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Derivation of imperfect interface models coupling damage and temperature

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Abstract In this paper we introduce a model describing a layered structure composed by two thermoelastic adherents and a thin adhesive subject to a degradation process. By an aymptotic expansion method, we derive a model of imperfect interface coupling damage and temperature evolution. Moreover, assuming that the behaviour of the adhesive is ruled by two different regimes, one in traction and one in compression, we derive a second limit model where unilateral contact conditions on the interface are also included.

Keywords Thin film · Bonding · Asymptotic analysis · Damage · Temperature · Imperfect interface

1 Introduction

In this paper, we perform a formal derivation of models of imperfect interface, coupling damage and temperature effects, as the formal asymptotic limits of models of a composite body made by two adherents with an adhesive substance located between them. The problem of finding effective models for imperfect interfaces is nowadays a subject of great interest both in engineering literature and in the analytical and numerical investigation of surface and bulk damage models [1, 8, 9, 11, 14–17, 19, 20]. Indeed, there are many applications of this

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kind of problems, in particular related to the development of layered composite structures. Moreover, it is known that interface regions between materials are fundamental to ensure strength and stability of structural elements. Thus, the theory of damage in (thermo)-elastic materials can be applied to derive models of contact with adhesion, assuming that the effectiveness of the adhesion between two bodies may be described in terms of a surface damage parameter (which is related to the active bonds in the adhesive substance on the contact interface).

Following the approach introduced in [3], we obtain models for a composite structure made by two thermoelastic bodies which are bonded together through an adhesive substance on a contact interface between them. It is assumed that microscopic damage in the interface may influence the strength of the adhesion and an unilateral condition is included ensuring non-penetrability between the bodies. We first consider a system describing the thermomechanical evolution of an adhesive substance located in a thin domain between two deformable bodies, subjected to the action of a damage process (described in terms of a damage parameter as in the phase transition theory), combined with thermal effects. The equations are recovered by the theory of Frémond for damage evolution of thermo-elastic materials [2,12,13], generalizing the principle of virtual powers and introducing the effects of microscopic forces, responsible for damage, in the whole energy balance of the system. Then, by using a formal asymptotic expansion method [18], letting the thickness of the adhesive substance go towards zero, we get limit models of imperfect interface coupling temperature and damage evolution. They are actually related with the models for contact with adhesion introduced and investigated, e.g., in [4–7]. Indeed, models describing contact with adhesion can be seen as surface damage models, in which the effectiveness of the adhesive bonds is related to the state of damage of microscopic cohesive links in the adhesive substance. In this sense a surface adhesive parameter actually corresponds to the local proportion of active bonds at the microscopic level of the adhesive substance, located on the contact surface. Note that some internal constraints on the damage parameter and its time derivative are considered, in order to guarantee physical consistency and to render the irreversible character of the degradation process, i.e. the material cannot repair itself once it results to be damaged. Moreover, assuming that the behaviour of the adhesive is ruled by two different regimes, one in traction and one in compression, in the limit model we include unilateral contact conditions on the interface, avoiding interpenetration between the adherents.

This kind of asymptotic analysis, introducing the interfaces as the limit of a thin medium bonding two adherents, has been investigated in the literature also to relate damage and delamination models (see e.g. [25] and [10,21]). Moreover, we recall [3] where, by the same approach used in the present paper, a model for imperfect interface with damage is obtained through an asymptotic analysis once the thickness ε of the adhesive substance between the adherents goes towards 0.

Here, in this contribution, we add the effects of the temperature, which is governed by evolution laws with different physical coefficients defined in the different regions and possibly depending on the thickness ε of the adhesive substance. Moreover, irreversibility of damage evolution is taken into account. The limit systems are complicated by the presence of internal constraints and by quadratic dissipative contributions coupling the equations. Actually, in the resulting limit models the jump of the temperatures ant the heat flux through the interface is activated by the evolution of surface damage. Consequently, the boundary conditions for the bulk equations of the temperatures are related to a dissipative evolution equation written on the interface for the damage parameter.

Now, let us make precise the outline of the paper. In Section 2 we introduce notation. In Section 3 we state the problem written in two main domains (the adherents) and on the thin layer located between them and corresponding to the adhesive substance with a given thickness $\varepsilon > 0$. In Section 4, we exploit the asymptotic expansion method to pass to the limit in the system as $\varepsilon \searrow 0$. Finally, in Section 5 we recover the unilateral condition in the limit system by use of an asymptotic analysis of some anisotropic property of the adhesive substance.

2 Nomenclature

In the following, a composite structure made by two adherents and a thin adhesive is considered. For the sake of simplicity, but without loss of generality, we simplify the geometry of the domain, as it is shown in Fig. 1. Then, the following notations are introduced:

- $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a Cartesian basis; the origin lies at the center of the adhesive midplane and the x_3-axis runs perpendicular to the plane $x_3 = 0$,
- (x_1, x_2, x_3) are the three coordinates of a particle,
- ε is the constant thickness of the adhesive,
- $B^\varepsilon = \{(x_1, x_2, x_3) \in \Omega^\varepsilon : \ |x_3| < \dfrac{\varepsilon}{2}\}$ is the domain of $\mathbb{R}^3$ occupied by the adhesive (or interphase),
- $\Omega^\varepsilon_\pm = \{(x_1, x_2, x_3) \in \Omega : \ \pm x_3 > \dfrac{\varepsilon}{2}\}$ are the two domains of $\mathbb{R}^3$ occupied by the adherents,
- $S^\varepsilon_\pm = \{(x_1, x_2, x_3) \in \Omega : \ x_3 = \pm\dfrac{\varepsilon}{2}\}$ are the interfaces between the adhesive and the adherents,
- $S_g \subset \partial\Omega^\varepsilon$ is the part of the boundary where an external load g is applied,
- $S_u \subset \partial\Omega^\varepsilon$ is the part of the boundary where the displacement is imposed to be equal to 0,
- f is a body force which is applied in $\Omega^\varepsilon_\pm$,
- $S = \{(x_1, x_2, x_3) \in \Omega^\varepsilon : \ x_3 = 0\}$ will be called in the following the interface, as it will formally correspond to the limit adhesive interface between the two bodies,

