down \to up} = N_0 v_1 = N_0 c \cos\theta \\ n_{escaped} = N_0 v_2 \end{cases} \tag{1.34}$$

Integrating over the solid angle, it is possible to obtain the total numbers associated to the previous densities:

$$\begin{cases} N_{down \to up} = \displaystyle\int_{\Omega+} N_0 c \cos\theta \frac{1}{4\pi} d\Omega = \frac{N_0 c}{4} \\ N_{escaped} = \displaystyle\int_{\Omega+} n v_2 \frac{1}{4\pi} d\Omega = N_0 v_2 \end{cases} \tag{1.35}$$

Finally the escape probability ϵ can be expressed in the form:

$$\epsilon = \frac{N_{escaped}}{N_{up \to down}} = \frac{N_{escaped}}{N_{down \to up} + N_{escaped}} = \frac{N_0 v_2}{N_0 \left(\frac{c}{4} + v_2\right)} \tag{1.36}$$

and consequently the probability to cross the shock again is:

$$1 - \epsilon = 1 - \frac{N_0 v_2}{N_0 \left(\frac{c}{4} + v_2\right)} = \frac{\frac{N_0 c}{4}}{\frac{N_0 c}{4} + N_0 v_2} \simeq \left(1 - \frac{4 v_2}{c}\right) \tag{1.37}$$

Substituting the value of $\xi = \frac{4}{3}\beta$, as evaluated by the DSA mechanism, and what just obtained for $1-\epsilon$, it is possible to compute the α exponent of equations (1.29) and (1.30) as:

$$\alpha = -\frac{\ln(1-\epsilon)}{\ln(1+\xi)} = -\frac{\ln(1 - \frac{4v_2}{c})}{\ln(1 + \frac{4}{3}\frac{v_1 - v_2}{c})} \simeq \frac{3 v_2}{v_1 - v_2} \tag{1.38}$$

where $\beta = \frac{v_1 - v_2}{c}$.
Using the relation (1.16) that depicts the ratio of the two velocities of the particles in upstream and downstream regions, easily it follows that the α exponent is

equal to:

$$\alpha = \frac{3v_2}{v_1 - v_2} = \frac{3v_2}{4v_2 - v_2} = 1 \tag{1.39}$$

The differential distribution number of cosmic rays, respect to energy, results therefore given by the inverse power law (1.32), with the exponent $\gamma = 2$:

$$\frac{dN(E)}{dE} \div E^{-2} \tag{1.40}$$

This relation justifies the shape of the energy spectrum of cosmic rays, but there is a discrepancy with the observational evidences. In fact the theoretically predicted value 2 for the γ exponent is lower than the observed one, that varies from 2.7 to about 3.2. The steepness of the inverse power law in fact changes at different energies, and the spectrum presents some "structures" where its slope modifies itself. The first of these, known as the *knee* for its shape, is located at an energy of about $4 \times 10^{15}\ eV$, where the γ exponent changes from a numeric value of 2.7 to about 3.0. Another *knee* is present at about $4 \times 10^{17}\ eV$ where γ reaches the value of 3.2. The final change of steepness takes place at an energy of about 3×10^{18} where the spectrum slope decreases and the exponent γ returns to the value of 2.7.

The fact that the spectrum is steeper can be justified taking into account that during the propagation for cosmic distances, a charged particle has to escape from the magnetic field traps of the astrophysical objects that it encounters. The probability to evade must be proportional to its energy and so the real shape of the energetic spectrum may be justified by the equality:

$$\left.\frac{dN(E)}{dE}\right|_{det} \div \left.\frac{dN(E)}{dE}\right|_{sou} \times E^{-\delta} \div E^{-\gamma-\delta} \tag{1.41}$$

where the additional term $E^{-\delta}$ accounts for the energy depending spectrum modification, the energy detected on Earth is E_{det}, E_0 is the original energy at sources. In fact if no particles annihilate during propagation, the flux is constant and only the spectrum shape is influenced by the energy losses suffered by CRs along their path, so it is necessary to write:

$$\int \left.\frac{dN(E)}{dE}\right|_{det} dE = \int \left.\frac{dN(E)}{dE}\right|_{det} dE_0 = \int \frac{dN(E_0)}{dE_0} dE_0 \tag{1.42}$$

From this, differentiating respect to the energy at the source E_0, it is possible to obtain the relation:

$$\left.\frac{dN(E)}{dE}\right|_{det} = \frac{dN(E_0)}{dE_0}\frac{dE_0}{dE} \tag{1.43}$$

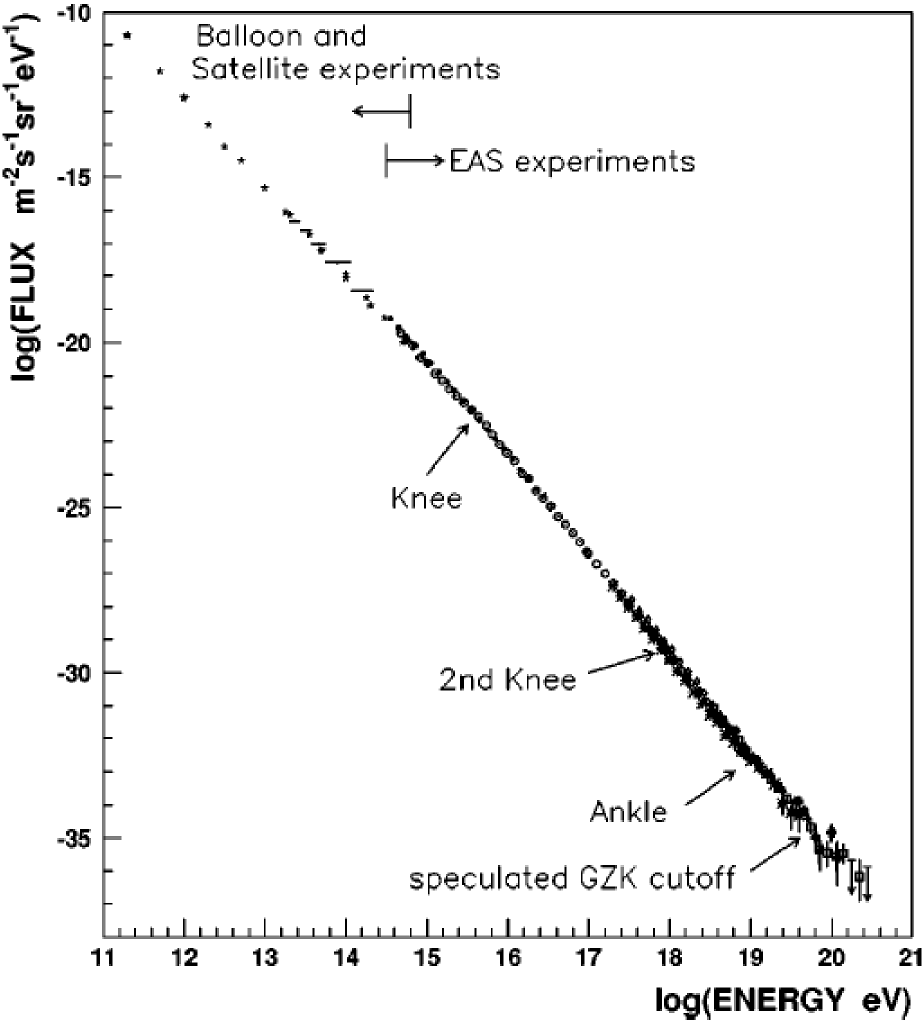


Figure 1.3: The differential distribution of cosmic rays energy observed on Earth by several experiments. All the changes in steepness are underlined. Figure taken from [1].

with the energy derivative respect to time given by the relation:

$$\frac{dE}{dt} = b(E) = E\beta(E) \tag{1.44}$$

From this:

$$\frac{dE}{dr} = \frac{1}{c}b(E) = \frac{1}{c}E\beta(E) \tag{1.45}$$

in this way it is possible to obtain from (1.43) the relation:

$$\left.\frac{dN(E)}{dE}\right|_{det} = \frac{dN(E_0)}{dE_0}\frac{b(E_0)}{b(E)} = \frac{dN(E_0)}{dE_0}\frac{E_0}{E}\frac{\beta(E_0)}{\beta(E)} \tag{1.46}$$

which makes clear the dependence of the spectrum shape on the energy dissipations suffered by propagating particles. In this way it is possible to justify the introduction of equation (1.41). More details on the effects of energy dissipations caused by the propagation of cosmic rays are given in the next chapter.
The presence of the *knees* is justified by the fact that there is a depression of the lighter particles in the cosmic rays flux at the higher energies. This fact with the estimation of the maximum attainable energy given by the equation (1.1) can therefore furnish an explanation of how the *knees* are generated [5]. In fact the increase of mean weight with the relative increase of charge number, causes an increment of energy, which modifies the steepness of the spectrum. The *ankle* [40] instead is explained by the fact that this structure is collocated where a transition from galactic to extragalactic origin of CRs takes place. This model predicts a transition from a heavy composition of CRs to a lighter one, principally constituted by protons, originated outside the galaxy and with a flat spectrum shape. At an energy of about $10^{17} \div 10^{19}\ eV$ a transition from galactic to extragalactic CRs is in fact expected to occur.

1.2.2 Maximum attainable energy

Even if there are no definitive proofs, nowadays the first order Fermi mechanism is supposed to be the responsible of the acceleration of cosmic rays till the higher energy values. The maximum energy that a charged cosmic ray can acquire thanks to such a mechanism is proportional to the energy gain rate multiplied by the time spent by the particle inside the shock wave region:

$$E_{max} \simeq \beta\, n = \beta\, T_{shock}\frac{E}{T_{cycle}} \tag{1.47}$$

Considering the accelerating region as a huge particle accelerator, it is necessary to confine the CR inside the region till it has gained the required energy. From this follows that the Lorentz force, acting on the charged particle and caused by

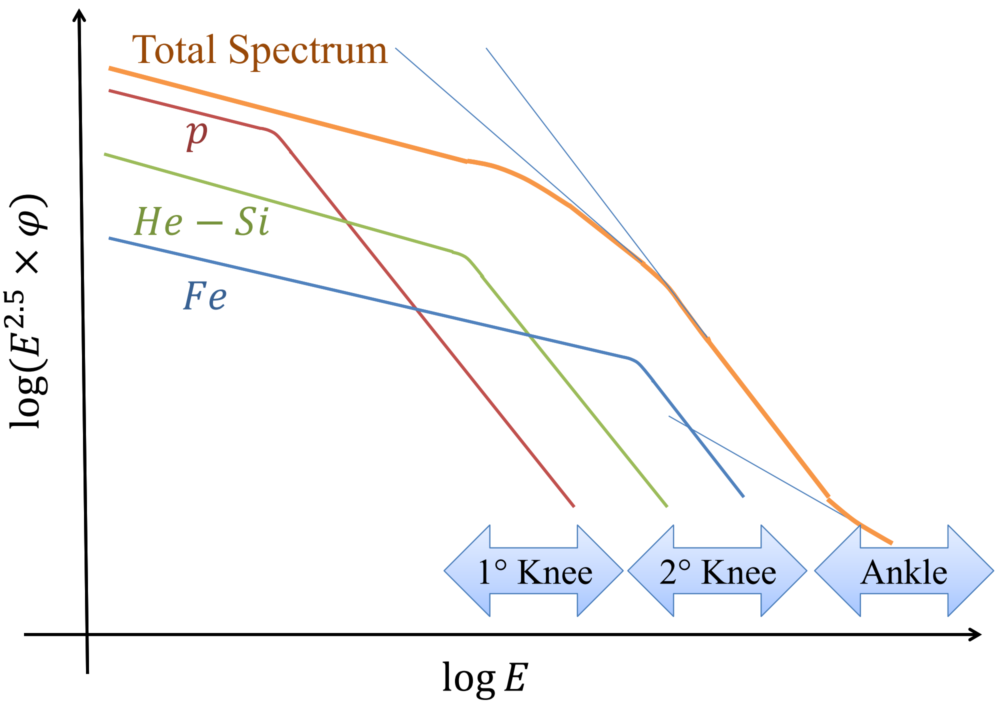


Figure 1.4: Modification in spectrum steepness justified by the change of cosmic rays mass composition, with the energy increase even the mean CRs mass grows up.

the magnetic field of the accelerating structure, must be equal to the centripetal acceleration necessary to trap the particle itself:

$$\frac{mv^2}{r} = q(\overrightarrow{v} \wedge \overrightarrow{B}) = qvB \;\Rightarrow\; r = \frac{mv}{qB} \tag{1.48}$$

For simplicity, the magnetic field is assumed orthogonal to the motion of the cosmic ray. In the previous equation r is the radius of the resulting motion and q the charge of the particle.
Now in the ultrarelativistic regime, with the velocity of the CR $v \simeq c$, it is possible to consider the energy $E = mc^2$, obtaining the expression of the Larmor radius:

$$r = R_L = \frac{E}{qcB} \tag{1.49}$$

and the time spent for a cycle of interactions becomes:

$$T_{cycle} \simeq \frac{R_L}{\beta c} \simeq \frac{E}{ZeB\beta c} \tag{1.50}$$

Finally, from (1.47):

$$E_{max} \simeq \beta^2 cZeBT_s \simeq ZeBR_s\beta \tag{1.51}$$

using the definition of the shock radius (radius of the accelerating space region): $R_s = T_s\beta c$, with T_s the time spent inside the shock wave portion of space. In this way it has been given a more precise explanation of the accelerating mechanism, reobtaining the relation (1.1). Another important consequence of this result is that the maximum attainable energy is proportional to the charge Z of the cosmic ray. As anticipated in the previous section, this model can therefore explains the *knees* of the spectrum with the mean mass increase and therefore the relative increase of the atomic numbers of the particles that constitute the cosmic rays flux at given energy. This prediction is supported by the observational evidences that with the increase of energy there is a depression of the lighter CRs component.
A more detailed derivation [41] of the maximum available energy, for a charged particle accelerated, is obtainable from the diffusion-convection equation:

$$\frac{d}{dt}N = Q + \overrightarrow{\nabla}(D\overrightarrow{\nabla}N) + \frac{\partial}{\partial E}(b(E)N) \tag{1.52}$$

N is the number of charged particles subject to the accelerating mechanism, Q is the term generated by the sources (injection term), D is the diffusion coefficient and $b(E)$ is a function that accounts for the energy spectrum changes due to the particles interactions.

Passing to the infinitesimal volume limit and considering the reference frame with the x axis oriented along the propagation of the shock wave front, it is possible to obtain the relation:

$$\frac{\partial}{\partial t}\rho = q + \frac{\partial}{\partial x}\left(D\frac{\partial}{\partial x}\rho - v\rho\right) + \frac{2}{3}\frac{dv}{dx}E\frac{\partial\rho}{\partial E} \tag{1.53}$$

where ρ is the particle density and q the injection term per volume unit.
The explicit form of the function $b(E)$ is obtained considering that the density change, as function of energy, is:

$$db(E) = \frac{\partial b(E)}{\partial E}dE = \frac{\partial\rho}{\partial E}dE \tag{1.54}$$

From relation (1.23) it follows that the infinitesimal energy increment for an infinitesimal movement, through the shock wave front becomes:

$$dE = \frac{2}{3}\frac{v_1 - v_2}{c\,dt}dx\,E \simeq \frac{2}{3}\frac{dv}{dx}E\,dx \tag{1.55}$$

and this represents the energy gain for Fermi first order mechanism half a cycle. From this equation (1.53) follows.
For simplicity it is assumed that the injection takes place only near the shock wave front, in the nearest part of the downstream region, so the injection term can be approximated with:

$$q(x) = q_0\delta(x) \tag{1.56}$$

Evaluating the integral of equation (1.48) in the transition region given by the shock front $x \in (0^-, 0^+)$, it follows the relation:

$$D\left.\frac{\partial\rho}{\partial x}\right|_1 - D\left.\frac{\partial\rho}{\partial x}\right|_2 + \frac{2}{3}(v_1 - v_2)E\frac{\partial\rho}{\partial E} + q_0(E) = 0 \tag{1.57}$$

In downstream region it is reasonable to suppose a homogeneous distribution of particles, so $\frac{\partial\rho}{\partial x}|_2 = 0$. Instead in the upstream region the change of velocity of a particle is null for infinitesimal changes of position, that is $\frac{dv}{dx} = 0$. The diffusion equation (1.51) takes the form:

$$\frac{\partial}{\partial x}\left(D\frac{\partial}{\partial x}\rho - v\rho\right) = 0 \tag{1.58}$$

and since the term inside the parenthesis at infinity vanishes, the solution is:

$$D\frac{\partial}{\partial x}\rho = v_1\rho \tag{1.59}$$

Substituting these results in relation (1.55), it takes the form:

$$\begin{aligned} v_1\rho = \frac{2}{3}(v_2 - v_1)E\frac{\partial\rho}{\partial E} + q_0\delta(E - E_{inj}) = 0 \;\Rightarrow \\ \Rightarrow \frac{3}{2}\frac{v_1}{v_2 - v_1} = E\frac{\partial\rho}{\partial E} + q_0\delta(E - E_{inj}) \end{aligned} \tag{1.60}$$

The solution of this equation is given by:

$$\rho(E) = q_0\left(\frac{E}{E_0}\right)^{\alpha} \tag{1.61}$$

where $r = \frac{v_1}{v_2}$ and:

$$\alpha = -\frac{3}{2}\frac{r}{r-1} \tag{1.62}$$

In case of strong shock, as in a supernova explosion, $r = 4$ and the density $\rho \div E^{-2}$, a result already obtained. In both upstream and downstream regions the flux of particles is given by equation (1.58) and in stationary condition:

$$D\frac{\partial\rho}{\partial x} = v\rho \;\Rightarrow\; \rho(x) = \rho_0 \exp\left(\frac{vx}{D}\right) \tag{1.63}$$

so the total number of charged particles is:

$$N_i = \int_{-\infty}^{0} \rho(x)dx = \frac{\rho_0 D_i}{v_i} \tag{1.64}$$

The time spent inside every region is given by:

$$t_i = \frac{N_i}{I_i} = \frac{4D_i}{v_i c} \tag{1.65}$$

where the particles flux is obtained from equation (1.35):

$$I_i = \frac{\rho_0 c}{4} \tag{1.66}$$

In this way the time spent for every cycle of interaction is given by:

$$t_{cycle} = t_1 + t_2 = \frac{4D_1}{v_1 c} + \frac{4D_2}{v_2 c} \tag{1.67}$$

Evaluating the ratio between the energy increment (1.23) and the time per cycle (1.60) it is possible to obtain:

$$\frac{\Delta E}{\Delta t} = \frac{4}{3}\frac{v_1 - v_2}{c}\frac{E}{t_{cycle}} = \frac{v_1 - v_2}{3}\frac{v_1 v_2}{D_1 v_2 + D_2 v_1} = \frac{E}{t_{acc}} \tag{1.68}$$

where the acceleration time t_{acc} has been introduced. From the previous relation it is simple to express this quantity as:

$$t_{acc} = \frac{3}{v_1 - v_2}\left(\frac{D_1}{v_1} + \frac{D_2}{v_2}\right) \tag{1.69}$$

In the case of strong shock, as for example a supernova deflagration, it is possible to use the *Bohm* diffusion coefficient. It is important to underline that its validity is limited to distances of the order of about 1 pc. For greater distances it would be more correct to use a diffusion equation that can account for the turbulence of the magnetic fields inside the interstellar medium. However, it is possible to approximate the real diffusion coefficient with the *Bohm* one, even for greater scales:

$$D = \frac{1}{3}\lambda v \tag{1.70}$$

where v is the particle velocity and λ is its mean free path during the diffusion process. In the ultra-relativistic approximation, with $v \simeq c$ and defining $\lambda = \frac{E}{ZeB}$ as the Larmor radius (1.44) associated to the particle, the diffusion coefficient becomes:

$$D = \frac{1}{3}\frac{Ec}{ZeB} \tag{1.71}$$

Consequently the acceleration times becomes:

$$t_{acc} = \frac{3}{v_1 - v_2}\frac{1}{3}\left(\frac{Ec}{ZeB_1v_1} + \frac{Ec}{ZeB_2v_2}\right) = \frac{cE}{Ze(v_1 - v_2)}\left(\frac{1}{v_1B_1} + \frac{1}{v_2B_2}\right) \tag{1.72}$$

Using again the approximation of strong shock wave, that implies $v_1 = 4v_2$ and assuming for the diffusion coefficients $D_1 \simeq D_2 \simeq D$ and for the magnetic fields $B_1 \simeq B_2 \simeq B$, the acceleration time can be written as:

$$t_{acc} = \frac{4}{3}\frac{cE}{ZeBv_1}\left(\frac{1}{v_1} + \frac{4}{v_1}\right) = \frac{20}{3}\frac{Ec}{ZeBv_1^2} \tag{1.73}$$

Finally it is possible to evaluate the maximum attainable energy E_{max} as:

$$E_{max} = \int_0^t \frac{\Delta E}{\Delta t}dt = \int_0^t \frac{E}{t_{acc}}dt = \frac{3}{20}\frac{ZeB}{c}v_1^2 t \tag{1.74}$$

where t represents the total time spent by the particle inside the accelerating region. From the equation obtained, it is simple to deduce that there is a relation between the Larmor radius of a charged particle and the dimension of the accelerating region for every specified energy value. In fact if the radius is much larger than the dimension of the region, the magnetic field is not strong enough to confine the charged particle and the cyclic mechanism can not take place. On the contrary, if the dimension of the accelerating region is much larger than the Larmor radius, the particle remains confined in one of the two regions separated by the shock wave and even in this case the cycle of interactions is forbidden. Even if the UHECRs sources are not known with absolute certainty, it is possible to identify some astrophysical objects as candidates. In fact a potential accelerator must fulfill some requirements, in order to let charged particles to reach the required energies. In fact the maximum attainable energy is evaluated, using equation (1.51), as proportional to $E \div ZeRB\beta$ where β represents the shock wave velocity in terms of units of light speed. Therefore extension and magnetic field strenght of candidates must be correlated. In the Hillas plot, here reported, are represented some astrophysical objects, as potentially cosmic rays accelerators. On the horizontal axis the linear extension of every object considered is indicated, instead on the vertical one the magnetic field strength. Another important topic of research about cosmic rays consists in discriminate between galactic and extragalactic origin. As already underlined, having a deeper insight on this phenomenon, for example, can explain the origin of the *ankle* in the spectrum. Establishing the origin point of CRs can appear arbitrary, because the lackness of a complete comprehension on how CRs are generated and accelerated. Nevertheless there is a general consensus in recognizing as a possible discrimination criterium the CRs energy. Using in the previous relation the charge of the proton $q \simeq 1.6 \times 10^{-19}\ C$, the speed of light $c \simeq 3 \times 10^{8}\ m/s$ and for the galactic magnetic field a mean value of $B \simeq 1\mu G \simeq 10^{-11}T$, the radius associated to the particle motion (1.44) becomes:

$$r = \frac{E_{eV} \times 1.6 \times 10^{-19}C}{1.6 \times 10^{-19}C \times 3 \times 10^{8}m/s} \times \frac{1}{10^{-11}T} \simeq \left(\frac{1}{3} \times 10^{3} \times E_J\right) m \tag{1.75}$$

Even if our galaxy shape is not globular, supposing the Milky Way average dimension $r_{MW} \simeq 33\ Kpc$, from (1.47) it is possible to find a transition from galactic to extragalactic proton constituted cosmic rays at an approximated energy of $10^{18} \div 10^{19}\ eV$. Even for heavier charged particle happens this kind of transition, but for smaller energy values. This demonstrates that almost all the CRs with an energy equal or superior to $10^{18}\ eV$ originate outside our galaxy. This type of Ultra High Energy Cosmic Rays (UHECR) are therefore very interesting because constitute an ideal probes for extragalactic astrophysical observations.

1.3 Composition and detection

Cosmic rays are composed principally by protons ($\sim 90\%$), but as already illustrated, the CRs average mass grows with the increasing of the energy [19, 42].

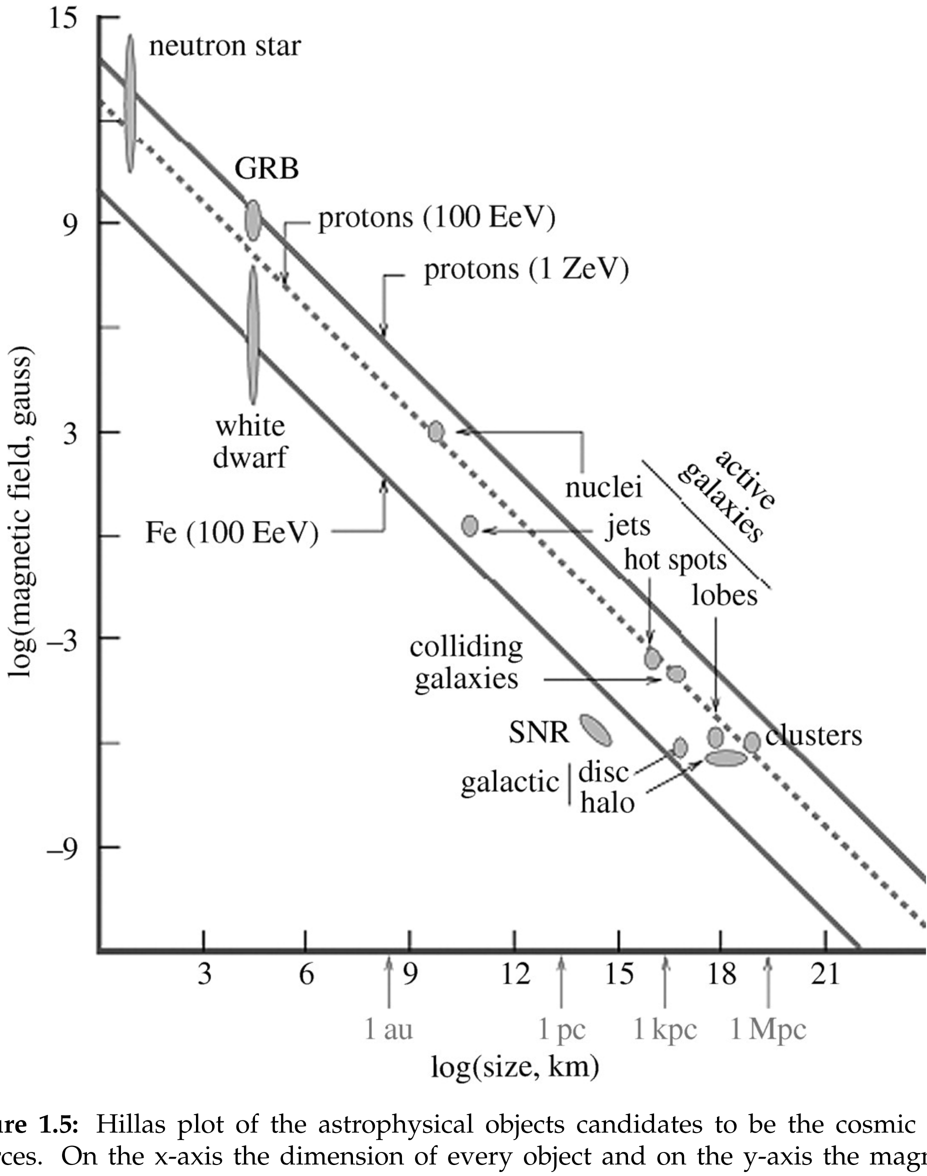


Figure 1.5: Hillas plot of the astrophysical objects candidates to be the cosmic rays sources. On the x-axis the dimension of every object and on the y-axis the magnetic field strenght are reported. A way to evaluate the maximum attainable energy is explained in the main text, and the result is $E_{max} = ZRB\beta$, with Z the charge number of the particle, R the extension of the astrophysical object, B the magnetic field strenght, β the shock velocity in speed of light units. The diagonal lines represent the maximal energies for different types of cosmic rays and different values of the shock wave propagation velocities. It is possible to accelerate to the same energy inside very large low field regions or in compact structures with large magentic field values.

Because only highly energetic protons are useful to conduct observational astrophysic, it is necessary to discriminate between protons made CRs from the heavier ones. Since the number of CRs revealed at soil decreases with energy, the arrival of UHECRs are very rare phenomena, about one event per century per square kilometer, at energy values of $E \geq 10^{19}\ eV$. For this reason it is impossible to use direct detections methods, such as satellites, to observe UHECRs. Instead it is necessary to use the atmosphere as a huge calorimeter, and then to collect the CRs print at soil. In fact, the only way to detect UHECRs is to to reveal the Extended Air Shower (EAS), the products cascade of secondary particles generated by the primary cosmic ray. The primary particle, propagating in the atmosphere, interacts with the oxygen and the nitrogen molecules, present in the air, generating a cascade of secondary products, the EAS, losing its energy until it is totally depleted. The secondary products cascade is made of a muonic component, which carries the 10% of the initital energy, a neutrinic one, which has the 1% of the total energy, an electromagnetic one, made of electrons, positrons and photons, with the 85% of energy, and the hadronic cascade, made of the fragments of the atomic nuclei involved in the interactions, with the 4% of energy. The produced particles are principally made of *pions* and *kaons*, which constitute the hadronic part of the EAS, together with the remnants of the atom destroyed in the interactions. These products may continue to interact, with other atoms, or decay, creating other hadrons or the remanent part of an EAS, that is muons, neutrinos or electro-magnetic secondary products (gamma rays or electron-positron pairs). During this process, pions decay before interacting with other particles, generating EM radiation or, with a lower probability, pairs. The EM component is constantly fed, because the larger part of particles, which compose the hadronic fraction of the cascade, interact again before decaying and most of the primary energy is dispersed in this channel. Gamma rays propagating in atmosphere generate electron and positron pairs, every particle of which carries away half of the original energy. Then these particles interact with the atmosphere atoms, generating again EM radiation by bremsstrahlung. This process dissipates energy until a critical value is reached, ionization becomes the principal dissipating mechanism for the pairs and no more particles are created. Instead, for the EM radiation, the Compton scattering starts to disperse the remaining energy, suppressing further pairs productions. Even a muonic component is present in every EAS and it evolves in a different manner. In fact they are created as decay products of pions and kaons in high atmosphere and are the only particles, together with neutrinos, which reach the ground level. In fact, because of their high energy, small cross section and long life time, muons survive enough to reach the ground level detectors. So they are used to reconstruct the original properties of the primary CR. Because the rarity of events involving UHECR, it is important to resort to huge facility to detect this kind of

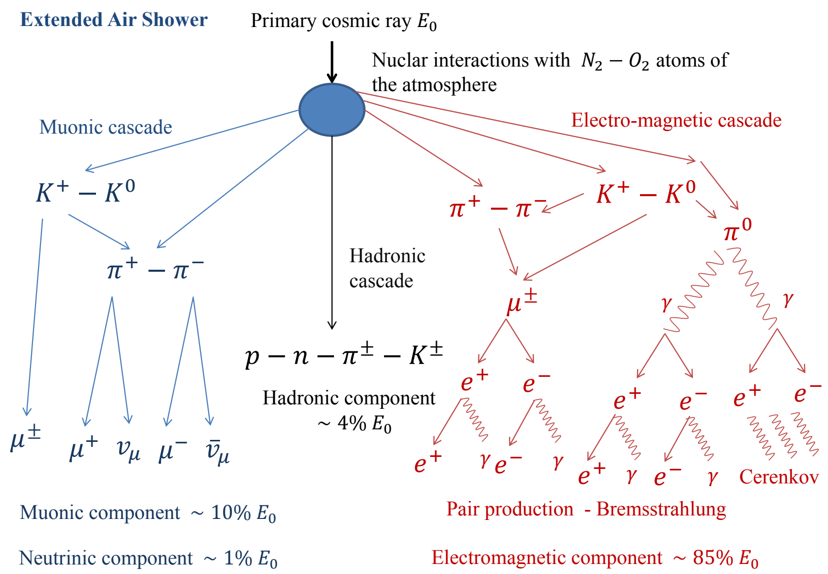


Figure 1.6: Developing of an Extended Air Shower in atmosphere of a primary cosmic ray with an initial energy E_0.

particles. For example Auger experiment is composed by 1660 Cerenkov Water Tanks (CWT) disposed on a triangular grid, optimized to detect UHECR generated EAS. It is necessary to obtain all the possible informations to reconstruct the original data of the primary particles, such as energy and arrival direction. To conduct this kind of research, experiments like Auger use the ground level CWT to collect data about the muonic component of each EAS. This process therefore consists in collecting what is called the EAS foot print at soil.

The discrimination of the lighter component of UHECR from the heavier one constitutes a great challenge for every experiment and it is very interesting because the great importance of the lighter component. A first method consists in measuring the penetration in atmosphere of the primary CR. This is made observing the altitude of the point where the EAS reaches the maximum production of fluorescence light, generated by the interaction of the primary charged particle with the atoms of nitrogen present in the atmosphere. Auger and Telescope Array experiments, for example, use fluorescence telescopes to detect the penetration depth of the primary, knowing that lighter particles penetrate much more, because they have a lower probability to interact. This method together with the reconstruction of the muonic print at soil guarantees a great precision in determining the energy and the original mass of the primary CR.

To increase this ability to discriminate the lighter component from the heavier one, Auger collaboration started recently an upgrade program [43], which will improve the reconstruction performance of the EAS muonic component. This provides the integration of a secondary fluorescence detector to the CWT, to discriminate the residue EM signal from the muonic one. Because lighter particles interact later, penetrating in the atmosphere, the ratio of the muonic and EM signal at soil changes from light to heavy particles. In fact if the EAS is generated at a big altitude in atmosphere, the electromagnetic component of the cascade can develop and reach its maximum production of particles. As consequence the EM component is predicted larger for heavier particles. But if the altitude exceeds a determined value, the effect of propagation in atmosphere determines an attenuation of this component. The further information obtained in this way can be integrated in the data analysis to better understand the nature of the original CR. In this way it results possible to discriminate the heavier component from the lighter one and answer to the question if there is an average mass increment with the detection energy.

CHAPTER 2

GZK effect

As already underlined, the ideal candidates, to conduct extragalactic astrophysical observations, are the most energetic cosmic rays, UHECR, those with an energy that exceeds a value of about $10^{19}\ eV$ [41]. In fact, it is well known that CRs, with such a high energy order of magnitude, are originated, with great probability, outside our galaxy, as already stressed at the end of the previous chapter. Observations of UHECR can therefore furnish interesting informations about extragalactic structures, together with other probes, like neutrinos and electromagnetic radiation, in what is called a *multi-messenger scenario*. But before conducing observational astrophysics with UHECR, it is necessary to have a deeper understanding of their propagation from the sources to the detection facilities.

2.1 Importance of UHECR for observational astronomy

To describe CRs physics, it is necessary to consider that they are constituted by charged particles, that interact with the Extragalactic and Galactic Magnetic Fields (EGMFs - GMFs respectively). These fields can deflect charged particles trajectories, making more difficult to identify their origin. The knowledge of these magnetic fields real magnitudes is still poor, but in some cases, such for example galaxy clusters, the strength is better known, with typical values of about $1\ \mu G$. Outside these structures, the magnetic field strength is not well known and it is possible to predict its value only by numerical simulations, about the formation of large scale structures, implementing a magneto-hydrodynamics evolution scenario. Other data are experimentally accessible via the observation of Faraday rotation of polarization, that poses an upper limit to the EGMFs intensity as: $B < 4\, nG$, in case of an inhomogeneous universe, with a coherence length of about $l_c \simeq 50\, Mpc$. These magnetic fields can deflect the path of charged CRs, with an intensity proportional to their electric charge. Therefore the trajectory is transformed, by this interaction, from straight to extremely chaotic, for low energy CRs or for heavy particles with a large electric charge. In fact, assuming an ideal configuration of a turbulent magnetic field with strength B, homogeneous inside regions with coherence length l_c, the Larmor radius (1.44) can be expressed as:

$$R_L \simeq \left(\frac{E_{PeV}}{Z B_{nG}} \right) \tag{2.1}$$

where the result is given in kpc, E is in units of $PeV = 10^{15}\, eV$ and B is in units of nG.
From this relation it follows, for example, that to aim to the center of our galaxy, distant $\sim\ 8\, kpc$ from the Earth, it is necessary to detect a proton with energy $E \simeq 10^{19}\, eV$.
Moreover what emerges from the equations (1.44) is that the CRs path deflection grows linearly with their electric charge, and consequently their mass, and is obviously bigger for low energy particles. More in detail it is possible to evaluate the distance on which a particle suffers a deflection of about $1\, rad$. This distance corresponds to the diffusion length l_d, the average path travelled by a particle between one scattering process and another. It can be evaluated using (1.65) and (1.67), and consequently, following [41]:

$$l_d \div D \tag{2.2}$$

where D represents the diffusion coefficient. Considering a diffusion mechanism for low energies, that is $R_L \ll l_c$, it results dominated by a resonant scattering phenomenon. The magnetic field strength is given by a magneto-hydro dynamic wave:

$$B_\lambda \exp \left(i \left(\overrightarrow{k}\, \overrightarrow{p} - \omega t \right) \right) \tag{2.3}$$

and is assumed constant for tiny λ wavelength. The resonant condition is implemented in the relation:

$$\omega = n \omega_g \tag{2.4}$$

with n a generic integer, ω the wave frequency and $\omega_g = \frac{eB}{\gamma mc}$ is the gyro-frequency. Using a Lorentz transformation the frequency becomes:

$$\omega - \overrightarrow{k}\, \overrightarrow{v} = \omega_g \tag{2.5}$$

and from this relation it is possible to compute the resonance wave number for $n = 1$:

$$k_{res} = \frac{\omega + \omega_g}{v\mu} \simeq \frac{\omega_g}{v\mu} = \frac{1}{R_L \mu} \tag{2.6}$$

where $\mu = \cos\theta$ and $R_L = \frac{v}{\omega_B}$.
Now from the normalized spectrum:

$$k\omega(k) = k_0\omega(k_0)\left(\frac{k}{k_0}\right)^{-\xi} \tag{2.7}$$

with the exponent $\xi = 0$ for Bohm, $\xi = \frac{1}{2}$ for Kraichnan, $\xi = \frac{2}{3}$ for Kolmogorov diffusion processes.
Following [12, 44] the frequencies of the scattering particle can be written as:

$$\nu(\mu, k_{res}) = 2\pi^2\omega_B k_{res}\omega(k_{res})\frac{1}{B_0^2} = 2\pi^2\omega_B k_0\omega_0\left(\frac{k}{k_0}\right)^{-\xi} \tag{2.8}$$

and to obtain the diffusion coefficient, considered for a process parallel to the magnetic field $\overrightarrow{B}_0$, it is necessary to perform the integral:

$$D = \frac{v^2}{4}\int_0^1 d\mu\frac{1-\mu}{\nu(\mu, k)} \tag{2.9}$$

The computed value is given finally by the summary formula:

$$D = \frac{1}{3}cl_D = \frac{1}{3}cl_c\left(\frac{E}{E_c}\right)^{\alpha} \tag{2.10}$$

and, for the diffusion length, the following relation is immediately obtained:

$$l_D \simeq l_c\left(\frac{E}{E_c}\right)^{\alpha} \tag{2.11}$$

The exponent is given by $\alpha = \frac{1}{3}$ for Kolmogorov, $\alpha = \frac{1}{2}$ for Kraichnan, $\alpha = 1$ for Bohm diffusion processes. Finally it is possible to deduce that α is related to the particle energy, which influences the way the particle itself feels the effects of the magnetic field. In fact, if the energy is such that the Larmor radius is smaller than the coherence length l_c, the particle scatters resonantly inside the turbulence with a resulting diffusion phenomenon. Instead, at bigger energies, when the Larmor radius is larger than the coherence length, that is $R_L \gg l_c$, it results $\xi = 1$ and all the previous computation can be repeated and the exponent of the equation (2.10) becomes $\alpha = 2$. At the highest energies the diffusion length l_D is even bigger than the average distances of the sources and in this case the particles do not feel influences by the interaction with the magnetic fields and the propagation is substantially rectilinear. This demonstrates that the lighter components of UHECRs flux, that is highly energetic protons, thanks to their huge energy and their low electric charge, are subject to a negligible interaction with magnetic fields.

Therefore they suffer tiny deflections from a rectilinear propagation. Direct consequence is that protons, with an energy equal or superior to $5 \cdot 10^{19}\ eV$, can be used to conduct anisotropy researches, because observing their arrival direction it is possible to aim to the sources. Therefore UHECRs result very interesting in the perspective of conducing observational astrophysics, in particular for extragalactic objects. Moreover, as already underlined, the importance of understanding the physics of UHECR is due to the fact that these ultra-energetic particles, which travel for cosmological distances, are the best candidates to investigate the physics of Lorentz Invariance Violation (LIV) and to probe the supposed microscopical quantum structure of space-time.

2.2 Cosmic rays propagation

2.2.1 Energy dissipation

A fundamental aspect of UHECRs physics is constituted by their propagation from the sources to the detection and the relative energy losses. In fact, Universe is not transparent to the propagation of particles, because of the presence of the Cosmic Microwave Background Radiation (CMBR). This radiation constitutes the electromagnetic relic of the big bang, a black body radiation distributed with an average temperature of $2.73\,^{\circ}K$. The interaction of a CR with this background depends on the particle nature [45, 46, 44]. A proton, for example, during its propagation in space, can interacts with this background via a photo-pion production process, passing through a delta resonance, or via a pair production:

$$\begin{aligned} & p + \gamma_{CMBR} \longrightarrow \Delta^+ \longrightarrow \begin{cases} p + \pi^0 \\ n + \pi^+ \end{cases} \\ & p + \gamma_{CMBR} \longrightarrow p + e^+ + e^- \end{aligned} \tag{2.12}$$

and dissipates part of its original energy for every particle creation process.
More massive particles interacting with the CMBR, can dissipate energy suffering a photo-dissociation. Through this process, the original cosmic ray breaks down in sub-particles, each of which carries away a part of the original energy:

$$(A,\, Z) + \gamma_{CMBR} \rightarrow (A - n,\, Z - n') + nN \tag{2.13}$$

A is the atomic number, Z is the charge number, n is the total number of stripped nucleons, n' the number of the stripped charged ones and N represents a nucleon.
Because their importance for astrophysics, in this section, first of all, only the pro-

tons propagation will be considered and therefore only the protons dissipation energy mechanisms will be analyzed. To understand the energy attenuation, it is possible to calculate the proton *optical depth*, defined, as the average distance over which a particle must travel in deep space to reduce its energy by a factor of $1/e$:

$$E = -\tau(E)Ecdt \;\Rightarrow\; -\frac{1}{E}\frac{dE}{cdt} = \tau(E) \tag{2.14}$$

This quantity is relevant in estimating the mean free path of a proton in our universe. For a given intrinsic spectrum of UHECR protons $(dF/dE)_{int}$, the observed spectrum $(dF/dE)_{obs}$ is given by:

$$\left(\frac{dF}{dE}\right)_{obs} = e^{-\tau_{p\gamma}(E_p,L)}\left(\frac{dF}{dE}\right)_{int} \tag{2.15}$$

where the optical depth is function of the energy E_p and the distance of propagation L of the particle. The *optical depth* is evaluated calculating the energy dissipation caused by the interaction of a proton, propagating in Universe, with the CMBR, for a given diffusion length. It is obtained integrating, for the given length, the product of the cross section, that accounts for the probability of the interaction, multiplied by the probability density function of the CMBR times the inelasticity of the process:

$$\tau_{p\gamma} = \frac{1}{l_{p\gamma}} = \int_{E_{thr}}^{+\infty} dE \int_{-1}^{+1} d\mu \frac{1-\mu}{2} n(E)\sigma_{p\gamma}(s)K(s) \tag{2.16}$$

In the previous equation $\mu = \cos(\theta)$, L is the length of propagation of the proton, $K(s)$ is the inelasticity, that is the fraction of the total incident energy lost in producing secondary particles, $\sigma_{p\gamma}$ is the cross section of the process analyzed, E_{thr} is the threshold energy for the interaction considered and $n(E)$ the probability density of CMBR, in other words the Planck's formula for the energy dependent photon density in black body radiation:

$$n(E) = \frac{1}{\pi^2}\frac{E^2}{e^{E/KT}-1} \tag{2.17}$$

The inverse of the mean free path now becomes:

$$\tau_{p\gamma} = \frac{1}{l_p} = \int_{E_{thr}}^{+\infty} dE\, n(E) \int_{-1}^{+1} (1 - v_p\cos\theta)\frac{d\cos\theta}{2}\sigma_{p\gamma}(s)K(s) \tag{2.18}$$

where $n(E)$ is the CBMR energetic distribution and v_p is the proton velocity. In the case of UHECR protons, it is possible to consider $v_p \simeq 1$ and taking:

$$ds = -2E_p \omega d\cos\theta \tag{2.19}$$

the optical depth path becomes:

$$\tau_{p\gamma} = \frac{1}{8p^2} \int_{E_{thr}}^{+\infty} dE\, \frac{n(E)}{E^2} \int_{s_{min}}^{s_{Max}} ds\, s\sigma_{p\gamma}(s)K(s) \tag{2.20}$$

Let now change the perspective and consider the Mandelstam variable s in the rest frame of the proton, where the proton four momentum is $(\omega', \ \overrightarrow{p}'_\gamma)$, and using the fact that

$$\begin{aligned} s &= (m_p + \omega')^2 - \overrightarrow{p}^2_\gamma = m_p^2 + 2m_p\omega' \\ \omega' &= \gamma\omega(1 - v_p\cos\theta) \simeq 2\omega\gamma \quad (head\, on\, collision) \end{aligned} \tag{2.21}$$

it is possible to obtain for the *optical depth* the equation:

$$\tau_{p\gamma} = \frac{-KT}{2\pi^2\gamma^2} \int_{\omega_0}^{+\infty} d\omega\, \sigma_{p\gamma}(\omega)K(\omega)\omega \ln(1 - e^{-\omega/2KT\gamma}) \tag{2.22}$$

The obtained formula accounts for both interaction processes of a proton with CMBR, but one by one. Each interaction optical depth can be computed using the previous formula with the correct choice of $\sigma_{p\gamma}$ cross section and K inelasticity. The total *optical depth* is given by the sum of that foreseen for photo-pion production with the one predicted for pair production.
To complete the analysis it is necessary to determine the energy threshold values for the two different interaction mechanisms of a proton with the CMBR. In the case of photo-pion production, during a collision of a proton with a photon, the free energy of the process, in the center of mass reference frame, is given by the Mandelstam variable s:

$$s = (E_p + \omega)^2 - (\overrightarrow{p}_p + \overrightarrow{\omega})^2 = m_p^2 + 2\omega E_p(1 - \cos\theta) \simeq m_p^2 + 4E_p\omega = m_\Delta^2 \tag{2.23}$$

The four momentum of the proton is $(E_p, \overrightarrow{p}_p)$, the four momentum of a CMBR photon is $(\omega, \overrightarrow{\omega})$, m_p denotes the proton mass and m_Δ denotes the delta resonance mass. In the previous computation the ultra-relativistic approximation $p \simeq E$ has been used.
In this case the threshold energy can be evaluated as:

$$E_{thr} = \frac{m_\Delta^2 - m_p^2}{2\omega(1 - \cos\theta)} \tag{2.24}$$

To compute the minimum energy required for the process it is sufficient to resort to the head-on collision approximation ($\theta = \pi \;\Rightarrow\; \cos\theta = -1$):

$$E_{thr} = \frac{m_\Delta^2 - m_p^2}{4\omega} \simeq 5 \cdot 10^{19}\, eV \tag{2.25}$$

with the average energy of CMBR $\omega \simeq 4 \cdot 10^{-4}\, eV$, the resonance delta mass is $m_\Delta = 1232\, MeV$ and the proton mass is $m_p = 938\, MeV$.
The computation for the pair production case follows the same procedure used before. The process free energy is given, as in (2.23), by the equation:

$$\begin{aligned} s = (E_p + \omega)^2 - (\overrightarrow{p}_p + \overrightarrow{\omega})^2 = m_p^2 + 2\omega E_p(1 - \cos\theta) \simeq \\ \simeq m_p^2 + 4E_p\omega = (m_p + 2m_e)^2 - m_p^2 \end{aligned} \tag{2.26}$$

Resorting to the same approximation, used in the previous computation, that is the head-on collision, with the introduction of m_e as the mass of the electron or the positron, the threshold energy for pair production can be written as:

$$E_{thr} = \frac{(m_p + 2m_e)^2 - m_p^2}{4\omega} = \frac{4(m_p m_e + m_e^2)}{4\omega} \simeq 2.5 \cdot 10^{18}\, eV \tag{2.27}$$

In performing the integration to compute the optical depth, the dominant contributions come in the region of the lower energy or, in other words, near the threshold energy. This results simple to deduce from equation (2.22) and from the shape of the black-body radiation spectrum, representing the CMBR distribution density. This means that pair production starts to cause proton energy dissipation only for energy $E \leq E_{thr} \simeq 2.5 \cdot 10^{18}\, eV$. In case of UHECR protons this process can be neglected and therefore only the photo-pion production can be considered influencing their propagation, until their energy goes under threshold.

