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Heterogeneous firms models and financial market frictions

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Part I

Introduction

The common thread in this thesis is represented by general equilibrium models with heterogeneous firms.

Initiated by Huggett (1993) and Aiyagari (1994), a strand of general equilibrium literature characterized by the distribution of heterogeneous individuals has been developed. Due to their complexity, even the simplest of these models has to be solved by means of numerical methods and require a not negligible computational power. In recent years, the introduction of heterogeneity in macroeconomics increased exponentially. This trend was supported both by a significant improvement in computers capacity but also by the growing availability of micro-data. More broadly, it became noticeable the large gains from trade between micro and macro. The evidence emerging by the analysis of micro-data provides invaluable information with which to evaluate the predictions of macro-models. On the other hand, the quantitative theory is a natural guide to interpreting and extrapolating the micro-evidence.

With this in mind, I decided to dedicate my thesis to acquire knowledge of this class of models and to learn the main mathematical methods and solution strategies required to solve them.

The thesis is composed of two papers

The first paper presents a general equilibrium model with heterogeneous agents. Agents in the model are subject to an occupational choice, such that each period they have to chose whether to be a worker or to be an entrepreneur. This mechanism closely links individuals to firms.

In this framework, we simulate a credit crunch, modeled as a restriction in credit conditions. First, we simulate the shock imposing that entrepreneurial productivity evolves according to a discretized version of an AR(1) process with an underlying lognormal distribution. Then, we evaluate the economy's response to the same shock by changing the individual productivity process. Specifically, we assume that each period individuals face a given probability to be hit by a productivity shock, in that case, the agent draws a new level of productivity from an invariant Pareto distribution.

We observe that, while in the AR(1) simulation the aggregate dynamics are significantly sensitive to the calibration of the persistence of the process, they are much less in the alternative case.

Going deeper in understanding the mechanism driving the aggregate results, we demonstrate that the key difference is related to how the assumptions concerning the productivity distribution shape the endogenous wealth distribution. In particular, in the AR(1) modelization, we report that the parametrization of the persistence of the process has a strong impact on determining the dispersion of the wealth distribution, with high level of persistence associated to more concentrated wealth distribution, Differently, in the case of the Pareto distribution for the productivity process, this link is much weaker

In the second paper, we provide a general equilibrium model, which is able to reconcile four important facts: the global decline in the relative price of investment goods, the global decline of the labor share of income, the increase in capital misallocation and the low total factor productivity growth.

The model is characterized by the presence of heterogeneous firms and financial frictions. Both are crucial to derive our theoretical results and they largely differentiate this analysis from the existing literature. Another fundamental ingredient in the model is CES production technology, which allows for variable factor shares. Starting from an exogenous fall in the relative price of investment, the model then generates an endogenous decline in the labor share and in the rental rate of capital. The presence of heterogeneous producers and borrowing constraint, generate and endogenous rise of capital misallocation and narrow TFP growth.

Our model offers an explanation for the decline of the real rental rate of capital based by the relative decline price of investment goods, which is thus alternative to the one offered by Gopinath et al (2017), among others, who link the decline in the rental rate to the Euro convergence process. Also, we claim that while the euro convergence process can be dated as starting in the early '90s, the long-run facts we aim to explain are well documented already in the early '80s in most of the countries.

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Part II

The dynamics of credit crunch under two alternative firm-specific productivity processes

Francesca Crucitti

Abstract

Although firms' heterogeneity in macro models is becoming common practice, there is still not absolute convergence on how to model such heterogeneity. This paper is aimed to highlight the importance of parametrization decisions in this framework and to make a step in understanding how it can affect model simulations' response accordingly. More specifically, we conduct a comparison between the Pareto and the Lognormal distribution in the particular context of a credit crunch. The results report that firms dynamics responses are highly dependent on the underlying process governing the evolution of firms' productivity.

1 Introduction

Introducing firms' heterogeneity in macroeconomic models is becoming common practice across the literature, especially since recent empirical evidence has been increasingly stressing that focusing on average outcomes could be misleading. Along this line, Bartelsman et al. (2013) show that the firm's size distribution is typically not clustered around the mean, observing that many small firms coexist with a smaller number of very large firms. The same holds for the productivity distribution, which is characterized by many below-average performers and a smaller number of star performers, as reported in Syverson (2004). Similarly, looking at the recent financial crisis, research is linking the aggregate slowdown observed in all the major economies to firm-level phenomena. Among others, Andrews et al. (2016) find a strong connection between the declining of aggregate TFP and the widening dispersion of productivity performance. Gopinath, et al. (2017) connect the same phenomenon to the rising resources misallocation across firms active in the market.

In conducting theoretical quantitative analysis, the literature has been relying on specific functional forms to characterize the distribution of firms' productivity. However, there is still not absolute convergence, neither empirically or theoretically, about which one is actually the closest to the firms' distribution observed in the data.

On one hand, among macro-stochastic models, with tractability being a key consideration, a widely used distribution is the Pareto distribution. It is the case in Rossi (2019) and in the majority of Ghironi and Melitz (2005) types of models.

On the other hand, macro models with a richer characterization of heterogeneity often assume that firms' productivity follows an AR(1) and it is distributed as a Log-Normal. In this line the work of Midrigan and Xu (2014), Gopinath, et al. (2017), Khan and Thomas (2013), among others. Into this class of models we count very few exceptions which use the Pareto distribution, as Buera and Moll (2015), and Buera et al. (2014).

In this paper, we want to analyze the implications of these two different modelization assumptions in term of model predictions. Namely, we want to compare the real economic effects of a credit crunch under two different parametrization of the firms' productivity process and distribution.

We decided to mainly focus on to the effects of a credit crunch rather than to a more standard TFP shock because, in the model, only a tightening in credit conditions can induce the composition of new entrants to fluctuate endogenously over the business cycle. This happens because the credit shock affects the profitability of different types of firms asymmetrically. The reason for the latter is that entrepreneurs differ in their productivity and initial wealth. This, in turn, generates heterogeneity in the sensitivity of firms to a shock that affects the possibility to borrow and then to invest. Differently, a standard TFP shock would not be able to generate such endogenous change in the composition of active firms, since this type of shock is fully symmetric across firms and hits all entrepreneurs equally. Later on, in the paper, we provide a quantitative example of this difference between the two types of shock.

The model economy is populated by a unit mass of heterogeneous agents who, each period, face an occupational choice: offering labor and working for the competitive wage, or being an entrepreneur and run a business. The economy is characterized by limited contract enforcement: so that the amount of credit that the firm can obtain is limited by entrepreneurs' wealth. Consequently, entry decision is not only determined by individual current and expected realization of productivity but also by individual wealth. This feature implies that extremely poor entrepreneurs, although highly productive, may not enter because they cannot borrow enough to pay the entry costs and to run the firm at a profitable scale. Finally, firms in the economy operate in a perfectly competitive market and also factors market is frictionless and perfectly competitive.