- $S_{\pm} = \{(x_1, x_2, x_3) \in \Omega : \; x_3 = \pm\frac{1}{2}\}$ are the rescaled interfaces between the adhesive and the adherents,
- u^ε is the displacement field,
- θ^ε is the temperature field,
- σ^ε is the Cauchy stress tensor field,
- q^ε is the thermal flux,
- $e(u^\varepsilon)$ is the strain tensor field which, under the small strain hypothesis, is defined by $e_{ij}(u^\varepsilon) = \frac{1}{2}(u^\varepsilon_{i,j} + u^\varepsilon_{j,i})$, where the comma denotes the partial derivative,
- $q^\varepsilon_{i,i}$ means $\operatorname{div} q^\varepsilon$, since repeated indexes are implicitly summed over; analogously, $\sigma^\varepsilon_{ij,j}$ means $\operatorname{div} \sigma^\varepsilon$,
- χ^ε is the damage variable field,
- λ^ε and μ^ε are the Lamé's coefficients of the adhesive, which depend a priori on the thickness ε,
- a^ε is the fourth order elasticity tensor of the adhesive, verifying the usual conditions of positivity and symmetry, which depends a priori on the thickness ε,
- α^ε, c^ε, ω^ε, η^ε and γ^ε are given material positive coefficients of the adhesive, which depend a priori on the thickness ε,
- $a^\pm$ is the fourth order elasticity tensor of the adherents, verifying the usual conditions of positivity and symmetry, which does not depend on the thickness ε,
- $\alpha^\pm$, $c^\pm$ and $\gamma^\pm$ are given material positive coefficients of the adherents, which do not depend on the thickness ε,
- $\mathbb{I}$ is the identity second order tensor in three dimensions,
- $\dot{f}$ is the time derivative of a function f,
- $[[f]]_\pm$ denotes the jump of f along $S^\varepsilon_\pm$ i.e. $f(x_1, x_2, (\pm\varepsilon/2)^\pm) - f(x_1, x_2, (\pm\varepsilon/2)^\mp)$. We recall that $f(a^+) = \lim_{x \longrightarrow a, x>a} f(x)$ and $f(a^-) = \lim_{x \longrightarrow a, x<a} f(x)$,
- $(\;)_+$ denote the positive part of a function, i.e. $(x)_+ = \max\{x, 0\}$,
- $(\;)_-$ denotes the negative part of a function, i.e. $(x)_- = \min\{x, 0\}$,
- $[f] = f(x_1, x_2, \frac{1}{2}) - f(x_1, x_2, -\frac{1}{2})$ denotes the jump of f along $S\pm$,
- $[] \, f \, [] = f(x_1, x_2, 0^+) - f(x_1, x_2, 0^-)$ denotes the jump of f along S,
- $\langle f \rangle_0 = \frac{1}{2}\big(f(x_1, x_2, 0^+) + f(x_1, x_2, 0^-)\big)$ denotes the average of a function f on S,
- I_A denotes the indicator function of the set A, i.e. $I_A(x) = 0$ if $x \in A$ and $I_A(x) = +\infty$ if $x \notin A$.

Moreover, it is assumed that

- $S_g \cap S_u = \emptyset$,
- $S_u \cap \partial B^\varepsilon = \emptyset$,
- $S_g \cap \partial B^\varepsilon = \emptyset$.

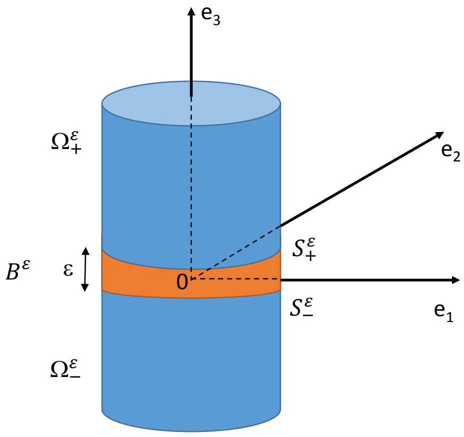

Fig. 1 Composite body: initial structure

3 The three-dimensional equations of the composite body

A composite structure made by two adherents and a thin adhesive is considered (see figure 1). The three solids are supposed to be deformable. The two adherents are supposed to be thermoelastic. We introduce the free energy functional, defined in the two adherents and in the domain occupied by the adhesive. For $\varepsilon > 0$, we let ψ^+ (respectively ψ^-) the free energy in Ω^ε_+ (respectively in Ω^ε_-) be defined as it follows

$$\psi^\pm(e(u^\varepsilon), \theta^\varepsilon) = \frac{1}{2} a^\pm e(u^\varepsilon) : e(u^\varepsilon) + \alpha^\pm \theta^\varepsilon tr(e(u^\varepsilon)) - c^\pm \theta^\varepsilon log \theta^\varepsilon \tag{1}$$

and, using the analogous notation, the pseudo-potential of dissipation is defined by

$$\phi^\pm(\nabla\theta^\varepsilon) = \frac{\gamma^\pm}{2\theta^\varepsilon} |\nabla\theta^\varepsilon|^2 \tag{2}$$

Moreover, we suppose that the adhesive is an isotropic thermoelastic material undergoing a damaging process. According to the Frémond theory on damage in thermo(visco)elasticity [2,12,13], we specify the free energy in the interphase B^ε as follows

$$\begin{aligned}\psi^\varepsilon(e(u^\varepsilon),\theta^\varepsilon,\chi^\varepsilon) = \frac{1}{2}\chi^\varepsilon a^\varepsilon e(u^\varepsilon):e(u^\varepsilon)+\omega^\varepsilon(1-\chi^\varepsilon)\\ +\alpha^\varepsilon\chi^\varepsilon\theta^\varepsilon tr(e(u^\varepsilon))-c^\varepsilon\theta^\varepsilon log\theta^\varepsilon+I_{[0,1]}(\chi^\varepsilon)\end{aligned} \tag{3}$$

where

$$a^\varepsilon e(u^\varepsilon):e(u^\varepsilon)=\lambda^\varepsilon(e_{kk}(u^\varepsilon))^2+2\mu^\varepsilon(e(u^\varepsilon))^2 \tag{4}$$

In (3) the indicator function $I_{[0,1]}$ yields the constraint on the damage parameter i.e $\chi^\varepsilon\in[0,1]$, where $\chi^\varepsilon=1$ and $\chi^\varepsilon=0$ correspond to the undamaged and completely damaged material, respectively. Next, we introduce the pseudo-potential of dissipation by

$$\phi^\varepsilon(\nabla\theta^\varepsilon,\dot\chi^\varepsilon)=\frac{1}{2}\frac{\gamma^\varepsilon}{\theta^\varepsilon}\left|\nabla\theta^\varepsilon\right|^2+\frac{1}{2}\eta^\varepsilon\left|\dot\chi^\varepsilon\right|^2+I_{\mathbb{R}^-}(\dot\chi^\varepsilon) \tag{5}$$

Note that the term $I_{\mathbb{R}^-}(\dot\chi^\varepsilon)$ forces $\dot\chi^\varepsilon$ to assume non-positive values and renders the irreversible character of the damage.