2.2.2 GZK effect

At the end of the previous section, it has been illustrated why the pair production can be neglected in computing the UHECR protons energy dissipation. So only the photo-pion interaction dominates the phenomenon for protons, which, after a long enough path, dissipate a consistent part of their initial energy and can be detected only under a definite energy threshold. This means that a proton, detected with energy exceeding determined high values, must be accelerate inside an opacity sphere, whose radius depends on the energy itself. This effect is called GZK cut-off and takes the name from the three physicists, who first predicted it: Greisen - Zatsepin -Kuz'min [47, 48]. The presence of this opacity sphere is of great importance, because, assuming its dimensions smaller than the homogene-

ity scale of the Universe, it let to conduct anisotropy researches of the sources of UHECRs. In fact, knowing that UHECR protons are almost not affected by extra-galactic magnetic fields, during their propagation, it is possible to aim to their sources, which must be confined inside a sphere, where Universe matter is not uniformly distributed.
A first estimation of the dimensions of the GZK sphere can be obtained, taking the integral average of the photon distribution:

$$\lambda_p \simeq \frac{1}{n_\gamma \sigma_{p\gamma}} \simeq 10\, Mpc \tag{2.28}$$

where the CMBR distribution is approximated with the mean value of $n_\gamma \simeq 410\, \gamma/cm^3$ and for each photo-pion production process, it is assumed an energy dissipation of the order of about $10\% \div 20\%$. In the energy threshold value range the cross section is almost constant and is $\sigma_{p\gamma} \simeq 0.5\ mb$. Finally the GZK sphere results to have an approximate radius of $r \sim 50 \div 100\, Mpc$.
To determine the mean free path of a proton, it is now necessary to define explicitly the *elasticity* factor $\eta = \left(\frac{E_{out}}{E_{in}}\right)$. It is defined as the ratio of the energy carried away by the most energetic particle emerging from the interaction, (E_{out}), divided by the energy of the incident particle, (E_{in}). From this it is possible to obtain the *inelasticity*, which represents the fraction of the total incident energy that is available for the production of secondary particles, and is defined as $K = (1 - \eta)$.
In fact, protons lose energy by the photo-pion production process, without annihilate. So, if they have enough energy, the process can repeat again, and the *inelasticity* K evaluates the fraction of initial proton energy transferred to each outgoing pion, after every interaction. To obtain the correct formula for the elasticity [45], it is necessary to start from the Mandelstam variable s, evaluated in the center of momenta (CM) frame of reference. Using the relativistic invariance of this quantity, evaluated for the proton-photon system, it is possible to obtain:

$$(\overrightarrow{p}^*_p + \overrightarrow{\omega}^*) = 0 \tag{2.29}$$

and so, in analogy with relation (2.23):

$$\begin{aligned} s = (E^*_p + \omega^*)^2 - (\overrightarrow{p}^*_p + \overrightarrow{\omega}^*)^2 = (E^*_p + \omega^*)^2 = \\ =(E_p + \omega_p)^2 - (\overrightarrow{p}_p + \overrightarrow{\omega})^2 = m_p^2 + 2m_p\omega \end{aligned} \tag{2.30}$$

With the exponent label $*$ are indicated the four-momenta in the CM frame.
Changing frame of reference requires the Lorentz transformation:

$$\gamma_{CM}(E^*_p + \omega^*) = (E_p + \omega) = \sqrt{s} \tag{2.31}$$

γ_{CM} is the Lorentz coefficient for transformations from the ordinary frame of reference to the center of momenta one.
From this it is possible to obtain:

$$\gamma_{CM} = \frac{E_p + \omega}{\sqrt{s}} \simeq \frac{E_p}{\sqrt{s}} = \frac{E_p}{m_p^2 + 2m_p\omega} \tag{2.32}$$

where, considering UHECR protons much more energetic than the average photons of the CMBR, the approximation $E_p >> \omega$ has been used.
Now it is possible to indicate the dispersion relations for the proton (p) and the pion (π) respectively as:

$$E_p = \sqrt{m_p^2 + \overrightarrow{p}_p^2} \qquad E_\pi = \sqrt{m_\pi^2 + \overrightarrow{p}_\pi^2} \tag{2.33}$$

equations that are valid even in the CM frame of reference.
It is simple to rewrite the dispersion relation for the pion, in the CM frame, as:

$$E_\pi^* = \sqrt{E_p^{*\,2} - m_p^2 + m_\pi^2} \tag{2.34}$$

where the relation $\overrightarrow{p}_p^* = \overrightarrow{p}_\pi^*$ has been used, as by definition of CM frame of reference. Considering the equation:

$$(E_p^* + E_\pi^*) = \sqrt{s} \quad \Rightarrow \quad E_p^* + \sqrt{E_p^{*\,2} - m_p^2 + m_\pi^2} = \sqrt{s} \tag{2.35}$$

where the proton and pion energies, after the photo-pion production process, are indicated, finally it is simple to derive the relation:

$$E_p^{*\,2} - m_p^2 + m_\pi^2 = s - 2\sqrt{s}E_p^* + E_p^{*\,2} \tag{2.36}$$

The last equation can be simplified, obtaining for the proton energy, after the interaction with the CMBR:

$$E_p^* = \frac{s + m_p^2 - m_\pi^2}{2\sqrt{2}} \tag{2.37}$$

From this relation, using the Lorentz transformations, it is possible to determine the proton energy in the laboratory reference frame:

$$E_p' = \gamma_{CM} E_p^* = \frac{1}{2}\left(1 + \frac{m_p^2 - m_\pi^2}{s}\right) E_p \tag{2.38}$$

Using the definition of *elasticity* η as the ratio of the energy conserved by the most energetic particle, after the interaction process, divided by the energy before the scattering, it is possible to arrive to:

$$\eta = \frac{E'_p}{E_p} = \frac{1}{2}\left(1 + \frac{m_p^2 - m_\pi^2}{s}\right) \tag{2.39}$$

From this equation and the definition of *inelasticity* $K = (1 - \eta)$, it is easy to derive the expression:

$$K = \frac{1}{2}\left(1 - \frac{m_p^2 - m_\pi^2}{s}\right) \tag{2.40}$$

Knowing the form of the inelasticity, it is possible to estimate the optical depth $\tau_{p\gamma}$ for a proton with an initial energy E_p. The relation (2.22), giving the *optical depth*, can be simplified to determine an approximate method, to evaluate numerically the mean free path of a proton in the UHECR energy region. Considering the delta resonance of determined energetic amplitude, is possible to write:

$$\frac{1}{l_p} = \sigma_{p\gamma} \int_{\frac{E_p - \Delta_{p\gamma}}{2}}^{\frac{E_p - \Delta_{p\gamma}}{2}} n(E)dE = \sigma_{p\gamma} \int_{\frac{E_p - \Delta_{p\gamma}}{2}}^{\frac{E_p - \Delta_{p\gamma}}{2}} n(E)dE \tag{2.41}$$

where $\Delta_{p\gamma} = 100\ MeV$ is the amplitude of the delta resonance.

2.2.3 More on spectrum shape

The energy dissipation, suffered by UHECRs during propagation, and caused by the interaction with the CMBR, via the two mechanisms before exposed, can influence their spectrum shape. So it is possible to find some traces of these dissipation phenomenons in modifications of the foreseen spectrum structure [49, 50, 51]. Considering, for example, protons originated with very high energies and detected after a long enough path, their energy dissipation is dominated by the photo-pion production. But this mechanism loses efficiency when energy decreases, so protons dissipation is slower when they reach region near a certain cut-off energy, about $10^{19}\ eV$. This effect, known as the *bump* model, produces in fact an accumulation point, that is a "hump", in the spectrum form, in the energy region indicated.

Another important feature of the spectrum is caused by the fact that, for energies equal or lower than the threshold energy $2.5 \cdot 10^{18}\, eV$, the dissipation is dominated by pair production. Most of the protons, that populate this spectrum region, with energies included in the range of $10^{18} \div 10^{19}\, eV$, are originated at distances of the order or larger than $1000\, Mpc$. At this dimension scale the Universe can be assumed, with great confidence, as homogeneous. For this reason this effect, known as *dip* model, predicts that the spectrum structure, for energies below about $10^{19}\, eV$, is not influenced by the sources distribution. This constitutes another argument at support of the utility of UHECR protons for conducing anisotropy researches.

2.3 Electromagnetic radiation propagation

In analogy of what exposed in the previous section, Universe is not totally transparent even to the propagation of photons. The only difference, respect to the previous case of protons consists in the fact that photons interacting with CMBR create a pair and disappear and the initial energy is carried away in equal parts by the electron and the positron generated.
The complete cross section for the photon-photon interaction for couple creation is given by the Breit-Wheeler formula:

$$\begin{aligned}\sigma_{\gamma\gamma} = &\frac{r_0^2\pi}{2}(1-\beta^2)\left((3-\beta^4)\ln\left(\frac{1+\beta}{1-\beta}\right)+2\beta(\beta^2-2)\right) = \\ =&\frac{3\sigma_T}{16}(1-\beta^2)\left((3-\beta^4)\ln\left(\frac{1+\beta}{1-\beta}\right)+2\beta(\beta^2-2)\right).\end{aligned} \tag{2.42}$$

with:

$$\begin{aligned}&r_0 = classical\, radius\, of\, the\, electron \\ &\beta = \sqrt{\left(1-\frac{E_{thr}}{E}\right)} \\ &\sigma_T = 6.65\times 10^{-25}\, cm^2\ \ Thompson\, cross\, section\end{aligned} \tag{2.43}$$

E_{thr} is the threshold energy for the process, obtained posing the free energy in the center of mass, that is the Mandelstam variable s, bigger or equal to the rest energy of two free electrons:

$$s = (E_\gamma + E_{CMB})^2 - (\overrightarrow{p}_\gamma + \overrightarrow{p}_{CMB})^2 = 2E_\gamma E_{CMB}(1-\cos(\theta)) \geq 4m_e^2 \tag{2.44}$$

so it is possible to obtain:

$$E_{thr} = \frac{4m_e^2}{E_{CMB}(1-\cos(\theta))} \tag{2.45}$$

In the ultra-relativistic limit the gamma-gamma cross section becomes:

$$\sigma_{\gamma\gamma} \simeq \pi r_0^2\left(\frac{m}{\omega}\right)^2\left(2\ln\left(\frac{2\omega}{m}-1\right)\right) \simeq \pi r_0^2\frac{1}{s}\ln\left(\frac{s}{2m_e}\right) \tag{2.46}$$

As in the protons case, it is possible to estimate the optical depth $\tau_{\gamma\gamma}$ of a photon with energy E_γ. This quantity is correlated with the spectrum of EM detected radiation by equation (2.22).
Even in this situation the optical depth is evaluated as the probability of interaction of the Cosmic Background Microwave Radiation (CMBR). It is obtained integrating the product of the cross section (2.42) multiplied by the probability density function of the CMBR, in analogy with the previous case:

$$\tau_{\gamma\gamma} = \frac{1}{l_{\gamma\gamma}} = \int_{E_{thr}}^{+\infty} dE \int_{-1}^{+1} d\mu \frac{1-\mu}{2} n(E)\sigma_{\gamma\gamma}(s) \tag{2.47}$$

where $\mu = \cos(\theta)$, L is the photon propagation length and $n(E)$ is the black body photon density (2.17).
It is relevant to underline the fact that a high energy photon, which interacts with the CMBR, is "killed" as an energetic particle, because creates a couple e^- e^+, disappearing.
Considering

$$ds = -2E_\gamma E_{CMB} d\cos(\theta) \tag{2.48}$$

is possible to obtain:

$$\begin{aligned} \tau_{\gamma\gamma} &= \frac{1}{8E_\gamma^2} \int_{E_{thr}}^{+\infty} dE \frac{n(E)}{E^2} \int_{s_{min}}^{s_{Max}} ds\, s\sigma_{\gamma\gamma}(s) = \\ &= \frac{1}{8E_\gamma^2} \int_{\omega_0(=m_e^2/E)}^{+\infty} d\omega \frac{1}{e^{\omega/KT}-1} \int_{s_{min}}^{s_{Max}} ds\, s\sigma_{\gamma\gamma}(s) \end{aligned} \tag{2.49}$$

where $s_{min} = 4m_e^2$ and $s_{Max} = 4E_\gamma E_{CMB}$.
Changing the reference frame to that of the center of momenta (CM):

$$\overrightarrow{p}_\gamma + \overrightarrow{p}_{CMB} = 0 \quad \Rightarrow \quad \overrightarrow{p}_\gamma = -\overrightarrow{p}_{CMB} \tag{2.50}$$

the energy (ω) of the incoming photon becomes equal to that of the CMBR photon. Taking into account the previous relation:

$$\omega_\gamma^2 - \overrightarrow{p}_\gamma^2 = \omega_{CMB}^2 - \overrightarrow{p}_{CMB}^2 = 0 \quad \Rightarrow \quad \omega_\gamma = \omega_{CMB} = \omega \tag{2.51}$$

so the Mandelstam variable s becomes:

$$s = (\omega_\gamma + \omega_{CMB})^2 - (\overrightarrow{p}_\gamma + \overrightarrow{p}_{CMB}) = 4\omega^2 \tag{2.52}$$

and for the optical depth is possible to obtain:

$$\begin{aligned} \tau_{\gamma\gamma} = \frac{1}{l_{\gamma\gamma}} = & \int_{E_{thr}}^{+\infty} dE \int_{-1}^{+1} d\mu \frac{1-\mu}{2} n(E) \sigma_{\gamma\gamma}(s) = \\ = & \frac{-4KT}{\pi^2 E^2} \int_{m_e^2}^{\infty} d\omega\, \omega^2 \sigma_{\gamma\gamma}(\omega) \log\left(1 - e^{-\omega^2/EKT}\right) \end{aligned} \tag{2.53}$$

2.4 Cosmological red shift

For UHECRs, energy dissipation, along their diffusion, is dominated by the interaction effects with the CMBR, as already underlined, and by the Universe expansion, which induces an adiabatic energy loss. Therefore, after the interaction with the CMBR, it is necessary to consider the effects induced by the cosmological red shift, in particular for particles, that propagates for cosmological distances.

2.4.1 Coordinates change

To introduce this topic, a Friedmann-Robertson-Walker (FRW) scenario is assumed, with the Universe supposed homogeneous and isotropic. The FRW metric takes the explicit form:

$$ds^2 = dt^2 - c^2 R^2(t) \left(\frac{dr^2}{1 - \Upsilon r^2} + r^2 (d\theta^2 + \sin\theta^2 d\phi^2) \right) \tag{2.54}$$

with Υ the parameter that accounts for the Universe curvature:

$$\begin{cases} \Upsilon > 0 & spherical\ geometry \Rightarrow closed \\ \Upsilon = 0 & flat\ geometry \\ \Upsilon < 0 & hyperbolic\ geometry \Rightarrow expansion \end{cases} \tag{2.55}$$

Let's consider now the effects of curvature on a light ray, emitted by the source with initial wave length λ_0 and initial frequency ω_0 and observed after a certain propagation. It is always possible to find a Killing vector ξ^μ, parallel to the projection of the wave vector $\overrightarrow{k}$ on the space hyper-surfaces Σ and Σ_0 at different times (t and t_0). In FRW scenario the space-time is foliated in different space-type hyper-surfaces tagged by the time t coordinate. The length of ξ^μ varies from time

t_0 to t, because of the change of the scale factor:

$$\frac{\sqrt{\xi^\mu \xi_\mu}\Big|_t}{\sqrt{\xi^\mu \xi_\mu}\Big|_{t_0}} = \frac{R(t)}{R(t_0)} \tag{2.56}$$

k^μ is a light type vector, so its internal product with the four velocity of the observer v^μ:

$$k_\mu v^\mu = \frac{\xi_\mu v^\mu}{\sqrt{\xi_\mu \xi^\mu}} \tag{2.57}$$

and thus it follows:

$$\omega_i = \frac{\xi_\mu v^\mu}{\sqrt{\xi_\mu \xi^\mu}}\Bigg|_{t_i} \tag{2.58}$$

The four vector k^μ is tangent to the light geodesic, so because the inner product of a Killing vector with a geodesic tangent one is constant, it follows that:

$$\xi_\mu v^\mu\Big|_t = \xi_\mu v^\mu\Big|_{t_0} \tag{2.59}$$

and therefore:

$$\frac{\omega(t_0)}{\omega(t)} = \frac{\sqrt{\xi_\mu \xi^\mu}\Big|_t}{\sqrt{\xi_\mu \xi^\mu}\Big|_{t_0}} = \frac{\lambda}{\lambda_0} = \frac{R(t)}{R(t_0)} \tag{2.60}$$

and, due to the dilatation of the space, the final effect is that every photon increases its wave length, effect called *cosmological red shift*.
To study this phenomenon it is possible to define a red shift parameter:

$$z = \frac{\Delta\lambda(t)}{\lambda(t_0)} = \left(\frac{\lambda(t)}{\lambda(t_0)} - 1\right) = \left(\frac{\omega(t_0)}{\omega(t)} - 1\right) \tag{2.61}$$

therefore immediately it follows:

$$1 + z = \frac{\lambda}{\lambda_0} = \frac{R(t)}{R(t_0)} \tag{2.62}$$

In case of sources collocated in nearby galaxies, it is possible to evaluate the previous equation via a Taylor series:

$$1 + z = \simeq \frac{R(t_0) + (t - t_0)\dot{R}(t_0) + \frac{1}{2}\ddot{R}(t_0)(t - t_0)^2 + \dots}{R(t_0)} = 1 + H(t - t_0) \tag{2.63}$$

where the *Hubble* constant has been defined as:

$$H_0 = \frac{\dot{R}(t_0)}{R(t_0)} \tag{2.64}$$

Defining now the coordinate:

$$x = ct = R(t)r \tag{2.65}$$

it is possible to obtain:

$$dr = \frac{dx}{R(t)} = \frac{cdt}{R(t)} = \frac{c(1+z)}{R(t_0)}dt = \frac{c(1+z)}{R(t_0)}\frac{dt}{dz}dz \tag{2.66}$$

Introducing the density parameters of radiation (Ω_R), matter (Ω_m), curvature (Ω_k) and the cosmological constant (Ω_Λ), which obey the relation:

$$\Omega_R + \Omega_m + \Omega_k + \Omega_\Lambda = 1 \tag{2.67}$$

it is possible to write the derivative in the form:

$$\frac{dz}{dt} = H_0(1+z)\sqrt{(1+z)^4\Omega_R + (1+z)^3\Omega_m + (1+z)^2\Omega_k + \Omega_\Lambda} \tag{2.68}$$

In case of an asymptotically flat Universe, dominated by matter[1] the previous derivative becomes:

$$\frac{dz}{dt} = H_0(1+z)\sqrt{(1+z)^3\Omega_m + \Omega_\Lambda} \tag{2.69}$$

Using the previous equalities it is possible to express the coordinate r in function of the red shift parameter:

$$r = \int_0^z \frac{dx}{R(t)} = \int_0^z \frac{dt}{dz}\frac{1+z}{R(t_0)}dz \tag{2.70}$$

and from this it follows the possibility to express the coordinate r in function of the red shift parameter z:

$$r = \int_0^z \frac{c}{H_0 R(t_0)} \frac{dz}{\sqrt{(1+z)^3\Omega_m + \Omega_\Lambda}} \tag{2.71}$$

[1]

- $H_0 \simeq 70 \frac{70\,km/s}{Mpc}$
- $\Omega_k \simeq 0$
- $\Omega_R \simeq 9.2 \cdot 10^{-5}$
- $\Omega_m \simeq 0.315$
- $\Omega_\Lambda \simeq 0.685$

2.4.2 Detection energy dependence on red shift parameter

To evaluate the red shift effects on the spectrum it is fundamental to find the explicit form of the derivative:

$$\frac{dE}{dE_0} \tag{2.72}$$

where E is the energy of the particle at detection and E_0 is the energy at source. This derivative can then be employed in equation (1.43) to clarify the effects of cosmological red shift.
First of all the cosmological expansion determines on the CMBR photons an increase of the number distribution and energy, given by:

$$\begin{aligned} n(E) &\longrightarrow (1+z)^3 n(E_0) \\ E &\longrightarrow (1+z)E_0 \end{aligned} \tag{2.73}$$

The Universe expansion causes an adiabatic energy dissipation for every particle propagating, which is given by:

$$-\frac{1}{E}\frac{dE}{dt} = -\frac{1}{E}\frac{dE}{dz}\frac{dz}{dt} = -\frac{1}{(1+z)E_0}E_0\frac{dz}{dt} = H_0\sqrt{(1+z)^3\Omega_m + \Omega_\Lambda} \tag{2.74}$$

There is even a red shift modification in the energy loss, caused by the interaction of a propagating particle with the CMBR, because the probability of the interaction with photons increases proportionally with their number and is function of the photons energy. Reconsidering the β function as introduced in in equation (1.44) and (1.45), it is possible to write:

$$\beta(E) \to (1+z)^3\beta_0((1+z)E) \tag{2.75}$$

From the previous one, and remembering the meaning of the β function in the definition of the time derivative of the detected energy, the dissipation assumes the form:

$$-\frac{1}{E}\frac{dE}{dt} = -\frac{1}{E}\frac{dE}{dz} = (1+z)^3\beta_0((1+z)E) \;\to\; \frac{dE}{dz} = -(1+z)^3E\frac{dt}{dz}\beta_0((1+z)E) \tag{2.76}$$

Summing the contributions due to the CMBR modification with the adiabatic expansion one, it is possible to evaluate the total derivative of the detection energy respect to the red shift parameter:

$$\frac{dE}{dz} = E\left(\frac{(1+z)^2\beta_0((1+z)E_0)E}{H_0\sqrt{(1+z)^3\Omega_m+\Omega_\Lambda}} + \frac{1}{1+z}\right) \tag{2.77}$$

Using the previous one together with the condition $E(0) = E_0$, it is possible to obtain the form for the energy for given z red shift parameter:

$$E(z) = E_0 - \int_0^z \frac{d\zeta}{1+\zeta} E(\zeta) - \int_0^z d\zeta \frac{(1+\zeta)^2}{H_0\sqrt{(1+z)^3\Omega_m + \Omega_\Lambda}} \beta_0((1+\zeta)E(\zeta)) \quad (2.78)$$

Defining the derivative:

$$\frac{dE(z)}{dE_0} = \chi(z) \quad (2.79)$$

the red shift Hubble function:

$$H(z) = H_0\sqrt{(1+z)^3\Omega_m + \Omega_\Lambda} \quad (2.80)$$

and the red shift energy:

$$\epsilon(z) = (1+z)E(z) \quad (2.81)$$

it is possible to obtain from (2.78) the differential equation:

$$\chi(z) = 1 - \int_0^z \frac{d\zeta}{1+\zeta} \chi(\zeta) - \int_0^z d\zeta \frac{(1+\zeta)^3}{H(\zeta)} \frac{\beta_0(\epsilon)}{d\epsilon} \quad (2.82)$$

From the previous relation, differentiating again respect to the energy E_0, the associated differential equation follows:

$$\frac{d}{dE_0} \ln \chi(z) = \frac{-1}{1+z} - \frac{(1+z)^3}{H(z)} \frac{d\beta_0(\epsilon)}{d\epsilon} \quad (2.83)$$

The solution of this equation gives the explicit form of the derivative of the detection energy respect to the red shift parameter z and gives an insight on the energy dependence from this parameter:

$$\chi(z) = \frac{dE(z)}{dE_0} = (1+z) \exp\left(\frac{1}{H_0} \int_0^z \frac{(1+\zeta)^3 d\zeta}{\sqrt{(1+z)^3\Omega_m + \Omega_\Lambda}} \frac{d\beta_0(\epsilon)}{d\epsilon} \right) \quad (2.84)$$

2.4.3 Red shift effect on the spectrum

Obtained the explicit dependence, from the red shift parameter z, of the energy at the detection point, it is possible now to integrate in the known expression of the spectrum the changes caused by the cosmological expansion. As illustrated before the effects are correlated to the red shift of the electro-magnetic waves of the CMBR. The adiabatic effects caused by the dimensional increase of the length scale determines an adiabatic energy loss. As already underlined, the total flux of particles is constant, because the energy loss mechanism does not predict annihilation. The expression of the energy spectrum at the detection point (1.46) becomes in this way:

$$\begin{aligned} \left.\frac{dN(E)}{dE}\right|_{det} &= \frac{dN(E_0)}{dE_0}\frac{d(E_0)}{dE} = \\ =&\frac{dN(E_0)}{dE_0}(1+z)\exp\left(\frac{1}{H_0}\int_0^z \frac{(1+\zeta)^3 d\zeta}{\sqrt{(1+z)^3\Omega_m+\Omega_\Lambda}}\frac{d\beta_0(\epsilon)}{d\epsilon}\right) \end{aligned} \tag{2.85}$$

The effects of red shift are visible even on the detected flux of particles, in fact the total number of emitted particles does not change, but the cosmic inflation modifies the number of detected ones. Giving the expression of the particle flux:

$$I(E,\, r)dE = \frac{Q_{inj}(E_0)}{4\pi r^2}dE_0 \tag{2.86}$$

where the number of the accelerated particles by the sources, that is the *injection spectrum*, is the function $Q_{inj}(E)$.
Defining the red shift length as:

$$d = R(t)r = \frac{R(t_0)r}{1+z} \tag{2.87}$$

the flux modified by the cosmological inflation can be written as:

$$I(E,\, z)dE = \frac{Q_{inj}(E_0)}{4\pi(1+z)(R(t_0)r)^2}dE_0 \to I(E,\, z) = \frac{Q_{inj}(E_0)}{4\pi(1+z)(R(t_0)r)^2}\frac{dE_0}{dE} \tag{2.88}$$

2.4.4 Red shift effect on optical depth

Even the optical depth is affected by the presence of the cosmological inflation. In fact, reconsidering the definition of this quantity, it depends on the density distribution of the CMBR photons, which is affected by the red shift. Furthermore this quantity depends even on the average energy of the photons and the energy of the proton and both of them are modified again by the red shift. Explicitly is possible to generalize the computation and, following [52], to obtain:

$$\begin{aligned} \tau_{p\gamma}(z) &= \int_L dx \int_{E'_{thr}}^{\infty} d\epsilon \sigma_{p\gamma}(\epsilon)\epsilon K(\epsilon)\ln\left(1-\exp\left(-\frac{\epsilon}{2KT\gamma}\right)\right) = \\ &= \int_0^z \frac{cdt}{d\zeta}d\zeta \int_{E'_{thr}}^{\infty} d\epsilon \sigma_{p\gamma}(\epsilon)\epsilon K(\epsilon)\ln\left(1-\exp\left(-\frac{\epsilon}{2KT\gamma}\right)\right) \end{aligned} \tag{2.89}$$

where the integral has been evaluated along the proton world line, and all the quantities have been modified by the cosmological inflation $T = (1+z)T_0$, $E = (1+z)E_0$, $\epsilon = (1+z)\epsilon_0$ and $E'_{thr} = \frac{E_{thr}}{1+z}$, so this equation reduces to:

$$\tau_{p\gamma}(z) = -\int_0^z \frac{d\zeta}{(1+\zeta)H(\zeta)} \int_{E'_{thr}}^{\infty} d\epsilon\sigma_{p\gamma}(\epsilon)\epsilon K(\epsilon)\ln\left(1-\exp\left(-\frac{\epsilon}{2KT\gamma}\right)\right) \quad (2.90)$$

where the explicit form of the derivative of time respect to red shift parameter z has been used. The computation of the previous equation simplifies if the threshold energy $E_{thr} \to 0$, so it is possible to perform the second integral as:

$$\int_0^{\infty} d\epsilon\sigma_{p\gamma}(\epsilon)\epsilon K(\epsilon)\ln\left(1-\exp\left(-\frac{\epsilon}{2KT\gamma}\right)\right) = \sigma_{p\gamma}(E)n_0(1+z)^3 \quad (2.91)$$

and the estimation of the optical depth simplifies and is analytically possible.

2.5 Modifying GZK effect by LIV

Lorentz invariance plays a key role in GZK cut-off computation [18]. Both the cross section and the inelasticity of the photo-pion process are evaluated in a Lorentz invariant scenario. For UHECR physics energy is so high that one must relies on extremely high Lorentz factors, with a magnitude of the order of $\gamma = \frac{1}{\sqrt{1-\beta^2}} \geq 10^{11}$, for particles with energy $E \geq 10^{19}eV$. For such extreme relativistic environments there are not definitive experimental observations if Lorentz Invariance is still preserved, as a fundamental nature symmetry, without modifications. One indirect proof of LI violation, or validity, can be obtained by the modification, or confirm, of the GZK predicted cut-off effect. Nowadays some astrophysical observation conducted on UHECR protons seem to correlate their origin with possible candidate sources located farther than the foreseen GZK opacity sphere [17]. One of the most favourite scenarios, used to justify such a modification, consists in resorting to Lorentz invariance violation, considering the importance that this symmetry plays in evaluating the GZK effect. The impact of such a violation can be understood following the original work of Coleman and Glashow [18, 22], where the introduction of LIV imposes a modification of the dispersion relation, that is a change of the kinematics or, more in detail, a change of the propagator of a massive particle.
Imposing a maximum attainable velocity, for a proton, different from that of light, the dispersion relation, expressed in natural measure units, becomes :

$$E_p^2 - \overrightarrow{p}^2(1-\epsilon)^2 = m_p^2(1+\epsilon)^4 \quad (2.92)$$

where ϵ is the correction factor that accounts for the maximum velocity and the light speed is $c = 1$.
The correction to the maximum velocity can be assumed very tiny compared to the light speed, namely $\epsilon << 1$. This is a plausible physical hypothesis due to the fact that the validity of the Lorentz symmetry is proved with great certainty for lower energies and therefore can be violated only by small perturbations for high energy values. Approximating $E_p \sim |\overrightarrow{p}|$, that is the case of a high energy proton, and considering that the term proportional to $|\overrightarrow{p}|$ is bigger than that proportional to m_p, the relation becomes:

$$E_p^2 - \overrightarrow{p}^2(1 + 2\epsilon) = m_p^2 \Rightarrow E_p^2 - \overrightarrow{p}^2 = m_p^2 + 2\epsilon\overrightarrow{p}^2 \simeq m_p^2 + 2\epsilon E_p^2 \tag{2.93}$$

Now considering the kinematic of the delta resonance creation process:

$$\begin{aligned} m_\Delta^2 \leq (E_P + \omega)^2 - (\overrightarrow{p} + \overrightarrow{\omega})^2 = E_p^2 + 2\omega E_p - 2\overrightarrow{p}\,\overrightarrow{\omega} - \overrightarrow{p}^2 = \\ =m_p^2 + 2\epsilon E_p^2 + 2\omega E_p(1 - \cos\theta) \end{aligned} \tag{2.94}$$

where $(E_p,\ \overrightarrow{p})$ and $(\omega,\ \overrightarrow{\omega})$ are respectively the proton and photon four-momenta, it is possible to compute the relation:

$$m_\Delta^2 - m_p^2 - 2\epsilon E_p^2 - 4\omega E_p \leq 0 \tag{2.95}$$

that must be satisfied in order to obtain a delta particle, condition necessary for the GZK effect.
Now there are two possible cases, one corresponding to $\epsilon > 0$, which means a maximum attainable velocity, for the proton, greater than the speed of light, even if this scenario is greatly improbable. The other case corresponds to a value of the correction parameter $\epsilon < 0$, that is the proton has a maximum attainable velocity lower than the speed of light. This second scenario is perhaps more probable, in fact it is presumable that a massive particle interacting with space-time background has a maximum personal attainable velocity lower than that of light. However, for completeness, both the cases have been analyzed:

1) $\epsilon > 0$

In this case to satisfy the relation (2.95), it is necessary that:

$$16\omega^2 + 8\epsilon(m_\Delta^2 - m_p^2) \geq 0 \tag{2.96}$$

and this condition is satisfied $\forall \epsilon > 0$.
In this case the GZK cut-off is obtained for $E_p \leq E_{min}$ or $E_p \geq E_{max}$, with:

$$
\begin{aligned}
E_{max} &= \frac{4\omega + \sqrt{16\omega^2 + 8\epsilon(m_\Delta^2 - m_p^2)}}{4\epsilon} \\
E_{min} &= \frac{4\omega - \sqrt{16\omega^2 + 8\epsilon(m_\Delta^2 - m_p^2)}}{4\epsilon}
\end{aligned}
\tag{2.97}
$$

$E_{min} < 0$ so the first condition is impossible, instead considering $< \omega > \sim 3\, meV$, $E_{max} > 10^{21}\, eV$,[2] so it can be possible to have a suppression of the GZK cut-off for protons with energy far beyond the energies observed till now.

2) $\epsilon < 0$

In this case the condition (2.95) imposes that:

$$16\omega^2 + 8\epsilon(m_\Delta^2 - m_p^2) \geq 0 \tag{2.98}$$

and so it must be satisfied the relation:

$$-\epsilon < \frac{2\omega^2}{m_\Delta^2 - m_p^2} \;\Rightarrow\; |\epsilon| < \frac{2\omega^2}{m_\Delta^2 - m_p^2} \tag{2.99}$$

and this imposes a limit to the value of the parameter ϵ, if the GZK effect is preserved.
In this case the energy of the proton must be $E_{min} < E_p < E_{max}$, so in this case there can be a suppression of the GZK effect, even at observed energies, if the ϵ parameter is not opportunely constrained.
The calculation of the kinematic has to consider even the angular distribution, so the real formula is:

$$-2\epsilon E_p^2 - 2\omega E_p(1 - \cos\theta) + m_\Delta^2 - m_p^2 \leq 0 \tag{2.100}$$

and it must be averaged on the angles, using the fact that:

$$\int_{-1}^{+1} \frac{1-\mu}{2} d\mu = 1 \tag{2.101}$$

[2]The value of the parameter ϵ is fixed by some experiments with values of the order of 10^{-27} or even less.

and arriving to the form used in the previous calculations:

$$-2\epsilon E_p^2 - 4\omega E_p + m_\Delta^2 - m_p^2 \leq 0 \tag{2.102}$$

All this section computation shows that if the maximum attainable proton velocity is sufficiently different (lower) than the photon one, supposed to be $c = 1$ (the speed of light), the GZK effect can result suppressed. Therefore the foreseen universe opacity sphere can be affected by new physics, introduced by LIV.

Part II

An isotropic LIV perspective

CHAPTER 3

A brief introduction to LIV

3.1 Introduction

As already underlined, new physical effects are expected as residual evidences of a more fundamental theory of nature, about the quantum structure of space-time. Some theories have been proposed as possible candidates to solve the so called ”quantum gravity problem”, that is the identification of a quantum unified theory of particles and gravity. The most known research fields, opened by the attempt to describe such a theory, consist in String Theory, Loop Quantum Gravity (LQG) and the space-time non-commutativity scenario. One of the greatest challenges in formulating this unified theory consists in the impossibility to obtain the energies needed to probe Planck scale space-time structure. In fact, it is commonly believed that the Planck lenght $\lambda_P = \sqrt{G\hbar c^{-3}}$ and the Planck energy $E_P = \sqrt{\hbar c^5 G^{-1}}$ represent the length and energy scales separating the classical gravity theory from the quantized one. Nevertheless, Planck-scale physics effects can possibly manifest themselves, in a lower energy scenario, as tiny classical physics laws violations. One therefore expects the possibility to detect these effects as residual signals. The investigation of quantum-gravity phenomenology is centered on the search for this new physics realm first evidences. But actually all these theories cannot provide precise testable predictions, since the complexity of the mathematical formalism they adopt. So the possibility to detect this kind of effects appears far from being reached in short time. Nevertheless there is an increasing and promising ideas exchange between theory and phenomenology in some of these investigation fields, such as Loop Quantum Gravity and space-time noncommutativity.
In the following paragraph are rapidly presented these models, following the [53] review.

3.2 String Theory

The most famous attempt to solve the quantum-gravity challenge consists in String Theory. In this scenario the effort to introduce quantization in the gravita-

tional sector is conducted from the particle-physics perspective, but still using a classical Minkowski background to formulate the theory. This corresponds to a relatively "little" departure from current theories, in particular the "fixed" background is far from the desired quantization. Moreover from the phenomenology perspective, the effects introduced should be very small and therefore undetectable, at the Planck energy. Extra dimensions with size comparable to the Planck scale could generate visible effects even in an accessible and testable energy scenario. But the suppression effects that emerge in String Theory are so intense that these induced effects remain undetectable. In fact String Theory introduced phenomenology does not involve any direct reference to the Planck scale and the connection with this quantity remains weak.
A recent attempt to boost the comprehension of the String phenomenology consists in introducing "large" extra dimensions [54, 55, 56, 57, 58]. Extra dimensions, with size much larger than scale Planck, can generate visible effects even at this magnitude energy. These kind of large extra dimensions are introduced in this theory, even not in a totally natural way.
Another important String Theory approach, that pay particular attention to the phenomenology, is given by the noncritical "Liouville String Theory" [59, 60, 61], which describes time in a totally new fashion.
Last and important thing to notice is that, since String Theory resorts to the "classical" space-time background, the Lorentz symmetry should be preserved. But it has been demonstrated that in some plausible String Theory scenarios [62], tensor fields exist, that could acquire a nonzero expectation value. One expects therefore that the Lorentz symmetry is spontaneously broken, in a Higgs mechanism fashion. For this reason it appears reasonable to put under examination the Lorentz symmetry even in String Theory, to find an effective theory describing the emerging physics.

3.3 Loop Quantum Gravity

The principal theoretical scenario, which provides a quantized description of space-time, consists in Loop Quantum Gravity [63, 64, 65, 66, 67]. One of the fundamental hypotheses, underlying all physical theories, is general covariance. This General relativity principle states that every physical law must be expressed in the same way in every reference frame. This idea implies that coordinates are not natural, they are an artificial creation used to describe nature. Physical laws must be invariant under diffeomorphisms transformations and must be background independent. LQG tries to solve this problem considering the space-time not as the container where physics takes place. The background is created by the gravitational interaction, interpreted as just another physical field. In this theoretical formulation, it is possible to quantize the background, via the discretization of volume and area [68, 69, 63]. The drawback of this formalism is caused by its complexity. The discretized space-time requires finite difference equation, rather than differential ones, to describe physics. This characteristic is expected to disappear in the low energy limit, but to become dominant for the Planck one. So the LQG comprehension state of the art seems to imply an intrinsic space-time quantization. This consequence is inferred from the theory general structure, even if one is obliged to resort to mathematical approximations to obtain some phenomenological results. This type of space-time discretization is even supposed to be responsible for a departure from the classical Lorentz symmetry in the high energy scenario. But till the work of Rovelli and Speziale [70], it is well known that LQG can be reconciled with the preservation of the Lorentz symmetry. Therefore the discover of Lorentz Invariance Violation can hint for the validity of LQG, but it could be not a definitive proof of this theory.

3.4 Space-time noncommutativity

Another attempt to furnish a quantum description of background consists in considering the space-time as not commutative. This idea is centered on the hypothesis that space non classical properties should be formalized in a similar fashion as classical quantum mechanics. This is obtained resorting to the mathematical formalism of quantum groups, and in particular to the non commutativity of coordinates [71, 72, 73, 74, 75]. The main focus of the actual studies about non commutativity are finalized to obtain a geometrical description of the particle standard model. Another important feature consists in preserving the Minkowski geometry as the low energy limit of the theory. In fact there is an increasing interest in introducing non commutativity in classical Minkowski space, using the canonical non commutativity, defined by the commutation relation:

$$[x_\mu,\, x_\nu] = i\,\theta_{\mu\nu} \tag{3.1}$$

and $\kappa-$Minkowski non commutativity, defined by:

$$[x_i,\, t] = \frac{i}{\kappa}\, x_i \qquad [x_i,\, x_j] = 0 \tag{3.2}$$

In this scenarios it results impossible, nowadays, to incorporate a quantization of the gravitational interaction, and the departures from classical physics are limited in the space-time structure description. Only Loop Quantum Gravity seems the ideal candidate to investigate both the quantum structure of the background with the quantization of the gravitational field.
The non commutative models share the feature of introducing a breaking, or a modification, of the classical space-time symmetries, resumed by the Lorentz-

Poincaré group. Therefore they contemplate the possibility of introduce Lorentz Invariance Violation.

3.5 Alternative proposed scenarios

Some of the previous proposals give a fuzzy description of space-time, such in the case of Loop Quantum Gravity. A usual way to describe the background fuzziness, induced by quantization, resorts to the Wheeler's idea of a space-time foam. Till now there is not a formal operative definition of this intuition. Therefore it is an open issue to find a formal definition of a geometry, where it is not possible to sharply define two space-time points distance.
Another approach, that investigates the background departure from the classical view is given by the "discrete casual sets" space-time discretization [76, 77]. This view point explores the possibility that a Lorentzian metric can determine geometry, up to conformal equivalences, and a casual structure.
A different scenario is given by the study of "Casual Dynamical Triangulations" [78, 79, 80, 81, 82], through which it is possible to give a non perturbative background independent formulation of a path integral formalism for gravitational interaction.
The last of the alternative scenarios is given by "decoherence", which consists in modification of the Heisenberg principle, caused by background departures from Minkowski geometry. Using the classical description of space-time, it is possible to formulate quantum mechanics and to obtain a lot of phenomenological predictions, which let to put the theory under scrutiny. Obviously, if the background space is discretized, it appears plausible to obtain some departures from standard quantum mechanics. This could manifest in the form of "decoherence" [83], such as a modification of the Heisenberg picture and of the de Broglie dispersion relation. In fact, this quantum mechanics formulation is influenced by the supposed continuity of the background and therefore by the usual calculus formalism. Instead a discretized space-time requires the introduction of a new calculus formalism [84]. In quantum mechanics the momentum observable is expressed using a differential operator, so in a modified scenario the relation between a wave packet and its wavelength must be reformulated [85, 86, 87].

3.6 LIV

As already exposed, currently it is believed that SM and GR are low energy limits of a more fundamental theory, which can provide an unified description of gravity and quantum physics. These two physical theories are conceived in a Lorentz symmetry preserving form, that is they are constructed on a classical Minkowski geometry background. Global Lorentz invariance is an approximate

space-time symmetry that emerges from particular Einstein field equations solutions, in particular low energies scenarios. Lorentz Invariance (LI) is nowadays at the roots of our nature understanding. In fact Special Relativity (SR) is a cornerstone of standard quantum field theory, that is the particle Standard Model (SM), and General Relativity (GR) formulation. In fact LI, as a global symmetry, is a fundamental assumption of SR and the associated SM. Instead in General Relativity the space-time symmetries are generated by classes of diffeomorphisms and LI is promoted from a global to a local symmetry, as in a gauge theory. So, even if there are no definitive evidences to sustain departures from LI, there are consistent hints indicating that *Lorentz Invariance Violation* can be a theoretical consequence of quantum gravity [88, 89, 90, 91]. Therefore there are consistent motivations to conduct systematic tests on this fundamental symmetry validity. As illustrated before, one LIV residual effect can modify the GZK photo-pion production process, influencing the UHE protons propagation. In fact, recent UHECR experimental observations, hint the possibility that the foreseen GZK Universe opacity, that influences this kind of highly energetic particles propagation, may be modified. Moreover UHECR for their high energy (the highest observed till now, about $10^{20}\,eV$) and their cosmic propagation length, are the ideal candidates to detect these residual LIV effects. This justifies the present work interest in studying LIV influence on UHECR physics.
There are different approach to identify these more fundamental physics residual signals. One consists in formulating Effective Field Theories (EFT), that can deal with all the predictable LIV perturbations. The other approach consists in modifying the LI, in order to identify a more general formulation of SR, which includes standard physics as a low energy limit and can deal with the new experimental foreseen effects. Following [92], for instance, a brief introduction to the most known LIV theories is reported below.

3.7 Maximum attainable velocities and Von Ignatowsi theory

Before introducing the most known LIV theoretical approaches, it is necessary to take some times to analyze Special Relativity and its mathematical formulation. SR was originally founded on two postulates:

- the relativity principle
- the constancy of light speed c

The first principle states the physics laws universality in every inertial reference frames. The second indicates that in SR, the vacuum light speed c is a universal constant, in all inertial frames. This velocity represents even the maximum limit speed for every material body. But the theory itself can not explain why c is such a fundamental constant for physics. In fact this constant emerges naturally in Maxwell's electrodynamical theory as a particular electromagnetic radiation property in void. After SR formulation, it was soon realized that the light speed postulate had not a universal character and there was a first attempt to obtain SR discarding this principle. This first attempt was made by Von Ignatowki [93, 94, 95, 96], who obtained the Lorentz transformation, starting from more general principles, like:

- homogeneity of space
- homogeneity of time
- isotropy of space
- the relativity principle

and discarding the universality of the light speed as maximum attainable velocity of the theory. A maximum speed emerges naturally from this model, but nothing is said about this limit velocity universal character. This constitutes a first important difference with the standard formulation of SR and the consequent Lorentz invariance.

3.7.1 Lorentz transformations without light postulate

As already exposed, the deduction is based on the relativity principle, homogeneity of space and time and isotropy of space [97]. If S and S' are two inertial frames, suppose S' moving respect S with v speed and initially (when $t = 0$) that the two frames origins coincide. The speed v is supposed directed along the x axis, so it is possible to consider only the coordinates $(t,\, x)$ and the space-time can be considered two-dimensional. The general coordinates transformations can be written as:

$$\begin{cases} t' = T(t,\, x,\, v) \\ x' = X(t,\, x,\, v) \end{cases} \tag{3.3}$$

where $(t,\, x)$ and $(t',\, x')$ are the coordinates respectively of the frames S and S'. Now using space-time homogeneity it is possible to demonstrate that the coordinates transformation rules must be linear functions. In fact the space homogeneity implies that a measured length does not depend on coordinates (position) in an inertial frame. So in S the length of a material body with ends x_1 and x_2 must be the same if the body is translated and its ends become $x_1 + \Delta x$ and $x_2 + \Delta x$. The same must be valid even in S', that is:

$$X(t,\, x_2 + \Delta x,\, v) - X(t,\, x_1 + \Delta x,\, v) = X(t,\, x_2,\, v) - X(t,\, x_1,\, v) \tag{3.4}$$

The previous relation can be rewritten as:

$$\frac{X(t,\,x_2+\Delta x,\,v)-X(t,\,x_2,\,v)}{\Delta x}=\frac{X(t,\,x_1+\Delta x,\,v)-X(t,\,x_1,\,v)}{\Delta x} \tag{3.5}$$

taking the limit $\Delta x \to 0$, it is possible to obtain the following:

$$\left.\frac{\partial X(t,\,x,\,v)}{\partial x}\right|_{x_2}=\left.\frac{\partial X(t,\,x,\,v)}{\partial x}\right|_{x_1} \tag{3.6}$$

The partial derivative are therefore constant, because the points x_1 and x_2 are arbitrary. The transformation function $X(t,\,x,\,v)$ must be linear in the variable x. Since even time is posed homogeneous, the function $X(t,\,x,\,v)$ must be linear even with respect to t. A identical method proofs that $T(t,\,x,\,v)$ too is linear with respect to the variables t and x. The general coordinate transformation therefore takes the form:

$$\begin{pmatrix} t' \\ x' \end{pmatrix}=\begin{pmatrix} \alpha_v & \beta_v \\ \gamma_v & \delta_v \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix} \tag{3.7}$$

where $\{\alpha_v,\,\beta_v,\,\gamma_v,\,\delta_v\}$ are functions only of the velocity v. A first constrain for these coefficients can be obtained from the fact that S' moves with a speed v respect S, therefore $x'=0$ when $x=vt$, so:

$$x'=0=\gamma_v\,t+\delta_v\,x \;\Rightarrow\; \gamma_v=-v\,\delta_v \tag{3.8}$$

Space isotropy implies that the transformations must retain their form when the x-axis is reversed, that is if x and v change sign, even x' must change its one. The following relations are derived, therefore, from the isotropy principle:

$$\begin{cases} \alpha_v=\alpha_{-v} \\ \beta_v=-\beta_{-v} \\ \gamma_{-v}=-\gamma_v \\ \delta_{-v}=\delta_v \end{cases} \tag{3.9}$$

Last constrains on the coefficients of equation (3.7) can be obtained from the relativity principle. This postulate in fact states that inverse coordinate transformations must assume the same functional form of the direct transformations. Together with isotropy, it implies reciprocity, that is if S' is moving with v speed respect to S, the velocity of S respect to S' must be $-v$. Finally, one obtains the following relations:

$$\begin{cases} \alpha_{-v} = \dfrac{-\delta_v}{\alpha_v\,\delta_v - \beta_v\,\gamma_v} \\ \beta_{-v} = \dfrac{-\beta_v}{\alpha_v\,\delta_v - \beta_v\,\gamma_v} \\ \gamma_{-v} = \dfrac{-\gamma_v}{\alpha_v\,\delta_v - \beta_v\,\gamma_v} \\ \delta_{-v} = \dfrac{\alpha_v}{\alpha_v\,\delta_v - \beta_v\,\gamma_v} \end{cases} \tag{3.10}$$

It is simple now to derive the following results, from equations (3.9) and (3.10):

$$\alpha_v = \delta_v \tag{3.11}$$

$$\beta_v = \frac{\delta_v^2 - 1}{\gamma_v} = -\frac{\delta_v^2 - 1}{v\,\delta_v} = -\frac{\alpha_v^2 - 1}{v\,\alpha_v^2} \tag{3.12}$$

Finally the transformations (3.7) take the form:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \alpha_v & -\frac{\alpha_v^2 - 1}{v\,\alpha_v} \\ -v\,\alpha_v & \alpha_v \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = \alpha_v \begin{pmatrix} 1 & -\frac{\alpha_v^2-1}{v\,\alpha_v^2} \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \tag{3.13}$$

Now one must consider a third inertial frame S'', moving at a speed u respect to S'. The coordinate transformations of S'' respect to S take the form:

$$\begin{aligned} \begin{pmatrix} x'' \\ t'' \end{pmatrix} &= \alpha_u\,\alpha_v \begin{pmatrix} 1 & -\frac{\alpha_u^2-1}{u\,\alpha_u^2} \\ -u & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\alpha_v^2-1}{v\,\alpha_v^2} \\ -v & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \\ &= \alpha_u\,\alpha_v \begin{pmatrix} 1 + u\,\frac{\alpha_v^2-1}{v\,\alpha_v^2} & -\frac{\alpha_u^2-1}{u\,\alpha_u^2} - \frac{\alpha_v^2-1}{v\,\alpha_v^2} \\ -(u+v) & 1 + v\,\frac{\alpha_u^2-1}{u\,\alpha_u^2} \end{pmatrix} \end{aligned} \tag{3.14}$$

To preserve the relativity principle, this coordinates change, from S'' to S, must assume the same form as the transformation (3.13), from S' to S, so it follows the relation:

$$1 + u\,\frac{\alpha_v^2 - 1}{v\,\alpha_v^2} = 1 + v\,\frac{\alpha_u^2 - 1}{u\,\alpha_u^2} \tag{3.15}$$

that immediately becomes:

$$\frac{\alpha_v^2 - 1}{v^2\,\alpha_v^2} = \frac{\alpha_u^2 - 1}{u^2\,\alpha_u^2} \tag{3.16}$$

This relation implies that both elements are constant, because u and v are velocities arbitrarily defined. Thus it follows:

$$\alpha_v^2 - 1 = K\,v^2\,\alpha_v^2 \;\Rightarrow\; \alpha_v = \frac{1}{\sqrt{1-K\,v^2}} \tag{3.17}$$

with K an opportune constant. The final form of the coordinates transformation (3.13) is therefore:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1-K\,v^2}} \begin{pmatrix} 1 & -K\,v \\ -v & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{3.18}$$

Now it is possible to obtain the velocity addition law. Matching the first diagonal terms of equations (3.13) and (3.14), using (3.13) for the final transformation, with resultant velocity w, it follows:

$$\alpha_w = \alpha_u\,\alpha_v \left(1 + u\,\frac{\alpha_v^2 - 1}{v\,\alpha_v^2}\right) \tag{3.19}$$

Using (3.17) for α coefficients, one obtains:

$$\frac{1}{\sqrt{1-K\,u^2}}\,\frac{1}{\sqrt{1-K\,v^2}}(1+K\,u\,v) = \frac{1}{\sqrt{1-K\,w^2}} \tag{3.20}$$

and solving for w:

$$w = \frac{u+v}{1+K\,u\,v} \tag{3.21}$$

discarding the solution with negative sign.
From this last relation it follows immediately that K must be a constant with squared speed dimension, that is $\sqrt{K}$ is an invariant velocity. Important to underline that when $K = 0$, the transformations (3.18) reduce to the Galilei ones. Only when $K \neq 0$ $(0 < K < \infty)$ one obtains the Lorentz group transformations. Finally it is fundamental to note that the need of a maximum velocity for every material body simply emerges from the postulates of space-time isotropy and homogeneity and from the relativity principle. Nothing is stated about the universality of the K constant and its nature, that is if $\sqrt{K} = c$, where c represents the light speed. Moreover only empirical observations can definitely state if this maximum velocity is finite (Einstein Special Relativity) or not (Galilei Relativity).

3.8 Very Special Relativity

One of the first EFT approach to LIV was introduced by Coleman and Glashow [18, 22]. They developed an isotropic perturbative framework to deal with tiny

LIV departures from classical quantum field theories. The Lagrangian modifications cause the maximum attainable velocities of massive particles to differ from the c speed of light. The perturbations are conceived so that the gauge symmetry $SU(3) \times SU(2) \times U(1)$ is preserved. Moreover this kind of perturbations are rotationally and translationally invariants, but in a preferred fixed inertial frame.
Considering scalar fields, as first example, the most general Lagrangian that preserves $U(1)$ symmetry has the form:

$$\mathcal{L} = \partial_\mu \overline{\psi}\, Z\, \partial^\mu \psi - \overline{\psi}\, M^2\, \psi \tag{3.22}$$

with Z and M^2 hermitian positive definite matrices. It is always possible to transform the fields so that $Z = \mathbb{I}$ and M becomes diagonal, obtaining in this way a n decoupled fields theory. This Lagrangian is perturbed with the addition of a LIV term with the form:

$$\partial_\mu \overline{\psi}\, \epsilon\, \partial^\mu \psi \tag{3.23}$$

with ϵ an hermitian matrix. This perturbation operator presence lets the single particle eigenstates to evolve from those of the M^2 matrix, in the infrared limit, to those of ϵ in the ultraviolet limit. This means that the maximum attainable velocity of a material particle changes continuously from a low energy limit to a high energy one.
The most general case Lagrangian can be constructed starting from the representations of the Lorentz group $SO(1,\, 3)$. Summarizing all the theory field operators in one vector Φ, the Lorentz invariance implies:

$$U^\dagger(\Lambda)\, \Phi(x)\, U(\Lambda) = D(\Lambda)\, \Phi(\Lambda^{-1} x) \tag{3.24}$$

where $\Lambda \in SO(1,\, 3)$, $D(\Lambda)$ is a representation of the Lorentz group.
The Lie algebra $\mathfrak{so}(1,\, 3)$ can be decomposed in a sum of $\overrightarrow{J}_+$ and $\overrightarrow{J}_-$ operators, so that the eigenvalues of the couple $(\overrightarrow{J}_+,\ \overrightarrow{J}_-)$ are:

- $(0,\, 0)$ for scalars,
- $(1/2,\, 0)$ or $(0,\, 1/2)$ for left or right Weyl spinors,
- $(1/2,\, 1/2)$ for four-vectors,
- $(1,\, 1)$ for traceless symmetric tensors,
- a direct sum of $(1,\, 0)$ and $(0,\, 1)$ for antisymmetric tensors.