In this environment, we simulate a credit crunch, modeled as an unexpected restriction in credit conditions. We run the first simulation by imposing that individual productivity follows an AR(1) process with an underlying log-normal distribution. This modelization follows Midrigan and Xu (2014), among the others. Subsequently, we simulate the same credit shock altering the assumption concerning the evolution of the individual's stochastic productivity. We follow Buera et al. (2014) for the design of the idiosyncratic process in the second simulation. In doing so, we impose that, each period, an agent has a given probability to lose her productivity. If this state realizes, she draws a new productivity from a time-invariant Pareto distribution.

In both experiments, for different calibrations, the model generates the standard dynamics for capital, output and interest rate that one would expect from a financial crisis. In a nutshell, immediately after the shock, the demand for capital collapses, and so do the interest rate and the output. Furthermore, by affecting firms' entry and exit decisions and by worsening capital misallocation among active entrepreneurs, the shock also generates an endogenous TFP fall.

In fact, moving the focus on firms dynamics, in both specification, when the credit crunch hits the economy, a fraction of entrepreneurs is forced to quit the business, due to the restrictions in borrowing possibilities. Contemporaneously, a fraction of individuals, who were workers before the shock, find it convenient to enter and become entrepreneurs. The agents who enter during the crunch are relatively less productive but wealthier. This prediction of the model is in line with a bunch of empirical findings that analyses the characteristics of entrepreneurs and firms entering during a recession. Sedlácek and Sterk (2017) show that cohorts born at different stages of the business cycle are composed of different types of firms, with the ones born during recessions reporting significantly lower growth rates. It goes in a similar direction also the result in Lee and Mukoyama (2015).

The element which significantly differentiates the two specifications is the connection between the exogenous productivity distribution and the endogenous wealth distribution. Precisely, in the first specification with log-normal distribution, we find that the calibration of the persistence of productivity's process has a strong impact on determining the dispersion of the wealth distribution, with high levels of persistence being associated to more concentrated wealth distributions. Differently, in the specification with Pareto distribution, this link is much weaker.

This aspect is clearly of primary importance in regulating the economic mechanisms. In fact, in the model, the composition and relative magnitude of the flows of firm's that enter and exit during the shock governs the aggregate dynamics of the economy and it is fundamentally related to the shape of the joint distribution of productivity and wealth.

In the log-normal specification, for low values of the persistence of the productivity shock, we observe an increase in entry during the recession which is so large to induce a rise in the equilibrium wage rather than a decline, as one should expect. In fact, the decision of agents to move from the labor market into entrepreneurship depresses the labor supply so badly that the price of labor input increases. Even adding positive entry costs, although it reduces the entry rate and then the increase in wage, it is not able to invert the sign of the wage dynamic. This counter-intuitive result on wage never shows up in the specification with Pareto distribution, independently from the value assigned to the persistence of the productivity process.

As we mentioned above, we document that the relative magnitude of the set of agents who enter and who exit during the cycle depends on the shape of the endogenous wealth distribution. Being now more specific, it depends on the density of the distribution around the marginal productivity levels, meaning the levels of productivity who are too low to let agents entering in steady state, but it is high enough to let her entering during the credit shock, when the rental rate of capital is lower. In the simulation with log-normal distribution, when the persistence of the productivity shock is high, the wealth distribution is very concentrated, so that there are few individuals with marginal productivity who have enough wealth to enter during the crunch. Differently, when the persistence is low, the wealth distribution is more spread, so that the fraction of agents who can afford to become an entrepreneur is significantly larger. At the contrary, in the simulation with Pareto distribution, the fraction of agents with marginal levels of productivity remains always limited since it does not vary significantly as the persistence of the productivity process decreases. Thus, we obtain that, while in the Log-Normal simulation the entry rate during the crunch and then dynamic of wage are significantly sensitive to the calibration of the persistence of the process, it is much less in the alternative case.

The paper is related to different strands of the literature. We model financial frictions closely following the work of Kiyotaki and Moore (1997), where entrepreneurs' borrowing is limited by a collateral constraint arising from a limited enforceability problem between financial institutions and firms. Similarly, Jermann and Quadrini (2009) adopt the same modeling strategy for financial frictions to study the role of credit as a driver of business cycles. The most salient difference between our work and theirs is that we introduce credit shocks in an economy with heterogeneous producers and, hence, the tightness of credit at any given point in time is not symmetric across producers. We show that this heterogeneity is a crucial element for the predictions of the model.

This feature links the paper to the broad class of models with heterogeneous firms. In the first part of the paper, concerning the specification with log-normal distribution, we design the idiosyncratic productivity process following the strategy in Khan and Thomas (2013) and Midrgan and Xu (2014). In the second specification, we follow a modelization strategy similar to the one proposed in Buera et al. (2014).

The paper also built on the literature which studies the quantitative implications of models of occupational choice as Cagetti and Nardi (2006) and Quadrini (2000). Indeed, this feature allows for endogenous characterization of agent's entry and exit decisions from entrepreneurship and it closely links these decisions to the state of financial markets. This mechanism connects the paper also to the literature of firms dynamics with endogenous TFP, as the paper of Rossi (2019).

Finally, moving toward the design of the shock, the model follows Buera and Moll (2015), assuming as a stressor of the crisis the rise of collateral requirement for firms' borrowing.

The rest of the paper is organized as follows. In section 2 we define the model with AR(1) process and lognormal distribution (2.1), present the relative calibration and solution strategy (2.2), reports the quantitative analysis of different simulations of the model (2.3). Section 3 presents the model (3.1) and simulation results (3.2) of the credit crunch under the alternative specification of individual productivity process with underlying Pareto distribution. Finally, section 4 concludes.

2 First specification: AR(1) with Lognormal distribution

2.1 The model

The model economy is populated by a unitary mass of indefinitely living agents who are characterized by heterogeneous wealth and ability. Individual ability takes the forms of entrepreneurial productivity, i.e. the capacity to invest factors of production more or less productively. Productivity follows an idiosyncratic stochastic process, while wealth is chosen endogenously by forward-looking saving decisions. Firms entry and exit decision is also endogenous and depends on the individual's occupational choice. Among active firms, new entrants pay a positive sunk cost in the first year of activity. Similarly, incumbents who want to dismiss their business pay a positive exit cost. The model is solved assuming that agents have perfect foresight, so there is no aggregate uncertainty.

All individuals have identical preferences, and they maximize their lifetime utility

$$U = E_t \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \tag{1}$$

where c is the level of consumption and β is the intertemporal discount factor.

2.1.1 Idiosyncratic productivity process

Individual entrepreneurial ability evolves according to an N-state Markov chain with transition matrix $P\left(N \times N\right)$. Each period, individual productivity takes one of the N values, indexed by $s \in S$. In particular, the set of possible productivities. is $S = \left[s^1 < s^2 < s^3 ... < s^N\right]$, with $p\left(s^j | s^i\right) > 0$ being the probability to move from state s^i to state s^j, so that $\sum_{j=1}^{N} p\left(s^j | s^i\right) = 1$ for each $i = 1, ..., N$. The productivity values and the associated probabilities matrix $P\left(NxN\right)$ are obtained from the discretization of an AR(1) process with Gaussian disturbances which is discretized using Tauchen method.