In the adherents, it is obtained the classical thermoelastic constitutive equation

$$\sigma^\varepsilon=\partial\psi^\pm_{,e}(e(u^\varepsilon),\theta^\varepsilon)=a^\pm e(u^\varepsilon)+\alpha^\pm\theta^\varepsilon\mathbb{I} \tag{6}$$

Moreover, the entropy is defined by

$$s^\varepsilon=-\partial\psi^\pm_{,\theta}(e(u^\varepsilon),\theta^\varepsilon)=c^\pm(log\theta^\varepsilon+1)-\alpha^\pm tr(e(u^\varepsilon)) \tag{7}$$

and the thermal flux by

$$q^\varepsilon=\theta^\varepsilon\partial\phi^\pm_{,\nabla\theta}(\nabla\theta^\varepsilon)=\gamma^\pm\nabla\theta^\varepsilon \tag{8}$$

In the same way, it is obtained in the adhesive

$$\sigma^\varepsilon=\partial\psi^\varepsilon_{,e}(e(u^\varepsilon),\theta^\varepsilon,\chi^\varepsilon)=\chi^\varepsilon a^\varepsilon e(u^\varepsilon)+\alpha^\varepsilon\theta^\varepsilon\chi^\varepsilon\mathbb{I}, \tag{9}$$

$$q^\varepsilon=\theta^\varepsilon\partial\phi^\varepsilon_{,\nabla\theta}(\nabla\theta^\varepsilon,\dot\chi^\varepsilon)=\gamma^\varepsilon\nabla\theta^\varepsilon \tag{10}$$

To simplify notations, in the following we use the indices $i,j=1,2,3$ while the notation α for the index when it is intended to vary just for $\alpha=1,2$.

The equilibrium problem of the composite structure is described by the following system

$$
\begin{cases}
\sigma^\varepsilon_{ij,j} + f_i = 0 & \text{in } \Omega^\varepsilon_\pm \\
c^\pm \theta^\varepsilon - q^\varepsilon_{i,i} = 0 & \text{in } \Omega^\varepsilon_\pm \\
\sigma^\varepsilon_{ij} n_j = g_i & \text{on } S_g \\
[[\sigma^\varepsilon_{i3}]]_\pm = [[q^\varepsilon_i]]_\pm = 0 & \text{on } S^\varepsilon_\pm \\
[[u^\varepsilon_i]]_\pm = [[\theta^\varepsilon]]_\pm = 0 & \text{on } S^\varepsilon_\pm \\
u^\varepsilon_i = 0 & \text{on } S_u \\
\sigma^\varepsilon_{ij} = a^\pm_{ijhk} e_{hk}(u^\varepsilon) + \alpha^\pm \theta^\varepsilon \delta_{ij} & \text{in } \Omega^\varepsilon_\pm \\
q^\varepsilon = \gamma^\pm \nabla \theta^\varepsilon & \text{in } \Omega^\varepsilon_\pm \\
q^\varepsilon_i n_i = 0 & \text{on } \partial\Omega^\varepsilon_\pm \setminus S^\varepsilon_\pm \\
\sigma^\varepsilon_{ij,j} = 0 & \text{in } B^\varepsilon \\
c^\varepsilon \dot{\theta}^\varepsilon - q^\varepsilon_{i,i} - \alpha^\varepsilon \theta^\varepsilon (\dot{\chi}^\varepsilon div u^\varepsilon + \chi^\varepsilon div \dot{u}^\varepsilon) = \eta^\varepsilon \left|\dot{\chi}^\varepsilon\right|^2 & \text{in } B^\varepsilon \\
\sigma^\varepsilon_{ij} = \chi^\varepsilon a^\varepsilon e(u^\varepsilon) + \alpha^\varepsilon \theta^\varepsilon \chi^\varepsilon \ \mathbb{I} & \text{in } B^\varepsilon \\
q^\varepsilon = \gamma^\varepsilon \nabla \theta^\varepsilon & \text{in } B^\varepsilon \\
\eta^\varepsilon \dot{\chi}^\varepsilon = \left(\omega^\varepsilon - \frac{1}{2} a^\varepsilon e(u^\varepsilon) : e(u^\varepsilon) - \alpha^\varepsilon \theta^\varepsilon tr(e(u^\varepsilon)) \right)_- & \text{in } B^\varepsilon \\
\chi^\varepsilon > 0 & \text{in } B^\varepsilon
\end{cases}
\tag{11}
$$

supplemented by given initial data. In particular, on the initial condition χ_0 for the damage variable we assume that $0 < \chi_0 \leq 1$. Note that this condition along with the irreversible character of the damage process ($\dot{\chi}^\varepsilon$ cannot increase) yields in particular the upper bound $\chi^\varepsilon \leq 1$.

In the following, the adhesive will be supposed as isotropic i.e. defined by the Lamé coefficients λ^ε and μ^ε.

4 The asymptotic expansion method

Since the thickness of the interphase is very small, it is natural to seek the solution of problem (11) using asymptotic expansions with respect to the parameter ε [20–23]. In particular, the following asymptotic series with integer powers are assumed:

$$
\begin{cases}
\mathbf{u}^\varepsilon = \mathbf{u}^0 + \varepsilon\, \mathbf{u}^1 + o(\varepsilon) \\
\sigma^\varepsilon = \sigma^0 + \varepsilon\, \sigma^1 + o(\varepsilon) \\
\theta^\varepsilon = \theta^0 + \varepsilon\, \theta^1 + o(\varepsilon) \\
q^\varepsilon = q^0 + \varepsilon\, q^1 + o(\varepsilon) \\
\chi^\varepsilon = \chi^0 + \varepsilon\, \chi^1 + o(\varepsilon)
\end{cases}
\tag{12}
$$

The domain is then rescaled (see figure 2) using a classical procedure:

- In the adhesive, the following change of variable is introduced

$$(x_1, x_2, x_3) \in B^\varepsilon \rightarrow (z_1, z_2, z_3) \in B, \text{ with } (z_1, z_2, z_3) = (x_1, x_2, \frac{x_3}{\varepsilon})$$

  and it is set $\hat{\mathbf{u}}^\varepsilon(z_1, z_2, z_3) = \mathbf{u}^\varepsilon(x_1, x_2, x_3)$ and $\hat{\sigma}^\varepsilon(z_1, z_2, z_3) = \sigma^\varepsilon(x_1, x_2, x_3)$, where $B = \{(x_1, x_2, x_3) \in \Omega : \ |x_3| < \frac{1}{2}\}$.

