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INTEGRABLE MULTIDIMENSIONAL COSMOLOGIES WITH MATTER AND A SCALAR FIELD

MAT/07

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1 Introduction

1.1 Some background

Scalar fields in cosmology. The scalar fields considered throughout this work are classical objects; the term "classical" is used in opposition to "second quantized" and describes all situations in which the field is not an operator (including first quantization). The consideration of scalar fields in cosmological models has a long story, and arises from different motivations. Hereafter we will describe two important motivations for cosmologies supported by standard general relativity.

On one hand, a scalar field ("inflaton") can be used as a model for the mechanism driving inflation. This approach originates from the work of some scholars at the beginning of the 1980's: let us mention, in particular, Linde [18], Madsen and Coles [19].

On another hand, one can use a scalar field as a model for dark energy. This idea seems to have appeared in a 1988 paper by Ratra and Peebles [27]; Caldwell, Dave and Steinhardt [5] are credited for introducing, ten years later, the term "quintessence" to indicate a scalar model of dark energy.

The experimental consolidation of the dark energy notion dates back to the same years: as well known, in 1998-99 the High-Z Supernova Search Team [28] and the Supernova Cosmology Project [24] published their observations on the redshift of Type 1a supernovae. The insertion of some dark energy term in Einstein's equations is necessary to explain the observed dependence of the luminosity distance from redshift; the cosmological constant term $\Lambda g_{\mu\nu}$ is the most obvious candidate, and has been used as a benchmark in the analysis of the astronomical data. Of course the scalar model of dark energy is much more attractive, since it possesses its own degrees of freedom (as expected for any physical entity). Shortly after the publication of [28] [24] Saini, Raychaudhury, Sahni and Starobinsky [31] first proposed to fit the data on luminosity distance and redshift via a scalar field model; an unspecified self-interaction potential was assumed for the field and the most probable shape of this potential was reconstructed from the astronomical data, for the epoch ranging from present time to the time when the spatial scale factor had half of its present value.

It is hardly the case to specificy that most investigations in the two areas mentioned before (inflation or dark energy described via scalar fields) rely on the paradigm of a homogeneous and isotropic universe at each time. So, the spacetime metric has the Robertson-Walker form (possibly, with non zero spatial curvature) and the scalar field depends only of time, like the density and pressure of matter and radiation (if these characters are present in the model). The evolution of the model is governed by a system of ODEs where the unknowns are, typically, the field an the spatial scale factor as functions of time.

In an inflationary model, the attention is focused on an infinitesimal lapse of time after the Big-Bang, during which the energy of the inflaton was dominating; for this reason, it is not necessary to include matter or radiation in the model.
The situation is different for dark energy models; these encompass the history of universe up to present time, and cannot neglect the roles of matter and radiation.
Let us point out some additional features typically possessed by models for inflation or dark energy. It is commonly assumed that the scalar field is minimally coupled to gravity, and that no direct interaction occurs between the actors of the model: the stress energy tensors of the field, of matter and/or radiation are conserved separately. All papers cited in this Introduction are tacitly assumed to fit to this paradigm (spatial homogeneity and isotropy, minimal coupling of the scalar field with gravity, no direct interaction between the field, matter and radiation), unless the contrary is explictly stated.
To conclude the present considerations, let us point out that scalar fields appear in some theories alternative to Einstein's general relativity and in their applications of cosmology: let us just mention the Brans-Dicke theory [4]. These subjects are completely outside the scope of the present thesis.

Integrable scalar cosmologies, and other issues. Since the late 1980's, the rising physical interest for cosmologies with scalar fields stimulated the search for *integrable* models, in which the evolution equations can be solved explicitly; it turned out that this is possible for models with certain features, like a special functional form for the self-interaction potential of the scalar field. The availability of exact solutions is an advantage with respect to numerical integration, since it allows to identify details and conceptual aspects that could be missed by a numerical approach.
From the very beginning of these investigations, it was understood that exact solutions can be obtained assuming an exponential form for the self-interaction potential of the scalar field; let us describe these results with the normalizations of this thesis, using a suitable dimensionless version ϕ of the scalar field (up to a purely numerical factor, ϕ is the scalar field multiplied by the square root of the gravitational constant; for the precise definition see section 2, especially Eq. (2.25)).
In 1987 Barrow [1] assumed a potential of the form const. $e^{-\lambda\phi}$ (with λ another arbitrary constant), and a vanishing spatial curvature; he presented a particular exact solution of the evolution equations (but *not* the general solution), for the case of a scalar field alone. In the already mentioned paper [27] of 1988, Ratra and Peebles considered the same exponential potential as in [1] (with $\lambda > 0$) and a vanishing spatial curvature; they presented some particular exact solutions of the evolution equations, both for the case of a scalar field alone and for a model with a scalar field and pressureless matter (dust; the authors pointed out that the solution derived in presence of dust was too peculiar to be physically significant). In the same year,

Burd and Barrow [3] considered again the exponential potential const. $e^{-\lambda\phi}$ (with $\lambda > 0$), with possibly non-zero spatial curvature in arbitrary spacetime dimension $d+1$; they proposed a detailed stability analysis of the models and presented some new exact solutions which exhibit the transition to power-law inflation at late times.
In 1990 de Ritis, Marmo, Platania, Rubano, Scudellaro and Stornaiolo [7] considered a cosmology with a scalar field and no matter/radiation (useful as an inflationary model), for zero spatial curvature; to analyze the evolution equations, they proposed a systematic use of the Lagrangian viewpoint. In this way they proved that the only potentials giving rise to a Noether symmetry for the system have the form (with the normalizations of this thesis) const. $e^{3\phi}$ + const. $e^{-3\phi}$ + const; moreover, they constructed the general solution of the evolution equations for this class of potentials. The same authors extended these results in [8] to the case of a field non minimally coupled to gravity.
In 1998 Chimento [6] investigated cosmological models driven by two scalar fields, one of them self-interacting with an exponential potential of the form const. $e^{-\lambda\phi}$ (as in [1] [27] [2]) and the other one free and massive. Exact general solutions were obtained and examined in detail; in particular these solutions show the transition from expansion dominated by the free scalar field to that dominated by the self-interacting field, yielding a power-law inflation.
The potential const. $e^{3\phi}$ + const. $e^{-3\phi}$ + const was reconsidered in 2002 by Rubano and Scudellaro [29], and in 2012 by Piedipalumbo, Scudellaro, Esposito and Rubano [26], again for zero spatial curvature but in presence of dust. These authors showed that the addition of dust kept the solvability of the evolution equations; they proposed this model for a description of dark energy and dust up to the present time, and started an analysis of the physical significance of the solutions.
All papers [1] [27] [7] [8] [29] [26] [6] considered a spacetime with the "physical" dimension $3+1$.
In 2013 Fré, Sagnotti and Sorin [12] considered the cosmologies with a scalar field alone and zero spatial curvature in arbitrary spacetime dimension $d+1$; their aim was, essentially, to determine all the potentials giving rise solvable equations. They produced several classes of solvable potentials, much of them of exponential type; for the purposes of this thesis it is important to mention the first two classes, which can be described as follows with the normalizations of the thesis. The first class in [12] contains potentials of the form const. $e^{d\phi}$ + const. $e^{-d\phi}$ + const, thus extending to any dimension the solvability result found in [7] for $d = 3$; the second class is formed by potentials of the form const. $e^{d\gamma\phi}$ + const. $e^{d(1+\gamma)\phi}$, with γ an arbitrary constant.
To conclude this survey of the literature, let us mention some references which have just a partial intersection with the subject of this thesis.
In the previous review and in the sequel, the attention focused on a "direct problem": find the general solution of a cosmological model with a pre-assigned potential for

the scalar field. The cited papers [7] [8] [29] [26] [12] are all about this direct problem, like the present thesis. However, there is also an "inverse problem": find the scalar field potential producing a time evolution with a prescribed feature. Nice results on this subject were obtained in 1993 by Easther [10], who found new scalar field cosmologies with possibly non-zero spatial curvature by deriving the potential that produces a specified form of the density or the Hubble parameter; the exponential potentials derived in this way were motivated by supergravity or superstring models. Other interesting results were obtained in 2016 by Dimakis, Karagiorgos, Zampeli, Paliathanasis, Christodoulakis and Terzis [9], and by Barrow and Paliathanasis [2]. (In many examples from these papers, the feature prescribed is a specified time dependence for the ratio between the pressure and the density produced jointly by matter and by the scalar field, or by the scalar field alone.)
Finally let us mention that one can give up solving explicitly a cosmological model with a scalar field and yet derive rigorous results on the qualitative behavior of the solutions, under minimal conditions on the self-interaction potential. This has been done in 2009 by Giambò, Giannoni and Magli [14].

1.2 The present thesis

This work is in the area of the integrable cosmologies with a scalar field and matter; from now on the term matter is intended in a generalized sense, and includes the case of radiation. The chief aim is to give a mathematical contribution to the issue of solvability; however, we do not forget the connections with the real world and, in the final part of the thesis, we set up some connections with physical cosmology.
Let us go into the details of the thesis.

The basic frame. This is introduced in section 2; it is connected to the paradigm for cosmology already mentioned before.
So, the universe is homogeneous and isotropic cosmology at any time. The metric is of the Robertson-Walker type, with an arbitrary value of the spatial sectional curvature; as usually, its essential character is the time dependent scale factor $a > 0$. The dimension of spacetime is $d+1$ for any $d \geqslant 2$; of course, $d = 3$ is the physically relevant case.
The universe contains a self-interacting scalar field Φ minimally coupled to gravity; in addition, there is matter represented as a perfect fluid with pressure $p^{(m)}$ and mass/energy density $\rho^{(m)}$ fulfilling an equation of state of $p^{(m)} = w\rho^{(m)}$. Here w is an arbitrary coefficient; especially interesting cases are $w = 0$ (dust) and $w = 1/d$ (radiation); the dust case provides an acceptable model for most of the history of the real universe, while the radiation case can be used to describe the initial part of this history. The scalar field is typically understood as representing dark energy.
As in all the previously mentioned works, we assume a separate conservation law

for the stress-energy tensor of the scalar field and of any type of matter.
We formulate all our results statements using dimensionless versions for all the involved physical quantities. We already mentioned the use of a dimensionless version ϕ of the scalar field, defined by Eq. (2.25); the same equation introduces a dimensionless self-interaction potential $\mathcal{V}(\phi)$. The spatial sectional curvature has a dimensionless variant, represented by a real number k (in the sequel, simply referred to as "the curvature").
We also employ a dimensionless "time" coordinate t, related to the physical or "cosmic" time τ by a suitable "gauge" relation. The simplest relation has the form $\tau = \theta t$ where θ is a constant with the dimension of a time; more general gauge relations of the form $d\tau = \theta b(t) dt$ are considered, since they turn out to be useful in the study of the evolution equations of the model.
Obviously enough, the above mentioned evolution equations are Einstein's equations and the Klein-Gordon equation for the scalar field; these give rise to a system of ODEs for the scale factor a and the scalar field ϕ, with a constraint on the initial data (note that the matter density is a power function of a with exponent depending on w, due to the conservation of the corresponding stress-energy tensor).
A Lagrangian reformulation of the evolution equations is systematically employed; in this approach, the scale factor a and the dimensionless version of Φ are the Lagrangian coordinates, and the initial data fulfill a constraint of zero energy.
In section 2 the curvature k, the field potential $\mathcal{V}(\phi)$ and the coefficient w in the state equation for matter are arbitrary; the rest of the thesis considers integrable special choices, allowing to compute the general solution of the evolution equations.

The case $k = 0$, $w = 0$, $\mathcal{V}(\phi) = \frac{1}{d^2}\left(V_1\, e^{d\,\phi} + V_2\, e^{-d\,\phi}\right)$. This is the subject of section 3. This cosmology with zero curvature and dust is the d-dimensional extension of the solvable model described for $d = 3$ in the already mentioned papers by by Rubano and Scudellaro [29], Piedipalumbo *et al.* [26]. (We already noted that the same potential with arbitrary d and no dust is in the list of solvable models obtained by Fré *et al.* [12]).
The solvability of this case is readily shown performing a coordinate change $(a, \phi) \to (x, y)$ that reduces the Lagrangian to a quadratic form, yielding linear evolution equations.
The reason why we discuss this case in the thesis is not just to extend the results of [29][26] to an arbitrary dimension; we also take the opportunity for a more in-depth analysis of the solution, both from a qualitative and a quantitative point of view. The qualitative analysis is proposed in the subsection 5.1. The quantitative analysis is set up in section 7, mainly for the physical case $d = 3$; we return to this point in the final part of this introduction.

The case $\mathcal{V}(\phi) = \frac{1}{d^2}\left(V_1\, e^{2\,d\,w\,\phi} + V_2\, e^{d\,(1+w)\,\phi}\right)$. This is treated in section 4,

which is the longest part of the thesis; the results presented therein are novel to the best of my knowledge. It was already mentioned in this Introduction that Fré *et al.* [12] have discovered the solvability of the potential $\mathcal{V}(\phi) = \text{const.}\, e^{d\gamma\phi} + \text{const.}\, e^{d(1+\gamma)\phi}$, for arbitrary values of the constant γ, for a cosmology with curvature $k = 0$ and a scalar field only. *The contribution of this thesis is to show that, adding matter with an equation of state $p^{(m)} = w\rho^{(m)}$ and setting $\gamma = w$ we have again a solvable model, even for non zero curvature; w can be chosen arbitrarily if $k = 0$, and must be fixed suitably if $k \neq 0$.*
Drawing inspiration from [12], these results are obtained via a suitable coordinate change $(a, \phi) \to (x, y)$ that reduces the Lagrangian system to a triangular form: this means that the Lagrange equation for x involves this variable only, and substituting the general solution for x into the second Lagrange equation, one gets an equation for y only. The initial equation for x is certainly solvable; if the same occurs for the final equation in y, we have an integrable model. The Lagrangian system is triangular in the following cases:

(i) $k = 0$, w arbitrary;

(ii) k arbitrary, $w = \frac{1}{d}$ (radiation gas);

(iii) k arbitrary, $w = \frac{2}{d} - 1$,

In cases (i)(iii) the initial equation for x describes a harmonic oscillator or repulsor, or a free particle; the final equation for y has the same structure with the addition of a time dependent forcing term, so both equations are explicitly solvable and we have an integrable system. A similar situation occurs in case (ii) for $d = 3$, while case (ii) for $d \neq 3$ gives a problematic equation for y, whose solvability is not evident.

Some comments on the previous choices of the potentials. Admittedly, the potentials $\mathcal{V}$ considered in sections 3, 4 of the thesis have a peculiar form. However, it should be kept in mind that we are searching for models in which the Lagrange equations are explicitly solvable; since integrability is an exceptional feature, it is not at all surprising that we can grant it just for very special choices of $\mathcal{V}$. In any case, the exponential potentials $\mathcal{V}$ considered here to grant integrabiliy are not so different from those considered in some of the previously cited works, which were argued to possess a minimum of physical realism.

Why a minimal coupling? Another feature of this thesis, deserving a brief comment, is the fact that the scalar field is always minimally coupled to gravity; the possibility of a coupling between the scalar curvature R of the spacetime metric and the scalar field Φ is never considered. Indeed, the case of minimal coupling is the one most frequently addressed in the literature on scalar cosmologies. However, there is a more important motivation for this choice: we are interested in integrable cosmologies which are exceptional even with a minimal coupling, and extending this investigation to models with curvature coupling is, to say the least, a non trivial affair. It is true that some transformations have been proposed in the literature [13] [20] to connect minimally coupled theories to systems with curvature coupling: however, these transformations refer to systems with a scalar field and no type of matter fluid, while matter is an essential character of the models in this thesis.

Qualitative analysis of the solutions of sections 3, 4. This is the subject of section 5. For the solvable case of section 3 (with $V_1, V_2 > 0$) and for some solvable cases of section 4, a qualitative analysis of the solutions is performed focusing on the following aspects: possible presence of a Big Bang or a Big Crunch (initial or final singularity of the solution, manifested by the vanishing of the scale factor a); possible divergence of the scale factor on long times; comparison between the energy densities of matter and of the scalar field near the Big Bang, or the Big Crunch, or on long times.

A partially solvable case with simultaneous presence of dust and radiation. Another, presumably original result of the thesis is presented in section 6; this gives some explicit solutions (but not the general solution) of the evolution equations of a cosmological model with spatial dimension $d = 2$ containing a scalar field with a self-interaction exponential potential and two distinct types of matter: dust ($w = 0$) and radiation ($w = 1/2$). Some qualitative features of the solutions are indicated.

Quantitative analysis of one case. This is the subject of the final section 7, where we return to the flat model with dust and potential $\mathcal{V}(\phi) = \frac{1}{d^2}\left(V_1\, e^{d\,\phi} + V_2\, e^{-d\,\phi}\right)$, discussed in section 3. Here we make the choice $V_1 = V_2 \equiv V > 0$, fixing some integration constants of the solution so that the energy density of dust dominates the energy density of the scalar field (dark energy) at the Big Bang; this requirement is put to get a model not too different from the "benchmark model" of cosmology were dark energy is described in terms of a cosmological constant. d is initially arbitrary, and ultimately set to 3 to make contact with physical reality.
In this section we discuss the following problem: is it possible to determine the constant V in the potential, the instant corresponding to present time and the residual integration constants of the solution so as to obtain any value ϕ_* for the scalar field at present time and prescribed values for Hubble's expansion rate and for the matter density at present time? This problem is solved explicitly.
We then set $d = 3$ and substitute for Hubble's parameter and the density of matter the commonly accepted values on the grounds of astronomical data and of the previous benchmark model. This gives a family of physically reasonable cosmological models, labelled by the value ϕ_* of the field at present time.
In this framework, the choice $\phi_* = 0$ (the minimum point of $\mathcal{V}$) implies $\phi = 0$ at all

times and is a equivalent to a model with cosmological constant. A case with non zero ϕ_* is treated quantitatively and compared with the previous choice $\phi_* = 0$. Perhaps, a similar quantitative analysis could be performed for the radiation solutions ($w = 1/d$) obtained in subsection 4.2 setting $d = 3$, so as to get a picture of the initial, radiation dominated part of the history of universe; in this case, reasonable values should be prescribed for Hubble's parameter and for the energy density of radiation at the end of this era.

On the Appendices. At the end of the thesis there are some Appendices that deal in more details some aspects. Appendix A treats the gravitational constant in arbitrary dimension $d + 1$; Appendix B is on the particle horizon; Appendix C gives some details about the calculation of certain integrals involved in the solutions; Appendix D is about some validity conditions for the solutions; Appendix E treats models with a cosmological constant and an arbitrary number of perfect fluids; Appendix F describes the "benchmark model" of cosmology (modelling the real universe in terms of dust, radiation and a cosmological constant term for dark energy), and also reports the values of some important constants; Appendix G gives some information which can be used to compare the potentials in this thesis with phenomenological potential derived by Saini, Raychaudhury, Sahni and Starobinsky [31].

1.3 Acknowledgements and final remarks.

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2 A general frame

2.1 Some conventions

Throughout this work we consider spacetimes of dimension $d+1$, with $d \geqslant 2$; of course, $d=3$ is the physically realistic choice. To treat the dimensions of physical quantities, we write $\mathbb{L}, \mathbb{T}$ and $\mathbb{M}$ to indicate length, time and mass, respectively. For any d, there are a speed of light c and a reduced Planck constants $\hbar$. We work in units where

$$c = 1, \qquad \hbar = 1 \; ; \tag{2.1}$$

since $[c] = \mathbb{L}/\mathbb{T}$ and $[\hbar] = \mathbb{M}\,(\mathbb{L}/\mathbb{T})^2\,\mathbb{T}$, from here we infer that $\mathbb{L} = \mathbb{T} = \mathbb{M}^{-1}$. Whenever necessary, and especially for the physical case $d=3$, we use the standard values of $c, \hbar$ so that

$$1 = c = 2.99792458 \times 10^{10}\,\frac{\text{cm}}{\text{sec}} \; , \quad 1 = \hbar = 1.05457168 \times 10^{-27}\,\frac{\text{cm}^2\,\text{gr}}{\text{sec}} \; ; \tag{2.2}$$

this implies

$$\text{cm} = 3.33564095 \times 10^{-11}\,\text{sec} = 2.84278882 \times 10^{-37}\,\text{gr}^{-1} \; . \tag{2.3}$$

We write Einstein's equations as

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu} R = d(d-1)\,\gamma_d\,G_d\,T_{\mu\nu} \; , \tag{2.4}$$

where $T_{\mu\nu}$ is the stress-energy tensor, G_d is a constant of dimension $\mathbb{L}^d\,\mathbb{T}^{-2}\,\mathbb{M}^{-1} = \mathbb{L}^{d-1}$ and γ_d is a numerical coefficient. For any $d \geqslant 3$, γ_d can be fixed so that, in the Newtonian limit, these equations reproduce the classical law $F = G_d\,m\,M/r^{d-1}$ for the gravitational force F between two particles of masses m and M at a distance r. As reviewed in Appendix A, which is inspired by [21], this occurs if

$$\gamma_d := \frac{\pi^{d/2}}{(d-2)\,\Gamma(d/2+1)} \qquad (d \geqslant 3) \; ; \tag{2.5}$$

note that γ_d is the volume of a unit ball in d-dimensional Euclidean space, divided by $d-2$; in particular, $\gamma_3 = 4\pi/3$. In the case of space dimension $d=2$, it is known that Einstein's equations do not possess a Newtonian limit (see, e.g., [15]); in this case, the value of γ_d is not fixed on physical grounds and can be chosen arbitrarily, so we set

$$\gamma_2 := \text{any real number} > 0 \; . \tag{2.6}$$

For any d, the constant G_d appearing in Eq. (2.4) is referred to as the gravitational constant. For $d=3$, we have the physical value

$$\begin{aligned} G_3 = 6.6742 \times 10^{-8}\,\frac{\text{cm}^3}{\text{sec}^2\,\text{gr}} = \\ = 2.6122 \times 10^{-66}\,\text{cm}^2 = 2.9065 \times 10^{-87}\,\text{sec}^2 = 2.1110 \times 10^{9}\,\text{gr}^{-2} \; . \end{aligned} \tag{2.7}$$

2.2 A general cosmological model with matter and a scalar field

Let us describe the universe via a spacetime of dimension $d+1$, for the moment arbitrary; this will have coordinate systems $(x^\mu)_{\mu=0,\dots,d}$ and a line element

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \, . \tag{2.8}$$

We assume the presence of matter/radiation that we represent as a perfect fluid, whose pressure $p^{(m)}$ and mass-energy density $\rho^{(m)}$ fulfill the equation of state

$$p^{(m)} = w\, \rho^{(m)} \tag{2.9}$$

for some suitable real constant w; in any spacetime coordinate system (x^μ), the associated stress-energy tensor reads

$$T^{(m)}{}_{\mu\nu} = \left(p^{(m)} + \rho^{(m)}\right) U_\mu\, U_\nu + p^{(m)}\, g_{\mu\nu} \ , \tag{2.10}$$

where U^μ is the four-velocity of the fluid. In the sequel, we often refer to the conditions

$$\rho^{(m)} \geqslant 0 \ , \qquad w \geqslant -1 \ , \tag{2.11}$$

$$\rho^{(m)} \geqslant 0 \ , \qquad -1 \leqslant w \leqslant 1 \ , \tag{2.12}$$

which are equivalent, respectively, to the weak and dominant energy conditions for $T^{(m)}{}_{\mu\nu}$ (see, e.g., [16]). In the case $w = 0$, the fluid is a *dust*; if $w = 1/d$ the trace $T^{(m)\mu}{}_{\mu}$ vanishes, as typical of a *radiation gas*.
The universe also contains a scalar field Φ, minimally coupled to gravity and with a self-interaction potential $\mathfrak{V}(\Phi)$. The corresponding stress-energy tensor is

$$T^{(\Phi)}{}_{\mu\nu} = \partial_\mu \Phi\, \partial_\nu \Phi - \frac{1}{2}\, g_{\mu\nu}\, \partial_\alpha \Phi\, \partial^\alpha \Phi - g_{\mu\nu}\, \mathfrak{V}(\Phi) \ ; \tag{2.13}$$

Φ and $\mathfrak{V}(\Phi)$ have dimensions $\mathbb{L}^{-d+1}$ and $\mathbb{L}^{d+1}$, respectively.
Einstein's equations are formulated according to the general scheme (2.4), and read

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = d(d-1)\, \gamma_d\, G_d \left(T^{(m)}{}_{\mu\nu} + T^{(\Phi)}{}_{\mu\nu}\right) \, . \tag{2.14}$$

In addition to these equations, we postulate the conservation law for the matter stress-energy tensor

$$\nabla_\mu T^{(m)\mu}{}_{\nu} = 0 \, . \tag{2.15}$$

Of course, Eq. (2.14) (along with the Bianchi identity) implies the conservation of the total stress-energy tensor $T^{(m)}{}_{\mu\nu} + T^{(\Phi)}{}_{\mu\nu}$ that, combined with Eq. (2.15), yields the relation

$$\nabla_\mu T^{(\Phi)\mu}{}_{\nu} = 0 \, . \tag{2.16}$$

The separate conservation laws (2.15) (2.16) mean that matter and the scalar field do not interact directly.
For the field we postulate the Klein-Gordon-type equation

$$\Box\Phi = \mathfrak{V}'(\Phi) \ , \tag{2.17}$$

where $\Box\Phi := \nabla_\mu \nabla^\mu \, \Phi = \frac{1}{\sqrt{|g|}} \, \partial_\mu \big(\sqrt{|g|} \, g^{\mu\nu} \, \partial_\nu \Phi \big)$ and $|g| := - \det(g_{\mu\nu})$. This equation is closely related to Eq. (2.16), since the mere expression (2.30) of $T^{(\Phi)}{}_{\mu\nu}$ ensures

$$\nabla_\mu T^{(\Phi)\mu}{}_\nu = \Big(\nabla_\mu \nabla^\mu \, \Phi - \mathfrak{V}'(\Phi) \Big) \, \partial_\nu \Phi \ ; \tag{2.18}$$

thus Eq. (2.16) implies (2.17), at least in the spacetime region where Φ has non-zero differential.

2.3 The reference model of the present work

Throughout this work, the general model of the previous subsection is specialized to the case of spatial homogeneity and isotropy. So we consider a Robertson-Walker spacetime, the product of the time line and of a Riemannian manifold $\mathcal{M}^d_{\mathsf{k}}$ of constant sectional curvature k ($[\mathsf{k}] = \mathbb{L}^{-2}$). Using the standard Robertson-Walker cosmic time τ and any system of coordinates $\mathbf{x} = (x^i)_{i=1,..,d}$ for $\mathcal{M}^d_{\mathsf{k}}$, we have

$$ds^2 = - \, d\tau^2 + a^2(\tau) \, d\ell^2 = - \, d\tau^2 + a^2(\tau) \, h_{ij}(\mathbf{x}) \, dx^i dx^j \ , \tag{2.19}$$

where $d\ell^2 = h_{ij}(\mathbf{x}) \, dx^i dx^j$ is the line element of $\mathcal{M}^d_{\mathsf{k}}$ ([1]) and $a(\tau) > 0$ is the dimensionless scale factor; typically, the latter is fixed so that $a(\tau_*) = 1$ at some reference time τ_* (e.g., the present time).
For our purposes, it is convenient to use in place of τ a dimensionless "time" coordinate t such that

$$d\tau = \theta \, b(t) \, dt \ , \tag{2.20}$$

where $b(t) > 0$ is a dimensionless "gauge function" to be determined and θ is a dimensional constant with $[\theta] = \mathbb{T}$; thus

$$ds^2 = - \, \theta^2 \, b(t)^2 dt^2 + a^2(t) \, h_{ij}(\mathbf{x}) \, dx^i dx^j \ , \tag{2.21}$$

[1]As well known, $\mathcal{M}^d_{\mathsf{k}}$ (minus a point) can be represented as the product of two factors: the interval $0 < r < r_{\mathsf{k}}$, with $r_{\mathsf{k}} := +\infty$ if $\mathsf{k} \leqslant 0$ and $r_{\mathsf{k}} := 1/\sqrt{\mathsf{k}}$ if $\mathsf{k} > 0$, and a $(d-1)$-dimensional sphere of unit radius and line element $d\sigma$. The line element of $\mathcal{M}^d_{\mathsf{k}}$ is $d\ell^2 = dr^2/(1 - \mathsf{k}\, r^2) + r^2 d\sigma^2$; in spherical coordinates $\varphi^2, \varphi^3, ..., \varphi^d$, it is $d\sigma^2 = \sum_{i=2}^d \Big(\prod_{\ell=2}^{i-1} \sin^2 \varphi_\ell \Big) \, d\varphi_i^2$. The family $\mathbf{x} \equiv (x^i)_{i=1,...,d} := (r, \varphi^2, ..., \varphi^d)$ is a system of coordinates for $\mathcal{M}^d_{\mathsf{k}}$ of very frequent use.

where $a(t)$ stands for $a(\tau)$ when τ is viewed as a function of t. Accordingly, we shall write the scalar curvature in terms of a dimensionless coefficient k, setting

$$\mathsf{k} = \frac{k}{\theta^2} \ . \tag{2.22}$$

From here to the end of this work we will use a spacetime coordinate system

$$(x^\mu) \equiv (x^0, x^i) := (t, \mathbf{x}) \ , \qquad (\mu = 0, ..., d;\ i = 1, ..., d) \tag{2.23}$$

where, as above, $\mathbf{x} = (x^i)$ are coordinates on $\mathcal{M}^d_{\mathsf{k}}$; Greek indexes always range from 0 to d, Latin indexes from 1 to d. From now on we indicate with U^μ the $(d{+}1)$-velocity of the Robertson-Walker frame; in the coordinate system (2.23) we have

$$U^\mu = \frac{\delta^\mu{}_0}{\theta\, b(t)} \ , \qquad U_\mu = -\,\theta\, b(t)\, \delta_{\mu 0} \ . \tag{2.24}$$

The perfect fluid describing the matter/radiation content of the universe is assumed to be at rest in the Robertson-Walker frame; it has pressure, mass-energy density and stress-energy tensor as in Eq.s (2.9) (2.10), with U^μ as in (2.24).
The second character other than this fluid is the scalar field Φ, with a stress-energy tensor as in Eq. (2.13). It is convenient to introduce dimensionless versions $\phi, \mathcal{V}$ of the field and of its potential, defined by

$$\Phi = \frac{\phi}{\sqrt{\gamma_d\, G_d}} \ , \qquad \mathfrak{V}(\Phi) = \frac{\mathcal{V}(\phi)}{\gamma_d\, G_d\, \theta^2} \ . \tag{2.25}$$

In the above θ is the time parameter of Eq. (2.20), G_d is the gravitational constant and γ_d the related numerical coefficient (see Eq.s (2.4-2.6); γ_d is inserted here for future convenience). From now on, the terms "field" and "potential" will be generally used to indicate the dimensionless objects $\phi, \mathcal{V}$. Eq.s (2.13) (2.25) imply

$$T^{(\Phi)}{}_{\mu\nu} = \frac{1}{\gamma_d\, G_d}\, \partial_\mu \phi\, \partial_\nu \phi - \frac{1}{\gamma_d\, G_d}\, g_{\mu\nu} \left(\frac{1}{2}\, \partial_\alpha \phi\, \partial^\alpha \phi + \frac{\mathcal{V}(\phi)}{\theta^2} \right) . \tag{2.26}$$

Obviously enough, to fulfill the homogeneity requirement of this cosmology we assume that the matter density and the field depend only on time:

$$\rho^{(m)} = \rho^{(m)}(t) \ , \qquad \phi = \phi(t) \ . \tag{2.27}$$

The coefficients of the stress-energy tensors for the fluid and the field in a coordinate system of the form (2.23) are (for $i, j = 1, ...d$)

$$T^{(m)}{}_{00} = \theta^2\, b^2 \rho^{(m)} \ , \quad T^{(m)}{}_{0i} = 0 \ , \quad T^{(m)}{}_{ij} = a^2\, p^{(m)}\, h_{ij} = w\, a^2 \rho^{(m)}\, h_{ij} \ ; \tag{2.28}$$

$$
\begin{gathered}
T^{(\Phi)}{}_{00} = \frac{1}{\gamma_d\, G_d} \left(\frac{1}{2}\,\dot{\phi}^2 + b^2\, \mathcal{V}(\phi) \right) , \qquad T^{(\Phi)}{}_{0i} = 0 \ , \\
T^{(\Phi)}{}_{ij} = \frac{1}{\gamma_d\, G_d\, \theta^2} \left(\frac{1}{2}\,\dot{\phi}^2 - b^2\, \mathcal{V}(\phi) \right) \frac{a^2}{b^2}\, h_{ij} \ ;
\end{gathered} \tag{2.29}
$$

here and in the sequel, $\dot{}$ stands for d/dt. It should be noted that the field stress-energy tensor can be represented in a fluid-like form

$$
\begin{gathered}
T^{(\Phi)}{}_{\mu\nu} = \big(p^{(\Phi)} + \rho^{(\Phi)}\big)\, U_\mu\, U_\nu + p^{(\Phi)}\, g_{\mu\nu} \ , \\
\rho^{(\Phi)} := \frac{1}{\gamma_d\, G_d\, \theta^2} \left(\frac{\dot{\phi}^2}{2\, b^2} + \mathcal{V}(\phi) \right) , \qquad p^{(\Phi)} := \frac{1}{\gamma_d\, G_d\, \theta^2} \left(\frac{\dot{\phi}^2}{2\, b^2} - \mathcal{V}(\phi) \right) ;
\end{gathered} \tag{2.30}
$$

for this reason, $\rho^{(\Phi)}$ and $p^{(\Phi)}$ will be referred to as the density and pressure of the field. Let us remark that $\rho^{(m)}$, $p^{(m)}$, $\rho^{(\Phi)}$ and $p^{(\Phi)}$ all have dimension $\mathbb{M}/\mathbb{L}^d = \mathbb{L}^{-(d+1)}$. The field Φ will be regarded as a model for the dark energy content of the universe; its role in the dynamics of the model will be similar to the one of the cosmological term $\Lambda\, g_{\mu\nu}$ usually added by hand in the Einstein equations. In the sequel, we often refer to the "equation of state coefficient"

$$
w^{(\Phi)} := \frac{p^{(\Phi)}}{\rho^{(\Phi)}} \ , \tag{2.31}
$$

depending on t (which makes sense at all times t such that $\rho^{(\Phi)}(t) \neq 0$).
The evolution equations of the model are Eq.s (2.14) (2.15) (2.17) (with the connections and implications already pointed out in the previous subsection, especially for what concerns Eq. (2.16)).
The present framework is a generalization of the one considered in [12], where it was assumed that $k = 0$ and no matter field was present. In the interpretation proposed above, the setting of [12] corresponds to a spatially flat universe filled exclusively with dark energy.