In case of rotations, the Lorentz group representation operator assumes the explicit form:

$$D(R(\overrightarrow{e}\theta)) = \exp\left(i\,(\overrightarrow{J}_+ + \overrightarrow{J}_-) \cdot \overrightarrow{e}\theta\right) \tag{3.25}$$

where θ is the rotation angle around the vector $\overrightarrow{e}$. A boost operator explicit form is:

$$D(B(\overrightarrow{e}\eta)) = \exp\left((\overrightarrow{J}_{+} - \overrightarrow{J}_{-}) \cdot \overrightarrow{e}\eta\right) \tag{3.26}$$

where η represents the boost rapidity in $\overrightarrow{e}$ direction.
Rotationally invariant theories require therefore that the $\overrightarrow{J}_{\pm}$ operators eigenvalues must satisfy the equation $j_{+} = j_{-} = j$. It follows that the LIV perturbation Lagrangian can be written as:

$$\mathcal{L}' = \sum_{m=-j}^{j} (-1)^m \, \Phi_{-m,\,m} \tag{3.27}$$

Analyzing a generic rapidity η boost effects on the Lagrangian operators:

$$\langle\psi|U^{\dagger}(B(\overrightarrow{e}\eta)\,\mathcal{L}'(0)\,U(B(\overrightarrow{e}\eta)|\psi\rangle \div E^{2j\eta}\langle\psi|\Phi_{-j,\,j}(0)|\psi\rangle + O(E^{(2j-2)\eta}) \tag{3.28}$$

it emerges that the perturbation increases as $E^{2j\eta}$ in the high energy limit. In a rotational invariant model the possibilities for renormalizable operators are limited to: $j = 0$ for scalars, $j = 1/2$ for CPT-odd vectors and $j = 1$ for CPT-even tensors. A non trivial CPT-even model requires therefore $j = 1$, so for scalar fields the LIV renormalizable perturbation operator must be constructed with two fields and two derivatives:

$$\sum_{a,\,b} \partial_{\mu}\phi^{a}\,\epsilon_{ab}\,\partial^{\mu}\phi^{b} \tag{3.29}$$

with ϵ_{ab} a real symmetric matrix.
For spinor fields, the most general Lorentz invariant Lagrangian can be written as:

$$u^{\dagger}(i\partial_0 - \overrightarrow{\sigma} \cdot \overrightarrow{\partial})\,Z\,u + \frac{1}{2}u^{t}\,\sigma_y\,M\,u + \frac{1}{2}u^{\dagger}\,\sigma_y\,M^{\dagger}\,u^{*} \tag{3.30}$$

where u represents a generic Weyl spinor. After an opportune field redefinition, the Lagrangian can be written, as function of ψ Dirac spinors, in the usual way:

$$\overline{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi \tag{3.31}$$

The LIV CPT-even renormalizable operator can be written, as function of u Weyl spinors:

$$\frac{i}{2}\,\epsilon_{ab}\,u^{a\dagger}\,\sigma^{\mu}\partial_{\mu}u^{b} \tag{3.32}$$

where ϵ_{ab} is a hermitian matrix. This term can be rewritten as function of ψ Dirac spinors as:

$$\frac{i}{2}\,\overline{\psi}\,\gamma^{\mu}\partial_{\mu}(\epsilon_{+}\,P_R + \epsilon_{-}\,P_L)\,\psi \tag{3.33}$$

with P_R and P_L the chirality projectors, defined as

$$\psi_L = P_L\,\psi = \frac{1}{2}(\mathbb{I} - \gamma_5)\,\psi \quad , \quad \psi_R = P_R\,\psi = \frac{1}{2}(\mathbb{I} + \gamma_5)\,\psi \tag{3.34}$$

and ϵ_{+} and ϵ_{-} positive LIV coefficients.
The kinematical effects of this model emerge in modifying the material particle propagators. In a Lorentz invariant theory the particle propagator has the general form:

$$D(p^2) = \frac{i}{(p^2 - m^2)\,A(p^2)} \tag{3.35}$$

where the function A is normalized via the relazion $A(m^2) = 1$. Adding a LIV interaction term, proportional to a small coefficient ϵ, one obtains:

$$D(p^2) = \frac{i}{(p^2 - m^2)\,A(p^2) + \epsilon\,p^2\,B(p^2)} \tag{3.36}$$

with B a generic function, normalizated so that $B(m^2) = 1$.
Reintroducing the maximum attainable velocity c, to substitute the light speed, at first order, the dispersion relation becomes:

$$E^2 = p^2\,c^2 + m^2\,c^4 \tag{3.37}$$

The perturbation terms, proportional to the mass, can be neglected, because smaller than the perturbative contribution proportional to the squared momentum. The dispersion relation becomes therefore in the high energy limit:

$$E^2 = p^2(1+\epsilon) + m^2 \tag{3.38}$$

3.9 Standard Model Extension

The most complete and coherent EFT framework to study the LIV phenomenology is referred as Standard Model Extension (SME) [26]. This theory explores the LIV scenarios by amending the particle SM, supplementing all the possible LIV operators, that preserve the gauge symmetry $SU(3)\times SU(2)\times U(1)$. The SME formulation is conceived even in order to preserve microcausality, positive energy and four-momentum conservation law. Moreover the quantization principles are conserved, in order to guarantee the existence of Dirac and Schrödinger equations, in the correct energy regime limit. Therefore the SME modifications consist

of perturbation operators, generated by the coupling of matter Lagrangian standard fields with background tensors. These tensors non-zero void expectation value and their constant non dynamical nature break the LI under active transformations of the observed system. It is important to underline that this model introduce a difference between active and passive reference frame transformations, not present in SR [98, 99, 100]. Active transformations refer to the transformation that affect the observed particle, instead the passive ones are those that affect the observer. The presence of couplings with a fixed background induces a Lorentz violation only for active transformations, that modify the coupling of the observed particle field with the background tensors. In this sense SME preserves the covariance of phisics formulation under passive transformations, that is observer rotations or boosts.
In SME framework the perturbation operators are classified according to their power counting renormalizability (their mass dimension) and their CPT symmetry behavior [101].

3.9.1 minimal SME

The minimal SME (mSME) consists of the SME subset that deals only with power counting renormalizable and super-renormalizable operators (mass dimension 3 and 4). The minimal SM extension is constructed from the SM standard Lagrangian [26]. Defining for every ψ field the left and right-handed component with the chirality projectors, it is possible to pose the left (L) and right-handed (R) lepton multiplets:

$$L_j = \begin{pmatrix} \nu_j \\ l_j \end{pmatrix}_L \quad , \quad R_j = (l_j)_R \tag{3.39}$$

where ν is the neutrino field and l the associated leptonic one. The analogous left and right-handed quark multiplets are:

$$Q_j = \begin{pmatrix} u_j \\ d_j \end{pmatrix}_L \quad , \quad U_j = (u_j)_R \quad , \quad D_j = (d_j)_R \tag{3.40}$$

where the index $j = \{1, 2, 3\}$ labels the particle flavor:

$$l_j = \{e, \mu, \tau\} \quad , \quad \nu_j = \{\nu_e, \nu_\mu, \nu_\tau\} \quad , \quad u_j = \{u, c, t\} \quad , \quad d_j = \{d, s, b\} \tag{3.41}$$

Moreover the Higgs doublet, in unitary gauge, is denoted as:

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ r_\phi \end{pmatrix} \tag{3.42}$$

and the $SU(3)$, $SU(2)$ and $U(1)$ gauge fields as respectively G_μ, W_μ and B_μ, with corresponding strengths $G_{\mu\nu}$, $W_{\mu\nu}$ and $B_{\mu\nu}$ and corresponding couplings g_S, g and g_0. The $U(1)$ electromagnetic charge is defined via the θ_W angle in the usual way as $q = g\,\sin\theta_W = g_0\,\cos\theta_W$ and the Yukawa couplings are $G_L,\ G_U,\ G_D$. The "classical" SM Lagrangian can therefore be written, splitting the single terms, as:

$$\mathcal{L}_{lepton} = \frac{i}{2}\,\overline{L}_j\,\gamma^\mu\,\overleftrightarrow{D}_\mu\,L_j + \frac{i}{2}\,\overline{R}_j\,\gamma^\mu\,\overleftrightarrow{D}_\mu\,R_j \tag{3.43}$$

$$\mathcal{L}_{quark} = \frac{i}{2}\,\overline{Q}_j\,\gamma^\mu\,\overleftrightarrow{D}_\mu\,Q_j + \frac{i}{2}\,\overline{U}_j\,\gamma^\mu\,\overleftrightarrow{D}_\mu\,U_j + \frac{i}{2}\,\overline{D}_j\,\gamma^\mu\,\overleftrightarrow{D}_\mu\,D_j \tag{3.44}$$

$$\mathcal{L}_{Yukawa} = -[(G_L)_{ij}\,\overline{L}_i\,\phi\,R_j + (G_U)_{ij}\,\overline{Q}_i\,\phi^c\,U_j + (G_D)_{ij}\,\overline{Q}_i\,\phi\,D_j] \,+\, h.c. \tag{3.45}$$

$$\mathcal{L}_{Higgs} = (D_\mu\phi)\,D^\mu\phi + \mu^2\,\phi^\dagger\phi - \frac{\lambda}{3!}\,(\phi^\dagger\phi)^2 \tag{3.46}$$

$$\mathcal{L}_{gauge} = -\frac{1}{2}\,Tr(G_{\mu\nu}G^{\mu\nu}) - \frac{1}{2}\,Tr(W_{\mu\nu}W^{\mu\nu}) - \frac{1}{4}\,B_{\mu\nu}B^{\mu\nu} \tag{3.47}$$

Now it is possible to write the new terms introduced by the LIV extension of the SM, starting from the fermion sector. The lepton contributions are written remembering that the new operators must have mass dimension 3 or 4 and are divided in CPT-even and CPT-odd contributions:

$$\mathcal{L}^{CPT-even}_{lepton} = \frac{i}{2}\,(c_L)_{\mu\nu ij}\,\overline{L}_i\,\gamma^\mu\,\overleftrightarrow{D}^\nu\,L_j + \frac{i}{2}\,(c_R)_{\mu\nu ij}\,\overline{R}_i\,\gamma^\mu\,\overleftrightarrow{D}^\nu\,R_j + \tag{3.48}$$

$$\mathcal{L}^{CPT-odd}_{lepton} = -(a_L)_{\mu ij}\,\overline{L}_i\,\gamma^\mu\,L_j - (a_R)_{\mu ij}\,\overline{R}_i\,\gamma^\mu\,R_j \tag{3.49}$$

The analogous extension terms for the quark sector are:

$$\begin{aligned}\mathcal{L}^{CPT-even}_{quark} =& \frac{i}{2}\,(c_Q)_{\mu\nu ij}\,\overline{Q}_i\,\gamma^\mu\,\overleftrightarrow{D}^\nu\,Q_j + \frac{i}{2}\,(c_U)_{\mu\nu ij}\,\overline{U}_i\,\gamma^\mu\,\overleftrightarrow{D}^\nu\,U_j + \\ &+ \frac{i}{2}\,(c_D)_{\mu\nu ij}\,\overline{D}_i\,\gamma^\mu\,\overleftrightarrow{D}^\nu\,D_j\end{aligned} \tag{3.50}$$

$$\mathcal{L}^{CPT-odd}_{quark} = -(a_Q)_{\mu ij}\,\overline{Q}_i\,\gamma^\mu\,Q_j - (a_U)_{\mu ij}\,\overline{U}_i\,\gamma^\mu\,U_j - (a_D)_{\mu ij}\,\overline{D}_i\,\gamma^\mu\,D_j \tag{3.51}$$

Important to underline that the a_μ coefficients are parameters with mass dimension, instead the $c_{\mu\nu}$ ones are assumed traceless. This is a general property of mSME coefficients with an even number of space-time indices. The trace com-

ponent is assumed equal to zero, because it corresponds to a redefinition of the maximum personal velocity of the particle. With a redefinition of the measure units, the effects produced by this coefficients can be reconciled with standard physics. This means that the $c_{\mu\nu}$ coefficients trace can produce physical effects only in processes that involve more than one particle species.
The mSME contains even lepton-Higgs coupling LIV terms, that have the usual Yukawa gauge structure, but these terms are all CPT-even:

$$\begin{aligned}\mathcal{L}_{Yukawa}^{CPT-even} =& -\frac{1}{2}\,[(H_L)_{\mu\nu ij}\,\overline{L}_i\,\phi\,\sigma^{\mu\nu}\,R_j + (H_U)_{\mu\nu ij}\,\overline{Q}_i\,\phi^c\,\sigma^{\mu\nu}\,U_j + \\ &+(H_D)_{\mu\nu ij}\,\overline{Q}_i\,\sigma^{\mu\nu}\,D_j] \;+\; h.c.\end{aligned} \tag{3.52}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma_\mu,\,\gamma_\nu]$.
The contributions to the Higgs sector have both CPT-even and CPT-odd terms:

$$\mathcal{L}_{Higgs}^{CPT-even} = \frac{1}{2}\,(k_{\phi\phi})^{\mu\nu}\,(D_\mu\phi)^\dagger D_\nu\phi + h.c. - \frac{1}{2}\,(k_{\phi i})^{\mu\nu}\,\phi^\dagger\phi\,B_{\mu\nu} - \frac{1}{2}\,(k_{\phi W})^{\mu\nu}\,\phi^\dagger W_{\mu\nu}\,\phi \tag{3.53}$$

$$\mathcal{L}_{Higgs}^{CPT-odd} = i(k_\phi)^\mu\phi^\dagger D_\mu\phi \;+\; h.c. \tag{3.54}$$

Finally even the gauge sector in the mSME is amended by the introduction of LIV operators, in the form:

$$\begin{aligned}\mathcal{L}_{gauge}^{CPT-even} =& -\frac{1}{2}\,(k_G)_{\mu\nu\alpha\beta}\,Tr(G^{\mu\nu}G^{\alpha\beta}) - \frac{1}{2}\,(k_W)_{\mu\nu\alpha\beta}\,Tr(W^{\mu\nu}W^{\alpha\beta}) + \\ &-\frac{1}{4}\,(k_B)_{\mu\nu\alpha\beta}\,B^{\mu\nu}B^{\alpha\beta}\end{aligned} \tag{3.55}$$

with the k coefficients real and with symmetries equal to the Riemann tensor. Moreover the coefficient have a null double trace, for the same reason exposed previously, about the trace of even space-time indices coefficients.

$$\begin{aligned}\mathcal{L}_{gauge}^{CPT-odd} =& (k_3)_\mu\epsilon^{\mu\nu\alpha\beta}\,Tr(G_\nu\,G_{\alpha\beta} + \frac{2i}{3}\,g_S\,G_\nu G_\alpha G_\beta) + (k_2)_\mu\,\epsilon^{\mu\nu\alpha\beta}\,Tr(W_\nu\,W_{\alpha\beta} + \\ &+\frac{2i}{3}\,g\,W_\nu\,W_\alpha\,W_\beta) + (k_1)_\mu\,\epsilon^{\nu\alpha\beta}\,B_\nu\,B_{\alpha\beta} + (k_0)_\mu\,B^\mu\end{aligned} \tag{3.56}$$

From the mSME Lagrangian, here reported, it is now possible to obtain a generalized quantum electrodynamic (QED), amended by the presence of LIV operators. In fact, as usual, after the spontaneous breaking of the $SU(2) \times U(1)$ symmetry, one can neglect all the gauge bosons, except the photons, setting them to zero. Moreover, only the charged fermions are taken into account, so the neutrinos are naturally discarded. The standard QED Lagrangian can be written, in the unitary gauge, used in order to introduce the lepton masses, as:

$$\mathcal{L}_{classic}^{QED} = \frac{i}{2}\,\bar{l}_j\,\gamma^\mu \overleftrightarrow{D}_\mu\, l_j - m_j\,\bar{l}_j\,l_j - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \tag{3.57}$$

where the index j represents the three lepton flavor $(e,\, \mu,\, \tau)$, the covariant derivative is $D_\mu = \partial_\mu + iqA_\mu$ and the field strenght is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The LIV terms are:

$$\mathcal{L}_{lepton}^{CPT-even} = -\frac{1}{2}\,(H_l)_{\mu\nu ij}\,\bar{l}_i\,\sigma^{\mu\nu}\,l_j + \frac{i}{2}\,(c_l)_{\mu\nu ij}\,\bar{l}_i\,\gamma^\mu \overleftrightarrow{D}^\nu\,l_j + \frac{i}{2}\,(d_l)_{\mu\nu ij}\,\bar{l}_i\,\gamma_5\,\gamma^\mu \overleftrightarrow{D}^\nu\,l_j \tag{3.58}$$

$$\mathcal{L}_{lepton}^{CPT-odd} = -(a_l)_{\mu ij}\,\bar{l}_i\,\gamma^\mu\,l_j - (b_l)_{\mu ij}\,\bar{l}_i\,\gamma_5\,\gamma^\mu\,l_j \tag{3.59}$$

The new LIV contributions to the gauge (photon) sector are:

$$\mathcal{L}_{photon}^{CPT-even} = -\frac{1}{4}\,(k_F)_{\mu\nu\alpha\beta}\,F^{\mu\nu}F^{\alpha\beta} \tag{3.60}$$

There is even a CPT-odd contribution with the form:

$$\mathcal{L}_{photon}^{CPT-odd} = \frac{1}{2}\,(k_{AF})_\mu\,\epsilon^{\mu\nu\alpha\beta}\,A_\nu\,F_{\alpha\beta} \tag{3.61}$$

but this term is discarded, posing $k_{AF} = 0$, because it can generate some theoretical problems, due to the negative energy contribute it can generate. In order to preserve the lepton number, all the LIV Lagrangian terms coefficients must be diagonal in flavor space indices. The lepton number conservation assumption can be justified by the conjecture that any quantum gravity theory have to reduce to "classical physics" (SR and GR) in the infrared regime. The QED LIV Lagrangian can be finally written as:

$$\begin{aligned}\mathcal{L}_{LIV}^{QED} =& \frac{i}{2}\,\overline{\psi}\,\gamma^\mu \overleftrightarrow{D}_\mu\,\psi - m\,\overline{\psi}\,\psi - \frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\,H_{\mu\nu}\,\overline{\psi}\,\sigma^{\mu\nu}\,\psi + \frac{i}{2}\,c_{\mu\nu}\,\overline{\psi}\,\gamma^\mu \overleftrightarrow{D}^\nu\,\psi + \\ &+ \frac{i}{2}\,d_{\mu\nu}\,\overline{\psi}\,\gamma_5\,\gamma^\mu \overleftrightarrow{D}^\nu\,\psi - a_\mu\,\overline{\psi}\,\gamma^\mu\,\psi - b_\mu\,\overline{\psi}\,\gamma_5\,\gamma^\mu\,\psi - \frac{1}{4}\,k_{\mu\nu\alpha\beta}\,F^{\mu\nu}F^{\alpha\beta}\end{aligned} \tag{3.62}$$

The a_μ proportional term represents another non physical contribution to the Lagrangian, in fact it can be reabsorbed in the phase factor of the fermion field, by shifting the phase itself.
New physical effects are introduced by the extended QED in the kinematics of the theory, in fact the photon and electron dispersion relations result modified. Fixed an opportune reference frame and limiting the analysis to rotationally

isotropic LIV operators, the dispersion relations can be written as:

$$\begin{aligned} E_e^2 &= p_e^2 + f_e\, p + g_e\, p^2 + m_e^2 \\ E_\gamma^2 &= (1+f_\gamma) p_\gamma^2 \end{aligned} \tag{3.63}$$

where $f_e = -2\, b\, s$, $g_e = -(c - d\, s)$, $f_\gamma = \frac{k}{2}$ and $s = \pm$ represents the helicity of the particle. Posed the rotational invariance of the theory, it is possible define the coefficients, in the previous equations, via the relations $b_\mu = b\, u_\mu$, $c_{\mu\nu} = c\, u_\mu\, u_\nu$, $d_{\mu\nu} = d\, u_\mu\, u_\nu$ and $k_{\mu\nu\alpha\beta} = k\, u_\mu\, u_\nu\, u_\alpha\, u_\beta$, where $\{u_\mu\}$ are opportune time-like vectors. Finally it is important to note the fact that the new phenomenology must appear in an energy regime comparable to the particle mass, that is $E \simeq m$. Since mSME deals with dimension 3 and 4 operators, the rotationally isotropic subset of this model coincides broadly with VSR scenario.

3.9.2 Generalized SME

Since SME is an effective field theory approach to LIV, it has become common to study even non power counting renormalizable terms [102, 103, 104, 105, 106] that is operators with mass dimension bigger than 4. In fact SM is commonly viewed as the low energy limit of a more general theory. Therefore its renormalizability can emerge, in the infrared regime, neglecting some higher order operators. These operators, assumed as generated by quantum gravity effects, can be neglected, since suppressed by an appropriate energy (or mass) scale. For simplicity here it is considered only the QED sector. The pure gauge (photon) sector action S must be quadratic in the gauge field A_μ and it can be written as sum of terms $S_{(d)}$, given by the integral of a Lagrangian density:

$$S = \sum_d S_{(d)} = \sum_d \int d^4x\, \kappa^{\mu_1\mu_2\ldots\mu_d} A_{\mu_1} \partial_{\mu_2} \ldots \partial_{\mu_{d-1}} A_{\mu_d} \tag{3.64}$$

d represents the tensor operator mass dimension and the coefficients $\kappa^{\mu_i\cdots}$ are the LIV background tensors, with mass dimension $d-4$. All the operators with even d dimension are CPT-even, while the other ones (with d odd) are CPT-odd. For simplicity the present analysis is truncated at mass dimension 5 and 6 operators. As before the analysis is limited to an isotropic case, where the LIV operators result rotationally invariant in the opportune fixed reference frame. The CPT-odd gauge operator with mass dimension 5 is:

$$-\kappa^{\mu\nu\alpha}\, F_{\mu\beta}\, \partial_\nu F_\alpha^{\ \beta} \tag{3.65}$$

instead the CPT-even 6 mass dimension operator is given by:

$$-\kappa^{\mu\nu\alpha\beta}\, F_{\mu\tau}\, \partial_\nu\, \partial_\alpha\, F_\beta^\tau \tag{3.66}$$

The photon dispersion relation results modified by the introduction of these two terms. In the first case the dispersion relation becomes:

$$\omega_\pm^2 = k^2 \pm \xi\, k^3 \tag{3.67}$$

where $(\omega,\ \overrightarrow{k})$ is the photon four-momentum, and ξ is a constant such that $\kappa^{\mu\nu\alpha} = \xi\, u^\mu\, u^\nu\, u^\alpha$ with $\{u^\mu\}$ unitary vectors. The $+$ or $-$ signs of this relation depend on the right or left photon polarization. Instead, in the second case, the dispersion relation is:

$$\omega^2 = k^2 + \beta\, k^4 \tag{3.68}$$

where β is defined, as in the previous case, so that $\kappa^{\mu\nu\alpha\beta} = \beta\, u^\mu\, u^\nu\, u^\alpha\, u^\beta$.
Even the matter-radiation interaction sector results amended by higher order operators. Truncating the analysis at the 5 and 6 mass dimension operators, one obtains for the CPT-odd terms:

$$c_{\mu\nu\alpha}\, \overline{\psi}\, \gamma^\mu\, (\zeta_1 + \gamma_5 \zeta_2)\, \partial^\nu\, \partial^\alpha\, \psi \tag{3.69}$$

and the related modified dispersion relation results:

$$E_\pm^2 = p^2 + m^2 + \eta_\pm\, p^3 \tag{3.70}$$

where $(E,\ \overrightarrow{p})$ is the lepton four-momentum and $\eta_\pm = 2(\zeta_1 \pm \zeta_2)$, with the $+$ and $-$ signs correlated with the particle helicity. The CPT-even terms are instead:

$$\begin{aligned} &-\overline{\psi}\, c_{\mu\nu}\, D^\mu D^\nu\, (\zeta_1^{(5)} + \zeta_2^{(5)}\, \gamma_5)\, \psi - i\, \overline{\psi}\, c_{\mu\nu\alpha\beta}\, D^\mu D^\nu D^\alpha \gamma^\beta\, (\zeta_1^{(6)} + \zeta_2^{(6)}\, \gamma_5)\, \psi + \\ &-i\, \overline{\psi}\, c_{\mu\nu}\, D^\mu\, \Box\, \gamma^\nu\, (\widetilde{\zeta}_1^{(6)} + \widetilde{\zeta}_2^{(6)}\, \gamma_5)\, \psi \end{aligned} \tag{3.71}$$

The lepton dispersion relation results even in this case modified and its form is:

$$E^2 = p^2 + m^2 + \eta^{(2)}\, p^2 + \eta^{(4)}\, p^4 \tag{3.72}$$

where $\eta^{(2)}$ and $\eta^{(4)}$ are dimension full coefficients, obtained as suitable combinations of the ζ parameters. This dispersion relation discriminates between fermion helicities, even if generated by CPT-even operators, because of the presence of terms odd to P and T operators.
The new physics, introduced by SME approach to LIV, emerges in the dispersion relations modification. Common feature of all the LIV theories, analyzed here, is the general form that the particles rotational isotropic dispersion relations as-

sume:

$$E^2 = m^2 + \left(1 + \sum_{n=1}^{\infty} \eta^{(n)}\, p^{n-2}\right) p^2 \tag{3.73}$$

where $\eta^{(n)}$ are dimension-full coefficients.

3.10 Hoȓava-Lifshitz Gravity

Hoȓava-Lifshitz gravity [107, 108, 109, 110, 111, 112] is an attempt to obtain a power counting renormalizable gravity theory , modifying the graviton propagator. In the high energy limit (ultra-violet regime) the theory action is amended, adding higher order spatial derivatives of the metric. Time derivatives are avoided, in order to preserve the theory unitarity. This discrimination, between space-like and time-like coordinates, foliates the space-time in surfaces, spanned by spatial coordinates, and parameterized by the time one. The set of invariant diffeomorphisms for the theory reduces to transformations that preserve this separation, that is:

$$\begin{cases} t \to \Phi(t) = \tilde{t}(t) \\ \overrightarrow{x} \to \Phi(\overrightarrow{x}) = \overrightarrow{y}(\overrightarrow{x}) \end{cases} \tag{3.74}$$

First consequence is the introduction of the ADM (Arnowitt - Deser - Misner) formalism, characterized by the metric:

$$ds^2 = c^2\, N^2\, dt^2 - \tilde{g}_{ij}\, (dx^i + N^i\, dt)\, (dx^j + N^j\, dt) \tag{3.75}$$

where N represents the lapse function, N^i the shift one and $\tilde{g}_{ij}$ is the three-dimensional metric induced on every time constant hypersurface. The Einstein-Hilbert action for this kind of theory is:

$$S = \kappa \int dt\, d^3x\, N\, \sqrt{\tilde{g}} \left(K_{ij}K^{ij} - K^2 + R^{(3)}\right) \tag{3.76}$$

where K_{ij} is the extrinsic curvature, given by:

$$K_{ij} = \frac{1}{2N}\left(\dot{\tilde{g}}_{ij} - \nabla_{(i}N_{j)}\right) \tag{3.77}$$

and $K = Tr(K_{ij})$.
The modified form of the action, after adding the LIV operators, that depend on metric spatial derivatives, is:

$$S_{HL} = \kappa \int dt\, d^3x\, N\, \sqrt{\tilde{g}} \left(L_2 + \frac{1}{M^2}L_4 + \frac{1}{M^4}L_6\right) \tag{3.78}$$

where L_2 is the operator:

$$L_2 = K_{ij}K^{ij} - \lambda\, K^2 + \xi\, R + \eta\, a_i a^i \tag{3.79}$$

with $a_i = \partial_i \ln N$. M denotes the operators suppressing mass scale, which a priori does not coincide with the Planck mass M_{Pl}. L_4 and L_6 represent all the 4th and 6th order operators respectively. In such a theory LIV is present at all orders, even L_2 presents LIV terms. Lorentz violations are regulated by the parameters λ, ξ and η, which are identically zero in General Relativity. Power counting renormalizability requires that the operators in 4 dimensions must contain 6 spatial derivatives at least. In this model it is possible to assume that LIV terms can not affect the lowest order operators. This implies the presence of a forbidding mechanism, that prevents LIV perturbations to appears, thanks to radiative corrections, at lower orders. Moreover it is possible to state the gravitational action is even under CPT and P transformations. In fact, since no CPT and P odd operators are present at tree level, it seems reasonable to expect that such perturbations are not induced by radiative corrections. Therefore in dispersion relations only even momentum powers are allowed and there is not distinction between particles helicity, that is matter and anti-matter have the same dispersion relation forms:

$$E^2 = p^2 + \eta\, \frac{p^4}{M^2} + O\left(\frac{p^6}{M^4}\right) + m^2 \tag{3.80}$$

3.11 New Special Relativity theories

The models illustrated till now share all the common feature to be effective field theories, the most conservative and practical way to approach LIV. They are all based on highly reasonable, but arbitrary "assumptions", not founded on fundamental physical principles. Another approach to LIV consists in attempting the construction of complete physical theories, alternative to the standard ones. The first motivation to construct such a new theory consists in the attempt to reconcile the existence of a second universal constant, the Planck length, with the relativity principle. In fact the distance contraction, induced by the Lorentz transformations, is in contrast with the idea of a minimum invariant length quantum. A second inertial observer invariant quantity, a energy scale, that implies a minimum length, is introduced in this SR modification scenario, known as Double Special Relativity (DSR) [113, 114, 115, 116]. First of all, it is necessary to modify the SR postulates, in order to be coherent with the introduction of this second universal constant. Standard SR principles are:

- Physiscs laws are the same in every inertial frame (physics laws covariance)

- Vacuum light speed is an inertial observer invariant quantity, and it represents the maximum attainable velocity

The last postulate must be modified to include the second universal quantity and can be reformulated as:

- The light speed becomes a particle energy function $c(E)$, its functional form is universal for every inertial observer and the constant light speed c is obtained as a limit when the physical scale is much bigger than the Planck length $l/l_{Pl} \to \infty$. This last length is the second universal invariant quantity.

Important to undeline that with the acronym DSR is not described a unique SR modification scenario, but rather a variety of models that include the new universal quantity in SR. These models share the introduction of Lorentz group modifications. The Lorentz group is, by definition, the isometries group of the standard Minkowski norm $\eta_{\mu\nu} = diag(1,\,-1,\,-1,\,-1)$, that is:

$$\eta_{\mu\nu} = \Lambda^{\alpha}_{\mu}\,\eta_{\alpha\beta}\,\Lambda^{\beta}_{\nu} \tag{3.81}$$

where $\Lambda^{\alpha}_{\mu} \in SO(1,\,3)$ and $SO(1,\,3)$ is the Minkowski norm ortogonal (isometric) group.
The standard generators of this group are:

$$M_{\mu\nu} = x_{\mu}\frac{\partial}{\partial x^{\nu}} - x_{\nu}\frac{\partial}{\partial x^{\mu}} \tag{3.82}$$

which satisfy the relation:

$$[M_{\mu\nu},\, M_{\alpha\beta}] = i\,(\eta_{\mu\alpha}\,M_{\nu\beta} - \{\mu \longleftrightarrow \nu\}) - \{\alpha \longleftrightarrow \beta\} \tag{3.83}$$

where $\{\mu \longleftrightarrow \nu\}$ means that the previous term is rewritten, inverting μ and ν indices.
From the previous one it is possible to write the explicit form of the $\mathfrak{so}(1,\,3)$ Lie algebra (the Lorentz Lie algebra):

$$\begin{cases} [J^i,\, J^j] = \epsilon^{ijk}\,J^k \\ [J^i,\, K^j] = \epsilon^{ijk}\,K^k \\ [K^i,\, K^j] = -\epsilon^{ijk}\,J^k \end{cases} \tag{3.84}$$

where $J^i = \epsilon^{ijk}\,M_{jk}$ is the rotation generator and $K^i = M^{0i}$ denotes the boost generator.
The Lorentz transformations between two inertial frames, moving one respect to

the other with velocity $\vec{v}$, in a generic direction, are:

$$\begin{cases} t' = \gamma \left(t - \dfrac{\vec{v} \cdot \vec{x}}{c^2} \right) \\ \vec{x}' = \vec{x}_\perp + \gamma \left(\vec{x}_\perp - \vec{v}\, t \right) = \vec{x} + \left(\dfrac{\gamma - 1}{v^2}\, \vec{x} \cdot \vec{v} - \gamma\, t \right) \vec{v} \end{cases} \tag{3.85}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Lorentz factor.

A standard strategy to deform the Lorentz group, in order to introduce the second invariant quantity (the Planck length), consists in amending the material particles dispersion relation:

$$E^2 = p^2 + m^2 \;\Rightarrow\; f_{(1)}^2(E,\, p,\, l_{Pl})\, E^2 = f_{(2)}^2(E,\, p,\, l_{Pl})\, p^2 + m^2 \tag{3.86}$$

The modification is made introducing perturbations that depends on four-momentum and on the invariant length scale, or equivalently on the invariant energy. From this modified dispersion relation (MDR) it is simple to obtain the explicit form of the maximum attainable velocity. Rewriting equation (3.86) in a more convenient form:

$$E^2 = \frac{f_{(2)}^2}{f_{(1)}^2}\, p^2 + \frac{1}{f_{(1)}^2}\, m^2 \;\Rightarrow\; E^2 = f_{(3)}^2\, p^2 + m^2 \tag{3.87}$$

and neglecting the correction terms proportional to the mass. Using the Hamilton-Jacobi equation, one finally obtains:

$$\begin{aligned} c(E) =& \frac{dE}{dp} = \frac{d}{dp}\sqrt{f_{(3)}^2\, p^2 + m^2} \simeq \frac{d}{dp}\left(f_{(3)}\, p + \frac{m^2}{2\, f_{(3)}\, p} \right) = \\ =& \left(1 + \frac{m^2}{2\, p^2\, f_{(3)}^2} \right) (f_{(3)} + p\, f'_{(3)}) \simeq \left(f_{(3)} + p\, f'_{(3)} \right) \end{aligned} \tag{3.88}$$

This demonstrates that the maximum attainable velocity $c(E)$ acquires an energy dependence and therefore it is no more an universal quantity, breaking the Lorentz invariance. The DSR theories objective is to modify the Lorentz group representations in order to obtain transformations that preserves the MDR form. Starting from a linear map defined by:

$$(U(E)E,\, U(E)p) = (f_{(1)}\, E,\, f_{(2)}\, p) \tag{3.89}$$

The $\mathfrak{so}(1,\, 3)$ boost generators must be modified according to:

$$K^i = U(E)\, M^{i0}\, U(E)^{-1} \tag{3.90}$$

in order to generate modified Lorentz group elements that are MDR isometries.

3.11.1 Energy dependent speed of light DSR

The first example of a DSR model, that modify boosts, in order to preserve the MDR form, with a maximum attainable velocity that depends on the energy, was proposed by Amelino-Camelia [117, 118]. In this model the MDR is constructed posing the perturbation functions of equation (3.86) in the following form:

$$f_{(1)} = 1 \qquad f_{(2)} = f_{(3)} = 1 + \xi\, E \tag{3.91}$$

so the light speed becomes:

$$c(E) = \frac{dE}{dp} = (1 + \xi\, E)^2 \tag{3.92}$$

and the MDRs for a massive and a massless particles:

$$\begin{cases} E^2 = (1 + \xi\, E)^2\, p^2 + m^2 \\ E^2 = (1 + \xi\, E)^2\, p^2 \end{cases} \tag{3.93}$$

Magueijo and Smolin proposed another way to introduce a varying speed of light DSR [28]. They introduced the linear map of equation (3.89) with the form:

$$U(E) = \exp\left(-\xi\, E^2\, \frac{\partial}{\partial E}\right) \tag{3.94}$$

This linear map action on the momentum space is given by:

$$U(E)\, E = \sum_{n=0}^{\infty} \frac{\left(-\xi\, E^2\, \frac{\partial}{\partial E}\right)}{n!}\, E = \sum_{n=0}^{\infty} (-\xi\, E)^n = \frac{E}{1 + \xi\, E} \tag{3.95}$$

$$U(E)\, p = p \tag{3.96}$$

The MDRs for mass and massless particles become:

$$\begin{cases} \dfrac{E^2}{(1 + \xi\, E)^2} = p^2 + m^2 \\ \dfrac{E^2}{(1 + \xi\, E)^2} = p^2 \end{cases} \tag{3.97}$$

and the light speed is given by the functional form:

$$c(E) = \frac{dE}{dp} = (1 + \xi\, E)^2 \tag{3.98}$$

These two models therefore share the same speed of light functional formulation and the same MDR for massless particles (photons).

3.11.2 Gravity's rainbow DSR

Another example of DSR model [119, 120], known as DSR2, follows an approach similar to the previous one, defining the $U(E)$ linear map (3.89) as:

$$U(E) = \exp\left(l_{Pl}\, E\, D\right) \tag{3.99}$$

where the operator D is:

$$D = p_\mu \frac{\partial}{\partial p_\mu} \tag{3.100}$$

The action of (3.99) on momentum space is given by:

$$U(E)\, p_\alpha = \sum_{n=0}^{\infty} \frac{\left(l_{Pl}\, E\, p_\mu \frac{\partial}{\partial p_\mu}\right)^n}{n!}\, p_\alpha = p_\alpha \sum_{n=0}^{\infty} \left(l_{Pl}\, E\right)^n = \frac{p_\alpha}{1 - l_{Pl}\, E} \tag{3.101}$$

The boost generators (3.90) become:

$$K^i = M^{i0} + l_{Pl}\, p_i\, D \tag{3.102}$$

and the modified Lorentz group generators are now given by:

$$\widetilde{M}_{\mu\nu} = p_\mu \partial_\nu - p_\nu \partial_\mu + l_{Pl}(\eta_{0\mu} p_\nu - \eta_{0\nu} p_\mu)\, D \tag{3.103}$$

The modified Lorentz group can be evaluated exponentiating the previous generators. The finite form of the Lorentz transformations in x direction is:

$$\begin{cases} E' = \dfrac{\gamma\,(E - v\, p_x)}{1 + l_{Pl}\,(\gamma - 1)\, E - l_{Pl}\, \gamma\, v\, p_x} \\ p'_x = \dfrac{\gamma\,(p_x - v\, E)}{1 + l_{Pl}(\gamma - 1)\, E - l_{Pl}\, \gamma\, v\, p_x} \\ p'_y = \dfrac{p_y}{1 + l_{Pl}(\gamma - 1)\, E - l_{Pl}\, \gamma\, v\, p_x} \\ p'_z = \dfrac{p_z}{1 + l_{Pl}(\gamma - 1)\, E - l_{Pl}\, \gamma\, v\, p_x} \end{cases} \tag{3.104}$$

In this case the MDR for material bodies becomes:

$$\frac{1}{(1 - l_{Pl}\, E)^2}\,(E^2 - p^2) = m^2 \tag{3.105}$$

that reduces to the usual $E^2 - p^2 = 0$ for massless particles. From relation (3.105) it follows immediately that the metric acquires an energy dependence and therefore in the UV limit it becomes a rainbow metric.

3.11.3 Modified composition laws DSR

The last approach to LIV is known as *Relative Locality* [121]. The central idea of this model consists in supposing the momentum space and not the space-time as the fundamental structure to describe physics. Space-time is considered only a local projection of the momentum space. The new proposal of this model is that the concept of absolute locality is relaxed and different observers feel a personal space-time structure, which is energy (or equivalently momentum) dependent. This model is based on simple semiclassical assumptions about the momentum space geometry, that determine departure from the classical space-time description, first of all the relativity principle is modified and acquires a local character. The new *Principle of local relativity* states that the momentum space is the fundamental structure at the basis of the physical processes description, instead space-time description is constructed by every observer in a personal, local way, loosing universality. Space-time becomes therefore an auxiliary concept, which emerges from the fundamental momentum space, where the real dynamics takes place. The modified momentum space geometry is supposed to be determined by the MDR, used to define the space metric:

$$MDR(p) = p_\mu\, h^{\mu\nu}(p)\, p_\nu = m^2 \tag{3.106}$$

The starting point to determine the momentum space connection consists instead in modifying the interaction processes kinematics. More in detail it consists in defining a modified composition rule for momenta:

$$(p,\, q) \rightarrow (p \oplus q) = p + q + f(p,\, q) \tag{3.107}$$

where $f(p,\, q)$ represents a perturbation of the usual momenta sum. Therefore the momentum space acquire an algebraic structure defined by the $\oplus$ operation. Contemporary it is necessary to introduce the inverse operation, which lets to obtain incoming momenta from the outgoing ones: $(\ominus p) \oplus p = 0$. These definitions correspond to the replacement of the momentum with a modified one, given by the relation:

$$\pi_\mu = M_\mu^{\ \nu}(p) p_{nu} \tag{3.108}$$

with the transformations $M_\mu^{\ \nu}(p)$ determined by the geometric features of the momentum space [105].
Finally the momentum space geometry can be determined from the algebraic properties generated by the modified composition rule, with the affine connection given by:

$$\frac{\partial}{\partial p_a}\frac{\partial}{\partial q_b}(p\oplus q)_c\big|_{p=q=0}=-\Gamma_c^{ab}(0) \tag{3.109}$$

The torsion is evaluated from the asymmetric part of the composition rule:

$$-\frac{\partial}{\partial p_a}\frac{\partial}{\partial q_b}\left((p\oplus q)_c-(q\oplus p)_c\right)\big|_{p=q=0}(0) \tag{3.110}$$

and the curvature is defined as a measure of the departure from associativity for the new composition rule:

$$2\frac{\partial}{\partial p_{[a}}\frac{\partial}{\partial q_{b]}}\frac{\partial}{\partial k_c}\left((p\oplus q)\oplus k-p\oplus(q\oplus k)\right)_d\big|_{p=q=k=0}=R_d^{abc}(0) \tag{3.111}$$

To evaluate all these quantities away from the momentum space origin, it is necessary to define a translation, as:

$$p\oplus_k q=k\oplus((\ominus k\oplus p)\oplus(\ominus k\oplus q)) \tag{3.112}$$

so, for instance, the curvature evaluated in a generic point of the momentum space can be written as:

$$\frac{\partial}{\partial p_a}\frac{\partial}{\partial q_b}(p\oplus_k q)_c\big|_{p=q=0}=-\Gamma_c^{ab}(k) \tag{3.113}$$

Now it is possible to define the parallel transport determined by the geometric connection created by the new composition law. Composing the momentum p of a particle with the infinitesimal one dq of a different particle, one obtains:

$$p_a\oplus dq_a=p_a+\tau_a^{\ b}(p)\,dq_b \tag{3.114}$$

where the tensor τ determines the parallel transport operation and it can be expanded around the $p=0$ as:

$$\tau_a^{\ b}(p)=\delta_a^{\ b}-\Gamma_{\ a}^{bc}\,p_c-\left(\frac{\partial}{\partial p_d}\Gamma_{\ a}^{bc}-\Gamma_{\ i}^{bd}\Gamma_{\ a}^{ic}-\Gamma_{\ i}^{dc}\Gamma_{\ a}^{bi}\right)p_c\,p_d \tag{3.115}$$

The corresponding conservation law acquires the explicit form:

$$\mathcal{P}_a(p) = \sum_J p_a^J - \sum_J C_{IJ}\, \Gamma_a^{bc}\, p_b^I\, p_c^J \tag{3.116}$$

where C_{IJ} are coefficients opportunely defined.
The dynamics can be described therefore by a variational principle, in fact it is possible to write the free particle action as:

$$S_{free} = \sum_J \int ds\, \left(x_J^a\, p_a^J + \lambda_J\, C^J(p) \right) \tag{3.117}$$

The index J represents the particle species, s is an arbitrary time parameter, for instance the proper time, λ is a Lagrange multiplier and $C(p)$ is defined as:

$$C^J(p) = MDR^J(p) - m^2 = p_\mu\, h^{\mu\nu}(p)\, p_\nu - m^2 \tag{3.118}$$

with $h_{\mu\nu}(p)$ obtained from (3.106).
The contraction $x_J^a\, p_a^J$ is defined using the standard metric, in order to preserve the Poisson brackets:

$$\{x_I^a,\, p_a^J\} = \delta_b^a\, \delta_I^J \tag{3.119}$$

Integrating by parts, it is simple to obtain the following relation:

$$\delta S_{free} = \sum_J \int ds \left[\delta x_J^a\, \dot{p}_a^J - \delta p_a^J \left(\dot{x}_J^a - \lambda_J\, \frac{\delta C^J(p)}{\delta p_a^J} \right) + \delta\lambda_J\, C^J(p) \right] \tag{3.120}$$

and the equation of motion have the desired form, given by:

$$\begin{cases} \dot{p}_a^J = 0 \\ \dot{x}_J^a = \lambda_J\, \dfrac{\delta C^J(p)}{\delta p_a^J} \\ C^J(p) = MDR^J(p) - m^2 = 0 \end{cases} \tag{3.121}$$

To determine what happens during particles interaction, it is necessary to consider even the interaction contribution to the action, which can be write as the product of the conservation law (3.116) times a Lagrange multiplier:

$$S_{int} = \mathcal{P}(p)_a\, \xi^a \tag{3.122}$$

The variation of this term is given by:

$$\mathcal{P}_a(p)\, \delta\chi^a - \left(x_J^a(0) - \chi^b\, \frac{\delta P_b}{\delta p_a^J} \right)\, \delta p_a^J \tag{3.123}$$

from which follows immediately, by the vanishing of the term proportional to δp^J_a:

$$x^a_J(0) = \chi^b \, \frac{\delta P_b}{\delta p^J_a} \tag{3.124}$$

From the previous one and the relation (3.116) one can obtain the equation:

$$x^a_I(0) = \chi^a - \chi^b \sum_J C_{IJ} \, \Gamma^{bc}_{\ a} \, p^J_c \tag{3.125}$$

This equation express the worldline coordinate for every observer. If the momentum space curvature is negligible, this expression reduce to the fact that every observer sees the same event coordinate. This fact is in accordance with the usual physical description of interacting particles, that takes place at a given space-time point. In this way it is possible to reconcile this model with the standard physical description at low enough energies.
To express the locality of the newly introduced relativity principle, it is sufficient to derive, from (3.125), the relation:

$$\Delta x^a_J(0) = -\chi^b \sum_J C_{IJ} \, \Gamma^{bc}_{\ a} \, p^J_c \tag{3.126}$$

This expression tells that different observers can detect the same interaction events separated by different space-time coordinates intervals. Only for a subset of privileged observers the interaction events take place in the reference frame origin and therefore are defined at the same set of coordinates. The relativity principle, therefore, acquires a local valence and the Lorentz invariance, as usually conceived, is modified.
In conclusion, the model constructed in this way substitutes the classical concept of space-time as fundamental background, where the physical interactions take place, with the idea that the invariant background is constituted by a curved momentum space. This modification is made at the price to renounce to the concept of an observer independent locality.
Important to underline that even in this case the material particles dispersion relations are modified and acquire a personal functional form. Finally the curved geometry of the momentum space acquires an energy dependence, this features remain a constant in most of the models describing LIV.

CHAPTER 4

LIV

4.1 Introduction

The great part of LIV theories share the feature of introducing a modification of the free particle kinematics. This effect is supposed to be caused by the interaction of the free propagating particle with the space-time quantized background structure. In fact one expects that Planck-scale interactions could manifest themselves in a ”low” energy scenario as tiny residual effects, that can modify standard physics. Several candidates theories, such as Standard Model Extension (SME), Double Special Relativity (DSR), Very Special Relativity (VSR), have been proposed. All of these theories share the feature of considering modified dispersion relations for free particles, with the amended form $E^2 - (1 - f(p))p^2 = m^2$ and $f(p) = \sum_{k=1} \alpha_k(E_P)p^k$. Some proposed scenarios main feature consists in providing background structures, that introduce preferred directions to violate Lorentz Invariance, as in SME. This characteristic implies that the space-time is no more isotropic and therefore an inertial observers privileged class must exist. In fact, spontaneous symmetry breaking is a useful concept of particle physics. One, therefore, can suppose that, even in the high energy limit, the Planck scenario, quantum gravity could present the same mechanism, breaking the Lorentz symmetry. The introduction of a privileged reference frame might not be a real problem, even if it might result conceptually difficult. In fact, nowadays, there is not a clear idea about these preferred inertial observers nature. For example, some studies attempt to correlate the privileged reference frame with the natural one, used for the description of the CMBR. But there are apparently not physical reasons that can justify any connection between a supposed quantum phenomenon of the Planck scale, with the CMBR classical physics description.
In this work, to preserve the idea of space-time isotropy, a possible way to introduce a LIV theory, without a preferred class of inertial observers, is explored. The Lorentz symmetry is therefore only modified, as in DSR theories [121, 24, 122]. Hence the idea of space-time isotropy results restored respect to the new amended Lorentz transformations, here introduced.

4.2 Modified Dispersion Relations

In this work, to geometrize the supposed interaction of massive particles with the background, LIV is introduced, exploiting the possibility to perturb the kinematics. This approach consists in modifying the dispersion relations describing the free particle propagation. Following [123, 124], the Dispersion Relations of standard physics, written using the Minkowski metric as $E^2 - |\overrightarrow{p}|^2 = m^2 \Rightarrow p_\mu \eta^{\mu\nu} p_\nu = m^2$, are modified to a more general case (Modified Dispersion Relations):

$$MDR(p) = E^2 - \left(1 - f\left(\frac{|\overrightarrow{p}|}{E}\right) - g\left(\frac{\overrightarrow{p}}{E}\right)\right)|\overrightarrow{p}|^2 = m^2 \tag{4.1}$$

The f perturbation function preserves the rotational invariance of the MDR. The g one instead breaks this symmetry, introducing a preferred direction in space-time. It is even important to stress that the lack of distinction between particles and antiparticles in MDR means that one is dealing with a CPT even theory. In fact to construct a CPT even theory, the dispersion relations must not present a dependence on particle helicity or spin. In order to contemplate a CPT odd model extension, it is therefore sufficient to include this kind of dependence in the MDR formulation. Since the publication of the Greenberg paper [125], it is well known that LIV does not imply CPT violation. The opposite statement was declared true in the same work [125], but this point is controversial. In fact the idea that CPT violation automatically implies LIV was confuted in [126] and the argument has been widely debated in literaure [127, 128, 129, 130].