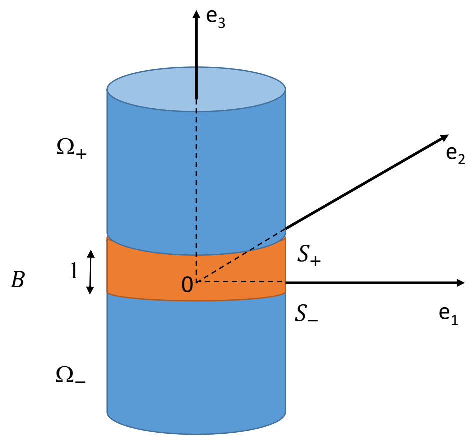

Fig. 2 Composite body: rescaled structure

– In the adherent, the following change of variable is introduced

$$(x_1, x_2, x_3) \in \Omega^\varepsilon_\pm \rightarrow (z_1, z_2, z_3) \in \Omega_\pm, \text{ with } (z_1, z_2, z_3) = (x_1, x_2, x_3 \pm 1/2 \mp \varepsilon/2)$$

and it is set $\bar{\mathbf{u}}^\varepsilon(z_1, z_2, z_3) = \mathbf{u}^\varepsilon(x_1, x_2, x_3)$ and $\bar{\sigma}^\varepsilon(z_1, z_2, z_3) = \sigma^\varepsilon(x_1, x_2, x_3)$, where $\Omega_\pm = \{(x_1, x_2, x_3) \in \Omega : \pm x_3 > \frac{1}{2}\}$. The external forces is assumed to be independent of ε. As a consequence, it is set $\bar{f}(z_1, z_2, z_3) = f(x_1, x_2, x_3)$ and $\bar{g}(z_1, z_2, z_3) = g(x_1, x_2, x_3)$.

The governing equations of the rescaled problem are as follows:

$$
\begin{cases}
\bar{\sigma}^{\varepsilon}_{ij,j} + \bar{f}_i = 0 & \text{in } \Omega_{\pm} \\
c^{\pm}\dot{\bar{\theta}}^{\varepsilon} - \bar{q}^{\varepsilon}_{i,i} = 0 & \text{in } \Omega_{\pm} \\
\bar{\sigma}^{\varepsilon}_{ij} n_j = \bar{g}_i & \text{on } \bar{S}_g \\
\bar{u}^{\varepsilon}_i = 0 & \text{on } \bar{S}_u \\
\bar{\sigma}^{\varepsilon}_{ij} = \bar{a}^{\pm}_{ijhk}\bar{e}_{hk}(\bar{u}^{\varepsilon}) + \bar{\alpha}^{\pm}\bar{\theta}^{\varepsilon}\delta_{ij} & \text{in } \Omega_{\pm} \\
\bar{q}^{\varepsilon} = \bar{\gamma}^{\pm}\nabla\bar{\theta}^{\varepsilon} & \text{in } \Omega_{\pm} \\
\bar{\sigma}^{\varepsilon}_{i3} = \hat{\sigma}^{\varepsilon}_{i3} & \text{on } S_{\pm} \\
\bar{u}^{\varepsilon}_i = \hat{u}^{\varepsilon}_i & \text{on } S_{\pm} \\
\bar{\theta}^{\varepsilon} = \hat{\theta}^{\varepsilon} & \text{on } S_{\pm} \\
\bar{q}^{\varepsilon}_i = \hat{q}^{\varepsilon}_i & \text{on } S_{\pm} \\
\bar{q}^{\varepsilon}_i n_i = 0 & \text{on } \partial\Omega_{\pm} \setminus S_{\pm} \\
\hat{\sigma}^{\varepsilon}_{ij,j} = 0 & \text{in } B \\
\hat{\sigma}^{\varepsilon}_{ij} = \hat{\chi}^{\varepsilon}\hat{a}_{ijhk}\hat{e}_{hk}(\hat{u}^{\varepsilon}) + \hat{\chi}^{\varepsilon}\hat{\alpha}^{\varepsilon}\hat{\theta}^{\varepsilon}\delta_{ij} & \text{in } B \\
\hat{c}^{\varepsilon}\dot{\hat{\theta}}^{\varepsilon} - \hat{q}^{\varepsilon}_{i,i} - \hat{\alpha}^{\varepsilon}\hat{\theta}^{\varepsilon}(\dot{\hat{\chi}}^{\varepsilon} div\hat{u}^{\varepsilon} + \hat{\chi}^{\varepsilon} div\dot{\hat{u}}^{\varepsilon}) = \hat{\eta}^{\varepsilon}\left|\dot{\hat{\chi}}^{\varepsilon}\right|^2 & \text{in } B \\
\hat{q}^{\varepsilon} = \hat{\gamma}^{\varepsilon}\nabla\hat{\theta}^{\varepsilon} & \text{in } B \\
\hat{\eta}^{\varepsilon}\dot{\hat{\chi}}^{\varepsilon} = \left(\hat{\omega}^{\varepsilon} - \dfrac{1}{2}\hat{a}^{\varepsilon}\hat{e}(\hat{u}^{\varepsilon}) : \hat{e}(\hat{u}^{\varepsilon}) - \hat{\alpha}^{\varepsilon}\hat{\theta}^{\varepsilon} tr(e(\hat{u}^{\varepsilon}))\right)_{-} & \text{in } B \\
\hat{\chi}^{\varepsilon} > 0 & \text{in } B
\end{cases}
\tag{13}
$$

where $\bar{\cdot}, \hat{\cdot}$ denote the rescaled operators in the adherents and in the adhesive, respectively.
In view of (12) the displacement field, stress field, flux field, temperature field and the damage field are written as asymptotic expansions

$$
\begin{cases}
\hat{\sigma}^{\varepsilon} = \hat{\sigma}^0 + \varepsilon\,\hat{\sigma}^1 + o(\varepsilon) \\
\hat{u}^{\varepsilon} = \hat{u}^0 + \varepsilon\,\hat{u}^1 + o(\varepsilon) \\
\hat{q}^{\varepsilon} = \hat{q}^0 + \varepsilon\,\hat{q}^1 + o(\varepsilon) \\
\hat{\chi}^{\varepsilon} = \hat{\chi}^0 + \varepsilon\,\hat{\chi}^1 + o(\varepsilon) \\
\hat{\theta}^{\varepsilon} = \hat{\theta}^0 + \varepsilon\,\hat{\theta}^1 + o(\varepsilon) \\
\bar{\sigma}^{\varepsilon} = \bar{\sigma}^0 + \varepsilon\,\bar{\sigma}^1 + o(\varepsilon) \\
\bar{u}^{\varepsilon} = \bar{u}^0 + \varepsilon\,\bar{u}^1 + o(\varepsilon) \\
\bar{q}^{\varepsilon} = \bar{q}^0 + \varepsilon\,\bar{q}^1 + o(\varepsilon) \\
\bar{\theta}^{\varepsilon} = \bar{\theta}^0 + \varepsilon\,\bar{\theta}^1 + o(\varepsilon)
\end{cases}
\tag{14}
$$

in the rescaled adhesive and adherents, respectively. In the following, only the soft case is considered i.e. $\lambda^{\varepsilon} = \varepsilon\lambda^0$, $\mu^{\varepsilon} = \varepsilon\mu^0$, $\gamma^{\varepsilon} = \varepsilon\gamma^0$ and $\alpha^{\varepsilon} = \varepsilon\alpha^0$. In addition, it is supposed that $\eta^{\varepsilon} = \varepsilon^{-1}\eta^{-1}$ and $\omega^{\varepsilon} = \varepsilon^{-1}\omega^{-1}$.