2.4 Explicit form of the evolution equations; considerations on independence

Consider again a coordinate system as in Eq. (2.23); one easily checks that

$$
\begin{gathered}
R_{00} = d \left(\frac{\dot{a}\, \dot{b}}{a\, b} - \frac{\ddot{a}}{a} \right) , \qquad R_{ij} = \frac{(d-1)\, k\, h_{ij}}{\theta^2} + \frac{a^2\, h_{ij}}{\theta^2\, b^2} \left(\frac{\ddot{a}}{a} - \frac{\dot{a}\, \dot{b}}{a\, b} + (d-1)\, \frac{\dot{a}^2}{a^2} \right) , \\
R_{0i} = 0, \qquad R = \frac{d(d-1)\, k}{\theta^2\, a^2} + \frac{2\, d}{\theta^2\, b^2} \left(\frac{\ddot{a}}{a} - \frac{\dot{a}\, \dot{b}}{a\, b} + \frac{d-1}{2}\, \frac{\dot{a}^2}{a^2} \right)
\end{gathered} \tag{2.32}
$$

($i,j=1,...,d$; note that $(d-1)\,k\,h_{ij}/\theta^2 = (d-1)\,\mathsf{k}\,h_{ij}$ is the Ricci tensor of $\mathcal{M}^d_{\mathsf{k}}$). Moreover, one finds that

$$\nabla_\mu T^{(m)\mu}{}_0 = -\left(\dot{\rho}^{(m)} + d\,(w+1)\,\frac{\dot{a}}{a}\,\rho^{(m)}\right) , \qquad \nabla_\mu T^{(m)\mu}{}_i = 0 \ . \tag{2.33}$$

Due to Eq. (2.33), Eq. (2.15) is equivalent to

$$\mathfrak{E}_m = 0 \ , \qquad \mathfrak{E}_m := \dot{\rho}^{(m)} + d\,(w+1)\,\frac{\dot{a}}{a}\,\rho^{(m)} \ ; \tag{2.34}$$

this equation is fulfilled if and only if $\rho^{(m)} = \rho^{(m*)}/a^{d(w+1)}$ where $\rho^{(m*)}$ is an integration constant with the dimension of $\rho^{(m)}$, i.e., $\mathbb{M}/\mathbb{L}^d = \mathbb{L}^{-(d+1)}$. For future convenience we set $\rho^{(m*)} = \Omega_{m*}/(2\,\gamma_d\,G_d\,\theta^2)$, where Ω_{m*} is a dimensionless constant; thus,

$$\rho^{(m)} = \frac{\Omega_{m*}}{2\,\gamma_d\,G_d\,\theta^2\,a^{d(w+1)}} \ . \tag{2.35}$$

Note that $\operatorname{sign}\rho^{(m)}(t) = \operatorname{sign}\Omega_{m*}$ at all times; we will tipically assume $\Omega_{m*} \geq 0$. Let us now consider Einstein's equations (2.14), that we write down using Eq.s (2.9) (2.28) (2.29) (2.32) and the expression (2.35) for $\rho^{(m)}$ (and so $p^{(m)}$); in this way, we obtain ([2])

$$\mathfrak{A} = 0 \ , \tag{2.36}$$

$$\mathfrak{A} := \frac{\ddot{a}}{a} + \frac{(d-2)\,\dot{a}^2}{2\,a^2} - \frac{\dot{a}\,\dot{b}}{a\,b} + d\left(\frac{w\,\Omega_{m*}\,b^2}{2\,a^{d(w+1)}} + \frac{\dot{\phi}^2}{2} - b^2\,\mathcal{V}(\phi)\right) + \frac{(d-2)\,k\,b^2}{2\,a^2} \ ;$$

$$\mathfrak{E} = 0 \ , \qquad \mathfrak{E} := \frac{\dot{a}^2}{a^2} - 2\left(\frac{\Omega_{m*}\,b^2}{2\,a^{d(w+1)}} + \frac{\dot{\phi}^2}{2} + b^2\,\mathcal{V}(\phi)\right) + \frac{k\,b^2}{a^2} \ . \tag{2.37}$$

Finally, the field equation (2.17) reads (with $\mathcal{V}' := d\mathcal{V}/d\phi$)

$$\mathfrak{F} = 0 \ , \qquad \mathfrak{F} := \ddot{\phi} + \left(d\,\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right)\dot{\phi} + b^2\,\mathcal{V}'(\phi) \ . \tag{2.38}$$

Summing up, the general evolution equations (2.14) (2.15) (2.17) give rise in the present case to the expression (2.35) for $\rho^{(m)}$ and to the system of equations

$$\mathfrak{A} = 0, \qquad \mathfrak{E} = 0, \qquad \mathfrak{F} = 0, \tag{2.39}$$

given explicitly by Eq.s (2.36) (2.37) (2.38). These three equations are not independent, as pointed out in the forthcoming items (i)(ii):

[2]To be precise: Eq. (2.14) with $(\mu,\nu) = (i,j) \in \{1,...,d\}^2$ is equivalent to $\mathfrak{A} = 0$, while Eq. (2.14) with $\mu = \nu = 0$ is equivalent to $\mathfrak{E} = 0$.

(i) We have already pointed out that the field equation (2.17) is a consequence of the other evolution equations (2.14) (2.15) in the spacetime region where the scalar field has a non-vanishing differential: this is due to Eq.s (2.16) (2.18). In the present setting one can check directly that $2\mathfrak{F}\dot{\phi} = -\dot{\mathfrak{E}} + 2\,(\dot{a}/a)\,\mathfrak{A} - (d\,\dot{a}/a - 2\,\dot{b}/b)\,\mathfrak{E}$; therefore,

$$\mathfrak{A} = 0,\ \mathfrak{E} = 0 \quad \Rightarrow \quad \mathfrak{F} = 0 \ \text{ when } \ \dot{\phi} \neq 0\,. \tag{2.40}$$

(ii) As a partial converse, let us consider the relations $\mathfrak{A} = 0, \mathfrak{F} = 0$ supplemented with the initial condition $\mathfrak{E}(t_0) = 0$, meaning that $\mathfrak{E}$ is required to vanish at a given time t_0; we claim that

$$\mathfrak{A} = 0,\ \mathfrak{F} = 0,\ \mathfrak{E}(t_0) = 0 \quad \Rightarrow \quad \mathfrak{E} = 0 \ \text{ at all times.} \tag{2.41}$$

To prove this, let us reconsider the identity already written in (i), and rephrase it as $\dot{\mathfrak{E}} = -(d\,\dot{a}/a - 2\,\dot{b}/b)\,\mathfrak{E} + 2\,(\dot{a}/a)\,\mathfrak{A} - 2\,\dot{\phi}\,\mathfrak{F}$. If $\mathfrak{A} = 0$, $\mathfrak{F} = 0$ (at all times), we infer $\dot{\mathfrak{E}} = -(d\,\dot{a}/a - 2\,\dot{b}/b)\,\mathfrak{E}$; the latter differential equation for $\mathfrak{E}$, supplemented with the initial condition $\mathfrak{E}(t_0) = 0$, gives $\mathfrak{E} = 0$ at all times.

2.5 Discussion on the domain of the solutions $(a(t), \phi(t))$; Big Bang and Big Crunch

Each solution $(a(t), b(t), \phi(t))$ of the system $\mathfrak{A} = 0, \mathfrak{E} = 0, \mathfrak{F} = 0$ is well defined and fulfills $a(t) > 0$ for t in a suitable interval $I \subset \mathbf{R}$; from now on, when we speak of a solution we always assume I to be maximal (i.e., that the solution cannot be extended to a larger interval). Let $I = (t_{in}, t_{fin})$, where $-\infty \leqslant t_{in} < t_{fin} \leqslant +\infty$; we recall that t and τ are related by Eq. (2.20), which is equivalent to

$$\tau(t) = \theta \int_{t_r}^{t} dt'\ b(t')\ , \tag{2.42}$$

where t_r is arbitrarily chosen in I. If $a(t) \to 0$ for $t \to t_{in}^{+}$ and $b(t)$ is integrable in a right neighborhood of t_{in}, we say that the model has a *Big Bang* at $t = t_{in}$. Needless to say, in this definition the condition $a(t) \to 0$ indicates an initial singularity and the integrability assumption for b ensures the cosmic time $\tau(t)$ to approach a finite limit τ_{in} for $t \to t_{in}^{+}$; of course we can set $\tau_{in} = 0$ stipulating Eq. (2.42) with $t_r = t_{in}$. If $a(t) \to 0$ for $t \to t_{fin}^{-}$ and $b(t)$ is integrable in a left neighborhood of t_{fin}, we say that the model has a *Big Crunch* at $t = t_{fin}$ (final singularity while the cosmic time approaches a finite value τ_{fin}). The occurrence of one (or both) of these features depends on the potential $\mathcal{V}$ for the scalar field and on the initial data assumed for the system at a suitable time; in the sequel we will meet many examples.

2.6 The particle horizon

Suppose the model has a Big Bang at $\tau_{in} = \tau(t_{in})$. We define the particle horizon at a cosmic time $\tau_1 = \tau(t_1)$ as (see Appendix B for more details)

$$\Theta(\tau_1) := \int_{\tau_{in}}^{\tau_1} \frac{d\tau}{a(\tau)} = \theta \int_{t_{in}}^{t_1} dt \, \frac{b(t)}{a(t)} \; ; \tag{2.43}$$

the above integral is finite or infinite, according to the behavior of $a(\tau)$ or $a(t), b(t)$ close to the Big Bang. The interpretation of $\Theta(\tau_1)$ is well known, and can be summarized as follows writing $\mathbf{p_0}, \mathbf{p_1}$, etc. for the points of $\mathcal{M}^d_{\mathsf{k}}$ and dist for the distance on $\mathcal{M}^d_{\mathsf{k}}$ corresponding to the metric $d\ell^2$ (see Eq. (2.19)): for each point $\mathbf{p_1}$, the ball $\mathcal{B}(\mathbf{p}_1, \tau_1) := \{\mathbf{p}_0 \in \mathcal{M}^d_{\mathsf{k}} \mid \text{dist}(\mathbf{p}_0, \mathbf{p}_1) < \Theta(\tau_1)\}$ is the portion of $\mathcal{M}^d_{\mathsf{k}}$ formed by the points $\mathbf{p_0}$ which had the time to interact causally with $\mathbf{p_1}$ from the Big Bang to τ_1 ([3]). This portion is the whole $\mathcal{M}^d_{\mathsf{k}}$ if and only if $\Theta(\tau_1) \geqslant \delta_{\mathsf{k}}$, where δ_{k} is the diameter of $\mathcal{M}^d_{\mathsf{k}}$, i.e., $\delta_{\mathsf{k}} := \sup\{\text{dist}(\mathbf{p}_0, \mathbf{p}_1) \mid \mathbf{p}_0 \in \mathcal{M}^d_{\mathsf{k}}\}$. The diameter δ_{k} is in fact independent of $\mathbf{p_0}$, and given by $\delta_{\mathsf{k}} = +\infty$ if $\mathsf{k} \leqslant 0$ and $\delta_{\mathsf{k}} = \pi/\sqrt{\mathsf{k}} = \theta\pi/\sqrt{k}$ if $\mathsf{k} > 0$. (Thus, for $\mathsf{k} \leqslant 0$, $\Theta(\tau_1) \geqslant \delta_{\mathsf{k}}$ just means $\Theta(\tau_1) = +\infty$.)
Of course the situation where $\Theta(\tau_1) \geqslant \delta_{\mathsf{k}}$ is of special interest, for it explains the homogeneity of the universe at time τ_1. As well known, many Robertson-Walker cosmologies violate this condition; this will happen, in particular, for many of the integrable cosmologies presented in this work.

2.7 Cosmological constant solutions

Let us search for a solution of the model with

$$\phi(t) = \text{const.} \equiv \phi_0 \, . \tag{2.44}$$

From Eq. (2.38) we see that this occurs if and only if

$$\mathcal{V}'(\phi_0) = 0 \, . \tag{2.45}$$

In the case (2.44) (2.45), Eq.s (2.36) (2.37) involve only the functions $a(t)$ and $b(t)$ and the constant $\mathcal{V}(\phi_0)$. Moreover, the field stress-energy tensor of Eq. (2.26) becomes

$$T^{(\Phi)}{}_{\mu\nu} = - \, g_{\mu\nu} \, \frac{\mathcal{V}(\phi_0)}{\gamma_d \, G_d \, \theta^2} \tag{2.46}$$

and bringing this term to the right-hand side of Einstein's equations (2.14) we obtain

$$R_{\mu\nu} - \frac{1}{2} \, g_{\mu\nu} R + \frac{d(d-1) \, \Omega_{\Lambda,*}}{2 \, \theta^2} \, g_{\mu\nu} = d \, (d-1) \, \gamma_d \, G_d \, T^{(m)}{}_{\mu\nu} \, , \qquad \Omega_{\Lambda,*} := 2 \, \mathcal{V}(\phi_0) \, . \tag{2.47}$$

[3]in fact, one shows that there exists a causal curve starting from $\mathbf{p_0}$ at a time $\tau_0 \in (\tau_{in}, \tau_1)$ and ending at $\mathbf{p_1}$ at time τ_1 if and only if $\text{dist}(\mathbf{p_0}, \mathbf{p_1}) < \boldsymbol{\Theta}(\boldsymbol{\tau_1})$.

So, in this case we have a model with a cosmological constant $\Lambda = d(d-1)\,\Omega_{\Lambda,*}/(2\,\theta^2)$ (note that $\Omega_{\Lambda,*}$ is dimensionless and $[\Lambda] = \mathbb{L}^{-2}$, as expected).
To conclude this subsection, let us characterize the cosmological constant solutions from the viewpoint of the field equation of state. To this purpose, let us consider any solution of the model. From Eq. (2.30) we see that $\phi(t) = \text{const.}$ (i.e., $\dot{\phi} = 0$) if and only if

$$p^{(\Phi)} = -\rho^{(\Phi)} \ . \tag{2.48}$$

Assuming in addition that $\rho^{(\Phi)} \neq 0$ at all times, and comparing with Eq. (2.31), we can readily infer that $\phi(t) = \text{const.}$ if and only if

$$w^{(\Phi)} = -1 \ . \tag{2.49}$$

2.8 Hubble's parameter and the deceleration parameter

The time dependent Hubble parameter is

$$H := \frac{1}{a}\frac{da}{d\tau} = \frac{\dot{a}}{\theta\, a\, b} \ ; \tag{2.50}$$

the first equality above is the standard definition in terms of the cosmic time τ, and the second identity is a reformulation in terms of the derivative $\dot{} = d/dt$ (recall Eq. (2.20)).
The dimensionless time dependent deceleration parameter is

$$q := -\frac{1}{a\,H^2}\frac{d^2 a}{d\tau^2} = -\frac{a\,\ddot{a}}{\dot{a}^2} + \frac{a\,\dot{b}}{\dot{a}\,b} \ . \tag{2.51}$$

2.9 The dimensionless density parameters

These are the time dependent quantities

$$\Omega_m := \frac{2\,\gamma_d\,G_d\,\rho^{(m)}}{H^2} \ , \qquad \Omega_\Phi := \frac{2\,\gamma_d\,G_d\,\rho^{(\Phi)}}{H^2} \ , \qquad \Omega_k := -\frac{k}{\theta^2 H^2\, a^2} \ , \tag{2.52}$$

From Eq.s (2.30) for $\rho^{(\Phi)}$, (2.35) for $\rho^{(m)}$ and (2.50) for H we get

$$\Omega_m = \frac{\Omega_{m*}\, b^2}{a^{d(w+1)-2}\,\dot{a}^2} \ , \qquad \Omega_\Phi = \left(\dot{\phi}^2 + 2\, b^2\, \mathcal{V}(\phi)\right)\frac{a^2}{\dot{a}^2} \ , \qquad \Omega_k = -\frac{k\, b^2}{\dot{a}^2} \ . \tag{2.53}$$

By comparison with Eq. (2.37), we see that

$$\mathfrak{E} = 0 \quad \Leftrightarrow \quad \Omega_m + \Omega_\Phi + \Omega_k = 1 \ . \tag{2.54}$$

The parameters Ω_m and Ω_k are standard objects in cosmology (see, e.g., [32]). Ω_Φ plays a role similar to the dimensionless parameter $\Omega_\Lambda := 2\,\Lambda/(d(d-1)H^2)$ usually considered when a cosmological term $\Lambda\, g_{\mu\nu}$ is present in Einstein's equations.

According to the remark after Eq. (2.19), we typically have $a(t_*) = 1$ at some reference time t_*; in addition, setting $\theta := 1/|H(t_*)|$, from Eq. (2.50) we obtain $b(t_*) = |\dot{a}(t_*)|$. By comparison with the first relation in (2.53), these facts give $\Omega_m(t_*) = \Omega_{m*}$.

2.10 Lagrangian formulation

A general cosmological model as in subsection 2.2 is described by the action functional ([4])

$$\mathcal{S} := \int d^{d+1}x \sqrt{-\det(g_{\mu\nu})} \left[\frac{R}{2d(d-1)\gamma_d\, G_d} - \frac{1}{2}\, g^{\mu\nu}\partial_\mu \Phi\, \partial_\nu \Phi - \mathfrak{V}(\Phi) - \rho^{(m)} \right], \tag{2.55}$$

which is dimensionless in our units with $\hbar = 1$; we now specify our considerations to the Robertson-Walker cosmology of the present work (subsection 2.3).
Let us use the coordinates $(x^\mu) = (t, \mathbf{x})$ introduced in Eq. (2.23), inserting in $\mathcal{S}$ the metric corresponding to the line element (2.21); so, $\det(g_{\mu\nu}) = -\,\theta^2\, a^{2d}\, b^2\, \det(h_{ij}(\mathbf{x}))$ and the scalar curvature R is given by Eq. (2.32). Moreover, let us express Φ and $\mathfrak{V}(\Phi)$ according to Eq. (2.25) with $\phi = \phi(t)$ and write $\rho^{(m)} = \rho^{(m)}(t)$ as in Eq. (2.35). In this way, we obtain

$$\mathcal{S} = \frac{1}{\gamma_d\, G_d\, \theta} \int d^d\mathbf{x} \sqrt{\det(h_{ij}(\mathbf{x}))} \int dt \left[\mathcal{L}(a, \dot{a}, \phi, \dot{\phi}, b) + \frac{1}{(d{-}1)} \frac{d}{dt}\left(\frac{a^{d-1}\, \dot{a}}{b} \right) \right], \tag{2.56}$$

$$\mathcal{L}(a, \dot{a}, \phi, \dot{\phi}, b) := \frac{1}{2\, b}\Big(-\, a^{d-2}\, \dot{a}^2 + a^d\, \dot{\phi}^2 \Big) - b \left(a^d\, \mathcal{V}(\phi) + \frac{\Omega_{m*}}{2\, a^{wd}} - \frac{k\, a^{d-2}}{2} \right). \tag{2.57}$$

In Eq. (2.56), the integral $\int d^d\mathbf{x} \sqrt{\det(h_{ij}(\mathbf{x}))}$ is an irrelevant multiplicative factor (even though infinite if $k \leqslant 0$); the total t-derivative in the integral is also irrelevant. In conclusion, we can regard $\mathcal{S}$ as associated to the (dimensionless) Lagrangian function $\mathcal{L}$ written in Eq. (2.57); this is a degenerate Lagrangian, since it does not depend on $\dot{b}$.
Independently of the previous considerations, it can be checked by direct inspection that the Lagrange equations induced by $\mathcal{L}$ are equivalent to the evolution equations of this model. In fact, the Lagrangian derivatives

$$\frac{\delta \mathcal{L}}{\delta q} := -\,\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) + \frac{\partial \mathcal{L}}{\partial q}\,, \qquad (q = a, \phi, b) \tag{2.58}$$

[4]The term $\rho^{(m)}$ in Eq. (2.55) must be regarded as depending on the *matter history* of the system; the notion of "matter history" can be defined following [16, page 69], where the matter density here denoted with $\rho^{(m)}$ is instead indicated with μ.

are such that

$$\frac{\delta \mathcal{L}}{\delta a} = \frac{a^{d-1}}{b} \mathfrak{A} \ , \qquad \frac{\delta \mathcal{L}}{\delta \phi} = - \frac{a^d}{b} \mathfrak{F} \ , \qquad \frac{\delta \mathcal{L}}{\delta b} = \frac{a^d}{2\, b^2} \mathfrak{E} \ , \tag{2.59}$$

which ensures the equivalence between the Lagrange equations $\delta \mathcal{L}/\delta q = 0$ ($q = a, \phi, b$) and the evolution equations $\mathfrak{A} = 0, \mathfrak{E} = 0, \mathfrak{F} = 0$ (see Eq.s (2.36) (2.37) (2.38)). We already noted that such evolution equations are not independent; from the present Lagrangian viewpoint, this is a consequence of the degeneracy of $\mathcal{L}$.

2.11 Gauge fixing. Energy constraint

From here to the end of this work it is assumed

$$b = \mathcal{B}(a, \phi) \ , \tag{2.60}$$

where $\mathcal{B} : (0, +\infty) \times \mathbf{R} \to (0, +\infty)$ is a suitable function, in the sequel referred to as the *gauge function*. Of course, our evolution equations are still $\mathfrak{A} = 0$, $\mathfrak{E} = 0$, $\mathfrak{F} = 0$; the results of the previous paragraphs continue to hold, with b as in Eq. (2.60) and

$$\dot{b} = \frac{\partial \mathcal{B}}{\partial a}(a, \phi)\, \dot{a} + \frac{\partial \mathcal{B}}{\partial \phi}(a, \phi)\, \dot{\phi} \ . \tag{2.61}$$

Under the same gauge fixing, the Lagrangian $\mathcal{L}$ of Eq. (2.57) becomes

$$\begin{gathered} \mathcal{L}(a, \dot{a}, \phi, \dot{\phi}) := \\ \frac{1}{2\mathcal{B}(a, \phi)} \Big(- a^{d-2}\, \dot{a}^2 + a^d\, \dot{\phi}^2 \Big) - \mathcal{B}(a, \phi) \left(a^d\, \mathcal{V}(\phi) + \frac{\Omega_{m*}}{2\, a^{wd}} - \frac{k\, a^{d-2}}{2} \right) . \end{gathered} \tag{2.62}$$

We note that $\mathcal{L}$ is a non-degenerate Lagrangian of mechanical type, whose kinetic part is induced by a metric of signature $(-, +)$ on the (a, ϕ) configuration space. Let us introduce the Lagrangian derivatives

$$\frac{\delta \mathcal{L}}{\delta q} := - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) + \frac{\partial \mathcal{L}}{\partial q} \qquad (q = a, \phi) \tag{2.63}$$

and the energy function

$$\begin{aligned} E &:= \sum_{q = a, \phi} \dot{q}\, \frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} \\ &= \frac{1}{2\, \mathcal{B}(a, \phi)} \Big(- a^{d-2}\, \dot{a}^2 + a^d\, \dot{\phi}^2 \Big) + \mathcal{B}(a, \phi) \left(a^d\, \mathcal{V}(\phi) + \frac{\Omega_{m*}}{2\, a^{wd}} - \frac{k\, a^{d-2}}{2} \right) , \end{aligned} \tag{2.64}$$

which is a constant of motion for the Lagrange equations $\delta\mathcal{L}/\delta q = 0$ $(q = a, \phi)$. It is easily checked that

$$\begin{pmatrix} \delta\mathcal{L}/\delta a \\ \delta\mathcal{L}/\delta\phi \\ E \end{pmatrix} = \frac{a^d}{2\,\mathcal{B}^2} \begin{pmatrix} 2\,\mathcal{B}/a & 0 & \mathcal{B}_a \\ 0 & -2\,\mathcal{B} & \mathcal{B}_\phi \\ 0 & 0 & -\mathcal{B} \end{pmatrix} \begin{pmatrix} \mathfrak{A} \\ \mathfrak{F} \\ \mathfrak{E} \end{pmatrix}, \tag{2.65}$$

$$\begin{pmatrix} \mathfrak{A} \\ \mathfrak{F} \\ \mathfrak{E} \end{pmatrix} = \frac{1}{a^d} \begin{pmatrix} a\,\mathcal{B} & 0 & a\,\mathcal{B}_a \\ 0 & -\mathcal{B} & -\mathcal{B}_\phi \\ 0 & 0 & -2\,\mathcal{B} \end{pmatrix} \begin{pmatrix} \delta\mathcal{L}/\delta a \\ \delta\mathcal{L}/\delta\phi \\ E \end{pmatrix}, \tag{2.66}$$

where $\mathfrak{A}, \mathfrak{F}, \mathfrak{E}$ are evaluated with $b, \dot{b}$ as in Eq.s (2.60) (2.61) and $\mathcal{B}, \mathcal{B}_a, \mathcal{B}_\phi$ are shorthand notations for $\mathcal{B}(a,\phi)$, $(\partial\mathcal{B}/\partial a)(a,\phi)$, $(\partial\mathcal{B}/\partial\phi)(a,\phi)$.
Therefore: after gauge fixing, *our evolution equations* $\mathfrak{A} = 0$, $\mathfrak{E} = 0$, $\mathfrak{F} = 0$ *are equivalent to the Lagrange equations* $\delta\mathcal{L}/\delta q = 0$ $(q = a, \phi)$*, supplemented by the energy condition* $E = 0$*; the latter is satisfied at all times if and only if it is fulfilled by the initial datum* $(a, \dot{a}, \phi, \dot{\phi})(t_0)$.
From now on, we systematically refer to the Lagrangian $\mathcal{L}$ of Eq. (2.62) and to the energy condition $E = 0$ to analyze the dynamics of our cosmological model. $E = 0$ will be referred to as the *(zero)-energy constraint*.
Whenever we speak of a solution of (one or all) these equations, we always tacitly assume the maximality of the interval where it is defined; this convention is consistent with the domain prescriptions of subsection 2.5, and will also be applied to the solutions obtained using Lagrangian coordinates different from (a, ϕ) (say, the coordinates (x, y) of the next sections).
The plan for the sequel is to consider specic choices for $\mathcal{V}$ allowing to solve explicitly the Lagrange equations. We will start from a known case, reviewed in section 3.

3 A solvable exponential potential for the flat, dust case $k = 0$, $w = 0$

Generalities. In this section we consider the flat, dust model $k = 0$, $w = 0$ in the case where the field potential has the form

$$\mathcal{V}(\phi) = \frac{1}{d^2}\left(V_1\,e^{d\,\phi} + V_2\,e^{-d\,\phi}\right) , \tag{3.1}$$

where V_1, V_2 are dimensionless constants; this case was already considered in [26], to which the present section is largely indebted. To treat the system in our setting, we fix the gauge putting

$$\mathcal{B}(a, \phi) = 1 ; \tag{3.2}$$

from here and from (2.20) (with $b(t) = \mathcal{B}(a(t), \phi(t))$ we get $d\tau = \theta\, dt$; so, in this case we can take

$$\tau = \theta\, t . \tag{3.3}$$

To go on we introduce a new pair of dimensionless Lagrangian coordinates $x, y > 0$, related to a, ϕ by

$$a = x^{\frac{1}{d}}\, y^{\frac{1}{d}} , \qquad \phi = \frac{1}{d}\,\log\!\left(\frac{x}{y}\right) . \tag{3.4}$$

In coordinates x, y as in Eq. (3.4), with the choice (3.2) for $\mathcal{B}$ and (3.1) for $\mathcal{V}$, the Lagrangian (2.62) with $k = 0$, $w = 0$ becomes

$$\mathcal{L}(x, \dot{x}, y, \dot{y}) = \frac{1}{d^2}\left(-\,2\,\dot{x}\,\dot{y} - V_1\,x^2 - V_2\,y^2 - \frac{d^2\,\Omega_{m*}}{2}\right) , \tag{3.5}$$

and the energy function (2.64) becomes

$$E(x, \dot{x}, y, \dot{y}) = \frac{1}{d^2}\left(-\,2\,\dot{x}\,\dot{y} + V_1\,x^2 + V_2\,y^2 + \frac{d^2\,\Omega_{m*}}{2}\right) . \tag{3.6}$$

The Lagrange equations $\delta\mathcal{L}/\delta y = 0$, $\delta\mathcal{L}/\delta x = 0$ are, respectively, equivalent to

$$\ddot{x} - V_2\,y = 0 , \tag{3.7}$$

$$\ddot{y} - V_1\,x = 0 ; \tag{3.8}$$

recall that they must be supplemented with the constraint $E = 0$.

Solving the equations. As an example, let us assume that

$$V_1, V_2 > 0\,. \tag{3.9}$$

In this case it is convenient to consider the geometric mean

$$V := \sqrt{V_1 V_2} \; ; \tag{3.10}$$

of course, for $V_1 = V_2$ we have $V = V_1 = V_2$. The general solution of Eq.s (3.7) (3.8), depending on four arbitrary integration constants A, B, C, D, is ([5])

$$\begin{aligned} x(t) = \frac{1}{2\sqrt{V_1}} \Big[& (A+C)\cosh\big(\sqrt{V}\,t\big) + (A-C)\cos\big(\sqrt{V}\,t\big) + \\ & + (B+D)\sinh\big(\sqrt{V}\,t\big) + (B-D)\sin\big(\sqrt{V}\,t\big) \Big] , \end{aligned} \tag{3.11}$$

$$\begin{aligned} y(t) = \frac{1}{2\sqrt{V_2}} \Big[& (A+C)\cosh\big(\sqrt{V}\,t\big) - (A-C)\cos\big(\sqrt{V}\,t\big) + \\ & + (B+D)\sinh\big(\sqrt{V}\,t\big) - (B-D)\sin\big(\sqrt{V}\,t\big) \Big] , \end{aligned} \tag{3.12}$$

$$\big(\, t \in I := \text{a maximal real interval s.t. } x(t), y(t) > 0 \;\; \forall t \in I \, \big) \; .$$

From Eq.s (3.6) (3.11) (3.12), by elementary computations we obtain

$$E\big(x(t), \dot{x}(t), y(t), \dot{y}(t)\big) = \frac{1}{d^2} \left(A^2 + C^2 - \, 2\,B\,D + \frac{d^2\,\Omega_{m*}}{2} \right) \; . \tag{3.13}$$

In view of this, to fulfill the zero-energy constraint we must require

$$\Omega_{m*} = \frac{2}{d^2} \left(2\,B\,D - A^2 - C^2 \right) \; . \tag{3.14}$$

Now, we can return to Eq. (3.4) for a, ϕ and insert therein the expressions (3.11), (3.12) for $x = x(t), y = y(t)$.
In the sequel of this work, we will frequently return to the solution presented here: in subsection 5.1, we will point out some of its features (Big Bang and long time behaviour); in subsection 7.1 we will propose it for $d = 3$ as a semi-realistic cosmological model, and discuss the determination of the related constants on physical grounds.

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[5]Notice that Eq.s (3.7) (3.8) imply

$$\ddot{\ddot{x}} = V_2\,\ddot{y} = V_2\,V_1\,x \; ,$$

$$\ddot{\ddot{y}} = V_1\,\ddot{x} = V_1\,V_2\,y \; .$$

The general solution of the above fourth-order differential equations for $x(t), y(t)$ is readily obtained; one then asks $x(t), y(t)$ to fulfill the original system (3.7) (3.8), thus obtaining Eq.s (3.11) (3.12) .

4 A solvable exponential potential for arbitrary k, w

Generalities. Let us consider, for arbitrary k and w, the case of a field potential

$$\mathcal{V}(\phi) = \frac{1}{d^2}\Big(V_1\, e^{2\,d\,w\,\phi} + V_2\, e^{d\,(1+w)\,\phi}\Big) \,, \tag{4.1}$$

where V_1, V_2 are constants. A potential with this structure has been considered in [12] for a cosmology with $k = 0$, a scalar field and no matter content; in that paper w is replaced by an arbitrary constant, not related to any material equation of state. Drawing inspiration from the cited work, we fix the gauge as

$$\mathcal{B}(a,\phi) \equiv \mathcal{B}(\phi) = e^{-\,d\,w\,\phi} \tag{4.2}$$

and we introduce a new pair of dimensionless Lagrangian coordinates $x, y > 0$, related to a, ϕ by

$$a = x^{\frac{1}{d\,(1+w)}}\, y^{\frac{1}{d\,(1-w)}} \,, \qquad \phi = \frac{1}{d}\,\log\Big(x^{\frac{1}{1+w}}\, y^{-\frac{1}{1-w}}\Big) \,. \tag{4.3}$$

Of course, Eq.s (4.3) make sense if and only if

$$w \neq \pm 1 \,, \tag{4.4}$$

a condition that we assume from here to the end of this work when dealing with this model. In coordinates x, y, and with the above choices for $\mathcal{V}$ and $\mathcal{B}$, the Lagrangian (2.62) becomes

$$\mathcal{L}(x,y,\dot{x},\dot{y}) = \tag{4.5}$$
$$\frac{1}{d^2}\bigg(-\frac{2\,\dot{x}\,\dot{y}}{1-w^2} - V_1\,x\,y - V_2\,x^{\frac{2}{1+w}} - \frac{d^2}{2}\,\Omega_{m*}\,x^{-\frac{2w}{1+w}} + \frac{d^2}{2}\,k\,x^{\frac{d(1-w)-2}{d(1+w)}}\,y^{\frac{d(1+w)-2}{d(1-w)}}\bigg) \,,$$

and the energy function (2.64) becomes

$$E(x,y,\dot{x},\dot{y}) = \tag{4.6}$$
$$\frac{1}{d^2}\bigg(-\frac{2\,\dot{x}\,\dot{y}}{1-w^2} + V_1\,x\,y + V_2\,x^{\frac{2}{1+w}} + \frac{d^2}{2}\,\Omega_{m*}\,x^{-\frac{2w}{1+w}} - \frac{d^2}{2}\,k\,x^{\frac{d(1-w)-2}{d(1+w)}}\,y^{\frac{d(1+w)-2}{d(1-w)}}\bigg) \,.$$

The Lagrange equations $\delta\mathcal{L}/\delta y = 0, \delta\mathcal{L}/\delta x = 0$ can be written, respectively, in the following way:

$$\ddot{x} - \frac{(1-w^2)\,V_1}{2}\,x \;=\; -\,\frac{d\,(1+w)\,\big(d\,(1+w)-2\big)}{4}\,k\,x^{\frac{d\,(1-w)-2}{d\,(1+w)}}\,y^{\frac{2\,(d\,w-1)}{d\,(1-w)}} \,, \tag{4.7}$$

$$\ddot{y} - \frac{(1-w^2)\,V_1}{2}\,y \;=\; (1-w)\,V_2\,x^{\frac{1-w}{1+w}} - \frac{d^2\,w\,(1-w)}{2}\,\Omega_{m*}\,x^{-\frac{1+3w}{1+w}} + \\ - \frac{d\,(1-w)\,\big(d\,(1-w)-2\big)}{4}\,k\,x^{-\frac{2\,(d\,w+1)}{d\,(1+w)}}\,y^{\frac{d\,(1+w)-2}{d\,(1-w)}} \;. \tag{4.8}$$

Let us recall that we are interested in the solutions of these equations fulfilling the positivity condition

$$x(t), y(t) > 0 \;, \tag{4.9}$$

and the energy constraint

$$E(x(t), \dot{x}(t), y(t), \dot{y}(t)) = 0 \;; \tag{4.10}$$

the latter holds at all times t if and only if it is fulfilled by the initial data. Once a solution has been obtained, one returns to the physical variables $a = a(t), \Phi = \Phi(t)$ and $b = \mathcal{B}(a, \Phi) = b(t)$ using Eq.s (4.2) (4.3). Let us also recall that the cosmic time τ is related to t by Eq. (2.42) .

Triangular cases for the system (4.7-4.8). In the cases

$$k = 0 \;, \qquad w \text{ arbitrary} \;, \tag{4.11}$$

$$k \text{ arbitrary} \;, \qquad w = 1/d \text{ (radiation gas)} \;, \tag{4.12}$$

$$k \text{ arbitrary} \;, \qquad w = 2/d - 1 \;, \tag{4.13}$$

the right-hand side of Eq. (4.7) is respectively zero, a function of x only, zero. Thus (4.7) is a differential equation involving only the unknown $x = x(t)$; once this equation has been solved, inserting the result for $x(t)$ into Eq. (4.8) we obtain a differential equation involving only $y = y(t)$, to be solved.
For the above reasons, the cases (4.11-4.13) are called *triangular*. The procedure outlined before to solve the system (4.7-4.8) is implemented in the forthcoming subsections 4.1, 4.2 and 4.3 where we consider, respectively, the cases (4.11), (4.12) and (4.13). Concerning the choices of w in Eq.s (4.7-4.8), let us recall that Eq.s (2.11) (2.12) give natural prescriptions, related to the energy conditions.

4.1 The triangular case $k = 0$, w arbitrary

Let us assume (4.11). For future convenience, we put

$$\epsilon := -\operatorname{sign}\big((1-w^2)\,V_1\big) \;, \tag{4.14}$$

$$\omega := \sqrt{\frac{|(1-w^2)\,V_1|}{2}} \;. \tag{4.15}$$

Due to this position and to the assumption $k = 0$, Eq.s (4.7-4.8) take the form

$$\ddot{x} + \epsilon\,\omega^2\,x = 0 \;, \tag{4.16}$$

$$\ddot{y} + \epsilon\,\omega^2\,y = (1-w)\,V_2\,x^{\frac{1-w}{1+w}} - \frac{d^2\,w\,(1-w)}{2}\,\Omega_{m*}\,x^{-\frac{1+3w}{1+w}}\ . \tag{4.17}$$

In the three subcases $\epsilon = -1$, $\epsilon = 0$ and $\epsilon = 1$, Eq. (4.16) describes, respectively, a harmonic repulsor, a free particle and a harmonic oscillator; Eq. (4.17) has a similar interpretation, with the addition of a forcing term.