2.1.2 Individuals

Individual productivity is positively correlated over time and not correlated among individuals. There is no within-period uncertainty regarding entrepreneurial ability. Indeed, it is observable and known by the agent at the beginning of the period. Every period, each individual chooses whether to be an entrepreneur or to be a worker. Workers offer their unit of labor time in the competitive labor market earning the equilibrium wage. They can save (but not borrow) at a riskless, constant rate of return r. Differently, entrepreneurs use their entire time endowment to run their own firm and make all the corresponding production decisions, taking into account that their investment is constrained by their financial wealth a. In particular, the amount of capital they can invest is $k \leq \lambda_t a$. Where λ is the exogenous variable driving the pattern of the credit crunch. This is a simple and standard modelization of limited contract enforcement. In the absence of market imperfections λ is equal to ∞. In this case, the optimal level of investment is only related to technological parameters. In the range $1 < \lambda < \infty$, investment in production also depends on the agent initial assets. As a result, not all potentially profitable projects receive appropriate funding. Agents with little wealth can borrow little, even if they have high ability as entrepreneurs. Since the entrepreneur forgoes his potential earnings as a worker, he will choose to become an entrepreneur only if the size of the firm that he can start is big enough; that is, he is rich enough to be able to borrow and invest a suitable amount of money in his firm.

At time $t,$ for each i entrepreneur, the production function is:

$$y = sk^{\alpha} l^{\eta} \tag{2}$$

The production technology is characterized by decreasing return to scale $\alpha + \eta < 1$. This assumption allows for a richer characterization of the occupational choice. Assuming constant return to scale would simplify the analysis since it would become a function of productivity only. With decreasing return to scale, instead, both the productivity and the initial wealth matter.

2.1.3 Occupational choice

Entrepreneurs employ the unit of time they are endowment with as manager of the firm and choose labor and capital in order to maximize profits.

Formally, incumbent entrepreneurs solve the following problem

$$\Pi\left(a, s, 1\right) = \max_{k,l} sk^{\alpha} l^{\eta} - \left(r_t + \delta\right) k - w_t l \tag{3}$$

subject to:

$$k \leq \lambda_t a \tag{4}$$

Similarly, new entrant entrepreneurs' profit maximization problem is

$$\Pi\left(a, s, 0\right) = \max_{k,l} sk^{\alpha}l^{\eta} - \left(r_t + \delta\right)k - w_t l \tag{5}$$

subject to:

$$k \leq \lambda_t \left(a - \phi\right) \tag{6}$$

The term ϕ is the entry cost. In this modelization, the entry costs constitutes a further restriction on the entrepreneur's possibility to borrow, since it reduces the availability of individual wealth that is used as collateral

Given capital market frictions and the consequent limited possibility to invest, the occupational choice is linked to individual initial wealth.

Differently, workers, sell their unit of time in the labor market, receiving as compensation the equilibrium wage

2.1.4 Individual problem

At the beginning of each period individuals choose the occupation and then how to allocate saving and consumption to maximize their lifetime utility. The value of being entrepreneur today differs according to the previous period occupation.

a) If in $t-1$ the agent was a worker, the value of being entrepreneur for her is

$$v_t^E\left(a, s, 0\right) = \max_{c,a'} u\left(c\right) + \beta \sum_{s'} p\left(s'|s\right) \max\left\{v_{t+1}^{W\prime}\left(a', s'\right); v_{t+1}^{E\prime}\left(a', s', 1\right)\right\} \tag{7}$$

s.t.

$$c + \phi + a' = \Pi\left(a, s, 0\right) + (1 + r_t)a \tag{8}$$

b) If in $t-1$ the agent was an entrepreneur, the value of being entrepreneur for her is

$$v_t^E\left(a, s, 1\right) = \max_{c,a'} u\left(c\right) + \beta \sum_{s'} p\left(s'|s\right) \max\left\{v_{t+1}^{W\prime}\left(a', s'\right); v_{t+1}^{E\prime}\left(a', s', 1\right)\right\} \tag{9}$$

s.t.

$$c + a' = \Pi\left(a, s, 1\right) + (1 + r_t)a \tag{10}$$

Instead, the value of being worker does not change according to the occupational choice in the previous period. It is equal to:

$$v_t^W(a,s) = \max_{c,a'} u(c) + \beta \sum_{s'} p(s'|s) \max\left\{v_{t+1}^{W\prime}(a',s') ; v_{t+1}^{E\prime}(a',s',0)\right\} \quad (11)$$

s.t.

$$c + a' = w_t + (1+r_t)a \quad (12)$$

2.1.5 Deeper on optimal choices

We generally define the solution of the individual problem $a' = g(a,s,o_{-1})$ and $o = occ(a,s,o_{-1})$

- Given $o_{-1} = 1$:
  - if $v^E(a,s,1) \geq v^W(a,s) \Rightarrow o(a,s,1) = 1\ (y(a,s,1), l(a,s,1), k(a,s,1) > 0)$
  - if $v^E(a,s,1) < v^W(a,s) \Rightarrow o(a,s,1) = 0\ (y(a,s,1), l(a,s,1), k(a,s,1) = 0)$
- Given $o_{-1} = 0$:
  - if $v^E(a,s,0) \geq v^W(a,s) \Rightarrow o(a,s,0) = 1, (y(a,s,0), l(a,s,0), k(a,s,0) > 0)$
  - $v^E(a,s,0) < v^W(a,s) \Rightarrow o(a,s,0) = 0, (y(a,s,0), l(a,s,0), k(a,s,0) = 0)$

2.1.6 Equilibrium

An equilibrium is a sequence of prices $\{r_t, w_t\}_{t=1}^{\infty}$, collateral constraint $\{\lambda_t\}_{t=1}^{\infty}$ and corresponding quantities such that, each periods:

- agents maximize their utility taking as given aggregate prices
- labor market clears :

$$L_t = \sum_{o_{-1}=1,0} \sum_{s} \int_a l_t(a,s,o_{-1}) \Omega_t(da,s,o_{-1}) = \quad (13)$$

$$1 - \sum_{o_{-1}=1,0} \sum_{s} \int_a o_t(a,s,o_{-1}) \Omega_t(da,s,o_{-1}) \quad (14)$$

- capital market clears:

$$K_t = \sum_{o_{-1}=1,0} \sum_s \int_a k_t(a,s,o_{-1})\,\Omega_t(da,s,o_{-1}) = \tag{15}$$

$$\sum_{o_{-1}=1,0} \sum_s \int_a a\Omega_t(da,s,o_{-1}) \tag{16}$$

- given the aggregate capital law of motion $K_t = (1-\delta)K_{t-1} + I_t$, the good market clears:

$$Y_t = \sum_{o_{-1}=1,0} \sum_s \int_a y_t(a,s,o_{-1})\,\Omega_t(da,s,o_{-1}) = \tag{17}$$

$$\sum_{o_{-1}=1,0} \sum_s \int_a c_t(a,s,o_{-1})\,\Omega_t(da,s,o_{-1}) + I_t$$

$$+\phi \sum_s \int_a o_t(a,s,0)\,\Omega_t(da,s,0) \tag{18}$$

- And a law of motion for the distribution:

$$\Omega_{t+1}(a',s',o) = \sum_{\{o_{-1}:o=occ(a,s,o_{-1})\}} \sum_s \int_{\{a:a'=g(a,s,o_{-1})\}} \Omega_t(a,s,o_{-1})\,p(s'|s) \tag{19}$$

2.2 Calibration and solution strategy

2.2.1 Calibration

The model is characterized by a small number of exogenous variables. Following the most standard practices in the literature, the period utility function is a CRRA $u(c_t) = \frac{c_t^{1-\mu}}{1-\mu}$ with $\mu = 1.5$. The depreciation rate of capital is $\delta = 0.06$.For the production function calibration, assuming diminishing return to scale $\alpha+\eta < 1$, I set $\alpha+\eta = 0.8$ as it was found to be in Gopinath et al. (2017). The value comes from the analysis over a sub-sample of euro-area countries. However, we think it is reliable also for the entire group of economies. In fact, it is very close to the calibration in Buera et al. (2015) for the US, where they set $\alpha+\eta = 0.81$ and it is also close to the value in Buera et al.(2014 and 2012) $\alpha+\eta = 0.79$.