In order to preserve the geometrical origin of the MDR, the perturbation functions are chosen homogeneous of degree 0, :

$$\begin{aligned} f\left(\frac{|\overrightarrow{p}|}{E}\right) &= \sum_{k=1}^{\infty} \alpha_k \left(\frac{|\overrightarrow{p}|}{E}\right)^k \\ g\left(\frac{\overrightarrow{p}}{E}\right) &= \sum_{k=1}^{\infty} \beta_k \left(\frac{\overrightarrow{p}}{E}\right)^k \end{aligned} \tag{4.2}$$

Therefore the Modified Dispersion Relations result defined via a Finsler pseudo-norm $F(p)$. In fact the perturbation function homogeneity hypotheses permits to write the MDRs (4.1) as:

$$\begin{aligned} &MDR(p) = F^2(p) = m^2 \\ &F(p) = \sqrt{E^2 - \left(1 - f\left(\frac{|\overrightarrow{p}|}{E}\right) - g\left(\frac{\overrightarrow{p}}{E}\right)\right)|\overrightarrow{p}|^2} \end{aligned} \tag{4.3}$$

Imposing the 0-degree homogeneity to the perturbation functions f and g, one obtains that the function F, defined in (4.3), results homogeneous of degree 1, condition to be a candidate Finsler pseudo-norm. Important to underline the difference between a Finsler structure and a pseudo-Finsler one. The first geometric structure is constructed using a positively defined metric to pose the norm, instead the second one resorts to a not positive one. Here the pseudo-Finsler geometry is used, because one is dealing with the space-time structure, where the standard metric is the Minkowski one, with signature $\{+, -, -, -\}$. From here on, the perturbation function g is posed equal to zero ($g = 0$) in order to try to construct an isotropic LIV theory. Hence only MDR, preserving rotational symmetry, will be considered:

$$\begin{aligned} MDR(p) &= F^2(p) = m^2 \\ F(p) &= \sqrt{E^2 - \left(1 - f\left(\frac{|\overrightarrow{p}|}{E}\right)\right)|\overrightarrow{p}|^2} \end{aligned} \tag{4.4}$$

It is important to underline that in literature the form of the MDRs is usually:

$$MDR(p) = E^2 - (1 - h(p))\,|\overrightarrow{p}|^2 = m^2 \tag{4.5}$$

with the perturbation $h(p) = \sum_{k=1}^{\infty} a_k \left(\frac{p}{M_{Pl}}\right)^k$, where M_{Pl} represents the Planck mass. The perturbation form introduced in this work can be justified because it is, for example, a subcase obtained in [104], so it constitutes an interesting physical eventuality per se. Moreover, it is possible to demonstrate that every Modified Dispersion Relation of the form (4.5) can be approximated with a MDR of type (4.4). In fact, the homogeneous MDR (4.1), that preserves rotational symmetry[1], can be rewritten in the form:

$$\begin{aligned} MDR(p) &= F(p)^2 = m^2 \\ F(p) &= \sqrt{E^2 - (1 - f(x))\,|\overrightarrow{p}|^2} = \sqrt{E^2 - \left(1 - \sum_{k=1}^{\infty} \alpha_k x^k\right)|\overrightarrow{p}|^2} \end{aligned} \tag{4.6}$$

Now, using the fact that the two MDR forms (4.4) and (4.11) must be equal, the perturbations equality $h(p) = f(x)$ follows. From this equation it is possible to obtain a series expansion for p as a function of x. In literature, for most physical cases, the $h(p)$ series terminate at first or second order. In these eventualities, it is always possible to find an approximation series for $p(x)$. Finally, it is simple to

[1]That is where the g perturbation function is posed equal to 0.

obtain the relation:

$$f(x) = \sum_{k=1}^{\infty} \alpha_k x^k = h(p(x)) \tag{4.7}$$

that permits to fix the expansion coefficients α_k in order to satisfy the equation itself. This procedure is analogous to map a function of the variable $p = |\overrightarrow{p}|$ on a function of the variable $x = \left(\frac{|\overrightarrow{p}|}{E}\right)$, noting that x and p have a biunivocal correspondence.

In order to verify that a dispersion relation defined with the introduction of the perturbation f is well posed, it is necessary to verify that the energy solution of the equation, obtained using (4.2) and (4.1):

$$E^2 = p^2 \left(1 - \sum_{k=1}^{n} \alpha_k \left(\frac{p}{E}\right)^k\right) + m^2 \tag{4.8}$$

is positive for every n value. Dividing the previous equation by E^2, it becomes:

$$1 = \left(\frac{p}{E}\right)^2 \left(1 - \sum_{k=1}^{n} \alpha_k \left(\frac{p}{E}\right)^k\right) + \frac{m^2}{E^2} \tag{4.9}$$

in the very high energy scenario, that is taking the limit for $E \to \infty$, one obtains $\frac{m^2}{E^2} \to 0$ and finally the equation (4.9) takes the form:

$$1 = \left(\frac{p}{E}\right)^2 \left(1 - \sum_{k=1}^{n} \left(\frac{p}{E}\right)^k\right) \tag{4.10}$$

with the introduction of the new variable $x = \left(\frac{p}{E}\right) = \left(\frac{|\overrightarrow{p}|}{E}\right)$, the previous relation becomes:

$$x^2 \left(1 - P_n(x)\right) = 1 \tag{4.11}$$

where $P_n(x)$ is a n degree polynomial. If the magnitude of this polynomial remains limited, that is this function represents a tiny perturbation, compared to the magnitude of p, the solution of the equation is $x \simeq 1$. So posing the correct constrain on the coefficients of the series (4.2), one can obtain a real value energy E from equation (4.9) and:

$$\lim_{p \to \infty} \frac{p}{E} = 1 + \delta \tag{4.12}$$

for δ a tiny positive constant. Therefore it is possible to resort to homogeneous perturbation functions, under appropriate general assumptions.

Now, fixed the MDR general form it is possible to investigate the space-time induced modified geometrical structure.

4.3 The Finsler geometric structure of space-time

The momentum space Finsler pseudo-norm determines the metric of the same space. Resorting to the hamiltonian formalism it results possible to construct the space-time structure, starting from the modified momentum space geometry. This approach is compatible with [131], where starting from the modified momentum geometry, the Hamiltonian is built. The explicit metric form, defined in momentum space, is obtained using the relation:

$$\widetilde{g}(p)^{\mu\nu} = \frac{1}{2}\frac{\partial}{\partial p^{\mu}}\frac{\partial}{\partial p^{\nu}} F^2(E,\ \overrightarrow{p}) \tag{4.13}$$

It remains a non-diagonal part, which does not give any contribution in computing the dispersion relations. It can be therefore eliminated by an opportune "gauge" choice. The final form of the metric becomes therefore:

$$\widetilde{g}^{\mu\nu}(p) = \begin{pmatrix} 1 & 0 \\ 0 & -(1 - f(p/E))\mathbb{I}_{3\times 3} \end{pmatrix} \tag{4.14}$$

Consistently with standard relativity, the free massive particle *Hamiltonian* is defined using the modified metric (4.14) as:

$$\mathcal{H} = \sqrt{\widetilde{g}(p)^{\mu\nu} p_\mu p_\nu} = F(p) = MDR(p) \tag{4.15}$$

written using the MDR, that is a pseudo-Finsler norm. Starting from this function, it is possible to compute the velocity, correlated to the momentum, resorting to the *Legendre* transformation[2] as:

$$\dot{x}^{\mu} = \frac{1}{2}\left(\frac{\partial}{\partial p_\mu} F^2(p)\right) = \frac{\widetilde{g}(p)^{\mu\nu}\, p_\nu}{\sqrt{\widetilde{g}(p)^{\alpha\beta}\, p_\alpha\, p_\beta}} \tag{4.16}$$

The homogeneity of the metric has been used to neglect its derivative by the momentum, in fact this derivative can be written as:

$$\frac{\partial}{\partial x^{\mu}} g_{\alpha\beta} = -\frac{\partial}{\partial x^{\mu}} f(p(x)) = -\frac{\partial}{\partial p} f(p) \frac{\partial}{\partial x^{\mu}} p(x) \tag{4.17}$$

Every term magnitude results to be small ($|\partial_{x_\mu} p(x)| \ll 1$), since the geometrized interaction of a massive particle, with the quantum structure of space-time, is supposed tiny, even at high energies. Moreover $|\partial_p f(p)| \ll 1$ because of the form

[2] In appendix it is possible to find the demonstration that it is always possible to define the Legendre transformation in case of Finsler geometry. Important to underline that in this work one is dealing with a pseudo-Finsler structure, the results presented in appendix remain valid even in this case

of the perturbation function (4.2). In fact, at high energies:

$$\begin{aligned}\partial_{p^j} f(p) = \partial_{p^j} \sum_k \alpha_k \frac{|\overrightarrow{p}|^k}{E^k} = \partial_{p^j} \sum_k \alpha_k \frac{|\overrightarrow{p}|^k}{(\sqrt{|\overrightarrow{p}|^2 + m^2})^k} = \\ = \sum_k \left(\alpha_k k \frac{|\overrightarrow{p}|^{k-2} p_j}{(\sqrt{|\overrightarrow{p}|^2 + m^2})^k} - \alpha_k k \frac{|\overrightarrow{p}|^k p_j}{(\sqrt{|\overrightarrow{p}|^2 + m^2})^{k+2}} \right) \to 0\end{aligned} \tag{4.18}$$

where it has been used the equivalence $E \simeq \sqrt{|\overrightarrow{p}|^2 + m^2}$.
Now the pseudo-Finsler norm, written as function of the new vector, becomes:

$$G(\dot{x}(p)) = F(p) \tag{4.19}$$

and the associated metric is given by the relation:

$$g(x,\, \dot{x}(p))_{\mu\nu} = \frac{1}{2} \left(\frac{\partial^2 G}{\partial \dot{x}^\mu \, \partial \dot{x}^\nu} \right) \tag{4.20}$$

where $g_{\mu\nu}$ is the inverse of the previous (4.15) metric:

$$g(x,\, \dot{x}(p))_{\mu\alpha} \, \widetilde{g}(x,\, p)^{\alpha\nu} = \delta_\mu^{\ \nu} \tag{4.21}$$

and can be written as:

$$g(x,\, \dot{x}(p))_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -(1 + f(p/E))\mathbb{I}_{3\times 3} \end{pmatrix} \tag{4.22}$$

To define the structure of space-time the Lagrangian formulation is used. In order to obtain this function, it is possible to resort to the Legendre transformation. Starting from the Hamiltonian (4.15), it is possible to compute the explicit form of the *Lagrangian*, that is:

$$\mathcal{L} = \overrightarrow{p}\, \overrightarrow{\dot{x}} - \mathcal{H} = -\dot{x}^\mu \, p_\mu = \left(\frac{\partial}{\partial \dot{x}^\mu} \mathcal{L} \right) \dot{x}^\mu = -m \sqrt{\dot{x}^\mu \, g_{\mu\nu}(p) \, \dot{x}^\nu} \tag{4.23}$$

The geometric structure of the obtained space-time, permits to preserve the Hamilton-Jacobi equations structure. In fact the momentum takes the explicit form:

$$p_\mu = \left(-\frac{\partial}{\partial \dot{x}^\mu} \mathcal{L} \right) = \frac{m \, g_{\mu\nu} \, \dot{x}^\nu}{\sqrt{\dot{x}^\mu \, g_{\mu\nu}(p) \, \dot{x}^\nu}} \tag{4.24}$$

where again the homogeneity of the metric $g_{\mu\nu}$ (4.22) has been used to justify the neglecting of the metric derivative. The momentum satisfies the mass-shell condition:

$$\widetilde{g}^{\mu\nu}(p)\, p_\mu\, p_\nu = \widetilde{g}^{\mu\nu}\, \frac{m\, g_{\mu\alpha}(p)\, \dot{x}^\alpha}{\sqrt{\dot{x}^\mu\, g_{\mu\nu}(p)\, \dot{x}^\nu}}\, \frac{m\, g_{\nu\beta}(p)\, \dot{x}^\beta}{\sqrt{\dot{x}^\mu\, g_{\mu\nu}(p)\, \dot{x}^\nu}} = g_{\mu\nu}(p)\dot{x}^\mu \dot{x}^\nu = m^2 \tag{4.25}$$

that is the MDR relation (4.4).
Now it is necessary to deal with the obtained pseudo-Finsler metric structure of the space-time, introducing the Cartan formalism, that is resorting to the *vierbein* or *thetrad*.
Remembering that two *vierbein* are equivalent if they originate the same metric:[3]:

$$e^a_{\ \mu}(x)\, \eta_{ab}\, e^b_{\ \nu}(x) = g_{\mu\nu}(x) = e'^a_{\ \mu}(x)\, \eta_{ab}\, e'^b_{\ \nu}(x) \tag{4.26}$$

from now on *thetrad* elements equivalence classes will be considered, identifying every class with one representative. To originate the (4.14) metric, the *vierbein* must have the following expression:

$$\begin{aligned} e^\mu_{\ a}(p) &= \begin{pmatrix} 1 & \overrightarrow{0} \\ \overrightarrow{0}^t & \sqrt{1-f(p)}\, \mathbb{I}_{3\times 3} \end{pmatrix} \\ e^a_{\ \mu}(p) &= \begin{pmatrix} 1 & \overrightarrow{0} \\ \overrightarrow{0}^t & \frac{1}{\sqrt{1-f(p)}}\, \mathbb{I}_{3\times 3} \end{pmatrix} \simeq \begin{pmatrix} 1 & \overrightarrow{0} \\ \overrightarrow{0}^t & \sqrt{1+f(p)}\, \mathbb{I}_{3\times 3} \end{pmatrix} \end{aligned} \tag{4.27}$$

where an explicit dependence on the momentum magnitude has been intoduced.
To analyze the geometric structure it is now necessary to introduce the *affine* connection:

$$\Gamma^{\ \alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right) \tag{4.28}$$

[3] A simple example is $e'^a_{\ \mu}(x) = -e^a_{\ \mu}(x)$.

Following [123] and [124], it is simple to evaluate the explicit forms of the connection components, starting from the metric tensor (1.14):

$$\begin{aligned}
&\Gamma^{0}_{\mu 0} = \Gamma^{i}_{00} = \Gamma^{i}_{\mu\nu} = 0 \qquad \forall \mu \neq \nu \\
&\Gamma^{0}_{ii} = -\frac{1}{2}\partial_0 f(p) \simeq 0 \\
&\Gamma^{0}_{0i} = \Gamma^{0}_{i0} = \frac{1}{2(1+f(p))}\partial_0 f(p) \simeq 0 \\
&\Gamma^{i}_{ii} = \frac{1}{2(1+f(p))}\partial_i f(p) \simeq 0 \\
&\Gamma^{i}_{jj} = -\frac{1}{2(1+f(p))}\partial_i f(p) \simeq 0 \qquad \forall i \neq j \\
&\Gamma^{i}_{ij} = \Gamma^{i}_{ji} = \frac{1}{2(1+f(p))}\partial_i f(p) \simeq 0 \qquad \forall i \neq j
\end{aligned} \tag{4.29}$$

As usual convention in General Relativity, the previous equation latin indices indicate spatial tensor components (they are defined in the values set $\{1, 2, 3\}$), instead the greek ones indicate all the four space-time components (they variate inside $\{0, 1, 2, 3\}$). The previous equalities approximations are possible, since the derivative $|\partial_p f(p)|$ can be neglected. This results possible under the assumption of tiny interaction with the space-time background structure and thanks to the homogeneity of the perturbation functions (4.2) and to equations (4.18). Introducing the local covariant derivative as:

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^{\nu}_{\mu\alpha} v^\alpha \simeq \partial_\mu v^\nu \tag{4.30}$$

it is immediate to compute the *Cartan* or *spinorial* connection, as:

$$\omega_{\mu ab} = e_a^{\ \nu} \nabla_\mu e_{b\nu} \simeq e_a^{\ \nu} \partial_\mu e_{b\nu} \tag{4.31}$$

Applying the first Cartan structural equation:

$$de = e \wedge \omega \tag{4.32}$$

to the external forms

$$\begin{aligned}
e_0^\mu &= dx^\mu \\
e_i^\mu &= \sqrt{1-f(p)}\,dx^\mu
\end{aligned} \tag{4.33}$$

it follows that even for the spinorial connection, the not null elements are negligible:

$$\frac{1}{2}\epsilon_{ijk}\omega^{ij} = \frac{1}{2}\frac{1}{1-f}\epsilon_{ijk}(\partial^i f dx^j - \partial^j f dx^i) \tag{4.34}$$

because, as in the previous case, they are proportional to perturbation functions derivatives. From relation (4.34), it follows that the only components of the connection differential forms ω, that are not identically equal to zero, are:

$$\begin{aligned}
\omega^{12} &= -\frac{1}{2}\frac{1}{1-f}\left(\partial_y f dx - \partial_x f dy\right) \\
\omega^{13} &= -\frac{1}{2}\frac{1}{1-f}\left(\partial_z f dx - \partial_x f dz\right) \\
\omega^{23} &= -\frac{1}{2}\frac{1}{1-f}\left(\partial_z f dy - \partial_y f dz\right)
\end{aligned} \tag{4.35}$$

So, even for the Cartan connection, the not null coefficients are proportional to terms like (4.18), hence, the connection is asymptotically zero ($\omega_{\mu ab} \simeq 0$). The tensor total covariant derivative results therefore:

$$D_\mu v_a^{\ \nu} = \partial_\mu v_a^{\ \nu} + \Gamma^{\nu}_{\mu\alpha} v_a^{\ \alpha} - \omega^{\ a}_{\mu\nu} v_b^{\ \nu} \simeq \partial_\mu v_b^{\ \nu} \tag{4.36}$$

At the end it is possible to conclude that a massive particle interaction with the "quantized" space-time background, if supposed negligible, determines an asymptotically flat Finslerian structure.

4.4 Modified Lorentz Transformations

Using the *tetrad*, it is possible to construct the explicit form of the modified Lorentz group. The obtained representation preserves the form of the MDR and the homogeneity of degree 0 of the perturbation functions.
In literature [132] it is possible to find the general form of the Lorentz transformations for General Relativity, defined as:

$$\Lambda_\mu^{\ \nu}(x) e_\nu(x) = e_\mu(\Lambda x) \tag{4.37}$$

Resorting to the *vierbein* it is possible to define projection from a tangent (local) space, parameterized by the metric $g_{\mu\nu}(x,\, v)$ to another local space, identified by a different metric tensor $\overline{g}(x',\, v')_{\mu\nu}$ as summarized in the following graph:

$$\begin{array}{ccc}
(TM,\, \eta_{ab},\, v) & \xrightarrow[\Lambda]{\hspace{3em}} & (TM,\, \eta_{ab},\, v') \\
\Big\downarrow{\scriptstyle e(x)} & & {\scriptstyle \overline{e}(x')}\Big\downarrow \\
(T_x M,\, g_{\mu\nu}(x), v) & \xrightarrow[\overline{e}\circ\Lambda\circ e^{-1}]{\hspace{3em}} & (T_x M,\, \overline{g}_{\mu\nu}(x'), v')
\end{array}$$

From now on, the dependence of thetrad and the metric tensor will be generalized from the space-time coordinates (x) to the coordinates of the phase space $(x,\, p)$. In this way a dependence on the momenta is included, like in Finsler geometry [133]. The dependence on the position is supposed trivial [123, 124] and therefore will be neglected to preserve the space homogeneity. Only the dependence on the momenta (velocities) is maintained. All the physical quantities are therefore generalized, acquiring an explicit dependence on the momenta. The graph of the transition from one tangent (local) space to the other becomes:

$$\begin{tikzcd} (TM,\, \eta_{ab},\, p) \arrow[rr, "\Lambda"'] \arrow[d, "e(p)"] & & (TM,\, \eta_{ab},\, p') \arrow[d, "\overline{e}(p')"'] \\ (T_xM,\, g_{\mu\nu}(p)) \arrow[rr, "\overline{e}\circ\Lambda\circ e^{-1}"'] & & (T_xM,\, \overline{g}_{\mu\nu}(p')) \end{tikzcd}$$

where is indicated the explicit dependence of the metric from momenta.
Using the vierbein to transform a latin (global index) in a greek one (local index), it is possible to write:

$$g_{\mu\nu}(p) = e^a_{\ \mu}(p)\,\eta_{ab}\,e^b_\nu(p) = e^a_{\ \mu}(p)\,\Lambda_a^{\ c}\,\eta_{cd}\,\Lambda_b^{\ d}\,e^b_\nu(p) \tag{4.38}$$

From the previous equation it follows:

$$\Lambda_a^{\ c} e^a_{\ \mu}(p) = e^c_{\ \mu}(p) \tag{4.39}$$

This result permits to correlate the global Lorentz transformations with the vierbein elements, indicating how a thetrad element transform under such transformations.
Now, using again the vierbein to transform local to global indices, it is possible to define the general modified Lorentz transformation as:

$$\Lambda_\mu^{\ \nu}(p) = e^a_{\ \mu}\,(\Lambda p)\Lambda_a^{\ b}\,e^b_{\ \nu}(p) \tag{4.40}$$

Using the equation (4.37), with the substitution of coordinate with momentum, and equation (4.40), it is possible to write:

$$\Lambda_\mu^{\ \nu}(p)e^a_{\ \nu}(p) = \underbrace{e_\mu^{\ b}(\Lambda p)\Lambda_b^{\ c}e_c^{\ \nu}(p)}_{\Lambda_\mu^{\ \nu}(p)}\,e^a_{\ \nu}(p) = e_\mu^{\ b}(\Lambda p)\delta_b^{\ a} = e_\mu^{\ a}(\Lambda p) \tag{4.41}$$

where in the last equality relation (4.39) has been used.
Considering the MDR (4.3) and remembering the momenta space metric is given by (4.14), it is now possible to verify that the Modified Lorentz Transformations (MLT) are isometries for the Modified Dispersion Relation, that is $MDR(\Lambda P) =$

$MDR(p)$. In fact:

$$MDR(\Lambda p) = \Lambda_{\mu}^{\ \alpha} p_\alpha g^{\mu\nu}(\Lambda p) \Lambda_{\nu}^{\ \beta} p_\beta = \underbrace{e^{a}_{\ \mu}(\Lambda p) \Lambda_{a}^{\ b} e_{b}^{\ \alpha}(p)}_{\Lambda_{\mu}^{\ \nu}(p)} \, p_\alpha \, g^{\mu\nu}(\Lambda p)$$
$$\underbrace{e^{c}_{\ \nu}(\Lambda p) \Lambda_{c}^{\ d} e_{d}^{\ \beta}(p)}_{\Lambda_{\nu}^{\ \beta}(p)} \, p_\beta = e^{a}_{\ \mu}(\Lambda p) \Lambda_{a}^{\ b} \, p_b \, g^{\mu\nu}(\Lambda p) \, e^{c}_{\ \nu}(\Lambda p) \Lambda_{c}^{\ d} p_d = \tag{4.42}$$
$$= \Lambda_{a}^{\ b} p_b \, \eta^{ac} \, \Lambda_{c}^{\ d} \, p_d = p_a \eta^{ab} \, p_b = e^{a}_{\ \mu}(p) \, p_a \, g^{\mu\nu}(p) e^{b}_{\ \nu}(p) \, p_b = MDR(p)$$

where the equalities (4.40) and (4.41), obtained before, have been used.
The Modified Lorentz Transformations, introduced in (4.40), are therefore the isometries of the Modified Dispersion Relations.
Moreover the amended Lorentz group transformations (4.40) acting on the 4-vector $p^\mu = (E, \overrightarrow{p})$, give, for the modification function f in the MDR:

$$f\left(\frac{|\overrightarrow{p}|}{E}\right) \rightarrow f\left(\frac{|\Lambda^{i}_{\ \mu}(p) p^\mu|}{\Lambda^{0}_{\ \mu}(p) p^\mu}\right) \tag{4.43}$$

It is simple to verify that this kind of transformations preserve the homogeneity of degree 0, because of the ratio present in the definition of the modification function f. Therefore the action of the modified Lorentz group preserves the homogeneity of the perturbation function f, preserving the MDR (4.4) form.

4.5 Relativistic Invariant Energy (Mandelstam variables)

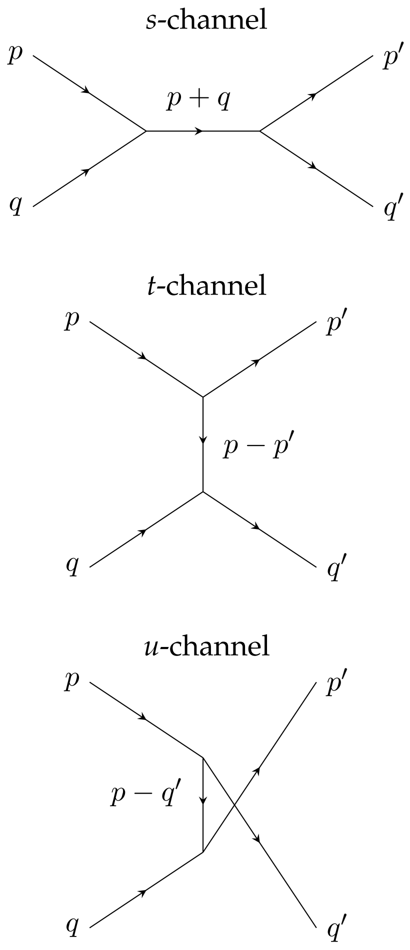
In this model every particle species has its own metric, with a personal maximum attainable velocity. Moreover every species presents its personal Modified Lorentz Transformations (MLT), which are the isometries for the Modified Dispersion Relation of the particle. The new physics, caused by LIV, emerges only in the interaction of two different species. That is every particle type physics is modified in a different way by the Lorentz symmetry violation. Therefore, to analyze the interaction of two particles, it is necessary to determine how the reaction invariants - that is the Mandelstam relativistic invariants - are modified.
Starting from the hypothesis of MDR, generated by a metric in the momentum space, it is necessary to resort to the *vierbein* to project the particles momenta on the Minkowski tangent space:

$$MDR(p) = p_\mu \, g^{\mu\nu}(p) \, p_\nu = p_\mu \, e_{a}^{\ \mu}(p) \, \eta^{ab} \, e_{b}^{\ \nu}(p) \, p_\nu = p_a \, \eta^{ab} \, p_b \tag{4.44}$$

For this reason it seems natural to generalize the definition of internal product of the sum of two different particle species momenta as:

$$\langle p+q|p+q\rangle = (p_\mu\, e^\mu_a(p) + q_\mu\, \tilde{e}^\mu_a(q))\, \eta^{ab}\, (p_\nu\, e^\nu_b(p) + q_\nu\, \tilde{e}^\nu_b(q)) \tag{4.45}$$

where with e is indicated the thetrad related to the first particle and with $\tilde{e}$ the vierbein related to the second one. With this internal product it is now possible to define the Mandelstam variables s, t and u, remembering that:



and considering the p and q momenta as belonging to different particle species. If the two interacting particles belong to the same species, the internal product and therefore the Mandelstam variables present no differences from standard Physics. That is the momenta of particles of the same kind, live in the same tangent (local) space, constructed with the Finsler metric. Instead, if the particles belong to different species, the definition of the internal product requires the necessity to correlate different local tangent spaces.

The new internal product can be generated introducing the concept of a generalized metric, written as:

$$G = \begin{pmatrix} g^{\mu\nu}(p) & e^{a\mu}(p)\tilde{e}_a^{\ \beta}(q) \\ \tilde{e}^{a\alpha}(q)e_a^{\ \nu}(p) & \tilde{g}^{\alpha\beta}(q) \end{pmatrix} \tag{4.46}$$

The Modified Lorentz Transfomations for this metric assumes the explicit form:

$$\Lambda = \begin{pmatrix} \Lambda_\mu^{\mu'} & 0 \\ 0 & \tilde{\Lambda}_\alpha^{\alpha'} \end{pmatrix} \tag{4.47}$$

using the MLT of the two particle species. The internal product (4.45) can be obtained as:

$$\begin{aligned} \langle p+q|p+q\rangle = \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} g^{\mu\nu}(p) & e^{a\mu}(p)\tilde{e}_a^{\ \beta}(q) \\ \tilde{e}^{a\alpha}(q)e_a^{\ \nu}(p) & \tilde{g}^{\alpha\beta}(q) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \\ = p_\mu\, g^{\mu\nu}(p)\, p_\nu + p_\mu\, e^{a\mu}\tilde{e}_a^{\ \beta}(q)\, q_\beta + q_\alpha\, \tilde{e}_{a\alpha}(q)\, e_a^{\ \nu}(p)\, p_\nu + q_\alpha \tilde{g}^{\alpha\beta}(q)\, q_\beta \end{aligned} \tag{4.48}$$

The new introduced MLT (4.47) are the isometries of the Mandelstam variables, in the same way as the MLT (4.40) are the isometries for the MDR for every particle species:

$$\begin{aligned} \langle p+q|p+q\rangle = \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} g^{\mu\nu}(p) & e_{a\mu}(p)\tilde{e}_a^{\ \beta}(q) \\ \tilde{e}^{a\alpha}(q)e_a^{\ \nu}(p) & \tilde{g}^{\alpha\beta}(q) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \\ = \langle \Lambda(p+q)|\Lambda(p+q)\rangle = \left[\begin{pmatrix} \Lambda_{\mu'}^{\ \mu} & 0 \\ 0 & \tilde{\Lambda}_{\alpha'}^{\ \alpha} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \right] \cdot \\ \begin{pmatrix} g^{\mu'\nu'}(\Lambda p) & e^{a\mu'}(\Lambda p)\tilde{e}_a^{\ \beta'}(\tilde{\Lambda} q) \\ \tilde{e}^{a\alpha}(\tilde{\Lambda} q)e_a^{\ \nu'}(\Lambda p) & \tilde{g}^{\alpha'\beta'}(\tilde{\Lambda} q) \end{pmatrix} \cdot \left[\begin{pmatrix} \Lambda_{\nu'}^{\ \nu} & 0 \\ 0 & \tilde{\Lambda}_{\beta'}^{\ \beta} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \right] \end{aligned} \tag{4.49}$$

where equation (4.40) has been repeatedly used and the internal product is defined using the (4.14) momentum depending metric tensor.
The complete physical description of interactions can be made using the formalism of the S matrix, which results to be an analytic function of the Madelstam variables. Since these quantities result covariant, respect to the amended Lorentz transformations (MLT), the concept of isotropy is restored. In this way the necessity of introducing a privileged class of inertial observers disappears.
In case of composition of three particles momenta, each of different type, the internal product (4.45) can be constructed by an analogous process. In fact it is possible to introduce a new metric, that integrates the different types of vierbeins

and metrics relative to the three particles species:

$$G = \begin{pmatrix} g^{\mu\nu}(p_1) & e^{a\mu}(p_1)\widetilde{e}_a^{\ \beta}(p_2) & e^{a\mu}(p_1)\overline{e}_a^{\ \rho}(p_3) \\ \widetilde{e}^{a\alpha}(p_2)e_a^{\ \nu}(p_1) & \widetilde{g}^{\alpha\beta}(p_2) & \widetilde{e}^{a\alpha}(p_2)\overline{e}_a^{\ \rho}(p_3) \\ \overline{e}^{a\theta}(p_3)e_a^{\ \nu}(p_1) & \overline{g}^{a\theta}(p_3)\widetilde{e}_a^{\ \beta}(p_2) & \overline{g}^{\theta\rho}(p_3) \end{pmatrix} \tag{4.50}$$

where g and e are related to the first particle species, $\widetilde{g}$ and $\widetilde{e}$ are related to the second one and $\overline{g}$ and $\overline{e}$ to the third one. All the processes, introduced for the two particles interaction, can be generalized in this way for generic n-particles (n-momenta) interactions.

4.6 Standard Model modifications

As underlined in [123, 124], the introduction of a deformed geometry, influences the form of the Dirac equation, with the result of modifying spinors and correlated currents. The deformed Dirac matrices can be computed, requiring that they satisfy the Clifford Algebra relation:

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\, g^{\mu\nu}(p) = 2\, e_\mu^{\ a}(p)\, \eta_{ab}\, e_\nu^{\ b}(p) \tag{4.51}$$

from which it is simple to obtain the equality:

$$\Gamma^\mu = e_a^{\ \mu}(p)\, \gamma^a \tag{4.52}$$

From the previous equation it is immediate to compute the modified Gamma matrices explicit forms:

$$\begin{aligned} &\Gamma_0 = \gamma_0 \qquad \Gamma_i = \frac{1}{\sqrt{1 - f(p(x,\, \dot{x}))}}\, \gamma_i \simeq \sqrt{1 + f(p(x,\, \dot{x}))}\, \gamma_i \\ &\Gamma^0 = \gamma^0 \qquad \Gamma^i = \sqrt{1 - f(p(x,\, \dot{x}))}\, \gamma^i \end{aligned} \tag{4.53}$$

The Γ_5 matrix can be introduced using the total antisymmetric tensor $\epsilon_{\mu\nu\alpha\beta}$, defined in curved space-time:

$$\begin{aligned} \Gamma_5 = &\frac{\epsilon^{\mu\nu\alpha\beta}}{4!}\Gamma_\mu\Gamma_\nu\Gamma_\alpha\Gamma_\beta = \frac{1}{\sqrt{\det g}}\Gamma_0\Gamma_1\Gamma_2\Gamma_3 = \\ = &\frac{1}{\sqrt{\det g}}\sqrt{\det g}\, \gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_5 \end{aligned} \tag{4.54}$$

As consequence the new constructed geometry preserves the standard chirality classifications of particles.
To determine the explicit form of spinors and associated conserved currents, first

it is essential to define the modified Dirac equation:

$$(i\Gamma^\mu \partial_\mu - m)\,\psi = 0 \tag{4.55}$$

From this equation, following a standard argumentation, present in literature [134], it i possible to obtain the modified spinors. Posing the possibility to develop the general spinor in plane waves:

$$\begin{aligned} \psi^+(x) &= u_r(p)e^{-ip_\mu x^\mu} \\ \psi^-(x) &= v_r(p)e^{ip_\mu x^\mu} \end{aligned} \tag{4.56}$$

and taking into account only the positive energy one (for the negative one the computation retains the same form), the modified spinors can be easily computed from the associated Dirac equation in momentum space. Applying this equation to the generic positive energy spinor, it is possible to obtain:

$$(i\Gamma^\mu \partial_\mu - m)u_r(p)e^{-ip_\mu x^\mu} \Rightarrow (\not{p} - m)u_r(p) = 0 \tag{4.57}$$

and the associated identity for spinors with null momentum $\overrightarrow{p} = 0$:

$$\begin{aligned} &(\not{p} - m)(\not{p} + m) = (p^\mu p_\mu) - m^2 = 0 \;\Rightarrow \\ &\Rightarrow (\not{p} - m)(\not{p} + m)u_r(m,\ \overrightarrow{0}) = 0 \end{aligned} \tag{4.58}$$

From this relation follows that generic momenta $\overrightarrow{p}$ spinors can be obtained from those with null momenta. From this statement, the possibility to compute modified positive energy not normalized spinor immediately follows. Starting from the null momentum positive energy spinor standard representation:

$$u_r(m,\ \overrightarrow{0}) = \chi_r = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.59}$$

it is simple to compute the generic spinor from the relation:

$$\begin{aligned} &(\Gamma^\mu p_\mu + m)\begin{pmatrix} \chi_r \\ 0 \end{pmatrix} \Rightarrow \\ &\Rightarrow \left(p^0 \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} - p^i \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \sqrt{1-f}\right) \begin{pmatrix} \chi_r \\ 0 \end{pmatrix} + \\ &+m \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} \chi_r \\ 0 \end{pmatrix} = \begin{pmatrix} (E+m)\chi_r \\ \overrightarrow{p}\,\overrightarrow{\sigma}\sqrt{1-f}\;\chi_r \end{pmatrix} \end{aligned} \tag{4.60}$$

and finally the modified spinor normalized form can be written as:

$$\begin{aligned}&\begin{pmatrix}(E+m)\chi_r\\ \vec{p}\,\vec{\sigma}\sqrt{1-f}\,\chi_r\end{pmatrix}\Rightarrow\\ &\Rightarrow u_r(m,\,\vec{p})=\frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix}(E+m)\,\chi_r\\ \vec{p}\,\vec{\sigma}\,\sqrt{1-f}\,\chi_r\end{pmatrix}\end{aligned}\tag{4.61}$$

Having defined the modified spinors from the plane-waves expansion, it is now possible to verify its compatibility with the MDR, in fact:

$$\begin{aligned}&(i\Gamma^{\mu}\partial_{\mu}+m)\,(i\Gamma^{\nu}\partial_{\nu}-m)\,u(p)e^{ip_{\mu}x^{\mu}}\Rightarrow\\ &\Rightarrow\left(\frac{1}{2}\{\Gamma^{\mu},\Gamma^{\nu}\}p_{\mu}p_{\nu}-m^2\right)u(p)=0\Rightarrow\\ &\Rightarrow\left(p_{\mu}p_{\nu}g^{\mu\nu}-m^2\right)u(p)=0\Rightarrow\\ &\Rightarrow E^2-|\vec{p}|^2(1-f(p))-m^2=0\end{aligned}\tag{4.62}$$

This proves that the free propagation of the introduced modified spinors is governed by the MDR (4.4).
To describe a physical theory like QED or SM (weak sector and QCD), it is essential to deal with interaction terms, that are described using conserved currents. Therefore it is necessary to introduce the theory modified currents. Starting from the simpler case of QED, the current must be defined as a spinor bilinear, in order to be contracted with the boson gauge vectorial field of the theory. Moreover spinorial bilinear and gauge boson field must live in the same tangent space, to permit this contraction. In this way the introduced theory contemplates a kinematical modification, but not a dynamical one. That is the new aspects are limited to the kinematics of the free particles, without modifying the known interactions. This can be achieved by the introduction of the generalized $\widetilde{\Gamma}$ matrices:

$$\widetilde{\Gamma}_{\mu}(p',\,p)=\begin{pmatrix}0 & \sigma_a e^{a}_{\ \mu}(p')\\ \overline{\sigma}_a e^{a}_{\ \mu}(p) & 0\end{pmatrix}\tag{4.63}$$

and the consequent modified current is given by:

$$J_{\mu}=e\sqrt{|\det[\widetilde{g}]|}\;\overline{\psi}\,\widetilde{\Gamma}_{\mu}(p,\,p')\,\psi\tag{4.64}$$

where e is the coupling constant (the electric charge), p represents the incoming spinor field momentum and p' the outgoing one, and the generalized metric has been introduced:

$$\{\widetilde{\Gamma_{\mu}}(p,\,p'),\,\widetilde{\Gamma_{\nu}}(p,\,p')\}=2\,\widetilde{g}_{\mu\nu}(p,\,p')\tag{4.65}$$

In this way, the LIV corrections present in the modified spinors compensate the modified matrix ones. The current lives in the tanget space $(TM,\, \eta_{\mu\nu})$ and therefore it is possible to write:

$$J^\mu = \eta^{\mu\nu}\, J_\nu \tag{4.66}$$

The interaction term can therefore be written as:

$$\mathcal{L}_{inter} = e\sqrt{|\det[\widetilde{g}]|}\;\; \overline{\psi}\, \widetilde{\Gamma}_\mu(p,\, p')\, \psi\, \overline{e}^\mu_\nu\, A^\nu \tag{4.67}$$

where $\overline{e}$ represents the *vierbein* correlated to the gauge field and the index μ, even if greek, represents a coordinate of the Minkowski space-time $(TM,\, \eta_{\mu\nu})$. The term, that in (4.64) and (4.67) multiplies the conserved current, is a generalization of the analogous term borrowed from curved space-time QFT, where its explicit form is given by: $\sqrt{|\det[g]|}$ [135]. With the previous definitions it is possible now to write the interaction Lagrangians of the LIV perturbed theories, that become for the QED:

$$\mathcal{L} = \sqrt{|\det[g]|}\;\; \overline{\psi}(i\Gamma^\mu\partial_\mu - m)\psi + e\sqrt{|\det[\widetilde{g}]|}\;\; \overline{\psi}\, \widetilde{\Gamma}_\mu(p,\, p')\, \psi\, \overline{e}^\mu_\nu\, A^\nu \tag{4.68}$$

where $\widetilde{g}(p)$ is obtained in (4.65) and $g(p)$ represents the metric computed in (4.14) that coincides with that used in (4.51).
It is important to underline that the modified Dirac matrices (4.52) are used to write the kinetic part of the Lagrangian, describing the free fermion propagation. This part determines the form of the propagator of the particle and therefore the dispersion relation and in particular the MDR (4.4). In the low energy scenario the perturbations result negligible. Instead, in case of high energy limit, it is possible to consider incoming and outgoing momenta with approximately the same magnitude, even after interaction. Therefore $\widetilde{\Gamma}$ matrices do not depend on the momenta and admit a constant form high energy limit. The definition of the current (4.64) reduces, as in [123, 124], to:

$$J_\mu = e\sqrt{|\det 1/2\{\Gamma_\mu,\, \Gamma_\nu\}|}\;\; \overline{\psi}\, \Gamma_\mu\, \psi = e\sqrt{|\det[g]|}\;\; \overline{\psi}\, \Gamma_\mu\, \psi \tag{4.69}$$

because $\widetilde{\Gamma}_\mu(p,\, p') \;\rightarrow\; \Gamma_\mu$ if $p \simeq p'$ and therefore $\widetilde{g}_{\mu\nu}(p,\, p') \;\rightarrow\; g_{\mu\nu}(p)$. Since the perturbation magnitude is supposed tiny and its effects are visible only for high energies, the last formulation can be considered as the main one. Moreover it is a reasonable physical hypothesis to suppose the quantum effects, caused by the interaction with the background, tiny for massless particles, it is possible to neglect this contribution for the gauge field A_μ. In this way the gauge field results Lorentz invariant, and preserves even the gauge symmetry. The Lagrangian becomes:

$$\mathcal{L} = \sqrt{\det[g]}\; \overline{\psi}(i\Gamma^\mu D_\mu - m)\psi \tag{4.70}$$

with the theory local covariant derivative defined as:

$$D_\mu = \partial_\mu - ie\,A_\mu \tag{4.71}$$

Important to stress that $\sqrt{\det[g]}\;\overline{\psi}\,\Gamma^\mu\,\psi = \sqrt{\det[g]}\;\overline{\psi}\,\Gamma_\nu\,\psi\,\eta^{\mu\nu}$, thanks to (4.66), because the current is defined in a Minkowskian space-time. In fact the spinor corrections cancel the gamma matrices ones, or only negligible corrections survive, permitting to suppose that the current itself lives in a flat space-time.
The same generalization can be applied to the SM Lagrangian. In fact, using the chirality projectors, it is possible to define, in the usual way, the left and right-hand component for every particle field:

$$\begin{aligned} \psi_L = P_L\,\psi = \frac{1}{2}\,(\mathbb{I} - \Gamma_5)\;\psi = \frac{1}{2}\,(\mathbb{I} - \gamma_5)\;\psi \\ \psi_R = P_R\,\psi = \frac{1}{2}\,(\mathbb{I} + \Gamma_5)\;\psi = \frac{1}{2}\,(\mathbb{I} + \gamma_5)\;\psi \end{aligned} \tag{4.72}$$

where the equality $\gamma_5 = \Gamma_5$ has been used.
The left-handed neutrino-lepton $(\nu - l)$ flavor f doublets can opportunely be defined as:

$$L_L^f = \begin{pmatrix} \nu_L^f \\ l_L^f \end{pmatrix} = \left(\begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_L^\mu \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_L^\tau \\ \tau_L \end{pmatrix} \right) \tag{4.73}$$

the right-handed leptons:

$$R^f = (l^f)_R = (e_R,\, \mu_R,\, \tau_R) \tag{4.74}$$

Analogously one can introduce the left-handed quark up-down $(u - d)$ flavor f doublets as:

$$Q_L^f = \begin{pmatrix} u_L^f \\ d_L^f \end{pmatrix} = \left(\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right) \tag{4.75}$$

and the right-handed up-down $(u - d)$ quark as:

$$\Big((u^f)_R,\; (d^f)_R\Big) \tag{4.76}$$

Starting from the leptonic part, the weak interaction Lagrangian can be written, starting from the free propagation part as:

$$\mathcal{L}_{free} = \sqrt{\det[g(L^f)]}\;\Big(i\,\overline{L}^f\,\Gamma(L^f)^\mu\,\partial_\mu\,L^f\Big) + \sqrt{\det[g(R^f)]}\;\Big(i\,\overline{R}^f\,\Gamma(R^f)^\mu\,\partial_\mu\,R^f\Big) \tag{4.77}$$

where the modified Γ matrices (4.52) have been used and are defined by the particle species of the interaction considered. Therefore they depends on the left-handed doublet or on the right-handed lepton flavor.
The neutral current interaction term can be written as:

$$\begin{aligned}\mathcal{L}_{n.c.} =&\sqrt{|\det \widetilde{g}(\psi^f)|}\,\Big(i\,\overline{\psi}^f\,g_0\,\frac{Y}{2}\,\widetilde{\Gamma}(\psi)^\mu\,\psi^f\,\widetilde{e}(B)_\mu^{\;\nu}\,B_\nu+\\ &+i\,\overline{\psi}^f\,g_1\,\frac{\tau^0}{2}\,\widetilde{\Gamma}(\psi)^\mu\,\psi^f\,\widetilde{e}(W^0)_\mu^{\;\nu}\,W^0_\nu\Big)\end{aligned} \tag{4.78}$$

where ψ represents every leptonic field, both left and right-handed, g_0 and g_1 are the coupling constants, Y and τ^0 are the usual matrices, in diagonal form, correlated respectively with the $U(1)$ and the $SU(2)$ gauge symmetries. $e(B)_\mu^{\nu}$ and $e(W^0)_\mu^{\nu}$ are the vierbein correlated respectively with the gauge fields B_μ and W^0_μ. As for QED, in order to guarantee that the neutral currents live in the tangent space $(TM,\,\eta_{\mu\nu})$, the interaction terms must be written using the generalized $\widetilde{\Gamma}$ matrices (4.63), that depend on the particle ψ.
The Lagrangian charged current interaction term instead acquires the explicit form:

$$\mathcal{L}_{c.c.} = g_1\,\sqrt{|\det \widetilde{g}(L^f)|}\,\Big(i\,\overline{L}^f\,\widetilde{\Gamma}(L^f)^\mu\,\tau^+\,L^f\,\widetilde{e}(W^+)_\mu^{\;\nu}\,W^+_\nu\Big)+h.c. \tag{4.79}$$

where the matrices $\tau^+ = \frac{1}{2}(\tau^1+i\tau^2)$ and $\tau^- = \frac{1}{2}(\tau^1-i\tau^2)$ are correlated respectively to the gauge fields $W^+ = \frac{1}{2}(W^1-iW^2)$ and $W^- = \frac{1}{2}(W^1+iW^2)$ and are again in diagonal form. The $\widetilde{\Gamma}$ matrices (4.63), depending from the field L^f, have been used to define the interaction terms, for the same reason illustrated before. As for the QED case, considering the high energy limit, when interaction term incoming and outgoing momenta are approximately the same, the generalized $\widetilde{\Gamma}$ matrices (4.63) reduce to the modified Γ (4.52). Moreover, supposing again the coupling of the gauge fields with the background negligible, the interaction preserves the gauge symmetry. Therefore it is possible to write the Standard Model leptonic Lagrangian as:

$$\begin{aligned}\mathcal{L}_{lept} =&\sqrt{|\det[g(L^f)]|}\,\Big(\overline{L}^f\,i\,\Gamma^\mu\,D_\mu\,L^f\Big)+\sqrt{|\det[g(R^f)]|}\,\Big(\overline{R}^f\,i\,\Gamma^\mu\,D_\mu\,R^f\Big)=\\ =&\sqrt{|\det[g^{(f)}]|}\,\Big(\overline{L}^f\,i\,\Gamma^\mu\,D_\mu\,L^f+\overline{R}^f\,i\,\Gamma^\mu\,D_\mu\,R^f\Big)\end{aligned} \tag{4.80}$$

introducing the $SU(2)\times U(1)$ covariant derivative D_μ:

$$D_\mu = \partial_\mu - ig_0\frac{Y}{2}B_\mu - ig_1\frac{\tau^i}{2}W^i_\mu \tag{4.81}$$

and posing the Γ matrices and the metric g dependents only on the particle flavor and not on the particle chirality.
The quark sector weak interaction Lagrangian can be written in an similar fashion, the free propagation term is given by:

$$\begin{aligned}\mathcal{L}_{free} =& \sqrt{\det\left[g(Q^f_L)\right]}\left(i\,\overline{Q}^f_L\,\Gamma(Q^f_L)^\mu\,\partial_\mu\,Q^f_L\right)+\sqrt{\det\left[g(u^f_R)\right]}\left(i\,\overline{u}^f_R\,\Gamma(u^f_R)^\mu\,\partial_\mu\,u^f_R\right)+\\ &+\sqrt{\det\left[g(d^f_R)\right]}\left(i\,\overline{d}^f_R\,\Gamma(d^f_R)^\mu\,\partial_\mu\,d^f_R\right)=\\ &=\sqrt{|\det\left[g^{(f)}\right]|}\left(\overline{Q}^f\,i\,\Gamma^\mu_{(f)}\,\partial_\mu\,Q^f+\overline{u}^f_R\,i\Gamma^\mu_{(f)}\,\partial_\mu\,u^f_R+\overline{d}^f_R\,i\,\Gamma^\mu_{(f)}\,\partial_\mu\,d^f_R\right)\end{aligned} \tag{4.82}$$

with the Γ matrices, and consequently the metric, depending only on the quark flavor and being equal for the same doublet left-handed quarks.
The neutral current interaction term can be written again as in (4.78) with the fields ψ that represents left and right-hand quarks. To write the charged current term one must take into account that this interaction is not diagonal in the chosen quark fields basis. The explicit form of this term becomes:

$$\mathcal{L}_{c.c.} = g_1\sqrt{|\det\widetilde{g}_{(fg)}|}\,\left(i\,\overline{Q}^f\,\widetilde{\Gamma}^\mu_{(fg)}\,T^i_{fg}\,Q^g\,\widetilde{e}(W^i)^\nu_\mu\,W^i_\nu+h.c.\right) \tag{4.83}$$

where T^i_{fg} are the interaction matrices correlated to the gauge fields W^i with $i = \pm$. The $\widetilde{\Gamma}$ matrices have been generalized to take into account the not diagonal coupling of quark doublets, and are defined as:

$$\widetilde{\Gamma}^\mu_{(fg)} = \begin{pmatrix} 0 & \sigma^a\,e^{\ \mu}_{a(f)}(p) \\ \overline{\sigma}^a\,e^{\ \mu}_{a(g)}(p') & 0\end{pmatrix} \tag{4.84}$$

where the vierbein $e^{\ \mu}_{a(f)}(p)$ is correlated to a quark doublet of flavor f, with momentum p. The generalized metric, generated by these modified matrices, takes the form:

$$\{\widetilde{\Gamma}^\mu_{(f)}(p),\,\widetilde{\Gamma}^\nu_{(g)}(p')\} = 2\,\widetilde{g}^{\mu\nu}_{(fg)}(p,\,p') \tag{4.85}$$

This metric again defines the space-time where the interaction takes place, that is where the conserved current propagates and the interaction vertex is defined.
Even for quark sector, the high energy limit can be treated considering that the $\widetilde{\Gamma}_{(fg)}$ matrices tend to a constant form, not depending on the momenta. They maintain only the dependence on the doublet flavor correlated. This permits to write this interaction term as:

$$\mathcal{L}_{c.c.} = g_1\sqrt{|\det g_{(fg)}|}\,\left(i\,\overline{Q}^f\,\Gamma^\mu_{(fg)}\,T^i_{fg}\,Q^g\,W^i_\mu\right) \tag{4.86}$$

again supposing the perturbation effects, correlated to the gauge fields, negligible. It is interesting to underline that it is possible to rewrite the modified Γ matrices with the form:

$$\Gamma^{\mu}_{(fg)} = c^{\mu\nu}_{(fg)}\gamma_{\nu} \tag{4.87}$$

as in [26], to show that the high energy limit corresponds to a redefinition of the metric, as in the cited work. Even the quark sector weak interaction Lagrangian can be written using the $SU(2) \times U(1)$ gauge covariant derivative D_μ (4.81), obtaining an explicit form, similar to eq. (4.80).
Following the same methodology used till now, it is possible to modify even the strong interaction Lagrangian, obtaining for the interaction term:

$$\mathcal{L}_{strong} = g_s \sqrt{|\det[\widetilde{g}(p,\, p')]|}\, \left(i\, \overline{Q}^{(f)}_i\, \widetilde{\Gamma}^{\mu}(p,\, p')\, t^a_{ij}\, Q^{(f)}_j\, G^a_{\mu} \right) \tag{4.88}$$

with t^a indicating the matrix form of the generators of $SU(3)$ gauge symmetry group, with i and j representing the colour indices of the quark fields and g_s the strong coupling constant. Even in this case the Lagrangian can be rewritten in the simpler form:

$$\mathcal{L}_{strong} = g_s \sqrt{|\det[g]|}\, \left(i\, \overline{Q}^{(f)}_i\, \Gamma^{\mu}\, t^a_{ij}\, Q^{(f)}_j\, G^a_{\mu} \right) \tag{4.89}$$

again using the constant high energy limit of the $\widetilde{\Gamma}$ matrices and the $\widetilde{g}$ metric.
The last Lagrangian part to be amended remains the gauge free propagation fields terms. This part can be modified in a similar way as done in [26] and can be written as:

$$\begin{aligned}\mathcal{L}_{gauge} =& \frac{1}{4}\, g^{(G)}_{\mu\nu}\, g^{(G)}_{\alpha\beta}\, Tr(G^{\mu\alpha}G^{\nu\beta}) \,+\, \frac{1}{4}\, g^{(W)}_{\mu\nu}\, g^{(W)}_{\alpha\beta}\, Tr(W^{\mu\alpha}W^{\nu\beta}) + \\ &+ \frac{1}{4}\, g^{(ph)}_{\mu\nu}\, g^{(ph)}_{\alpha\beta}\, B^{\mu\alpha}B_{\nu\beta}\end{aligned} \tag{4.90}$$

where the metric $g^{f}_{\mu\nu}$ depends on the gauge field f species considered, and $\{G_{\mu\nu}$, $W_{\mu\nu}$, $B_{\mu\nu}\}$ represent the gauge fields strength. The similarity with [26] is given by the tensor that appears in the perturbation term $k^{(f)}_{\mu\nu\alpha\beta} = g^{(f)}_{\mu\nu}\, g^{(f)}_{\alpha\beta} - \eta_{\mu\nu}\eta_{\alpha\beta}$. Supposing the gauge field interaction with the background negligible, this Lagrangian term reduces to the standard form:

$$\mathcal{L}_{gauge} = \frac{1}{4}\, Tr(G^{\mu\nu}G_{\mu\nu}) + \frac{1}{4}\, Tr(W^{\mu\nu}W_{\mu\nu}) + \frac{1}{4} B^{\mu\nu}B_{\mu\nu} \tag{4.91}$$

Finally, the complete formulation of the amended Standard Model formulation can be simplified, introducing the $SU(3) \times SU(2) \times U(1)$ covariant derivative D_μ:

$$D_\mu = \partial_\mu - ig_0 \frac{Y}{2} B_\mu - ig_1 \frac{\tau^i}{2} W^i_\mu - ig_s t^i G^i_\mu \tag{4.92}$$

and resorting to the modified Dirac matrices, preserving the gauge formulation of the theory.