4.1.1 The subcase $\epsilon = -1$

This corresponds to

$$(1-w^2)\,V_1 > 0\ . \tag{4.18}$$

In this case, Eq.s (4.16) (4.17) read

$$\ddot{x} - \omega^2\,x = 0\ , \tag{4.19}$$

$$\ddot{y} - \omega^2\,y = (1-w)\,V_2\,x^{\frac{1-w}{1+w}} - \frac{d^2\,w\,(1-w)}{2}\,\Omega_{m*}\,x^{-\frac{1+3w}{1+w}}\ . \tag{4.20}$$

After a time translation $t \to t + \text{const.}$ and possibly a time reflection $t \to -t$, any positive solution of Eq. (4.19) can be written in one of the following ways:

$$x(t) = A\,\sinh(\omega\,t)\ ,\quad A > 0\ ,\quad t \in (0, +\infty)\ , \tag{4.21}$$

$$x(t) = A\,\cosh(\omega\,t)\ ,\quad A > 0\ ,\quad t \in (-\infty, +\infty)\ , \tag{4.22}$$

$$x(t) = A\,e^{\omega\,t}\ ,\quad A > 0\ ,\quad t \in (-\infty, +\infty)\ , \tag{4.23}$$

where A is a constant. In the sequel we suppose to have chosen the time coordinate t so that the solution of Eq. (4.19) can be represented in one of the forms (4.21) (4.22) (4.23). For the general solution of Eq. (4.20) (on the intervals mentioned above), we have the familiar representation

$$y(t) = C\cosh(\omega\,t) + D\sinh(\omega\,t) \tag{4.24}$$

$$+\frac{1}{\omega}\int_0^t ds\ \sinh\big(\omega(t-s)\big)\Big[(1-w)\,V_2\,x(s)^{\frac{1-w}{1+w}} - \frac{d^2\,w\,(1-w)}{2}\,\Omega_{m*}\,x(s)^{-\frac{1+3w}{1+w}}\Big]\ ,$$

where C, D are arbitrary constants; this holds, at least, for the values of w ensuring convergence of the above integral, but we shall see later that the result can be extended to a larger set of values of w by analytic continuation. The calculation of the integral in (4.24) with x (and t) as in Eq.s (4.21) (4.22) (4.23) is reduced to the evaluation, for $\eta = \frac{1-w}{1+w}$ or $\eta = -\,\frac{1+3w}{1+w}$, of some more basic integrals. More precisely, we can use the following relations (where ω, t, η are real numbers and we always assume $\omega > 0$):

$$\frac{1}{\omega}\int_0^t ds\ \sinh\big(\omega\,(t-s)\big)\,\sinh^\eta(\omega\,s) \tag{4.25}$$

$$= \frac{\sinh^{\eta+2}(\omega\, t)}{\omega^2} \left[\frac{1}{1+\eta} - \frac{\cosh(\omega\, t)}{2+\eta}\ {}_2F_1\left(\frac{1}{2}\,, 1+\frac{\eta}{2}\,, 2+\frac{\eta}{2}\,; -\sinh^2(\omega\, t) \right) \right]$$

$$(t > 0\,,\ \ \eta > -1)\ ;$$

$$\frac{1}{\omega} \int_0^t ds\ \sinh\big(\omega\,(t-s)\big) \cosh^\eta(\omega\, s) \tag{4.26}$$

$$= \frac{\cosh(\omega\, t)\left(1-\cosh^{\eta+1}(\omega\, t)\right)}{\omega^2\,(1+\eta)} + \frac{\sinh^2(\omega\, t)}{\omega^2}\ {}_2F_1\left(\frac{1}{2}\,, -\frac{\eta}{2}\,, \frac{3}{2}\,; -\sinh^2(\omega t) \right)\ ;$$

$$\frac{1}{\omega} \int_0^t ds\ \sinh\big(\omega\,(t-s)\big)\, e^{\eta\,\omega\, s} = \frac{\cosh(\omega\, t) + \eta \sinh(\omega\, t) - e^{\eta\,\omega\, t}}{\omega^2\,(1-\eta^2)}\ . \tag{4.27}$$

Here and in the sequel, ${}_2F_1$ denotes the hypergeometric function (see, e.g., [22]); for the computation of the integrals (4.26) and (4.27), see Appendix C. The right-hand sides of Eq.s (4.26) (4.27) must be intended in a limit sense for certain values of η; more precisely, we understand that

$$\left.\frac{1-\cosh^{\eta+1}(\omega\, t)}{1+\eta}\right|_{\eta=-1} := \lim_{\eta\to-1} \frac{1-\cosh^{\eta+1}(\omega\, t)}{1+\eta} = -\ \log\big(\cosh(\omega\, t)\big)\ , \tag{4.28}$$

$$\left.\frac{\cosh(\omega\, t) + \eta\ \sinh(\omega\, t) - e^{\eta\,\omega\, t}}{1-\eta^2}\right|_{\eta=\pm1} := \tag{4.29}$$

$$\lim_{\eta\to\pm1} \frac{\cosh(\omega\, t) + \eta \sinh(\omega\, t) - e^{\eta\,\omega\, t}}{1-\eta^2} = \pm\, \frac{\omega\, t\, e^{\pm\,\omega\, t} -\ \sinh(\omega\, t)}{2}\ .$$

Using the above results, Eq.s (4.21)-(4.27) yield the following solutions (for the values of w discussed in the sequel):

$$x(t) = A \sinh(\omega\, t)\ , \tag{4.30}$$

$$y(t) = C \cosh(\omega\, t) + D \sinh(\omega\, t)$$

$$+ \frac{V_2}{V_1}\, A^{\frac{1-w}{1+w}} \sinh^{\frac{3+w}{1+w}}(\omega\, t) \left[1 - \frac{2\cosh(\omega\, t)}{3+w}\ {}_2F_1\left(\frac{1}{2}\,, \frac{3+w}{2+2w}\,, \frac{5+3w}{2+2w}\,; -\sinh^2(\omega\, t) \right) \right]$$

$$+ \frac{d^2\,\Omega_{m*}}{2\,V_1}\, A^{-\frac{1+3w}{1+w}} \sinh^{\frac{1-w}{1+w}}(\omega\, t) \left[1 + \frac{2\, w\cosh(\omega\, t)}{1-w}\ {}_2F_1\left(\frac{1}{2}\,, \frac{1-w}{2+2w}\,, \frac{3+w}{2+2w}\,; -\sinh^2(\omega\, t) \right) \right]$$

$$\left(\begin{array}{c} A > 0\,,\ \omega \text{ as in Eq. (4.15)},\ \ w \neq -\dfrac{3+2h}{1+2h}\ \ \text{for all } h \in \{0,1,2,\ldots\} \\ t \in I := \text{a maximal real interval s.t. } y(t)\!>\!0\ \ \forall\, t \in I \end{array} \right);$$

$$x(t) = A \cosh(\omega\, t)\ , \tag{4.31}$$

$$y(t) = C \cosh(\omega\, t) + D \sinh(\omega\, t)$$

$$+\frac{V_2}{V_1}\,A^{\frac{1-w}{1+w}}\left[\cosh(\omega\,t)\big(1-\cosh^{\frac{2}{1+w}}(\omega\,t)\big)+\frac{2\,\sinh^2(\omega\,t)}{1+w}\;{}_2F_1\!\left(\frac{1}{2}\,,-\,\frac{1-w}{2+2w}\,,\frac{3}{2}\,;-\sinh^2(\omega\,t)\right)\right]$$

$$+\frac{d^2\,\Omega_{m*}}{2\,V_1}\,A^{-\frac{1+3w}{1+w}}\left[\cosh(\omega\,t)\big(1-\cosh^{-\frac{2w}{1+w}}(\omega\,t)\big)-\frac{2\,w\,\sinh^2(\omega\,t)}{1+w}\;{}_2F_1\!\left(\frac{1}{2}\,,\frac{1+3w}{2+2w}\,,\frac{3}{2}\,;-\sinh^2(\omega\,t)\right)\right]$$

$\big(\,A>0,\ \omega$ as in Eq. (4.15), $t\in I:=$ a maximal real interval s.t. $y(t)\!>\!0\ \ \forall\,t\in I\,\big)$;

$$x(t)=A\,e^{\omega\,t}\ , \tag{4.32}$$

$$y(t)=C\cosh(\omega\,t)+D\sinh(\omega t)$$
$$+\frac{1+w}{2\,w}\,\frac{V_2}{V_1}\,A^{\frac{1-w}{1+w}}\left[\cosh(\omega\,t)+\frac{1-w}{1+w}\,\sinh(\omega\,t)-e^{\frac{(1-w)}{1+w}\,\omega\,t}\right]$$
$$+\frac{1+w}{1+2w}\,\frac{d^2\,\Omega_{m*}}{4\,V_1}\,A^{-\frac{1+3w}{1+w}}\left[\cosh(\omega\,t)-\frac{1+3w}{1+w}\,\sinh(\omega\,t)-e^{-\frac{(1+3w)}{1+w}\,\omega\,t}\right]$$

$\big(\,A>0,\ \omega$ as in Eq. (4.15), $t\in I:=$ a maximal real interval s.t. $y(t)>0\ \ \forall\,t\in I\,\big)$.

The derivation of Eq. (4.30) under the general condition $w\neq-\frac{3+2h}{1+2h}$ $(h=0,1,2,\ldots)$ contains some subtleties, since it requires some considerations on analytic continuations (see Appendix D). Eq.s (4.30) (4.31) (4.32) implicity assume $w\neq\pm1$, since this was prescribed for the whole section (see Eq. (4.4)). Eq. (4.32) must be intended in the limit sense of Eq. (4.29) for $w=0$ and $w=-1/2$ (corresponding, respectively, to $\eta:=-\frac{1+3w}{1+w}=-1$, $\eta:=\frac{1-w}{1+w}=1$ and to $\eta:=-\frac{1+3w}{1+w}=1$).
To go on, let us recall the expression (4.6) for the energy. From here and from Eq.s (4.30) (4.31) (4.32) we obtain ([6]), respectively,

$$E\big(x(t),\dot{x}(t),y(t),\dot{y}(t)\big)=-\,\frac{V_1\,A\,D}{d^2}\ , \tag{4.33}$$

$$E\big(x(t),\dot{x}(t),y(t),\dot{y}(t)\big)=\frac{A}{d^2}\left(\frac{d^2\,\Omega_{m*}}{2}\,A^{-\frac{1+3w}{1+w}}+V_2\,A^{\frac{1-w}{1+w}}+V_1\,C\right)\ , \tag{4.34}$$

$$E\big(x(t),\dot{x}(t),y(t),\dot{y}(t)\big)=\frac{A}{d^2}\left(\frac{d^2\,\Omega_{m*}}{2}\,A^{-\frac{1+3w}{1+w}}+V_2\,A^{\frac{1-w}{1+w}}+V_1\,(C-D)\right)\ . \tag{4.35}$$

Taking the above relations into account and recalling that in the present case $V_1\neq0$ (see Eq. (4.18)), we conclude the following: to fulfill the zero-energy constraint (4.10) we must put

$$D=0\ , \tag{4.36}$$

[6] For the energy computation, it can be useful to recall that E is a constant of motion, so it does not depend on t; on account of this, it suffices to compute $E\equiv E(x(t),\dot{x}(t),y(t),\dot{y}(t))$ for a given t or in a suitable limit, e.g. for $t\to0^+$. This remark applies to all subsequent energy computations in this work, but it will never be repeated.

$$C = -\frac{V_2}{V_1} A^{\frac{1-w}{1+w}} - \frac{d^2\,\Omega_{m*}}{2\,V_1} A^{-\frac{1+3w}{1+w}} \ , \tag{4.37}$$

$$C = D - \frac{V_2}{V_1} A^{\frac{1-w}{1+w}} - \frac{d^2\,\Omega_{m*}}{2\,V_1} A^{-\frac{1+3w}{1+w}} \tag{4.38}$$

in Eq.s (4.30) (4.31) (4.32), respectively.

4.1.2 The subcase $\epsilon = 0$

We consider the case (4.11) with $\epsilon = 0$, i.e.,

$$V_1 = 0 \tag{4.39}$$

(we recall again that we are assuming $w \neq \pm 1$; see Eq. (4.4)). Eq.s (4.16) (4.17) read

$$\ddot{x} = 0 \ , \tag{4.40}$$

$$\ddot{y} = (1-w)\,V_2\,x^{\frac{1-w}{1+w}} - \frac{d^2\,w\,(1-w)}{2}\,\Omega_{m*}\,x^{-\frac{1+3w}{1+w}} \ . \tag{4.41}$$

After a time translation $t \to t + \text{const.}$ and possibly a time reflection $t \to -t$, any positive solution of Eq. (4.40) can be written in one of the following ways:

$$x(t) = A\,t \ , \quad A > 0, \quad t \in (0, +\infty) \ , \tag{4.42}$$

$$x(t) = A \ , \quad A > 0, \quad t \in (-\infty, +\infty) \ , \tag{4.43}$$

where A is a constant. In the sequel we suppose to have chosen the time coordinate t so that the solution of Eq. (4.40) can be represented in one of the forms (4.42) (4.43). Substituting these expressions in Eq. (4.41), one obtains the subsequent pair of solutions, where C, D are constants:

$$x(t) = A\,t \ , \tag{4.44}$$

$$y(t) = C + D\,t + \frac{V_2\,(1+w)^2\,(1-w)}{2\,(3+w)}\,A^{\frac{1-w}{1+w}}\,t^{\frac{3+w}{1+w}} + \frac{d^2\,(1+w)^2}{4}\,\Omega_{m*}\,A^{-\frac{1+3w}{1+w}}\,t^{\frac{1-w}{1+w}}$$

$$\Big(\,A > 0,\ t \in I := \text{a maximal subinterval of } (0, +\infty) \text{ s.t. } y(t) > 0 \ \ \forall\, t \in I\,\Big) \ ;$$

$$x(t) = A \ , \tag{4.45}$$

$$y(t) = C + D\,t + \frac{t^2}{2}\left(V_2\,(1-w)\,A^{\frac{1-w}{1+w}} - \frac{d^2\,w\,(1-w)}{2}\,\Omega_{m*}\,A^{-\frac{1+3w}{1+w}}\right)$$

$$\Big(\,A > 0,\ t \in I := \text{a maximal real interval s.t. } y(t) > 0 \ \ \forall\, t \in I\,\Big) \ .$$

Besides assuming $w \neq \pm 1$, let us observe that Eq. (4.44) requires $w \neq -3$; the special case $w = -3$ should be treated separately, but this will be avoided for brevity. From Eq.s (4.6) (4.44) (4.45) one obtains, respectively,

$$E\big(x(t), \dot{x}(t), y(t), \dot{y}(t)\big) = -\frac{2\,A\,D}{d^2\,(1-w^2)}\ , \tag{4.46}$$

$$E\big(x(t), \dot{x}(t), y(t), \dot{y}(t)\big) = \frac{A^{-\frac{2w}{1+w}}}{d^2}\left(V_2\,A^2 + \frac{d^2\,\Omega_{m*}}{2}\right)\ . \tag{4.47}$$

In view of this, to fulfill the zero-energy constraint we must require

$$D = 0\ , \tag{4.48}$$

$$V_2 < 0 \ \text{ and } \ A = \sqrt{\frac{d^2\,\Omega_{m*}}{2\,|V_2|}} \qquad \text{or} \qquad V_2 = 0 \ \text{ and } \ \Omega_{m*} = 0 \tag{4.49}$$

in the cases (4.44) (4.45), respectively.

4.1.3 The subcase $\epsilon = 1$

We consider the case (4.11) with $\epsilon = 1$, i.e.

$$V_1\,(1-w^2) < 0\ . \tag{4.50}$$

Eq.s (4.16) (4.17) read

$$\ddot{x} + \omega^2 x = 0\ , \tag{4.51}$$

$$\ddot{y} + \omega^2 y = (1-w)\,V_2\,x^{\frac{1-w}{1+w}} - \frac{d^2\,w\,(1-w)}{2}\,\Omega_{m*}\,x^{-\frac{1+3w}{1+w}}\ . \tag{4.52}$$

After a time translation $t \to t + \text{const.}$, any positive solution of Eq. (4.51) can be written as

$$x(t) = A\,\sin(\omega\,t)\ , \quad A > 0\ , \quad t \in (0, \pi/\omega)\ , \tag{4.53}$$

where A is a constant. For the general solution of Eq. (4.52) (on the interval mentioned above), we have the familiar representation

$$y(t) = C\,\cos(\omega\,t) + D\,\sin(\omega\,t) \tag{4.54}$$

$$+\frac{1}{\omega}\int_0^t ds\ \sin\big(\omega\,(t-s)\big)\left[V_2\,(1-w)\,x(s)^{\frac{1-w}{1+w}} - \frac{d^2\,w\,(1-w)}{2}\,\Omega_{m*}\,x(s)^{-\frac{1+3w}{1+w}}\right]\ ,$$

where C, D are constants. The calculation of the integral in Eq. (4.54) with x (and t) as in Eq. (4.53) is reduced to the evaluation, for $\eta = \frac{1-w}{1+w}$ or $\eta = -\frac{1+3w}{1+w}$, of an integral of the form

$$\frac{1}{\omega}\int_0^t ds\ \sin\big(\omega\,(t-s)\big)\ \sin^\eta(\omega\,s) \tag{4.55}$$

$$\frac{\sin^{\eta+2}(\omega\,t)}{\omega^2}\left[\frac{1}{\eta+1}-\frac{\cos(\omega\,t)}{\eta+2}\;{}_2F_1\!\left(\frac{1}{2}\,,1+\frac{\eta}{2}\,,2+\frac{\eta}{2}\,;\,\sin^2(\omega\,t)\right)\right]$$
$$\left(\omega>0\,,\ 0<t<\frac{\pi}{\omega}\,,\ \eta>-1\right)\,,$$

where, as previously, ${}_2F_1$ denotes the hypergeometric function; for the calculation of the above integral, see Appendix C.
Summing up, Eq.s (4.53)-(4.55) yield the solution

$$x(t)=A\,\sin(\omega\,t)\;, \tag{4.56}$$

$$y(t)=C\cos(\omega\,t)+D\sin(\omega\,t)$$
$$+\frac{V_2}{V_1}\,A^{\frac{1-w}{1+w}}\,\sin^{\frac{3+w}{1+w}}(\omega\,t)\left[1-\frac{2\cos(\omega\,t)}{3+w}\;{}_2F_1\!\left(\frac{1}{2}\,,\frac{3+w}{2+2w}\,,\frac{5+3w}{2+2w}\,;\sin^2(\omega\,t)\right)\right]$$
$$-\frac{d^2\,\Omega_{m*}}{2\,V_1}\,A^{-\frac{1+3w}{1+w}}\,\sin^{\frac{1-w}{1+w}}(\omega\,t)\left[1+\frac{2\,w\cos(\omega\,t)}{1-w}\;{}_2F_1\!\left(\frac{1}{2}\,,\frac{1-w}{2+2w}\,,\frac{3+w}{2+2w}\,;\sin^2(\omega\,t)\right)\right]$$

$\big(A>0\,,\ \omega$ as in Eq. (4.15), $t\in I:=$ a maximal subinterval of $(0,\pi/\omega)$ s.t. $y(t)>0\ \ \forall\,t\in I\,\big)$.
A posteriori, it is found that the result (4.56) holds for $w\neq-\frac{3+2h}{1+2h}$ $(h=0,1,2,\ldots)$ (recall again that we are assuming $w\neq\pm1$; see Eq. (4.4)). From Eq.s (4.6) (4.56) we infer

$$E\big(x(t),\dot{x}(t),y(t),\dot{y}(t)\big)=\frac{V_1\,A\,D}{d^2}\;. \tag{4.57}$$

In view of this, to fulfill the zero-energy constraint we must require

$$D=0 \tag{4.58}$$

in Eq. (4.56) . Some features of the solution (4.56) (4.58), in particular its Big Bang behavior, will be analyzed in subsection 5.2 (under the conditions $V_1<0$, $-1<w<1$ which are a specialization of (4.50)).

4.1.4 The dust choice $w=0$

Let us now consider Eq. (4.11) with $w=0$. In principle, there are three subcases $\epsilon=-1,0,1$. For the sake of brevity we only analyze the subcase $\epsilon=-1$, i.e., $V_1>0$; moreover, we choose the solution (4.30). For $w=0$, the first hypergeometric function in Eq. (4.30) becomes (see, e.g., [22, Ch. 15])

$${}_2F_1\left(\frac{1}{2}\,,\frac{3}{2}\,,\frac{5}{2}\,;s\right)=-\,\frac{3}{2}\left[\frac{\sqrt{s\,(1-s)}-\arcsin\sqrt{s}}{s\sqrt{s}}\right]\;; \tag{4.59}$$

the second hypergeometric function is regular at $w=0$ and is multiplied by w, so it gives no contribution for $w=0$. Due to these facts, after simple algebraic manipulations, for $w = 0$ Eq. (4.30) gives $y(t) = \Big(D + \frac{d^2\,\Omega_{m*}}{2\,V_1\,A} - \frac{V_2\,A}{V_1}\Big)\sinh\Big(\sqrt{\frac{V_1}{2}}\,t\Big) + \Big(C + \frac{V_2\,A\,t}{\sqrt{2\,V_1}}\Big)\cosh\Big(\sqrt{\frac{V_1}{2}}\,t\Big)$ ([7]) . We now put the constraint of zero energy that, according to Eq. (4.36), is fulfilled if and only if $D = 0$. In conclusion, Eq. (4.30) and the zero-energy constraint give the following for $w = 0$:

$$x(t) = A\,\sinh\Big(\sqrt{\frac{V_1}{2}}\,t\Big)\ , \tag{4.60}$$

$$y(t) = \Big(\frac{d^2\,\Omega_{m*}}{2\,V_1\,A} - \frac{V_2\,A}{V_1}\Big)\sinh\Big(\sqrt{\frac{V_1}{2}}\,t\Big) + \Big(C + \frac{V_2\,A\,t}{\sqrt{2\,V_1}}\Big)\cosh\Big(\sqrt{\frac{V_1}{2}}\,t\Big)$$

$$\big(\,A > 0,\ \ t \in I := \text{a maximal real interval s.t. } y(t) > 0\ \ \forall\, t \in I\,\big)\ .$$

We note that Eq.s (2.20) (4.2) with $w = 0$ give

$$\mathcal{B} = 1 \quad \Rightarrow \quad d\tau = \theta\,dt\ ;$$

so, as in the dust case of section 3, the cosmic time τ coincides with the dimensionless coordinate t multiplied by the time parameter θ.

4.2 The triangular case k arbitrary, $w = 1/d$ (mainly, for $d = 3$)

In this subsection we consider the triangular case (4.12), representing a radiation gas; this can be used to model the radiation dominated part in the history of the universe. We put

$$\epsilon := -\,\mathrm{sign}\,V_1\ , \tag{4.61}$$

$$\omega := \sqrt{\frac{(d^2-1)\,|V_1|}{2\,d^2}}\ . \tag{4.62}$$

Due to these positions, Eq.s (4.7-4.8) take the form

$$\ddot{x} + \epsilon\,\omega^2\,x = -\,\frac{k\,(d^2-1)}{4}\,x^{\frac{d-3}{d+1}}\ , \tag{4.63}$$

$$\ddot{y} + \Big(\epsilon\,\omega^2 + \frac{k\,(d-3)(d-1)}{4}\,x^{-\frac{4}{d+1}}\Big)\,y = \frac{V_2\,(d-1)}{d}\,x^{\frac{d-1}{d+1}} - \frac{d-1}{2}\,\Omega_{m*}\,x^{-\frac{d+3}{d+1}}\ . \tag{4.64}$$

[7] This expression for $y(t)$ seems to depend on Ω_{m*}, while Eq. (4.20) does not contain Ω_{m*} for $w = 0$. However, the dependence on Ω_{m*} in the expression of $y(t)$ is fictitious if D is regarded as an arbitrary constant, since $\tilde{D} := D + \frac{d^2\,\Omega_{m*}}{2\,V_1\,A} - \frac{V_2\,A}{V_1}$ is arbitrary as well. The situation changes if we stipulate the zero-energy constraint, which fixes D (and so $\tilde{D}$); note that, according to Eq. (4.6), the energy depends on Ω_{m*} even for $w = 0$.

Eq. (4.63) can be reduced to quadratures; this allows to determine $x(t)$ ([8]). When $x(t)$ is known, Eq. (4.64) is an inhomogeneous, linear second order differential equation for $y(t)$. If $d \neq 3$, the quadrature of Eq. (4.63) is not so simple; moreover the coefficient of y in Eq. (4.64) is time dependent and the reducibility to quadratures of this equation is, to say the least, a non trivial affair, since one should refer to the Picard-Vessiot theory [25].
In the sequel, to simplify the treatment we put $d = 3$ (which, of course, is also the most interesting choice from the physical viewpoint). In this case, we have

$$\omega = \frac{2}{3}\sqrt{|V_1|}\ , \tag{4.65}$$

$$\ddot{x} + \epsilon\,\omega^2\,x = -\,2\,k\ , \tag{4.66}$$

$$\ddot{y} + \epsilon\,\omega^2\,y = \frac{2\,V_2}{3}\,x^{1/2} - \,\Omega_{m*}\,x^{-3/2}\ . \tag{4.67}$$

Let us recall that ϵ is given by Eq. (4.61). In the three subcases $\epsilon = -1$, $\epsilon = 0$ and $\epsilon = 1$, both Eq.s (4.66) (4.67) describe, respectively, a harmonic repulsor, a free particle and a harmonic oscillator with a forcing term. Due to the structural similarity with Eq.s (4.16) (4.17), we omit the details of the related calculations and just report the solutions in the three cases.

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[8]Indeed, Eq. (4.63) has the form

$$\ddot{x} = -\,\mathcal{U}'(x)\ , \qquad \mathcal{U}(x) := \frac{1}{2}\,\epsilon\,\omega^2\,x^2 + \frac{k\,(d+1)^2}{8}\,x^{\frac{2(d-1)}{d+1}}\ .$$

Thus,

$$\frac{1}{2}\,\dot{x}^2 + \mathcal{U}(x) = \text{const} \equiv \mathfrak{E}\ \Rightarrow\ \int_{x(t_1)}^{x(t_2)} \frac{dx}{\sqrt{2\,(\mathfrak{E} - \mathcal{U}(x))}} = \sigma\,(t_2 - t_1)\ \text{if sign}\,\dot{x}(t) = \sigma \in \{\mp 1\}\,\forall\,t \in (t_1, t_2)\ .$$

For completeness, we mention the existence of a special solution for $k < 0$, with $\mathfrak{E} = 0$; this is the elementary function

$$x(t) = \left(\frac{d-1}{2}\sqrt{\frac{k\,(d+1)}{\epsilon\,\omega^2\,(d-2)}}\ \sinh\Big(\frac{\sqrt{-\,\epsilon\,\omega^2}}{d-1}\,t\Big)\right)^{d-1}\ .$$

(where $\sqrt{u} := i\,\sqrt{|u|}$ for real $u < 0$). More explicitly:

$$x(t) = \left(\sqrt{-\frac{k\,d^2(d-1)}{2\,(d-2)\,V_1}}\ \sinh\Big(\sqrt{\frac{(d+1)\,V_1}{2\,d^2(d-1)}}\,t\Big)\right)^{d-1}\ , \qquad \text{if} \qquad \epsilon = -1\ ;$$

$$x(t) = \left(\sqrt{\frac{k\,d^2(d-1)}{2\,(d-2)\,V_1}}\ \sin\Big(\sqrt{-\frac{(d+1)\,V_1}{2\,d^2(d-1)}}\,t\Big)\right)^{d-1}\ , \qquad \text{if} \qquad \epsilon = 1\ .$$

4.2.1 The subcase $d = 3$, $\epsilon = -1$

We consider the case (4.12) with $d = 3$ and $\epsilon = -1$, i.e.

$$V_1 > 0 \ . \tag{4.68}$$

Eq.s (4.66) (4.67) read

$$\ddot{x} - \omega^2 x = -\,2\,k \ , \tag{4.69}$$

$$\ddot{y} - \omega^2 y = \frac{2\,V_2}{3}\, x^{1/2} - \Omega_{m*}\, x^{-3/2} \ . \tag{4.70}$$

After a time translation $t \to t + \text{const.}$ and possibily a time reflection $t \mapsto -t$, any positive solution of Eq.s (4.69) (4.70) can be written in one of the following ways, with t ranging in an interval I whose features are indicated in the sequel:

$$x(t) = A\,\sinh(\omega\,t) + \frac{2\,k}{\omega^2} \ , \tag{4.71}$$

$$y(t) = C\,\cosh(\omega\,t) + D\,\sinh(\omega\,t)$$
$$+\frac{1}{\omega}\int_{t_0}^{t} ds\ \sinh\big(\omega\,(t{-}s)\big)\left[\frac{2\,V_2}{3}\left(A\,\sinh(\omega\,s) + \frac{2\,k}{\omega^2}\right)^{1/2} - \Omega_{m*}\left(A\,\sinh(\omega\,s) + \frac{2\,k}{\omega^2}\right)^{-3/2}\right],$$

$$x(t) = A\,\cosh(\omega\,t) + \frac{2\,k}{\omega^2} \ , \tag{4.72}$$

$$y(t) = C\,\cosh(\omega\,t) + D\,\sinh(\omega\,t)$$
$$+\frac{1}{\omega}\int_{t_0}^{t} ds\ \sinh\big(\omega\,(t{-}s)\big)\left[\frac{2\,V_2}{3}\left(A\,\cosh(\omega\,s) + \frac{2\,k}{\omega^2}\right)^{1/2} - \Omega_{m*}\left(A\,\cosh(\omega\,s) + \frac{2\,k}{\omega^2}\right)^{-3/2}\right],$$

$$x(t) = A\,e^{\omega\,t} + \frac{2\,k}{\omega^2} \ , \tag{4.73}$$

$$y(t) = C\,\cosh(\omega\,t) + D\,\sinh(\omega\,t)$$
$$+\frac{1}{\omega}\int_{t_0}^{t} ds\ \sinh\big(\omega(t-s)\big)\left[\frac{2\,V_2}{3}\left(A\,e^{\omega\,s} + \frac{2\,k}{\omega^2}\right)^{1/2} - \Omega_{m*}\left(A\,e^{\omega\,s} + \frac{2\,k}{\omega^2}\right)^{-3/2}\right].$$

In each one of Eq.s (4.71)-(4.73) A, C, D are constants and t_0 is a distinguished point of the interval I; it is required that $x(t), y(t) > 0$ for all $t \in I$, and that I is maximal with respect to these conditions.
From Eq.s (4.6) (4.71) (4.72) (4.73) one obtains, respectively,

$$E\big(x(t), \dot{x}(t), y(t), \dot{y}(t)\big) = \tag{4.74}$$

$$\frac{1}{9}\left[-\,V_1\,A\,D + V_2\left(A\,\sinh(\omega\,t_0) + \frac{9\,k}{2\,V_1}\right)^{3/2} + \frac{9\,\Omega_{m*}}{2}\left(A\,\sinh(\omega\,t_0) + \frac{9\,k}{2\,V_1}\right)^{-1/2}\right],$$

$$E\big(x(t),\dot{x}(t),y(t),\dot{y}(t)\big) = \tag{4.75}$$

$$\frac{1}{9}\left[V_1\,A\,C + V_2\left(A\,\cosh(\omega\,t_0)+\frac{9\,k}{2\,V_1}\right)^{3/2}+\frac{9\,\Omega_{m*}}{2}\left(A\,\cosh(\omega\,t_0)+\frac{9\,k}{2\,V_1}\right)^{-1/2}\right],$$

$$E\big(x(t),\dot{x}(t),y(t),\dot{y}(t)\big) = \tag{4.76}$$

$$\frac{1}{9}\left[V_1\,A\,(C-D) + V_2\left(A\,e^{\omega\,t_0}+\frac{9\,k}{2\,V_1}\right)^{3/2}+\frac{9\,\Omega_{m*}}{2}\left(A\,e^{\omega\,t_0}+\frac{9\,k}{2\,V_1}\right)^{-1/2}\right],$$

(Note that Eq.s (4.74) (4.75) (4.76) contain terms of the form $x(t_0)^{3/2}$ and $x(t_0)^{-1/2}$, well defined since $x(t_0) > 0$). So, the zero-energy constraint is satisfied if and only if the parameters in Eq.s (4.74) (4.75) (4.76) fulfill, respectively, the following conditions:

$$A\,D = \frac{V_2}{V_1}\left(A\,\sinh(\omega\,t_0)+\frac{9\,k}{2\,V_1}\right)^{3/2}+\frac{9\,\Omega_{m*}}{2\,V_1}\left(A\,\sinh(\omega\,t_0)+\frac{9\,k}{2\,V_1}\right)^{-1/2}, \tag{4.77}$$

$$A\,C = -\,\frac{V_2}{V_1}\left(A\,\cosh(\omega\,t_0)+\frac{9\,k}{2\,V_1}\right)^{3/2}-\,\frac{9\,\Omega_{m*}}{2\,V_1}\left(A\,\cosh(\omega\,t_0)+\frac{9\,k}{2\,V_1}\right)^{-1/2}, \tag{4.78}$$

$$A\,(C-D) = -\,\frac{V_2}{V_1}\left(A\,e^{\omega\,t_0}+\frac{9k}{2V_1}\right)^{3/2}-\,\frac{9\,\Omega_{m*}}{2\,V_1}\left(A\,e^{\omega\,t_0}+\frac{9\,k}{2\,V_1}\right)^{-1/2}. \tag{4.79}$$

4.2.2 The general subcase $d = 3$, $\epsilon = 0$

We consider the case (4.12) with $d = 3$ and $\epsilon = 0$, i.e.

$$V_1 = 0\ . \tag{4.80}$$

Eq.s (4.66) (4.67) read

$$\ddot{x} = -\,2\,k\ , \tag{4.81}$$

$$\ddot{y} = \frac{2\,V_2}{3}\,x^{1/2} - \,\Omega_{m*}\,x^{-3/2}\ . \tag{4.82}$$

To continue, we must distinguish among the three possible choices for the sign of k.

4.2.3 The subcase $d = 3$, $\epsilon = 0$, $k < 0$

Any positive solution of Eq.s (4.81) (4.82) can be written as

$$x(t) = A + B\,t - k\,t^2\ , \tag{4.83}$$

$$y(t) = C + D\,t\ +\ \frac{\sqrt{x(t)}\,(\alpha - 4\,V_2\,B\,\Delta\,k\,t + 4\,V_2\,\Delta\,k^2\,t^2)}{36\,\Delta\,k^2}$$

$$- \frac{\Delta\,(B - 2\,k\,t)}{24\,|k|^{5/2}}\,V_2\,\log\!\left(\frac{B - 2\,k\,t}{|k|^{1/2}}\ +\ 2\,\sqrt{x(t)}\right)$$

$$\left(\begin{array}{c} \Delta := B^2 + 4\,k\,A\,, \quad \alpha := \big(3\,B^4 + 20\,k\,A\,B^2 + 32\,k^2\,A^2\big)\,V_2 + 144\,k^2\,\Omega_{m*}\,, \\ t \in I := \text{a maximal real interval s.t. } x(t),\ y(t) > 0\ \ \forall\, t \in I \end{array}\right)$$

where A, B, C, D are constants. Note that, differently from the previous subcases $d = 3$, $\epsilon = \pm 1$, here we have elementary expressions for both $x(t)$ and $y(t)$. A posteriori, it is clear that the result (4.83) requires $\Delta \neq 0$; the case $\Delta = 0$ could be treated similarly, but we omit its discussion for brevity.
From Eq.s (4.6) (4.83) we infer

$$E\big(x(t), \dot{x}(t), y(t), \dot{y}(t)\big) = -\,\frac{1}{4}\,\big(B\,D + 2\,k\,C\big)\ . \tag{4.84}$$

In view of this, to fulfill the zero-energy constraint we must require

$$C = -\,\frac{B\,D}{2\,k} \tag{4.85}$$

in Eq. (4.83) .