The capital share is such that $\alpha = 0.33$.

In the first specification, entrepreneurial ability can takes 8 values $z = \{s_1, s_2, ..., s_8\}$, equally spaced from 0.617 to 1.621. The range of values and the associated probabilities matrix are obtained from the discretization of an AR(1) process with first-order autoregressive coefficient $\rho = 0.59$ and standard deviation $\sigma = 0.13$. Also the calibration of the productivity process, ρ and σ, is based on the results of Gopinath et al. (2017).

In the second specification, we assume that entrepreneurial productivity follows a Pareto distribution, with cumulative density given by $\mu(z) = 1 - s^{-\eta}$ for $s \geq 1$. Each period, an individual retains his s with probability ψ while a new entrepreneurial productivity should be drawn with the complementary probability $1-\psi$. We solve the model for different values of ψ, specifically $\psi = 0, 0.3, 0.9$. To calibrate the parameters η we follow Buera et al. (2015) and set $\eta = 5.25$.

The intertemporal discount factor is $\beta = 0.966$, and $\lambda_0 = 6$, which implies a real annual interest rate of 3.5 in the initial steady state.

In the specification with positive entry sunk costs, we calibrate the entry costs following the data World Bank's Doing Business surveys 2004-2009. The data reports that world wide the value of entry costs range from 0 to 764 percent of output per worker, with an average value of 32 percent of output per worker and a standard deviation of 78 percent. Given the large variation in the data we decide to set $\phi = 1.6$ such that entry costs in the benchmark model are equal to one unit of output per worker.

It is assumed that at time 1 the economy is in equilibrium, for a given level of $\lambda_{ss} = \lambda_1$. The aggregates variables are at the stationary level $Y_{ss}(\lambda_{ss})$, $r_{ss}(\lambda_{ss})$, $w_{ss}(\lambda_{ss})$, $C_{ss}(\lambda_{ss})$, $K_{ss}(\lambda_{ss})$. In period 2, the credit crunch hits the economy and, from this period on, agents learn that λ will be lower than λ_0 for several periods in the future. At the end of each period the agents see the realization of their productivity for the following period while and their wealth a comes from the saving choice of the previous period. Thus, she has all the information she needs to make the occupational choice and the saving choice. Finally, aggregates will be obtained from the sum of individual decisions and prices will be such that all markets clear.

2.2.2 Solution strategy

The model is solved numerically and the code is written in $MATLBA\ R2018a$. Given the discrete occupational choice, the individual problem is solve using

Value Function Iteration on a discrete grid for capital. The grid has a minimum value of $a_{\max} = 0$, since it is assumed that agents cannot borrow for consumption purpose. The grid is constructed locating a relatively greater number of points at the lower asset levels where the curvature of the value function is higher and there is greater population density. More precisely, the distance between nodes increases exponentially at a power of 2. Different grid limits and densities of points have been considered, but results are not sensitive to grid changes, provided the grid is sufficiently fine at lower asset levels. Experiments with higher-limited grids indicate an upper limit of 1200 is sufficient for good results (especially as most of the interesting dynamics concern much lower levels of wealth).

The idiosyncratic AR(1) productivity shock process is discretized using Tauchen's method, obtaining a Markov process with transition probability matrix $P(N \times N)$ with $N = 8$ the number of states.

Algorithm for the initial stationary equilibrium

1. Guess a value for the aggregate prices r_0^0, w_0^0.
2. Taking as given the prices, maximize profits of entrepreneurs and compute the occupational choice. At the end of this step we have the occupational choice and the demand for labor and capital of active entrepreneurs $(O\,(nk \times N)\,,\, L^{dem}\,(nk \times N)\,,\, K^{dem}\,(nk \times N))$
3. Solve the individual utility maximization problem through value function iteration and find the optimal saving choice and use the budget constraint to compute consumption.
4. Construct the transition matrix and find the stationary distribution through fixed point iteration
5. Use the distribution and compute aggregates.
6. Check market clearing conditions and iterate over prices until they are all satisfied, $r = r_0^1$, $w = w_0^1$

Algorithm for the transitional dynamics The model is a perfect foresight model, the change in the financial parameter λ is unexpected, however, once it occurred all the agents can perfectly anticipate the return path to its steady state. Thus, the main steps of the algorithm to compute the transitional dynamics the following:

1. Guess an initial time series for $\{r\}_{t=1}^{T}$ and $\{w\}_{t=1}^{T}$ with $r_{t=1}$ and $w_{t=1}$ are the initial steady state equilibrium prices.

2. Given the time path of λ, and the guess path for the input prices, solve, for each t the profit maximization problem, find capital and labor demand and the occupational choice.

3. Solve the individual problem backward, using the value function computed at the initial steady state

4. For each t, compute the joint distribution of wealth and entrepreneurial productivity iterating forward using the aggregate law of motion

5. Compute the time series of aggregates

6. Adjust r and w until market clears in each period.

2.3 Quantitative analysis

2.3.1 Credit crunch

We simulate a tightening of the collateral constraint in the economy and solve for the aggregate dynamics. We assume that the initial drop in λ is completely unexpected - up to that point everyone in the economy expected λ_t to be constant over time - but, once this shock hits the economy, its path is deterministic and perfectly known by all agents.

We present the response of the economy for different levels of persistence of the idiosyncratic productivity process.