4.7 Allowed symmetries

The SM modifications, introduced in this work, are conceived in order to preserve space-time homogeneity and isotropy, but even the standard physics interactions. As consequence, the same $SU(3) \times SU(2) \times U(1)$ internal symmetries are preserved. To prove this statement it is possible to verify that the Coleman-Mandula theorem [136] is still valid. In this way it results that the allowed symmetries are restricted to the direct product of internal ones with those generated by the modified Lorentz group, introduced before. A less rigorous proof can be obtained generalizing a Witten argument [137] about the fact that any additional kinematic and non internal symmetry would overconstrain the scattering amplitude. Therefore any further symmetry generator beyond Lorentz group would allow nontrivial scattering amplitude only for a discrete set of scattering angles. It is possible to start from admitting the existence of a symmetry generator $Q_{\mu\nu}$, symmetric, traceless and such that:

$$[Q_{\mu\nu},\, P_\alpha] \neq 0 \;\; and \;\; Q_{\mu\nu} \neq J_{\mu\nu} \in \mathfrak{so}(1,\,3) \tag{4.93}$$

$\forall P_\alpha$ generator of the Poincaré group.
The symmetry and tracelessness of $Q_{\mu\nu}$ let to write:

$$\langle p|Q_{\mu\nu}|p\rangle \div p_\mu p_\nu - \frac{1}{4} g_{\mu\nu}(p)\, p^2 \tag{4.94}$$

where the tracelessness is evaluated using the metric $g_{\mu\nu}(p)$ (4.22). Moreover, assuming that this operator acts like a tensor, for orthonormal states $|p_{(1)}\rangle$ and $|p_{(2)}\rangle$, one obtains the equality:

$$\langle p_{(1)},\, p_{(2)}\, |Q_{\mu\nu}|\, p_{(1)},\, p_{(2)}\rangle = \langle p_{(1)}\, |Q_{\mu\nu}|\, p_{(1)}\rangle + \langle p_{(2)}\, |Q_{\mu\nu}|\, p_{(2)}\rangle \tag{4.95}$$

If the momenta after the interaction are defined as: $q_{(1)\mu} = p_{(1)\mu} + a_\mu$ and $q_{(2)\mu} = p_{(2)\mu} + b_\mu$, from the momentum conservation for elastic scattering $p_{(1)\mu} + p_{(2)\mu} = q_{(1)\mu} + q_{(2)\mu}$, it implies that $a_\mu = -b_\mu$. From the $Q_{\mu\nu}$ conservation it follows now that:

$$\langle p_{(1)},\, p_{(2)}\, |Q_{\mu\nu}|\, p_{(1)},\, p_{(2)}\rangle = \langle q_{(1)},\, q_{(2)}\, |Q_{\mu\nu}|\, q_{(1)},\, q_{(2)}\rangle \tag{4.96}$$

and from this relation, using (4.94) and making a series expansion for the metric $g_{\mu\nu}(p)$:

$$a_\mu\,(p_{(1)\nu}-p_{(2)\nu})+a_\nu\,(p_{(1)\mu}-p_{(2)\mu})+2\,a_\mu\,a_\nu-\frac{1}{4}\,\partial_\alpha g_{\mu\nu}(p_{(1)})\,a^\alpha\,p^2_{(1)}+ \\ -\frac{1}{4}\,\partial_\alpha g_{\mu\nu}(p_{(2)})\,a^\alpha\,p^2_{(2)}=0 \tag{4.97}$$

Since the derivative $\partial_\alpha g_{\mu\nu}(p)$ are negligible, from this equation it follows that $a_\mu = 0$ and this means that only trivial scattering is allowed.

4.8 Coleman-Mandula theorem generalization

Following the demonstration present in [138], it is possible to verify that the theorem is still valid, even replacing the underlying Minkowski geometry with the pseudo-Finsler, considered in this work. The Lorentz group must be modified, acquiring a dependence on the particle momentum and the theorem hypothesis must be modified respect to those present in [138] and can be written as:

1. Lorentz invariance respect to the Modified Lorentz Transformations
2. Particle number finitness: $\forall M > 0\ \exists n < \infty$ number of particles with mass $m < M$
3. Elastic scattering is an analytic function of the Mandelstam variables
4. Nontrivial scattering happens for almost all energies
5. $\forall g \in G$, where G is the symmetry group, the element $g \in U(1)$ is representable in a identity neighbourhood via an integral operator, with distribution kernel

The S matrix is expressed as a function of the modified Mandelstam variables and is therefore invariant under the action of the modified Lorentz group.
The first part of the demonstration regards the subset of symmetry operators, that commute with the Poincaré group. These operators satisfy therefore the relation:

$$[B_\alpha,\,P_\mu]=0\quad \forall B_\alpha \in G_{sym},\ \forall P_\mu \in \mathcal{P} \tag{4.98}$$

where G_{sym} is the symmetry group. These operators act like tensors on single particle states:

$$\begin{aligned} B_\alpha|p,\, m,\, q,\, n\ldots\rangle = &\sum_{m'}[b_\alpha(p)]_{mm'}|p,\, m',\, q,\, n\ldots\rangle + \\ &+\sum_{n'}[b_\alpha(p)]_{nn'}|p,\, m,\, q,\, n'\ldots\rangle + \ldots \end{aligned} \tag{4.99}$$

where $[b_\alpha(p)]_{mm'}$ is the matrix representation of the operator B_α. Since the operators commute with the Poincaré group generators, as in the classical case, they satisfy the Lie algebra commutation rules:

$$\begin{aligned} &[B_\alpha,\, B_\beta] = i\, C^\tau_{\alpha\beta}\, B_\tau \\ &[b_\alpha(p),\, b_\beta(p)] = i\, C^\tau_{\alpha\beta}\, b_\tau(p) \quad \forall p \end{aligned} \tag{4.100}$$

Now starting from this relation it is possible to follow the classic demonstration, to prove that the correspondence $B_\alpha \to [b_\alpha(p)]$ is a bijection. It is only necessary to be careful to replace the on-shell condition of a particle with the MDR (4.4) and considering that:

$$\langle p',\, m',\, q',\, n'|[B_\alpha,\, S]|p,\, m,\, q,\, n\rangle = 0 \tag{4.101}$$

since the operators B_α are symmetry generators and commute with the S matrix, function of the new defined Mandelstam variables. Moreover the computation of the particle number with mass lower than a given number is:

$$N\left(\sqrt{p_\mu p^\mu}\right) = N\left(\sqrt{p_\mu\, g^{\mu\nu}(p)\, p_\nu}\right) \tag{4.102}$$

so the particle number finitness is still preserved. Now, as in the classical version of the theorem, it is possible to find operators:

$$B^\sharp_\alpha = B_\alpha - a^\mu_\alpha P_\mu \tag{4.103}$$

for opportune coefficients a^μ_α. This operator commutes with P_μ and $[P_\mu,\, J(p)]$, where $J(p)$ is a generator of the modified Lorentz group, for given momentum. The last statement is true because $[P_\mu,\, J(p)]$ is given by a linear combination of P_μ momenta, so the Jacobi identity:

$$[P_\mu,\, [J(p),\, B^\sharp]] + [J(p),\, [B^\sharp,\, P_\mu]] + [B^\sharp,\, [P_\mu,\, J(p)]] = 0 \tag{4.104}$$

is still valid. Now it is possible to show that:

$$[B_\alpha^\sharp, J(P)] = 0 \tag{4.105}$$

proving the theorem for the case of operators belonging to this particular subalgebra.
Considering now the symmetry generators subgroup, made of operators that do not commute with the Poincaré group: $[A_\alpha,\, P_\beta] \neq 0$, one can write the action of a generic element of this group, on a single particle state, as:

$$A_\alpha|p,\, n\rangle = \sum_{n'} \int d^4p' \left[\mathcal{A}_\alpha(p,\, p')\right]_{nn'} |p',\, n'\rangle \tag{4.106}$$

The classical theorem demonstration version focuses on the fact that this kind of operators have integral kernel null for $p \neq p'$. This remains valid even in the modified case, considering again the modified version of the mass-shell definition. Now the argumentation remains the same, arriving to demonstrate that such an operator can be written as:

$$A_\alpha = -\frac{i}{2} a(p)_\alpha^{\mu\nu} J_{\mu\nu} + B_\alpha \tag{4.107}$$

with an opportune coefficient $a(p)_\alpha^{\mu\nu}$, proving that this type of symmetry generators are given by the direct product of Poincaré group elements times internal symmetry generators:

$$A = P(p) \otimes G_{sym} \tag{4.108}$$

Finally it is possible to state that the allowed symmetries of the scenario, proposed in this work, are given by the direct product $\mathcal{P}(p) \otimes G_{int}$, where $\mathcal{P}(p)$ is the modified Poincaré group, that depends explicitly on the particle energy (momentum) and G_{int} is the internal symmetries group (in this case $SU(3) \times SU(2) \times U(1)$).

4.9 VSR correspondence

The MDR (4.1) can be generalized in a form, which includes energy dependent corrections, as for example in [139, 119]:

$$f_1^2 E^2 - f_2^2 |\overrightarrow{p}|^2 = m^2\,, \tag{4.109}$$

where f_i are four-momentum p functions. These functions can be written in a perturbative fashion as $f_i = 1 - h_i$, where $h_i \ll 1$ are the velocities modification parameters. From this relation, it is possible to derive an explicit equality for the energy:

$$E = \sqrt{\frac{m^2}{f_1^2} + \frac{f_2^2}{f_1^2}\,|\overrightarrow{p}|^2} \simeq p f_3 \;, with\, f = \frac{f_2}{f_1} \tag{4.110}$$

The velocity of the particle can be obtained using Hamilton-Jacobi equation (3.88):

$$c'(E) = \frac{\partial}{\partial p} E\bigg|_{max} = (f_{(3)} + p\, f'_{(3)}) = \left(f_{(3)} + f'_{(3)}\, p\left(\frac{1}{E} - \frac{p}{E^2}\right)\right) \tag{4.111}$$

Therefore every massive lepton feels a local space-time foliation, depending on its momentum. From this the necessity follows to resort to Finsler geometry, that can deal with this local space-time momentum depending parametrization.
Returning now to the homogeneous perturbation f, introduced in this work (4.2), if its magnitude remains negligible, compared to the momentum, the ratio $\frac{|\overrightarrow{p}|}{E} \longrightarrow 1 + \delta$ have a finite limit for $p \longrightarrow \infty$. As consequence, even the functions admits finite limit, $f(1+\delta) = \epsilon$. In this way the perturbation f_3, for $p \longrightarrow \infty$, tends to $\lim_{p\longrightarrow\infty} f_3 = 1 - f(1+\delta) = 1 - \epsilon$. Therefore it is possible to obtain the Coleman and Glashow's "Very Special Relativity" (VRS) scenario as a high energy (high momenta) limit. In this case it is possible to recover from equation (4.111) a massive particle "personal" *maximum attainable velocity* c':

$$c'(E) = f_3 = 1 - \epsilon \tag{4.112}$$

because $f'_{(3)} = 0$ for $p \longrightarrow \infty$, reobtaining a result provided in [18].
It is also possible to show that the modified Lorentz group, introduced in this work, is compatible with the special relativity transformations computed introducing a personal maximum attainable velocity different for every particle species. It is well known that this corresponds to ignore the speed of light universality postulate, in computing the Lorentz transformations [97]. In fact, supposing the case of Lorentz boost along the "x" direction, the explicit form of the Modified Lorentz transformations is given, using equation (4.40), by:

$$\begin{aligned}
&\Lambda^{\mu}_{\nu}(p', p) = e_a^{\ \mu}(p')\Lambda^a_{\ b} e^b_{\ \nu}(p) = e_a^{\ \mu}(\Lambda p)\Lambda^a_{\ b} e^b_{\ \nu}(p) = \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \chi & 0 & 0 \\ 0 & 0 & \chi & 0 \\ 0 & 0 & 0 & \chi \end{pmatrix}
\begin{pmatrix} \gamma & -\beta\,\gamma & 0 & 0 \\ -\beta\,\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\xi & 0 & 0 \\ 0 & 0 & 1/\xi & 0 \\ 0 & 0 & 0 & 1/\xi \end{pmatrix} = \\
&= \begin{pmatrix} \gamma & -\beta/\xi\,\gamma & 0 & 0 \\ -\beta\,\chi/\gamma & \chi/\xi\,\gamma & 0 & 0 \\ 0 & 0 & \chi/\xi & 0 \\ 0 & 0 & 0 & \chi/\xi \end{pmatrix} \simeq
\begin{pmatrix} \gamma & -\beta/\xi\,\gamma & 0 & 0 \\ -\beta\,\chi/\gamma & \chi/\xi\,\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{4.113}$$

where $\beta = v/c$, $\chi = \sqrt{1 - f(p')}$ and $\xi = \sqrt{1 - f(p)}$. The ratio χ/ξ can be approximated with 1 because this term corrections are negligible, compared with the other matrix coefficients.
The transformations obtained are correlated with the natural coordinate units of measure. The maximum attainable velocity in the two reference frames, denoted by the momenta p and p' are:

$$\begin{cases} c(p') = \chi(p')\, c_0 \\ c(p) = \xi(p)\, c_0 \end{cases} \tag{4.114}$$

To convert this MLT to the usual coordinates $\{t,\, x,\, y,\, z\}$, it is necessary to determine the value of x' in the transformed reference frame, noting that:

$$\gamma' = \frac{\chi}{\xi}\,\gamma \tag{4.115}$$

This is compatible with the form of the coefficients:

$$\gamma = \frac{c(p)}{\sqrt{c(p)^2 - v^2}} \qquad \gamma' = \frac{c(p')}{\sqrt{c(p')^2 - v'^2}} \tag{4.116}$$

from which it follows that $\gamma' \simeq \chi/\xi\,\gamma$ and therefore equation (4.115) is correct. The final MLT form for the usual standard coordinates $\{t,\, x,\, y,\, z\}$ results therefore:

$$\Lambda^{\mu}_{\ \nu}(p) = \begin{pmatrix} \chi/\xi\,\gamma & -v\,\xi/\chi\,\gamma & 0 & 0 \\ -v\,\chi/\xi\,\gamma & \chi/\xi\,\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.117}$$

where the second term of the first row has been divided by c^2, to convert the measure units. These results are compatible with the ones obtained in equations (3.17) and (3.18), proving that the construction here introduced is coherent with the special relativity constructed without the light speed postulate, that is with personal maximum attainable velocities.

4.10 DSR correspondence

In Double Special Relativity (or κ-deformed relativity) [121] the starting point consists again in modifying the kinematics of the interaction processes, requiring the invariance of the formulation respect to new introduced (modified) Lorentz transformations. To obtain this principle, in these theories the geometry of the momentum space is modified, introducing a *modified composition rule* for the momenta (3.107):

$$(p,\, q) \to (p \oplus q) = p + q + f(p,\, q) \tag{4.118}$$

where $f(p,\, q)$ represents a perturbation of the usual momenta sum. Contemporary it is introduced the inverse operation, which lets to obtain incoming momenta from the outgoing ones: $(\ominus p) \oplus p = 0$. These definitions correspond to the replacement of the momentum with a modified one, given by the relation (3.108):

$$\pi_\mu = M_\mu^{\ \nu}(p) p_{nu} \tag{4.119}$$

with the transformations $M_\mu^{\ \nu}(p)$ determined by the geometric features of the momentum space [105]. The geometry of the momentum space can be determined from the algebraic properties generated by the modified composition rule [121], with the affine connection given by (3.109):

$$\frac{\partial}{\partial p_a} \frac{\partial}{\partial q_b} (p \oplus q)|_c = \Gamma_c^{ab} \tag{4.120}$$

This class of Relativity modification theories present the advantage of preserving covariance respect to the introduced Modified Lorentz Transformations. In this work an analogous idea is explored, modifying the momentum space geometry with the introduction of what can compare to the modified composition rule of momenta. From the definition of the internal product (4.45) it is possible to recognize the new introduced composition law for momenta. In fact the only space where two different species momenta can "live together" is the Minkowski one, that underlies all the personal spaces of every particle. From this it is possible to obtain a modified composition rule for the momenta. Considering their projection on the Minkowski space:

$$(p,\, q) \to (p \oplus q) = (p_a\, e^a_{\ \mu}(p) + q_a\, \tilde{e}^a_{\ \mu}(q)) \tag{4.121}$$

and the generalization for the composition of a generic number of different species momenta:

$$(p,\, q,\, k \ldots) \to (p \oplus q \oplus k \oplus \ldots) = (p_a\, e^a_{\ \mu}(p) + q_a\, e'^a_{\ \mu}(q) + k_a\, e''^a_{\ \mu}(k) + \ldots) \tag{4.122}$$

The model proposed in this work presents therefore an analogy with DSR theories [121], [24], but it is important to underline a difference. In fact the modified composition rule does not present an universal character, instead it is species depending, and moreover it is associative and abelian. The new physics emerges by the comparison of different particle species that have different Modified Lorentz Transformations. The construction of the modified physics can therefore predict physical effects, experimentally detectable. Instead in case of a physics modification with universal character, independent from the particle species, the new physical effects correspond to a redefinition of the units of measure - that is the speed of light [140].

4.11 SME correspondence

The introduction of species depending MLT permits to introduce new Physics, generated by the different way particles are affected by LIV. As already underlined in [105], this idea is compatible with SME, where different particles can break the Lorentz symmetry differently. LIV is introduced in the present work, starting from a kinematical modification, that can be investigated by the isotropic coefficients of the SME. In this work only MDRs that are equal for particles and antiparticles have been considered. This corresponds to modify the Standard Model introducing only CPT-even therms, as illustrated in [26]. Furthermore the MDRs form selected does not distinguish between particle polarizations. In fact, considering a SM extension with CPT even terms of the form:

$$\frac{1}{2} i\, c_{\mu\nu} \overline{\psi} \gamma^{\mu} \overleftrightarrow{D}^{\nu} \psi + \frac{1}{2} i\, d_{\mu\nu} \overline{\psi} \gamma_5 \gamma^{\mu} \overleftrightarrow{D}^{\nu} \psi \tag{4.123}$$

it is possible to define the modified Dirac matrices:

$$\Gamma^{\mu} = \gamma^{\mu} + c^{\mu\nu} \gamma_{\nu} + d^{\mu\nu} \gamma_5 \gamma_{\nu} \tag{4.124}$$

obtaining an effective Lagrangian that induces an MDR with a difference, taking into account that one is dealing with real fermions (particles with spin). The present work considers a subset of SME, the one generated by the isotropic coefficient $c_{\mu\nu}$. Moreover it introduces isometry transformations for this subclass of violation cases, in order to preserve space-time isotropy. The only difference with the SME theory consists in posing the trace of this coefficient not null: $Tr\,(c_{\mu\nu}) \neq 0$. This hypothesis is not considered in SME, because it represents a simple scaling of the kinetic term and therefore is only part of the definition of the normalization of the field. In other words the trace of this tensor represents a universal modification of the maximum attainable velocity, eventuality that in SME is supposed to not generate visible physical effects. Instead in this work it is proved that the species depending character of the MLT can generate visible effects.

CHAPTER 5

De Sitter projective relativity

5.1 Introduction

It is well known that the fundamental symmetry group governing low energy physics is the Galilei one. As the energy increases this effective symmetry fails in describing kinematics and must be replaced by the Poincaré group. This means that Galilei relativity must be sostituted by the Einstein one. Some recent experimental evidences, as in the propagation of Ultra High Energy Cosmic Rays (UHECR), seem to implicate that even Einstein relativity must be considered an effective symmetry, not completely respected at the highest energies. This would implicate the necessity to search another *special relativity*. Since the work of von Ignatowsky [93, 94, 95, 96], it is evident that, with only the hypothesis of a homogeneous and isotropic space-time, it is possible to determine the relativity groups, Galilei and Poincaré, which present a fundamental difference, the first one admits velocities that can diverge to infinity, while the second one implies a finite parameter, which represents the maximum attainable velocity. From the Poincaré group it is possible to reobtain the Galilei one taking the infinite limit for the maximum velocity parameter. So the introduction of a new symmetry group to generalize the Galilei one corresponds to the introduction of a finite scale, the maximum speed, in the theory. In the same way the necessity to resort to a new group to generalize the Poincaré one corresponds to the introduction of a new fixed scale, the Planck length, at which the Einstein relativity starts to fail. Henry Bacry and Jean Marie Levy Leblond [141], starting from very general assumptions on the space-time structure, have investigated all the permitted forms of kinematics groups, obtaining the de Sitter one as a possible candidate. Following Fantappié and Arcidiacono [142], it is possible to construct a projective relativity, which is an example of *doubly special relativity* [143, 144]. But as already underlined by Aldrovandi and Pereira [145, 146, 147], in all special relativity models, the Lorentz symmetry is violated when a sufficient energy is reached, and Einstein relativity is no more valid. Instead de Sitter relativity is valid at all energy scales, giving a universal theory, a property shared by all fundamental theories.

5.2 Possible kinematical groups

Recent results seem to question the validity of the Poincaré group as the relativity one, at the highest energies, when the quantum structure of space-time can be accessible. This seems to suggest the necessity to consider another kinematical group as the symmetry group of a more general physical theory. A manner to introduce a generalization of this group, in order to fix the problem, is the one followed by Bacry and Levy Leblond [141], starting from very general assumptions on the structure of space-time and the symmetry group itself. First of all the kinematics group is supposed to be continuous, so it is possible to use Lie-algebraic methods. The generators of the kinematical group are the usual ones:

$$\{\mathcal{H},\ P,\ J,\ K\} \tag{5.1}$$

where $\mathcal{H}$ is the Hamiltonian, P the spatial translations generator, J the rotations generator and K the boosts generator.
From space isotropy one can deduce the way as the action of rotations transform the infinitesimal group generators, obtaining:

$$[J, H] = 0 \qquad [J, J] = 0 \qquad [J, P] = 0 \qquad [J, K] = 0 \tag{5.2}$$

Generalizing the Bacry and Levy Leblond work, parity is supposed not to be a fundamental symmetry of the kinematical group, since it is violated in weak interaction, instead time reversal is supposed as another fundamental symmetry. The action of parity can be summarized as:

$$\Pi : \{\mathcal{H} \to \mathcal{H},\ P \to -P,\ J \to J, K \to -K\} \tag{5.3}$$

and the action of the time reversal operator on the kinematical generators is given by:

$$\Theta : \{\mathcal{H} \to -\mathcal{H},\ P \to P,\ J \to J, K \to -K\} \tag{5.4}$$

The space-time is posed isotropic, that is invariant under rotations, and the inertial transformations, boosts along a definite direction, form a noncompact subgroup. In order to satisfy these properties, the previous commutations relations (5.2) and the action of the time reversal operator (5.4), one can impose the following commutation Lie brackets:

$$
\begin{aligned}
&[\mathcal{H},\, P_i] = i\,\alpha\, K_i \\
&[\mathcal{H},\, P_i] = i\,\lambda\, P_i + i\,\zeta\, J_i \\
&[P_i,\, P_j] = i\,\beta\,\epsilon_{ijk}\, J_k \\
&[K_i,\, K_j] = i\,\mu\,\epsilon_{ijk}\, J_k \\
&[P_i,\, K_j] = \rho\,\delta_{ij}\,\mathcal{H} + \tau\,\epsilon_{ijk}\, K_k
\end{aligned}
\tag{5.5}
$$

which can be written only with generators linear combinations, using the covariant tensors δ_{ij} and ϵ_{ijk}.
Taking into account the Jacobi identities, all the ones that contain at least one J generator are automatically satisfied, since space-time is rotationally invariant. The other Jacobi identities that are satisfied are $[\mathcal{H}\, P\, P]$, $[P\, P\, P]$ and $[K\, K\, K]$, instead the others identities let to obtain constrains to the coefficients that appear in (5.5):

$$[\mathcal{H}\, K\, K] \;\Rightarrow\; \zeta + \lambda\,\tau = 0 \tag{5.6}$$

$$[\mathcal{H}\, P\, K] \;\Rightarrow\; \begin{cases} \tau\,\lambda + \zeta = 0 \\ \zeta\,\tau - \lambda\,\beta - \alpha\,\mu = 0 \end{cases} \tag{5.7}$$

$$[P\, P\, K] \;\Rightarrow\; \rho\,\alpha + \tau^2 - \beta = 0 \tag{5.8}$$

$$[P\, K\, K] \;\Rightarrow\; \begin{cases} \rho\,\zeta\tau\,\mu = 0 \\ \lambda\,\rho + \mu = 0 \end{cases} \tag{5.9}$$

From equation (5.7) one can obtain the following:

$$\tau^2 = -\beta - \alpha\,\frac{\mu}{\lambda} \tag{5.10}$$

Furthermore from relation (5.6) one can compute:

$$\zeta = -\lambda\,\tau \tag{5.11}$$

and from (5.9) it is possible to obtain:

$$\mu = -\lambda\,\rho \tag{5.12}$$

Using (5.10), (5.11) and (5.12), one can arrive to the relation:

$$\tau^2 = -\beta - \alpha\,\frac{\mu}{\lambda} = -\beta + \alpha\,\rho \tag{5.13}$$

Now from equation (5.8) and the previous one it results possible to write:

$$\tau^2 = \beta - \alpha\,\rho = -\beta + \alpha\,\rho \;\Rightarrow\; \tau = 0 \tag{5.14}$$

and from (5.11) it follows the equation:

$$\zeta = -\lambda\,\tau = 0 \tag{5.15}$$

Retaining only the time reversal symmetry and rejecting the parity one, therefore, one can arrive to the same final result present in [141] about the admitted kinematical groups. In fact these groups can be classified on the base of the remaining structure constants:
in case of $\rho \neq 0$:

- $\{\alpha \neq 0,\, \lambda \neq 0\} \;\Rightarrow\; \{\beta \neq 0,\, \mu \neq 0\}$ the Lie algebras are those of $SO(5)$, $SO(1,4)$, $SO(2,3)$, $SO(5)$ must be rejected because the boosts subgroup is compact
- $\{\alpha = 0,\, \lambda \neq 0\} \;\Rightarrow\; \{\beta = 0,\, \mu \neq 0\}$ Poincaré group Lie algebra
- $\{\alpha \neq 0,\, \lambda = 0\} \;\Rightarrow\; \{\beta \neq 0,\, \mu = 0\}$ this Lie algebra satisfies all the requests, but the role of translations and boosts are inverted
- $\{\alpha = 0,\, \lambda = 0\} \;\Rightarrow\; \{\beta = 0,\, \mu = 0\}$ this is the Carroll group Lie algebra

in case of $\rho = 0$:

- $\{\alpha \neq 0,\, \lambda \neq 0\} \;\Rightarrow\; \{\beta \neq 0,\, \mu \neq 0\}$ Galilei group Lie algebra
- $\{\alpha = 0,\, \lambda \neq 0\} \;\Rightarrow\; \{\beta = 0,\, \mu \neq 0\}$ Newton group Lie algebra
- $\{\alpha \neq 0,\, \lambda = 0\} \;\Rightarrow\; \{\beta \neq 0,\, \mu = 0\}$ para Galilei group Lie algebra
- $\{\alpha = 0,\, \lambda = 0\} \;\Rightarrow\; \{\beta = 0,\, \mu = 0\}$ static group Lie algebra

The interesting cases are those represented by the Galilei group, the Poincaré one and the deSitter $SO(1,4)$ and Anti-deSitter $SO(2,3)$ groups.

5.3 DeSitter projective Relativity

Using the deSitter $SO(1,4)$ as the fundamental space-time symmetry group, it is possible to construct what is known as projective dS relativity [148, 149]. The fundamental principle of General Relativity is the equivalence one, which states that it is always possible to find a local frame of reference, isomorphic to Minkowski one, to describe physics. This space $\mathbb{M}$ is constructed as the quotient between Poincaré $\mathbb{P}$ and Lorentz $\mathcal{L}$ groups:

$$\mathbb{M} = \mathbb{P}/\mathcal{L} \tag{5.16}$$

and results a solution of the sourceless Einstein equation:

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = 0 \tag{5.17}$$

Similarly the deSitter space is obtained as a solution of the Einstein equation modified with the adjoint of the cosmological $\Lambda > 0$ constant:

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R - \Lambda\, g_{\mu\nu} = 0 \tag{5.18}$$

and it is given by the ratio of the deSitter group $SO(1,4)$ and the Lorentz one $\mathcal{L}$:

$$d\mathbb{S}(1,4) = SO(1,4)/\mathcal{L} \tag{5.19}$$

The curvature scalar of this space-time results negative $R < 0$. Considering the deSitter space as immersed in a five dimensional space and resorting to the stereographic (conformally Minkowski) chart [150], the deSitter space coordinates (projective coordinates) can be written as function of the physical coordinates:

$$\begin{cases} \chi^\mu = \Omega(x)\, x^\mu \\ \chi^4 = -l\, \dfrac{\Omega(x)}{\Omega'(x)} \end{cases} \tag{5.20}$$

The index $\mu \in \{0,\, 1,\, 2,\, 3\}$ is the usual Minkowski space one and the constant l is obtained from the cosmological one, posing $\Lambda = \frac{3}{l^2}$. The functions Ω and Ω' are given by:

$$\begin{cases} \Omega(x) = \dfrac{1}{1 - \frac{x_\mu x^\mu}{4\, l^2}} \\ \Omega'(x) = \dfrac{1}{1 + \frac{x_\mu x^\mu}{4\, l^2}} \end{cases} \tag{5.21}$$

Using the new defined coordinates, it results possible to write the definitory equation of the deSitter space:

$$\eta_{AB}\, \chi^A\, \chi^B = -l^2 \tag{5.22}$$

where the metric is $\eta_{AB} = diag\{+1,\, -1,\, -1,\, -1,\, -1\}$ and the latin indicex $A \in \{0,\, 1,\, 2,\, 3,\, 4\}$ are relative to the new $d\mathbb{S}(1,4)$ coordinates.

The Killing vectors of $d\mathbb{S}(1,4)$ are given by the equation:

$$L^{\mu}_{AB} = \frac{1}{\Omega(x)}\,\eta^{\mu\nu}\,\eta_{AC}\,\eta_{BD}\,\left(\chi^{C}\frac{\partial\chi^{D}}{\partial x^{\nu}} - \chi^{D}\frac{\partial\chi^{C}}{\partial x^{\nu}}\right) \tag{5.23}$$

They are the internal symmetry generators of the deSitter space-time and satisfy the commutation relation:

$$[L_{AB},\, L_{CD}] = -\eta_{AC}\,L_{BD} + \eta_{AD}\,L_{BC} - \eta_{BD}\,L_{AC} + \eta_{BC}\,L_{AD} \tag{5.24}$$

The explicit form of the symmetry generators is given by:

$$\begin{aligned} L_{\mu\nu} &= \eta_{\mu\alpha}\,x^{\alpha}\,\partial_{\nu} - \eta_{\nu\alpha}\,x^{\alpha}\,\partial_{\mu} = \eta_{\mu\alpha}\,x^{\alpha}\,P_{\nu} - \eta_{\nu\alpha}\,x^{\alpha}\,P_{\mu} \\ L_{4\mu} &= l\,\partial_{\mu} - \frac{1}{4\,l}\left(2\,\eta_{\mu\nu}\,x^{\nu}\,x^{\alpha} - \frac{x_{\nu}x^{\nu}\delta^{\alpha}_{\mu}}{4\,l^{2}}\right)\partial_{\alpha} = l\,P_{\mu} - \frac{1}{4\,l}\,K_{\mu} \end{aligned} \tag{5.25}$$

Now from equations (5.25) it is possible to recognize, as generators of the kinematical group, the elements:

$$\begin{aligned} P_{\mu} &= \partial_{\mu} \\ \Pi_{\mu} &= \frac{L_{4\mu}}{l} = P_{\mu} - \frac{1}{4\,l^{2}}\,K_{\mu} \end{aligned} \tag{5.26}$$

The $d\mathbb{S}(1,4)$ algebra structure can be resumed in the following commutation relations:

$$\begin{aligned} &[L_{\mu\nu},\, L_{\alpha\beta}] = \eta_{\mu\beta}\,L_{\nu\alpha} - \eta_{\mu\alpha}\,L_{\nu\beta} + \eta_{\nu\alpha}\,L_{\mu\beta} - \eta_{\nu\beta}\,L_{\mu\alpha} \\ &[\Pi_{\mu},\, L_{\nu\alpha}] = \eta_{\mu\alpha}\,\Pi_{\nu} - \eta_{\nu\alpha}\,\Pi_{\mu} \\ &[\Pi_{\mu},\, \Pi_{\nu}] = \frac{1}{l^{2}}\,L_{\mu\nu} \end{aligned} \tag{5.27}$$

From equation (5.26), taking the limit $l \to \infty$, that corresponds to take $\Lambda \to 0$, the limit $\Pi_{\mu} \to K_{\mu}$ follows. This means that the deSitter algebra (5.27) reduces to the Poincaré one and therefore one recovers the space-time solution of the Einstein equation without the cosmological constant, that is the Minkowski space-time.

5.4 Modified dispersion relations in deSitter Relativity

To obtain the energy explicit form and consequently the MDR in deSitter projective relativity scenario, it is useful to resort to the Beltrami chart [151]. The $d\mathbb{S}$

coordinates result correlated to the physical ones by the relation:

$$\chi^\mu = \frac{x^\mu}{\Xi(x)} \qquad \chi^4 = \frac{l}{\Xi(x)} \tag{5.28}$$

where the function Ξ is given by:

$$\Xi(x) = \sqrt{1 - \frac{x_\mu x^\mu}{l^2}} \tag{5.29}$$

The projective metric, that is the metric associated to the immersed variety, can be computed from the relation:

$$ds^2 = d\chi_A d\chi^A = d\chi^A \, d\chi^B \, \eta_{AB} \tag{5.30}$$

From the equality:

$$l\,\chi^\mu = x^\mu \, \chi^4 \tag{5.31}$$

it is possible to obtain , differentiating:

$$l\,d\chi^\mu = x^\mu \, d\chi^4 + \chi^4 \, dx^\mu \tag{5.32}$$

Now, from (5.30) and (5.31) it is possible to obtain the equation:

$$ds^2 = \frac{1}{l^2} \left((dx_\mu dx^\mu)\,(\chi^4)^2 + (-l^2 + x_\mu x^\mu)(d\chi^4)^2 - 2\,x_\mu dx^\mu \, \chi^4 d\chi^4 \right) \tag{5.33}$$

and finally one can obtains the relation:

$$ds^2 = \frac{1}{\Xi(x)^2} \left(\Xi(x)^2 (dx_\mu dx^\mu) + \frac{1}{l^2} (x_\mu dx^\mu)^2 \right) \tag{5.34}$$

and the explicit metric tensor form is given by:

$$g_{\mu\nu} = \frac{1}{\Xi(x)^4} \left(\Xi(x)^2 \delta_{\mu\nu} + \frac{1}{l^2}\, x_\mu x_\nu \right) \tag{5.35}$$

To obtain the deSitter projective relativity kinematics, it is necessary now to construct the projective velocity, given by:

$$u^A = \frac{d\chi^A}{ds} \tag{5.36}$$

where the proper time obtained from (5.34) has been used. The projective acceleration is obtained differentiating the projective velocity respect to the proper time and results:

$$\alpha^A = \frac{du^A}{ds} = \frac{d^2\chi^A}{ds^2} \tag{5.37}$$

Now it results simple to verify the relations:

$$u_A u^A = 1 \qquad \alpha_A u^A = 0 \tag{5.38}$$

The projective momentum is defined as:

$$\pi^A = m_0\, u^A = m_0\, \frac{d\chi^A}{ds} \tag{5.39}$$

Using the relation (5.31) it is possible to obtain the corresponding physical variables, so the physical velocity results:

$$v^\mu = \frac{dx^\mu}{ds} = \frac{l}{(\chi^4)^2}\,(\chi^4 u^\mu - \chi^\mu u^4) \tag{5.40}$$

and the physical momentum is given by:

$$p^\mu = \frac{l}{(\chi^4)^2}\,(\chi^4 \pi^\mu - \chi^\mu \pi^4) \tag{5.41}$$

Defining the projective angular momentum as:

$$M^{AB} = \chi^A \pi^B - \chi^B \pi^A \tag{5.42}$$

one can obtain the following relations:

$$\begin{aligned} M^{4\mu} &= \frac{l}{\Xi(x)^2}\, p^\mu \\ M^{\mu\nu} &= \frac{1}{\Xi(x)^2}\, m^{\mu\nu} \end{aligned} \tag{5.43}$$

where $m^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ represents the physical angular momentum. To obtain relation (5.43) it is necessary to differentiate equation (5.28) respect to the proper time, obtaining:

$$\begin{aligned} u^\mu &= \frac{1}{\Xi(x)^3}\left(\Xi(x)^2 \delta^\mu_\nu + \frac{x^\mu x_\nu}{l^2}\right) v^\nu \\ u^4 &= -\frac{1}{\Xi(x)^3}\,\frac{u_\mu x^\mu}{l} \end{aligned} \tag{5.44}$$

then these relations must be used in (5.42).
Now using the projective angular momentum, it is possible to write the relation:

$$M_{AB}M^{AB} = 2\,l^2\,\pi_A\pi^A \tag{5.45}$$

which can be expanded, giving the result:

$$-2\,m_0^2\,\Xi(x)^4 = \frac{m_{\mu\nu}m^{\mu\nu}}{l^2} - 2\,p_\mu p^\mu \tag{5.46}$$

From the previous one, it is simple to obtain the modified dispersion relation:

$$E^2 = p^2 + m_0^2\,\Xi(x)^4 + \frac{m_{\mu\nu}m^{\mu\nu}}{2\,l^2} \tag{5.47}$$

demonstrating that even in deSitter projective relativity the effect on physics are of kinematical nature and emerge in modifying the dispersion relations.

5.5 Projective Anti-deSitter Relativity

It is possible to repeat the same procedure, made using the deSitter group, with the Anti-deSitter $SO(2,3)$ group, to produce a similar projective relativity. In this case the space-time emerges as a solution of the modified Einstein equation (5.18), with the cosmological constant $\Lambda < 0$, and is given by the ratio of the Anti-deSitter group $SO(2,3)$ and the Lorentz group $\mathcal{L}$:

$$Ad\mathbb{S} = SO(2,3)/\mathcal{L} \tag{5.48}$$

On the contrary of deSitter space, in Anti-deSitter one the scalar curvature of the space-time results positive $R > 0$ and the definitory equation of the space-time is:

$$\overline{\eta}_{AB}\chi^A\chi B = -l^2 \tag{5.49}$$

with the metric $\overline{\eta}_{AB} = diag\{+1,\,-1,\,-1,\,-1,\,+1\}$.
Using, as in the previous case, the stereographic (conformally Minkowski) chart [150], the projective coordinates can be written as function of the physical ones as:

$$\begin{cases} \chi^\mu = \Omega'(x)\,x^\mu \\ \chi^4 = l\,\dfrac{\Omega(x)'}{\Omega(x)} \end{cases} \tag{5.50}$$

where the functions $\Omega(x)$ and $\Omega'(x)$ are given by equations (5.21).

$$\begin{cases} \Omega(x) = \dfrac{1}{1 - \frac{x_\mu x^\mu}{4\,l^2}} \\ \Omega'(x) = \dfrac{1}{1 + \frac{x_\mu x^\mu}{4\,l^2}} \end{cases} \tag{5.51}$$

The Killing vectors are again computed, using the relation (5.23), and the symmetry generators assume the explicit form:

$$\begin{aligned} L_{\mu\nu} &= \eta_{\mu\alpha}\, x^\alpha\, \partial_\nu - \eta_{\nu\alpha}\, x^\alpha\, \partial_\mu = \eta_{\mu\alpha}\, x^\alpha\, P_\nu - \eta_{\nu\alpha}\, x^\alpha\, P_\mu \\ L_{4\mu} &= -l\,\partial_\mu - \frac{1}{4\,l}\left(2\,\eta_{\mu\nu}\, x^\nu\, x^\alpha + \frac{x_\nu x^\nu \delta_\mu^{\,\alpha}}{4\,l^2}\right)\partial_\alpha = l\,P_\mu - \frac{1}{4\,l}\,K_\mu \end{aligned} \tag{5.52}$$

recognizing even in this case the kinematical group generators, again with the form of equation (5.26).

$$\begin{aligned} &[L_{\mu\nu},\, L_{\alpha\beta}] = \eta_{\mu\beta}\, L_{\nu\alpha} - \eta_{\mu\alpha}\, L_{\nu\beta} + \eta_{\nu\alpha}\, L_{\mu\beta} - \eta_{\nu\beta}\, L_{\mu\alpha} \\ &[\Pi_\mu,\, L_{\nu\alpha}] = \eta_{\mu\alpha}\,\Pi_\nu - \eta_{\nu\alpha}\,\Pi_\mu \\ &[\Pi_\mu,\, \Pi_\nu] = \frac{1}{l^2}\, L_{\mu\nu} \end{aligned} \tag{5.53}$$

Even in this case the $d\mathbb{S}(2,3)$ algebra verifies the commutation relations (5.27). So even in case of Anti-deSitter space-time the cosmological constant $\Lambda \to 0$ limit, that is the limit $l \to \infty$, is given by the Minkowski space.

5.6 Modified dispersion relations in Anti-deSitter Relativity

The energy explicit form and consequently the MDR in Anti-deSitter projective relativity scenario can be computed in an analogous way as in deSitter case. It results useful modify the Beltrami chart formulation [151] of the previous case. The $Ad\mathbb{S}$ coordinates result correlated to the physical ones by the relation:

$$\begin{aligned} \chi^\mu &= \frac{x^\mu}{\Xi'(x)} \\ \chi^4 &= \frac{l}{\Xi'(x)} \end{aligned} \tag{5.54}$$

where the function Ξ' is defined as:

$$\Xi'(x) = \sqrt{1 + \frac{x_\mu x^\mu}{l^2}} \tag{5.55}$$

The projective metric, associated to the immersed variety, can be evaluated using the equation:

$$ds^2 = d\chi_A d\chi^A = d\chi^A \, d\chi^B \, \overline{\eta}_{AB} \tag{5.56}$$

From the relation:

$$l \, \chi^\mu = x^\mu \, \chi^4 \tag{5.57}$$

differentiating it is possible to obtain the following:

$$l \, d\chi^\mu = x^\mu \, d\chi^4 + \chi^4 \, dx^\mu \tag{5.58}$$

similar to the relation (5.32).
Now, from (5.55) and (5.57) it is possible to obtain the equation:

$$ds^2 = \frac{1}{l^2} \left((dx_\mu dx^\mu) \, (\chi^4)^2 + (-l^2 + x_\mu x^\mu)(d\chi^4)^2 - 2 \, x_\mu dx^\mu \, \chi^4 d\chi^4 \right) \tag{5.59}$$

and finally:

$$ds^2 = \frac{1}{\Xi(x)^2} \left(\Xi(x)^2 (dx_\mu dx^\mu) - \frac{1}{l^2} (x_\mu dx^\mu)^2 \right) \tag{5.60}$$

The explicit metric tensor becomes:

$$g_{\mu\nu} = \frac{1}{\Xi'(x)^4} \left(\Xi'(x)^2 \delta_{\mu\nu} - \frac{1}{l^2} \, x_\mu x_\nu \right) \tag{5.61}$$

analogous to the explicit expression (5.35), valid for the deSitter case.
The AntideSitter projective relativity kinematics can be determined computing the projective velocity and the projective acceleration, given again respectively by equations (5.36) and (5.37). The orthogonality relation (5.38) is still verified. The projective momentum is defined, as in the previous case (5.39). Using the relation (5.57) it is possible to obtain the physical velocity, that has an expression analogous to (5.40). The corresponding physical momentum is given by (5.41) and the projective angular momentum is again defined as (5.42). Even in AdS projective relativity one has to differentiate equation (5.54) respect to the proper time to obtain:

$$\begin{aligned} u^\mu &= \frac{1}{\Xi'(x)^3} \left(\Xi'(x)^2 \delta^\mu_\nu - \frac{x^\mu x_\nu}{l^2} \right) v^\nu \\ u^4 &= -\frac{1}{\Xi'(x)^3} \, \frac{u_\mu x^\mu}{l} \end{aligned} \tag{5.62}$$

Again, using the projective angular momentum, it is possible to obtain:

$$M_{AB} M^{AB} = 2 \, l^2 \, \pi_A \pi^A \tag{5.63}$$

Expanding this result it is possible to compute the result:

$$2\, m_0^2\, \Xi(x)^4 = \frac{m_{\mu\nu} m^{\mu\nu}}{l^2} + 2\, p_\mu p^\mu \tag{5.64}$$

From the previous one, it is simple to obtain the modified dispersion relation:

$$E^2 = p^2 + m_0^2\, \Xi(x)^4 - \frac{m_{\mu\nu} m^{\mu\nu}}{2\, l^2} \tag{5.65}$$

demonstrating that even in deSitter projective relativity the effect on physics are of kinematical nature and emerge in modifying the dispersion relations.

5.7 Massive particles kinematics in deSitters/Anti-deSitter Relativity

In every LIV theory massive particles are supposed to interact with the quantum structure of the background, during free propagation. Furthermore this effect is supposed increasing with the particle energy. Even in projective relativity this idea can be considered valid, thinking that the quantum effects of space-time manifest themselves when a particle acquires enough energy to feel even the exceeding spatial dimension. A similar idea is considered in ([145]), where it is assumed that the propagating photons energy can cause tiny space-time fluctuations, described using the deSitter relativity.
Considering the deSitter space-time, resorting to the Lemaitre-Robertson-Walker coordinates, its squared line element, as function of the physical coordinates, becomes [152], as function of the physical coordinates:

$$ds^2 = d\tau^2 - n(\tau)\delta_{ij} dx^i dx^j \tag{5.66}$$

where the latin indices are the Minkowski space ones and belong to the set $\{1,\, 2,\, 3\}$. The function $n(\tau)$ is given by:

$$n(\tau) = \exp\left[\sqrt{\Lambda/3}\tau\right] \tag{5.67}$$

The cosmological constant Λ is assumed as depending on the particle species and its way of interacting with the quantum structure of the background. The τ variable in [145] is identified with the particle wavelenght, that is:

$$\tau = \frac{p}{E} \tag{5.68}$$

using de de Broglie relation. Moreover τ corresponds to a time variable and is therefore inversely proportional to the particle velocity, in the sense that bigger is the particle speed and smaller is the time necessary to obtain the same length element (5.66):

$$\tau \div \frac{1}{v} \simeq \frac{1}{\partial_p E} \simeq \frac{p}{E} \tag{5.69}$$

using the group velocity relation to obtain the particle speed. In this way the line element (5.66) can be written in the form:

$$\begin{aligned} ds^2 &= d\tau^2 - \exp\left(\alpha\,\frac{p}{E}\right)\delta_{ij}dx^i dx^j = d\tau^2 - \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\left(\frac{p}{E}\right)^n \delta_{ij}dx^i dx^j = \\ &= d\tau^2 - \sum_{n=0}^{\infty}\alpha_n\left(\frac{p}{E}\right)^n \delta_{ij}dx^i dx^j \end{aligned} \tag{5.70}$$

with α and α_n opportunely defined constants. From this relation it is possible to recognize the 0 degree homogeneous perturbation function (4.2):

$$f\left(\frac{|\overrightarrow{p}|}{E}\right) = \sum_{k=1}^{\infty}\alpha_n\left(\frac{|\overrightarrow{p}|}{E}\right)^n \tag{5.71}$$

and the metric (4.22):

$$g(x,\,\dot{x}(p))_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -(1+f(p/E))\mathbb{I}_{3\times 3} \end{pmatrix} \tag{5.72}$$

Therefore, as for physics constructed starting from kinematical modifications of the dispersion relations, the deSitter scenario admits the Finsler geometry as the underlying one.

In the Anti-deSitter case, following an identical procedure, it is possible to find an opportune coordinate system [153] that allows to write the function $n(\tau)$ (5.67) as:

$$n(\tau) = \cos\sqrt{\Lambda/3}\tau \tag{5.73}$$

The line element (5.66) becomes therefore:

$$\begin{aligned} ds^2 &= d\tau^2 - \cos{(\alpha\tau)}\delta_{ij}dx^i dx^j = d\tau^2 - \sum_{n=0}^{\infty}(-1)^n\frac{\alpha^{2n}}{(2n)!}\left(\frac{p}{E}\right)^{2n}\delta_{ij}dx^i dx^j = \\ &= d\tau^2 - \sum_{n=0}^{\infty}(-1)^n\alpha_{2n}\left(\frac{p}{E}\right)^{2n}\delta_{ij}dx^i dx^j \end{aligned} \tag{5.74}$$

Even in this case the perturbation consists in a 0 degree homogeneous function, as in (4.2):

$$f\left(\frac{|\overrightarrow{p}|}{E}\right) = \sum_{k=1}^{\infty}(-1)^n \alpha_{2n}\left(\frac{|\overrightarrow{p}|}{E}\right)^{2n} \tag{5.75}$$

The resultant metric is again in the form of (5.72), therefore even in the Anti-deSitter case the underlying geometry is of Finsler type. Finally it is important to notice that in the deSitter projective relativity scenario, the foreseen maximum attainable velocity, for massive particles, results lower than the light speed, as assumed in the rest of this work. In fact the first term (dominant one) of the expansion series of the perturbation is positive. Instead in the Anti-deSitter scenario the first term results negative, therefore the predicted maximum speed for massive bodies results superluminal.

Part III

LIV Phenomenology on Cosmic Messengers

CHAPTER 6

Cosmic messengers LIV phenomenology

6.1 Introduction

It is well known that the optical depth of a $10^{20}\,eV$ UHECR should be only of few Mpc. Inside this opacity sphere there should be enough astrophysical objects with the necessary characteristics to accelerate a particle to such high energies (1.4). Moreover this kind of highly energetic particles propagates on almost straight lines, in void, so it would be possible to identify the candidate sources of these UHECRs. Inside the foreseen opacity sphere it is not possible to identify enough sources, but some recent works [17] seem to indicate the possibility to find some candidates outside this sphere. That is recent experimental observations hint the possibility that the predicted Universe opacity to the propagation of UHECR may be modified, expanding its radius. UHECR are therefore ideal candidates for probing a look inside the quantum gravity phenomenology.