4.2.4 The subcase $d = 3$, $\epsilon = 0$, $k > 0$

In this case, any positive solution of Eq. (4.81) (4.82) can be written as

$$x(t) = A + B\,t - k\,t^2\ , \tag{4.86}$$

$$y(t) = C + D\,t\ +\ \frac{\sqrt{x(t)}\,(\beta + V_2\,\Delta\,k\,t - V_2\,B\,\Delta\,t^2)}{9\,\Delta\,k}$$

$$+\,\frac{V_2\,\Delta}{24\,k^3}\,\left[2\,k\,\sqrt{x(t)} + \sqrt{k}\,(B - 2\,k\,t)\,\arctan\!\left(\frac{B - 2\,k\,t}{2\sqrt{k\,x(t)}}\right)\right]$$

$$\left(\begin{array}{c} \Delta := B^2 + 4\,k\,A\,, \quad \beta := 36\,k\,\Omega_{m*} - V_2\,A\,\Delta\,, \\ t \in I := \text{a maximal real interval s.t. } x(t),\ y(t) > 0\ \ \forall\, t \in I \end{array}\right)$$

where A, B, C, D are constants. As in the subcase $d = 3$, $\epsilon = 0$, $k < 0$, we have elementary expressions for both $x(t)$ and $y(t)$. Also in this case, the result (4.86) appears to makes sense only for $\Delta \neq 0$. On the other hand, since we are assuming that $k > 0$, from Eq. (4.86) it can be readily inferred that the requirement $x(t) > 0$ cannot be fulfilled if $\Delta = 0$ (namely, I reduces to the empty set in this case).

From Eq.s (4.6) (4.86) we obtain

$$E\big(x(t),\dot{x}(t),y(t),\dot{y}(t)\big) = -\,\frac{1}{4}\left(B\,D + 2\,k\,C\right)\,. \tag{4.87}$$

In view of this, to fulfill the zero-energy constraint we must require

$$C = -\,\frac{B\,D}{2\,k} \tag{4.88}$$

in Eq. (4.86) .

4.2.5 The subcase $d = 3,\ \epsilon = 0,\ k = 0$

After a time translation $t \to t + \text{const.}$ and possibly a time reflection $t \to -t$, any positive solution of Eq.s (4.81) (4.82) can be written in one of the following ways (with A, C, D constants):

$$x(t) = A\,t\ , \tag{4.89}$$

$$y(t) = C + D\,t + \frac{8\,V_2\sqrt{A}}{45}\ t^{5/2} + \frac{4\,\Omega_{m*}}{A^{3/2}}\ t^{1/2}$$

$$\big(\,A > 0,\ \ t \in I := \text{a maximal subinterval of } (0, +\infty) \ \ \text{ s.t. } \ y(t) > 0\ \ \forall\, t \in I\,\big)\ ;$$

$$x(t) = A\ , \tag{4.90}$$

$$y(t) = C + D\,t + \left(\frac{V_2\sqrt{A}}{3} - \frac{\Omega_{m*}}{2\,A^{3/2}}\right) t^2$$

$$\big(\,A > 0,\ \ t \in I := \text{a maximal real interval s.t. } \ y(t) > 0\ \ \forall\, t \in I\,\big)\ .$$

From Eq.s (4.6) (4.89) (4.90) one obtains, respectively,

$$E\big(x(t),\dot{x}(t),y(t),\dot{y}(t)\big) = -\,\frac{A\,D}{4}\ , \tag{4.91}$$

$$E\big(x(t),\dot{x}(t),y(t),\dot{y}(t)\big) = \frac{1}{9\sqrt{A}}\left(V_2\,A^2 + \frac{9\,\Omega_{m*}}{2}\right). \tag{4.92}$$

In view of this, to fulfill the zero-energy constraint we must require

$$D = 0\ , \tag{4.93}$$

$$V_2 < 0 \ \text{ and } \ A = \sqrt{\frac{9\,\Omega_{m*}}{2\,|V_2|}} \qquad \text{or} \qquad V_2 = 0 \ \text{ and } \ \Omega_{m*} = 0 \tag{4.94}$$

in Eq.s (4.89) (4.90), respectively. Notice that, in the second case, to fulfill the zero-energy constraint there must hold $V_2 \leqslant 0$; since $V_1 = 0$, this means that we have to consider a negative definite potential $\mathcal{V}(\phi)$.
In subsection 5.3 we will return to the solution (4.89) (4.93) to analyze some of its features, such as its Big Bang behavior.

4.2.6 The subcase $d = 3$, $\epsilon = 1$

We consider the case (4.12) with $d = 3$ and $\epsilon = 1$, i.e.

$$V_1 < 0 \; . \tag{4.95}$$

Eq.s (4.66) (4.67) read

$$\ddot{x} + \omega^2 \, x = -2 \, k \; , \tag{4.96}$$

$$\ddot{y} + \omega^2 \, y = \frac{2 \, V_2}{3} \, x^{1/2} - \Omega_{m*} \, x^{-3/2} \; . \tag{4.97}$$

After a time translation $t \to t + \text{const.}$, any positive solution of Eq.s (4.96) (4.97) can be written as follows in terms of three constants A, C, D:

$$x(t) = A \, \sin(\omega \, t) - \frac{2 \, k}{\omega^2} \; , \tag{4.98}$$

$$y(t) = C \, \cos(\omega \, t) + D \, \sin(\omega \, t)$$

$$+ \frac{1}{\omega} \int_0^t ds \; \sin\big(\omega \, (t - s)\big) \left[\frac{2 \, V_2}{3} \left(A \, \sin(\omega \, t) - \frac{2 \, k}{\omega^2} \right)^{1/2} - \Omega_{m*} \left(A \sin(\omega \, t) - \frac{2 \, k}{\omega^2} \right)^{-3/2} \right]$$

(ω as in Eq. (4.65), $t \in I :=$ a maximal real interval s.t. $x(t), y(t) > 0 \;\; \forall t \in I$) .

From Eq.s (4.6) (4.98) one obtains

$$E\big(x(t), \dot{x}(t), y(t), \dot{y}(t)\big) = \frac{1}{9} \left[V_1 \, A \, D + V_2 \left(\frac{9 \, k}{2 \, V_1} \right)^{3/2} + \frac{9 \, \Omega_{m*}}{2} \left(\frac{9 \, k}{2 \, V_1} \right)^{-1/2} \right] . \tag{4.99}$$

Notice that the requirement $x(t) > 0$ and the explicit expression for $x(t)$ written in Eq. (4.98) (along with Eq.s (4.65) (4.95)) imply $k \leqslant 0$ (whence, $k/V_1 \geqslant 0$); thus, the above expression for $E\big(x, \dot{x}, y, \dot{y}\big)$ makes sense.
Then, keeping in mind that $V_1 < 0$ (due to Eq. (4.95)), to fulfill the zero-energy constraint we must require the parameters A, D in Eq. (4.98) to satisfy

$$A \, D = -\frac{V_2}{V_1} \left(\frac{9 \, k}{2 \, V_1} \right)^{3/2} - \frac{9 \, \Omega_{m*}}{2 \, V_1} \left(\frac{9 \, k}{2 \, V_1} \right)^{-1/2} . \tag{4.100}$$

4.3 The triangular case k arbitrary, $w = 2/d - 1$

The case (4.13) considered here is perhaps less interesting than the previous triangular cases (4.11) (4.12), since it gives $w < 0$ for $d \geqslant 3$ (for $d = 2$ one has a dust case $w = 0$). However, if $w = 2/d - 1$ and we assume $\Omega_{m*} \geqslant 0$ in Eq. (2.35) (non negative matter density), the requirements (2.11) (2.12) corresponding to the weak and dominant energy conditions are both fulfilled for any $d \geqslant 2$ (indeed, for $d \geqslant 1$). In the sequel we put

$$\epsilon := -\,\mathrm{sign}(V_1)\ , \tag{4.101}$$

$$\omega := \sqrt{\frac{2\,(d-1)\,|V_1|}{d^2}}\ . \tag{4.102}$$

Due to these positions and to the assumption $w = 2/d - 1$, Eq.s (4.7-4.8) take the form

$$\ddot{x} + \epsilon\,\omega^2\,x = 0\ , \tag{4.103}$$

$$\ddot{y} + \epsilon\,\omega^2\,y = \frac{2\,(d-1)\,V_2}{d}\ x^{d-1} + (d-2)\,(d-1)\,\big(\Omega_{m*} - k\big)\ x^{d-3}\ . \tag{4.104}$$

As in subsection 4.1, in the three subcases $\epsilon = -1$, $\epsilon = 0$ and $\epsilon = 1$, Eq. (4.103) describes, respectively, a harmonic repulsor, a free particle and a harmonic oscillator; Eq. (4.104) has a similar interpretation, with the addition of a forcing term.
In the following paragraphs 4.3.1, 4.3.2 and 4.3.3, we always assume

$$d \geqslant 3\ ; \tag{4.105}$$

due to the similarities with other situations considered previously, we just report the final expressions for the solutions. The case $d = 2$ can be treated similarly, but we omit its discussion for the sake of brevity (note that, for $d = 2$, the term proportional to x^{d-3} in the right-hand side of Eq.(4.104) vanishes identically).

4.3.1 The subcase $\epsilon = -1$

This corresponds to

$$V_1 > 0\ . \tag{4.106}$$

Eq.s (4.103) (4.104) read

$$\ddot{x} - \omega^2\,x = 0\ , \tag{4.107}$$

$$\ddot{y} - \omega^2\,y = \frac{2\,(d-1)\,V_2}{d}\ x^{d-1} + (d-2)\,(d-1)\,\big(\Omega_{m*} - k\big)\ x^{d-3}\ . \tag{4.108}$$

After a time translation $t \to t + \mathrm{const.}$ and possibly a time reflection $t \to -t$, any positive solution of Eq.s (4.107) (4.108) can be written in one of the following ways (with A, C, D constants)

$$x(t) = A\,\sinh(\omega\,t)\ , \tag{4.109}$$

$$y(t) = C\,\cosh(\omega\,t) + D\,\sinh(\omega\,t)$$

$$+\,\frac{V_2}{V_1}\,A^{d-1}\,\sinh^{d+1}(\omega\,t)\left[1 - \frac{d\,\cosh(\omega\,t)}{d+1}\ {}_2F_1\!\left(\frac{1}{2}\,,\,\frac{d+1}{2}\,,\,\frac{d+3}{2}\,;-\sinh^2(\omega\,t)\right)\right]$$

$$+\,\frac{d^2\big(\Omega_{m*} - k\big)}{2\,V_1}\,A^{d-3}\sinh^{d-1}(\omega\,t)\left[1 - \frac{(d-2)\,\cosh(\omega\,t)}{d-1}\ {}_2F_1\!\left(\frac{1}{2},\,\frac{d-1}{2},\,\frac{d+1}{2};-\sinh^2(\omega\,t)\right)\right]$$

$\big(A > 0,\ \omega$ as in Eq. (4.102), $t \in I :=$ a maximal subinterval of $(0, +\infty)$ s.t. $y(t) > 0\ \ \forall t \in I \big)$;

$$x(t) = A \cosh(\omega t) \ , \tag{4.110}$$

$$y(t) = C \cosh(\omega t) + D \sinh(\omega t)$$

$$+ \frac{V_2}{V_1} A^{d-1} \left[\cosh(\omega t)(1 - \cosh^d(\omega t)) + d \sinh^2(\omega t)\ {}_2F_1 \left(\frac{1}{2}, -\frac{d-1}{2}, \frac{3}{2}; -\sinh^2(\omega t) \right) \right]$$

$$+ \left[\cosh(\omega t)\big(1 - \cosh^{d-2}(\omega t)\big) + (d-2) \sinh^2(\omega t)\ {}_2F_1 \left(\frac{1}{2}, -\frac{d-3}{2}, \frac{3}{2}; -\sinh^2(\omega t) \right) \right] \times$$

$$\times\ \frac{d^2\, (\Omega_{m*} - k)}{2\, V_1}\ A^{d-3}$$

$\big(A > 0,\ \omega$ as in Eq. (4.102), $t \in I :=$ a maximal real interval s.t. $y(t) > 0\ \ \forall t \in I \big)$;

$$x(t) = A\, e^{\omega t} \ , \tag{4.111}$$

$$y(t) = C \cosh(\omega t) + D \sinh(\omega t)$$

$$- \frac{V_2}{V_1} \frac{A^{d-1}}{d-2} \Big[\cosh(\omega t) + (d-1) \sinh(\omega t) - e^{(d-1)\,\omega t} \Big]$$

$$- \frac{d^2 (\Omega_{m*} - k)}{2\, V_1} \frac{A^{d-3}}{d-4} \Big[\cosh(\omega t) + (d-3) \sinh(\omega t) - e^{(d-3)\,\omega t} \Big]$$

$\big(A > 0,\ \omega$ as in Eq. (4.102), $t \in I :=$ a maximal real interval s.t. $y(t) > 0\ \ \forall t \in I \big)$.

The last term in Eq. (4.111) should be intended as follows for $d = 4$:

$$\frac{1}{d-4} \Big[\cosh(\omega t) + (d-3) \sinh(\omega t) - e^{(d-3)\,\omega t} \Big] \Big|_{d=4} \tag{4.112}$$

$$:= \lim_{d \to 4} \frac{1}{d-4} \Big[\cosh(\omega t) + (d-3) \sinh(\omega t) - e^{(d-3)\,\omega t} \Big] = \sinh(\omega t) - \omega t\, e^{\omega t} \ .$$

From Eq.s (4.6) (4.109) (4.110) (4.111) one obtains, respectively,

$$E\big(x(t), \dot{x}(t), y(t), \dot{y}(t)\big) = -\frac{V_1\, A\, D}{d^2} \ , \tag{4.113}$$

$$E\big(x(t), \dot{x}(t), y(t), \dot{y}(t)\big) = \frac{A}{d^2} \left(V_1\, C + V_2\, A^{d-1} + \frac{d^2 (\Omega_{m*} - k)}{2} A^{d-3} \right) , \tag{4.114}$$

$$E\big(x(t), \dot{x}(t), y(t), \dot{y}(t)\big) = \frac{A}{d^2} \left(V_1\, (C - D) + V_2\, A^{d-1} + \frac{d^2 (\Omega_{m*} - k)}{2} A^{d-3} \right) . \tag{4.115}$$

Taking the above relations into account and recalling that in the present case $V_1 \neq 0$ (see Eq. (4.106)), to fulfill the zero-energy constraint we must require

$$D = 0 \ , \tag{4.116}$$

$$C = -\frac{V_2\,A^{d-1}}{V_1} - \frac{d^2(\Omega_{m*}-k)}{2\,V_1}\,A^{d-3}\ , \tag{4.117}$$

$$C = D - \frac{V_2\,A^{d-1}}{V_1} - \frac{d^2(\Omega_{m*}-k)}{2\,V_1}\,A^{d-3} \tag{4.118}$$

in Eq.s (4.109) (4.110) (4.111), respectively.

4.3.2 The subcase $\epsilon = 0$

We consider the case (4.13) with $\epsilon = 0$, i.e.

$$V_1 = 0\ . \tag{4.119}$$

Eq.s (4.103) (4.104) read

$$\ddot{x} = 0\ , \tag{4.120}$$

$$\ddot{y} = \frac{2(d-1)\,V_2}{d}\,x^{d-1} + (d-2)\,(d-1)\,\big(\Omega_{m*}-k\big)\,x^{d-3}\ . \tag{4.121}$$

After a time translation $t \to t + \text{const.}$ and possibly a time reflection $t \to -t$, any positive solution of Eq. (4.120) (4.121) can be written in one of the following ways (with A, C, D constants):

$$x(t) = A\,t\ , \tag{4.122}$$

$$y(t) = C + D\,t + \frac{2\,(d-1)\,V_2}{d^2\,(d+1)}\,A^{d-1}\,t^{d+1} + \big(\Omega_{m*}-k\big)\,A^{d-3}\,t^{d-1}$$

$$\big(\,A > 0,\ \ t \in I := \text{a maximal subinterval of } (0,+\infty) \text{ s.t. } y(t) > 0\ \ \forall\, t \in I\,\big)\ ;$$

$$x(t) = A\ , \tag{4.123}$$

$$y(t) = C + D\,t + \frac{d-1}{2}\left(\frac{2\,V_2}{d}\,A^{d-1} + (d-2)\,\big(\Omega_{m*}-k\big)\,A^{d-3}\right)t^2$$

$$\big(\,A > 0,\ \ t \in I := \text{a maximal real interval s.t. } y(t) > 0\ \ \forall\, t \in I\,\big)\ .$$

From Eq.s (4.6) (4.122) (4.123) one obtains, respectively,

$$E\big(x(t),\dot{x}(t),y(t),\dot{y}(t)\big) = -\,\frac{A\,D}{2\,(d-1)}\ , \tag{4.124}$$

$$E\big(x(t),\dot{x}(t),y(t),\dot{y}(t)\big) = \frac{A^{d-2}}{d^2}\left(V_2\,A^2 + \frac{d^2(\Omega_{m*}-k)}{2}\right)\ . \tag{4.125}$$

In view of this, to fulfill the zero-energy constraint we must require

$$D = 0\ , \tag{4.126}$$

$$V_2 \neq 0,\ \ \text{sign}\ (\Omega_{m*}-k) = \text{sign}\ V_2\ \ \text{and}\ \ A = \sqrt{\frac{d^2\,(\Omega_{m*}-k)}{2\,|V_2|}} \tag{4.127}$$

$$\text{or}\qquad V_2 = 0\ \ \text{and}\ \ \Omega_{m*} = k$$

in Eq.s (4.122) (4.123), respectively.

4.3.3 The subcase $\epsilon = 1$

We consider the case (4.13) with $\epsilon = 1$, i.e.

$$V_1 < 0 \ . \tag{4.128}$$

Eq.s (4.103) (4.104) read

$$\ddot{x} + \omega^2 x = 0 \ , \tag{4.129}$$

$$\ddot{y} + \omega^2 y = \frac{2(d-1)\, V_2}{d}\, x^{d-1} + (d-2)\,(d-1)\,\big(\Omega_{m*} - k\big)\, x^{d-3} \ . \tag{4.130}$$

After a time translation $t \to t + \text{const.}$ and possibly a time reflection $t \to -t$, any positive solution of Eq.s (4.129) (4.130) (for $d \geqslant 3$ as in Eq. (4.105)) can be written as follows (with A, C, D constants):

$$x(t) = A\, \sin(\omega\, t) \ , \tag{4.131}$$

$$y(t) = C\, \cos(\omega\, t) + D\, \sin(\omega\, t)$$
$$- \frac{V_2}{V_1}\, A^{d-1}\, \sin^{d+1}(\omega\, t) \left[1 - \frac{d\, \cos(\omega\, t)}{d+1}\; {}_2F_1\!\left(\frac{1}{2}, \frac{d+1}{2}, \frac{d+3}{2}; \sin^2(\omega\, t)\right)\right]$$
$$+ \frac{d^2 \big(\Omega_{m*} - k\big)}{2\, V_1}\, A^{d-3} \sin^{d-1}(\omega\, t) \left[1 - \frac{(d-2)\, \cos(\omega\, t)}{d-1}\; {}_2F_1\!\left(\frac{1}{2}, \frac{d-1}{2}, \frac{d+1}{2}; \sin^2(\omega\, t)\right)\right]$$

$\big(A > 0,\ \omega$ as in Eq. (4.102), $t \in I :=$ a maximal subinterval of $(0, \pi/\omega)$ s.t. $y(t) > 0\ \ \forall\, t \in I \big)$.

From Eq.s (4.6) (4.131) one obtains

$$E\big(x(t), \dot{x}(t), y(t), \dot{y}(t)\big) = \frac{V_1\, A\, D}{d^2} \ . \tag{4.132}$$

In view of this, to fulfill the zero-energy constraint we must require

$$D = 0 \tag{4.133}$$

in Eq. (4.131) .

4.4 Tables of the solutions of subsections 4.1 , 4.2 , 4.3

For the reader's convenience, this section summarizes all the solutions found in sections 4.1, 4.2 and 4.3. We report here the field potential and the description of the model solutions given by Eq.s (4.1) (4.2) (4.3) (4.4) (4.14) (4.15) (4.30) (4.31) (4.32) (4.44) (4.45) (4.56) (4.71) (4.72) (4.73) (4.83) (4.86) (4.89) (4.90) (4.98) (4.109) (4.110) (4.111) (4.122) (4.123) (4.131) and by the energy constraints (4.36) (4.37) (4.38) (4.48) (4.49) (4.58) (4.77) (4.78) (4.79) (4.85) (4.88) (4.93) (4.94) (4.100) (4.116) (4.117) (4.133) (4.126) (4.127) (4.133) (involving the constants $V_1, V_2, A, C, D, \Omega_{m*}$):

	$\mathcal{V}(\phi)=\frac{1}{d^2}\left(V_1\,e^{2\,d\,w\,\phi}+V_2\,e^{d\,(1+w)\,\phi}\right)\ ,\ a=x^{\frac{1}{d\,(1+w)}}\ y^{\frac{1}{d\,(1-w)}}\ ,\ \phi=\frac{1}{d}\,\log\!\left(x^{\frac{1}{1+w}}\ y^{-\frac{1}{1-w}}\right)$ $k=0\ ,\ w\ \text{arbitrary}\neq\pm1\ ,\ \epsilon:=-\,\text{sign}\big((1-w^2)\,V_1\big)\ ,\ \omega:=\sqrt{\frac{\lvert(1-w^2)\,V_1\rvert}{2}}$
$\epsilon=-1$	$x(t)=A\sinh(\omega\,t)$ $y(t)=C\cosh(\omega\,t)+D\sinh(\omega\,t)$ $+\frac{V_2}{V_1}\,A^{\frac{1-w}{1+w}}\sinh^{\frac{3+w}{1+w}}(\omega\,t)\Big[1-\frac{2\cosh(\omega\,t)}{3+w}\ {}_2F_1\Big(\frac{1}{2}\,,\frac{3+w}{2+2w}\,,\frac{5+3w}{2+2w}\,;-\sinh^2(\omega\,t)\Big)\Big]$ $+\frac{d^2\,\Omega_{m*}}{2\,V_1}\,A^{-\frac{1+3w}{1+w}}\ \sinh^{\frac{1-w}{1+w}}(\omega\,t)\Big[1+\frac{2\,w\cosh(\omega\,t)}{1-w}\ {}_2F_1\Big(\frac{1}{2}\,,\frac{1-w}{2+2w}\,,\frac{3+w}{2+2w}\,;-\sinh^2(\omega\,t)\Big)\Big]$ $\left(\begin{array}{c}A>0\,,\ D=0,\ w\neq-\dfrac{3+2h}{1+2h}\ \ \text{for all}\ h\in\{0,1,2,\ldots\}\\ t\in I:=\text{a maximal real interval s.t.}\ y(t)>0\ \ \forall\,t\in I\end{array}\right)$
	$x(t)=A\cosh(\omega\,t)$ $y(t)=C\cosh(\omega\,t)+D\sinh(\omega\,t)$ $+\frac{V_2}{V_1}\,A^{\frac{1-w}{1+w}}\ \Big[\cosh(\omega\,t)\big(1-\cosh^{\frac{2}{1+w}}(\omega\,t)\big)+\frac{2\ \sinh^2(\omega\,t)}{1+w}\ {}_2F_1\Big(\frac{1}{2}\,,-\,\frac{1-w}{2+2w}\,,\frac{3}{2}\,;-\sinh^2(\omega\,t)\Big)\Big]$ $+\frac{d^2\,\Omega_{m*}}{2\,V_1}\,A^{-\frac{1+3w}{1+w}}\ \Big[\cosh(\omega\,t)\big(1-\cosh^{-\frac{2w}{1+w}}(\omega\,t)\big)-\frac{2\,w\ \sinh^2(\omega\,t)}{1+w}\ {}_2F_1\Big(\frac{1}{2}\,,\frac{1+3w}{2+2w}\,,\frac{3}{2}\,;-\sinh^2(\omega\,t)\Big)\Big]$ $\left(\begin{array}{c}A>0\,,\ C=-\,\dfrac{V_2}{V_1}\,A^{\frac{1-w}{1+w}}-\ \dfrac{d^2\,\Omega_{m*}}{2\,V_1}\,A^{-\frac{1+3w}{1+w}}\\ t\in I:=\text{a maximal real interval s.t.}\ y(t)>0\ \ \forall\,t\in I\end{array}\right)$
	$x(t)=A\,e^{\omega\,t}$ $y(t)=C\cosh(\omega\,t)+D\sinh(\omega t)$ $+\frac{1+w}{2\,w}\ \frac{V_2}{V_1}\,A^{\frac{1-w}{1+w}}\Big[\cosh(\omega\,t)+\frac{1-w}{1+w}\ \sinh(\omega\,t)-e^{\frac{(1-w)}{1+w}\,\omega\,t}\Big]$ $+\frac{1+w}{1+2w}\ \frac{d^2\,\Omega_{m*}}{4\,V_1}\,A^{-\frac{1+3w}{1+w}}\Big[\cosh(\omega\,t)-\frac{1+3w}{1+w}\ \sinh(\omega\,t)-e^{-\frac{(1+3w)}{1+w}\,\omega\,t}\Big]$ $\left(\begin{array}{c}A>0\,,\ C=D-\dfrac{V_2}{V_1}\,A^{\frac{1-w}{1+w}}-\ \dfrac{d^2\,\Omega_{m*}}{2\,V_1}\,A^{-\frac{1+3w}{1+w}}\\ t\in I:=\text{a maximal real interval s.t.}\ y(t)>0\ \ \forall\,t\in I\end{array}\right)$
$\epsilon=0$	$x(t)=A\,t$ $y(t)=C+D\,t+\frac{V_2\,(1+w)^2\,(1-w)}{2\,(3+w)}\,A^{\frac{1-w}{1+w}}\,t^{\frac{3+w}{1+w}}+\frac{d^2\,(1+w)^2}{4}\ \Omega_{m*}\,A^{-\frac{1+3w}{1+w}}\ t^{\frac{1-w}{1+w}}$ $\big(\,A>0,\ D=0,\ t\in I:=\text{a maximal subinterval of}\ (0,+\infty)\ \text{s.t.}\ y(t)>0\ \ \forall\,t\in I\,\big)$
	$x(t)=A$ $y(t)=C+D\,t+\frac{t^2}{2}\left(V_2\,(1-w)\,A^{\frac{1-w}{1+w}}-\frac{d^2\,w\,(1-w)}{2}\ \Omega_{m*}\,A^{-\frac{1+3w}{1+w}}\right)$ $\left(\begin{array}{c}A>0\,,\ V_2<0\ \ \text{and}\ \ A=\sqrt{\dfrac{d^2\,\Omega_{m*}}{2\,\lvert V_2\rvert}}\ \ \text{or}\ \ V_2=0\ \ \text{and}\ \ \Omega_{m*}=0\\ t\in I:=\text{a maximal real interval s.t.}\ y(t)>0\ \ \forall\,t\in I\end{array}\right)$
$\epsilon=1$	$x(t)=A\ \sin(\omega\,t)$ $y(t)=C\cos(\omega\,t)+D\sin(\omega\,t)$ $+\frac{V_2}{V_1}\,A^{\frac{1-w}{1+w}}\ \sin^{\frac{3+w}{1+w}}(\omega\,t)\Big[1-\frac{2\cos(\omega\,t)}{3+w}\ {}_2F_1\Big(\frac{1}{2}\,,\frac{3+w}{2+2w}\,,\frac{5+3w}{2+2w}\,;\sin^2(\omega\,t)\Big)\Big]$ $-\frac{d^2\,\Omega_{m*}}{2\,V_1}\,A^{-\frac{1+3w}{1+w}}\ \sin^{\frac{1-w}{1+w}}(\omega\,t)\Big[1+\frac{2\,w\cos(\omega\,t)}{1-w}\ {}_2F_1\Big(\frac{1}{2}\,,\frac{1-w}{2+2w}\,,\frac{3+w}{2+2w}\,;\sin^2(\omega\,t)\Big)\Big]$ $\big(\,A>0\,,\ D=0,\ t\in I:=\text{a maximal subinterval of}\ (0,\pi/\omega)\ \text{s.t.}\ y(t)>0\ \ \forall\,t\in I\,\big)\ .$

	$\mathcal{V}(\phi) = \frac{1}{d^2}\left(V_1\, e^{2\, d\, w\, \phi} + V_2\, e^{d\,(1+w)\,\phi}\right)\ ,\ a = x^{\frac{1}{d\,(1+w)}}\ y^{\frac{1}{d\,(1-w)}}\ ,\ \phi = \frac{1}{d}\ \log\left(x^{\frac{1}{1+w}}\ y^{-\frac{1}{1-w}}\right)$ k arbitrary $,\ w = 1/d\ ,\ \epsilon := -\,\mathrm{sign}\, V_1\ ,\ \omega := \sqrt{\frac{(d^2-1)\,\lvert V_1\rvert}{2\,d^2}}$
$d = 3$ $\epsilon = -1$	$x(t) = A\ \sinh(\omega\, t) + \frac{2\,k}{\omega^2}$ $y(t) = C\ \cosh(\omega\, t) + D\ \sinh(\omega\, t)$ $+\ \frac{1}{\omega}\int_{t_0}^{t} ds\ \sinh\left(\omega\,(t-s)\right)\left[\frac{2\,V_2}{3}\left(A\ \sinh(\omega\, s) + \frac{2\,k}{\omega^2}\right)^{1/2} - \Omega_{m*}\left(A\ \sinh(\omega\, s) + \frac{2\,k}{\omega^2}\right)^{-3/2}\right]$ $\left(A\, D = \frac{V_2}{V_1}\left(A\ \sinh(\omega\, t_0) + \frac{9\,k}{2\,V_1}\right)^{3/2} + \frac{9\,\Omega_{m*}}{2\,V_1}\left(A\ \sinh(\omega\, t_0) + \frac{9\,k}{2\,V_1}\right)^{-1/2}\right.$ $\left. t \in I :=\text{ a maximal real interval s.t. } x(t), y(t) > 0\ \ \forall\, t \in I \right)$
	$x(t) = A\ \cosh(\omega\, t) + \frac{2\,k}{\omega^2}$ $y(t) = C \cosh(\omega\, t) + D \sinh(\omega\, t)$ $+\ \frac{1}{\omega}\int_{t_0}^{t} ds\ \sinh\left(\omega\,(t-s)\right)\left[\frac{2\,V_2}{3}\left(A\ \cosh(\omega\, s) + \frac{2\,k}{\omega^2}\right)^{1/2} - \Omega_{m*}\left(A\ \cosh(\omega\, s) + \frac{2\,k}{\omega^2}\right)^{-3/2}\right],$ $\left(A\, C = -\,\frac{V_2}{V_1}\left(A\ \cosh(\omega\, t_0) + \frac{9\,k}{2\,V_1}\right)^{3/2} - \frac{9\,\Omega_{m*}}{2\,V_1}\left(A\ \cosh(\omega\, t_0) + \frac{9\,k}{2\,V_1}\right)^{-1/2}\right.$ $\left. t \in I :=\text{ a maximal real interval s.t. } x(t), y(t) > 0\ \ \forall\, t \in I \right)$
	$x(t) = A\, e^{\omega\, t} + \frac{2\,k}{\omega^2}$ $y(t) = C\ \cosh(\omega\, t) + D\ \sinh(\omega\, t)$ $+\ \frac{1}{\omega}\int_{t_0}^{t} ds\ \sinh\left(\omega(t-s)\right)\left[\frac{2\,V_2}{3}\left(A\, e^{\omega\, s} + \frac{2\,k}{\omega^2}\right)^{1/2} - \Omega_{m*}\left(A\, e^{\omega\, s} + \frac{2\,k}{\omega^2}\right)^{-3/2}\right]$ $\left(A\,(C-D) = -\,\frac{V_2}{V_1}\left(A\, e^{\omega\, t_0} + \frac{9k}{2V_1}\right)^{3/2} - \frac{9\,\Omega_{m*}}{2\,V_1}\left(A\, e^{\omega\, t_0} + \frac{9\,k}{2\,V_1}\right)^{-1/2}\right.$ $\left. t \in I :=\text{ a maximal real interval s.t. } x(t), y(t) > 0\ \ \forall\, t \in I \right)$
$d = 3$ $\epsilon = 0$	$x(t) = A + B\, t - k\, t^2$ $y(t) = C + D\, t\ +\ \frac{\sqrt{x(t)}\,(\alpha - 4\,V_2\, B\,\Delta\, k\, t + 4\, V_2\,\Delta\, k^2\, t^2)}{36\,\Delta\, k^2}$ $-\ \frac{\Delta\,(B - 2\,k\,t)}{24\,\lvert k\rvert^{5/2}}\ V_2\ \log\left(\frac{B - 2\,k\,t}{\lvert k\rvert^{1/2}}\ +\ 2\,\sqrt{x(t)}\right)$ $\left(k < 0\,,\quad \Delta := B^2 + 4\,k\,A\,,\quad \alpha := \left(3\,B^4 + 20\,k\,A\,B^2 + 32\,k^2\,A^2\right)V_2 + 144\,k^2\,\Omega_{m*}\right.$ $\left. C = -\,\frac{B\,D}{2\,k}\,,\quad t \in I :=\text{ a maximal real interval s.t. } x(t),\ y(t) > 0\ \ \forall\, t \in I \right)$
	$x(t) = A + B\, t - k\, t^2$ $y(t) = C + D\, t\ +\ \frac{\sqrt{x(t)}\,(\beta + V_2\,\Delta\, k\, t - V_2\, B\,\Delta\, t^2)}{9\,\Delta\, k}$ $+\ \frac{V_2\,\Delta}{24\,k^3}\ \left[2\,k\,\sqrt{x(t)} + \sqrt{k}\ (B - 2\,k\,t)\ \arctan\left(\frac{B - 2\,k\,t}{2\sqrt{k\,x(t)}}\right)\right]$ $\left(k > 0\,,\quad \Delta := B^2 + 4\,k\,A\,,\quad \beta := 36\,k\,\Omega_{m*} - V_2\,A\,\Delta\,,\quad C = -\,\frac{B\,D}{2\,k}\right.$ $\left. t \in I :=\text{ a maximal real interval s.t. } x(t),\ y(t) > 0\ \ \forall\, t \in I \right)$
	$x(t) = A\, t$ $y(t) = C + D\, t + \frac{8\,V_2\,\sqrt{A}}{45}\ t^{5/2} + \frac{4\,\Omega_{m*}}{A^{3/2}}\ t^{1/2}$ $\left(k = 0,\ A > 0,\ D = 0,\ t \in I :=\text{ a maximal subinterval of } (0, +\infty)\ \ \text{ s.t. } y(t) > 0\ \ \forall\, t \in I\right)$
	$x(t) = A$ $y(t) = C + D\, t + \left(\frac{V_2\,\sqrt{A}}{3} - \frac{\Omega_{m*}}{2\,A^{3/2}}\right) t^2$ $\left(k = 0\,,\quad A > 0\,,\quad V_2 < 0\ \text{ and }\ A = \sqrt{\frac{9\,\Omega_{m*}}{2\,\lvert V_2\rvert}}\ \text{ or }\ V_2 = 0\ \text{ and }\ \Omega_{m*} = 0\right.$ $\left. t \in I :=\text{ a maximal real interval s.t. } y(t) > 0\ \ \forall\, t \in I \right)$
$d = 3$ $\epsilon = 1$	$x(t) = A\ \sin(\omega\, t) - \frac{2\,k}{\omega^2}$ $y(t) = C\ \cos(\omega\, t) + D\ \sin(\omega\, t)$ $+\frac{1}{\omega}\int_{0}^{t} ds\ \sin\left(\omega\,(t-s)\right)\left[\frac{2\,V_2}{3}\left(A\ \sin(\omega\, t) - \frac{2\,k}{\omega^2}\right)^{1/2} - \Omega_{m*}\left(A \sin(\omega\, t) - \frac{2\,k}{\omega^2}\right)^{-3/2}\right]$ $\left(A\, D = -\frac{V_2}{V_1}\left(\frac{9\,k}{2\,V_1}\right)^{3/2} - \frac{9\,\Omega_{m*}}{2\,V_1}\left(\frac{9\,k}{2\,V_1}\right)^{-1/2},\ t \in I :=\text{ a maximal real interval s.t. } x(t), y(t) > 0\ \ \forall\, t \in I\right)\ .$