Figure 1 shows how the shock was designed and its direct effects on the aggregate credit-to-capital ratio. The credit ratio is defined as:

$$ratio = \frac{1}{K_t}\left[\begin{array}{c}\sum_s \int_a \max\left[\left(k_t\left(a,s,0\right)\right)-a,0\right]\Omega_t\left(da,s,0\right)+ \\ \sum_s \int_a \max\left[\left(k_t a,s,1\right)-a,0\right]\Omega_t\left(da,s,1\right)\end{array}\right]$$

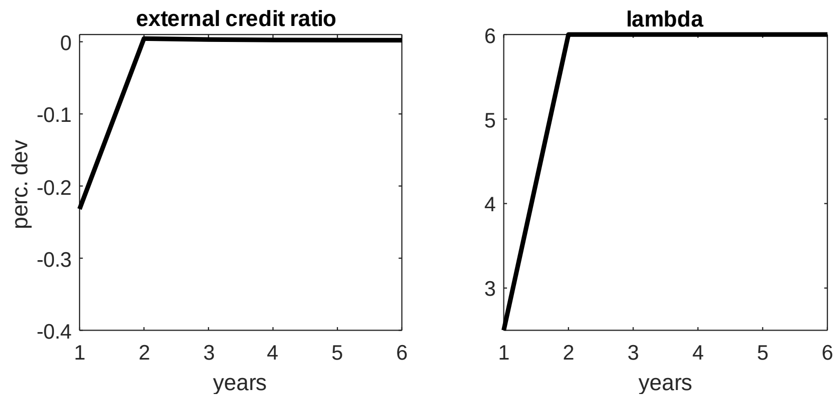


Figure 1: The shock modellization

We calibrated the shock in order to achieve a reduction in the credit ratio which is close to the one observed during the recent financial crisis.

Figure 2 illustrates the effects of the shock on the main aggregate variables. All variables are reported in percentage deviation from their initial steady-state level. The only exception is the rental rate of capital, which is reported in levels.

As expected, the credit shock generates a recession in the model economy. Output falls by more than 5 percentage points. The decline in aggregate production is driven by the simultaneous fall in aggregate capital and in aggregate productivity. Indeed, as the credit shock hits the economy, the fraction of constrained entrepreneurs increases, so that they are forced to reduce the amount of capital employed in production.

Capital flows from relatively poorer entrepreneurs toward unconstrained entrepreneurs, who want to increase their capital demand in order to take advantage of the drop in the rental rate. Hence, the shock induces a reallocation of capital from productive but constrained entrepreneurs to entrepreneurs who are unconstrained but relatively unproductive, the important consequence is the reduction in TFP.

We present in figure 3 the dynamics of rental rate and wage.

Although the drop in the rental rate of capital is expected and easy to understand in light of the discussion above, it may be instead surprising the rise in the wage.

Since the increase in wage is contemporaneous to the decline of aggregate labor, we can say that it is driven by a reduction in the aggregate labor supply

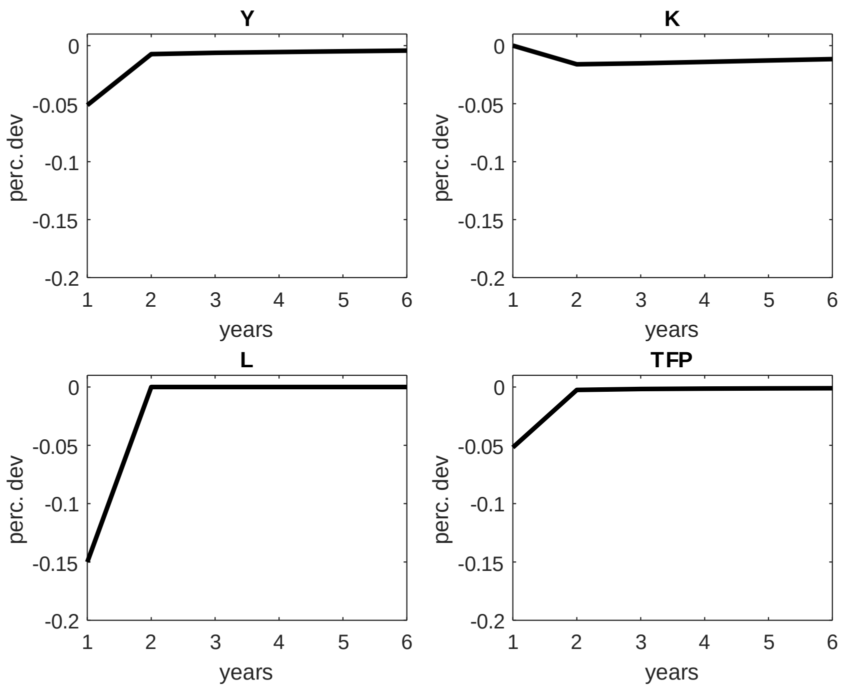


Figure 2: Aggregate dynamics

rather than an increase in the aggregate demand.

Furthermore, given the model's design, fluctuations in aggregate labor always reflects fluctuations in the number of firms operating in the economy. Accordingly, in order to understand the counter-intuitive dynamic of wage, we should look at how the credit crunch affects firms dynamics. In particular, note that when the credit crunch hits the economy there is a fraction of entrepreneurs who are forced to quit the business, due to the restrictions in borrowing possibilities. However, contemporaneously, there is also a fraction of individuals who, instead, find it convenient to enter and become entrepreneurs. This happens because the credit shock affects the profitability of different types of firms asymmetrically. The reason for the latter is that entrepreneurs differ in their productivity and initial wealth.

What is more intuitive is understanding the decision to exit and the identification of the agents who move in this direction. As the shock realizes and credit condition worsens, a fraction of individuals who were entrepreneurs in steady state is forced to close the business because they are too poor to run the firm at a profitable scale. This set of agents, moving from entrepreneurship into the labor market, increase the labor supply generating downward pressure on wages.

We have to make an additional effort here to recognize who are instead the agents who decide to enter when the shock hits. These agents are the wealthiest and relatively less productive so that they do not suffer from the restrained credit condition but rather they can take advantage of the drop in the rental rate of capital. Their occupation decision to move from the labor market into entrepreneurship decrease the labor supply and increases demand, generating upward pressure on wages.

In the model, given the shape of the ergodic distribution, the second mechanism overcome the first one and this explains why we observe a rise of wage and the decline of aggregate labor in general equilibrium. We will document in the following subsection that the parameter of the persistence of the idiosyncratic productivity process is crucial in driving this result.

2.3.2 Credit crunch and entry costs

Given the peculiar result we presented in the previous section, we run a simulation of the same shock introducing in the economy positive entry costs. Entry costs serve as a barrier on firms dynamics so that they should lessen the entry rate during the credit crunch.

Figure 4 illustrates the effects of the fall in the collateral constraint on the main aggregate variables. All variables are reported in percentage deviation from their initial steady-state level.

The blue lines are the dynamics of variables in the benchmark model with entry sunk costs. The black lines are, instead, the dynamics of the variables in a simulated model with zero costs $\phi = 0$.

Similarly as above, also in the model with entry cost, the credit shock generates a recession.

Focusing on the differences in the dynamics between the model with entry costs and the one without entry costs, it is noticeable how the presence of entry costs generates a slower recovery.

Total factor productivity returns to its initial level with a year of lag when $\phi > 0$, while output takes even longer. Particularly interesting are the differences reported in the dynamics of aggregate capital and labor.