6.2 Effects induced on protons UHECR phenomenology

Since the Cosmic Microwave Background Radiation (CMBR) interacts with cosmic rays, Universe is not transparent for the propagation of these particles, which in fact dissipate energy during their path. In fact high energy particles lose part of their energy after a determined propagation length, which depends on their energy itself and on their nature (type of the particle). UHECR are mostly constituted by heavy nuclei (iron type) or protons and the ways they interact with the background radiation depends on their nature. Heavy nuclei with sufficient energy can suffer a photo-dissociation process caused by CMB:

$$A + \gamma \rightarrow (A - 1) + n \tag{6.1}$$

where A represents the atomic number of the bare nucleus considered.
Instead protons interacts with the CMB through a delta resonance, which decay-

ing generates a photo-pion:

$$
\begin{aligned}
&p + \gamma \to \Delta \to p + \pi^0 \\
&p + \gamma \to \Delta \to n + \pi^+
\end{aligned}
\tag{6.2}
$$

effect known as GZK cut-off. First, in this work, the effects of LIV on the propagation of protons will be analyzed.
In order to obtain a delta resonance, the free energy of the proton and the photon interacting must be bigger than the rest energy of the delta particle and this poses a constrain on the magnitude of LIV. In the approximation of head on collision, the reaction free energy becomes:

$$
\begin{aligned}
&s = (E_p + E_\gamma)^2 - (\overrightarrow{p}_p[e^{-1}] + \overrightarrow{p}_\gamma)^2 \geq m_\Delta^2 \ \Rightarrow \\
&\Rightarrow (E_p + E_\gamma)^2 - \left(\frac{\overrightarrow{p}_p}{\sqrt{1 - f_p(p_p)}} + \overrightarrow{p}_\gamma\right)^2 \geq m_\Delta^2 \ \Rightarrow \\
&\Rightarrow E_p^2 - \overrightarrow{p}_p(1 - f_p(p_p)) - 2f_p(p_p)\overrightarrow{p}_p + 2E_pE_\gamma + \\
&\quad - 2\overrightarrow{p}_p \cdot \overrightarrow{p}_\gamma \left(1 + \frac{1}{2}f_p(p_p)\right) \geq m_\Delta^2
\end{aligned}
\tag{6.3}
$$

where the four momentum of the proton is $(E_p, \overrightarrow{p}_p)$, the four momentum of a CMBR photon is $(\omega, \overrightarrow{\omega})$, m_p denotes the proton mass and m_Δ denotes the delta resonance mass. In the previous calculus it has been used MDR (4.4) and the approximation:

$$
\frac{1}{\sqrt{1 - f_p(p_p)}} \simeq 1 + \frac{1}{2}f_p(p_p) \tag{6.4}
$$

In the first line the momentum of the proton has been projected from its space of definition $(T_xM,\ g_{\mu\nu})$ to the space where the interaction between the massive lepton and the photon takes place, $(T_xM,\ \eta_{\mu\nu})$, using the thetrad formalism.
From (6.3) one obtains the inequality:

$$
2f_p(p_p)E_p^2 - E_p(4E_\gamma + f_p(p_p)E_\gamma) + m_\Delta^2 - m_p^2 \leq 0 \tag{6.5}
$$

The GZK effect is suppressed as a consequence of LIV, if this second grade inequality is not satisfied. From the study of this inequality, one obtains the constrain, that limits the GZK existence:

$$
f_p(p_p) < \frac{\Delta M^2 - 4E_\gamma - \sqrt{(\Delta M)^2 - 8E_\gamma \Delta M^2}}{4E_\gamma^2} \tag{6.6}
$$

where $\Delta M^2 = (m_\Delta^2 - m_p^2)$.
Substituting these physical quantities with the average values of $E_\gamma \simeq 7.0 \times$

$10^{-4}\ eV$, $m_\Delta \simeq 1232\ MeV$, $m_p \simeq 938\ MeV$, one obtains the constrain value:

$$f_p(p_p) < 6 \cdot 10^{-23} \tag{6.7}$$

to guarantee the existence of the GZK effect[1].
Protons lose energy by the photo-pion production process, without annihilate in what can be considered a dissipation mechanism. So this process can repeat again if they have enough energy. Therefore it becomes necessary to evaluate the fraction of initial proton momentum transferred to the outgoing pion. As already underlined, to describe this phenomenon it is necessary to introduce the *elasticity* factor $\eta = \left(\frac{E_{out}}{E_{in}}\right)$. That is the ratio of the energy carried away by one of the particles emerging from the interaction, (E_{out}), divided by the energy of the incident particle, (E_{in}). It is even important the *inelasticity*, which represents the fraction of the total incident energy that is avaiable for the production of secondary particles, and is defined as $K = (1-\eta)$. Now it is possible to determine the *attenuation length* or *optical depth* of a proton, defined in eq. (2.22) as the average length of propagation that a proton has to travel to see its energy reduced by a factor of e^{-1}. Introducing the LIV, it is important to notice that the optical depth computation is executed in a flat reference frame, because the propagation of UHECR happens in an asymptotically flat space-time. In fact, in this work the theory underlying the interaction between UHECR protons with CMBR is the asymptotically flat SM minimal extension introduced before. So the most evident effect of the introduction of LIV on the photo-pion process is limited to the modification of the inelasticity function, that is a modification of the allowed phase space for this kind of reaction. The inelasticity, calculated without introducing Lorentz violation in the theory, is given [45] by the formula (2.40). The modified kinematics, caused by the introduction of LIV, introduces some changes in its calculation. Following the computation of [27, 28] one starts from the definition of the center of momenta reference frame:

$$\overrightarrow{p}^*_p + \overrightarrow{p}^*_\pi = 0 \tag{6.8}$$

where these vectors are defined in $(TM,\ \eta_{ab})$. Now considering the free energy of the photo-pion production:

$$\sqrt{s} = (E^*_p + E^*_\pi) \tag{6.9}$$

it is possible to obtain the γ_{CM} factor, correlated with the modified Lorentz transformations. In this way it is possible to evaluate the effects of the change of ref-

[1]This constrain is comparable to the limit $4.5 \cdot 10^{-23}$, numerically computed in [27, 28]

erence frame, from the center of momenta to a generic one.

$$\gamma_{CM}(E_p^* + E_\pi^*) = \gamma_{CM}\sqrt{s} = (E_p + E_\pi) \;\Rightarrow\; \gamma_{CM} = \frac{E_p * E_\pi}{\sqrt{s}} = \frac{E_{tot}}{\sqrt{s}} \tag{6.10}$$

Now computing the free energy ($\sqrt{s}$ Mandelstam variable) necessary for the creation of a photo-pion in the CM frame of reference, and using the CM definition, so $\overrightarrow{p}_p^* = \overrightarrow{p}_\pi^*$:

$$\begin{aligned} &(\sqrt{s} - E_p^*)^2 - ([e_\pi]p_p^*)^2 = m_\pi^2 \Rightarrow \\ \Rightarrow &(s - 2\sqrt{s}E_p^*) + E_p^{*2} - p_p^{*2}(1 - f_p) - p_p^{*2} f_p + p_p^{*2} f_\pi = m_\pi^2 \end{aligned} \tag{6.11}$$

f_p and f_π represent the LIV correction functions, introduced in eq.(4.1) and (4.2) for the proton and the pion respectively.
From the previous relation follows:

$$E_p^* = \frac{s + m_p^2 - f_p p_p^{*2} - m_\pi^2 + f_\pi p_\pi^{*2}}{2\sqrt{s}} = F(s) \tag{6.12}$$

and one can approximate:

$$\begin{aligned} p_p^* &\simeq E_p' = (1 - k_\pi(\theta))\sqrt{s} \\ p_\pi^* &\simeq E_\pi' = k_\pi(\theta)\sqrt{s} \end{aligned} \tag{6.13}$$

The final energies, belonging to the final products after the interaction, are used and $\sqrt{s}$ represents the initial free total energy.
From the change of reference frame and approximating at the first order the coordinate change equations with the Lorentz invariant ones, it is possible to write:

$$E_p = \gamma_{CM}(E_p^* + \beta\cos\theta p_p) \tag{6.14}$$

where $E_p = (1 - k_\pi(\theta))E_{tot}$, using the pion inelasticity.
Substituting γ_{CM} with the value computed in eq. (6.10) and approximating the three-momentum magnitude with the energy and the velocity factor β with 1, in the hypothesis of ultra-relativistic particles, one obtains the following equation:

$$(1 - k_\pi(\theta)) = \frac{1}{\sqrt{s}}\left(F(s) + \cos\theta\sqrt{F(s)^2 - m_p^2 + 2f_p}\right) \tag{6.15}$$

From this it is possible to obtain the inelasticity as a function of the collision angle θ. The quantity must then be averaged on the interval $\theta \in [0,\ \pi]$ to obtain the inelasticity used in the computation:

$$k_\pi = \frac{1}{\pi}\int_0^\pi k_\pi(\theta)\,d\theta \qquad (6.16)$$

In fig. 6.1 an example is illustrated of the effects of a tiny perturbation, inferior to constrain (6.7), due to the introduction of LIV, on the value of the inelasticity of the photo-pion process. This quantity is expected to become $1/2$ for high energies. The consequent proton expected optical depth modification, as a function of its energy and of the LIV magnitude, is plotted in fig. 6.3, 6.4. This plots depend on the difference between the magnitude of the perturbation correlated with the proton and the pion: $f_{p\pi} = f_p - f_\pi$. In this work only LIV perturbations are considered, which imply that every particle has a maximum attainable velocity lower than c. It is a physically reasonable hypothesis to expect the more massive one having smaller velocity, that is a bigger violation, which corresponds to $f_p > f_\pi$. It is important to underline that even for very tiny violations of Lorentz Invariance, the effects can be absolutely relevant, implying a consistent dilatation of the predicted GZK opacity sphere. In computing the solution for the optical depth, the proton-photon cross section used is that obtained by the ZEUSS collaboration [154]. LIV modifications of the cross section are very tiny and therefore neglected in the computation. In fact, as underlined in [27, 28], the computation of the classical optical depth (without LIV) suffers of an uncertainty of almost $10-15\%$. So even the red shift correction are neglected in computation of the standard opacity sphere. Following the same logic, in this work the cross section is not modified and the corrections due to red shift are not considered.

6.3 Effects induced on heavy UHECR phenomenology

Natural generalization of the work, conducted on the propagation of UHE protons, is to expand the analysis to heavy cosmic rays. These UHECR are made of bare iron type nuclei and the principal way, they interact with CMBR and dissipate energy, is the photo-dissociation process (2.13). In fact even Compton scattering and pair production should be taken into account, but these processes are relevant only for cosmic rays with energy under $10^{19}\,eV$, above this limit the photo-dissociation contribution is dominant. During this process a photo-absorbtion happens, when the nucleus interacts with the CMBR. Then the compound, with excited energy level, decays, through nucleons emission. The process can be resumed by the relation:

$$^{A}_{Z}N + \gamma \longrightarrow ^{A'}_{Z'} N' + \, ^{A''}_{Z''}N'' \qquad (6.17)$$

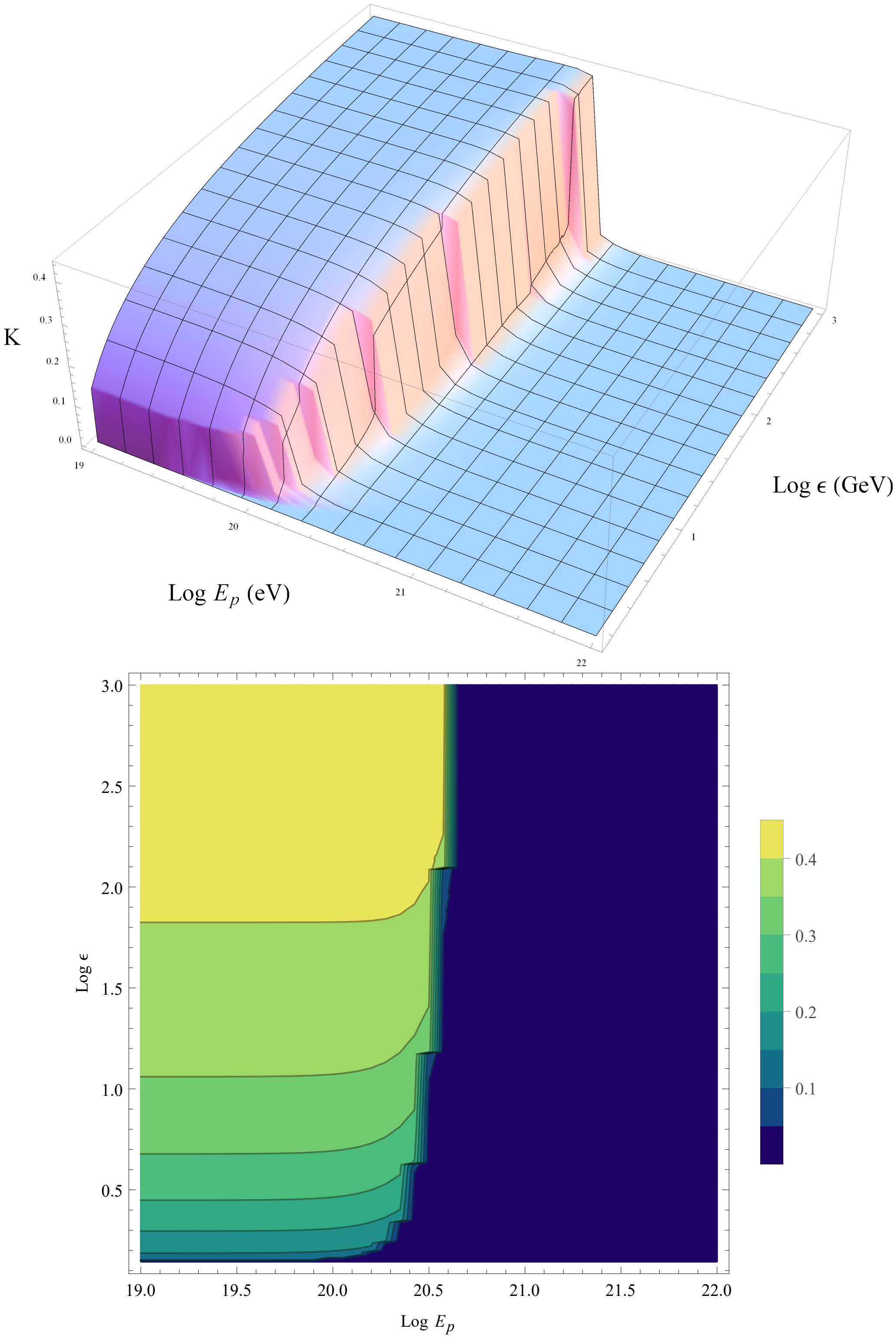


Figure 6.1: Inelasticity for perturbation $f_{p\pi} \simeq 9 \cdot 10^{-23}$

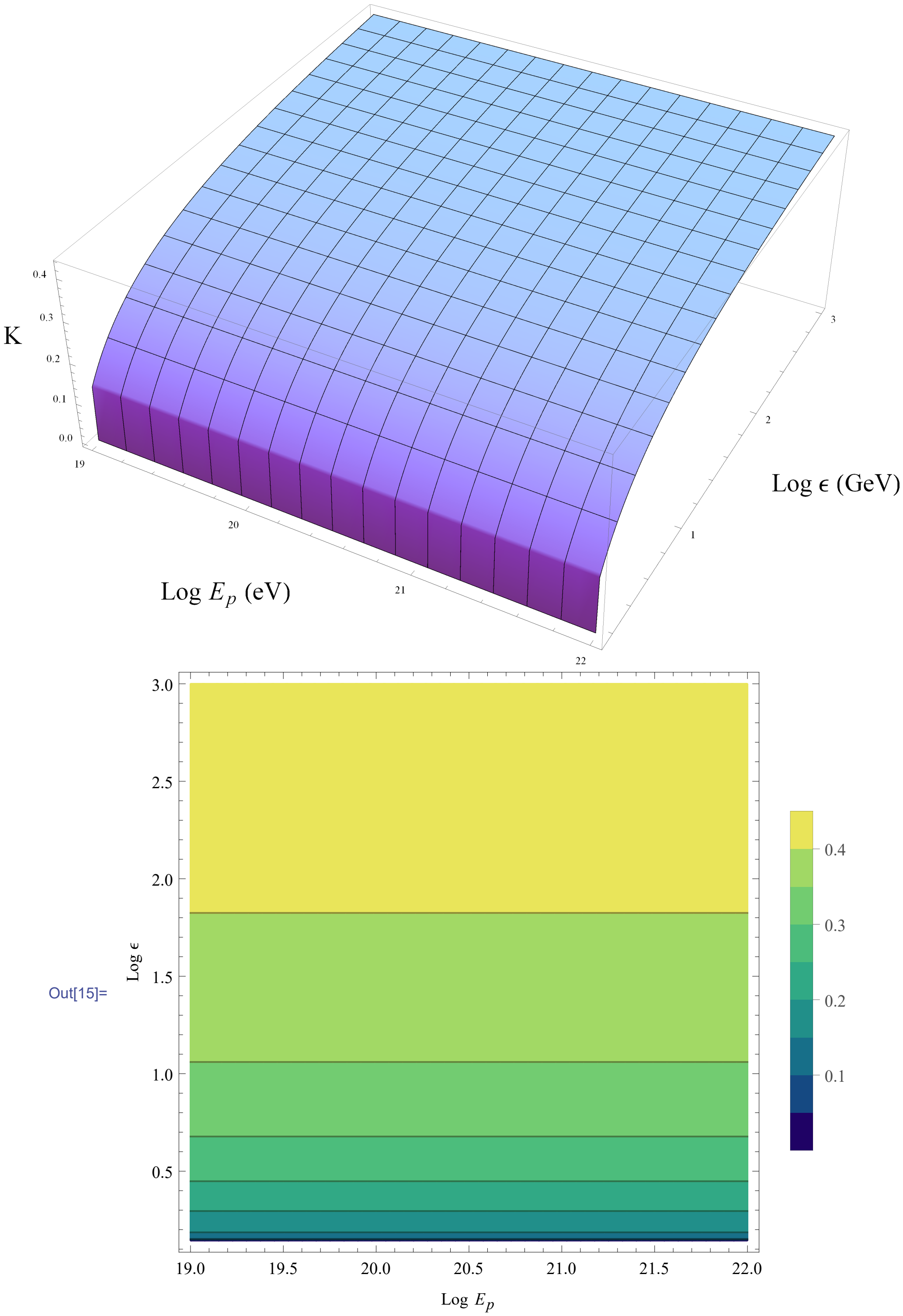


Figure 6.2: Inelasticity for perturbation $f_{p\pi} \simeq 0$

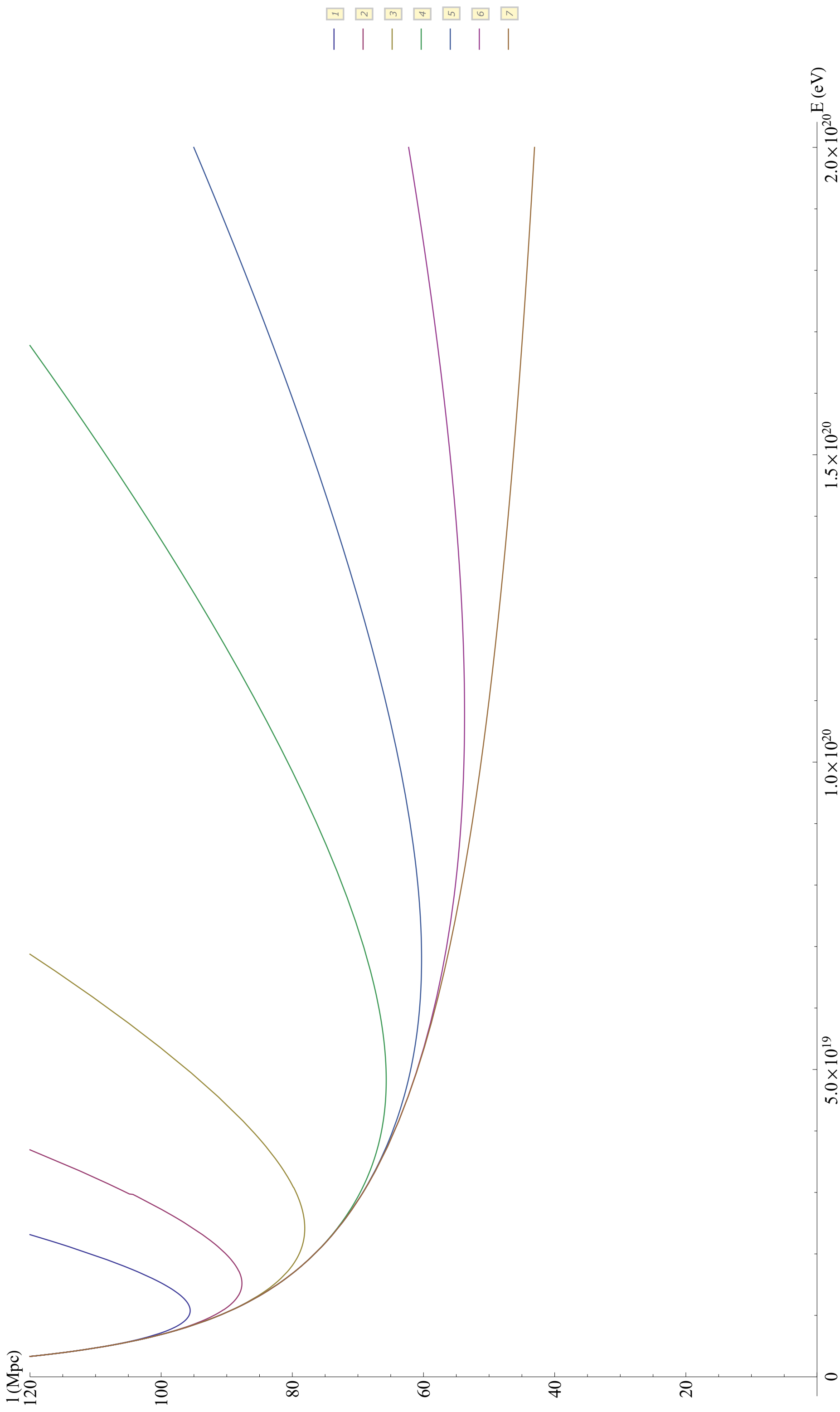


Figure 6.3: Optical depth as function of energy and LIV parameters, respectively: 1) $f_{p\pi} \simeq 9 \cdot 10^{-23}$, 2) $f_{p\pi} \simeq 6 \cdot 10^{-23}$, 3) $f_{p\pi} \simeq 3 \cdot 10^{-23}$, 4) $f_{p\pi} \simeq 9 \cdot 10^{-24}$, 5) $f_{p\pi} \simeq 6 \cdot 10^{-24}$, 6) $f_{p\pi} \simeq 3 \cdot 10^{-24}$, 7) $f_{p\pi} \simeq 0$

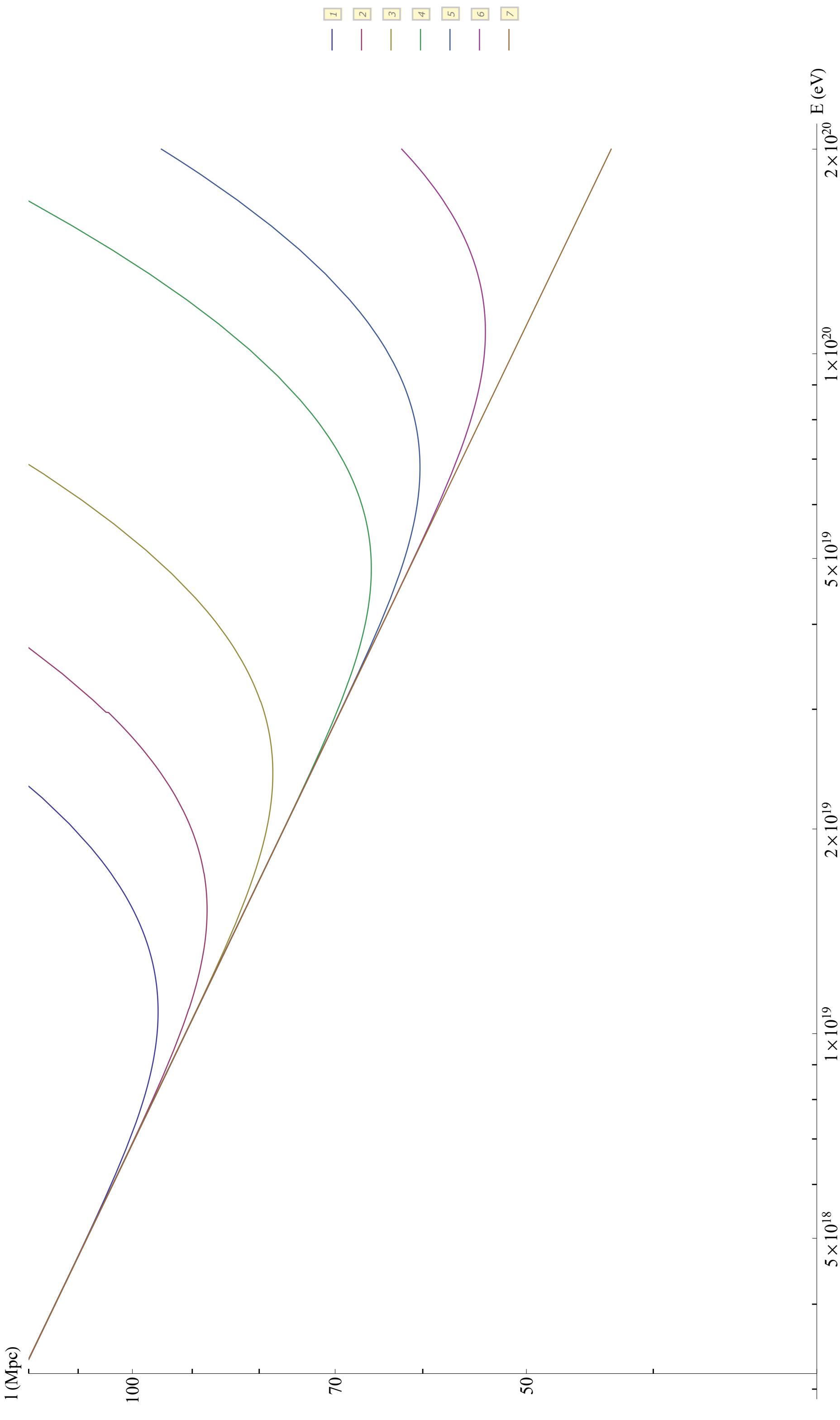


Figure 6.4: Optical depth as function of energy and LIV parameters, respectively: 1) $f_{p\pi} \simeq 9{\cdot}10^{-23}$, 2) $f_{p\pi} \simeq 6{\cdot}10^{-23}$, 3) $f_{p\pi} \simeq 3{\cdot}10^{-23}$, 4) $f_{p\pi} \simeq 9 \cdot 10^{-24}$, 5) $f_{p\pi} \simeq 6 \cdot 10^{-24}$, 6) $f_{p\pi} \simeq 3 \cdot 10^{-24}$, 7) $f_{p\pi} \simeq 0$

where $A = A' + A''$ and $Z = Z' + Z''$. ${}^{A}_{Z}N$ represents the initial nucleus, ${}^{A'}_{Z'}N'$ is the residual one and ${}^{A''}_{Z''}N''$ represents the nucleons emitted during the decay process.
As already underlined in [155], the LIV effects can manifest themselves modifying the reaction thresholds of the interaction processes. Using the MDR (4.4) it is possible to write for the free energy of the photo-dissociation process:

$$\begin{aligned}
&(E+\omega)^2 - (p[e^{-1}]+\omega)^2 = (E'+E'')^2 - (p'[e'^{-1}] + p''[e''^{-1}])^2 \Rightarrow \\
&\Rightarrow E^2 + \omega^2 + 2E\,\omega - p^2\left(1 - f\left(\frac{p}{E}\right)\right) - \omega^2 - 2p\,\omega\left(1 - \frac{1}{2}f\left(\frac{p}{E}\right)\right) = \\
&= E'^2 + E''^2 + 2E'E'' - p'^2\left(1 - f'\left(1 - \frac{p'}{E'}\right)\right) - p''^2\left(1 - f''\left(\frac{p''}{E''}\right)\right) + \\
&+2p'\,p''\left(1 - \frac{1}{2}f'\left(\frac{p'}{E'}\right)\right)\left(1 - \frac{1}{2}f''\left(\frac{p''}{E''}\right)\right)
\end{aligned} \tag{6.18}$$

where the thetrad explicit form (4.27) and the approximation $\sqrt{1-f} = 1 - \frac{1}{2}f$ have been repeatedly used. Resorting again to the relation (4.4), one can obtain the following:

$$\begin{aligned}
&m^2 + 2\,f\left(\frac{p}{E}\right)p^2 + 2\,E\,\omega + 2\,p\,\omega\left(1 - \frac{1}{2}f\left(\frac{p}{E}\right)\right) = m'^2 + 2\,f'\left(\frac{p'}{E'}\right)p'^2 + m''^2 + \\
&+2\,f''\left(\frac{p''}{E''}\right)p''^2 + 2\,E'\,E'' - 2\,p'\,p''\left(1 - \frac{1}{2}f'\left(\frac{p'}{E'}\right)\right)\left(1 - \frac{1}{2}f''\left(\frac{p''}{E''}\right)\right)
\end{aligned} \tag{6.19}$$

The large nuclei initial energy ($\sim 10^{20}\,eV$) implies that the decay products have momenta almost parallel to the direction of the original particle momentum, so from the momentum conservation one can obtain the following relations:

$$\begin{cases} p' = (1-y)\,p \\ p'' = y\,p \end{cases} \Longrightarrow \begin{cases} E' = (1-y)\,E \\ E'' = y\,E \end{cases} \tag{6.20}$$

where y represents the process inelasticity. From relation (6.19) and using the relations:

$$\begin{cases} E'E'' = E'E\,y = E'^2\left(\dfrac{y}{1-y}\right) = E''E\,(1-y) = E''^2\left(\dfrac{1-y}{y}\right) \\ p'p'' = p'p\,y = p'^2\left(\dfrac{y}{1-y}\right) = p''p\,(1-y) = p''^2\left(\dfrac{1-y}{y}\right) \end{cases} \tag{6.21}$$

it is simple to obtain the following equation:

$$
\begin{aligned}
& m^2 + 2\,f\left(\frac{p}{E}\right)p^2 + 4\,E\,\omega = m'^2 + 2\,f'\left(\frac{p'}{E'}\right)p'^2 + m''^2 + 2\,f''\left(\frac{p''}{E''}\right)p''^2 + \\
& + \left(E' - p'^2\left(1 - f'\left(\frac{p'}{E'}\right)\right)\right)\frac{y}{1-y} + \left(E'' - p''^2\left(1 - f''\left(\frac{p''}{E''}\right)\right)\right)\frac{1-y}{y} \;\Rightarrow \\
& \Rightarrow m^2 + 2\left(f\left(\frac{p}{E}\right) - f'\left(\frac{p'}{E'}\right)(1-y)^2 - f''\left(\frac{p''}{E''}\right)y^2\right)p^2 + 4\,E\,\omega + \\
& - \frac{m'^2}{1-y} - \frac{m''^2}{y} = 0
\end{aligned}
\tag{6.22}
$$

Now it has already proved that the momentum-energy ratio admits a finite limit $\frac{p}{E} \longrightarrow 1 + \delta$. Using the relations (6.20), it is possible to obtain that:

$$
\frac{p}{E} \simeq \frac{(1-y)p}{(1-y)E} \simeq \frac{p'}{E'} \simeq \frac{y\,p}{y\,E} \simeq \frac{p''}{E''} \tag{6.23}
$$

so in the high energy scenario, it results possible to substitute, in equation (6.21), the perturbation functions $\{f,\, f',\, f''\}$ with their high energy limits, denoted as $\{\epsilon,\, \epsilon',\, \epsilon''\}$. In this way one can obtain the following result:

$$
4\,\omega\,p + 2\left(-\epsilon + (1-y)^2\,\epsilon' + y^2\epsilon''\right) + m^2 - \frac{m'^2}{1-y} - \frac{m''^2}{y} = 0 \tag{6.24}
$$

which can be rewritten in the following way:

$$
\Upsilon(\epsilon,\, \epsilon',\, \epsilon'')\,p^2 + 4\,\omega\,p - \Delta M = 0 \tag{6.25}
$$

where:

$$
\begin{aligned}
\Upsilon(\epsilon,\, \epsilon',\, \epsilon'') &= 2(-\epsilon + (1-y)^2\epsilon' + y^2\epsilon'') \\
\Delta M &= \left(\frac{m'^2}{1-y} + \frac{m''^2}{y} - m^2\right)
\end{aligned}
\tag{6.26}
$$

Posing the new variable $x = \frac{4\,\omega}{\Delta M}$, it is possible to obtain, from equation (6.24), the following:

$$
\Omega\,x^2 + x - 1 = 0 \tag{6.27}
$$

where

$$
\Omega = \left(\frac{\Delta M\,\Upsilon(\epsilon,\, \epsilon',\, \epsilon'')}{(4\,\omega)^2}\right) \tag{6.28}
$$

This last equation admits 1 positive (allowed) solution for $\Omega \geq 0$ and 2 positive solutions for $\Omega < -\frac{1}{4}$. This limits the possible parameters kinematical configurations to obtain photodissociation. In this way it results possible to pose LIV caused constrains to the reaction. The LIV coefficients are supposed, as in the rest of this work, positive and ordered in a natural way, that is the bigger coefficients are associated to the heavier elements.
Now it is possible to evaluate the optical depth even for heavy cosmic rays. This quantity is evaluated as in the proton case, considering the photodissociation as an energy dissipation phenomenon. This approximation is possible because, even if after a photodissociation process the cosmic ray changes its chemical nature, the effect is caused by the emission of one or almost few nucleons. Moreover this process is not very frequent, so an ultra high energy heavy CR "does not change too much", after free propagation. Therefore if one considers heavy CR divided into families, with similar chemical composition, they will belong to the same family even after propagation. The optical depth is therefore evaluated as made previously, obtaining an equation similar to (2.20):

$$\tau_{opt}(p) = \frac{m_{AZ}^2}{2\,\beta\,p^2} \int_{E_{thr}}^{+\infty} \frac{d\omega}{\omega^2}\, n(\omega) \int_{\underline{E}}^{\overline{E}} dE \left(E - \frac{1}{2\,m_{AZ}} f_{AZ}\left(\frac{p}{E}\right)\right) k(E,\, p)\, \sigma(E) \tag{6.29}$$

where:

$$\begin{aligned} \underline{E} &= \frac{p}{m_{AZ}}(1-\beta)\,E + \frac{1}{2\,m_{AZ}} f_{AZ}\left(\frac{p}{E}\right) \\ \overline{E} &= \underline{E} + 2\,\beta \frac{p}{2\,m_{AZ}} E \end{aligned} \tag{6.30}$$

The $k(E,\, p)$ function represents the process inelasticity, amended again by the kinematics modifications induced by the LIV perturbation terms, in a similar way to the one computed for the light cosmic rays (protons). Therefore it is possible to conclude that in case of LIV, the optical depth for heavy nuclei presents a behavior similar to that evaluated in the proton cosmic rays case.

CHAPTER 7

LIV and Neutrino oscillations

7.1 LIV and Neutrino oscillations in an Hamiltonian approach

Let's focus now on the analysis of the eventual Lorentz violation effects impact on neutrino phenomenology. The introduction of LIV can in fact modify the flavor oscillation probabilities. The existence of this quantum phenomenon, the flavor oscillation, has been proved in experimental observations with natural neutrino sources (mainly solar [156, 157, 158, 159, 160, 161] and atmospheric [162]). Even with artificial neutrinos, short [163, 164, 165, 166, 167, 168] and long baseline [169, 170] reactor antinteutrinos, long-baseline [171, 172, 173] and the discussed LSND [174, 175] and MiniBOONE [176, 177] results confirmed this phenomenon. The evidences of oscillation have been further reinforced in the last decade by appearance experiments, like the CNGS beam [178], T2K [179] and NoνA [180], which collects neutrino signals with a flavor changed respect to the production one.

The LIV perturbation, introduced in this work, can account just for tiny perturbative effects, respect to the standard physics predictions. The presence only of Lorentz invariance violating interaction terms, determines a modification of the Hamiltonian H that rules the evolution of neutrino wave function. During its propagation, from the production point to the detector, the neutrino wave function evolves, in fact, according to Schroedinger equation: $i\partial_t|\psi\rangle = H|\psi\rangle$.

Following the SME approach to LIV, the extended Standard Model Lagrangian can be written in the general form [26]:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{LIV} \tag{7.1}$$

with

$$\mathcal{L}_{LIV} = -(a_L)_\mu \overline{\psi}_L \gamma^\mu \overline{\psi}_L - (c_L)_{\mu\nu} \overline{\psi}_L \gamma^\mu \partial^\nu \overline{\psi}_L \tag{7.2}$$

The first proportional to (a_L) term, in eq.(7.2), violates CPT and consequently the Lorentz invariance, while the second contribution, proportional to (c_L), breaks "only" Lorentz Invariance. Consequently, it is possible to build the effective LIV

Hamiltonian with the explicit form:

$$H_{eff} = H_0 + H_{LIV} \tag{7.3}$$

where H_0 denotes the usual Lorentz invariant Hamiltonian and H_{LIV} indicates the correction introduced by the LIV violating perturbative terms (7.2). Neglecting the standard part of the Hamiltonian (H_0) that (for a fixed momentum neutrino beam) contributes identically to all the three mass eigenvalues oscillations probabilities, it is possible to use a perturbative approach. The remaning part of the extended Hamiltonian becomes therefore:

$$H = \frac{1}{2E}\left(M^2 + 2(a_L)_\mu p^\mu + 2(c_L)_{\mu\nu} p^\mu p^\nu\right) \tag{7.4}$$

where M^2 is a 3×3 matrix, that in the mass eigenvalues basis assumes the form:

$$\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} \tag{7.5}$$

Resorting to the quantum mechanic perturbation theory, the new eigenstates can be written as:

$$|\widetilde{\nu_i}\rangle = |\nu_i\rangle + \sum_{i \neq j} \frac{\langle \nu_j | H_{LIV} | \nu_i \rangle}{E_i - E_j} |\nu_j\rangle \tag{7.6}$$

Now one can introduce the perturbed time evolution operator:

$$\begin{aligned} S(t) &= \left(e^{-(iH_0 + H_{LIV})t} e^{iH_0 t}\right) e^{-iH_0 t} = \\ &= \left(e^{-i(H_0 + H_{LIV})t} e^{iH_0 t}\right) S^0(t) \end{aligned} \tag{7.7}$$

and the oscillation probability can be evaluated as:

$$\begin{aligned} &P(\nu_\alpha \to \nu_\beta) = |\langle \beta(t) | \alpha(0) \rangle|^2 = \\ &\left| \sum_n \left[\langle \beta(t) | \left(|n_0\rangle\langle n_0| + \sum_{j \neq n} \frac{\langle j_0 | H_{LIV} | n_0 \rangle}{E_n^0 - E_j^0} |j_0\rangle\langle j_0| \right) |\alpha(0)\rangle + \ldots \right] \right|^2 \\ &= P^0(\nu_\alpha \to \nu_\beta) + P^1(\nu_\alpha \to \nu_\beta) + \ldots \end{aligned} \tag{7.8}$$

In eq.(7.8) $P^0(\nu_\alpha \to \nu_\beta)$ represents the standard foreseen oscillation probability, the remaining term is given by:

$$\begin{aligned} & P^1(\nu_\alpha \to \nu_\beta) = \\ & = \sum_{ij} \sum_{\rho\sigma} 2L\, \mathfrak{Re} \left(\left(S^0_{\alpha\beta}\right)^* U_{\alpha i} U^*_{\rho i} H^{LIV}_{\rho\sigma} U_{\sigma j} U^*_{\beta j} \tau_{ij} \right) \end{aligned} \tag{7.9}$$

with:

$$U_{\alpha i} = \langle \alpha | i \rangle \tag{7.10}$$

where $|\alpha\rangle$ represents a generic flavor eigenstate and $|j\rangle$ denotes a H_0 one, that is a mass eigenstate. Moreover in (7.9):

$$\tau_{ij} = \begin{cases} (-i)e^{-iE_i t} & i = j \\ \frac{e^{-iE_i t} - e^{-iE_j t}}{E_i - E_j} & i \neq j \end{cases} \tag{7.11}$$

with the constrains on the Hamiltonian matrix:

$$\begin{cases} H^{LIV}_{\alpha\beta} = \left(H^{LIV}_{\beta\alpha}\right)^* & \alpha \neq \beta \\ H^{LIV}_{\alpha\alpha} \in \mathbb{R} \end{cases} \tag{7.12}$$

Hence, as expected, also the flavor transition probability can be expanded perturbatively. In case of a general treatment of H_{LIV}, assuming a direction depending perturbation, it would be necessary to specify a privileged frame of reference when reporting this kind of results. But for the LIV model here introduced, isotropy is preserved and a privileged class of inertial observers is not required.

7.2 LIV and Neutrino oscillations in the isotropic scenario

Using directly the MDR constitutes an equivalent way to introduce LIV even in neutrino oscillations phenomenology. This is the way followed geometrizing the neutrino interactions with the background. In this work neutrino MDRs are supposed spherically symmetric also because until now there are no experimental evidences against this assumption. In this way the eq.(4.4) for MDRs reduces to the form:

$$E^2 = |\overrightarrow{p}|^2 \left(1 - f\left(\frac{|\overrightarrow{p}|}{E}\right)\right) + m^2 \tag{7.13}$$

Furthermore, since the perturbation function f is supposed homogeneous of degree 0, the MDR is originated by a metric in the momentum space and this guarantees the validity of Hamiltonian dynamics, as already illustrated. The ultra-relativistic particle propagation in vacuum, such as in case of neutrino, is governed by the Schrödinger equation, whose solutions are written in the form of generic plane waves:

$$e^{i(p_\mu x^\mu)} = e^{i(Et-\overrightarrow{p}\cdot\overrightarrow{x})} = e^{i\phi} \tag{7.14}$$

The effects of the modified metric do not appear, because the correction terms simplify, since the contraction is between a covariant and a controvariant vector. To give the explicit form of the solution, it is possible to start from the MDR (4.4), and using the approximation of ultrarelativistic particle $|\overrightarrow{p}| \simeq E$, we obtain:

$$\begin{aligned} |\overrightarrow{p}| &= \sqrt{|\overrightarrow{p}|^2\left(1-f\left(\frac{|\overrightarrow{p}|}{E}\right)\right)+m^2} \simeq \\ &\simeq\ E\left(1-\frac{1}{2}f\left(\frac{|\overrightarrow{p}|}{E}\right)\right)+\frac{m^2}{2E} \end{aligned} \tag{7.15}$$

In this way it is possible to evaluate the phase ϕ of the plane wave of eq.(7.14) for a given mass eigenstate, using the natural measure units, for which $t = L$:

$$\phi = Et - EL + \frac{f}{2}EL - \frac{m^2}{2E}L = \left(fE - \frac{m^2}{E}\right)\frac{L}{2}\,. \tag{7.16}$$

Hence the same energy E two mass neutrino eigenstates phase difference can be written as:

$$\begin{aligned} \Delta\phi_{kj} &= \phi_j - \phi_k = \frac{(f_j - f_k)}{2}EL - \left(\frac{m_j^2}{2E} - \frac{m_k^2}{2E}\right)L = \\ &= \left(\frac{\Delta m_{kj}^2}{2E} - \frac{\delta f_{kj}}{2}E\right)L \end{aligned} \tag{7.17}$$

The oscillation probability shows therefore a dependence on the phase differences $\Delta\phi_{kj}$, in addition to the usual 3×3 unitary matrix PMNS. The transition probability from a flavor $|\alpha\rangle$ to a flavor $|\beta\rangle$, in the most general case, that includes even the CP violating phase, can be written in the usual form:

$$\begin{aligned} P(\nu_\alpha\to\nu_\beta) =& \delta_{\alpha\beta} - 4\sum_{i>j}\mathfrak{Re}\left(U_{\alpha i}U^*_{\beta i}U^*_{\alpha j}U_{\beta j}\sin^2(\Delta\phi_{ij})\right) + \\ &+2\sum_{i>j}\mathfrak{Im}\left(U_{\alpha i}U^*_{\beta i}U^*_{\alpha j}U_{\beta j}\sin^2(\Delta\phi_{ij})\right) \end{aligned} \tag{7.18}$$

The modified oscillation probability results modified and this effect is caused by the appearance in the phase differences defined in eq.(7.17) of the LIV violating perturbation term, proportional to $\delta f_{kj} = f_k - f_j$. This term is different from zero only if the coefficients f_i, ruling the LIV violations are different for all the

three mass eigenstates. Otherwise the expression of equation (7.18) reduces to the usual three flavor oscillation probability, valid in absence of Lorentz invariance violation.
It is essential to notice that in this LIV including theory, MDR induced and CPT even, oscillation effects result caused by the difference of perturbations between different mass eigenstates ([92]). The fundamental assumption, that represents a reasonable physical hypothesis, is that every mass state presents a personal maximum attainable velocity, because interacts in a peculiar personal way with the background. It is even important to underline that the form of Lorentz invariance violation, introduced in this model, could not explain the neutrino oscillation, without the introduction of masses. In fact, the perturbative mass term, introduced in this LIV theory, is proportional to the energy of the particle, and this would be in contrast with the evidences of neutrino oscillations. In fact this phenomenon is ruled by a dominant mass term, that does not show such a dependence. Neutrino oscillations are well described by phase, depending only on squared masses differences, divided by the energy:

$$\Delta\phi_{jk} = \left(\frac{m_j^2}{2E} - \frac{m_k^2}{2E}\right) = \frac{\Delta m_{jk}^2}{2E} L \tag{7.19}$$

and LIV effects, of the type here introduced, could only appear at high energies as tiny perturbations (7.17). Therefore this theory can account only for relatively little deviations from what is considered "standard physics" and, in neutrino oscillation sector, could generate, at the highest observable energies, only tiny effects. Nevertheless these effects are very interesting experimentally, because they could open a window on what can be new fundamental physics, the realm of quantum gravity.
Other LIV theories can explain oscillations, without resorting to the classical concept of neutrino masses [181]. They usually introduce terms in the Standard Model Lagrangian that generate masses by the interaction with background fields, as in [182], where the modified Dirac equation can be written using the modified Dirac matrices:

$$\begin{aligned} \Gamma^{\mu}_{AB} =& \gamma^{\mu}\delta_{AB} + c^{\mu\nu}_{AB}\gamma_{\nu} + d^{\mu\nu}_{AB}\gamma_5\gamma_{\nu} + \\ &+ e^{\mu}_{AB} + if^{\mu}_{AB}\gamma_5 + \frac{1}{2}g^{\mu\nu\tau}_{AB}\sigma_{\nu\tau} \end{aligned} \tag{7.20}$$

and the modified mass matrix:

$$\begin{aligned} M_{AB} =& m_{AB} + im_{5AB}\gamma_5 + a^{\mu}_{AB}\gamma_{\mu} + \\ &+ b^{\mu}_{AB}\gamma_5\gamma_{\mu} + \frac{1}{2}H^{\mu\nu}_{AB}\sigma_{\mu\nu} \end{aligned} \tag{7.21}$$

In the previous equations m and m_5 are Lorentz and CPT conserving masses. The CPT conserving Lorentz violating terms are: $c,\ d,\ H$, while $a,\ b,\ e,\ f,\ g$ are CPT violating. It is important to underline that in this case, the LIV introduced mass terms would constitute a theoretical justification for the oscillations. However this kind of LIV introduced masses (differently from the CPT even LIV corrections present this model and also in [26]) would not spoil the general dependence of the oscillation probabilities on the neutrino energy. Therefore, it would not modify the "standard" oscillation pattern with the introduction of new effects that could be experimentally used to confirm or not the validity of LIV hypothesis.