	$\mathcal{V}(\phi) = \frac{1}{d^2}\left(V_1\, e^{2\,d\,w\,\phi} + V_2\, e^{d\,(1+w)\,\phi}\right)\,,\ a = x^{\frac{1}{d\,(1+w)}}\, y^{\frac{1}{d\,(1-w)}}\,,\ \phi = \frac{1}{d}\,\log\!\left(x^{\frac{1}{1+w}}\, y^{-\frac{1}{1-w}}\right)$ k arbitrary $,\ w = 2/d-1\,,\ d \geqslant 3\,,\ \epsilon := -\,\mathrm{sign}\, V_1\,,\ \omega := \sqrt{\frac{2\,(d-1)\,\lvert V_1\rvert}{d^2}}$
$\epsilon = -1$	$x(t) = A\,\sinh(\omega\, t)$ $y(t) = C\,\cosh(\omega\,t) + D\,\sinh(\omega\,t)$ $+\,\frac{V_2}{V_1}\,A^{d-1}\,\sinh^{d+1}(\omega\,t)\left[1 - \frac{d\,\cosh(\omega\,t)}{d+1}\;{}_2F_1\!\left(\frac{1}{2}\,,\frac{d+1}{2}\,,\frac{d+3}{2}\,;-\sinh^2(\omega\,t)\right)\right]$ $+\,\frac{d^2\left(\Omega_{m*}-k\right)}{2\,V_1}\,A^{d-3}\sinh^{d-1}(\omega\,t)\left[1 - \frac{(d-2)\,\cosh(\omega\,t)}{d-1}\;{}_2F_1\!\left(\frac{1}{2},\,\frac{d-1}{2},\,\frac{d+1}{2};-\sinh^2(\omega\,t)\right)\right]$ $\left(A>0,\ D=0,\ t\in I :=\right.$ a maximal subinterval of $(0,+\infty)$ s.t. $\left. y(t)>0\ \ \forall\, t\in I\right)$
	$x(t) = A\,\cosh(\omega\,t)$ $y(t) = C\,\cosh(\omega\,t) + D\,\sinh(\omega\,t)$ $+\,\frac{V_2}{V_1}\,A^{d-1}\left[\cosh(\omega\,t)(1-\cosh^d(\omega\,t)) + d\sinh^2(\omega\,t)\;{}_2F_1\left(\frac{1}{2}\,,-\frac{d-1}{2}\,,\frac{3}{2}\,;-\sinh^2(\omega\,t)\right)\right]$ $+\left[\cosh(\omega\,t)\big(1-\cosh^{d-2}(\omega\,t)\big) + (d-2)\,\sinh^2(\omega\,t)\;{}_2F_1\left(\frac{1}{2}\,,-\frac{d-3}{2}\,,\frac{3}{2}\,;-\sinh^2(\omega\,t)\right)\right]\times$ $\times\,\frac{d^2\,(\Omega_{m*}-k)}{2\,V_1}\,A^{d-3}$ $\left(\begin{array}{c} A>0\,,\ C = -\,\dfrac{V_2\,A^{d-1}}{V_1} - \dfrac{d^2(\Omega_{m*}-k)}{2\,V_1}\,A^{d-3} \\ t\in I := \text{a maximal real interval s.t. } y(t)\!>\!0\ \ \forall\,t\in I \end{array}\right)$
	$x(t) = A\,e^{\omega\,t}$ $y(t) = C\cosh(\omega\,t) + D\sinh(\omega t)$ $-\,\frac{V_2}{V_1}\,\frac{A^{d-1}}{d-2}\left[\cosh(\omega\,t) + (d-1)\,\sinh(\omega\,t) - e^{(d-1)\,\omega\,t}\right]$ $-\,\frac{d^2(\Omega_{m*}-k)}{2\,V_1}\,\frac{A^{d-3}}{d-4}\left[\cosh(\omega\,t) + (d-3)\sinh(\omega\,t) - e^{(d-3)\,\omega\,t}\right]$ $\left(\begin{array}{c} A>0\,,\ C = D - \dfrac{V_2\,A^{d-1}}{V_1} - \dfrac{d^2(\Omega_{m*}-k)}{2\,V_1}\,A^{d-3} \\ t\in I := \text{a maximal real interval s.t. } y(t)\!>\!0\ \ \forall\,t\in I \end{array}\right)$
$\epsilon = 0$	$x(t) = A\,t$ $y(t) = C + D\,t + \frac{2\,(d-1)\,V_2}{d^2\,(d+1)}\,A^{d-1}\,t^{d+1} + \big(\Omega_{m*}-k\big)\,A^{d-3}\,t^{d-1}$ $\left(A>0,\ D=0,\ t\in I :=\right.$ a maximal subinterval of $(0,+\infty)$ s.t. $\left. y(t)>0\ \ \forall\, t\in I\right)$
	$x(t) = A$ $y(t) = C + D\,t + \frac{d-1}{2}\left(\frac{2\,V_2}{d}\,A^{d-1} + (d-2)\,\big(\Omega_{m*}-k\big)\,A^{d-3}\right)t^2$ $\left(\begin{array}{c} A>0\,,\ \ V_2\neq 0,\ \ \text{sign}\ (\Omega_{m*}-k) = \text{sign}\ V_2\ \ \text{and}\ \ A = \sqrt{\dfrac{d^2\,(\Omega_{m*}-k)}{2\,\lvert V_2\rvert}}\ \ \text{or}\ \ V_2 = 0\ \ \text{and}\ \ \Omega_{m*} = k \\ t\in I := \text{a maximal real interval s.t. } y(t)\!>\!0\ \ \forall\,t\in I \end{array}\right)$
$\epsilon = 1$	$x(t) = A\,\sin(\omega\,t)$ $y(t) = C\cos(\omega\,t) + D\sin(\omega\,t)$ $-\,\frac{V_2}{V_1}\,A^{d-1}\,\sin^{d+1}(\omega\,t)\left[1 - \frac{d\,\cos(\omega\,t)}{d+1}\;{}_2F_1\left(\frac{1}{2}\,,\frac{d+1}{2}\,,\frac{d+3}{2}\,;\sin^2(\omega\,t)\right)\right]$ $+\,\frac{d^2\left(\Omega_{m*}-k\right)}{2\,V_1}\,A^{d-3}\sin^{d-1}(\omega\,t)\left[1 - \frac{(d-2)\,\cos(\omega\,t)}{d-1}\;{}_2F_1\left(\frac{1}{2},\,\frac{d-1}{2},\,\frac{d+1}{2};\sin^2(\omega\,t)\right)\right]$ $\left(A>0\,,\ D=0,\ t\in I :=\right.$ a maximal subinterval of $(0,\pi/\omega)$ s.t. $\left. y(t)>0\ \ \forall\, t\in I\right)$

5 Qualitative analysis of some solutions

Many of the solutions constructed in Sections 3 and 4 exhibit a Big Bang; some of them also exhibit a Big Crunch (intendig these expressions in the precise sense defined in subsection 2.5).

If there is a Big Bang one should understand how a and ϕ approach it, evaluate the particle horizon (subsection 2.6) and also analyze the limiting behavior of the dimensionless densities of subsection 2.9. If there is a Big Crunch, one should make a similar analysis of the behavior of a, ϕ and of the densities. In absence of this, one should anyhow discuss the long time behavior of the previously mentioned characters.

In the following we will discuss the above issues (or some of them) for few examples. In each one of these examples, the solution depends on a set of integration constants, and different choices of these constants can produce qualitatively different behaviors. All these cases have vanishing scalar curvature, i.e.,

$$k = 0 \ , \tag{5.1}$$

and possess a Big Bang at $t = 0$ (to which we devote most our attention). We always define the cosmic time as

$$\tau(t) := \theta \int_0^t dt' \, b(t') \tag{5.2}$$

(recalling that the integrability of b in a right neighborhood of zero is required by the very definition of Big Bang, see subsection 2.5); of course $\tau(t) \to 0^+$ for $t \to 0^+$, and we can speak of the inverse function $t = t(\tau)$. In each case analyzed, it is important to ascertain whether the particle horizon

$$\Theta(\tau_1) := \int_0^{\tau_1} \frac{d\tau}{a(\tau)} = \theta \int_0^{t_1} dt \, \frac{b(t)}{a(t)} \tag{5.3}$$

is finite or infinite ar any time $\tau_1 = \tau(t_1)$.

Since $k = 0$, from Eq.s (2.53) (2.54) we infer

$$\Omega_m + \Omega_\Phi = 1 \ . \tag{5.4}$$

In the sequel, we will say that *matter dominates at the Big Bang* if $\Omega_m(t) \to 1$ (or equivalently, $\Omega_\Phi(t) \to 0$) when $t \to 0^+$; we will say that *the scalar field (or dark energy) dominates at the Big Bang* if $\Omega_\Phi(t) \to 0$ (or equivalently, $\Omega_m(t) \to 1$) in the same limit. (One can define similarly the cases where *matter* or *the scalar field dominate at the Big Crunch*, if this exists).

5.1 The dust solution of section 3

Consider the flat dust model ($k = 0, w = 0$) of Section 3. For convenience of the reader, we report here the field potential and the description of the model solution given by Eq.s (3.1) (3.2) (3.3) (3.4) (3.9) (3.10) (3.11) (3.12) and by the energy constraint (3.14) (involving the constants $V_1, V_2, V, A, B, C, D, \Omega_{m*}$):

$$\mathcal{V}(\phi) = \frac{1}{d^2}\left(V_1\, e^{d\,\phi} + V_2\, e^{-d\,\phi}\right) \qquad \left(V_1 > 0,\ V_2 > 0,\ V := \sqrt{V_1\, V_2}\right)\ ; \tag{5.5}$$

$$b(t) = 1\ , \quad \tau = \theta\, t\ ;$$

$$a(t) = x(t)^{\frac{1}{d}}\, y(t)^{\frac{1}{d}}\ , \quad \phi(t) = \frac{1}{d}\,\log\left(\frac{x(t)}{y(t)}\right)\ ;$$

$$x(t) = \frac{1}{2\sqrt{V_1}}\Big[(A+C)\cosh\left(\sqrt{V}\,t\right) + (A-C)\cos\left(\sqrt{V}\,t\right) +$$
$$+ (B+D)\sinh\left(\sqrt{V}\,t\right) + (B-D)\sin\left(\sqrt{V}\,t\right)\Big]\ ,$$

$$y(t) = \frac{1}{2\sqrt{V_2}}\Big[(A+C)\cosh\left(\sqrt{V}\,t\right) - (A-C)\cos\left(\sqrt{V}\,t\right) +$$
$$+ (B+D)\sinh\left(\sqrt{V}\,t\right) - (B-D)\sin\left(\sqrt{V}\,t\right)\Big]$$

$$\Omega_{m*} = \frac{2}{d^2}\left(2\,B\,D - A^2 - C^2\right)$$

(the above relation involving Ω_{m*} is the already mentioned energy constraint). One accepts as a domain for the solution a maximal interval where $x(t), y(t) > 0$. For the sequel, let us also mention that Eq.s (2.30) (2.31) (2.53) and the previous expressions for b, a give the following representations for the coefficient $w^{(\Phi)}$ in the field equation of state and for the dimensionless matter density Ω_m:

$$w^{(\Phi)} = \frac{\dot{\phi}^2 - 2\,\mathcal{V}(\phi)}{\dot{\phi}^2 + 2\,\mathcal{V}(\phi)} = \frac{(\dot{x}\,y - x\,\dot{y})^2 - 2\,(V_1\,x^3 y + V_2\,x\,y^3)}{(\dot{x}\,y - x\,\dot{y})^2 + 2\,(V_1\,x^3 y + V_2\,x\,y^3)}\ , \tag{5.6}$$

$$\Omega_m = \frac{\Omega_{m*}}{a^{d-2}\,\dot{a}^2} = \frac{d^2\,\Omega_{m*}\,x\,y}{(x\,\dot{y} + \dot{x}\,y)^2}\ . \tag{5.7}$$

5.1.1 Big Bang analysis

Let us wonder under which conditions the solution (5.5) gives a Big Bang at some instant, that we conventionally choose as the time origin $t = 0$. The conditions for a Big Bang at $t = 0$ are that $a(t) \to 0$ for $t \to 0^+$ and, even prior to this, that $a(t)$ is well defined in a right neighborhood of $t = 0$; in terms of the above functions $x(t), y(t)$, this amounts to ask

$$x(t)\,y(t) \to 0, \qquad x(t),\ y(t) > 0 \text{ for } t \to 0^+ \tag{5.8}$$

(here and in the sequel, an expression of the form $f(t) > 0$ for $t \to 0^+$ means that $f(t) > 0$ for all t in some interval $(0, \epsilon)$, $\epsilon > 0$).
From the previous expressions, it is clear that $x(t) \to A/\sqrt{V_1}$, $y(t) \to C/\sqrt{V_2}$ for $t \to 0$, so the first condition in (5.8) holds if and only if

$$A\,C = 0 \; ; \tag{5.9}$$

in the sequel we will distinguish three subcases of (5.9).

The case $\boldsymbol{A = 0, C \neq 0}$. Let us first remark that for $B = 0$ the zero-energy constraint (3.14) yields in the present case $\Omega_{m*} = -2\,C^2/d^2 \leqslant 0$; a non-positive matter density is not interesting, so from now we assume $B \neq 0$.
Since $x(t) = B\,(V_2/V_1)^{1/4}\; t + O(t^2)$, $y(t) = C/\sqrt{V_2} + O(t)$ for $t \to 0$, the positivity requirement for these functions as $t \to 0^+$ yields the conditions $B > 0, C > 0$. To go on, we note that the previous expressions for $a, \phi, w^{(\Phi)}, \Omega_m$ imply the following, for $t \to 0^+$:

$$a(t) = \left(\frac{B\,C}{\sqrt{V}}\right)^{1/d} t^{1/d} + O\big(t^{(d+1)/d}\big) \; , \tag{5.10}$$

$$\phi(t) = \frac{1}{d}\,\log t + \frac{1}{d}\,\log\left(\frac{B\,V_2^{3/4}}{C\,V_1^{1/4}}\right) + O(t) \; , \tag{5.11}$$

$$w^{(\Phi)}(t) = 1 - \frac{4\,C}{B}\,\sqrt{V}\; t + O(t^2) \; , \tag{5.12}$$

$$\Omega_m(t) = 2\left(\frac{2\,B\,D - C^2}{B\,C}\right)\sqrt{V}\; t + O(t^2) \; ; \tag{5.13}$$

we have similar expansions in terms of the cosmic time τ, recalling that $t = \tau/\theta$. From Eq.s (5.3) (5.10), noting that $1/d < 1$ for any $d \geqslant 2$ we infer that the particle horizon is finite at any time. Eq. (5.13) indicates that $\Omega_m(t) \to 0$, so the scalar field dominates at the Big Bang.

The case $\boldsymbol{A \neq 0, C = 0}$. This is qualitatively very similar to the previous case. For $D = 0$, the zero-energy constraint yields a non-positive matter density, so we assume $D \neq 0$. The positivity requirement for $x(t), y(t)$ as $t \to 0^+$ yields the conditions $A > 0$, $D > 0$. For $t \to 0^+$ we have

$$a(t) = \left(\frac{A\,D}{\sqrt{V}}\right)^{1/d} t^{1/d} + O\big(t^{(d+1)/d}\big) \; , \tag{5.14}$$

$$\phi(t) = -\frac{1}{d}\,\log t + \frac{1}{d}\,\log\left(\frac{A\,V_2^{1/4}}{D\,V_1^{3/4}}\right) + O(t) \; , \tag{5.15}$$

$$w^{(\Phi)}(t) = 1 - \frac{4\,A}{D}\,\sqrt{V}\; t + O(t^2) \; , \tag{5.16}$$

$$\Omega_m(t) = 2\left(\frac{2\,B\,D - A^2}{A\,D}\right)\sqrt{V}\ t + O(t^2)\ ; \tag{5.17}$$

the particle horizon is finite and the scalar field dominates at the Big Bang.

The case $A = C = 0$. Let us first remark that, when either $B = 0$ or $D = 0$, we get $\Omega_m \equiv 0$ (whence, $\Omega_\Phi \equiv 1$) for all times; this means that the the scalar field is the only content of the universe described by this model. From now on we assume $B \neq 0$ and $D \neq 0$. Since $x(t) = B\,(V_2/V_1)^{1/4}\,t + O(t^3)$ and $y(t) = D\,(V_1/V_2)^{1/4}\,t + O(t^3)$, the positivity requirement for these functions as $t \to 0^+$ yields the conditions $B > 0, D > 0$. Again for $t \to 0^+$, we obtain

$$a(t) = (B\,D)^{1/d}\,t^{2/d} + O\big(t^{2(d+1)/d}\big)\ , \tag{5.18}$$

$$\phi(t) = \frac{1}{d}\,\log\left(\frac{B}{D}\sqrt{\frac{V_2}{V_1}}\right) + O(t^2)\ , \tag{5.19}$$

$$w^{(\Phi)}(t) = -1 + \frac{(B^2 - D^2)^2}{9\,B\,D\,(B^2 + D^2)}\ V\,t^2 + O(t^4)\ , \tag{5.20}$$

$$\Omega_m(t) = 1 - \frac{B^2 + D^2}{2\,B\,D}\ V\,t^2 + O(t^4)\ . \tag{5.21}$$

(Similar expansions could be given in terms of the cosmic time writing $t = \tau/\theta$.) From Eq.s (5.3) (5.18) we infer that, at any time, the particle horizon is finite if $d \geqslant 3$, and infinite if $d = 2$; in the latter case the integral in (5.3) diverges logaritmically. Eq. (5.20) indicates a field equation of state close to a model with cosmological constant (recall Eq. (2.49)). Eq. (5.21) tells us that $\Omega_m(t) \to 1$; thus, differently from the previous cases, matter dominates at the Big Bang.

5.1.2 Long time behavior

We refer again to the solution (5.5), introducing the abbreviation

$$F := A + B + C + D\ . \tag{5.22}$$

We wonder under which conditions $x(t), y(t)$ are positive in a neighborhood of $+\infty$ (i.e, for t in some interval $(t_{min}, +\infty)$). In response to this, we note that we have the $t \to +\infty$ expansions $x(t) = (F/4\sqrt{V_1})\,e^{\sqrt{V}\,t} + O(1)$, $y(t) = (F/4\sqrt{V_2})\,e^{\sqrt{V}\,t} + O(1)$. Thus, $x(t), y(t) > 0$ for $t \to +\infty$ as soon as

$$F > 0\ , \tag{5.23}$$

which we assume from now on. With this assumption the solution (5.5) is defined, at least, in neighborhood of infinity and we have the following expansions for $t \to +\infty$:

$$a(t) = \left(\frac{F}{4\sqrt{V}}\right)^{2/d} e^{(2/d)\,V\,t} + O\left(e^{(2/d-1)\,V\,t}\right)\ , \tag{5.24}$$

$$\phi(t) = \frac{1}{2d} \log \frac{V_2}{V_1} + O(e^{-V\,t}) \tag{5.25}$$

$$w^{(\Phi)}(t) = -1 + O(e^{-V\,t})\ , \tag{5.26}$$

$$\Omega_m(t) = \frac{4d^2 \Omega_{m*}}{F^2}\, e^{-2\,V\,t} + O(e^{-3\,V\,t}) \tag{5.27}$$

(recall once more that $t = \tau/\theta$). Thus, for large time the scale factor diverges, the field equation of state resembles the cosmological constant case (2.49) and the scalar field dominates ($\Omega_\Phi(t) \to 1$ since $\Omega_m(t) \to 0$); all these features are attained with exponential speed.

Of course, the most interesting situation occurs if the solution (5.5) is defined on the whole interval $(0, +\infty)$, with a Big Bang at $t = 0$; this happens for suitable choices of all the involved constants.

5.2 The solution of subsection 4.1.3

First of all, let us recall that in the cited subsection we fixed $k = 0$ and we constructed a solution assuming $V_1\,(1 - w^2) < 0$ (see (4.50)); hereafter we will make the more specific assumption $V_1 < 0$, $-1 < w < 1$. Let us summarize hereafter the description of this solution arising from this stronger requirement, from Eq.s (4.1) (4.2) (4.3) (4.15) (4.56) and from the energy constraint (4.58) (which involves the constants V_1, V_2, w, A, C):

$$\mathcal{V}(\phi) = \frac{1}{d^2} \Big(V_1\, e^{2\,d\,w\,\phi} + V_2\, e^{d\,(1+w)\,\phi} \Big) \qquad (V_1 < 0,\ -1 < w < 1)\ ; \tag{5.28}$$

$$a(t) = x(t)^{\frac{1}{d\,(1+w)}}\, y(t)^{\frac{1}{d\,(1-w)}}\ , \qquad \phi(t) = \frac{1}{d}\, \log\Big(x(t)^{\frac{1}{1+w}}\, y(t)^{-\frac{1}{1-w}} \Big)\ ,$$

$$b(t) = e^{-\,d\,w\,\phi(t)} = x(t)^{-\frac{w}{1+w}}\, y(t)^{\frac{w}{1-w}}\ ;$$

$$x(t) = A\,\sin(\omega\,t)\ ,$$

$$y(t) = C\cos(\omega\,t)$$

$$+ \frac{V_2}{V_1}\, A^{\frac{1-w}{1+w}} \sin^{\frac{3+w}{1+w}}(\omega\,t) \Big[1 - \frac{2\cos(\omega\,t)}{3+w}\ {}_2F_1\Big(\frac{1}{2}, \frac{3+w}{2+2w}, \frac{5+3w}{2+2w}; \sin^2(\omega\,t) \Big) \Big]$$

$$- \frac{d^2\,\Omega_{m*}}{2\,V_1}\, A^{-\frac{1+3w}{1+w}} \sin^{\frac{1-w}{1+w}}(\omega\,t) \Big[1 + \frac{2\,w\cos(\omega\,t)}{1-w}\ {}_2F_1\Big(\frac{1}{2}, \frac{1-w}{2+2w}, \frac{3+w}{2+2w}; \sin^2(\omega\,t) \Big) \Big]$$

$$\left(A > 0\,, \quad \omega := \sqrt{\frac{(1-w^2)\,|V_1|}{2}} \right)\ .$$

As a domain for this solution, we accept a maximal subinterval of $(0, \pi/\omega)$ where $y(t) > 0$. In the present case, Eq.s (2.30) (2.31) (2.53) for the coefficient $w^{(\Phi)}$ in the field equation of state and for the dimensionless matter density Ω_m give:

$$w^{(\Phi)} = \frac{\dot{\phi}^2 - 2\,b^2\mathcal{V}(\phi)}{\dot{\phi}^2 + 2\,b^2\mathcal{V}(\phi)} = \frac{\big((1-w)\,\dot{x}\,y + (1+w)\,x\,\dot{y}\big)^2 - 2\,(1-w^2)\,\big(V_1\,x^2 y^{\frac{2(1-w)}{1+w}} + V_2\,x^{\frac{3+w}{1+w}}\,y^{\frac{3+w}{1-w}}\big)}{\big((1-w)\,\dot{x}\,y + (1+w)\,x\,\dot{y}\big)^2 + 2\,(1-w^2)\,\big(V_1\,x^2 y^{\frac{2(1-w)}{1+w}} + V_2\,x^{\frac{3+w}{1+w}}\,y^{\frac{3+w}{1-w}}\big)}\,, \tag{5.29}$$

$$\Omega_m = \frac{\Omega_{m*}\,b^2}{a^{d(w+1)-2}\,\dot{a}^2} = \frac{d^2\,\Omega_{m*}\,(1-w^2)^2\,x^{\frac{1-w}{1+w}}\,y}{\big((1-w)\,\dot{x}\,y + (1+w)\,x\,\dot{y}\big)^2}\,. \tag{5.30}$$

5.2.1 Big Bang at $t = 0$

Let us consider the expression for $a(t), x(t), y(t)$ in Eq. (5.28), and notice that $x(t) > 0$, $x(t) \to 0$ for $t \to 0^+$; taking this into account, it appears that in order to have a Big Bang at $t = 0$ we must require $y(t) > 0$ and $b(t)$ to be integrable in a right neighborhood of zero. Keeping in mind that we are assuming $-1 < w < 1$, from the explicit expression for $y(t)$ written in Eq. (4.56) we infer that $y(t) = \Big(C + \frac{d^2\,\Omega_{m*}}{2|V_1|}\,\frac{1+w}{1-w}\,A^{-\frac{1+3w}{1+w}}\,(\omega\,t)^{\frac{1-w}{1+w}}\Big)\big(1 + O(t^2)\big)$ for $t \to 0^+$; so, to fulfill the restriction $y(t) > 0$ we must assume that either $C > 0$ or $C = 0$, $\Omega_{m*} > 0$. Hereafter we proceed to the analysis of these two cases (and find b to be integrable in both of them).

The case $C > 0$. For $t \to 0^+$ we find:

$$a(t) = (A\,\omega)^{\frac{1}{d(1+w)}}\,C^{\frac{1}{d(1-w)}}\,t^{\frac{1}{d(1+w)}} + o\big(t^{\frac{1}{d(1+w)}}\big)\,, \tag{5.31}$$

$$b(t) = (A\,\omega)^{-\frac{w}{1+w}}\,C^{\frac{w}{1-w}}\,t^{-\frac{w}{1+w}} + o\big(t^{-\frac{w}{1+w}}\big)\,, \tag{5.32}$$

$$\phi(t) = \frac{1}{d(1+w)}\,\log t + \frac{1}{d}\log\Big((A\,\omega)^{\frac{1}{1+w}}\,C^{-\frac{1}{1-w}}\Big) + o(1)\,, \tag{5.33}$$

$$w^{(\Phi)}(t) = 1 - \left(\frac{2\,V_1}{(1-w)^2\,C^{\frac{4w}{1+w}}}\,t^2 + \frac{4\,d^2\,\Omega_{m*}}{A^{\frac{1+3w}{1+w}}\,C}\,\frac{1+w}{1-w}\,(\omega\,t)^{\frac{1-w}{1+w}}\right)\Big(1 + o(1)\Big)\,, \tag{5.34}$$

$$\Omega_m(t) = \frac{d^2\,\Omega_{m*}\,(1+w)^2}{C\,(A\,\omega)^{\frac{1+3w}{1+w}}}\,t^{\frac{1-w}{1+w}} + o\big(t^{\frac{1-w}{1+w}}\big) \tag{5.35}$$

(here and in the following, $o(t^\alpha)$ indicates a generic reminder term such that $o(t^\alpha)/t^\alpha \to 0$ for $t \to 0$). In view of our assumption $-1 < w < 1$, Eq. (5.32) shows that $b(t)$ is integrable for $t \to 0^+$; besides, we get

$$\tau(t) = (A\,\omega)^{-\frac{w}{1+w}}\,C^{\frac{w}{1-w}}\,(1+w)\,\theta\,t^{\frac{1}{1+w}} + o\big(t^{\frac{1}{1+w}}\big) \qquad \text{for } t \to 0^+\,, \tag{5.36}$$

$$t(\tau) = \frac{(A\,\omega)^w\, C^{-\frac{w(1+w)}{1-w}}}{(1+w)^{1+w}} \left(\frac{\tau}{\theta}\right)^{1+w} + o\left(\left(\frac{\tau}{\theta}\right)^{1+w}\right) \qquad \text{for } \tau \to 0^+ \ ,$$

Eq.s (5.31) (5.32) imply

$$\frac{b(t)}{a(t)} = (A\,\omega)^{-\frac{1+dw}{d(1+w)}}\, C^{-\frac{1-dw}{d(1-w)}}\; t^{-\frac{1+dw}{d(1+w)}} + o\big(t^{-\frac{1+dw}{d(1+w)}}\big) \qquad \text{for } t \to 0^+ \ . \tag{5.37}$$

and this fact, with Eq. (2.43), ensures finiteness of the particle horizon at each time, since $\frac{1+dw}{d(1+w)} < 1$ in our case with $d \geq 2$, $-1 < w < 1$. Eq. (5.35) indicates that $\Omega_m(t) \to 0$ for $t \to 0^+$, thus showing that the scalar field is dominant at the Big Bang. The expansions (5.31-5.35) are easily reformulated in terms of the cosmic time, using Eq. (5.36) for $t(\tau)$; for example, Eq. (5.31) gives

$$a(t) = \left(\frac{C\,A\,\omega}{1+w}\right)^{1/d} \left(\frac{\tau}{\theta}\right)^{1/d} + o\left(\left(\frac{\tau}{\theta}\right)^{1/d}\right) \qquad \text{for } \tau \to 0^+ \ . \tag{5.38}$$

The case $\boldsymbol{C = 0, \Omega_{m*} > 0}$. For $t \to 0^+$ we find:

$$a(t) = \frac{(d^2\,\Omega_{m*}\,(1+w)^2)^{\frac{1}{d(1-w)}}}{2^{\frac{2}{d(1-w)}}\,(A\,\omega)^{\frac{4w}{d(1-w^2)}}}\; t^{\frac{2}{d(1+w)}} + o\big(t^{\frac{2}{d(1+w)}}\big) \ , \tag{5.39}$$

$$b(t) = \left(\frac{d^2\,\Omega_{m*}\,(1+w)}{4\,(A\,\omega)^2}\right)^{\frac{w}{1-w}} + o(1) \ , \tag{5.40}$$

$$\phi(t) = -\,\frac{1}{d(1-w)} \log\left(\frac{d^2\,\Omega_{m*}(1+w)^2}{4\,(A\,\omega)^2}\right) + o(1) \ , \tag{5.41}$$

$$w^{(\Phi)}(t) = 1 + o(1), \tag{5.42}$$

$$\Omega_m(t) = 1 + o(1) \ . \tag{5.43}$$

Eq. (5.2) (5.40) imply

$$\tau(t) = \left(\frac{d^2\,\Omega_{m*}\,(1+w)^2}{4\,(A\,\omega)^2}\right)^{\frac{w}{1-w}} \theta\, t + o(t) \qquad \text{for } t \to 0^+, \tag{5.44}$$

$$t(\tau) = \left(\frac{d^2\,\Omega_{m*}\,(1+w)^2}{4\,(A\,\omega)^2}\right)^{-\frac{w}{1-w}} \frac{\tau}{\theta} + o\left(\frac{\tau}{\theta}\right) \ \text{ for } \tau \to 0^+,$$

On the other hand, Eq.s (5.39) (5.40) give

$$\frac{b(t)}{a(t)} = \frac{(d^2\,\Omega_{m*})^{\frac{dw-1}{d(1-w)}}}{(A\,\omega)^{\frac{2w(d(1+w)-2)}{d(1-w^2)}}} \left(\frac{2}{1+w}\right)^{\frac{2}{d(1+w)}} t^{-\frac{2}{d(1+w)}} + o\big(t^{-\frac{2}{d(1+w)}}\big) \ ; \tag{5.45}$$

this, together with Eq. (2.43), allow us to we infer that the particle horizon is finite if and only if $\frac{2}{d(1+w)} < 1$, which in our case with $d \geq 2$, $-1 < w < 1$ happens if and only if

$$w > \frac{2}{d} - 1 \ . \tag{5.46}$$

Notice that the above condition is fulfilled, in particular, in the radiation case where $w = 1/d$. Besides, Eq. (5.43) indicates that $\Omega_m(t) \to 1$ for $t \to 0^+$, so the radiation dominates at the Big Bang. The expansions (5.39-5.43) can be reformulated in terms of the cosmic time, using Eq. (5.44); for example,

$$a(\tau) = \left(\frac{d^2\,\Omega_{m*}\,(1+w)^2}{4}\right)^{\frac{1}{d(1+w)}} \left(\frac{\tau}{\theta}\right)^{\frac{2}{d(1+w)}} + o\left(\left(\frac{\tau}{\theta}\right)^{\frac{2}{d(1+w)}}\right) \quad \text{for } \tau \to 0^+. \tag{5.47}$$

5.2.2 Big Crunch

Let us wonder under which conditions we have a Big Crunch at $t = t_{cr} > 0$. Noting that $x(t) \to 0$ if and only if $t \to \frac{\pi\,n}{\omega}$, $n \in \mathbf{Z}$ and $x(t) > 0$ for all $t \in (0, \frac{\pi}{\omega})$ (in our case with $A > 0$), watching the previous expressions for $a(t)$, we see that we have a Big Crunch at $t = t_{cr} = \frac{\pi}{\omega}$ if and only if $y(t) > 0$ for $t \in (0, \frac{\pi}{\omega})$ and $b(t)$ is integrable in a left neighborhood of t_{cr} (on this point, see the comment after Eq. (5.2)). Otherwise, if $y(t) = 0$ for $t_{cr} \in (0, \frac{\pi}{\omega})$ and $b(t)$ is integrable in a left neighborhood of t_{cr}, we have a Big Crunch at $t = t_{cr} < \frac{\pi}{\omega}$.

5.3 The radiation solution of subsection 4.2.5

The cited section assumes $k = 0$, $d = 3$, $w = 1/3$. We refer to the field potential and to the solutions described by Eq.s (4.1) (with $V_1 = 0$) (4.2) (4.3) (4.89) (4.93) which are as follows (with V_1, A, C constants):

$$\mathcal{V}(\phi) = \frac{V_2}{9}\, e^{4\,\phi} \ ; \tag{5.48}$$

$$a(t) = x(t)^{1/4}\, y(t)^{1/2} \ , \quad \phi(t) = \frac{1}{4}\,\log\left(\frac{x(t)}{y(t)^2}\right) \ , \quad b(t) = e^{-\phi(t)} = \frac{y(t)^{1/2}}{x(t)^{1/4}} \ ;$$

$$x(t) = A\,t \quad (A > 0) \ , \qquad y(t) = C + \frac{8\,V_2\,\sqrt{A}}{45}\, t^{5/2} + \frac{4\,\Omega_{m*}}{A^{3/2}}\, t^{1/2} \ ;$$

The zero-energy constraint is fulfilled, and the domain of this solution is a maximal subinterval of $(0, +\infty)$ where $y(t) > 0$.
With these expressions for a, b, ϕ, Eq.s (2.30) (2.31) (2.53) for the coefficient $w^{(\Phi)}$ in the field equation of state and for the dimensionless matter density Ω_m give:

$$w^{(\Phi)} = \frac{\dot{\phi}^2 - 2\,b^2\,\mathcal{V}(\phi)}{\dot{\phi}^2 + 2\,b^2\,\mathcal{V}(\phi)} = \frac{9(\dot{x}\,y - 2\,x\,\dot{y})^2 - 32\,V_2\,x^{5/2}\,y}{9(\dot{x}\,y - 2\,x\,\dot{y})^2 + 32\,V_2\,x^{5/2}\,y} \ , \tag{5.49}$$

$$\Omega_m = \frac{\Omega_{m*}\, b^2}{a^2\, \dot{a}^2} = \frac{16\, \Omega_{m*}\, x^{1/2}\, y}{(\dot{x}\, y + 2\, x\, \dot{y})^2}\ . \tag{5.50}$$

5.3.1 Big Bang at $t = 0$

Let us wonder under which conditions we have a Big Bang at $t = 0$. Noting that $x(t) \to 0$ for $t \to 0$, watching the previous expressions for $a(t)$, we see that we have a Big Bang if and only if $y(t) > 0$ and $b(t)$ is integrable in a right neighborhood of zero (on this point, see the comment after Eq. (5.2)). From the explicit expression of y, we see that $y(t) > 0$ for $t \to 0^+$ if either $C > 0$, or $C = 0, \Omega_{m*} > 0$, or $C = 0, \Omega_{m*} = 0, V_2 > 0$. The third case with $\Omega_{m*} = 0$ (zero radiation density) is not interesting for us; in the sequel we will consider the first two cases (recalling that $A > 0$ anyway), and find that $b(t)$ is integrable for $t \to 0^+$ in both of them.