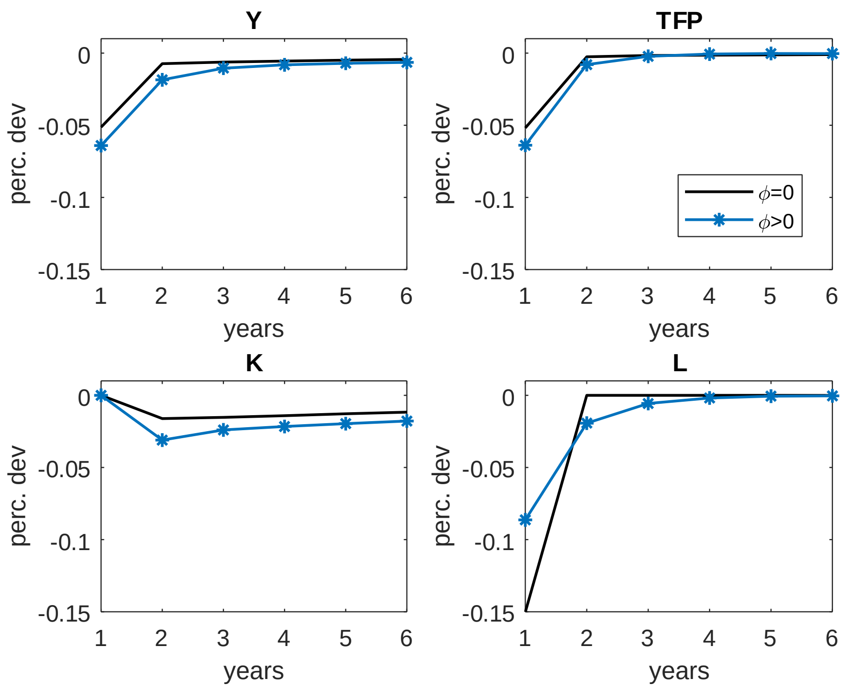
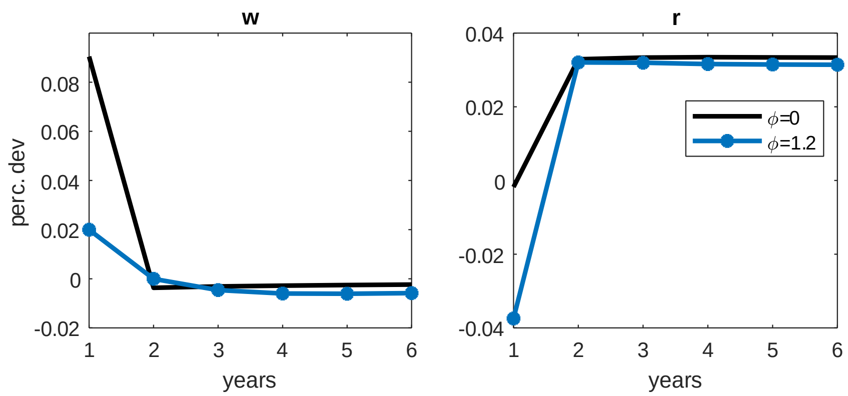


Figure 3: Aggregate dynamics with entry costs

As shown in the figure, the fall in aggregate capital is deeper in the economy with positive entry costs. Furthermore, capital is also the variable which reports a slower recovery path. With $\phi > 0$ even after more than 5 years, it lags behind its initial level. Indeed, the fraction of entrepreneurs who were forced to exit by the tightening in credit conditions needs more time to accumulate the necessary wealth to pay the sunk costs again and return to be entrepreneurs. Then, the presence of entry costs delays the entrance of marginal entrepreneurs who exit during the shock, having a direct consequence of the delay in the recovery of both production inputs and, consequently output.

Even labor takes longer to revert to its pre-shock level when there are entry costs. However, the initial drop is smaller in this case relative to the case with no entry costs.

As said above, the contraction of labor reflects a rise in the number of entrepreneurs. This rise is substantial in the economy with zero entry costs. Con-



versely, positive entry costs diminish the incentive to become an entrepreneur and fewer firms enter. The number of workers who become an entrepreneur during the shock is smaller relative to the economy with no costs. This leads to a smaller decline in labor supply and, given that the economy is in equilibrium, a smaller decline in aggregate labor. In turn, also the rise in wage is relatively lower. (Figure 6)

2.3.3 Credit crunch higher persistence

In this section we simulate the same credit crunch shock for different level of persistence of the idiosyncratic productivity process. We test for 3 alternative values of ρ and compare the dynamics obtained with the ones obtained in the benchmark calibration. The in Figure 7 we present dynamics for $\rho = \{0, 0.3, 0.59, 0.9\}$

As reported in the figure, for all calibration values, the credit crunch generates a recession. Furthermore, we observe that the recession is more severe when the individual productivity persistence is relatively low. Indeed, agents rely more on external credit when the autocorrelation of their productivity shock is very low. In the limiting case that the shock is completely i.i.d. ($\rho = 0$) agents face significant probability to shift between very different level of productivity from one period to another. For example, this implies that an agent who has high productivity today may have been a worker yesterday so that she was not able to accumulate enough wealth and, consequently, it makes her highly exposed to external credit. Conversely, if productivity moves slowly over time, individuals can accumulate wealth and progressively substitute external credit with self-financing.

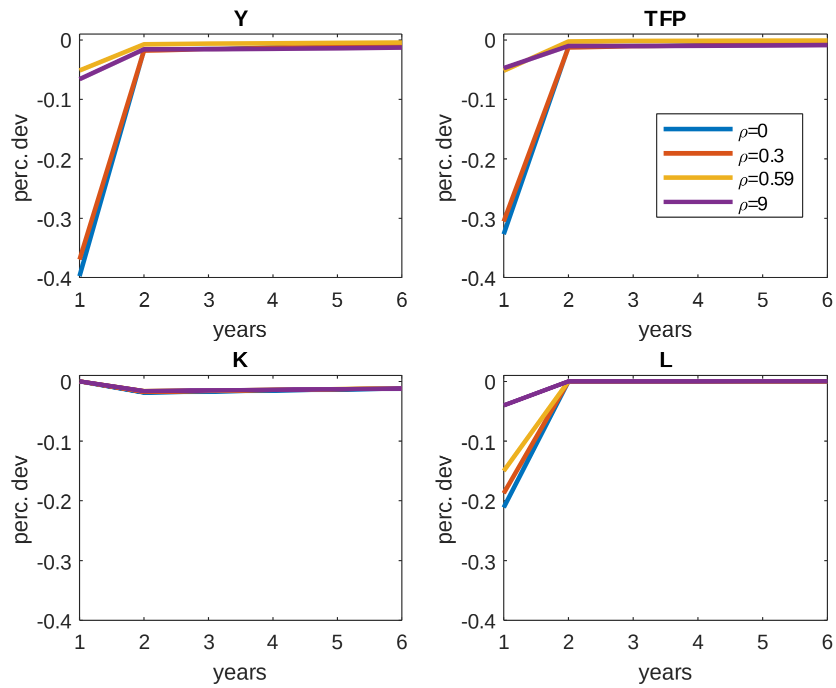


Figure 4: Aggregate dynamics for different level of persistency

In other words, if productivity shocks are sufficiently correlated, agents are more resilient to credit conditions, and the negative effects of the credit crunch are relatively moderated.

Moving the attention toward labor and firms dynamics, we can also say that the number of entrepreneurs entering during the shock is lower when the persistence is higher.

Figure 8 reports the dynamics of the rental rate of capital and wages. Noticeable, in this case, we observe the standard result of a drop in the equilibrium wage. This means that in this specification, although there is an increase in the number of active firms and then a reduction of aggregate labor supply, it is not enough to compensate the reduction of labor demand generated by the exit of the more productive entrepreneurs.