4.1 Expansions of the equilibrium equations in the adherents

Substituting (14) into the first to sixth equations of (13) and into the eleventh one, it is obtained at the first order of expansion (power 0 in ε)

$$\begin{cases} \bar{\sigma}^0_{ij,j} + \bar{f}_i = 0 & \text{in } \Omega_\pm \\ \bar{\sigma}^0_{ij} n_j = \bar{g}_i & \text{on } \bar{S}_g \\ \bar{u}^0_i = 0 & \text{on } \bar{S}_u \\ \bar{\sigma}^0_{ij} = \bar{a}^\pm_{ijhk} \bar{e}_{hk}(\bar{u}^0) + \bar{\alpha}^\pm \bar{\theta}^0 \delta_{ij} & \text{in } \Omega_\pm \\ c^\pm \dot{\bar{\theta}}^0 - \bar{q}^0_{i,i} = 0 & \text{in } \Omega_\pm \\ \bar{q}^0 = \bar{\gamma}^\pm \nabla \bar{\theta}^0 & \text{in } \Omega_\pm \\ \bar{q}^0_i n_i = 0 & \text{on } \partial\Omega_\pm \setminus S_\pm . \end{cases} \tag{15}$$

A quite classical problem of thermo-elasticity is obtained.

4.2 Expansions of the equilibrium equations in the adhesive

Substituting (14) into the twelfth equation of (13) it is deduced that the following conditions hold in B (power -1 in the expansions):

$$\hat{\sigma}^0_{i3,3} = 0, \tag{16}$$

i.e. $\hat{\sigma}^0_{i3}$ does not depend on z_3, that it can be expressed as

$$\left[\hat{\sigma}^0_{i3}\right] = 0 \tag{17}$$

In the adhesive the strain field becomes:

$$\hat{e}(\hat{u}^\varepsilon) = \varepsilon^{-1}\hat{e}^{-1} + \hat{e}^0 + \varepsilon \hat{e}^1 + o(\varepsilon) \tag{18}$$

where

$$\begin{aligned} \hat{e}^{-1}_{33} &= \hat{u}^0_{3,3} \\ \hat{e}^{-1}_{\alpha 3} &= \frac{1}{2}\hat{u}^0_{\alpha,3}, \end{aligned} \tag{19}$$

Thus it is obtained

$$\hat{\sigma}^0_{\alpha 3} = \hat{\chi}^0 \mu^0 \hat{u}^0_{\alpha,3} \tag{20}$$

and

$$\hat{\sigma}^0_{33} = \hat{\chi}^0 (\lambda^0 + 2\mu^0) \hat{u}^0_{3,3} \tag{21}$$

It is observed that $\hat{\sigma}^0_{i3}$ does not depend on z_3, thus

$$\begin{aligned} \mu^0 \hat{u}^0_{\alpha,3} &= \left(\hat{\chi}^0\right)^{-1} \hat{\sigma}^0_{\alpha 3} \\ (\lambda^0 + 2\mu^0)\hat{u}^0_{3,3} &= \left(\hat{\chi}^0\right)^{-1} \hat{\sigma}^0_{33} \end{aligned} \tag{22}$$

and thus by integration in z_3

$$\begin{aligned} \hat{\sigma}^0_{\alpha 3} &= \left\langle\left\langle \hat{\chi}^0 \right\rangle\right\rangle \mu^0 \left[\hat{u}^0_\alpha\right] \\ \hat{\sigma}^0_{33} &= \left\langle\left\langle \hat{\chi}^0 \right\rangle\right\rangle (\lambda^0 + 2\mu^0) \left[\hat{u}^0_3\right] \end{aligned} \tag{23}$$

where $\langle\langle\hat{\chi}^0\rangle\rangle = \left(\int\limits_{-1/2}^{1/2} \left(\hat{\chi}^0\right)^{-1} dz_3\right)^{-1}$.

It is obtained a soft model of imperfect interface with damage.

In the following lines, we will prove that $\hat{\chi}^0$ does not depend on z_3.

If we consider that the damage parameter $\hat{\chi}^\varepsilon$ decreases (i.e. $\dot{\hat{\chi}}^\varepsilon \leq 0$), the first term in the expansion gives (power -1 in ε)

$$\eta^{-1}\dot{\hat{\chi}}^0 = \omega^{-1} - \frac{1}{2}\left(\mu^0((\hat{u}^0_{1,3})^2 + (\hat{u}^0_{2,3})^2) + (\lambda^0 + 2\mu^0)(\hat{u}^0_{3,3})^2\right)$$

or equivalently

$$\eta^{-1}\dot{\hat{\chi}}^0 = \omega^{-1} - \frac{1}{2}\left(\hat{\sigma}^0_{13}\hat{u}^0_{1,3} + \hat{\sigma}^0_{23}\hat{u}^0_{2,3} + \hat{\sigma}^0_{33}\hat{u}^0_{3,3}\right)\left(\hat{\chi}^0\right)^{-1}$$

or equivalently

$$\eta^{-1}\dot{\hat{\chi}}^0 = \omega^{-1} - \frac{1}{2}\left(\left(\hat{\sigma}^0_{13}\right)^2/\mu^0 + \left(\hat{\sigma}^0_{23}\right)^2/\mu^0 + \left(\hat{\sigma}^0_{33}\right)^2/(\lambda^0 + 2\mu^0)\right)\left(\hat{\chi}^0\right)^{-2}$$

Now, these three last equations can be integrated along the third direction. Let $\langle\hat{\chi}^0\rangle = \int\limits_{-1/2}^{1/2} \hat{\chi}^0 dz_3$ and $\langle\langle\hat{\chi}^0\rangle\rangle_2 = \int\limits_{-1/2}^{1/2} \left(\hat{\chi}^0\right)^{-2} dz_3$. Particularly, it is obtained

$$\eta^{-1}\left\langle\dot{\hat{\chi}}^0\right\rangle = \omega^{-1} - \frac{1}{2}\left(\mu^0\left[\hat{u}^0_1\right]^2 + \mu^0\left[\hat{u}^0_2\right]^2 + (\lambda^0 + 2\mu^0)\left[\hat{u}^0_3\right]^2\right)\langle\langle\hat{\chi}^0\rangle\rangle_2\langle\langle\hat{\chi}^0\rangle\rangle^2$$

or

$$\eta^{-1}\left\langle\dot{\hat{\chi}}^0\hat{\chi}^0\right\rangle = \omega^{-1}\langle\hat{\chi}^0\rangle - \frac{1}{2}\left(\mu^0\left[\hat{u}^0_1\right]^2 + \mu^0\left[\hat{u}^0_2\right]^2 + (\lambda^0 + 2\mu^0)\left[\hat{u}^0_3\right]^2\right)\langle\langle\hat{\chi}^0\rangle\rangle$$