The case $C > 0$. For $t \to 0^+$ we find:

$$a(t) = A^{1/4}\, C^{1/2}\, t^{1/4} + O\big(t^{3/4}\big)\ , \tag{5.51}$$

$$b(t) = \frac{C^{1/2}}{A^{1/4}}\, t^{-1/4} + O\big(t^{1/4}\big)\ , \tag{5.52}$$

$$\phi(t) = \frac{1}{4}\, \log t + \frac{1}{4}\, \log\!\left(\frac{A}{C^2}\right) + O(t^{1/2})\ , \tag{5.53}$$

$$w^{(\Phi)}(t) = 1 - \frac{64\, V_2\, A^{1/2}}{9\, C}\, t^{5/2} + O(t^3)\ , \tag{5.54}$$

$$\Omega_m(t) = \frac{16\, \Omega_{m*}}{A^{3/2}\, C}\, t^{1/2} - \frac{192\, \Omega_{m*}^2}{A^3\, C^2}\, t + O\big(t^{3/2}\big)\ . \tag{5.55}$$

Eq. (5.52) ensures integrability of $b(t)$ near zero; from this equation and from (5.2) we get

$$\tau(t) = \frac{4\, C^{1/2}}{3\, A^{1/4}}\, \theta\, t^{3/4} + O(t^{5/4}) \text{ for } t \to 0^+, \tag{5.56}$$

$$t(\tau) = \frac{3^{4/3}\, A^{1/3}}{4^{4/3}\, C^{2/3}} \left(\frac{\tau}{\theta}\right)^{4/3} + O\!\left(\left(\frac{\tau}{\theta}\right)^2\right) \text{ for } \tau \to 0^+,$$

Eq.s (5.51) (5.52) imply $b(t)/a(t) = 1/\sqrt{A\, t} + O(1)$ for $t \to 0^+$ and this fact, with Eq. (5.3), ensures finiteness of the particle horizon at each time. Eq. (5.55) indicates that $\Omega_m(t) \to 0$ for $t \to 0^+$, so the scalar field dominates at the Big Bang. The expansions (5.51-5.55) are easily reformulated in terms of the cosmic time, using Eq. (5.56) for $t(\tau)$; for example, Eq.s (5.51) (5.56) give

$$a(\tau) = \left(\frac{3}{4}\, A\, C\right)^{1/3} \left(\frac{\tau}{\theta}\right)^{1/3} + O\left(\frac{\tau}{\theta}\right) \text{ for } \tau \to 0^+. \tag{5.57}$$

The case $C = 0, \Omega_{m*} > 0$. For $t \to 0^+$ we find:

$$a(t) = \frac{2\sqrt{\Omega_{m*}}}{\sqrt{A}}\, t^{1/2} + O\big(t^{5/2}\big) \tag{5.58}$$

$$b(t) = \frac{2\sqrt{\Omega_{m*}}}{A} + O(t^2)\ , \tag{5.59}$$

$$\phi(t) = \log\left(\frac{A}{2\sqrt{\Omega_{m*}}}\right) + O(t^2)\ , \tag{5.60}$$

$$w^{(\Phi)}(t) = 1 - \frac{16\, V_2\, A^2}{9\, \Omega_{m*}}\, t^2 + O(t^3)\ , \tag{5.61}$$

$$\Omega_m(t) = 1 - \frac{2\, V_2\, A^2}{9\, \Omega_{m*}}\, t^2 + O(t^4)\ . \tag{5.62}$$

Eq. (5.2) (5.59) imply

$$\tau(t) = \frac{2\sqrt{\Omega_{m*}}}{A}\, \theta\, t + O(t^3) \text{ for } t \to 0^+, \tag{5.63}$$

$$t(\tau) = \frac{A}{2\sqrt{\Omega_{m*}}}\, \frac{\tau}{\theta} + O\left(\left(\frac{\tau}{\theta}\right)^3\right) \text{ for } \tau \to 0^+,$$

Eq.s (5.58) (5.59) give $b(t)/a(t) = 1/(\sqrt{A\, t}\,) + O(t^{3/2})$ and this fact, with Eq. (5.3), implies finiteness of the particle horizon. Eq. (5.62) indicates that $\Omega_m(t) \to 1$ for $t \to 0^+$, so the radiation dominates at the Big Bang. The expansions (5.58-5.62) can be reformulated in terms of the cosmic time, using Eq. (5.63); for example,

$$a(\tau) = \sqrt{2}\, \Omega_{m*}^{1/4} \left(\frac{\tau}{\theta}\right)^{1/2} + O\left(\left(\frac{\tau}{\theta}\right)^{5/2}\right) \text{ for } \tau \to 0^+. \tag{5.64}$$

5.3.2 Big Crunch

Let us wonder under which conditions we have a Big Crunch at $t = t_{cr} > 0$. Noting that $x(t) \to 0$ if and only if $t \to 0$ and $x(t) > 0$ for all $t > 0$ (in our case with $A > 0$), watching the previous expressions for $a(t)$, we see that we have a Big Crunch at $t = t_{cr} > 0$ if and only if $y(t_{cr}) = 0$ and $b(t)$ is integrable in a left neighborhood of t_{cr} (on this point, see the comment after Eq. (5.2)). Note that, for semplicity, we can write the solution $y(t) = C + \frac{8\, V_2 \sqrt{A}}{45}\, t^{5/2} + \frac{4\, \Omega_{m*}}{A^{3/2}}\, t^{1/2}$ as

$$y(z) = C_1\, z^5 + C_2\, z + C\ , \qquad z := \sqrt{t}\ , \quad C_1 := \frac{8\, V_2 \sqrt{A}}{45}\ , \quad C_2 := \frac{4\, \Omega_{m*}}{A^{3/2}}\ . \tag{5.65}$$

This is a polynomial of grade five, and we can use *Descartes' rule of signs* for determining an upper bound on the number of positive or negative real roots for

this polynomial, corresponding to the presence of a possible Big Crunch.
In our set we have

$$C_2 > 0 \ , \qquad C_1 > 0 \ \text{ if and only if } \ V_2 > 0 \ . \tag{5.66}$$

Let analyse the two cases $C > 0$, or $C = 0, \Omega_{m*} > 0$ in which we have shown that we have a Big bang at $t = 0$.

The case $C > 0$.

i) If $V_2 > 0$, there are no sign changes in the coefficients of the polynomial $y(z)$, and so in this case there is not a Big Crunch.

ii) If $V_2 < 0$, there is one sign change in the coefficients of the polynomial $y(z)$, so there is exactly a (positive) Big Crunch and the solution is define for $t \in (0, t_{cr})$.

The case $C = 0, \Omega_{m*} > 0$. In this case one can write $y(z)$ as

$$y(z) = z(C_1\, z^4 + C_2) \ ; \tag{5.67}$$

i) If $V_2 > 0$, there is not a positive solution for $y(z)$, and so in this case there is not a Big Crunch.

ii) If $V_2 < 0$, there is exactly one positive (explicit) solution for $y(z)$, and so there is a Big Crunch at

$$z = \sqrt[4]{-\frac{C_2}{C_1}} \quad \to \quad t_{cr} = \frac{3}{A}\sqrt{-\frac{5\,\Omega_{m*}}{2\,V_2}} \ . \tag{5.68}$$

6 Some cases with two matter fields

We now present an integrable model including two matter fields, besides the usual scalar field. The matter fields have equations of state

$$p^{(m_1)} = w_1\, \rho^{(m_1)} \ , \tag{6.1}$$

$$p^{(m_2)} = w_2\, \rho^{(m_2)} \ , \tag{6.2}$$

and we indicate their stress-energy tensors, respectively, with ${T^{(m_1)}}_{\mu\nu}$, ${T^{(m_2)}}_{\mu\nu}$. Accordingly, the total stress-energy tensor for this model is given by $T_{\mu\nu} = {T^{(m_1)}}_{\mu\nu} + {T^{(m_2)}}_{\mu\nu} + {T^{(\Phi)}}_{\mu\nu}$. In addition, we postulate the separate conservation laws

$$\nabla_\mu {T^{(m_1)\mu}}_\nu = 0 \ , \tag{6.3}$$

$$\nabla_\mu {T^{(m_2)\mu}}_\nu = 0 \ . \tag{6.4}$$

On the one hand, the above assumptions and the Bianchi identity $\nabla_\mu {T^\mu}_\nu = 0$ imply the conservation of the field stress-energy tensor, i.e., $\nabla_\mu {T^{(\Phi)\mu}}_\nu = 0$. On the other hand, due to Eq.s (6.3) (6.4) we have two copies of Eq.s (2.33) (2.34) (2.35) with appropriate constants $w_1, \Omega_{m_1 *}$ and $w_2, \Omega_{m_2 *}$.
The Einstein equations describing the system under analysis are Eq.s (2.36) (2.37) with the term $(w\, \Omega_{m*}\, b^2)/(2a^{d(w+1)})$ replaced by the sum $(w_1\, \Omega_{m_1*}\, b^2)/(2a^{d(w_1+1)}) + (w_2\, \Omega_{m_2*}\, b^2)/(2a^{d(w_2+1)})$; the Lagrangian is

$$\mathcal{L}(a, \dot{a}, \phi, \dot{\phi}) := \tag{6.5}$$

$$\frac{1}{2\mathcal{B}(a,\phi)} \Big(- a^{d-2}\, \dot{a}^2 + a^d\, \dot{\phi}^2 \Big) - \mathcal{B}(a,\phi) \left(a^d\, \mathcal{V}(\phi) + \frac{\Omega_{m_1*}}{2\, a^{w_1 d}} + \frac{\Omega_{m_2*}}{2\, a^{w_2 d}} - \frac{k\, a^{d-2}}{2} \right) ,$$

to be compared with the Lagrangian (2.62) describing a system formed by gravity, a scalar field and a single matter field.
From now on, we make the particular choice

$$w_1 = w \ , \qquad w_2 = 0 \text{ (dust)} \ ; \tag{6.6}$$

the Lagrangian (6.5) becomes

$$\mathcal{L}(a, \dot{a}, \phi, \dot{\phi}) := \tag{6.7}$$

$$\frac{1}{2\mathcal{B}(a,\phi)} \Big(- a^{d-2}\, \dot{a}^2 + a^d\, \dot{\phi}^2 \Big) - \mathcal{B}(a,\phi) \left(a^d\, \mathcal{V}(\phi) + \frac{\Omega_{m_1*}}{2\, a^{w d}} + \frac{\Omega_{m_2*}}{2} - \frac{k\, a^{d-2}}{2} \right) .$$

Introducing a pair of coordinates $x, y > 0$ as in Eq. (4.3), and considering the choices (4.1) (4.2) for the potential $\mathcal{V}$ and the gauge function $\mathcal{B}$, the Lagrangian (6.7) becomes

$$\begin{aligned} \mathcal{L}(x,y,\dot{x},\dot{y}) &= \frac{1}{d^2}\Big(-\frac{2}{1-w^2}\,\dot{x}\,\dot{y} - V_1\,x\,y - V_2\,x^{\frac{2}{1+w}} \\ &- \frac{d^2}{2}\,\Omega_{m_1*}\,x^{-\frac{2w}{1+w}} - \frac{d^2}{2}\,\Omega_{m_2*}\,x^{-\frac{w}{1+w}}\,y^{\frac{w}{1-w}} + \frac{d^2}{2}\,k\,x^{\frac{d(1-w)-2}{d(1+w)}}\,y^{\frac{d(1+w)-2}{d(1-w)}}\Big)\,, \end{aligned} \tag{6.8}$$

and the corresponding energy function (compare with Eq. (2.64)) is

$$\begin{aligned} E(x,y,\dot{x},\dot{y}) &= \frac{1}{d^2}\Big(-\frac{2}{1-w^2}\,\dot{x}\,\dot{y} + V_1\,x\,y + V_2\,x^{\frac{2}{1+w}} \\ &+ \frac{d^2}{2}\,\Omega_{m_1*}\,x^{-\frac{2w}{1+w}} + \frac{d^2}{2}\,\Omega_{m_2*}\,x^{-\frac{w}{1+w}}\,y^{\frac{w}{1-w}} - \frac{d^2}{2}\,k\,x^{\frac{d(1-w)-2}{d(1+w)}}\,y^{\frac{d(1+w)-2}{d(1-w)}}\Big)\,, \end{aligned} \tag{6.9}$$

The Lagrange equations $\delta\mathcal{L}/\delta y = 0, \delta\mathcal{L}/\delta x = 0$ can be written, respectively, in the following way:

$$\begin{aligned} \ddot{x} - \frac{V_1\,(1-w^2)}{2}\,x \;=\; & \frac{d^2 w\,(1+w)}{4}\,\Omega_{m_2*}\,x^{-\frac{w}{1+w}}\,y^{\frac{2w-1}{1-w}} \\ & - \frac{d\,(1+w)\big(d\,(1+w)-2\big)}{4}\,k\,x^{\frac{d(1-w)-2}{d(1+w)}}\,y^{\frac{2(dw-1)}{d(1-w)}}\;, \end{aligned} \tag{6.10}$$

$$\begin{aligned} \ddot{y} - \frac{V_1\,(1-w^2)}{2}\,y \;=\; & V_2\,(1-w)\,x^{\frac{1-w}{1+w}} - \frac{d^2 w\,(1-w)}{2}\,\Omega_{m_1*}\,x^{-\frac{1+3w}{1+w}} + \\ & - \frac{d^2 w\,(1-w)}{4}\,\Omega_{m_2*}\,x^{-\frac{1+2w}{1+w}}\,y^{\frac{w}{1-w}} - \frac{d\,(1-w)\big(d\,(1-w)-2\big)}{4}\,k\,x^{-\frac{2(dw+1)}{d(1+w)}}\,y^{\frac{d(1+w)-2}{d(1-w)}}\;. \end{aligned} \tag{6.11}$$

In the particular cases

$$k \text{ arbitrary}\;, \quad w = 1/2\;, \quad d = 2 \qquad \text{(radiation gas)}\;, \tag{6.12}$$

$$k = 0\;, \quad w = 1/2\;, \quad d \text{ arbitrary}\;, \tag{6.13}$$

the system (6.10)-(6.11) is triangular. In the following we proceed to give more details on the specific cases described in Eq.s (6.12) (6.13).

6.1 The case k arbitrary, $w = 1/2\,, d = 2$

Eq.s (6.10) (6.11) are

$$\ddot{x} - \frac{3\,V_1}{8}\,x = \frac{3}{4}\,\big(\Omega_{m_2*} - k\big)\,x^{-1/3}\;, \tag{6.14}$$

$$\ddot{y} + \frac{1}{4}\Big((\Omega_{m_2 *} - k)\, x^{-\frac{4}{3}} - \frac{3\,V_1}{2} \Big)\, y = \frac{V_2}{2}\, x^{1/3} - \frac{1}{2}\, \Omega_{m_1 *}\, x^{-5/3} \ . \tag{6.15}$$

Eq. (6.14) can be reduced to quadratures; indeed, it is equivalent to

$$\ddot{x} = -\,\mathcal{U}'(x) \ , \qquad \mathcal{U}(x) := -\,\frac{3\,V_1}{16}\, x^2 - \frac{9}{8}\, (\Omega_{m_2 *} - k)\, x^{\frac{2}{3}} \ . \tag{6.16}$$

Thus,

$$\frac{1}{2}\,\dot{x}^2 + \mathcal{U}(x) = \text{const} \equiv \mathfrak{E} \quad \Rightarrow \quad \int_{x(t_1)}^{x(t_2)} \frac{dx}{\sqrt{2\,(\mathfrak{E} - \mathcal{U}(x))}} = \sigma\,(t_2 - t_1) \ , \tag{6.17}$$

if sign $\dot{x}(t) = \sigma \in \{\mp 1\}$, for all $t \in (t_2, t_1)$. Eq. (6.17) describes implicitly the general solution $x(t)$ of (6.14); the integral therein could be expressed as a combination of *elliptic integrals*.

Let us point out the existence of a special solution of Eq. (6.17) for $\mathfrak{E} = 0$ and $\Omega_{m_2 *} > k$, which is an elementary function:

$$x(t) = \left(\sqrt{\frac{6\,(\Omega_{m_2 *} - k)}{V_1}}\ \sinh\!\Big(\sqrt{\frac{V_1}{6}}\ t \Big) \right)^{3/2} .$$

Once $x(t)$ has been determined, Eq. (6.15) becomes an inhomogeneous, linear second order differential equation for $y(t)$. If $\Omega_{m_2 *} \neq k$, the coefficient of y is time dependent and the treatment of this equation is, to say the least, nontrivial.

6.1.1 The subcase k arbitrary, $w = 1/2$, $d = 2$, $V_1 = 0$, $\Omega_{m_2 *} > k$

If $V_1 = 0$, Eq. (6.17) yields

$$\int_{x(t_1)}^{x(t_2)} \frac{2\; dx}{\sqrt{8\,\mathfrak{E} + 9\,(\Omega_{m_2 *} - k)\, x^{2/3}}} \ = \sigma(t_2 - t_1) \ , \tag{6.18}$$

(recall that $\sigma = \text{sign}\dot{x}(t) \in \{\mp 1\}$ for all $t \in (t_2, t_1)$). Assuming in addition that $\Omega_{m_2 *} > k$, the integral on the l.h.s. of Eq. (6.18) can be evaluated by elementary means; this gives

$$- \left. \frac{8\,\mathfrak{E}\ \log\Big(6\,\sqrt{(\Omega_{m_2 *} - k)\,\big(8\,\mathfrak{E} + 9\,(\Omega_{m_2 *} - k)\, x^{2/3}\big)} + 18\,(\Omega_{m_2 *} - k)\, x^{1/3} \Big)}{9\,(\Omega_{m_2 *} - k)^{3/2}} \right|_{x(t_2)}^{x(t_1)}$$

$$+ \left. \frac{\sqrt{8\,\mathfrak{E} + 9\,(\Omega_{m_2 *} - k)\, x^{2/3}}}{3\,(\Omega_{m_2 *} - k)}\ x^{1/3} \right|_{x(t_2)}^{x(t_1)} = \sigma\,(t_2 - t_1) \ . \tag{6.19}$$

In general, Eq. (6.19) cannot be solved explicitly with respect to x. However, let us mention that in the case $\mathfrak{E} = 0$ we have the simple solution

$$x(t) = \big(\Omega_{m_2*} - k\big)^{3/4}\, t^{3/2}\ . \tag{6.20}$$

Substituting the above expression for $x(t)$ in Eq. (6.15) for $y(t)$ we get an explicitly solvable equation with general solution

$$y(t) = A\,\sqrt{t} + \frac{B}{2}\,\sqrt{t}\,\log t - \frac{\Omega_{m_1*}}{2\,(\Omega_{m_2*} - k)^{5/4}}\,\frac{1}{\sqrt{t}} + \frac{V_2}{8}\,\big(\Omega_{m_2*} - k\big)^{1/4}\, t^{5/2}\ . \tag{6.21}$$

It can be checked that the zero-energy constraint $E(x(t), \dot{x}(t), y(t), \dot{y}(t)) = 0$ (with $E(x, y, \dot{x}, \dot{y})$ as in Eq. (6.9)) is fulfilled if and only if

$$B = 0\ . \tag{6.22}$$

Introducing the constant

$$\Upsilon_1 := \big(\Omega_{m_2*} - k\big)^{1/4} > 0 \tag{6.23}$$

and implementing the zero-energy constraint (6.22), we can write $x(t), y(t)$ as

$$x(t) = \Upsilon_1^3\, t^{3/2}\ , \tag{6.24}$$

$$y(t) = \frac{1}{8\,\Upsilon_1^5 \sqrt{t}}\Big(V_2\,\Upsilon_1^6\, t^3 + 8\,\Upsilon_1^5 A\, t - 4\,\Omega_{m_1*}\Big)\ . \tag{6.25}$$

Eq.s (6.24) (6.25) make evident that $x(t) > 0$ as soon as $t > 0$ and that the positivity of $y(t)$ depends, in particular, on the sign of the parameters A, V_2. As well known from Cardano's formula (see, e.g., [17, §10.2, Ex.s 10.14 and 10.17]), the cubic polynomial

$$\alpha\, t^3 + \beta\, t^2 + \gamma\, t + \delta\ , \tag{6.26}$$

has three complex roots $t = t_i$ $(i = 1, 2, 3)$, where

$$t_1 = S + T - \frac{\beta}{3\alpha}\ , \tag{6.27}$$

$$t_2 = -\,\frac{S + T}{2} - \frac{\beta}{3\alpha} + i\,\frac{\sqrt{3}}{2}\,(S - T)\ , \tag{6.28}$$

$$t_3 = -\,\frac{S + T}{2} - \frac{\beta}{3\alpha} - i\,\frac{\sqrt{3}}{2}\,(S - T)\ , \tag{6.29}$$

$$S = \Big(R + \sqrt{Q^3 + R^2}\Big)^{1/3}\ , \qquad T = \Big(R - \sqrt{Q^3 + R^2}\Big)^{1/3}\ , \tag{6.30}$$

$$Q = \frac{3\alpha\gamma - \beta^2}{9\,\alpha^2}\ , \qquad R = \frac{9\alpha\beta\gamma - 27\alpha^2\delta - 2\,\beta^3}{54\alpha^3}\ . \tag{6.31}$$

Introducing the discriminant

$$\Delta = 18\,\alpha\beta\gamma\delta - 4\,\beta^3\delta + \beta^2\gamma^2 - 4\,\alpha\gamma^3 - 27\,\alpha^2\delta^2\ . \tag{6.32}$$

We can distinguish three cases:

i) If $\Delta > 0$, Eq. (6.26) has three distinct real roots;

ii) If $\Delta = 0$, Eq. (6.26) has one multiple, real root;

iii) If $\Delta < 0$, Eq. (6.26) has one real root and two complex conjugate roots.

In our case, the discriminant of the cubic polynomial $V_2\,\Upsilon_1^6\,t^3 + 8\,\Upsilon_1^5 A\,t - 4\,\Omega_{m_1*}$ is

$$\Delta = -16\,\Upsilon_1^{12}\,V_2\,(128\,\Upsilon_1^9 A^3 + 27\,V_2\,\Omega_{m_1*}^2)\ . \tag{6.33}$$

Recalling that $\Upsilon_1 > 0$ and $\Omega_{m_1*} \geq 0$, we have

i) $\Delta > 0$ if and only if one of two following conditions holds:

$$V_2 > 0 \qquad \text{and} \qquad A < -\frac{3}{4\Upsilon_1^3}\left(\frac{V_2\,\Omega_{m_1*}^2}{2}\right)^{1/3}\ , \tag{6.34}$$

$$V_2 < 0 \qquad \text{and} \qquad A > -\frac{3}{4\Upsilon_1^3}\left(\frac{V_2\,\Omega_{m_1*}^2}{2}\right)^{1/3}\ ; \tag{6.35}$$

ii) $\Delta = 0$ if and only if

$$V_2 = 0 \qquad \text{and/or} \qquad A = -\frac{3}{4\Upsilon_1^3}\left(\frac{V_2\,\Omega_{m_1*}^2}{2}\right)^{1/3}\ ; \tag{6.36}$$

iii) $\Delta < 0$ if and only if

$$V_2 > 0 \qquad \text{and} \qquad A > -\frac{3}{4\Upsilon_1^3}\left(\frac{V_2\,\Omega_{m_1*}^2}{2}\right)^{1/3}\ ; \tag{6.37}$$

$$V_2 < 0 \qquad \text{and} \qquad A < -\frac{3}{4\Upsilon_1^3}\left(\frac{V_2\Omega_{m_1*}^2}{2}\right)^{1/3}\ ; \tag{6.38}$$

Let us consider, for example, the case (6.37). From Eq. (6.27), we infer

$$t_1 := \left(R + \sqrt{Q^3 + R^2}\right)^{1/3} + \left(R - \sqrt{Q^3 + R^2}\right)^{1/3}, \quad Q = \frac{8\,A}{3\,V_2\Upsilon_1}\ , \quad R = \frac{2\,\Omega_{m_1*}}{V_2\Upsilon_1^6}\ ; \tag{6.39}$$

From Eq. (6.39) it is easy to see that $t_1 > 0$ if and only if

$$Q = 0 \text{ and } R > 0 \qquad \text{or} \qquad Q < 0 \text{ and } R \geq \sqrt{-Q^3}\ , \tag{6.40}$$

that means, recalling Eq. (6.37)

$$A = 0 \text{ and } \Omega_{m_1*} \neq 0 \qquad \text{or} \qquad -\frac{3}{4\Upsilon_1^3}\left(\frac{V_2\,\Omega_{m_1*}^2}{2}\right)^{1/3} < A < 0\ , \tag{6.41}$$

To go on, let us recall that in this case we have k arbitrary, $d = 2, w = 1/2$, $V_1 = 0$ and $\Omega_{m_2*} > k$. Then, according to Eq.s (2.52) (2.53), in the case with two matter fields with Lagrangian function (6.7), we get

$$\begin{aligned}\Omega_{m1} &:= \frac{2\,\gamma_d\, G_d\, \rho^{(m_1)}}{H^2} = \Omega_{m_1*}\, \frac{b^2}{a\,\dot{a}^2}\ , \qquad \Omega_{m2} := \frac{2\,\gamma_d\, G_d\, \rho^{(m_2)}}{H^2} = \Omega_{m_2*}\, \frac{b^2}{\dot{a}^2}\ , \\ \Omega_{\Phi} &:= \frac{2\,\gamma_d\, G_d\, \rho^{(\Phi)}}{H^2} = \big(\dot{\phi}^2 + 2\, b^2\, \mathcal{V}(\phi)\big)\, \frac{a^2}{\dot{a}^2}\ , \qquad \Omega_k := -\, \frac{k}{\theta^2 H^2\, a^2} = -\, \frac{k\, b^2}{\dot{a}^2}\ ,\end{aligned} \tag{6.42}$$

with a, $b \equiv \mathcal{B}(\phi)$ as in Eq.s (4.3) (4.2), respectively. This gives

$$\Omega_{m1} = \frac{9\,\Omega_{m_1*}\, x^{1/3}\, y}{(\dot{x}\, y + 3\, x\, \dot{y})^2}\ , \qquad \Omega_{m2} = \frac{9\,\Omega_{m_2*}\, x^{2/3}\, y^2}{(\dot{x}\, y + 3\, x\, \dot{y})^2}\ , \tag{6.43}$$

$$\Omega_{\Phi} = \frac{9\, V_2\, x^{7/3}\, y + 2\, y^2\, \dot{x}^2 - 12\, x\, y\, \dot{x}\, \dot{y} + 18\, x^2\, \dot{y}^2}{2(\dot{x}\, y + 3\, x\, \dot{y})^2}\ , \qquad \Omega_k = -\, \frac{9\, k\, x^{2/3}\, y^2}{(\dot{x}\, y + 3\, x\, \dot{y})^2}\ ,$$

with $x(t), y(t)$ as in Eq. (6.24) (6.25). In view of the related Eq. (4.3), with the constraint given by (6.37) (6.41), we have $x(t), y(t) > 0$ for $t \in I = (t_1, +\infty)$; it appears that a Big Bang singularity occurs at $t = t_1$, with t_1 as in Eq. (6.39).
In the following we analyze the limit

$$\tilde{t} := t - t_1 \to 0^+\ . \tag{6.44}$$

From Eq.s (6.24) (6.25), for $\tilde{t} \to 0^+$ we infer

$$\begin{aligned} x(t) &= c_1 + O\big(\tilde{t}\big)\ , \qquad c_1 := \Upsilon_1^3\, t_1^{3/2}\ , \\ y(t) &= c_2\, \tilde{t} + O\big(\tilde{t}^2\big) \to 0\ , \qquad c_2 := \frac{A}{\sqrt{t_1}} + \frac{3}{8}\, \Upsilon_1\, V_2\, t_1^{\frac{3}{2}}\ ,\end{aligned} \tag{6.45}$$

and so

$$\begin{aligned} \Omega_{m1}(\tilde{t}) &= \frac{\Omega_{m_1*}}{c_1^{5/3}\, c_2}\ \tilde{t} + O(\tilde{t}^2) \to 0\ , \qquad \Omega_{m2}(\tilde{t}) = \frac{\Omega_{m_2*}}{c_1^{4/3}}\ \tilde{t}^2 + O(\tilde{t}^3) \to 0\ , \\ \Omega_{\Phi}(\tilde{t}) &= 1 + O\big(\tilde{t}\big) \to 1\ , \qquad \Omega_k(\tilde{t}) = -\, \frac{k}{c_1^{4/3}}\ \tilde{t}^2 + O(\tilde{t}^3) \to 0\ ,\end{aligned} \tag{6.46}$$

showing that the scalar field dominates for small times. Furthermore, for $t \to 0^+$ we have

$$b(t) \equiv \mathcal{B}(\phi(t)) = \frac{c_2}{c_1^{1/3}}\ \tilde{t} + O(\tilde{t}^2) \to 0\ , \tag{6.47}$$

$$a(t) = c_1^{1/3} c_2\ \tilde{t} + O(\tilde{t}^2) \to 0\ , \tag{6.48}$$

whence

$$\frac{b(t)}{a(t)} = \frac{1}{c_1^{2/3}} + O(\tilde{t})\ . \tag{6.49}$$

From here and from Eq. (2.43) we infer that the particle horizon at any given time t_1 is finite.

6.1.2 The subcase k arbitrary, $w = 1/2$, $d = 2$, $\Omega_{m_2*} = k$.

With the additional assumption $\Omega_{m_2*} = k$, Eq.s (6.14) (6.15) read

$$\ddot{x} - \frac{3\,V_1}{8}\,x = 0 \ , \tag{6.50}$$

$$\ddot{y} - \frac{3\,V_1}{8}\,y = \frac{V_2}{2}\,x^{1/3} - \frac{1}{2}\,\Omega_{m_1*}\,x^{-5/3} \ . \tag{6.51}$$

Recalling that we are assuming with $w = 1/2$ and $d = 2$, it can be easily checked that Eq.s (6.50) (6.51) coincide with Eq.s (4.16) (4.17). Due to this fact, we refer to subsection 4.1 for the analysis of the solutions.

6.2 The case $k = 0$, $w = 1/2$, d arbitrary.

In this case, Eq.s (6.10) (6.11) are equivalent to

$$\ddot{x} - \frac{3\,V_1}{8}\,x \ = \ \frac{3\,d^2}{16}\,\Omega_{m_2*}\,x^{-1/3} \ , \tag{6.52}$$

$$\ddot{y} + \frac{d^2}{16}\left(\Omega_{m_2*}\,x^{-\frac{4}{3}} - \frac{6\,V_1}{d^2}\right)y \ = \ \frac{V_2}{2}\,x^{1/3} - \frac{d^2}{8}\,\Omega_{m_1*}\,x^{-5/3} \ . \tag{6.53}$$

Of course, for $d = 2$ the above pair (6.52) (6.53) is equivalent to Eq.s (6.14) (6.15).

7 A fully quantitative analysis of one of the previous cases

In this section, we reconsider the model of section 3 and fix all the constants it contains so as to make contact with physical reality.

7.1 The dust model of Section 3

For the moment we consider any $d \geqslant 2$; later on we will focus on the case $d = 3$. Moreover, recall that $b = \mathcal{B}(a, \phi) = 1$, so that the parametric time t and the proper time τ are related by $\tau = \theta\, t$.
It appears that the general solution described in Section 3 depends on the (so far unspecified) parameters $\theta, V_1, V_2, A, B, C, D, \Omega_{m*}$.
First of all, let us recall that the above mentioned solution of Section 3 possesses a Big Bang singularity at $t = 0$ if and only if $A\,C = 0$ (see Eq. (5.9)); to say more, in order to have a dominant matter contribution near the Big Bang (i.e., $\Omega_m(t) \to 1$ for $t \to 0^+$), we must assume that (see subsection 5.1.1 and, in particular, Eq. (5.21))

$$A = 0 \qquad \text{and} \qquad C = 0 \; .$$

To proceed let us remark that, on account of the gauge invariance $\phi \mapsto \phi + \text{const.}$, we can always reduce to the analysis of the case where (recall that $V := \sqrt{V_1\, V_2}$)

$$V_1 = V_2 = V \; ; \tag{7.1}$$

then, the potential (3.1) reads

$$\mathcal{V}(\phi) = \frac{V}{d^2}\left(e^{d\,\phi} + e^{-d\,\phi}\right) = \frac{2\,V}{d^2}\; \cosh(d\,\phi) \; . \tag{7.2}$$

The corresponding solution is

$$x(t) = \frac{1}{2\sqrt{V}}\left[(B+D)\sinh\left(\sqrt{V}\,t\right) + (B-D)\sin\left(\sqrt{V}\,t\right)\right] \; , \tag{7.3}$$

$$y(t) = \frac{1}{2\sqrt{V}}\left[(B+D)\sinh\left(\sqrt{V}\,t\right) - (B-D)\sin\left(\sqrt{V}\,t\right)\right] \; , \tag{7.4}$$

$$a(t) = x(t)^{\frac{1}{d}}\, y(t)^{\frac{1}{d}} = \left[\frac{(B+D)^2\sinh^2\left(\sqrt{V}\,t\right) - (B-D)^2\sin^2\left(\sqrt{V}\,t\right)}{4\,V}\right]^{\frac{1}{d}} \; , \tag{7.5}$$

$$\phi(t) = \frac{1}{d}\,\log\!\left(\frac{x(t)}{y(t)}\right) = \frac{1}{d}\,\log\!\left(\frac{(B+D)\sinh\left(\sqrt{V}\,t\right) + (B-D)\sin\left(\sqrt{V}\,t\right)}{(B+D)\sinh\left(\sqrt{V}\,t\right) - (B-D)\sin\left(\sqrt{V}\,t\right)}\right) \; , \tag{7.6}$$

$$\Omega_m(t) = \frac{d^2\,\Omega_{m*}\,x\,y}{(x\,\dot{y}+\dot{x}\,y)^2} = d^2\,\Omega_{m*}\,\frac{(B+D)^2\sinh^2\left(\sqrt{V}\,t\right)-(B-D)^2\sin^2\left(\sqrt{V}\,t\right)}{\left[(B+D)^2\sinh\left(2\sqrt{V}\,t\right)-(B-D)^2\sin\left(2\sqrt{V}\,t\right)\right]^2}\ , \tag{7.7}$$

$$w^{(\Phi)}(t) = \frac{(\dot{x}\,y-x\,\dot{y})^2-2\,(V_1\,x^3y+V_2\,x\,y^3)}{(\dot{x}\,y-x\,\dot{y})^2+2\,(V_1\,x^3y+V_2\,x\,y^3)} = \frac{N(t)}{D(t)}\ , \tag{7.8}$$

$$\begin{aligned} N(t) := {}& \sin^2\left(\sqrt{V}\,t\right)\left[(B^2-D^2)^2\cosh^2\left(\sqrt{V}\,t\right)+(B-D)^4\sin^2\left(\sqrt{V}\,t\right)\right] \\ &+\sinh^2\left(\sqrt{V}\,t\right)\left[(B^2-D^2)^2\cos^2\left(\sqrt{V}\,t\right)-(B+D)^4\sinh^2\left(\sqrt{V}\,t\right)\right] \\ &-\frac{1}{2}(B^2-D^2)^2\sin\left(2\sqrt{V}\,t\right)\sinh\left(2\sqrt{V}\,t\right)\ , \\ D(t) := {}& (B^2-D^2)^2\left[\cosh\left(\sqrt{V}\,t\right)\sin\left(\sqrt{V}\,t\right)-\cos\left(\sqrt{V}\,t\right)\sinh\left(\sqrt{V}\,t\right)\right]^2 \\ &+(B+D)^4\sinh^4\left(\sqrt{V}\,t\right)-(B-D)^4\sin^4\left(\sqrt{V}\,t\right)\ , \end{aligned}$$

$$\mathcal{V}(t) = \frac{2\,V}{d^2}\ \cosh\left(\log\left(\frac{x(t)}{y(t)}\right)\right) = \frac{2\,V}{d^2}\ \frac{(B+D)^2\sinh^2\left(\sqrt{V}\,t\right)+(B-D)^2\sin^2\left(\sqrt{V}\,t\right)}{(B+D)^2\sinh^2\left(\sqrt{V}\,t\right)-(B-D)^2\sin^2\left(\sqrt{V}\,t\right)} \tag{7.9}$$

and the zero-energy constraint (3.14) yields

$$B\,D = \frac{d^2\,\Omega_{m*}}{4}\ . \tag{7.10}$$

Next, we introduce a reference time t_* and set

$$\begin{cases} a(t_*) = 1\ , \\ \phi(t_*) = \phi_*\ , \\ \theta = 1/H(t_*) \end{cases} \tag{7.11}$$

where ϕ_* is an arbitrary parameter; in the following we will discuss various reasonable choices of ϕ_*.
Recalling the expressions of a, ϕ and H in terms of the Lagrangian variables x, y, the above conditions (7.11) imply

$$\begin{cases} x(t_*)^{1/d}y(t)^{1/d} = 1\ , \\ x(t_*)/y(t_*) = e^{d\,\phi_*}\ , \\ \dot{x}(t_*)\,y(t_*)+x(t_*)\,\dot{y}(t_*) = d\,x(t_*)^{\frac{d-1}{d}}\,y(t_*)^{\frac{d-1}{d}}\ . \end{cases} \tag{7.12}$$

These allow us to infer that

$$\begin{cases} x(t_*) = e^{\frac{d}{2}\,\phi_*}\ , \\ y(t_*) = e^{-\frac{d}{2}\,\phi_*}\ , \\ e^{-\frac{d}{2}\,\phi_*}\,\dot{x}(t_*)+e^{\frac{d}{2}\,\phi_*}\,\dot{y}(t_*) = d\ . \end{cases} \tag{7.13}$$

Considering the explicit expressions (7.3) (7.4) for $x(t_*)$ and $y(t_*)$, it appears that the first two relations in Eq. (7.13) can be viewed as a linear system for the two unknowns B, D; introducing the notation

$$s_* := \sqrt{V}\, t_* \tag{7.14}$$

and solving the said linear system yields

$$B = \sqrt{V}\left[\frac{\cosh(\frac{d}{2}\,\phi_*)}{\sinh(s_*)} + \frac{\sinh(\frac{d}{2}\,\phi_*)}{\sin(s_*)}\right], \quad D = \sqrt{V}\left[\frac{\cosh(\frac{d}{2}\,\phi_*)}{\sinh(s_*)} - \frac{\sinh(\frac{d}{2}\,\phi_*)}{\sin(s_*)}\right]. \tag{7.15}$$

Substituting the above expressions of B, D in the zero-energy constraint (7.10) and solving for V, we obtain

$$V = \frac{d^2\,\Omega_{m*}\ \sinh^2(s_*)\sin^2(s_*)}{4\big[\cosh^2(\frac{d}{2}\,\phi_*)\sin^2(s_*) - \sinh^2(\frac{d}{2}\,\phi_*)\sinh^2(s_*)\big]}\ . \tag{7.16}$$

Finally, the above relations (7.15) (7.16) and the last identity in Eq. (7.13) give

$$\begin{gathered}\cosh^2\left(\frac{d}{2}\,\phi_*\right)\cosh(s_*)\sin(s_*) - \sinh^2\left(\frac{d}{2}\,\phi_*\right)\sinh(s_*)\cos(s_*) = \\ \frac{1}{\sqrt{\Omega_{m*}}}\left[\cosh^2\left(\frac{d}{2}\,\phi_*\right)\sin^2(s_*) - \sinh^2\left(\frac{d}{2}\,\phi_*\right)\sinh^2(s_*)\right]^{1/2}\end{gathered} \tag{7.17}$$

For assigned values of d, ϕ_*, Ω_{m*}, one can look for a solution (assuming it exists) of the above equality by numerical methods.