In figure 9 we present how the TFP distribution of active entrepreneurs changes during the shock. We compare only the two cases with the extreme values assigned to ρ. We would simply observe something in between for the

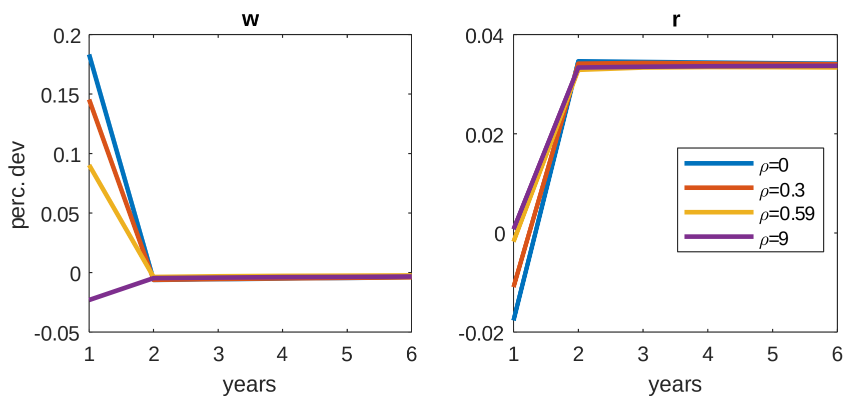


Figure 5: Equilibrium wage and intererst rate for different levels of persistency

other intermediate values.

In both panels we see the productivity distribution shifting to the right. It occurs because the collapse of the rental rate of capital, caused by the credit shock, reduced the cut-off level of productivity. Therefore, individuals who were not productive enough to be an entrepreneur in the stationary economy can instead become entrepreneurs during the crunch.

Notwithstanding, the shift is larger when the persistence is lower. In turn, this means that the fraction of wealthy but less productive entrepreneurs entering during the crunch is larger in the case with low ρ (left panel). This mechanism is related to the differences in the shape of the wealth distribution in the two cases. In fact, when persistence is high the wealth distribution is very concentrated so that few individuals (among the ones with marginal productivity) have enough wealth to enter during the crunch, even though they would be sufficiently productive. Differently, when the persistence is low, the wealth distribution is more spread, so that there are more individuals who can afford to enter.

In figure 10 we report the distribution of wealth of the agents with marginal productivity, i.e. agents whose productive is above the cut-off level during the crunch but it is below the cut-off in the stationary economy. The black line draws the distribution corresponding to the equilibrium with low ρ, the blue line draws distribution when ρ is high. Lastly, in the figure, we also report the minimum wealth level required for the agent with marginal productivity to enter (the two dashed vertical lines).

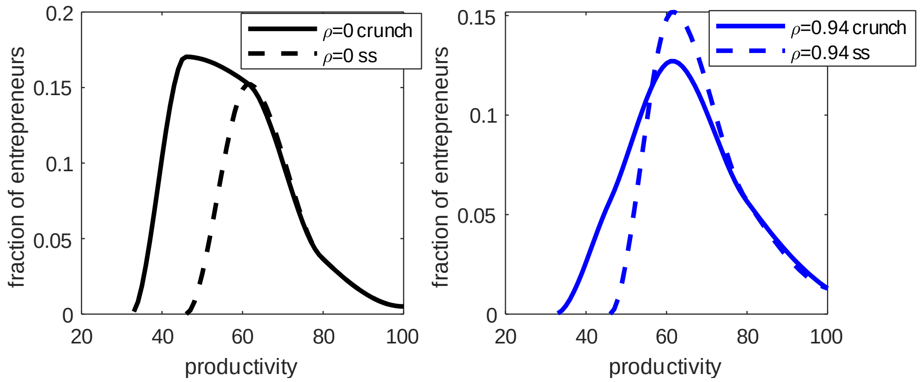


Figure 6: Productivity distribution of active entrepreneurs with lowe persistency (left panel). Productivity distribution of active entrepreneurs with high persistency (right panel)

As we can see, taking into consideration the same set of productivity levels, in the economy with low ρ the fraction of individual above the wealth threshold is much larger than in the economy with high ρ. Consequently, in the simulation, the fraction of agents entering during the shock is larger if the persistence of the idiosyncratic shock is lower.

2.3.4 TFP shock

In this section, we present the dynamics of the main aggregates following a standard TFP shock. We compare again the responses of the economy with entry costs and the ones in the model without entry costs. This exercise is useful to show how heterogeneity can be less crucial when the aggregate shock does not activate reallocations dynamics. In order to implement it, we slightly modify the individual production function by adding an aggregate productivity component.

The modified individual production function is:

$$y = A_t z k^{\alpha} l^{\theta}$$

Then we define a path for A_t, assuming that the shock only lasts one period, as in the simulation above. We model a decline of A by 10% and we use the benchmark calibration for ρ. Figure 9 shows these dynamics.

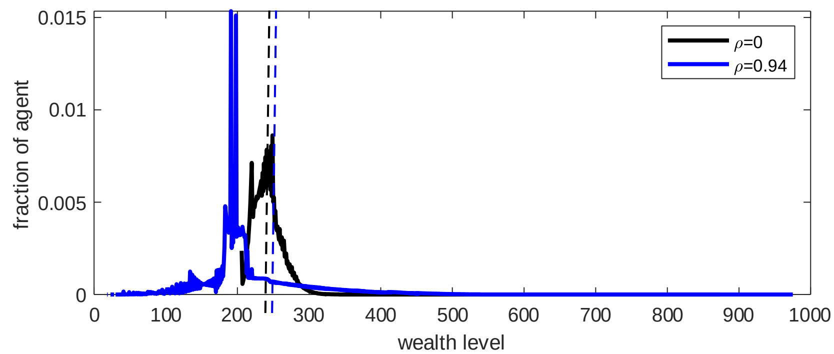


Figure 7: Wealth distribution for the two levels of persistency

As depicted in the figure, the TFP shock reports a much different pattern showing practically no differences between the model with entry costs and the model without.

Both aggregate output and TFP follow exactly the path of the exogenous variable A. The dynamic of capital is similar to the one observed in the simulation above, while labor does not move at all. This means that the shock, although it generates an important recession (output drop by 10%), does not generate interesting pattern at the firm level. The fundamental difference between the credit crunch and the standard TFP shock is that the credit crunch is much less democratic. During a credit tightening, only the subset of entrepreneurs who have little wealth is actually affected. The remaining entrepreneurs, who were wealthy enough to be unconstraint, are not affected by the shock. On the contrary, a TFP shock hits all entrepreneurs symmetrically, so that prices changes can fully compensate for the lower productivity level without inducing a reallocation of factors or changes in the occupational choice of agents.

In this context, the presence of entry costs does not add any difference to the model's dynamics because this kind of shock does not entail changes in firms decision to enter or exit.