The two last equations are verified simultaneously for any choice of η^{-1} and ω^{-1} if and only if $\left\langle\dot{\hat{\chi}}^0\hat{\chi}^0\right\rangle = \left\langle\dot{\hat{\chi}}^0\right\rangle\langle\hat{\chi}^0\rangle$. This latter (assuming that the initial value of $\hat{\chi}^0$ does not depend on z_3) is equivalent to $\left(\langle\hat{\chi}^0\rangle\right)^2 = \int\limits_{-1/2}^{1/2}(\hat{\chi})^2 dz_3$ and this holds if and only if $\hat{\chi}^0$ does not depend on z_3.

It is obtained

$$\eta^{-1}\dot{\hat{\chi}}^0 = \omega^{-1} - \frac{1}{2}\left(\mu^0\left[\hat{u}^0_1\right]^2 + \mu^0\left[\hat{u}^0_2\right]^2 + (\lambda^0 + 2\mu^0)\left[\hat{u}^0_3\right]^2\right)$$

This equation can be decomposed, as classical, into normal and tangential parts

$$\eta^{-1}\dot{\hat{\chi}}^0 = \omega^{-1} - \frac{1}{2}\left(\lambda^0 + 2\mu^0\right)\left[\hat{u}^0_N\right]^2 - \frac{1}{2}\mu_0\left[\hat{u}^0_T\right].\left[\hat{u}^0_T\right]$$

Now, we are interested by the two penultimate equations in (13). Let us consider the case where c^ε does not depend on ε. It is obtained

$$-\hat{q}^0_{3,3} = \eta^{-1}\left|\dot{\hat{\chi}}^0\right|^2$$

$$\hat{q}^0_{\alpha,\alpha} = 0, \ \ \alpha = 1,2$$

$$\hat{q}^0_3 = \gamma^0 \hat{\theta}^0_{,3}$$

and

$$[\hat{q}^0_3] = -\eta^{-1}\left|\dot{\hat{\chi}}^0\right|^2$$

4.3 Matching between the adhesive and the adherents

Substituting (16) into the seventh to tenth equations of (13), it is deduced that the following conditions hold on $S_\pm$:

$$\begin{aligned}
\hat{\sigma}^0_{i3}(z_1,z_2,\pm\frac{1}{2}) &= \bar{\sigma}^0_{i3}(z_1,z_2,\pm\frac{1}{2}) = \sigma^0_{i3}(x_1,x_2,\pm\frac{\varepsilon}{2}) \approx \sigma^0_{i3}(x_1,x_2,0)\\
\hat{u}^0_i(z_1,z_2,\pm\frac{1}{2}) &= \bar{u}^0_i(z_1,z_2,\pm\frac{1}{2}) = u^0_i(x_1,x_2,\pm\frac{\varepsilon}{2}) \approx u^0_i(x_1,x_2,0^\pm)\\
\hat{\theta}^0(z_1,z_2,\pm\frac{1}{2}) &= \bar{\theta}^0(z_1,z_2,\pm\frac{1}{2}) = \theta^0(x_1,x_2,\pm\frac{\varepsilon}{2}) \approx \theta^0(x_1,x_2,0^\pm)\\
\hat{q}^0(z_1,z_2,\pm\frac{1}{2}) &= \bar{q}^0(z_1,z_2,\pm\frac{1}{2}) = q^0(x_1,x_2,\pm\frac{\varepsilon}{2}) \approx q^0(x_1,x_2,0^\pm)
\end{aligned} \tag{24}$$

In conclusion, it is obtained on the final configuration (figure 3)

$$\left\{\begin{array}{ll}
\sigma^0_{ij,j} + f_i = 0 & \text{in } \Omega_\pm\\
\sigma^0_{ij} n_j = g_i & \text{on } S_g\\
u^0_i = 0 & \text{on } S_u\\
\sigma^0_{ij} = a^\pm_{ijhk} e_{hk}(u^0) + \alpha^\pm \theta^0 \delta_{ij} & \text{in } \Omega_\pm\\
c^\pm \dot{\theta}^0 - q^0_{i,i} = 0 & \text{in } \Omega_\pm\\
q^0 = \gamma^\pm \nabla\theta^0 & \text{in } \Omega_\pm\\
q^0_i n_i = 0 & \text{on } \partial\Omega_\pm \setminus S\\
[\!]\, \sigma^0_{i3}\, [\!] = 0 & \text{on } S\\
\sigma^0_{\alpha 3} = \chi^0 \mu^0 \,[\!]\, u^0_\alpha \,[\!] & \text{on } S\\
\sigma^0_{33} = \chi^0 \left(\lambda^0 + 2\mu^0\right) [\!]\, u^0_3 \,[\!] & \text{on } S\\
\eta^{-1}\dot{\chi}^0 = \left(\omega^{-1} - \left(\mu^0 \,[\!]\, u^0_1 \,[\!]^2 + \mu^0 \,[\!]\, u^0_2 \,[\!]^2 + (\lambda^0 + 2\mu^0) \,[\!]\, u^0_3 \,[\!]^2\right)\right)_- & \text{on } S\\
\chi^0 > 0 & \text{on } S\\
[\!]\, q^0_3 \,[\!] = -\eta^{-1}\left|\dot{\chi}^0\right|^2 & \text{on } S\\
[\!]\, \theta^0 \,[\!] = (\gamma^0)^{-1} \left\langle q^0_3 \right\rangle_0 & \text{on } S
\end{array}\right. \tag{25}$$

It is obtained a model of imperfect soft interface with damage evolution which takes into account thermal variations. The temperature is activated by the damage evolution.