7.1.1 The case $\phi_* = 0$

In this special case, corresponding to the minimum of the potential (7.2), Eq. (7.17) reduces to

$$\cosh(s_*) = \frac{1}{\sqrt{\Omega_{m*}}} \tag{7.18}$$

which, for $\Omega_{m*} < 1$, can be resolved analytically, yielding

$$s_* = \operatorname{arccosh}\left(\frac{1}{\sqrt{\Omega_{m*}}}\right) \tag{7.19}$$

(note that this solution does not depend on the dimension d).
Substituting the above solution in the explicit expressions (7.15) (7.16) we obtain

$$B = D = \frac{d\sqrt{\Omega_{m*}}}{2}\ , \qquad V = \frac{d^2\,(1-\Omega_{m*})}{4}\ . \tag{7.20}$$

Besides, we have

$$t_* = \frac{2}{d\sqrt{(1-\Omega_{m*})}} \operatorname{arccosh}\left(\frac{1}{\sqrt{\Omega_{m*}}}\right) . \tag{7.21}$$

Note that this solution is in agreement with the solution described in Appendix E in the case of one matter field and a cosmological term: in fact, in the case $\phi_* = 0$, the scalar field behaves like a cosmological constant (in particular, Eq. (7.21) is exactly the same of Eq. (E.43)).
To make contact with reality, we fix $d = 3$ and choose the matter density parameter as (see, e.g., [23, p. 252])

$$\Omega_{m*} := 0.308 . \tag{7.22}$$

Then, the previous relations give

$$s_* = 1.19416 , \quad B = D = 0.832466 , \quad V = 1.557 , \quad t_* = 0.957016 . \tag{7.23}$$

In particular, if we fix (see, e.g., [23])

$$H(t_*) := H_* \simeq 67.74 \frac{Km}{Mpc \cdot s} \simeq 2.1953 \times 10^{-18} \, s^{-1} \tag{7.24}$$

one finds out that the age of the universe is

$$\theta \, t_* = \frac{t_*}{H_*} \simeq 13.8 \times 10^9 \, \text{years} . \tag{7.25}$$

7.1.2 The case $\phi_* = 1/4$

In this case, once we have fixed $d = 3, \Omega_{m*} := 0.308$ as in the previous case, Eq. (7.17) depends only on s_* and it can be solved numerically, giving

$$s_* \simeq 1.00344 . \tag{7.26}$$

Substituting the above solution in the explicit expressions (7.15) (7.16) we obtain

$$B \simeq 1.44972 , \qquad D \simeq 0.481126 , \qquad V \simeq 1.13212 . \tag{7.27}$$

Besides, from Eq. (7.14) we have

$$t_* \simeq 0.943076 , \tag{7.28}$$

that correspond to an age of the universe of about (with H_* as in Eq. (7.24))

$$\theta \, t_* = \frac{t_*}{H_*} \simeq 13.5 \times 10^9 \, \text{years} . \tag{7.29}$$

7.1.3 Plotting the dust solutions in the two cases $\phi_* = 0, \phi_* = 1/4$

In what follows, we have represented the dimensionless scale factor $a(t)$, the equation of state coefficient of the scalar field $w^{(\Phi)}(t)$, the time dependent matter density parameter $\Omega_m(t)$ and the scalar field $\phi(t)$ in the two previous cases corresponding to the choices $\phi_* = 0$ and $\phi_* = 1/4$. In figures 1, 2 we have represented the scale factor $a(t)$ against two different time intervals; at the beginning one has $a(0) = 0$ (Big Bang), than we have an initial phase where the expansion of the universe is decelerating, followed by a phase where the expansion of the universe become accelerated and in the future it continues in this way.

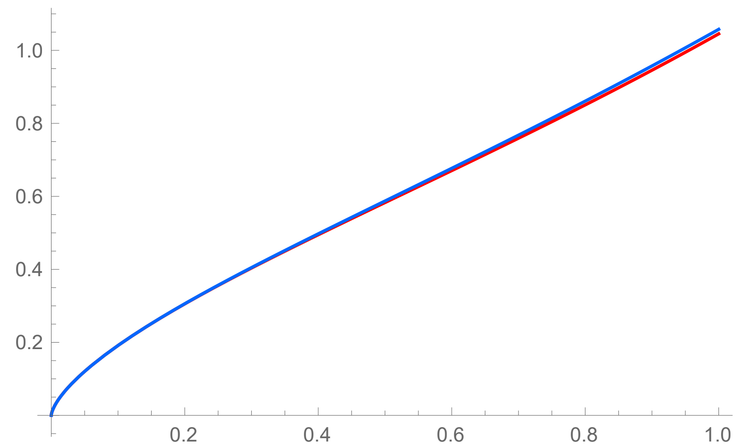

Figure 1: $a(t)$ in the two cases $\phi_* = 0$, $\phi_* = \frac{1}{2}$

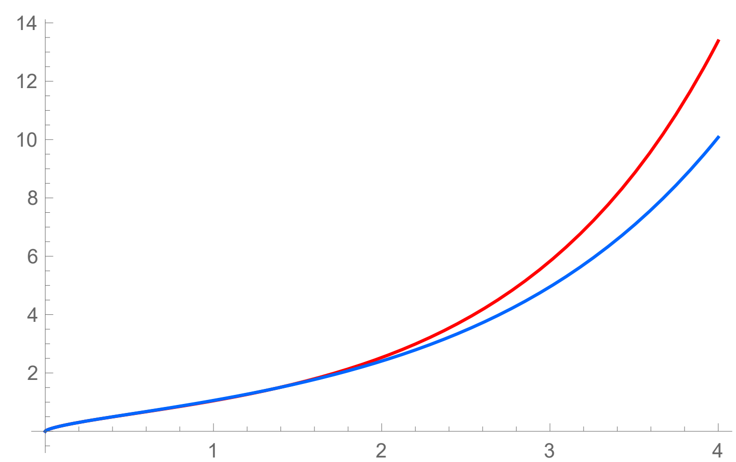

Figure 2: $a(t)$ in the two cases $\phi_* = 0$, $\phi_* = \frac{1}{2}$

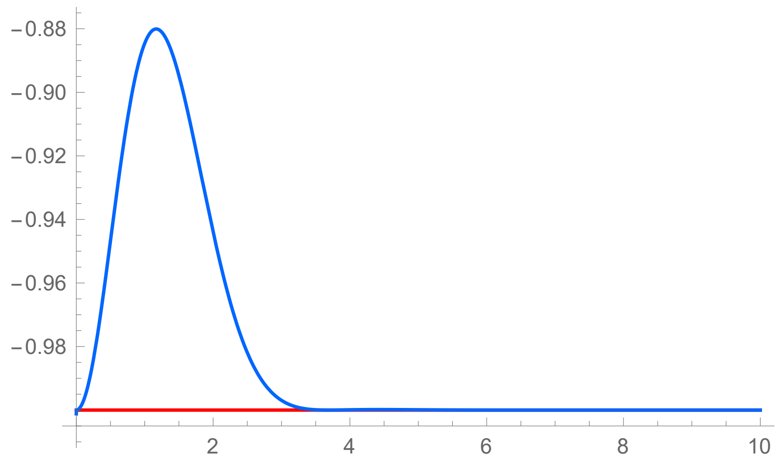

Figure 3: $w^{(\Phi)}(t)$ in the two cases $\phi_* = 0$, $\phi_* = \frac{1}{2}$

In Figure 3 we have plotted the equation of state coefficient of the scalar field

$$w^{(\Phi)} = \frac{p^{(\Phi)}}{\rho^{(\Phi)}} = \frac{\dot{\phi}^2 - 2\mathcal{V}(\phi)}{\dot{\phi}^2 + 2\mathcal{V}(\phi)} \ .$$

Note that, in the case with $\phi_* = 0$, one has $w^{(\Phi)} \equiv -1$, as expected (recall that in this case the scalar field behaves like a cosmological constant); besides, in the case $\phi_* = 1/4$, $w^{(\Phi)} \to -1$ for $t \to 0$, at the present time we have $w^{(\Phi)}(t_*) \simeq -0.89$ and $w^{(\Phi)} \to -1$ for $t \to \infty$; so, in the future the scalar field continues forever to behave as a cosmological constant (see the discussion after Eq. (2.31)).

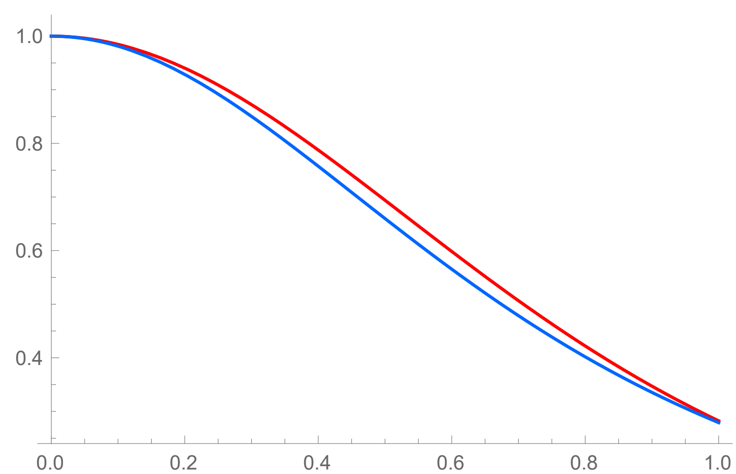

Figure 4: $\Omega_m(t)$ in the two cases $\phi_* = 0$, $\phi_* = \frac{1}{2}$

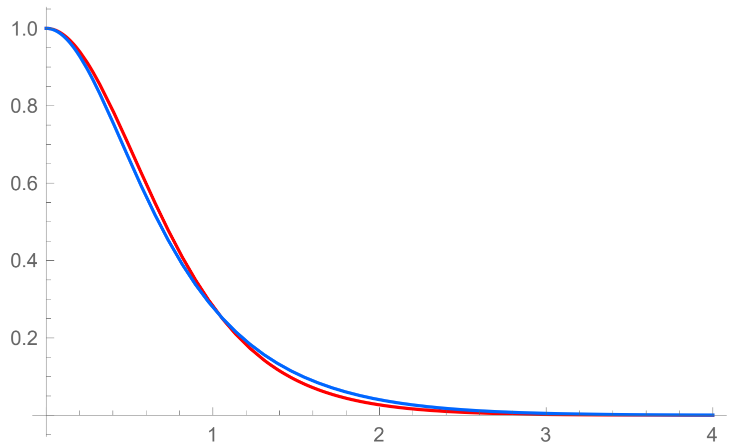

Figure 5: $\Omega_m(t)$ in the two cases $\phi_* = 0$, $\phi_* = \frac{1}{2}$

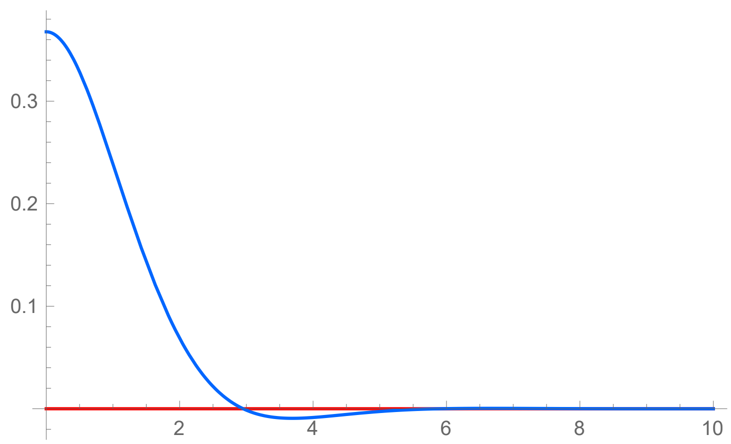

Figure 6: $\phi(t)$ in the two cases $\phi_* = 0$, $\phi_* = \frac{1}{2}$

In Figures 4, 5 we have plotted the time dependent matter density parameters

$$\Omega_m(t) = \frac{\Omega_{m*}}{a^{d-2}\,\dot{a}^2} \ ,$$

against two different time intervals. As made apparent by the graphs, the universe is initially filled almost exclusively with matter ($\Omega_m \to 1, \Omega_\Phi \to 0$ for $t \to 0$); matter continues to dominate over dark energy until $t = \bar{t}$ when $\Omega_m(\bar{t}) = \Omega_\Phi(\bar{t}) = 0.5$, where $\bar{t} = 0.70736 \to \bar{\tau} := \frac{\bar{t}}{H_*} \simeq 10.21 \times 10^9$ years for $\phi_* = 0$, $\bar{t} = 0.67422 \to \bar{\tau} := \frac{\bar{t}}{H_*} \simeq 9.73 \times 10^9$ years for $\phi_* = 1/4$. This is in agreement with the results obtained in Appendix F in the case of the benchmark model. At the present time $\Omega_m(t_*) = 0.31$ ($\Omega_\Phi(t_*) = 0.69$). In the future dark energy continues to dominate and completely fills the universe ($\Omega_m(t) \to 0, \Omega_\Phi(t) \to 1$ for $t \to \infty$). In Figure 6 we have plotted the dimensionless scalar field ϕ. Note that, in the case with $\phi_* = 0$, one has $\phi(t) = \text{const.} \equiv 0$ (recall that in this case the scalar field behaves like a cosmological constant, see subsection 2.7); in the case $\phi_* = 1/4$, $\phi \to 0.3676$ for $t \to 0$, at the present time we have $\phi(t_*) \simeq 0.25$ and $\phi \to 0$ for $t \to \infty$.

7.1.4 Some considerations on the potentials

Both in the cases $\phi_* = 0, \phi_* = 1/4$, the potentials are qualitatively not so far from the phenomenological self-interaction potential reconstructed by Saini, Raychaudhury, Sahni and Starobinsky [31] (i.e FIG. 3.) starting from the empirical redshift-luminosity distance curve [24]. To make a quantitave comparison, it is necessary to clarify the relations between the dimensionless scalar field ϕ and its potential $\mathcal{V}(\phi)$ of this work and the ones of the paper of [31]. All the necessary informations are provided in the Appendix G.

A Appendix. The gravitational constant

Let us first recall that, in d-dimensional ($d \geqslant 2$) Newtonian gravity, the gravitational constant G_d is fixed by the following requirement: the gravitational field $\vec{\mathfrak{G}}$ produced at distance r by a point mass M ought to fulfil (see, e.g., [21])

$$|\vec{\mathfrak{G}}| = \frac{G_d\, M}{r^{d-1}} \ .$$

In presence of an arbitrary mass distribution of density ρ, one has

$$\vec{\mathfrak{G}} = -\,\mathrm{grad}\,\varphi \ ,$$

where the gravitational potential field φ fulfills the Poisson equation

$$\Delta\,\varphi = \frac{2\,\pi^{\frac{d}{2}}\,G_d}{\Gamma(d/2)}\,\rho \ . \tag{A.1}$$

Next, let us pass to general relativity and write Einstein's equations as in Eq. (2.4), i.e.,

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,R = d(d-1)\,\gamma_d\,G_d\,T_{\mu\nu} \ ;$$

here, G_d is the gravitational constant and γ_d is some numerical coefficient that we proceed to determine below, requiring Einstein's equations to reproduce Eq. (A.1) in a suitable Newtonian regime.

We already mentioned in Section 2 that Einstein's equations do not possess any sensible Newtonian limit in the case of space dimension $d = 2$ (see, e.g., [15]); because of this, for $d = 2$ the value of γ_d can be chosen arbitrarily (see Eq. (2.6)).

On account of the above remarks, in the following we fix $d \geqslant 3$ and consider the stationary, almost-flat case described by the line element

$$ds^2 = -\big(1 + 2\varphi(\mathbf{x})\big)\,dt^2 + \big(\delta_{ij} + \psi_{ij}(\mathbf{x})\big)\,dx^i dx^j \ ,$$

where $\varphi(\mathbf{x})$, $\psi_{ij}(\mathbf{x})$ are small in a suitable sense. By analysis of geodetic motions, φ is found to be an analogue of the Newtonian potential. More precisely, let us assume the universe to be filled with a dust of (small) density ρ, at rest in the coordinate system $(t, \mathbf{x})$, so that

$$T_{\mu\nu} = \mathrm{diag}(\rho, \underbrace{0, \ldots, 0}_{d-times}) \ .$$

Recalling our assumption $d \geqslant 3$ and linearizing the Einstein's equations (2.4) (to first order in φ, $\psi_{i,j}$, ρ), we obtain

$$\Delta\,\varphi = d\,(d-2)\,\gamma_d\,G_d\,\rho \ , \tag{A.2}$$

which agrees with the Newtonian analogue written in Eq. (A.1) if we fix γ_d as in Eq. (2.5), i.e.,

$$\gamma_d = \frac{\pi^{d/2}}{(d-2)\,\Gamma(d/2+1)}$$

(note that $(d/2)\,\Gamma(d/2) = \Gamma(d/2+1)$).

B Appendix. On the particle horizon

B.1 Preliminaries

Let us recall the following definitions:

i) a *curve* in any manifold $\mathcal{M}$ is a smooth map $\lambda : [\sigma_0, \sigma_1] \to \mathcal{M}$, $\sigma \rightsquigarrow \lambda(\sigma)$ defined on a compact interval;

ii) if $\mathcal{M}$ is a spacetime of any dimension $d{+}1$, a *causal curve* in $\mathcal{M}$ is a smooth map $\lambda : [\sigma_0, \sigma_1] \to \mathcal{M}$, $\sigma \rightsquigarrow \lambda(\sigma)$ such that the tangent vector $\lambda'(\sigma)$ is causal for each $\sigma \in [\sigma_0, \sigma_1]$ (here and in the sequel, $'$ is the derivative). In addition, if $\mathcal{M}$ is time oriented, a *future directed causal curve* is a casual curve $\lambda : [\sigma_0, \sigma_1] \to \mathcal{M}$, $\sigma \rightsquigarrow \lambda(\sigma)$ such that $\lambda'(\sigma)$ is future directed for each $\sigma \in [\sigma_0, \sigma_1]$.

Let us focus on the case of a $(d+1)-$dimensional Robertson-Walker spacetime $\mathcal{M}$ with Big Bang, say, at cosmic time $\tau = 0$ and space curvature k. Thus

$$\mathcal{M} = \mathcal{T} \times \mathcal{M}^d_{\mathsf{k}} \tag{B.1}$$

where $\mathcal{T} = (0, \tau_{fin})$ is the time interval and $\mathcal{M}^d_{\mathsf{k}}$ has the usual meaning.
A point of $\mathcal{M}$ is a pair $(\tau, \mathbf{p})$ with $\tau \in \mathcal{T}$, $\mathbf{p} \in \mathcal{M}^{\mathbf{d}}_{\mathsf{k}}$; the tangent space $T_{(\tau,\mathbf{p})}\mathcal{M}$ is the set of pairs $X = (\zeta, z)$ where ζ is a real number ([9]) and $z \in T_{\mathbf{p}}\mathcal{M}^d_{\mathsf{k}}$. Let us write g for the Lorentzian metric of $\mathcal{M}$ and h for the Riemannian metric of $\mathcal{M}^d_{\mathsf{k}}$ corresponding, respectively, to the line elements indicated in sect. 2 with $ds^2, d\ell^2$. The relation $ds^2 = -\,d\tau^2 + a^2(\tau)\,d\ell^2$ means that

$$g_{(\tau,\mathbf{p})}((\zeta, z), (\zeta', z')) = -\zeta\zeta' + a^2(\tau)\, h_{\mathbf{p}}(z, z')\ , \tag{B.2}$$

for all $(\zeta, z), (\zeta', z') \in T_{(\tau,\mathbf{p})}\mathcal{M}$. Of course, causal vectors (ζ, z) are characterized by the condition

$$0 \geq g_{(\tau,\mathbf{p})}((\zeta, z), (\zeta, z)) = -\zeta^2 + a^2(\tau)\, h_{\mathbf{p}}(z, z)\ . \tag{B.3}$$

By definition, a causal vector (ζ, z) is future directed if $\zeta \geq 0$. Let us also remark that a curve in $\mathcal{M}$ is a smooth map $\lambda = (\theta, \pi) : [\sigma_0, \sigma_1] \to \mathcal{T} \times \mathcal{M}^d_{\mathsf{k}}$.

B.2 Particle horizon

At any time $\tau \in \mathcal{T}$, this is defined by Eq. (2.43)

$$\Theta(\tau_1) := \int_0^{\tau_1} \frac{d\tau}{a(\tau)}\ . \tag{B.4}$$

[9]more precisely: ζ is an element of the $1-$dimensional, oriented vector space of time durations. This space can be identified with $\mathbf{R}$ choosing a reference, positively oriented time duration δ and confusing each duration δ with the real number ζ/δ.

The considerations of subsection 2.6 rely on the equivalence of the following statements $(i)(ii)$, for all $\mathbf{p_0} \neq \mathbf{p_1} \in \mathcal{M}_{\mathsf{k}}^{\mathbf{d}}$ and $\tau_1 \in \mathcal{T}$:

(i) there are $\tau_0 \in (0, \tau_1)$ and a future directed causal curve $\lambda : [\sigma_0, \sigma_1] \to \mathcal{T} \times \mathcal{M}_{\mathsf{k}}^d$ such that $\lambda(\sigma_0) = (\sigma_0, \mathbf{p_0})$ and $\lambda(\sigma_1) = (\sigma_1, \mathbf{p_1})$.

(ii) $\mathrm{dist}(\mathbf{p_0}, \mathbf{p_1}) < \boldsymbol{\Theta}(\tau_\mathbf{1})$, where dist is the distance on $\mathcal{M}_{\mathsf{k}}^d$ induced by the metric h.

Let us derive this equivalence.

Proof of the implication *(i)* ⇒ *(ii)*. Let $\lambda = (\theta, \pi) : [\sigma_0, \sigma_1] \to \mathcal{T} \times \mathcal{M}_{\mathsf{k}}^d$ be a future directed causal curve with end points $(\sigma_0, \mathbf{p_0})$ and $(\sigma_1, \mathbf{p_1})$ $(0 < \tau_0 < \tau_1)$, so that $\theta(\sigma_i) = \tau_i$ and $p((\sigma_i) = \mathbf{p_i}$ for $i = 0, 1$. The causal nature of λ ensures $(\,' := d/d\sigma)$

$$0 \geq g_{\lambda(\sigma)}(\lambda'(\sigma), \lambda'(\sigma)) = -\theta'^{\,2}(\sigma) + a^2(\theta(\sigma))\, h_{\pi(\sigma)}(\pi'(\sigma), \pi'(\sigma)) \ , \tag{B.5}$$

whence

$$h_{\pi(\sigma)}(\pi'(\sigma), \pi'(\sigma)) \leq \frac{\theta'^{\,2}(\sigma)}{a^2(\theta(\sigma))} \ ; \tag{B.6}$$

the assumption that λ is future directed tells us

$$\theta'(\sigma) \geq 0 \tag{B.7}$$

for all σ.
Since π is a curve in $\mathcal{M}_{\mathsf{k}}^d$ with end points $\mathbf{p_1}, \mathbf{p_2}$ we have

$$\mathrm{dist}(\mathbf{p_0}, \mathbf{p_1}) \leq \text{Lenght of } \pi = \int_{\sigma_\mathbf{0}}^{\sigma_\mathbf{1}} \mathbf{d}\sigma \, \sqrt{\mathbf{h}_{\pi(\sigma)}(\pi'(\sigma), \pi'(\sigma))} \ . \tag{B.8}$$

But (B.6) (B.7) give $\sqrt{h_{\pi(\sigma)}(\pi'(\sigma), \pi'(\sigma))} \leq \frac{\theta'(\sigma)}{a(\theta(\sigma))}$, thus

$$\mathrm{dist}(\mathbf{p_0}, \mathbf{p_1}) \leq \int_{\sigma_\mathbf{0}}^{\sigma_\mathbf{1}} \mathbf{d}\sigma \, \frac{\theta'(\sigma)}{\mathbf{a}(\theta(\sigma))} = \int_{\tau_\mathbf{0}}^{\tau_\mathbf{1}} \frac{\mathbf{d}\tau}{\mathbf{a}(\tau)} < \int_{\mathbf{0}}^{\tau_\mathbf{1}} \frac{\mathbf{d}\tau}{\mathbf{a}(\tau)} = \boldsymbol{\Theta}(\tau_\mathbf{1}) \ , \tag{B.9}$$

(where the first equality above is obtain with a change of variable $\tau = \theta(\sigma)$ in the integral).

Proof of the implication *(ii)* ⇒ *(i)*. Let

$$\delta := \mathrm{dist}(\mathbf{p_0}, \mathbf{p_1}) \ . \tag{B.10}$$

We are assuming

$$\delta < \Theta(\tau_1) = \int_0^{\tau_1} \frac{d\tau}{a(\tau)} \ , \tag{B.11}$$

so there is $\tau_0 \in (0, \tau_1)$ such that

$$\delta = \int_{\tau_0}^{\tau_1} \frac{d\tau}{a(\tau)} \ . \tag{B.12}$$

hereafter we will construct a future directed causal curve $\lambda = (\theta, \pi)$ with end points $(\tau_0, \mathbf{p_0})$ and $(\tau_1, \mathbf{p_1})$.
To this purpose, let γ denote a minimizing geodesic in $\mathcal{M}^d_{\mathsf{k}}$ parametrized by arc length and with end points $\mathbf{p_0}, \mathbf{p_1}$, so that

$$\gamma : [0, \delta] \to \mathcal{M}^d_{\mathsf{k}} \ , \ s \rightsquigarrow \gamma(s) \ , \qquad h(\gamma'(s), \gamma'(s)) = 1 \ , \tag{B.13}$$

$$\gamma(0) = \mathbf{p_0} \ , \qquad \gamma(\delta) = \mathbf{p_1} \ . \tag{B.14}$$

Let us also put

$$f : [\tau_0, \tau_1] \to \mathbf{R} \ , \ \tau \rightsquigarrow f(\tau) := \int_{\tau_0}^{\tau} \frac{d\tau'}{a(\tau')} \ . \tag{B.15}$$

We have $f'(\tau) = \frac{1}{a(\tau)} > 0$, $f(\tau_0) = 0, f(\tau_1) = \delta$ by Eq. (B.12). Using f with the previous geodesic γ, we define

$$\pi : [\tau_0, \tau_1] \to \mathcal{M}^d_{\mathsf{k}} \ , \ \tau \rightsquigarrow \pi(\tau) := \gamma(f(\tau)) \tag{B.16}$$

and

$$\lambda : [\tau_0, \tau_1] \to \mathcal{T} \times \mathcal{M}^d_{\mathsf{k}} \ , \ \tau \rightsquigarrow \lambda(\tau) := (\tau, \pi(\tau)) \ . \tag{B.17}$$

Then

$$\begin{aligned} \pi(\tau_0) &= \gamma(f(\tau_0)) = \gamma(0) = \mathbf{p_0} \\ \pi(\tau_1) &= \gamma(f(\tau_1)) = \gamma(\delta) = \mathbf{p_1} \end{aligned} \tag{B.18}$$

so that

$$\lambda(\tau_0) = (\tau_0, \mathbf{p_0}) \ , \qquad \lambda(\tau_\mathbf{1}) = (\tau_\mathbf{1}, \mathbf{p_1}) \ . \tag{B.19}$$

Moreover

$$\pi'(\tau) = \gamma'(f(\tau))\, f'(\tau) = \frac{\gamma'(f(\tau))}{a(\tau)} \ , \tag{B.20}$$

$$\lambda'(\tau) = (1, \frac{\gamma'(f(\tau))}{a(\tau)}) \ , \tag{B.21}$$

$$\begin{aligned} g_{\lambda(\tau)}(\lambda'(\tau), \lambda'(\tau)) &= -1 + a^2(\tau)\, h_{\gamma(f(\tau))} \left(\frac{\gamma'(f(\tau))}{a(\tau)}, \frac{\gamma'(f(\tau))}{a(\tau)} \right) \\ &= -1 + h_{\gamma(f(\tau))} \left(\gamma'(f(\tau)), \gamma'(f(\tau))\right) = 0 \end{aligned} \tag{B.22}$$

(the last equality follows from Eq. (B.13)). Eq. (B.22) tells us that λ is a light-like curve, and therefore a causal curve. We know from Eq. (B.19) that λ has endpoints $(\tau_0, \mathbf{p_0})$ and $(\tau_1, \mathbf{p_1})$, so the thesis (i) is proved.

C Appendix. Derivation of Eq.s (4.25)(4.26)(4.55)

First of all, let us recall a well-known integral identity for the hypergeometric function ${}_2F_1(\alpha, \beta, \gamma; z)$, holding true for any $\alpha, \beta, \gamma, z \in \mathbf{R}$ with $\gamma > \beta > 0$ (see, e.g., [22, Eq. 15.6.1], keeping in mind [22, 15.1.2]):

$$\int_0^1 dv \, \frac{v^{\beta-1} (1-v)^{\gamma-\beta-1}}{(1-z\,v)^{\alpha}} \;=\; \frac{\Gamma(\beta)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma)} \; {}_2F_1(\alpha, \beta, \gamma; z) \; . \tag{C.1}$$

In the following we shall employ systematically the above identity to provide a derivation of Eq.s (4.25) (4.26) (4.55). In the sequel t, η, ω are real numbers with $\omega > 0$.

Derivation of Eq. (4.25)**.** Let $t > 0\,,\ \eta > -1$ (the second condition is required for the convergence of the subsequent integrals). By means of trivial trigonometric identities and by evaluation of elementary integrals, we get

$$\begin{aligned} &\frac{1}{\omega}\int_0^t ds\, \sinh\big(\omega\,(t-s)\big)\, \sinh^{\eta}(\omega\,s) = \\ &= \frac{\sinh(\omega\,t)}{\omega}\int_0^t ds\, \cosh(\omega\,s)\, \sinh^{\eta}(\omega\,s) - \frac{\cosh(\omega\,t)}{\omega}\int_0^t ds\, \sinh^{\eta+1}(\omega\,s) = \\ &= \frac{\sinh^{\eta+2}(\omega\,t)}{\omega^2\,(\eta+1)} - \frac{\cosh(\omega\,t)\, \sinh^{\eta+2}(\omega\,t)}{2\,\omega^2}\int_0^1 dv\, \frac{v^{\eta/2}}{\sqrt{1+\sinh^2(\omega\,t)\,v}} \end{aligned} \tag{C.2}$$

(the last identity follows making the change of variable $s = \frac{1}{\omega}\,\mathrm{arcsinh}\big(\sinh(\omega\,t)\sqrt{v}\big)$ in the second integral). Then, using the relation (C.1) with $\alpha = 1/2$, $\beta = 1+\eta/2$, $\gamma = 2+\eta/2$ and $z = -\sinh^2(\omega\,t)$ (along with the identities $\Gamma(1) = 1$, $\Gamma(2+\eta/2) = (1+\eta/2)\,\Gamma(1+\eta/2)$), we obtain Eq. (4.25), i.e.,

$$\begin{aligned} &\frac{1}{\omega}\int_0^t ds\, \sinh\big(\omega\,(t-s)\big)\, \sinh^{\eta}(\omega\,s) = \\ &= \frac{\sinh^{\eta+2}(\omega\,t)}{\omega^2}\left[\frac{1}{\eta+1} - \frac{\cosh(\omega\,t)}{\eta+2}\; {}_2F_1\!\left(\frac{1}{2}\,, 1+\frac{\eta}{2}\,, 2+\frac{\eta}{2}\,;\, -\sinh^2(\omega\,t)\right)\right] . \end{aligned}$$

Derivation of Eq. (4.26)**.** Again, by obvious trigonometric identities and evaluation of simple integrals, for $\eta \neq -1$ we get

$$\begin{aligned} &\frac{1}{\omega}\int_0^t ds\, \sinh\big(\omega\,(t-s)\big)\, \cosh^{\eta}(\omega\,s) = \\ &= -\,\frac{\cosh(\omega\,t)}{\omega}\int_0^t ds\, \sinh(\omega\,s)\, \cosh^{\eta}(\omega\,s) + \frac{\sinh(\omega\,t)}{\omega}\int_0^t ds\, \cosh^{\eta+1}(\omega\,s) = \\ &= \frac{\cosh(\omega\,t)\,\big(1-\cosh^{\eta+1}(\omega t)\big)}{\omega^2\,(\eta+1)} + \frac{\sinh^2(\omega\,t)}{2\,\omega^2}\int_0^1 dv\; v^{-1/2}\big(1+\sinh^2(\omega\,t)\,v\big)^{\eta/2} \end{aligned} \tag{C.3}$$

(consider again the change of integration variable $s = \frac{1}{\omega}\operatorname{arcsinh}\big(\sinh(\omega\, t)\sqrt{v}\big)$). Using the identity (C.1) with $\alpha = -\eta/2$, $\beta = 1/2$, $\gamma = 3/2$ and $z = -\sinh^2(\omega\, t)$ (along with the identities $\Gamma(1/2)/\Gamma(3/2) = 2$ and ${}_2F_1(\beta, \alpha, \gamma; z) = {}_2F_1(\alpha, \beta, \gamma; z)$), we obtain Eq. (4.26), i.e.,

$$\frac{1}{\omega}\int_0^t ds\ \sinh\big(\omega\,(t-s)\big)\ \cosh^\eta(\omega\, s) =$$

$$= \frac{1}{\omega^2}\left[\frac{\cosh(\omega\, t)\,\big(1-\cosh^{\eta+1}(\omega t)\big)}{\eta+1} + \sinh^2(\omega\, t)\ {}_2F_1\!\left(\frac{1}{2}\,, -\frac{\eta}{2}\,, \frac{3}{2}\,; -\sinh^2(\omega\, t)\right)\right]$$

One checks directly that the above relation holds as well for $\eta = -1$, in the limit sense explained by Eq. (4.28) .