7.3 Phenomenological analysis of the LIV effects on neutrino oscillations

Neutrino physics is an ideal playground to search for deviations from Lorentz invariance [183], thanks to its various set of experiments, covering a wide spectrum of energies and baselines.
The three oscillation probabilities, ruling the neutrino oscillations ($P_{\nu_e\nu_\mu}$, $P_{\nu_e\nu_\tau}$ and $P_{\nu_\mu\nu_\tau}$) are evaluated by means of equations (7.17) and (7.18)) in presence of LIV. This is made in order to evaluate the impact on neutrino phenomenology of the possible Lorentz invariance violations studied here. Comparing the results with the standard oscillation probabilities one gets if Lorentz invariance is satisfied.
This analysis has been pursued in the realistic three flavor scenario, differently from previous studies, that adopted the two flavor oscillation approximation. The values of the Δm^2_{ij} and of the various PMNS matrix elements ($U_{\alpha,i}$), used for the computations, have been taken from the most recent global fits, including all the different neutrino experiments [184, 185]. For simplicity, the value $\delta = 0$ is assumed for the Dirac CP violation phase, because in this case the study of CP violating effects would not spoil our results. This effect could be reintroduced, modifying in a simple way the analysis.
The outcome of this study is reported in the following series of figures, where the different oscillation probabilities $P_{\nu_\alpha\nu_\beta}$ are plotted. The graphs are obtained in absence and in presence of Lorentz violating terms, evaluated for fixed neutrino energy values, as a function of the baseline L, that is the distance between the neutrino production and detection points. The first two plots report the probabilities for a muonic neutrino to oscillate, respectively, into an electronic and a tauonic one. The probability $P_{\nu_\mu\nu_\tau}$ is the most relevant one for the atmospheric neutrinos study and for long-baseline accelerator neutrino experiments. The knowledge of $P_{\nu_\mu\nu_e}$ over a wide range of L (from 1 up to $10^5 - 10^6$ km) covers the regions of interest both for short- and long-baseline accelerator experiments

and, under the assumption of CPT invariance, also for reactor antineutrino experiments, because $P_{\bar{\nu}_e\bar{\nu}_\mu} = P_{\nu_\mu\nu_e}$. The remaining oscillation probability $P_{\nu_e\nu_\tau}$ is shown in fig..7.3. The energy value considered in this series of 3 figures (E = 1 GeV) has been chosen to have a similar magnitude, relevant for atmospheric and for long-baseline accelerator neutrino experiments. The oscillation probability

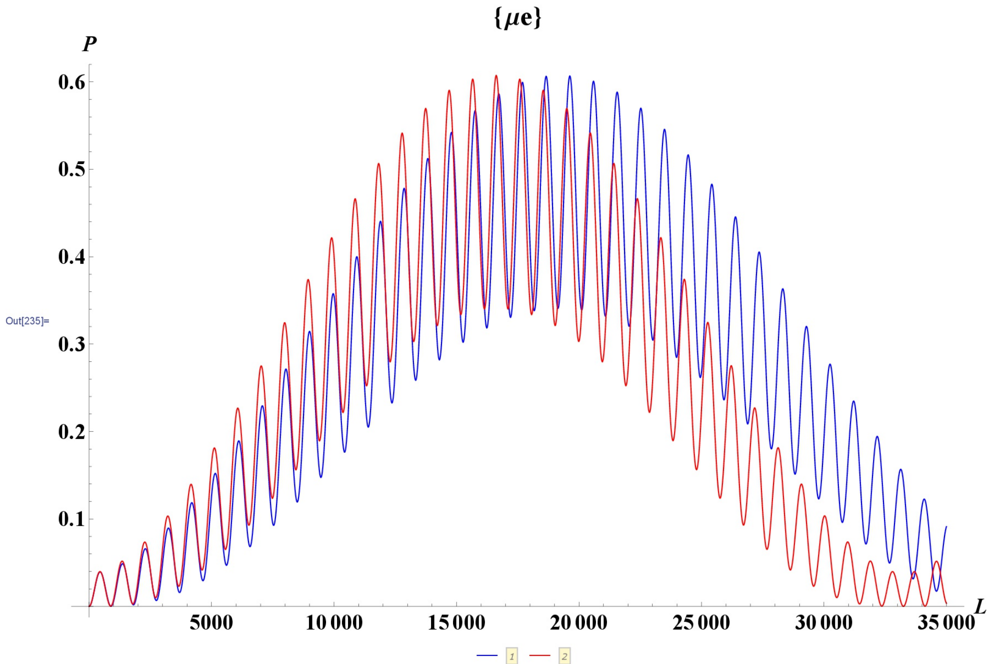


Figure 7.1: Comparison of the oscillation probability $\nu_\mu \to \nu_e$, computed, as function of baseline L, for neutrino energy $\mathrm{E} = 1\,\mathrm{GeV}$, "standard theory" (red curve) and LIV (blue curve), for LIV parameters $\delta f_{32} = \delta f_{21} = 1 \times 10^{-23}$.

corrections order of magnitude is determined by the values chosen for the three parameters f_k, inducing the LIV perturbation and, consequently, for their differences δf_{kj}, as shown in eq.(7.18). For simplicity the 3 parameters f_k are assumed of same order of magnitude and are ordered in a "natural" way, with the highest LIV parameter correction associated to the highest mass eigenvalue (that is: $f_1 < f_2 < f_3$ and $\delta f_{32} \simeq \delta f_{21}$). In figs.7.1-7.3 the values $\delta f_{32} = \delta f_{21} = 1 \times 10^{-23}$ are adopted. These limits have the same order of magnitude of the constrains derived in the phenomenological studies, for LIV violation, one could find in literature up to 2015 [186], or even more conservative. As one can see clearly from figs. 7.1-7.3, for $\delta f_{ki} = 1 \times 10^{-23}$ the presence of LIV would modify in a visible way the oscillation probabilities patterns.
Recently the SuperKamiokande collaboration, however, performed a test of Lorentz invariance, analyzing atmospheric neutrinos and derived more stringent con-

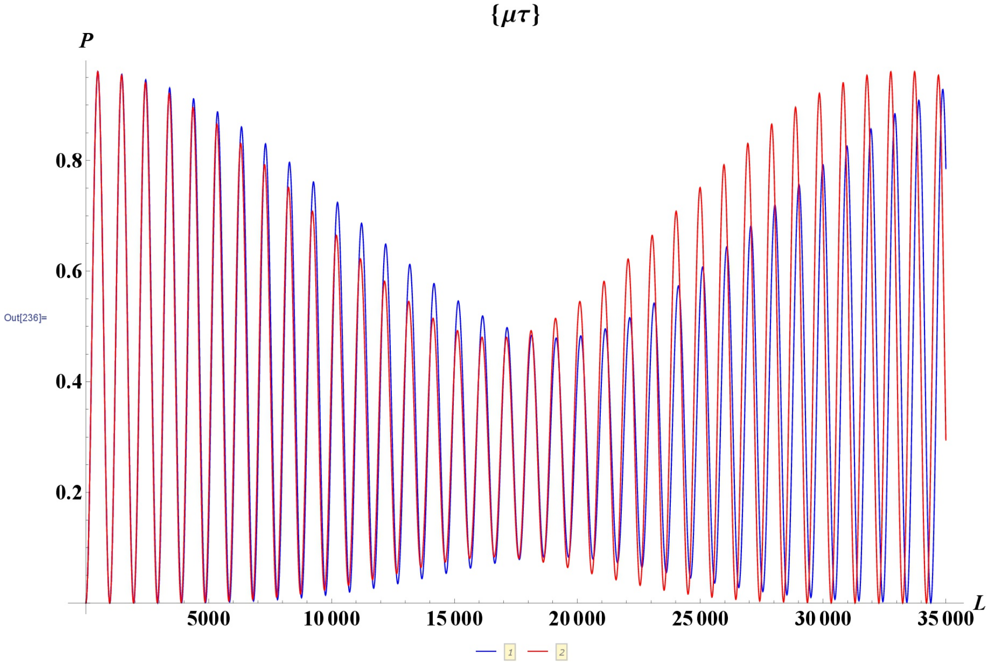


Figure 7.2: Same analysis of fig.7.1, but for the oscillation $\nu_\mu \to \nu_\tau$.

straints on the possible coefficients values for the Hamiltonian Lorentz invariant violating corrections [187]. In particular limits of the order of $10^{-26} - 10^{-27}$ were derived for the coefficient of the isotropic CPT even term. This terms introduces corrections to the oscillation probabilities proportional to $L \times E$ and would correspond to the kind of Lorentz invariance violation here presented. But it is necessary to underline that the parameter, introduced in this work, presents some differences from other works, it is just correlated to the isotropic coefficient of SME. As a matter of fact, the comparison between this model and the Hamiltonian, derived in SME and used, as reference, for the SuperK analysis, is not so immediate. In that Hamiltonian are present also other kind of LIV violating corrections and in particular CPT odd terms, that introduce corrections to $P_{\nu_\alpha \nu_\beta}$, not proportional to the neutrino energy, of the order of 10^{-23}.
The fig. 7.4 reports the comparison of the $\nu_\mu - \nu_e$ oscillation probabilities with and without LIV for values of our parameters $\delta f_{kj} = 10^{-25}$. In this case the two curves are practically superimposed and the situation is essentially the same also for $P_{\nu_\mu \nu_\tau}$ and for $P_{\nu_e \nu_\tau}$. The effects of LIV corrections are no more visible and the percentage variations of $P_{\nu_\alpha \nu_\beta}$ are lower than 1% in all the regions where P is significantly different from zero. Therefore the LIV effects on the oscillations probabilities are observable only for higher neutrino energies, for values of the δf_{kj} coefficients of the same order derived by SuperKamiokande, for the CPT

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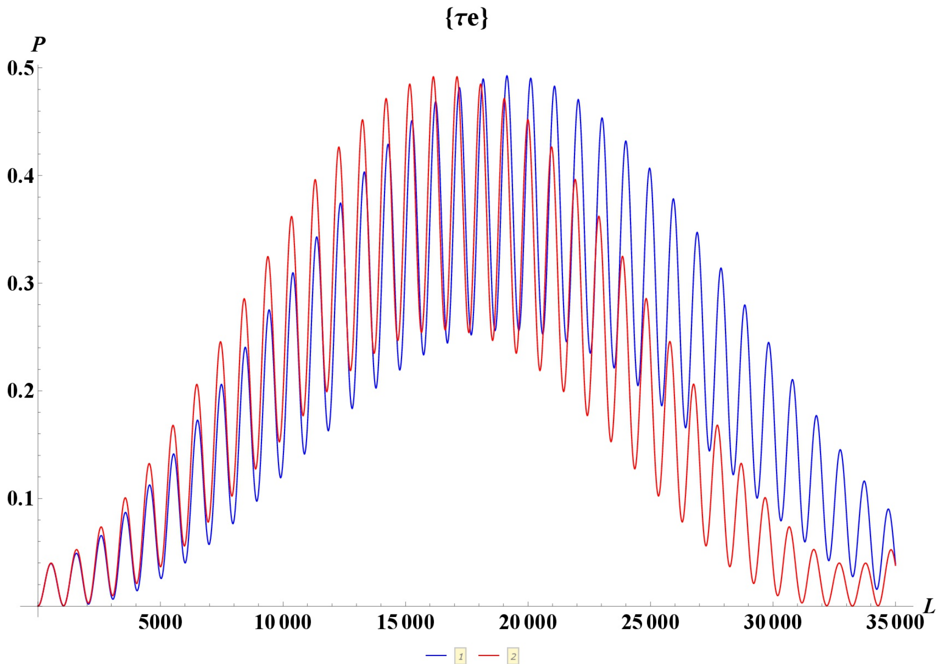

Figure 7.3: Same analysis of fig.7.1, but for the oscillation $\nu_e \to \nu_\tau$.

even isotropic LIV corrections ($\delta f_{kj} \simeq 10^{\cdot -26} - 10^{-27}$).
In the figs. 7.5-7.8 the results for the three oscillation probabilities and for the total ν_μ survival probability $(1 - P_{\nu_\mu . \nu_e} - P_{\nu_\mu , \nu_\tau})$ are presented energy studied for atmospheric neutrinos by SuperKamiokande and neutrino telescopes, in case of 100 GeV neutrino. In these graphs $\delta f_{32} = \delta f_{21} = 4.5 \times 10^{-27}$ is assumed, that is the same magnitude order derived for the corresponding parameter by SuperKamiokande. For these values of the LIV coefficients, perturbation effects are visible and they induce probabilities variations, at least of a few percent for most values of L, fig.7.9. In this figure simultaneously the percentage variations for all the 3 probabilities $P_{\nu_\alpha , \nu_\beta}$ are present. The LIV corrections percentage are computed as $2\,\frac{P_{LIV} - P_{NOLIV}}{P_{LIV} + P_{NOLIV}} \times 100$ and are evaluated over a restricted baseline values set, for which the oscillation probabilities are observable. The LIV induced percentage variations are higher than $5 - 10\%$ for P_{ν_μ , ν_e} for almost all the considered baseline values and, for the two other oscillation probabilities, above $2 - 3\%$. The LIV corrections become particularly significant for $\mathrm{L} > 60000\,\mathrm{km}$ (more than $15\,\%$ for P_{ν_μ , ν_e}) [1] in the energetic range considered. The impact

[1]The interpretation of these percentage variations must be conducted with caution. It must be evaluated considering also the absolute value of the oscillation probabilty, used to "normalize" these variations. For some values of L, higher percentage variations sometimes are mainly due to the fact that the corresponding absolute value of $P_{\nu_\alpha , \nu_\beta}$ is extremely small.

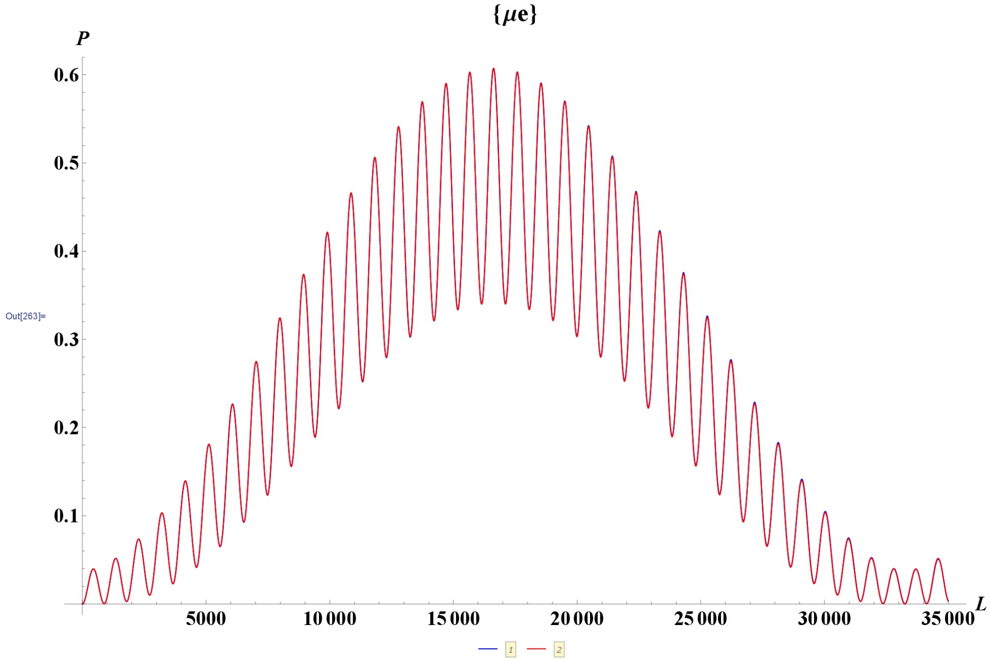

Figure 7.4: Same analysis of fig.7.1, but for LIV parameters $\delta f_{kj} \simeq 10^{-25}$.

of the LIV corrections increases if one considers higher energy neutrino beams. For instance, neutrino energies in the region from TeV to PeV, are interesting for present and future neutrino telescopes, like ANTARES [188], KM3NET [189] and (for the higher energies mainly) IceCube [190]. Even higher energies can be of great interest, as in the case of Ultra High Energy (above EeV) cosmic neutrinos, investigated, for instance, by Auger [191, 192]. These cosmic neutrinos will play a relevant role in a multimessenger approach, in future astrophysical researches, stimulated by the recent gravitational waves observation [193, 194, 195]. The effect of Lorentz violation for a 1 TeV neutrino is studied, starting from the analysis characterized by 3 different sets of possible values for the δf_{kj} parameters. In the first case $\delta f_{32} = \delta f_{21} = 4.5 \times 10^{-27}$, corresponding to the present limit derived by SuperKamiokande, while in the other 2 cases δf_{kj} values are taken lower, respectively, of one and two orders of magnitude ($4.5 * 10^{-28}$ and $4.5 * 10^{-29}$). The promising results for the 3 oscillation probabilities (P_{ν_μ,ν_e}, P_{ν_μ,ν_τ}, P_{ν_e,ν_τ}) are reported in the series of graphs of figs. 7.10-7.12. In fig. 7.13 the total survival probability of muonic neutrino is plotted. The curves corresponding to the LIV expressions, obtained for $\delta f_{32} = \delta f_{21} = 4.5 \times 10^{-27}$ (blue lines), are signiflcantly different from the ones obtained in absence of LIV violations (orange curves). Moreover, the corrections, caused by LIV, remain significant also for δf_{kj} parameters one order of magnitude lower (red). They are in any case apprecciable even

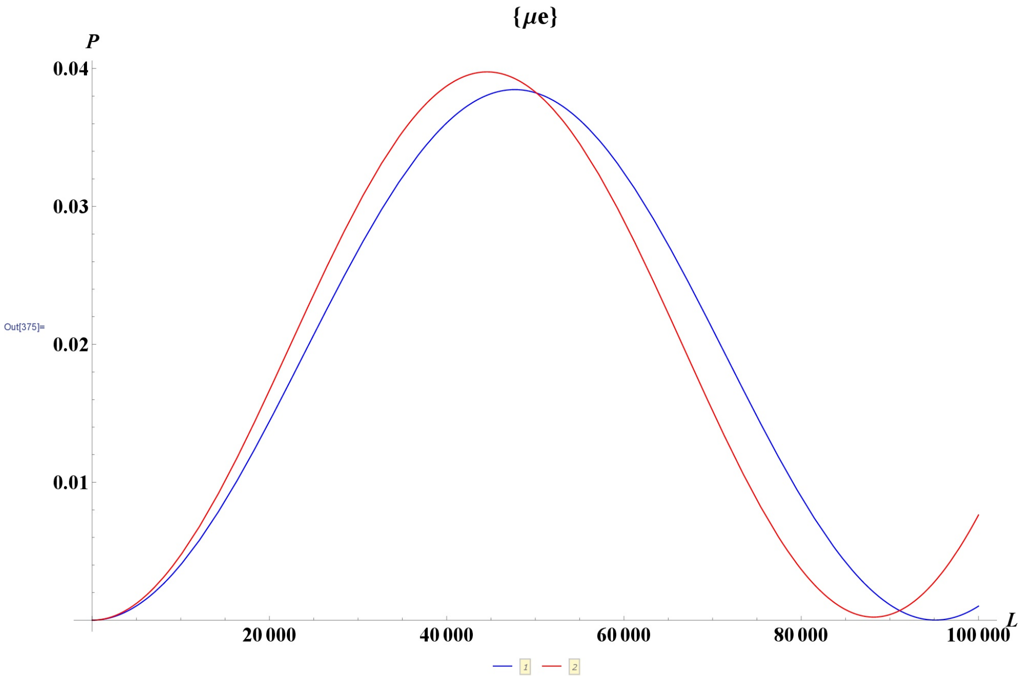

Figure 7.5: Same analysis of fig.7.1, but for LIV parameters $\delta f_{32} = \delta f_{21} = 4.5 \times 10^{-27}$ and for neutrino energy E = 100 GeV.

for $\delta f_{32} = \delta f_{21} = 4.5 \times 10^{-29}$ (green curve), at least for baseline values sufficiently high. Hence, selecting the appropriate experimental context, in future one could use the detailed study of high energy neutrinos to further constraint the LIV coefficients. Significant deviations from the standard probabilities values could appear already for energies around 1 GeV, considering LIV coefficients of magnitude around 10^{-23}, usually analyzed in literature [186, 196, 197, 198, 199, 200]. Instead, if the magnitude of LIV corrections is significantly limited, the effect of LIV on the oscillation probabilities starts to become evident only for higher energies. That is around 100 GeV, for instance considering the values obtained recently by SuperKamiokande [187], for the CPT even LIV coefficients. The situation for 1 TeV neutrinos is analyzed, taking into account the improvement that should be possible to obtain on LIV coefficients, in a scenario more promising, the ultra high energy neutrinos (like cosmic ones). To obtain a full phenomenological analysis, usable in any realistic experimental situation, the information about the oscillation probability must be complemented by an accurate knowledge of the expected fluxes, for every flavor neutrinos, and the knowledge of the different interaction cross sections. The number $N_{\alpha,\beta}$ of detected transition

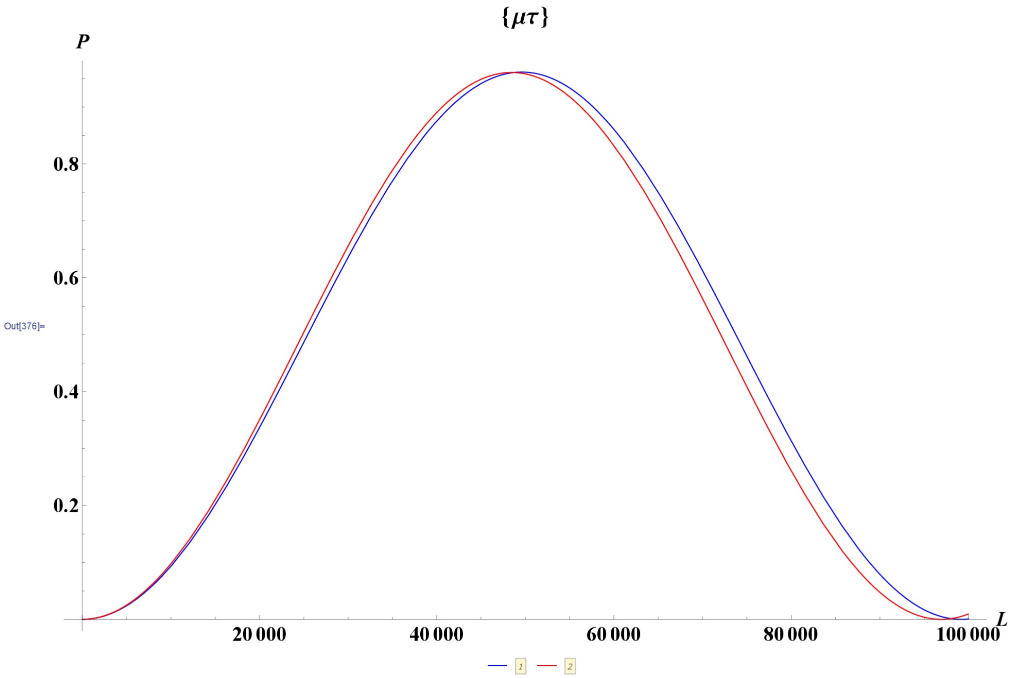

Figure 7.6: Same of fig.7.5, but for the oscillation probability $P_{\nu_\mu \nu_\tau}$.

events caused by the $\nu_\alpha \to \nu_\beta$ flavor oscillation, will be given by:

$$N_{\alpha,\beta} \propto \Phi_\alpha(L, E)\, P_{\nu_\alpha,\nu_\beta}(L, E)\, \sigma_\beta(E) \tag{7.22}$$

$\Phi_\alpha(L, E)$ represents the predicted flux of an α flavor neutrino, in oscillation absence, at given energy E, $\sigma_\beta(E)$ is the interaction cross section of a β neutrino with the detector, again energy dependent, and L represents the distance travelled, by neutrinos, from the production to the detection point. This information must be integrated over the neutrino energies and eventually also over the distances L. Finally the integrals has to be convoluted with functions describing the detector resolution and efficiencies.
From the comparison between the experimental results and the theoretical predictions, one can extract the information about the impact of this model supposed LIV violations. Otherwise one can put constraints on the magnitude order of the LIV coefficients.

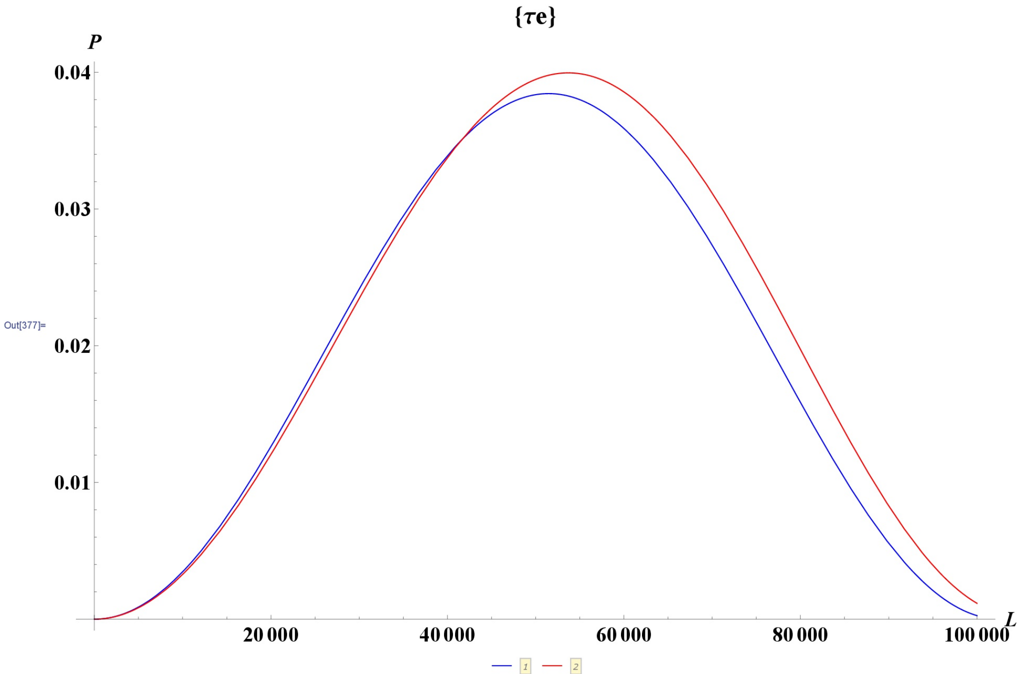

Figure 7.7: Same of fig.7.5, but for $P_{\nu_e \nu_\tau}$.

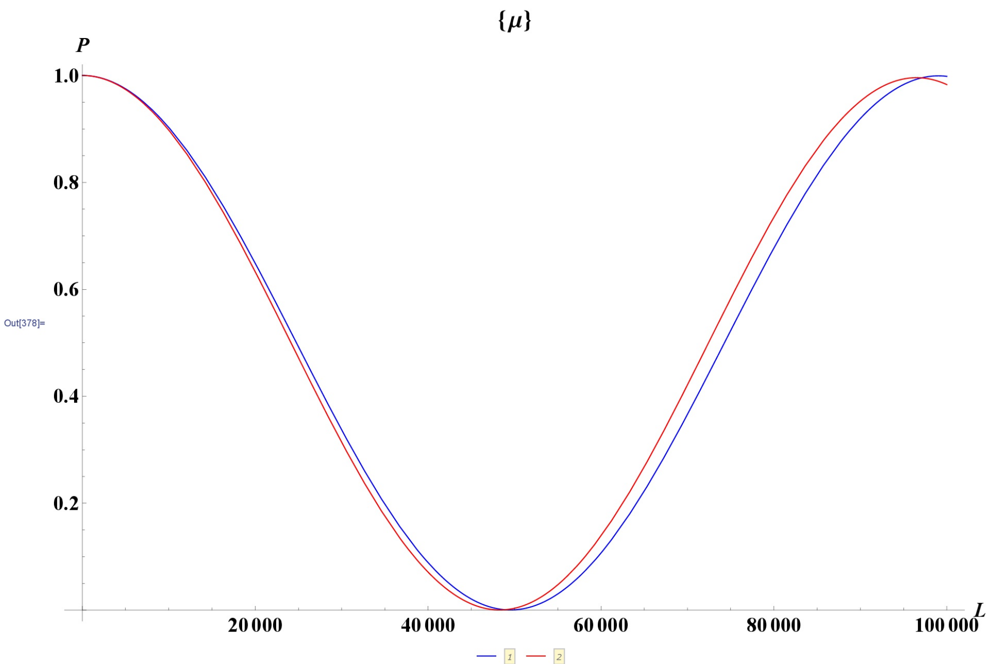

Figure 7.8: Survival probability for muonic neutrino, for the same conditions of fig.7.5

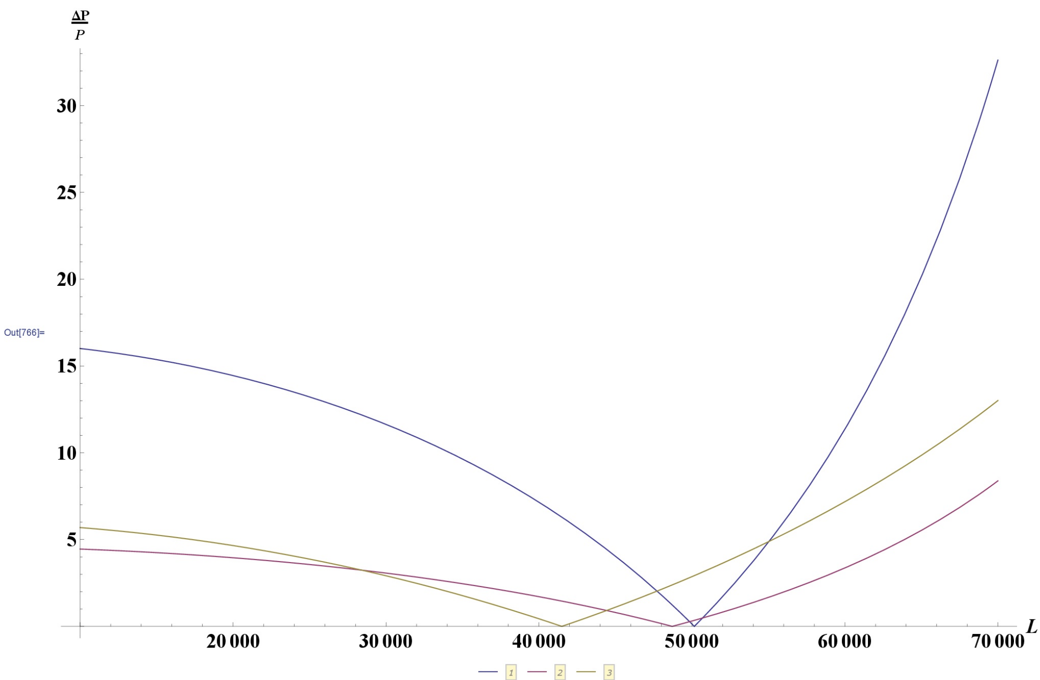

Figure 7.9: Percentage variations induced in neutrino oscillation probabilities by LIV corrections. On the vertical axis, as function of the baseline L, the percentage differences between the oscillation probabilities, for a 100 GeV neutrino, in presence and in absence of LIV, normalized respect to their average value. The 3 different curves correspond to the percentage differences for the 3 oscillation probabilities: $P_{\nu_e \nu_\mu}$ (blue), $P_{\nu_\mu \nu_\tau}$ (violet) and $P_{\nu_e \nu_\tau}$ (green curve).

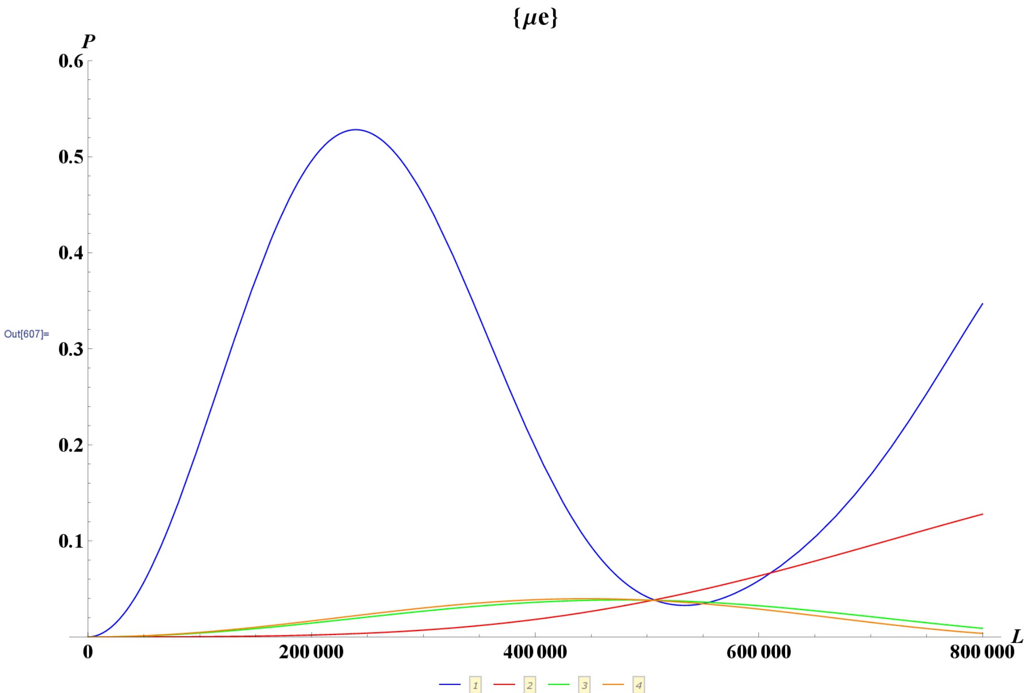

Figure 7.10: Comparison of the $P_{\nu_\mu \nu_e}$ oscillation probability, as function of baseline L, for neutrino energy $\mathrm{E} = 1\,\mathrm{TeV}$, for ”classical theory”, preserving Lorentz Invariance (orange curve) and for LIV including models, with parameters equal, respectively, to $\delta f_{32} = \delta f_{21} = 4.5 \times 10^{-27}$ (blue), $\delta f_{32} = \delta f_{21} = 4.5 \times 10^{-28}$ (red) and $\delta f_{32} = \delta f_{21} = 4.5 \times 10^{-29}$ (green curve).

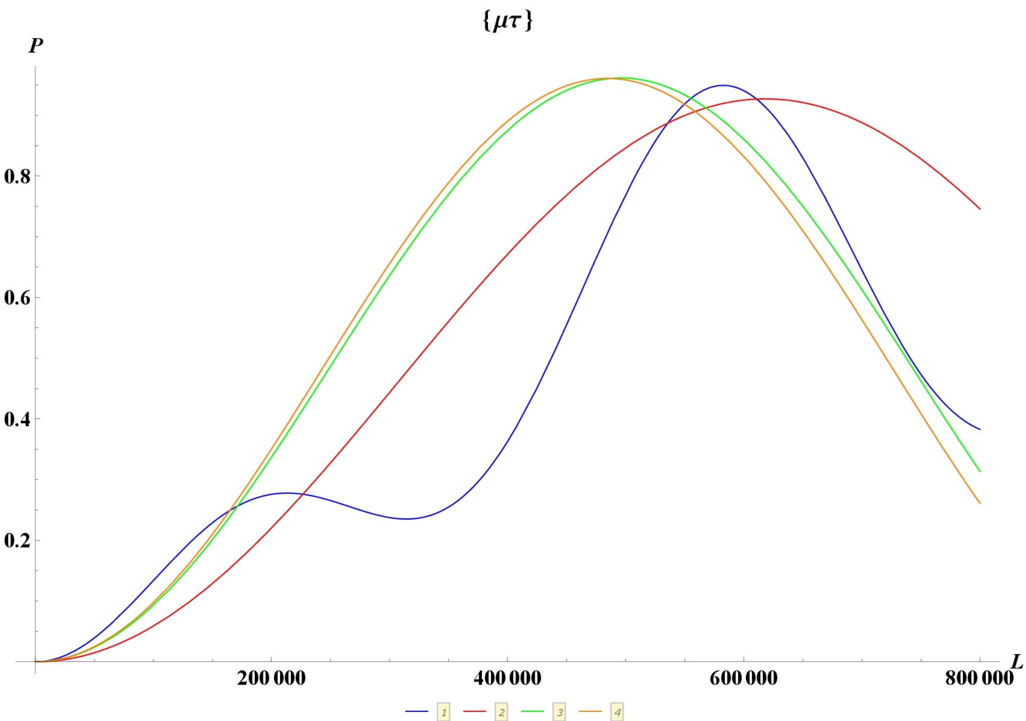

Figure 7.11: Same analysis of fig.7.10, but for the case of $P_{\nu_\mu \nu_\tau}$.

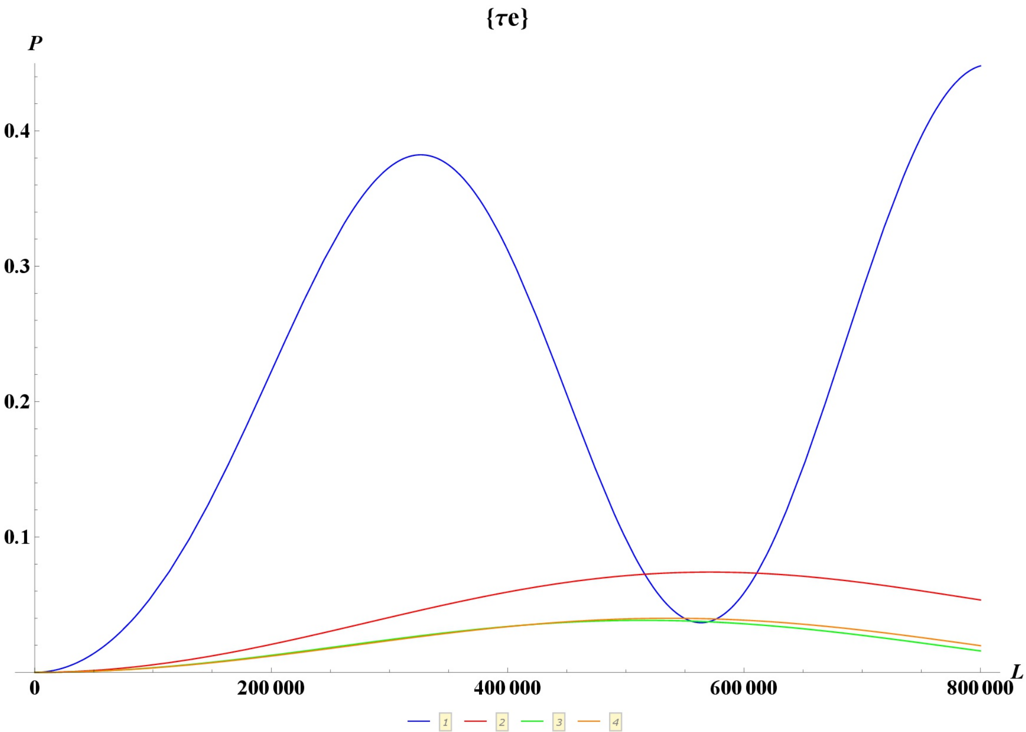

Figure 7.12: Same analysis of fig.7.10, but for $P_{\nu_e \nu_\tau}$

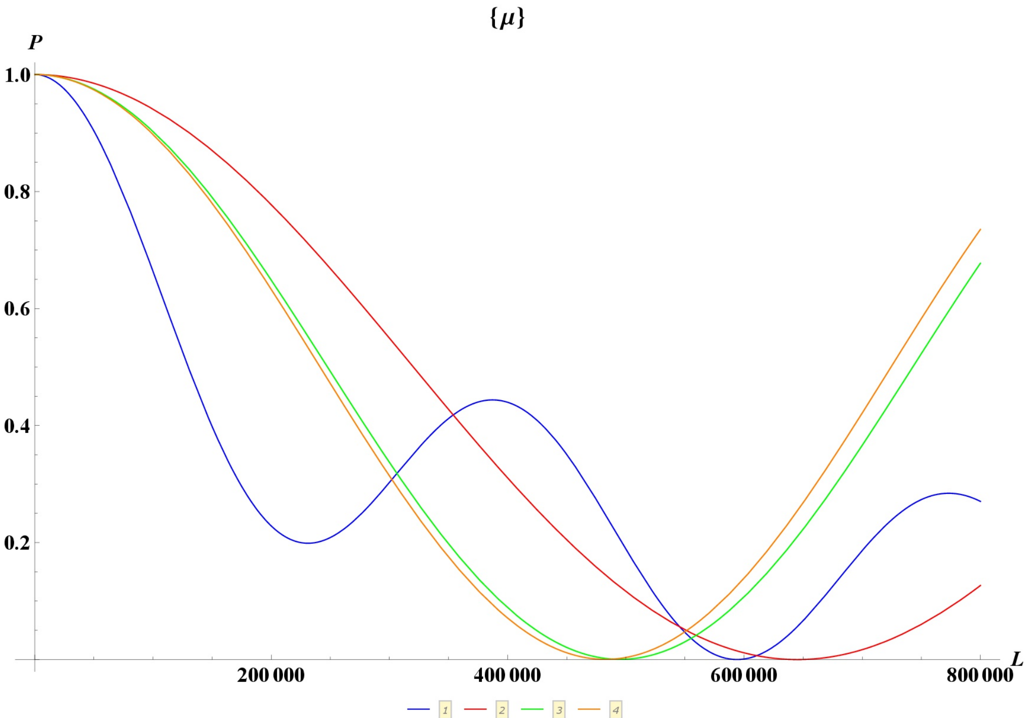

Figure 7.13: Comparison of the results for the muonic neutrino survival probability in a classical theory and in models with LIV corrections, corresponding to three different values of the δf_{kj} parameters, as illustrated in fig.7.10. Also the color code is the same adopted in figs.7.10-7.12.

Conclusion

Lorentz covariance constitutes a fundamental property that underlies all standard physics theoretical models. However some departures from conventional physics are expected, as residual effects of a more fundamental physical theory, that provides a quantized description of space-time. Since UHECR are the most energetic particles, nowadays accessible, their physics can open a window on quantum gravity effects phenomenology. Over the last twenty years some experimental evidences ([17]) emerged, indicating the eventuality that the GZK opacity sphere is modified, increasing its dimensions. Until the pioneering work of Coleman and Glashow ([18]), Lorentz invariance violation is indicated as a possible explanation of this "cosmic ray puzzle". In fact a little departure from exact Lorentz covariance can introduce dramatic changes in UHECRs behaviour, thanks to their huge energies and their propagation on cosmological distances [27, 28]. The investigation on the predicted universe opacity to UHECR propagation can therefore represent a probe for the validity of Lorentz invariance, or at less can furnish a useful way to increase the constrain knowledge about an eventual Lorentz invariance violation. Most of the models considered to introduce LIV, even in UHECR propagation physics, share the feature of a preferred reference frame, breaking the Lorentz covariance and the space-time isotropy, in an Effective Field Theory scenario ([26]).

The central idea of this work consists in indicating a possible way to introduce a standard model extension, that preserves the idea of isotropy. As already highlighted, this is possible taking into account some concepts of the SME [26] with some ideas borrowed from DSR [121, 24]. The key point consists in constructing the space-time, starting from a pure kinematical modification, that results depending on the particle species, interacting with the background. The kinematical modification is introduced modifying the dispersion relations, as in other works ([139, 119, 120], but the LIV perturbation is introduced using a 0 degree homogeneous function of the ratio $\frac{p}{E}$. In this way the geometrical derivation of

the dispersion relation is preserved and this leads to a new space-time structure, that depends on the propagating material body momentum, the Finsler geometry [23, 25, 201, 202, 203, 204]. The Lorentz invariance is not broken, but modified, introducing an amended Lorentz group, in order to redefine the concept of spatial symmetries and reconcile the introduced perturbations of space-time with the idea of symmetry conservation. Furthermore the perturbation considered have only a kinetic character, so the dynamic is not affected and new exotic interactions are not introduced, preserving the internal $SU(3) \times SU(2) \times U(1)$ particle standard model symmetry. In fact the modified Lorentz group represents the isometric group of the amended dispersion relations. Moreover it results possible to modify the definition of the Mandelstam variables, that result invariant under the action of the modified Lorentz group. The S matrix, that in every interaction physical theory correlates initial states with the final ones, is a function of the Mandelstam variables, therefore it results covariant under the action of the amended Lorentz group. In this way it results simple to generalize the concept of isotropy, respect to the new generalized personal Lorentz transformations.
This way of introducing LIV is analogous to consider not null the trace of the isotropic tensor $c_{\mu\nu}$ of the Standard Model minimal extension [26]. This term is usually neglected, because it corresponds to a field normalization redefinition. That corresponds to a redefinition of the maximum attainable velocity, effect very difficult to be detected if it presents an universal character. On the contrary, following this work theory, the modified maximum attainable velocity presents a species depending nature. Physical effects can emerge only in interaction processes where different particle species are involved, as in case of GZK cut-off effect. Furthermore the LIV introduced in this work is compatible with the deSitter projective relativity, giving therefore another theoretical support idea to the hypothesis of 0 degree homogeneous perturbation functions in MDR.
In this work the predicted opacity GZK sphere dimension increase has been determined, as function of some LIV parameter values. The impact of LIV on the UHECR interaction, with the CMBR, manifests as a kinematical effect. In fact the larger effect appears in the inelasticity computation, demonstrating that the phase space for the photo-pion production can result dramatically reduced even by tiny Lorentz violating perturbations. The computation has been conducted following the isotropic LIV model, introduced in this work, demonstrating that a privileged class of inertial observers is not required to justify a modification of the predicted GZK opacity sphere, result already known in the DSR scenario [205].
Neutrino constitute another ideal candidate to conduct observation on the preservation of Lorentz covariance. In this work is analyzed, in fact, even the impact of LIV perturbative corrections on neutrino phenomenology, by means of an oscillation probabilities detailed study. The modification of the dispersion relations,

caused by Lorentz invariance violating perturbation terms, determines a change of the "phase differences" $\Delta\phi_{ij}$. These terms enter as contribution of the i, j mass eigenstates into the explicit form of the oscillation probability functions (7.18). As consequence, in addition to the usual term $\frac{\Delta m_{ij}^2 L}{E}$, another contribution appears in the expression for $\Delta\phi_{ij}$, dependent on the LIV coefficients differences and proportional to $L \times E$. This means that this LIV model again predicts detectable effects only if the perturbation terms are not identical for all the mass generations. Moreover, the LIV corrections are proportional to E, instead of $\frac{1}{E}$. This implies that, to be consistent with the data from the different oscillation experiments, these corrections must represent only tiny perturbations. That is the perturbations must be small in order to do not change the general pattern of neutrino oscillation. Nevertheless, in particular experimental situations, these corrections could be significant. Therefore it could be possible to further constrain the possible values of the LIV coefficients, with an appropriate choice of the experimental tests. It is important to underline that other works, based on EFT, such as in [92], predicted this kind of oscillation corrections. This model faculty to reproduce these predictions constitutes a validity test for the geometrical approach, here introduced, which gives a theoretical background to the introduction of MDRs.

Appendices

APPENDIX A

Finsler Geometry

A.1 Introduction

As already underlined, the necessity to resort to a more general geometry emerges naturally in many theoretical models that describe LIV [206, 207, 208]. This new geometry must be able to deal with a local space-time structure dependence on velocity, or equivalently energy. This kind of geometry is represented by the Finsler one. A Finsler manifold is a space where the geometrical structure is not parameterized only by points, but even by vectors defined on the tangent space. This means that this manifold type is parameterized not only by points but even by directions. In this sense a Finsler manifold is a generalization of the analogous Riemann structure, where the geometry depends only on position coordinates. The Finsler geometry presented here is the natural generalization of the euclidean Riemann one, a geometric structure where the global underlying metric is the euclidean one $diag(1,\,1,\,1,\,1)$. Instead in all physical models of interest the Finsler structure is a pseudo-Finsler geometry, a generalization of the Minkowski Riemann geometry, where the underlying global metric is the Minkowski one $diag(1,\,-1,\,-1,\,-1)$. All the main results presented here remain valid even in this scenario, with the metric that is not positive definite. This introduction follows the books of Shen [133], based on the original idea of Paul Finsler [209].

A.2 Minkowski norm

First property of Finsler geometry is that every manifold is equipped with a Minkowski norm:

Definition 1. (**Minkowski norm**)
A Minkowski norm is a function defined on a vector space V*:*
$F\,:\,V\,\longrightarrow\,[0,\,\infty)$*, satisfying the following properties:*

1. $F \in C^{\infty}$ *in* $V \setminus \{0\}$

2. $F(\lambda\, v) = \lambda\, F(v)$ *that is* F *is homogeneous of degree* 1
3. $\forall\, v\, \in\, V \setminus \{0\}$ *the hessian* $g_{ij}(v) = \frac{1}{2}\, \frac{\partial^2 F}{\partial v^i\, \partial v^j}$ *is positive definite.*

Now it is possible to demonstrate the following propositions:

Proposition 1. $\forall\, u,\, v \in V$ *and* $y \in V \setminus \{0\}$ *it follows that:*

$$g_{\lambda\, y}(u,\, v) = g_y(u,\, v) \quad \forall\, \lambda > 0 \tag{A.1}$$

Proposition 2. *If* $\|.\|$ *is every norm on the vectorial space* V *then* $S_E = \{v\, \in\, V\, :\, \|v\| = 1\}$ *is a compact subsect of* V*. Posing* $m = \min\{F(v) :\, v \in\, S_E\}$ *and* $M = m = \max\{F(v) :\, v \in\, S_E\}$*, then:*

$$m\|v\| \leq F(v) \leq M\|v\| \forall v \in V \tag{A.2}$$

From the last proposition it follows that any two norms on a Finsler manifold are equivalent, in fact $\forall\, \widetilde{F},\ F$ norms $\exists\, m,\, M > 0$ such that:

$$m\, \widetilde{F}(v) \leq F(v) \leq M\, \widetilde{F}(v) \tag{A.3}$$

Finally the next important theorem is direct consequence of the previous results:

Theorem 1. *If* F *is a norm on the Finsler manifold* M*, the following statements are equivalent*

1. $F > 0\, \forall v \in V \setminus \{0\}$
2. *the indicatrix is strictly convex*
3. $F^2(v) = g_v(v,\, v) = g_{ij}(v)\, v^i\, v^j$

where the indicatrix is the unitay radius ball: $I = \{x \in V\, :\, F(x) \leq 1\}$.
Last results of this section are the following:

Proposition 3. $(\textbf{Triangle inequality})$
$\forall v,\, w \in V$ *it follows that:*

$$F(v + w) \leq F(v) + F(w) \tag{A.4}$$

equality is true if and only if $w = \lambda\, v$ *with* $\lambda \geq 0$*.*

Theorem 2. $(\textbf{Cauchy} - \textbf{Schwartz inequality})$
$\forall v,\, w \in V$ *it follows that:*

$$|g_{ij}(u)\, v^i\, w^j| \leq F(v)F(w) \tag{A.5}$$

as before, equality if and only if $w = \lambda\, v$ *with* $\lambda \geq 0$*.*

From the last theorem it is simple to derive the definition of the internal product as:

$$g(v,\, w) = F(v)\, F(w)\, \cos\theta \leq F(v)\, F(w) \tag{A.6}$$

A.3 Legendre transformation

To define the Legendre transformation in a Finsler manifold, it is necessary, first of all, to introduce the concept of dual Minkowski norm:

Definition 2. $(\mathbf{Dual\ Minkowski\ \ norm})$
the dual Minkowski norm is a function defined on the dual vector space $F^* :\, V^* \longrightarrow \mathbb{R}$, defined as:

$$F^*(\chi) = \max\left\{\chi(x) :\, x \in V,\, F(x) = 1\right\} \quad \chi \in V^* \tag{A.1}$$

This definition is well posed because $\{x \in V \,:\, F(x) = 1\}$ is compact and therefore the dual norm results finite.

Now it is possible to introduce the principal concept of this section:

Definition 3. $(\mathbf{Legendre\ \ transformation})$
the Legendre transformation on a Finsler manifold M is a function $l \,:\, V \longrightarrow V^*$ defined as:

$$l(x) = g_x(x,\, \cdot) \qquad \forall x \in V \setminus \{0\} \tag{A.2}$$

with the property that $l(0) = 0$.

Proposition 4.

1. $F = F^* \circ l$
2. *The Legendre transformation l is a bijection*

Proof*:*
The first point can be demonstrate considering the fact that:

$$F(v) = \frac{g(y)_{ij}\, y^i\, y^j}{F(y)} = l_y\left(\frac{y}{F(y)}\right) \leq F^* \circ l(y) \tag{A.3}$$

and considering the symmetric inequality, written as:

$$F^* \circ l(y) = \sup_{v \neq 0} l_y\left(\frac{v}{F(v)}\right) = \sup_{v \neq 0}\left(\frac{g(y)_{ij}\, y^i\, v^j}{F(v)}\right) \leq F(y) \tag{A.4}$$

for $y \neq 0$, for $y = 0$ it is immediately true, therefore the first enunciate is verified $\forall y \in V$. The Legendre transformation injectivity follows from:

$$g(y)_{ij} y^i wj = g(v)_{ij} v^i w^j \quad \forall w \in V \;\Rightarrow v = y \tag{A.5}$$

in fact, posing $w = v$ and $w = y$ and using the Cauchy-Schwarz inequality, one obtains:

$$\begin{aligned} F^2(v,v) = g(v)_{ij}\, v^i\, v^j = g(y)_{ij}\, y^i\, v^j \leq F(v)F(y) \\ F^2(y,y) = g(y)_{ij}\, y^i\, y^j = g(v)_{ij}\, v^i\, y^j \leq F(y)F(v) \end{aligned} \tag{A.6}$$

The surjectivity can be verified noting that, if $\xi \in V^*\backslash\{0\}$, $\lambda = F^*(\xi)$ and $y \in V$ it follows that $F(y) = 1$ and $\xi(y) = \lambda$. Considering now the smooth curve $\gamma : \mathbb{I} \longrightarrow F^{-1}(1)$:

$$\gamma(t) = \frac{y + tw}{F(y + tw)}, \quad t \in I \tag{A.7}$$

where the vector w is defined such that $w \in \{w \in V : g(y)_{ij}y^i w^j = 0\}$, because y is a stationary point of the function $v \to \xi(v)$, it follows that:

$$0 = \frac{d}{dt}\xi(\gamma(t)) \mid_{t=0} = \xi\left(\frac{w}{F(y)} - \frac{y}{F^2(y)}\frac{\partial F}{\partial y^i}(y)w^i\right) \tag{A.8}$$

so, because $g(y)_{ij}y^i w^j = 0$, finally it follows that $\xi(w) = 0$. From this, one finds that $\forall v \in V$ the following decomposition is correct:

$$v = g(y)_{ij}\, y^i\, v^j + w \tag{A.9}$$

From these results $\xi = l(\lambda y)$ follows, proving the surjectivity.

Posing g^{ij} as the inverse metric of g_{ij}, $\{\theta^i\}$ the dual basis correlated to the standard basis $\{e_i\}$, it is possible to define $l_i(y)$ as the i-esim $l(y)$ component, whose explicit form is:

$$l_i(y) = l(y)(e_i) = \frac{1}{2}\frac{\partial F^2}{\partial y^i}(y) \tag{A.10}$$

Now one can proofs the following:

Proposition 5.