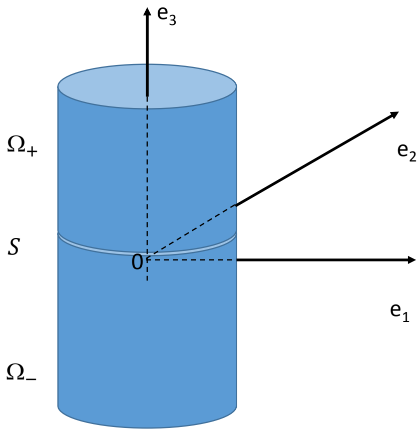

Fig. 3 Composite body: final configuration

5 Introducing unilateral contact

In this section, it is shown how to introduce unilateral conditions [15,11]. The initial system of constitutive equations

$$\begin{cases} \sigma^{\varepsilon} = \chi^{\varepsilon} a^{\varepsilon} e(u^{\varepsilon}) + \alpha^{\varepsilon}\theta^{\varepsilon}\chi^{\varepsilon}\ \mathbb{I} & \text{in } B^{\varepsilon} \\ \eta^{\varepsilon}\dot{\chi}^{\varepsilon} = \left(\omega^{\varepsilon} - \dfrac{1}{2} a^{\varepsilon} e(u^{\varepsilon}) : e(u^{\varepsilon}) - \alpha^{\varepsilon}\theta^{e} tr(e(u^{\varepsilon}))\right)_{-} & \text{in } B^{\varepsilon} \end{cases} \tag{26}$$

is replaced by constitutive equations with two regimes

$$\begin{cases} \sigma^{\varepsilon} = \chi^{\varepsilon} a^{\varepsilon} e(u^{\varepsilon}) + \alpha^{\varepsilon}\theta^{\varepsilon}\chi^{\varepsilon}\ \mathbb{I} \quad \text{if } tre(u^{\varepsilon}) \geq 0 & \text{in } B^{\varepsilon} \\ \sigma^{\varepsilon} = \chi^{\varepsilon} b^{\varepsilon} e(u^{\varepsilon}) + \alpha^{\varepsilon}\theta^{\varepsilon}\chi^{\varepsilon}\ \mathbb{I} \quad \text{if } tre(u^{\varepsilon}) \leq 0 & \text{in } B^{\varepsilon} \\ \eta^{\varepsilon}\dot{\chi}^{\varepsilon} = \left(\omega^{\varepsilon} - \dfrac{1}{2} a^{\varepsilon} e(u^{\varepsilon}) : e(u^{\varepsilon}) - \alpha^{\varepsilon}\theta^{e} tr(e(u^{\varepsilon}))\right)_{-} & \text{if } tre(u^{\varepsilon}) \geq 0 \text{ in } B^{\varepsilon} \\ \eta^{\varepsilon}\dot{\chi}^{\varepsilon} = \left(\omega^{\varepsilon} - \dfrac{1}{2} b^{\varepsilon} e(u^{\varepsilon}) : e(u^{\varepsilon}) - \alpha^{\varepsilon}\theta^{e} tr(e(u^{\varepsilon}))\right)_{-} & \text{if } tre(u^{\varepsilon}) \leq 0 \text{ in } B^{\varepsilon} \end{cases} \tag{27}$$

where b^{ε} is a fourth order tensor of elasticity with the usual conditions of symmetry and positivity. In addition, the material is supposed to be isotropic

and the Lamé' coefficients associated to b^ε are $\lambda^\varepsilon = \lambda^1$ and $\mu^\varepsilon = \varepsilon\mu^0$.
The asymptotic method presented above is used. Let us consider the term $tre(\hat{u}^\varepsilon)$ whose first term in the expansion (order -1) is $\hat{u}^0_{3,3}$. By integration in z_3 it becomes $\left[\hat{u}^0_3\right]$. Thus $tre(\hat{u}^\varepsilon) \geq 0$ (resp. $tre(\hat{u}^\varepsilon) \leq 0$) gives $\left[\hat{u}^0_3\right] \geq 0$ (resp. $\left[\hat{u}^0_3\right] \leq 0$). Note that the second term in the expansion gives $\hat{u}^0_{1,1} + \hat{u}^0_{2,2} + \hat{u}^1_{3,3}$. Proceeding as in the previous section, in the case $tre(\hat{u}^\varepsilon) \geq 0$ (leading to $\left[\hat{u}^0_3\right] \geq 0$ and $[]\, u^0_3\, [] \geq 0$), we have

$$\sigma^0 \mathbf{e}_3 = \chi^0 \left(\mu^0 \,[]\, u^0_1 \,[]\, , \mu^0 \,[]\, u^0_2 \,[]\, , (\lambda^0 + 2\mu^0) \,[]\, u^0_3 \,[]\right)$$

and

$$\eta^{-1}\dot{\chi}^0 = \left(\omega^{-1} - \left(\mu^0 \,[]\, u^0_1 \,[]^2 + \mu^0 \,[]\, u^0_2 \,[]^2 + (\lambda^0 + 2\mu^0) \,[]\, u^0_3 \,[]^2\right)\right)_-$$

Now, the case $tre(\hat{u}^\varepsilon) \leq 0$ (corresponding to $\left[\hat{u}^0_3\right] \leq 0$ or $[]\, u^0_3\, [] \leq 0$) is studied. The (constitutive) second equation of (27) leads to

$$\hat{u}^0_{3,3} = 0$$

or equivalently

$$\left[\hat{u}^0_3\right] = 0$$

or

$$[]\, u^0_3\, [] = 0$$

and also to

$$\hat{\sigma}^0_{\alpha 3} = \hat{\chi}^0 \mu^0 \hat{u}^0_{\alpha,3}, \;\; \alpha = 1, 2$$

Moreover, it holds

$$\hat{\sigma}^0_{33} = \hat{\chi}^0 \lambda^1 (\hat{u}^0_{1,1} + \hat{u}^0_{2,2} + \hat{u}^1_{3,3}) + 2\mu^0 \hat{u}^0_{3,3} = \hat{\chi}^0 \lambda^1 (\hat{u}^0_{1,1} + \hat{u}^0_{2,2} + \hat{u}^1_{3,3}),$$

since $\hat{u}^0_{3,3} = 0$. Then, letting $\hat{\tau}^0 = \hat{\chi}^0 \lambda^1 (\hat{u}^0_{1,1} + \hat{u}^0_{2,2} + \hat{u}^1_{3,3})$, we recall that $\hat{\tau}^0 \leq 0$.