Derivation of Eq. (4.55)**.** Let $0 < t < \frac{\pi}{\omega}$, $\eta > -1$; also in this case, by elementary arguments similar to those employed previously, we get

$$\frac{1}{\omega}\int_0^t ds\ \sin\big(\omega\,(t-s)\big)\ \sin^\eta(\omega\, s) =$$

$$= \frac{\sin(\omega\, t)}{\omega}\int_0^t ds\ \cos(\omega\, s)\ \sin^\eta(\omega\, s) - \frac{\cos(\omega\, t)}{\omega}\int_0^t ds\ \sin^{\eta+1}(\omega\, s) = \tag{C.4}$$

$$= \frac{\sin^{\eta+2}(\omega\, t)}{\omega^2\,(\eta+1)} - \frac{\cos(\omega\, t)\ \sin^{\eta+2}(\omega\, t)}{2\,\omega^2}\int_0^1 dv\ \frac{v^{\eta/2}}{\sqrt{1-\sin^2(\omega\, t)\, v}}$$

(the last identity can be derived with the change of variable $s = \frac{1}{\omega}\arccos\big(\sqrt{1-\sin^2(\omega\, t)v}\big)$). On account of Eq. (C.1), here employed with $\alpha = 1/2$, $\beta = 1+\eta/2$, $\gamma = 2+\eta/2$ and $z = \sin^2(\omega\, t)$ (along with the identities $\Gamma(1) = 1$, $\Gamma(2+\eta/2) = (1+\eta/2)\,\Gamma(1+\eta/2)$), we obtain Eq. (4.55), i.e.,

$$\frac{1}{\omega}\int_0^t ds\ \sin\big(\omega\,(t-s)\big)\ \sin^\eta(\omega\, s) =$$

$$= \frac{\sin^{\eta+2}(\omega\, t)}{\omega^2}\left[\frac{1}{\eta+1} - \frac{\cos(\omega\, t)}{\eta+2}\ {}_2F_1\!\left(\frac{1}{2}\,, 1+\frac{\eta}{2}\,, 2+\frac{\eta}{2}\,; \sin^2(\omega\, t)\right)\right] .$$

D Appendix. Validity conditions for Eq. (4.30)

Our starting point is Eq. (4.24), assuming implicitly the convergence of the integral therein. To compute this integral in the case yielding Eq. (4.30), we must use the identity (4.25) with $\eta = \frac{1-w}{1+w}$ or $\eta = -\frac{1+3w}{1+w}$ and the integral therein converges only for $\eta > -1$. Due to this, the expression for $y(t)$ in Eq. (4.30) seems to hold only for $\frac{1-w}{1+w} > -1$ and $-\frac{1+3w}{1+w} > -1$, which happens if and only if $-1 < w < 0$. However for our purposes, it suffices that the expression in Eq. (4.30) for $y(t)$ is a solution of Eq. (4.20); we know that this occurs for $-1 < w < 0$ and so, by elementary consideration based on analytic continuation, the same will hold on the full region where this term is analytic in w. To find this region, let us recall that for any fixed $z \in (-\infty, 1)$, ${}_2F_1(a, b, c\,; z)$ is analytic in a, b, c for $a, b \in \mathbf{R}$ and $c \in \mathbf{R}\backslash\{0, -1, -2, ...\}$ (see, e.g., [22]). In Eq. (4.30) we have two hypergeometric terms with $c = \frac{3+w}{2+2w}$ and $c = \frac{5+3w}{2+2w} = \frac{3+w}{2+2w} + 1$ which are both different from $0, -1, -2, \ldots$ if and only if

$$w \neq -\frac{3+2h}{1+2h}\,, \qquad \text{for all } h \in \{0, 1, 2, \ldots\}\,. \tag{D.1}$$

E Appendix. Models with a cosmological constant and an arbitrary number of perfect fluids

Let us consider Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R + \Lambda\, g_{\mu\nu} = d\,(d-1)\,\gamma_d\, G_d\, \sum_{A=1}^{N} T^{(A)}{}_{\mu\nu} \ , \tag{E.1}$$

where Λ is the cosmological constant and, for $A = 1, ..., N$, $T^{(A)}{}_{\mu\nu}$ is the stress-energy tensor of a perfect fluid of $(d+1)$ velocity $U^{(A)\mu}$, whose pressure $p^{(A)}$ and mass-energy density $\rho^{(A)}$ fulfill the equation of state

$$p^{(A)} = w^{(A)}\, \rho^{(A)} \tag{E.2}$$

for some suitable real constant $w^{(A)}$; this reads

$$T^{(A)}{}_{\mu\nu} = \left(p^{(A)} + \rho^{(A)}\right) U^{(A)}_{\mu}\, U^{(A)}_{\nu} + p^{(A)}\, g_{\mu\nu} \ , \tag{E.3}$$

In addition, we postulate the conservation law for the fluid stress-energy tensor:

$$\nabla_\mu T^{(A)\mu}{}_{\nu} = 0 \ . \tag{E.4}$$

Note that Eq.s (E.1) can be written as

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = d\,(d-1)\,\gamma_d\, G_d\, \left(\sum_{A=0}^{N} T^{(A)}{}_{\mu\nu} + T^{(\Lambda)}{}_{\mu\nu} \right) \ , \tag{E.5}$$

where

$$T^{(\Lambda)}{}_{\mu\nu} = -\, \frac{\Lambda}{d(d-1)\,\gamma_d\, G_d}\, g_{\mu\nu} \ ; \tag{E.6}$$

Note that $T^{(\Lambda)}{}_{\mu\nu}$ has the form (E.3) characterizing a perfect fluid, with an arbitrary $(d+1)$ velocity $U^{(\Lambda)\mu}$ and with pressure and density

$$p^{(\Lambda)} = -\, \frac{\Lambda}{d(d-1)\,\gamma_d\, G_d} \ , \qquad \rho^{(\Lambda)} = \frac{\Lambda}{d(d-1)\,\gamma_d\, G_d} \ . \tag{E.7}$$

The equation of state of this fluid is

$$p^{(\Lambda)} = w^{(\Lambda)}\, \rho^{(\Lambda)} \ , \qquad w^{(\Lambda)} := -1 \ . \tag{E.8}$$

Due to (E.6), Eq. (E.4) holds trivially even for $(A) = (\Lambda)$.

E.1 Generalities

Let us specialise the previous model to the case of a Robertson-Walker spacetime with line element

$$ds^2 = -\,d\tau^2 + a^2(\tau)\,d\ell^2 = -\,d\tau^2 + a^2(\tau)\,h_{ij}(\mathbf{x})\,dx^i dx^j \ , \tag{E.9}$$

as in subsection 2.3. Moreover, assume all perfect fluids to be at rest in the Robertson Walker frame; this means that, for each A,

$$U^{(A)\mu} = U^\mu \tag{E.10}$$

where U^μ is the $(d+1)$-velocity of the Rovertson-Walker frame (in coordinates $x^0 := \tau$ and x^i one has $U^0 = 1$, $U^i = 0$ for $i = 1, ..., d$). From Eq. (E.4) one has

$$\rho^{(A)} = \frac{\rho_*^{(A)}}{a^{d(w^{(A)}+1)}} \ , \qquad \rho_*^{(A)} = \text{const.} \qquad (A = 1, \ldots, N). \tag{E.11}$$

(Moreover, recall that $\rho^{(\Lambda)} = \text{const.} = \rho_*^{(\Lambda)} = \frac{\Lambda}{d(d-1)\,\gamma_d\,G_d}$). The only independent Einstein Equation is

$$H^2 - 2\,\gamma_d\,G_d \left(\sum_{A=1}^{N} \rho^{(A)} + \rho^{(\Lambda)} \right) + \frac{\mathsf{k}}{a^2} = 0 \ , \qquad \left(H := \frac{1}{a}\frac{da}{d\tau} \right) \tag{E.12}$$

or, equivalently

$$2\,\gamma_d\,G_d \left(\sum_{A=1}^{N} \frac{\rho^{(A)}}{H^2} + \frac{\rho^{(\Lambda)}}{H^2} \right) - \frac{\mathsf{k}}{a^2 H^2} = 1 \ . \tag{E.13}$$

This can be written as

$$\sum_{A=1}^{N} \Omega_A + \Omega_\Lambda + \Omega_k = 1 \tag{E.14}$$

where

$$\Omega_A := 2\,\gamma_d\,G_d \frac{\rho^{(A)}}{H^2} = 2\,\gamma_d\,G_d \frac{\rho_*^{(A)}}{a^{d(w^{(A)}+1)}\,H^2} \ , \qquad (A = 1, \ldots, N) \tag{E.15}$$

$$\Omega_\Lambda := 2\,\gamma_d\,G_d \frac{\rho^{(\Lambda)}}{H^2} = \frac{2\,\Lambda}{d(d-1)\,H^2} \ , \qquad \Omega_k := -\frac{\mathsf{k}}{a^2\,H^2} = -\frac{\mathsf{k}}{\left(\frac{da}{d\tau}\right)^2} \ ,$$

are the (usual) dimensionless density parameters.

E.2 Dimensionless formulation

Let assume the following:

- θ a characteristic time;
- τ_* is some reference time when $a(\tau_*) = 1$ (e.g., the present time in a model for the physical universe);
- if there is a Big Bang we take it as the origin of the cosmic time; so that the Big Bang time istant is $\tau = 0$.

Furthermore, let us introduce an adimensional time t related to τ by

$$\tau = \theta\, t \; ; \tag{E.16}$$

then $t_* = \frac{\tau_*}{\theta}$ is the reference time and $t = 0$ is the Big Bang (if any). From now on, $a = a(t)$, $\dot{} = d/dt$; it follows

$$H = \frac{1}{a}\frac{da}{d\tau} = \frac{1}{a\,\theta}\frac{da}{dt} = \frac{1}{\theta}\frac{\dot a}{a} \; . \tag{E.17}$$

We define the dimensionless parameters $k, \Omega_{A*}, \Omega_{\Lambda*}$, setting

$$\mathsf{k} = \frac{k}{\theta^2} \; , \qquad \rho^{(A)}_* = \frac{\Omega_{A*}}{2\,\gamma_d\, G_d\, \theta^2} \; , \qquad \Lambda = \frac{d(d-1)\,\Omega_{\Lambda*}}{2\,\theta^2} \; ; \tag{E.18}$$

from Eq. (E.11) (E.15) one has

$$\rho^{(A)} = \frac{\Omega_{A*}}{2\,\gamma_d\, G_d\, \theta^2\, a^{d(w^{(A)}+1)}} \; , \qquad (A = 1, \ldots, N) \; ; \qquad \rho^{(\Lambda)} = \frac{\Omega_{\Lambda*}}{2\,\gamma_d\, G_d\, \theta^2} \; , \tag{E.19}$$

$$\Omega_A = \frac{\Omega_{A*}}{\theta^2\, a^{d(w^{(A)}+1)}\, H^2} = \frac{\Omega_{A*}}{a^{d(w^{(A)}+1)-2}\, \dot a^2} \; , \qquad (A = 1, \ldots, N) \tag{E.20}$$

$$\Omega_\Lambda = \frac{\Omega_{\Lambda*}}{\theta^2\, H^2} = \frac{\Omega_{\Lambda*} a^2}{\dot a^2} \; , \qquad \Omega_k = -\frac{k}{\theta^2\, a^2\, H^2} = -\frac{k}{\dot a^2} \; .$$

Note that for $t = t_*$ one has $a(t_*) = 1$, and Eq. (E.20) gives

$$\Omega_A(t_*) = \Omega_{A*}, \;\; \Omega_\Lambda(t_*) = \Omega_{\Lambda*} \;\; \text{if } \theta := \frac{1}{|H(t_*)|} \; ; \tag{E.21}$$

the choice for θ considered in (E.21) is very natural, and will often be considered. The dimensionless version of Eq. (E.12) is

$$\frac{\dot a^2}{a^2} - 2\,\gamma_d\, G_d\, \theta^2 \left(\sum_{A=1}^{N} \rho^{(A)} + \rho^{(\Lambda)} \right) + \frac{k}{a^2} = 0 \; , \tag{E.22}$$

or, equivalently

$$\frac{\dot{a}^2}{a^2} - \sum_{A=1}^{N} \frac{\Omega_{A*}}{a^{d(w^{(A)}+1)}} - \Omega_{\Lambda*} + \frac{k}{a^2} \ . \tag{E.23}$$

If we put $\theta := \frac{1}{H(t_*)}$, Eq. (E.12) becomes

$$H^2 = H^2(t_*) \left(\sum_{A=1}^{N} \frac{\Omega_{A*}}{a^{d(w^{(A)}+1)}} + \Omega_{\Lambda*} - \frac{k}{a^2} \right) \ ; \tag{E.24}$$

if the scale factor $a = a(t)$ is known, this last relation allows us to calculate the Hubble parameter $H = H(t)$.
On the other hand, Eq. (E.23) can be written as

$$\dot{a}^2 - \mathfrak{U}(a) = 0 \ , \qquad \mathfrak{U}(a) = \sum_{A=1}^{N} \frac{\Omega_{A*}}{a^{d(w^{(A)}+1)-2}} + \Omega_{\Lambda*}\, a^2 - k \ . \tag{E.25}$$

This equation allows a qualitative study of the function $a = a(t)$, based on usual methods for one-dimensional, conservative mechanical systems. Let us just say the following: since $\mathfrak{U}(a(t)) = \dot{a}^2(t) \geqslant 0$, the range of the function $t \to a(t)$ must be contained in the set $\{a > 0 \mid \mathfrak{U}(a) \geqslant 0\}$ (more precisely, in a connected component of this set). To make explicit the implications of this fact, it is necessary to analyse the graph of $\mathfrak{U}$, whose shape depends on all the involved constants w_A, Ω_{A*}, $\Omega_{\Lambda*}$, k.
Or course, Eq. (E.25) can be reduced to quadratures. Indeed, let sign $\dot{a}(t) = \sigma \in \{\pm 1\}$ for $t \in (t_1, t_2)$; then, the relation $da/\sqrt{\mathfrak{U}(a)} = \sigma dt$ in this interval gives one has

$$\int_{a(t_1)}^{a(t_2)} \frac{d\mathfrak{a}}{\sqrt{\mathfrak{U}(\mathfrak{a})}} = \sigma(t_2 - t_1) \ . \tag{E.26}$$

In particular, assume there is a Big Bang at time zero, so that $a(0) = 0$ ([10]). If $\dot{a}(t') > 0$ for $t' \in (0, t)$, Eq. (E.26) with $t_1 = 0$ and $t_2 = t$ gives

$$\int_{0}^{a(t)} \frac{d\mathfrak{a}}{\sqrt{\mathfrak{U}(\mathfrak{a})}} = t \quad \Rightarrow \quad \theta \int_{0}^{a(\tau)} \frac{d\mathfrak{a}}{\sqrt{\mathfrak{U}(\mathfrak{a})}} = \tau \ . \tag{E.27}$$

For future use, let us also remark that Eq. (E.20) for the dimensionless densities, with the relation $\dot{a}^2 = \mathfrak{U}(a)$ coming from (E.25), gives

$$\Omega_A = \frac{\Omega_{A*}}{a^{d(w^{(A)}+1)-2}\, \mathfrak{U}(a)} \ , \ (A = 1, \ldots, N); \ \ \Omega_\Lambda = \frac{\Omega_{\Lambda*} a^2}{\mathfrak{U}(a)} \ ; \ \Omega_k = -\frac{k}{\mathfrak{U}(a)} \ . \tag{E.28}$$

[10] here and in the sequel, $a(0)$ always means $\lim_{t \to 0^+} a(t)$.

E.3 Qualitative features close to the Big Bang

Assume there is $B \in \{1, ..., N\}$ such that

$$w^{(B)} > w^{(A)} \text{ for } A = 1, ..., N,\ A \neq B\ ; \quad \Omega_{B*} > 0\ ; \tag{E.29}$$

$$w^{(B)} > -1 \text{ if } k = 0\,,\ w^{(B)} > -1 + 2/d \text{ if } k \neq 0\ .$$

Then, we see from (E.25) that

$$\mathfrak{U}(a) \sim \frac{\Omega_{B*}}{a^{d(w^{(A)}+1)-2}} \text{ for } a \to 0^+. \tag{E.30}$$

The small a behavior of $\mathfrak{U}$, along with the qualitative considerations mentioned after Eq. (E.25), ensures that there are the conditions for a Big Bang at $t = 0$.
From here to the end of this subsection we consider the $t \to 0^+$ limit, in which $a(t) \to 0^+$. From Eq. (E.25) and from the asymptotics (E.30) for $\mathfrak{U}$ we obtain $t = \int_0^{a(t)} \frac{d\mathfrak{a}}{\sqrt{\mathfrak{U}(\mathfrak{a})}} \sim \frac{2}{d(w^{(B)}+1)\sqrt{\Omega_{B*}}} a(t)^{d(w^{(B)}+1)/2}$, whence

$$a(t) \sim \left(\frac{d}{2}\,(w^{(B)} + 1)\Omega_{B*}\, t\right)^{\frac{2}{d(w^{(B)}+1)}} \tag{E.31}$$

(of course, we can rephrase this result in terms of the cosmic time τ writing $t = \tau/\theta$). Let us mention that the integrability or non integrability of $1/a$ in a right neigborhood of zero corresponds, respectively, to a finite or infinite particle horizon (recall the discussion of Section 2.6, to be used here with $b = 1$). According to (E.31), the particle horizon is finite if $w^{(B)} + 1 > 2/d$, and infinite if $w^{(B)} + 1 \leqslant 2/d$.
From Eq.s (E.28) and the asymptotics (E.30) we also obtain (we repeat it, for $t \to 0^+$):

$$\Omega_B(t) \to 1; \quad \Omega_A(t) \sim \frac{\Omega_{A*}}{\Omega_{B*}} a(t)^{d(w^{(B)}-w^{(A)})} \to 0 \text{ for } A = 1, ..., N,\ A \neq B; \tag{E.32}$$

$$\Omega_\Lambda(t) \sim \frac{\Omega_{\Lambda*}}{\Omega_{B*}} a(t)^{d(w^{(B)}+1)} \to 0\ ;\ \ \Omega_k(t) \sim -\frac{k}{\Omega_{B*}} a(t)^{d(w^{(B)}+1)-2} \to 0 \text{ if } k \neq 0$$

(of course, $\Omega_k(t) = 0$ identically if $k = 0$). Thus, the energy density of the type B matter dominates at the Big Bang.

E.4 The case of one matter field (and a cosmological term)

Let us consider the case with only one matter field ($N = 1$) with matter density $\rho^{(1)} \equiv \rho^{(m)}$, pressure $p^{(1)} \equiv p^{(m)}$, density $\Omega^{(1)} \equiv \Omega_m$ and $\Omega_{A*} \equiv \Omega_{m*}$. Eq. (E.14) reads

$$\Omega_m + \Omega_\Lambda + \Omega_k = 1\ , \tag{E.33}$$

and Eq. (E.26) can be explicity written as

$$\int_{a(t_1)}^{a(t_2)} \frac{d\mathfrak{a}}{\sqrt{\Omega_{m*}\, \mathfrak{a}^{2-d(1+w)} + \Omega_{\Lambda*}\, \mathfrak{a}^2 - k}} = \sigma(t_2 - t_1) \tag{E.34}$$

(if $\operatorname{sign} \dot{a} = \sigma \in \{\pm 1\}$ on (t_1, t_2)). Let us furtherly specialize our considerations assuming

$$\Omega_{m*} > 0 \quad w > -1\ , \quad \Omega_{\Lambda*} \geqslant 0, \quad k = 0\ . \tag{E.35}$$

Then, by a qualitative analysis of the kind mentioned after Eq. (E.25), one finds that there are Big Bang solutions $t \to a(t)$ defined for $t \in (0, +\infty)$, with $a(t) \to 0$ for $t \to 0^+$ and $\dot{a}(t) > 0$ for all $t \in (0, +\infty)$.
For a solution with these features, we can apply Eq. (E.34) with $t_2 \equiv t \in (0, +\infty)$, $t_1 = 0$, $a(0) = 0$ and $\sigma = 1$ to get

$$t = \int_0^{a(t)} \frac{d\mathfrak{a}}{\sqrt{\Omega_{m*}\, \mathfrak{a}^{2-d(1+w)} + \Omega_{\Lambda*}\, \mathfrak{a}^2}} = \frac{2}{d(w+1)\sqrt{\Omega_{\Lambda*}}} \operatorname{arcsinh}\left(\sqrt{\frac{\Omega_{\Lambda*}}{\Omega_{m*}}}\, a(t)^{\frac{d(1+w)}{2}}\right) \tag{E.36}$$

which, in turn, implies

$$a(t) = \left[\sqrt{\frac{\Omega_{m*}}{\Omega_{\Lambda*}}}\ \sinh\left(\frac{d}{2}(w+1)\sqrt{\Omega_{\Lambda*}}\, t\right)\right]^{\frac{2}{d(1+w)}} \tag{E.37}$$

(If $\Omega_{\Lambda*} = 0$, this equation must be understood in a limit sense: send $\Omega_{\Lambda*}$ to 0^+ in the written formula, so that $\sqrt{\frac{\Omega_{m*}}{\Omega_{\Lambda*}}}\ \sinh\big(\frac{d}{2}(w+1)\sqrt{\Omega_{\Lambda*}}\, t\big) \to \frac{d}{2}(w+1)\sqrt{\Omega_{m*}}\, t)$.
From Eq. (E.37) and from the general expressions (E.20) for the dimensionless densities, we get

$$\Omega_m(t) = \operatorname{sech}^2\left(\frac{d}{2}(w+1)\sqrt{\Omega_{\Lambda*}}\, t\right)\ ,\ \Omega_\Lambda(t) = \tanh^2\left(\frac{d}{2}(w+1)\sqrt{\Omega_{\Lambda*}}\, t\right) \tag{E.38}$$

(and $\Omega_m + \Omega_\Lambda = 1$, as expected from (E.33) with $k = 0$, since $\operatorname{sech}^2 x + \tanh^2 x = 1$).
From Eq.s (E.37) (E.38) we obtain, for $t \to 0^+$:

$$a(t) \sim \left(\frac{d}{2}\,(w+1)\Omega_{m*}\, t\right)^{\frac{2}{d(w+1)}}\ , \tag{E.39}$$

$$\Omega_m(t) \to 1\ , \quad \Omega_\Lambda(t) \to 0\ ; \tag{E.40}$$

so, matter is dominant at the Big Bang. (As expected, these results agree with the general relations (E.31) (E.32)).

All the statements of this section can be applied, in particular, to the cases of *dust* ($w = 0$) and *radiation* ($w = 1/d$); note that (E.39) gives

$$a(t) \sim \left(\frac{d}{2}\Omega_{m*}\, t\right)^{\frac{2}{d}} \text{ for } w = 0; \quad a(t) \sim \left(\frac{d+1}{2}\,\Omega_{m*}\, t\right)^{\frac{2}{d+1}} \text{ for } w = \frac{1}{d}\ . \tag{E.41}$$

Final remark. From Eq. (E.36), it is easy to find the reference time t_* when $a(t_*) = 1$; this is

$$t_* = \frac{2}{d(w+1)\sqrt{\Omega_{\Lambda*}}}\,\text{arcsinh}\sqrt{\frac{\Omega_{\Lambda*}}{\Omega_{m*}}}\ . \tag{E.42}$$

In particular, let us consider the choice $\theta = 1/H(t_*)$ implying $\Omega_{m*} = \Omega(t_*)$ and $\Omega_{\Lambda*} = \Omega_\Lambda(t_*)$ (recall Eq. (E.21)); then $\Omega_{m*} + \Omega_{\Lambda*} = 1$, so we can write $t_* = \frac{2}{d(w+1)\sqrt{1-\Omega_{m*}}}\,\text{arcsinh}\sqrt{\frac{1}{\Omega_{m*}} - 1}$, and the elementary identity $\text{arcsinh}\, x = \text{arccosh}\sqrt{1+x^2}$ gives

$$t_* = \frac{2}{d(w+1)\sqrt{1-\Omega_{m*}}}\,\text{arccosh}\,\frac{1}{\sqrt{\Omega_{m*}}}\ . \tag{E.43}$$

E.5 Critical density

Let us consider now the case with only one matter field ($N = 1$) and no cosmological constant ($\Lambda = 0$); the matter is a dust ($w^{(1)} \equiv w = 0$) with matter density $\rho^{(1)} \equiv \rho$, pressure $p^{(1)} \equiv p = 0$ and density $\Omega^{(1)} \equiv \Omega$. Eq. (E.14) reads

$$\Omega + \Omega_k = 1\ , \qquad \Omega := 2\,\gamma_d\, G_d \frac{\rho}{H^2}\ , \qquad \Omega_k := -\frac{\mathsf{k}}{a^2\, H^2} \tag{E.44}$$

and so

$$\Omega_k = 1 - \Omega \quad \Rightarrow \quad -\frac{\mathsf{k}}{a^2\, H^2} = 1 - 2\,\gamma_d\, G_d \frac{\rho}{H^2}\ . \tag{E.45}$$

From the last realation one has

$$\text{sign}(\mathsf{k}) = -\text{sign}(\Omega_k) = -\text{sign}\left(1 - 2\,\gamma_d\, G_d \frac{\rho}{H^2}\right) = \text{sign}\left(2\,\gamma_d\, G_d \frac{\rho}{H^2} - 1\right)\ , \tag{E.46}$$

for all istants t; in particular, let consider

$$t_* = \text{present time}\ .$$

From Eq. (E.46) with $t = t_*$ one has

$$\text{sign}(\mathsf{k}) = \text{sign}\left(2\,\gamma_d\, G_d \frac{\rho(t_*)}{H^2(t_*)} - 1\right)\ . \tag{E.47}$$

We define a *critical density* ρ_{cr} as a solution of the equation

$$2\,\gamma_d\, G_d \frac{\rho_{cr}}{H^2(t_*)} = 1\ , \tag{E.48}$$

from which it follows that

$$\rho_{cr} = \frac{H^2(t_*)}{2\,\gamma_d\,G_d}\ . \tag{E.49}$$

Note that for $d = 3$ is $\gamma_d = \gamma_3 = \frac{4}{3}\pi$, from which one has $\rho_{cr} = \frac{3H^2(t_*)}{8\,\pi\,G}$. Furthermore, from Eq. (E.47) we have

$$\text{sign}(\mathsf{k}) = \text{sign}\left(\frac{\rho(t_*)}{\rho_{cr}} - 1\right) = \text{sign}\left(\frac{\rho(t_*) - \rho_{cr}}{\rho_{cr}}\right)\ , \tag{E.50}$$

from which, being $\rho_{cr} > 0$,

$$\text{sign}(\mathsf{k}) = \text{sign}(\rho(t_*) - \rho_{cr})\ . \tag{E.51}$$

Note that, if we define the constant θ as

$$\theta = \frac{1}{H(t_*)}$$

then

$$\rho_{cr} = \frac{1}{2\,\gamma_d\,G_d\,\theta^2}\ . \tag{E.52}$$

F Appendix. Some useful constants. The benchmark model

F.1 The Hubble constant

First of all, let us give the best available value of the Hubble parameter at the present time (see, e.g., [23]):

$$H_* \simeq 67.74 \, \frac{Km}{Mpc \cdot s} \simeq 2.1953 \times 10^{-18} \, s^{-1} \ ;$$

note that the last relation holds in our unit with $c = 1$. In the sequel we illustrate the benchmark model and we specify some useful constants.

F.2 The benchmark model

Let us consider now the spatially flat case with a cosmological constant and two matter fields: dust and radiation; this model is known as the *benchmark model* [30], and is often used to fit the observational data. The benchmark model is just a particular case of the general model of Appendix E with $d = 3$, $k = 0$ and $N = 2$ types of perfect fluids: dust ($w = 0$) and radiation ($w = 1/3$). In the sequel we will often refer to the framework of Appendix E, to be used here with these prescription and with t_* the present time; the constant θ of the cited Appendix, with the dimension of a time, is chosen here as the reciprocal of Hubble's parameter at present age:

$$\theta = \frac{1}{H_*} \simeq 14.4 \times 10^9 \text{ years.} \tag{F.1}$$

In the sequel, following a standard use we indicate dust with the term "matter" and use for it the subscript m; radiation is indicated with the subscript r.
Due to the previous positions, Eq. (E.14) reads

$$\Omega_m + \Omega_r + \Omega_\Lambda = 1 \ . \tag{F.2}$$

The radiation is assumed to consists of photons (indicated with γ) and neutrinos (indicated with ν); its current contribution is

$$\Omega_{\gamma\star} \simeq 5 \times 10^{-5} \ , \qquad \Omega_{\nu\star} \simeq 3.4 \times 10^{-5} \quad \Rightarrow \quad \Omega_{r\star} \simeq 8.4 \times 10^{-5} \ . \tag{F.3}$$

The matter content consists of baryonic matter (usual atomic matter composed by protons and neutrons, with associated electrons) and of nonbaryonic dark matter. This last form of matter is the most present in the universe: the present density parameter $\Omega_{dm\star}$ of the nonbaryonic dark matter is six times greater than the present density parameter $\Omega_{bar\star}$ of the ordinary baryonic matter:

$$\Omega_{dm\star} \simeq 0.268 \ , \qquad \Omega_{bar\star} \simeq 0.04 \quad \Rightarrow \quad \Omega_{m*} \simeq 0.308 \ . \tag{F.4}$$

Nowadays, the cosmological content is dominant; in fact, from Eq. (F.2) one has

$$\Omega_{\Lambda*} = 1 - \Omega_{m*} - \Omega_{r\star} \simeq 0.692 \ . \tag{F.5}$$

Eq. (E.27) can be explicity written as

$$\tau = \frac{1}{H_*} \int_0^{a(\tau)} \frac{d\mathfrak{a}}{\sqrt{\Omega_{r\star}\, \mathfrak{a}^{-2} + \Omega_{m*}\, \mathfrak{a}^{-1} + \Omega_{\Lambda*}\, \mathfrak{a}^2}} \tag{F.6}$$

and this can be used to estimate the age of the universe at a given time τ.

F.2.1 The radiation-dominated era

In the past, there was initially a radiation-dominated era. In fact, we know that the energy density of the type B matter with the greatest $w^{(B)}$ dominates at the Big Bang (see Appendix E, subsect. E.3); in this case, $w^{(B)} = 1/3$. This radiation-dominated era ended when the density of radiation $\rho^{(r)} = \frac{\Omega_{r\star}}{2\,\gamma_d\, G_d\, \theta^2\, a^4}$ dropped to that of matter $\rho^{(m)} = \frac{\Omega_{m*}}{2\,\gamma_d\, G_d\, \theta^2\, a^3}$ (see Eq. (E.19)); this happened at a scale factor

$$a_{rm} = \frac{\Omega_{r\star}}{\Omega_{m*}} \simeq 2.8 \times 10^{-4} \ . \tag{F.7}$$

Using the scale factor ratio (F.7) as the upper limit in the integral (F.6), we find out that the radiation era ended approximately when

$$\tau_r = \frac{1}{H_*} \int_0^{a_{rm}} \frac{d\mathfrak{a}}{\sqrt{\Omega_{r\star}\, \mathfrak{a}^{-2} + \Omega_{m*}\, \mathfrak{a}^{-1} + \Omega_{\Lambda*}\, \mathfrak{a}^2}} \simeq 50953 \text{ years} \ . \tag{F.8}$$

F.2.2 The matter-dominated era

After the radiation-dominated era, matter dominated until its density $\rho^{(m)} = \frac{\Omega_{m*}}{2\,\gamma_d\, G_d\, \theta^2\, a^3}$ dropped to the constant mass density $\rho^{(\Lambda)} = \frac{\Omega_{\Lambda*}}{2\,\gamma_d\, G_d\, \theta^2}$; this happened at a scale factor

$$a_{m\Lambda} = \left(\frac{\Omega_{m*}}{\Omega_{\Lambda\star}}\right)^{\frac{1}{3}} \simeq 0.76458 \ . \tag{F.9}$$

Using the scale factor ratio (F.9) as the upper limit in the integral (F.6), we find out that the matter era ended approximately when

$$\tau_m = \frac{1}{H_*} \int_0^{a_{m\Lambda}} \frac{d\mathfrak{a}}{\sqrt{\Omega_{r\star}\, \mathfrak{a}^{-2} + \Omega_{m*}\, \mathfrak{a}^{-1} + \Omega_{\Lambda*}\, \mathfrak{a}^2}} \simeq 10.1928 \times 10^9 \text{ years} \ . \tag{F.10}$$

This last estimate tells us that the dark-energy dominated era began until 3.6×10^9 years ago.

F.2.3 The age of the universe

Using the present value of scale factor $a(t_*) = 1$ as the upper limit in the integral (F.6), we find out that the estimated age of the universe is

$$\tau_* = \frac{1}{H_*} \int_0^1 \frac{d\mathfrak{a}}{\sqrt{\Omega_{r\star}\,\mathfrak{a}^{-2} + \Omega_{m*}\,\mathfrak{a}^{-1} + \Omega_{\Lambda*}\,\mathfrak{a}^2}} \simeq 13.792 \times 10^9 \text{ years} . \tag{F.11}$$

F.2.4 The horizon

One can prove that the benchmark model has a finite horizon at the present time; this is

$$\Theta(\tau_*) = \int_0^{\tau_*} \frac{d\tau}{a(\tau)} \simeq 14\, Gpc \simeq 4.32 \times 10^{26}\, m \ ; \tag{F.12}$$

Eq. (F.12) tells us that if the banchmark model is a good realistic model for our universe, then we can't see objects more than 14 gigaparsecs away.

G Appendix. A comparison between the potentials in the thesis and the one considered by Saini, Raychaudhury, Sahni and Starobinsky [31]

As in the rest of the thesis, recall that we are working in units where

$$c = 1, \qquad \hbar = 1 \; ; \tag{G.1}$$

furthermore, we now consider the case $d = 3$. In section 2 we have defined dimensionless versions $\phi, \mathcal{V}$ of the field Φ and of its potential $\mathfrak{V}$, given by

$$\Phi = \frac{\phi}{\sqrt{\gamma_3 \, G_3}} \; , \qquad \mathfrak{V}(\Phi) = \frac{\mathcal{V}(\phi)}{\gamma_3 \, G_3 \, \theta^2} \; . \tag{G.2}$$

Saini et al. [31] define instead a dimensionless scalar field ϕ^s and a dimensionless potential $\mathcal{V}^s$ through the equations

$$\frac{\Phi}{m_p} = \phi^s \; , \qquad \frac{\mathfrak{V}(\Phi)}{\rho_{cr}} = \mathcal{V}^s(\phi^s) \; , \tag{G.3}$$

where $m_p := \frac{1}{\sqrt{G_3}}$ is the Planck mass (in our units), $\rho_{cr} = \frac{1}{2 \, \gamma_3 \, G_3 \, \theta^2}$ is the critical density as in Eq. (E.52). From Eq.s (G.3) one has

$$\Phi = m_p \, \phi^s = \frac{\phi^s}{\sqrt{G_3}} \; , \qquad \mathfrak{V}(\Phi) = \rho_{cr} \, \mathcal{V}^s(\phi^s) = \frac{\mathcal{V}^s(\phi^s)}{2 \, \gamma_3 \, G_3 \, \theta^2} \; ; \tag{G.4}$$

comparing Eq.s (G.2) (G.4), we find out that

$$\phi^s = \frac{\phi}{\sqrt{\gamma_3}} \; , \qquad \mathcal{V}^s(\phi^s) = 2 \, \mathcal{V}(\phi) = 2 \, \mathcal{V}(\sqrt{\gamma_3} \, \phi^s) \; . \tag{G.5}$$

As an example, if we consider the potential

$$\mathcal{V}(\phi) = \frac{1}{9} \left(V_1 \, e^{3 \, \phi} + V_2 \, e^{-3 \, \phi} \right) \; , \tag{G.6}$$

of subsection 3, then

$$\mathcal{V}^s(\phi^s) = 2 \, \mathcal{V}(\sqrt{\gamma_3} \, \phi^s) = \frac{2}{9} \left(V_1 \, e^{3 \sqrt{\gamma_3} \, \phi^s} + V_2 \, e^{-3 \sqrt{\gamma_3} \, \phi^s} \right) \; . \tag{G.7}$$

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