1. *The dual Minkowski norm is a norm on V^**

2. *If*

$$\widetilde{g}^{ij}(\chi) = \frac{1}{2}\frac{\partial^2 F^{*2}}{\partial \chi_i \partial \chi_j}(\chi) \qquad \forall \chi \in V^* \setminus \{0\} \tag{A.11}$$

Then it follows:

$$\begin{cases} l(y) = l_j(y)\,\theta^j = g_{ij}(y)\,y^i\,\theta^j & \forall y \in V \setminus \{0\} \\ l^{-1}(\chi) = \widetilde{g}^{ij}(\chi)\,\chi_i\,e_j & \forall \chi \in V^* \setminus \{0\} \\ g^{ij}(y) = \widetilde{g}^{ij} \circ l(y) & \forall y \in V \setminus \{0\} \end{cases} \tag{A.12}$$

Proof*:*
From the first equation of (A.12), trivial to be proved, one obtains that l is smooth. Its Jacobian is defined as $\partial_i \partial_j l = g_{ij}$, hence even the inverse l^{-1} results smooth. It follows that $l : V \setminus \{0\} \longrightarrow V^* \setminus \{0\}$ is a diffeomorphism. This last statement, together with the first point of the previous proposition, are sufficient to demonstrate that F^* is a smooth function on $V^* \setminus \{0\}$. F^* is homogeneous of degree 1, so only the F^* positive definite condition remains to be verified to prove that it is a Minkowski norm.
Differentiating the function $\frac{1}{2}F^2 = \frac{1}{2}F^{*2} \circ l$ respect to y^i one obtains:

$$\frac{1}{2}\frac{\partial F^2}{\partial y^i}(y) = \frac{1}{2}\frac{\partial F^{*2}}{\partial \chi_k} \circ l(y)\,g_{ki}(y) \tag{A.13}$$

where $\chi_k = g_{ki}\,y^i$ have been used.
Equation (A.14) implies that $l_i(y) = (\widetilde{g}^{kj} \circ l)(y)\,l_j(y)\,g_{ki}(y)$ and from this follows:

$$y^j = \widetilde{g}^{jk} \circ l(y)\,l_k(y) \tag{A.14}$$

Differentiating (A.13) respect to y^j one obtains:

$$g_{ij}(y) = (\widetilde{g}^{kl} \circ l)(y)\,g_{ki}(y)\,g_{lj}(y) + \frac{1}{2}\frac{\partial F^{*2}}{\partial \chi_k} \circ l(y)\,\frac{\partial g_{ki}}{\partial y^j}(y) \tag{A.15}$$

using (A.14), the last term of equation (A.15) becomes:

$$\frac{1}{2}\frac{\partial F^{*2}}{\partial \chi_k} \circ l(y)\,\frac{\partial g_{ki}}{\partial y^j}(y) = (\widetilde{g}^{km} \circ l)(y)\,l_m(y)\,\frac{\partial g_{ki}}{\partial y^j}(y) = y^k\,\frac{\partial g_{ij}}{\partial y^k}(y) = 0 \tag{A.16}$$

so from (A.16) it follows the validity of the third relation of (A.12). From this it follows that $\widetilde{g}^{ij}$ is positive definite, because g_{ij} is positive definite and so even its inverse g^{ij}. Therefore F^* has positive definite Jacobian and results a Minkowski norm.
The second relation of (A.12) follows from the fact that $l^{-1} \circ l = \iota_V$ and $l \circ l^{-1} = \iota_{V^*}$.

A.4 Finsler manifold

In differential geometry a *n-dimensional manifold M* is a topological Hausdorff space, first numerable (that is with a countable base), locally homeomorphic to the euclidean space $\mathbb{R}^n$. In addition, a *smooth manifold* has C^∞ transition functions and here only this manifold type is considered. Assigned a manifold M with a coordinate basis $\{x^i\}$, it is possible to define the concept of curve as:

$$\begin{aligned} &\gamma(t) \in M \quad \forall t \in [0,\, 1] \\ &\gamma(t) :\ [0,\, 1] \longrightarrow M \end{aligned} \tag{A.1}$$

Now it is possible to define the *tanget space* as:

$$T_x M = \{\gamma'(t) \,|\, \gamma :\ [0,\, 1] \longrightarrow M,\ \gamma(0) = x\} = \left\{ \sum_j a_j \frac{\partial}{\partial x^j} \right\} \tag{A.2}$$

and finally the *tangent bundle*:

$$TM = \bigcup_{x \in M} T_x M \tag{A.3}$$

All these definitions are mutued from standard Riemann differential geometry. The new geometrical structure is defined as:

Definition 4. (**Finsler manifold**)
A Finsler manifold M is a geometrical manifold associated to a function F, such that:

- $F \in C^\infty$ *on* $TM \setminus \{0\}$
- $F\big|_{T_x M} :\ T_x M \longrightarrow [0,\, \infty)$ *is* $\forall x \in M$ *a Minkowski norm*

where $TM \setminus \{0\} = \bigcup \{T_x M \setminus \{0\} :\ x \in M\}$

In a Finsler manifold it is possible to define a vector $v \in TM$ lenght as:

$$F(v) = \sqrt{g_{ij}(x,\, v)\, v^i\, v^j} \tag{A.4}$$

For every curve $\gamma(t) :\ [0,\, 1] \longrightarrow M$ it is possible to define the lenght of the curve as:

$$L(\gamma) = \int_0^1 F(\gamma'(t))\, dt \tag{A.5}$$

which results independent from the parametrization. Now it is possible to define the concept of distance between two points x and y as:

$$d(x,\, y) = \inf_{\{\gamma :\, x \to y\}} \{L(\gamma)\} \tag{A.6}$$

This definition does not guarantee the distance symmetry, that is $d(x,\, y) = d(y,\, x)$ does not result necessarily true. A Finsler manifold is said *reversible* if $F(v) =$

$F(-v)$, moreover this property implies:

$$\begin{aligned} d(x,\, y) &= \inf_{\{\gamma:\, x \to y\}} \{L(\gamma)\} = \int_0^1 F(\gamma'(t))\, dt = \\ &= \int_0^1 F(\gamma'(-t))\, dt = \inf_{\{\gamma:\, y \to x\}} \{L(\gamma)\} = d(y,\, x) \end{aligned} \tag{A.7}$$

A.5 Global Legendre transformation

Definition 5. $(\mathbf{co-Finsler\ norm})$
The co-Finsler norm, defined on a Finsler manifold M, is a function $H : T^*M \longrightarrow [0,\, \infty)$ such that:

- *$H \in C^\infty$ on $T^*M \setminus \{0\}$*
- *$H\big|_{T_x^*M} : T_x^*M \longrightarrow [0,\, \infty)$ $\forall x \in M$ is a Minkowski norm*

Given the pointwise Legendre transformation:

$$l_x : T_x^*M \longrightarrow T_x^{**}M \tag{A.1}$$

induced by the Minkowski norm, obtained from the restriction of a co-Finsler norm H as: $H\big|_{T_x^*M}$, it is possible to define the global Legendre transformation:

Definition 6. $(\mathbf{Global\ \ Legendre\ \ transformation})$
The global Legendre transformation is given by:

$$\begin{aligned} \mathcal{L} : T^*M &\longrightarrow TM \\ \chi &\longmapsto \iota^{-1} \circ l_{\pi(\chi)}(\chi) \end{aligned} \tag{A.2}$$

where $\pi : T^*M \to M$ is the canonical projection and $\iota : T_xM \to T_x^{**}M$ is the canonical linear isomorphism.

Posing, in local coordinates, the Jacobian of the co-Finsler norm H as:

$$h^{ij} = \frac{1}{2}\frac{\partial^2 H^2}{\partial \chi_i \partial \chi_j}(\chi) \qquad \forall \chi \in T^*M \setminus \{0\} \tag{A.3}$$

it is possible to demonstrate the following:

Proposition 6. *If $\mathcal{L}$ is the Legendre transformation induced by H, co-Finsler norm, then:*

1. *$\mathcal{L} : T^*M \longrightarrow TM$ is a bijection and $\mathcal{L} : T^*M \setminus \{0\} \longrightarrow TM \setminus \{0\}$ is a diffeomorphism*

2. *$F = H \circ \mathcal{L}^{-1}$ is a Finsler norm on the manifold M*

3. *if $g_{ij} = \frac{1}{2}\frac{\partial^2 F}{\partial y^i \partial y^j}$ and h_{ij} is the inverse of (A.2), then:*

$$\begin{cases} \mathcal{L}(\chi) = h^{ij}(\chi)\,\chi_i\,\dfrac{\partial}{\partial x^j} & \forall \chi \in T^*M \setminus \{0\} \\ \mathcal{L}^{-1}(y) = g_{ij}(y)\,y^i\,dx^j & \forall y \in TM \setminus \{0\} \\ g_{ij}(y) = h_{ij} \circ \mathcal{L}^{-1}(y) & \forall y \in TM \setminus \{0\} \end{cases} \tag{A.4}$$

Proof*:*
The proof starts from first equation of part 3 (A.4). Fixed a point $x \in M$, $\{e_i\}$ is the usual basis in a local coordinate system, $\{\theta^i\}$ is the associated dual basis for $T^*_x M$ and Δ_i is the correlated base of $T^{**}_x M$. Therefore the relations $\theta^i(e_j) = \delta^i_{\ j}$, $\Delta_i(\theta^j) = \delta^j_i$ are valid, hence $\iota(e_i) = \Delta_i$, $\iota^{-1}(\Delta_i) = e_i$ and $\{\theta^i\} = \{dx^i\}$. From equation (A.12) of previous section, it follows:

$$\mathcal{L} = \iota^{-1}(h^{ij}(\chi)\,\chi_i\,\Delta_j) = h^{ij}(\chi)\,\chi_i\,\frac{\partial}{\partial x^j} \tag{A.5}$$

so the first of (A.4) is proved.
The second equation of section 3 follows from the fact that if $\{w_i\}$ are coordinates of $T^{**}_x M$, then:

$$g_{ij}(y) = \frac{1}{2}\,\frac{\partial^2 H^2 \circ l_x^{-1}}{\partial w^i \partial w^j}(\iota(y)) \tag{A.6}$$

and from equation (A.12) of previous section:

$$\mathcal{L}^{-1}(y) = l_x^{-1} \circ \iota(y) = \frac{1}{2}\,\frac{\partial^2 H^2 \circ l_x^{-1}}{\partial w^i \partial w^j}(\iota(y))(\iota \circ y)^i\,\theta^j = g_{ij}(y)\,y^i\,dx^j \tag{A.7}$$

so the relation results demonstrated.
The third relation of (A.4) follows from (A.12) of previous section and the previous one:

$$h_{ij}(\chi) = \frac{1}{2}\,\frac{\partial^2 H^2 \circ l_x^{-1}}{\partial w^i \partial w^j} \circ l_x(\chi) = g_{ij} \circ \mathcal{L}(\chi) \tag{A.8}$$

The bijectivity of $\mathcal{L}$ follows from the fact that its Jacobian is not singular, because it has the form:

$$D\mathcal{L} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & h^{ij} \end{pmatrix} \tag{A.9}$$

computed from the first relation of section 3. From the inverse function theorem it follows that even the Jacobian of $\mathcal{L}^{-1}$ is not singular and therefore $\mathcal{L}$ results a bijection.
Property 2 can be proved showing that F is homogeneous of degree 1. In fact, taking $y \in TM$, then $\exists\, \chi \in T^*M$ such that $y = \mathcal{L}(\chi)$. Since $\mathcal{L}$ is homogeneous of degree 1, it

follows that:

$$\mathcal{L}^{-1}(\lambda y) = \mathcal{L}^{-1}(\mathcal{L}(\lambda y)) = \lambda y = \lambda \mathcal{L}^{-1}(y) \qquad \forall \lambda > 0 \tag{A.10}$$

Since $F = H \circ \mathcal{L}^{-1}$, H and even $\mathcal{L}^{-1}$ are homogeneous of degree 1, F results homogeneous of degree 1. Moreover F results positive definite thanks to the third relation of point 3, because h^{ij} is definite positive and so results even g_{ij}, proving that F is a Finsler norm on M.

The global Legendre transformation $\mathcal{L} : TM \longrightarrow T^*M$ can be introduced, following the same procedure used before. Assigned a Finsler norm F, one can define a pointwise Legendre transformation:

$$l'_x : T_xM \longrightarrow T_xM \tag{A.11}$$

induced by the Minkowski norm $F\big|_{T_xM}$. In the same way it is possible to introduce the global Legendre transformation as:

Definition 7. (**Global Legendre transformation**)
The global Legendre transformation is given by:

$$\begin{aligned} \mathcal{L}' : TM &\longrightarrow T^*M \\ \chi &\longmapsto l_{\pi(y)}(y) \end{aligned} \tag{A.12}$$

where $\pi : TM \longrightarrow M$ is the canonical projection.

Now one can demonstrate a proposition analogous to (6):

Proposition 7. *If $\mathcal{L}'$ is the Legendre transformation induced by F, Finsler norm, then:*

1. *$\mathcal{L}' : TM \longrightarrow T^*M$ is a bijection and $\mathcal{L}' : TM \setminus \{0\} \longrightarrow T^*M \setminus \{0\}$ is a diffeomorphism*
2. *$H = F \circ \mathcal{L}'^{-1}$ is a Finsler norm on the manifold M*
3. *if $g_{ij} = \frac{1}{2}\frac{\partial^2 F}{\partial y^i \partial y^j}$ and h_{ij} is the inverse of (A.2), then:*

$$\begin{cases} \mathcal{L}'(y) = g_{ij}(y)\, y^i\, dx^j & \forall \chi \in T^*M \setminus \{0\} \\ \mathcal{L}'^{-1}(\chi) = g_{ij}(y)\, y^i\, dx^j & \forall y \in TM \setminus \{0\} \\ h^{ij}(\chi) = g^{ij} \circ \mathcal{L}'^{-1}(\chi) & \forall y \in TM \setminus \{0\} \end{cases} \tag{A.13}$$

Proof*: The proof is analogous to that of (A.2).*

The next two results demonstrate that $\mathcal{L}^{-1} = \mathcal{L}'$ and Finsler an co-Finsler norms are in biunivocal correspondence.

Proposition 8. *If $\mathcal{L}$ is the Legendre transformation induced by the co-Finsler norm H and $\mathcal{L}'$ is the one induced by the Finsler norm $F = H \circ \mathcal{L}$, then:*

$$\mathcal{L}^{-1} = \mathcal{L}' \tag{A.14}$$

In the same way, if $\mathcal{L}'$ is the transformation induced by the norm F and $\mathcal{L}$ is the one induced by the co-Finsler norm $H = F \circ \mathcal{L}'$, it follows that:

$$\mathcal{L}'^{-1} = \mathcal{L} \tag{A.15}$$

Proof*: The demonstration follows immediately from the first equations of (A.4) and (A.13).*

Proposition 9. *Finsler norm F and co-Finsler norm H, defined on a fixed Finsler manifold M, are in biunivocal correspondence, via the Legendre transformations.*
Proof*: Define T as the function that maps a Finsler norm to a co-Finsler norm $T : F \longrightarrow F \circ \mathcal{L}'^{-1}$ and S the function that maps a co-Finsler norm on a Finsler one: $S : H \longrightarrow H \circ \mathcal{L}^{-1}$. Therefore, it follows:*

$$T \circ S(H) = H = T(F) = F \circ \mathcal{L}'^{-1} = H \circ \mathcal{L}^{-1} \circ \mathcal{L}'^{-1} = H \tag{A.16}$$

where the relation $F = H \circ \mathcal{L}^{-1}$ has been used. Finally it results $S \circ T = \iota = T \circ S$ and therefore T and S are bijections, that is $F \leftrightharpoons H$.

Bibliography

[1] M. Nagano and A. A. Watson. Observations and implications of the ultrahigh-energy cosmic rays. *Rev. Mod. Phys.*, 72:689–732, Jul 2000.

[2] F. W. Stecker. Origin of Cosmic Rays. *Phys. Rev. Lett.*, 35:188–191, 1975.

[3] F. W. Stecker. On the Origin of Cosmic Rays. In *14th International Cosmic Ray Conference (ICRC 1975) Munich, Germany, August 15-29, 1975*, pages 70–73, 1975.

[4] Pierre Auger, P. Ehrenfest, R. Maze, J. Daudin, and A. Freon Robley. Extensive cosmic ray showers. *Rev. Mod. Phys.*, 11:288–291, 1939.

[5] W. D. Apel. Energy Spectra of Elemental Groups of Cosmic Rays: Update on the KASCADE Unfolding Analysis. *Astropart. Phys.*, 31:86–91, 2009.

[6] Matthew S. J. Marshall, David T. Newell, David J. Payne, Russell G. Egdell, and Martin R. Castell. Atomic and electronic surface structures of dopants in oxides: Stm and xps of nb- and la-doped srtio$_3$(001). *Phys. Rev. B*, 83:035410, Jan 2011.

[7] M. et al. Aguilar. Precision measurement of the helium flux in primary cosmic rays of rigidities 1.9 gv to 3 tv with the alpha magnetic spectrometer on the international space station. *Phys. Rev. Lett.*, 115:211101, Nov 2015.

[8] M. et al. Aguilar. Precision measurement of the proton flux in primary cosmic rays from rigidity 1 gv to 1.8 tv with the alpha magnetic spectrometer on the international space station. *Phys. Rev. Lett.*, 114:171103, Apr 2015.

[9] The pierre auger cosmic ray observatory. *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, 798:172 – 213, 2015.

[10] C. Barth. Evolution of silicon sensor characteristics of the cms silicon strip tracker. *Nuclear Instruments and Methods in Physics Research, Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, 699:178–183, 2013.

[11] Pasquale Blasi, Elena Amato, and Pasquale D. Serpico. Spectral breaks as

a signature of cosmic ray induced turbulence in the galaxy. *Phys. Rev. Lett.*, 109:061101, Aug 2012.

[12] Roberto Aloisio and Pasquale Blasi. Propagation of galactic cosmic rays in the presence of self-generated turbulence. *Journal of Cosmology and Astroparticle Physics*, 2013(07):001, 2013.

[13] Floyd W. Stecker, M. A. Malkan, and S. T. Scully. Intergalactic photon spectra from the far ir to the uv lyman limit for 0 ¡ Z ¡ 6 and the optical depth of the universe to high energy gamma-rays. *Astrophys. J.*, 648:774–783, 2006.

[14] F. W. Stecker. Origin and composition of ultrahigh energy cosmic rays. *Nature*, 234:28–29, 1971.

[15] F. W. Stecker. Photodisintegration of ultrahigh-energy cosmic rays by the universal radiation field. *Phys. Rev.*, 180:1264–1266, 1969.

[16] J. L. Puget, F. W. Stecker, and J. H. Bredekamp. Photonuclear Interactions of Ultrahigh-Energy Cosmic Rays and their Astrophysical Consequences. *Astrophys. J.*, 205:638–654, 1976.

[17] Elisa Resconi, Stefan Coenders, Paolo Padovani, Paolo Giommi, and Lorenzo Caccianiga. Connecting blazars with ultrahigh-energy cosmic rays and astrophysical neutrinos. *Mon. Not. Roy. Astron. Soc.*, 468(1):597–606, 2017.

[18] Sidney R. Coleman and Sheldon L. Glashow. High-energy tests of Lorentz invariance. *Phys. Rev.*, D59:116008, 1999.

[19] Alexander Aab et al. Evidence for a mixed mass composition at the 'ankle' in the cosmic-ray spectrum. *Phys. Lett.*, B762:288–295, 2016.

[20] Daisuke Ikeda. Recent results from Telescope Array. *J. Phys. Conf. Ser.*, 375:052007, 2012.

[21] Alexander Aab et al. The Pierre Auger Observatory Upgrade - Preliminary Design Report. 2016.

[22] Andrew G. Cohen and Sheldon L. Glashow. Very special relativity. *Phys. Rev. Lett.*, 97:021601, 2006.

[23] Alan Kostelecky. Riemann-Finsler geometry and Lorentz-violating kinematics. *Phys. Lett.*, B701:137–143, 2011.

[24] Giovanni Amelino-Camelia, Leonardo Barcaroli, Giulia Gubitosi, Stefano Liberati, and Niccoló Loret. Realization of doubly special relativistic symmetries in Finsler geometries. *Phys. Rev.*, D90(12):125030, 2014.

[25] Florian Girelli, Stefano Liberati, and Lorenzo Sindoni. Planck-scale modified dispersion relations and Finsler geometry. *Phys. Rev.*, D75:064015, 2007.

[26] Don Colladay and V. Alan Kostelecky. Lorentz violating extension of the standard model. *Phys. Rev.*, D58:116002, 1998.

[27] S. T. Scully and F. W. Stecker. Lorentz Invariance Violation and the

Observed Spectrum of Ultrahigh Energy Cosmic Rays. *Astropart. Phys.*, 31:220–225, 2009.

[28] Floyd W. Stecker and Sean T. Scully. Searching for New Physics with Ultrahigh Energy Cosmic Rays. *New J. Phys.*, 11:085003, 2009.

[29] Maurizio Spurio. *Particles and Astrophysics, a Multi-Messenger Approach*. Astronomy and Astrophysics Library. Springer International Publishing, 2015.

[30] Maurizio Spurio. *The Probes of Multimessenger Astrophysics, Charged cosmic rays, neutrinos, γ-rays and gravitational waves*. Astronomy and Astrophysics Library. Springer International Publishing, 2018.

[31] Mário Pimenta Alessandro De Angelis. *Introduction to Particle and Astroparticle Physics, Questions to the Universe*. Undergraduate Lecture Notes in Physics. Springer-Verlag Mailand, 2015.

[32] Pijushpani Bhattacharjee and Gunter Sigl. Origin and propagation of extremely high-energy cosmic rays. *Phys. Rept.*, 327:109–247, 2000.

[33] G. A. Medina Tanco and A. A. Watson. On the possible galactic sources of the ultra-high energy cosmic ray anisotropy at 1-EeV. In *27th International Cosmic Ray Conference (ICRC 2001) Hamburg, Germany, August 7-15, 2001*, pages 531–533, 2001.

[34] J. Abraham et al. Upper limit on the cosmic-ray photon flux above 10^{19} eV using the surface detector of the Pierre Auger Observatory. *Astropart. Phys.*, 29:243–256, 2008.

[35] A. R. Bell. The acceleration of cosmic rays in shock fronts. I. *Mon. Not. R. Astron. Soc.*, 182:147–156, January 1978.

[36] ENRICO Fermi. On the origin of the cosmic radiation. *Phys. Rev.*, 75:1169–1174, Apr 1949.

[37] A. R. Bell. The acceleration of cosmic rays in shock fronts. II. *Mon. Not. Roy. Astron. Soc.*, 182:443–455, 1978.

[38] Anthony Raymond Bell. Particle acceleration by shocks in supernova remnants. *Braz. J. Phys.*, 44:415–425, 2014. [,1286(2013)].

[39] R. D. Blandford and J. P. Ostriker. Particle Acceleration by Astrophysical Shocks. *Astrophys. J.*, 221:L29–L32, 1978.

[40] Daniel De Marco and Todor Stanev. On the shape of the ultrahigh energy cosmic ray spectrum. *Phys. Rev. D*, 72:081301, Oct 2005.

[41] Roberto Aloisio, Pasquale Blasi, Ivan De Mitri, and Sergio Petrera. Selected Topics in Cosmic Ray Physics. In Roberto Aloiso, Eugenio Coccia, and Francesco Vissani, editors, *Multiple Messengers and Challenges in Astroparticle Physics*, pages 1–95. 2018.

[42] D. Allard, N. G. Busca, G. Decerprit, A. V. Olinto, and E. Parizot. Implications of the cosmic ray spectrum for the mass composition at the highest

energies. *JCAP*, 0810:033, 2008.

[43] Alexander Aab et al. The Pierre Auger Observatory Upgrade - Preliminary Design Report. 2016.

[44] Roberto Aloisio. Acceleration and propagation of ultra high energy cosmic rays. *PTEP*, 2017(12):12A102, 2017.

[45] F. W. Stecker. Effect of photomeson production by the universal radiation field on high-energy cosmic rays. *Phys. Rev. Lett.*, 21:1016–1018, Sep 1968.

[46] D. Harari, S. Mollerach, and Esteban Roulet. On the ultrahigh energy cosmic ray horizon. *JCAP*, 0611:012, 2006.

[47] Kenneth Greisen. End to the cosmic-ray spectrum? *Phys. Rev. Lett.*, 16:748–750, Apr 1966.

[48] G. T. Zatsepin and V. A. Kuz'min. Upper Limit of the Spectrum of Cosmic Rays. *Soviet Journal of Experimental and Theoretical Physics Letters*, 4:78, August 1966.

[49] R. U. et al. Abbasi. First observation of the greisen-zatsepin-kuzmin suppression. *Phys. Rev. Lett.*, 100:101101, Mar 2008.

[50] J. et al. Abraham. Observation of the suppression of the flux of cosmic rays above 4×10^{19} eV. *Phys. Rev. Lett.*, 101:061101, Aug 2008.

[51] IHEP. *Proceedings, 32nd International Cosmic Ray Conference (ICRC 2011)*, Beijing, China, 2011. IHEP.

[52] R. Ruffini, G. V. Vereshchagin, and S. S. Xue. Cosmic absorption of ultra high energy particles. *Astrophys. Space Sci.*, 361:82, 2016.

[53] Giovanni Amelino-Camelia. Quantum-Spacetime Phenomenology. *Living Rev. Rel.*, 16:5, 2013.

[54] Ignatios Antoniadis. A Possible new dimension at a few TeV. *Phys. Lett.*, B246:377–384, 1990.

[55] Joseph D. Lykken. Weak scale superstrings. *Phys. Rev.*, D54:R3693–R3697, 1996.

[56] Edward Witten. Strong coupling expansion of Calabi-Yau compactification. *Nucl. Phys.*, B471:135–158, 1996.

[57] Nima Arkani-Hamed, Savas Dimopoulos, and G. R. Dvali. The Hierarchy problem and new dimensions at a millimeter. *Phys. Lett.*, B429:263–272, 1998.

[58] V. A. Rubakov. Large and infinite extra dimensions: An Introduction. *Phys. Usp.*, 44:871–893, 2001. [Usp. Fiz. Nauk171,913(2001)].

[59] John R. Ellis, N. E. Mavromatos, and Dimitri V. Nanopoulos. String theory modifies quantum mechanics. *Phys. Lett.*, B293:37–48, 1992.

[60] John R. Ellis, N. E. Mavromatos, and Dimitri V. Nanopoulos. A Noncritical string approach to black holes, time and quantum dynamics. In *From*

supersymmetry to the origin of space-time. Proceedings, International School of Subnuclear Physics, Erice, Italy, July 4-12, 1993, pages 0001–66, 1994.

[61] G. Amelino-Camelia, John R. Ellis, N. E. Mavromatos, and Dimitri V. Nanopoulos. Distance measurement and wave dispersion in a Liouville string approach to quantum gravity. *Int. J. Mod. Phys.*, A12:607–624, 1997.

[62] V. Alan Kostelecky and Stuart Samuel. Spontaneous Breaking of Lorentz Symmetry in String Theory. *Phys. Rev.*, D39:683, 1989.

[63] Carlo Rovelli. Loop quantum gravity. *Living Rev. Rel.*, 1:1, 1998.

[64] Abhay Ashtekar and Jerzy Lewandowski. Background independent quantum gravity: A Status report. *Class. Quant. Grav.*, 21:R53, 2004.

[65] Lee Smolin. How far are we from the quantum theory of gravity? 2003.

[66] Thomas Thiemann. Lectures on loop quantum gravity. *Lect. Notes Phys.*, 631:41–135, 2003. [,41(2002)].

[67] Abhay Ashtekar. Introduction to loop quantum gravity and cosmology. *Lect. Notes Phys.*, 863:31–56, 2013.

[68] Carlo Rovelli and Lee Smolin. Discreteness of area and volume in quantum gravity. *Nucl. Phys.*, B442:593–622, 1995. [Erratum: Nucl. Phys.B456,753(1995)].

[69] Abhay Ashtekar and Jerzy Lewandowski. Quantum theory of geometry. 1: Area operators. *Class. Quant. Grav.*, 14:A55–A82, 1997.

[70] Carlo Rovelli and Simone Speziale. Reconcile Planck scale discreteness and the Lorentz-Fitzgerald contraction. *Phys. Rev.*, D67:064019, 2003.

[71] S. Majid and H. Ruegg. Bicrossproduct structure of kappa Poincare group and noncommutative geometry. *Phys. Lett.*, B334:348–354, 1994.

[72] Jerzy Lukierski, Henri Ruegg, and Wojciech J. Zakrzewski. Classical quantum mechanics of free kappa relativistic systems. *Annals Phys.*, 243:90–116, 1995.

[73] John Madore, Stefan Schraml, Peter Schupp, and Julius Wess. Gauge theory on noncommutative spaces. *Eur. Phys. J.*, C16:161–167, 2000.

[74] Giovanni Amelino-Camelia and Shahn Majid. Waves on noncommutative space-time and gamma-ray bursts. *Int. J. Mod. Phys.*, A15:4301–4324, 2000.

[75] A. P. Balachandran, G. Mangano, A. Pinzul, and S. Vaidya. Spin and statistics on the Groenwald-Moyal plane: Pauli-forbidden levels and transitions. *Int. J. Mod. Phys.*, A21:3111–3126, 2006.

[76] Luca Bombelli, Joohan Lee, David Meyer, and Rafael Sorkin. Space-Time as a Causal Set. *Phys. Rev. Lett.*, 59:521–524, 1987.

[77] D. P. Rideout and R. D. Sorkin. Evidence for a continuum limit in causal set dynamics. *Phys. Rev.*, D63:104011, 2001.

[78] Jan Ambjorn, Jerzy Jurkiewicz, and Charlotte F. Kristjansen. Quantum

gravity, dynamical triangulations and higher derivative regularization. *Nucl. Phys.*, B393:601–632, 1993.

[79] J. Ambjorn, J. Jurkiewicz, and R. Loll. Emergence of a 4-D world from causal quantum gravity. *Phys. Rev. Lett.*, 93:131301, 2004.

[80] J. Ambjorn, J. Jurkiewicz, and R. Loll. Reconstructing the universe. *Phys. Rev.*, D72:064014, 2005.

[81] J. Ambjorn, J. Jurkiewicz, and R. Loll. Causal Dynamical Triangulations and the Quest for Quantum Gravity. In *Proceedings, Foundations of Space and Time: Reflections on Quantum Gravity: Cape Town, South Africa*, pages 321–337, 2010.

[82] Renate Loll. Discrete approaches to quantum gravity in four-dimensions. *Living Rev. Rel.*, 1:13, 1998.

[83] G. Amelino-Camelia. Limits on the measurability of space-time distances in the semiclassical approximation of quantum gravity. *Mod. Phys. Lett.*, A9:3415–3422, 1994.

[84] Andrzej Sitarz. Noncommutative differential calculus on the kappa Minkowski space. *Phys. Lett.*, B349:42–48, 1995.

[85] Achim Kempf and Gianpiero Mangano. Minimal length uncertainty relation and ultraviolet regularization. *Phys. Rev.*, D55:7909–7920, 1997.

[86] A. Blaut, M. Daszkiewicz, J. Kowalski-Glikman, and S. Nowak. Phase spaces of doubly special relativity. *Phys. Lett.*, B582:82–85, 2004.

[87] Giovanni Amelino-Camelia. Doubly-Special Relativity: Facts, Myths and Some Key Open Issues. *Symmetry*, 2:230–271, 2010.

[88] V. Alan Kostelecky and Stuart Samuel. Spontaneous Breaking of Lorentz Symmetry in String Theory. *Phys. Rev.*, D39:683, 1989.

[89] Alejandro Jenkins. Spontaneous breaking of Lorentz invariance. *Phys. Rev.*, D69:105007, 2004.

[90] S. L. Dubovsky and S. M. Sibiryakov. Spontaneous breaking of Lorentz invariance, black holes and perpetuum mobile of the 2nd kind. *Phys. Lett.*, B638:509–514, 2006.

[91] Akikazu Hashimoto. A Note on Spontaneously Broken Lorentz Invariance. *JHEP*, 08:040, 2008.

[92] Stefano Liberati. Tests of Lorentz invariance: a 2013 update. *Class. Quant. Grav.*, 30:133001, 2013.

[93] W. A. von Ignatowsky. Einige allgemeine bemerkungen zum relativitätsprinxzip. *Physikalische Zeitschrift*, (11):972–976, 1910.

[94] W. A. von Ignatowsky. Einige allgemeine bemerkungen zum relativitatsprinxzip. *Verh. Deutsch. Phys. Ges.*, (12):788–796, 1910.

[95] W. A. von Ignatowsky. Das relativitätprinzip. *Arch. Math. Phys.*, (17):1–24,

1911.

[96] W. A. von Ignatowsky. Das relativitätprinzip. *Arch. Math. Phys.*, (18):17–41, 1911.

[97] Jinwen Hu. The non-Lorentz transformation corresponding to the symmetry of inertial systems and a possible way to the quantization of time-space. *Physics Essays*, (30):322–327, 2017.

[98] V. Alan Kostelecky, editor. *CPT and Lorentz symmetry. Proceedings, 4th Meeting, Bloomington, USA, Aug 8-11, 2007*, Singapore, 2008. World Scientific, World Scientific.

[99] V. Alan Kostelecký, editor. *Proceedings, 6th Meeting on CPT and Lorentz Symmetry (CPT 13)*, Singapore, 2014. World Scientific, World Scientific.

[100] V. Alan Kostelecky, editor. *Proceedings, 7th Meeting on CPT and Lorentz Symmetry (CPT 16)*, Singapore, 2017. World Scientific, World Scientific.

[101] V. Alan Kostelecky and Neil Russell. Data Tables for Lorentz and CPT Violation. *Rev. Mod. Phys.*, 83:11–31, 2011.

[102] Robert C. Myers and Maxim Pospelov. Experimental challenges for quantum gravity. In *Proceedings, 3rd International Symposium on Quantum theory and symmetries (QTS3): Cincinnati, USA, September 10-14, 2003*, pages 732–744, 2004.

[103] David Mattingly. Have we tested Lorentz invariance enough? In *Proceedings, Workshop on From quantum to emergent gravity: Theory and phenomenology (QG-Ph): Trieste, Italy, June 11-15, 2007*, 2008.

[104] Alan Kostelecky and Matthew Mewes. Fermions with Lorentz-violating operators of arbitrary dimension. *Phys. Rev.*, D88(9):096006, 2013.

[105] V. Alan Kostelecky and Matthew Mewes. Electrodynamics with Lorentz-violating operators of arbitrary dimension. *Phys. Rev.*, D80:015020, 2009.

[106] Alan Kostelecky and Matthew Mewes. Neutrinos with Lorentz-violating operators of arbitrary dimension. *Phys. Rev.*, D85:096005, 2012.

[107] Petr Horava. Quantum Gravity at a Lifshitz Point. *Phys. Rev.*, D79:084008, 2009.

[108] Thomas P. Sotiriou. Horava-Lifshitz gravity: a status report. *J. Phys. Conf. Ser.*, 283:012034, 2011.

[109] Silke Weinfurtner, Thomas P. Sotiriou, and Matt Visser. Projectable Horava-Lifshitz gravity in a nutshell. *J. Phys. Conf. Ser.*, 222:012054, 2010.

[110] Daniele Vernieri and Thomas P. Sotiriou. Hořava-Lifshitz gravity with detailed balance. *J. Phys. Conf. Ser.*, 453:012022, 2013.

[111] Daniele Vernieri and Thomas P. Sotiriou. Horava-Lifshitz Gravity: Detailed Balance Revisited. *Phys. Rev.*, D85:064003, 2012.

[112] Niayesh Afshordi. Cuscuton and low energy limit of Horava-Lifshitz grav-

ity. *Phys. Rev.*, D80:081502, 2009.

[113] Giovanni Amelino-Camelia. Doubly special relativity. *Nature*, 418:34–35, 2002.

[114] Giovanni Amelino-Camelia. Doubly special relativity: First results and key open problems. *Int. J. Mod. Phys.*, D11:1643, 2002.

[115] R. Aloisio, A. Galante, A. F. Grillo, E. Luzio, and F. Mendez. A Note on DSR-like approach to space-time. *Phys. Lett.*, B610:101–106, 2005.

[116] R. Aloisio, A. Galante, A. F. Grillo, E. Luzio, and F. Mendez. Approaching space time through velocity in doubly special relativity. *Phys. Rev.*, D70:125012, 2004.

[117] Giovanni Amelino-Camelia. Relativity in space-times with short distance structure governed by an observer independent (Planckian) length scale. *Int. J. Mod. Phys.*, D11:35–60, 2002.

[118] Giovanni Amelino-Camelia. Testable scenario for relativity with minimum length. *Phys. Lett.*, B510:255–263, 2001.

[119] Joao Magueijo and Lee Smolin. Generalized Lorentz invariance with an invariant energy scale. *Phys. Rev.*, D67:044017, 2003.

[120] Joao Magueijo and Lee Smolin. Lorentz invariance with an invariant energy scale. *Phys. Rev. Lett.*, 88:190403, 2002.

[121] Giovanni Amelino-Camelia, Laurent Freidel, Jerzy Kowalski-Glikman, and Lee Smolin. The principle of relative locality. *Phys. Rev.*, D84:084010, 2011.

[122] Giovanni Amelino-Camelia. Kinematical solution of the UHE - cosmic - ray puzzle without a preferred class of inertial observers. *Int. J. Mod. Phys.*, D12:1211–1226, 2003.

[123] Marco Danilo Claudio Torri, Stefano Bertini, Marco Giammarchi, and Lino Miramonti. Lorentz Invariance Violation effects on UHECR propagation: A geometrized approach. *JHEAp*, 18:5–14, 2018.

[124] Vito Antonelli, L. Miramonti, and M. D. C. Torri. Neutrino oscillations and Lorentz Invariance Violation in a Finslerian Geometrical model. *Eur. Phys. J.*, C78(8):667, 2018.

[125] O. W. Greenberg. CPT violation implies violation of Lorentz invariance. *Phys. Rev. Lett.*, 89:231602, 2002.

[126] Masud Chaichian, Alexander D. Dolgov, Victor A. Novikov, and Anca Tureanu. CPT Violation Does Not Lead to Violation of Lorentz Invariance and Vice Versa. *Phys. Lett.*, B699:177–180, 2011.

[127] Anca Tureanu. CPT and Lorentz Invariance: Their Relation and Violation. *J. Phys. Conf. Ser.*, 474:012031, 2013.

[128] Masud Chaichian, Kazuo Fujikawa, and Anca Tureanu. Electromagnetic interaction in theory with Lorentz invariant CPT violation. *Phys. Lett.*,

B718:1500–1504, 2013.

[129] Michael Duetsch and Jose M. Gracia-Bondia. On the assertion that PCT violation implies Lorentz non-invariance. *Phys. Lett.*, B711:428–433, 2012.

[130] O. W. A Greenberg. Remarks on a Challenge to the Relation between CPT and Lorentz Violation. 2011.

[131] Leonardo Barcaroli, Lukas K. Brunkhorst, Giulia Gubitosi, Niccoló Loret, and Christian Pfeifer. Hamilton geometry: Phase space geometry from modified dispersion relations. *Phys. Rev.*, D92(8):084053, 2015.

[132] Sean M. Carroll. *Spacetime and geometry: An introduction to general relativity*. 2004.

[133] D. Shen, D. Bao, S.S. Chern, and Z. Shen. *An Introduction to Riemann-Finsler Geometry*. Graduate Texts in Mathematics. Springer New York, 2000.

[134] F. Schwabl. *Advanced quantum mechanics (QM II)*. 1997.

[135] Leonard E. Parker and D. Toms. *Quantum Field Theory in Curved Spacetime*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2009.

[136] Sidney R. Coleman and J. Mandula. All Possible Symmetries of the S Matrix. *Phys. Rev.*, 159:1251–1256, 1967.

[137] Philip C. Argyres. An Introduction to Global Supersymmetry. 2001.

[138] Steven Weinberg. *The quantum theory of fields. Vol. 2: Modern applications*. Cambridge University Press, 2013.

[139] Joao Magueijo and Lee Smolin. Gravity's rainbow. *Class. Quant. Grav.*, 21:1725–1736, 2004.

[140] Stefano Liberati, Sebastiano Sonego, and Matt Visser. Interpreting doubly special relativity as a modified theory of measurement. *Phys. Rev.*, D71:045001, 2005.

[141] H. Bacry and J. Levy-Leblond. Possible kinematics. *J. Math. Phys.*, 9:1605–1614, 1968.

[142] Giuseppe Arcidiacono. La relatività di Fantappiè. *Collectanea Mathematica*, 10:85–124, 1958.

[143] Jerzy Kowalski-Glikman and Sebastian Nowak. Doubly special relativity and de Sitter space. *Class. Quant. Grav.*, 20:4799–4816, 2003.

[144] S. Mignemi. Doubly special relativity in de Sitter spacetime. *Annalen Phys.*, 522:924–940, 2010.

[145] R. Aldrovandi and J. G. Pereira. de Sitter relativity: A New road to quantum gravity. *Found. Phys.*, 39:1–19, 2009.

[146] R. Aldrovandi, J. P. Beltran Almeida, C. S. O. Mayor, and J. G. Pereira. Lorentz transformations in de Sitter relativity. 2007.

[147] R. Aldrovandi, J. P. Beltran Almeida, and J. G. Pereira. Some implica-

tions of the cosmological constant to fundamental physics. *AIP Conf. Proc.*, 910:381–395, 2007. [,361(2007)].

[148] R. Aldrovandi, J. P. Beltran Almeida, and J. G. Pereira. de Sitter special relativity. *Class. Quant. Grav.*, 24:1385–1404, 2007.

[149] R. Aldrovandi and J. G. Pereira. *An Introduction to geometrical physics*. 1996.

[150] Gabriel Pascu. Atlas of Coordinate Charts on de Sitter Spacetime. 2012.

[151] Ignazio Licata, Leonoardo Chiatti, and Elmo Benedetto. *De Sitter Projective Relativity*. SpringerBriefs in Physics. Springer, Cham, 2017.

[152] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2011.

[153] M. Ibison. On the conformal forms of the Robertson-Walker metric. *J. Math. Phys.*, 48:122501, 2007.

[154] S. Chekanov et al. Measurement of the photon proton total cross-section at a center-of-mass energy of 209-GeV at HERA. *Nucl. Phys.*, B627:3–28, 2002.

[155] Andrey Saveliev, Luca Maccione, and Guenter Sigl. Lorentz Invariance Violation and Chemical Composition of Ultra High Energy Cosmic Rays. *JCAP*, 1103:046, 2011.

[156] Arthur B. McDonald. Nobel Lecture: The Sudbury Neutrino Observatory: Observation of flavor change for solar neutrinos. *Rev. Mod. Phys.*, 88(3):030502, 2016.

[157] G. Bellini et al. Final results of Borexino Phase-I on low energy solar neutrino spectroscopy. *Phys. Rev.*, D89(11):112007, 2014.

[158] G. Bellini et al. Neutrinos from the primary proton–proton fusion process in the Sun. *Nature*, 512(7515):383–386, 2014.

[159] V. Antonelli, L. Miramonti, C. Pena Garay, and A. Serenelli. Solar Neutrinos. *Adv. High Energy Phys.*, 2013:351926, 2013.

[160] Johannes Bergstrom, M. C. Gonzalez-Garcia, Michele Maltoni, Carlos Pena-Garay, Aldo M. Serenelli, and Ningqiang Song. Updated determination of the solar neutrino fluxes from solar neutrino data. *JHEP*, 03:132, 2016.

[161] Francesco Vissani. Joint analysis of Borexino and SNO solar neutrino data and reconstruction of the survival probability. *Nucl. Phys. Atom. Energy*, 18(4):303–312, 2017.

[162] Takaaki Kajita. Nobel Lecture: Discovery of atmospheric neutrino oscillations. *Rev. Mod. Phys.*, 88(3):030501, 2016.

[163] F. P. An et al. Observation of electron-antineutrino disappearance at Daya Bay. *Phys. Rev. Lett.*, 108:171803, 2012.

[164] F. P. An et al. New Measurement of Antineutrino Oscillation with the Full

Detector Configuration at Daya Bay. *Phys. Rev. Lett.*, 115(11):111802, 2015.

[165] J. K. Ahn et al. Observation of Reactor Electron Antineutrino Disappearance in the RENO Experiment. *Phys. Rev. Lett.*, 108:191802, 2012.

[166] Myoung Youl Pac. Recent Results from RENO. *PoS*, NuFact2017:038, 2018.

[167] Y. Abe et al. Indication of Reactor $\bar{\nu}_e$ Disappearance in the Double Chooz Experiment. *Phys. Rev. Lett.*, 108:131801, 2012.

[168] Stefan Schoppmann. Latest results of Double Chooz. *PoS*, HQL2016:010, 2017.

[169] K. Eguchi et al. First results from KamLAND: Evidence for reactor antineutrino disappearance. *Phys. Rev. Lett.*, 90:021802, 2003.

[170] M. P. Decowski. KamLAND's precision neutrino oscillation measurements. *Nucl. Phys.*, B908:52–61, 2016.

[171] Yu. G. Kudenko. A study of neutrino oscillations in long baseline accelerator experiments. *Usp. Fiz. Nauk*, 181:569–594, 2011. [Phys. Usp.54,549(2011)].

[172] M. H. Ahn et al. Measurement of Neutrino Oscillation by the K2K Experiment. *Phys. Rev.*, D74:072003, 2006.

[173] P. Adamson et al. Improved search for muon-neutrino to electron-neutrino oscillations in MINOS. *Phys. Rev. Lett.*, 107:181802, 2011.

[174] C. Athanassopoulos et al. Evidence for anti-muon-neutrino —¿ anti-electron-neutrino oscillations from the LSND experiment at LAMPF. *Phys. Rev. Lett.*, 77:3082–3085, 1996.

[175] C. Athanassopoulos et al. Evidence for nu(mu) —¿ nu(e) neutrino oscillations from LSND. *Phys. Rev. Lett.*, 81:1774–1777, 1998.

[176] A. A. Aguilar-Arevalo et al. A Combined $\nu_\mu \to \nu_e$ and $\bar{\nu}_\mu \to \bar{\nu}_e$ Oscillation Analysis of the MiniBooNE Excesses. 2012.

[177] A. A. Aguilar-Arevalo et al. Improved Search for $\bar{\nu}_\mu \to \bar{\nu}_e$ Oscillations in the MiniBooNE Experiment. *Phys. Rev. Lett.*, 110:161801, 2013.

[178] N. Agafonova et al. Search for $\nu_\mu \to \nu_e$ oscillations with the OPERA experiment in the CNGS beam. *JHEP*, 07:004, 2013. [Addendum: JHEP07,085(2013)].

[179] K. Abe et al. Measurements of neutrino oscillation in appearance and disappearance channels by the T2K experiment with 6.8×10^{20} protons on target. *Phys. Rev.*, D91(7):072010, 2015.

[180] P. Adamson et al. First measurement of electron neutrino appearance in NOvA. *Phys. Rev. Lett.*, 116(15):151806, 2016.

[181] P. Arias, J. Gamboa, J. Lopez-Sarrion, F. Mendez, and Ashok K. Das. CPT / Lorentz invariance violation and neutrino oscillation. *Phys. Lett.*, B650:401–406, 2007.

[182] Alan Kostelecky and Matthew Mewes. Neutrinos with Lorentz-violating operators of arbitrary dimension. *Phys. Rev.*, D85:096005, 2012.

[183] Jorge S. Diaz. Neutrinos as probes of Lorentz invariance. *Adv. High Energy Phys.*, 2014:962410, 2014.

[184] Francesco Capozzi, Eleonora Di Valentino, Eligio Lisi, Antonio Marrone, Alessandro Melchiorri, and Antonio Palazzo. Global constraints on absolute neutrino masses and their ordering. *Phys. Rev.*, D95(9):096014, 2017.

[185] P. F. de Salas, D. V. Forero, C. A. Ternes, M. Tortola, and J. W. F. Valle. Status of neutrino oscillations 2018: 3σ hint for normal mass ordering and improved CP sensitivity. *Phys. Lett.*, B782:633–640, 2018.

[186] Shimin Yang and Bo-Qiang Ma. Lorentz violation in three-family neutrino oscillation. *Int. J. Mod. Phys.*, A24:5861–5876, 2009.

[187] K. Abe et al. Test of Lorentz invariance with atmospheric neutrinos. *Phys. Rev.*, D91(5):052003, 2015.

[188] M. Ageron et al. ANTARES: the first undersea neutrino telescope. *Nucl. Instrum. Meth.*, A656:11–38, 2011.

[189] S. Adrian-Martinez et al. Letter of intent for KM3NeT 2.0. *J. Phys.*, G43(8):084001, 2016.

[190] M. G. Aartsen et al. Observation of High-Energy Astrophysical Neutrinos in Three Years of IceCube Data. *Phys. Rev. Lett.*, 113:101101, 2014.

[191] Alexander Aab et al. Improved limit to the diffuse flux of ultrahigh energy neutrinos from the Pierre Auger Observatory. *Phys. Rev.*, D91(9):092008, 2015.

[192] Enrique Zas. Searches for neutrino fluxes in the EeV regime with the Pierre Auger Observatory. *PoS*, ICRC2017:972, 2018. [,64(2017)].

[193] B. P. Abbott et al. GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence. *Phys. Rev. Lett.*, 116(24):241103, 2016.

[194] B. P. Abbott et al. Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.*, 116(6):061102, 2016.

[195] A. Albert et al. Search for High-energy Neutrinos from Binary Neutron Star Merger GW170817 with ANTARES, IceCube, and the Pierre Auger Observatory. *Astrophys. J.*, 850(2):L35, 2017.

[196] Sheldon Lee Glashow. Atmospheric neutrino constraints on Lorentz violation. 2004.

[197] G. Battistoni et al. Search for a Lorentz invariance violation contribution in atmospheric neutrino oscillations using MACRO data. *Phys. Lett.*, B615:14–18, 2005.

[198] Dean Morgan, Elizabeth Winstanley, Lee F. Thompson, Jurgen Brunner, and Lee F. Thompson. Neutrino telescope modelling of Lorentz invariance

violation in oscillations of atmospheric neutrinos. *Astropart. Phys.*, 29:345–354, 2008.

[199] P. Adamson et al. A Search for Lorentz Invariance and CPT Violation with the MINOS Far Detector. *Phys. Rev. Lett.*, 105:151601, 2010.

[200] P. Adamson et al. Search for Lorentz invariance and CPT violation with muon antineutrinos in the MINOS Near Detector. *Phys. Rev.*, D85:031101, 2012.

[201] Benjamin R. Edwards and Alan Kostelecky. Riemann-Finsler Geometry and Lorentz-Violating Scalar Fields. 2018.

[202] Claus Lämmerzahl and Volker Perlick. Finsler geometry as a model for relativistic gravity. *Int. J. Geom. Meth. Mod. Phys.*, 15(supp01):1850166, 2018.

[203] Laurenţiu Bubuianu and Sergiu I. Vacaru. Axiomatic formulations of modified gravity theories with nonlinear dispersion relations and Finsler–Lagrange–Hamilton geometry. *Eur. Phys. J.*, C78(11):969, 2018.

[204] M. Schreck. Classical Lagrangians and Finsler structures for the non-minimal fermion sector of the Standard-Model Extension. *Phys. Rev.*, D93(10):105017, 2016.

[205] Giovanni Amelino-Camelia. Kinematical solution of the UHE - cosmic - ray puzzle without a preferred class of inertial observers. *Int. J. Mod. Phys.*, D12:1211–1226, 2003.

[206] George Yu. Bogoslovsky. Subgroups of the group of generalized Lorentz transformations and their geometric invariants. 2005.

[207] G. Yu. Bogoslovsky. Lorentz symmetry violation without violation of relativistic symmetry. *Phys. Lett.*, A350:5–10, 2006.

[208] G. W. Gibbons, Joaquim Gomis, and C. N. Pope. General very special relativity is Finsler geometry. *Phys. Rev.*, D76:081701, 2007.

[209] Paul Finsler. Über Kürven und Fläschen in Allgemeiner Rämen, Dissertation, University of Göttingen. 1918.

List of Publications

As of .....

Refereed publications

Lorentz Invariance Violation effects on UHECR propagation: A geometrized approach [123] - M. D.C. Torri, S. Bertini, M. Giammarchi, L. Miramonti - **JHEAP** 18 (2018) - doi:10.1016/j.jheap.2018.01.001

Neutrino oscillations and Lorentz Invariance Violation in a Finslerian Geometrical model [124] - V. Antonelli, L. Miramonti, M.D.C. Torri - **EPJ-C** 78 (2018) - doi:10.1140/epjc/s10052-018-6124-2

Publications in preparation

A possible way to introduce an isotropic Lorentz Invariance Violation

Lorentz Invariance Violation: from deSitter relativity to Finsler geometry

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