In conclusion, also considering the two situations for $tre(\hat{u}^\varepsilon) \geq 0$ and $tre(\hat{u}^\varepsilon) \leq 0$, the following system is written, coupling bulk and surface equations and including Signorini type unilateral conditions

$$
\begin{cases}
\sigma^0_{ij,j} + f_i = 0 & \text{in } \Omega_\pm \\
\sigma^0_{ij} n_j = g_i & \text{on } S_g \\
u^0_i = 0 & \text{on } S_u \\
\sigma^0_{ij} = a^\pm_{ijhk} e_{hk}(u^0) + \alpha^\pm \theta^0 \delta_{ij} & \text{in } \Omega_\pm \\
c^\pm \dot{\theta}^0 - q^0_{i,i} = 0 & \text{in } \Omega_\pm \\
q^0 = \gamma^\pm \nabla \theta^0 & \text{in } \Omega_\pm \\
q^0_i n_i = 0 & \text{on } \partial\Omega_\pm \setminus S \\
[\![\sigma^0_{i3}]\!] = 0 & \text{on } S \\
\sigma^0_{\alpha 3} = \mu^0 [\![u^0_\alpha]\!] & \text{on } S \\
\sigma^0_{33} = (\lambda^0 + 2\mu^0) [\![u^0_3]\!]_+ + \tau_0 & \text{on } S \\
[\![u^0_3]\!] \geq 0, \; \tau_0 \leq 0, \; [\![u^0_3]\!] \tau_0 = 0 & \text{on } S \\
\eta^{-1} \dot{\chi}^0 = \left(\omega^{-1} - \left(\mu^0 [\![u^0_1]\!]^2 + \mu^0 [\![u^0_2]\!]^2 + (\lambda^0 + 2\mu^0) [\![u^0_3]\!]^2_+ \right) \right)_- & \text{on } S \\
\chi^0 > 0 & \text{on } S \\
[\![q^0_3]\!] = -\eta^{-1} \left| \dot{\chi}^0 \right|^2 & \text{on } S \\
[\![\theta^0]\!] = (\gamma^0)^{-1} \langle q^0_3 \rangle_0 & \text{on } S
\end{cases}
\tag{28}
$$

Note that the resulting system may be read as a model for contact with adhesion between two deformable solids including thermal effects. In particular, conditions on the traces of the temperatures of the two adherents on the contact interface provide boundary conditions for the evolution of the temperature inside the domains which turns out to be activated by damage evolution.

6 Conclusion

In this paper a model of imperfect interface is derived by an asymptotic expansion method. The model takes into account damage evolution and thermal couplings. It is shown how it is possible to introduce unilateral conditions by the same methodology.

In the future, we intend to implement this model in a numerical software [10, 11, 26] to study a larger family of parameters and to propose more general damage evolution.

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References

1. Benveniste, Y. and Miloh, T., *Imperfect soft and stiff interfaces in two-dimensional elasticity*, Mechanics of Materials, 33 (6), 309-323 (2001).
2. Bonetti E., Bonfanti G., *Well-posedness results for a model of damage in thermoviscoelastic materials*, Ann. I. H. Poincaré - AN, 25 1187-1208 (2008).

3. Bonetti E., Bonfanti G., Lebon F., Rizzoni R., *A model of imperfect interface with damage*, Meccanica, 52 (8), 1911-1922, (2017).
4. Bonetti E., Bonfanti G., Rossi R., *Global existence for a contact problem with adhesion*, Math. Meth. Appl. Sci., 31, 1029–1064, (2008).
5. Bonetti E., Bonfanti G., Rossi R., *Thermal effects in adhesive contact: modelling and analysis*, Nonlinearity, 22, 2967–2731, (2009).
6. Bonetti E., Bonfanti G., Rossi R., *Analysis of a temperature-dependent model for adhesive contact with friction*, Phyisca D, 285, 29-42, (2014).
7. Bonetti E., Bonfanti G., Rossi R., *Modeling via internal energy balance and analysis of adhesive contact with friction in thermoviscoelasticity*, Nonlin. Anal. RWA, 22, 473–507, (2015).
8. Bornert P.G., Bretheau, T., Gilormini P., *Homogénéisation en mécanique des matériaux, Tome 1 : Matériaux aléatoires élastiques et milieux périodiques* . Hermes Sciences, Paris (2001).
9. Challamel, N., Girhammar, U.A., *Boundary-layer effect in composite beams with interlayer slip*, Journal of Aerospace Engineering, 24 (2), 199-209 (2011).
10. Dumont, S.,Lebon, F., Rizzoni, R., *An asymptotic approach to the adhesion of thin stiff films*, Mech. Res. Com., 58, 24-35 (2014).
11. Dumont, S.,Lebon, F., Raffa, M.L., Rizzoni, R., *Towards nonlinear imperfect interface models including micro-cracks and smooth roughness*, Ann. Solid Struct. Mech., 9, 13-27 (2017).
12. Frémond M., *Adhrence des solides*, Journal de Mcanique Thorique et Applique, 6, 383-407 (1987).
13. Frémond M., *Non-Smooth Thermo-mechanics*, Springer (2001).
14. Fouchal F., Lebon F., Titeux, I., *Contribution to the modelling of interfaces in masonry construction*, Const. Buld. Mat., 23, 24282441 (2009).
15. Fouchal F., Lebon F., Raffa M.L., Vairo G., *An Interface Model Including Cracks and Roughness Applied to Masonry*, The Open Civ. Eng. J., 8, 263-271 (2014).
16. Geymonat, G., Krasucki, F., Lenci, S., *Mathematical analysis of a bonded joint with a soft thin adhesive*, Math. Mech. Sol, 4 (2), 201-225 (1999).
17. Hashin, Z., *Thin interphase/imperfect interface in elasticity with application to coated fiber composites*, Journal of the Mechanics and Physics of Solids, 50 (12), 2509-2537 (2002).
18. Klarbring A., *Derivation of the adhesively bonded joints by the asymptotic expansion method*, International Journal of Engineering Science, 29, 493-512 (1991).
19. Lebon, F., Ould Khaoua, A., Licht, C., *Numerical study of soft adhesively bonded joints in finite elasticity*, Comp. Mech.., 21, 134-140 (1998).
20. Lebon, F., Rizzoni, R., *Asymptotic study on a soft thin layer: The non-convex case*, Mech. Adv. Mat. Struct., 15, 12-20 (2008).
21. Lebon, F., Rizzoni, R., *Asymptotic behavior of a hard thin linear elastic interphase: An energy approach*, Int. J. Sol. Struct., 48, 441-449 (2011).
22. Lebon, F., Rizzoni, R., *Modeling a hard, thin curvilinear interface*, Discr. Cont. Dyn. Syst. Series S, 6 (6), 1569-1586 (2013).
23. Lebon, F., Rizzoni, R., Ronel-Idrissi, S., *Asymptotic analysis of some non-linear soft thin layers*, Comp. and Struct., 82, 1929-1938-486 (2004).
24. Lebon, F., Ronel-Idrissi, S., *Asymptotic analysis of Drucker-Prager and Mohr-Coulomb soft thin interfaces*, Steel Comp. Struct., 4, 133-147 (2004).
25. Mielke A., Roubiceck T., Thomas M., *From damage to delamination in nonlinear elastic materials at small strains*, J. Elast., 109 (2), 235-273, (2012).
26. Pelissou, C., Lebon, F., *Asymptotic modeling of quasi-brittle interfaces*, Comp. Struct., 87, 1216-1223 (2009).