The TFP shock generates a fall in both interest rate and wage. This is standard and it is due to the decline in the marginal productivity of both production factors, which is in turn driven by the decline in the aggregate productivity. Moreover, here there is no room for an increase in the wage since there no variation at the extensive margin of entrepreneurship. As said, the shock does not affect the occupational decision of individuals, so that in this case the drop in

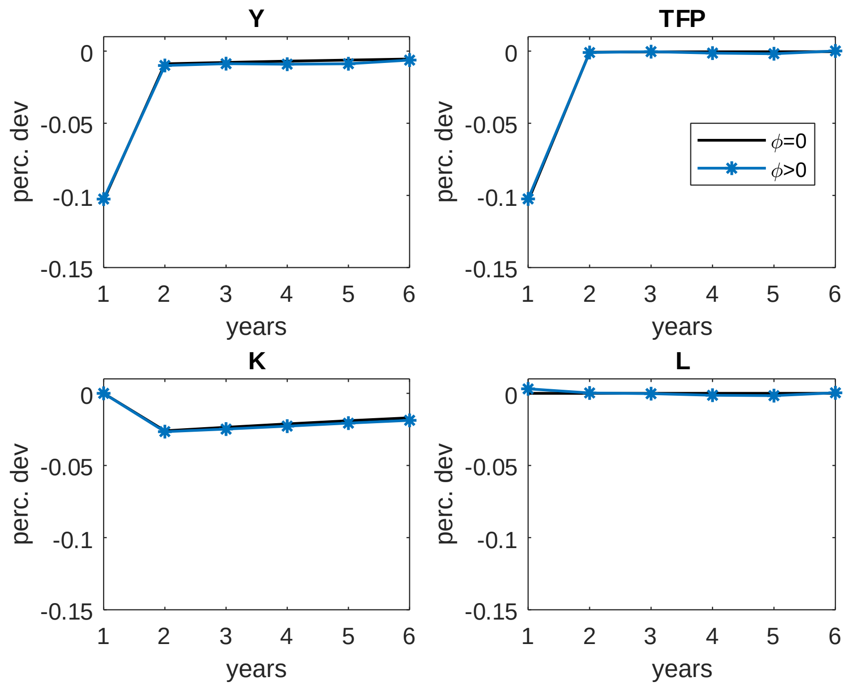


Figure 8: Aggregate dynamics after TFP shock with entry costs

wage completely reflect the drop in the marginal productivity of labor. In fact, it falls by exactly 10%, which is the exogenous change in TFP. In this context, we never observe the rise in wage, independently on the level of persistence, because TFP does not entail any entry-exit dynamics on firms' side.

3 Second specification: Poisson shock with Pareto distribution

In this section we present the simulation of the credit crunch in the model with an alternative stochastic process for individual productivity. Besides we solve here the model in continuos time.

3.1 The model

Individual wealth evolves endogenously according to the individual optimal saving decision, while the entrepreneurial ability follows a stochastic process. We assume that each individual retains her productivity with probability ψ while with probability $(1-\psi)$ she loses the current productivity and has to draw a new one. The new draw is from a time-invariant distribution with a cumulative density $\mu(s) = 1 - s^{-\eta}$ and it is independent of her previous productivity level.

On the demand side, all the individuals have the same utility function:

$$U = E_t \int_{t=0}^{\infty} e^{-\rho t} u(c)\, dt \tag{20}$$

where ρ is the intertemporal discount factor and c is the level of consumption.

Individuals can purchases consumption and investment goods x, from final good producers at their relative price. They use investment good to accumulate wealth such that $\dot{a} = x - \delta a$.

3.1.1 Individual problem

Agents $i(a,s)$ who decide to be entrepreneur obtain as income the realized profit $M(a,s) = \Pi(a,s)$. The occupational choice of the agent is then defined as $oc(a,s) = 1$ and labor and capital demand is $l(a,s), k(a,s) > 0$. Differently, the income of an agent $i(a,s)$ who decide to be worker is given by the wage $M(a,s) = w$. Her occupational choice, capital and labor demand are $oc(a,s), l(a,s), k(a,s) = 0$.

The agent chooses consumption c and investment x in order to maximize her utility, subject to the period budget constraint.

$$M(a,s) = \max[w, \Pi(a,s)] \tag{21}$$

Utility maximization problem

$$\max_c E_t \int_{t=0}^{\infty} e^{-\rho t} u(c)\, dt \tag{22}$$

s.t. the budget constraint:

$$c(a,s) + \dot{a}(a,s) = M(a,s) + ar \tag{23}$$

The first order condition of the problem is:

$$u'(c) = v'(a,s) \tag{24}$$

3.1.2 Equilibrium

As in standard Ayagari model, individuals' consumption-saving decision and the evolution of the joint distribution of their income and wealth can be summarized with two differential equations:

- Hamilton-Jacobi-Bellman (HJB) equation

$$\rho v(a,s) = \max_c \left[\left\{ \begin{array}{c} u(c(a,s)) + \\ \partial_a v(a,s)(M(a,s) + ra - c(a,s)) \\ + (1-\psi)\left[\sum_{s'} p(s')(v(a,s') - v(a,s))\right] \end{array} \right\} \right] \tag{25}$$

- Kolmogorov Forward (or Fokker-Planck) equation

$$0 = -\partial_a [g(a,s)\, sav(a,s)] - (1-\psi)\left[g(a,s) - p(s) \sum_{s'} g(a,s') \right]$$

  where $p(s)$ is the probability distribution of productivity shock.

Finally capital and labor market clearing condition and the aggregate resource constraint are:

$$
\begin{aligned}
K &= \int_{(a,s):oc(a,s)=1} k\left(a,s\right)g\left(a,s\right)dads = \int_{(a,z)} ag\left(a,s\right)dads &(26)\\
L &= \int_{(a,s):oc(a,s)=1} l\left(a,s\right)g\left(a,s\right)dads \\
&= 1-\int_{(a,s):oc(a,s)=1} g\left(a,s\right)dads &(27)\\
Y &= \int_{(a,s):oc(a,s)=1} y\left(a,s\right)g\left(a,s\right)dads \\
&= \int_{(a,s)} c\left(a,s\right)g\left(a,s\right)dads + \delta\int_{(a,s)} a\left(a,s\right)g\left(a,s\right)dadz &(28)
\end{aligned}
$$

3.2 Calibration and solution strategy

3.2.1 Calibration

We calibrate the stochastic process following Buera et al. (2014). We set the parameter $\eta = 5.25$ of the Pareto distribution.

In the life-time utility function, the discout factor is set $\rho = 0.067$ is set to obtain an initial interest rate equal to 0.035, as in the previous sepcification.

All the others parameters remain the same as in the previous specification.

3.2.2 Solution strategy

To solve the model in continuous time we follow the method proposed in Achdou et al. (2017). The steps of the algorithm are similar to the ones reported above for the model in discrete time, both for the solution of the steady state and of the dynamics. The main differences concern the computational method used to compute the individual value function (the HJB equation) and the ergodic distribution (the KF equation).

Similar to what we did above, the first step of the algorithm is to solve the HJB equation for a given time path of prices. Then we solve the KF equation for the evolution of the joint distribution of income and wealth. The third step is to iterate and repeat the first two steps until an equilibrium fixed point for the time path of prices is found. For the first step, as explained in Achdou et al. (2017), we solve HJB equation using finite a difference method.

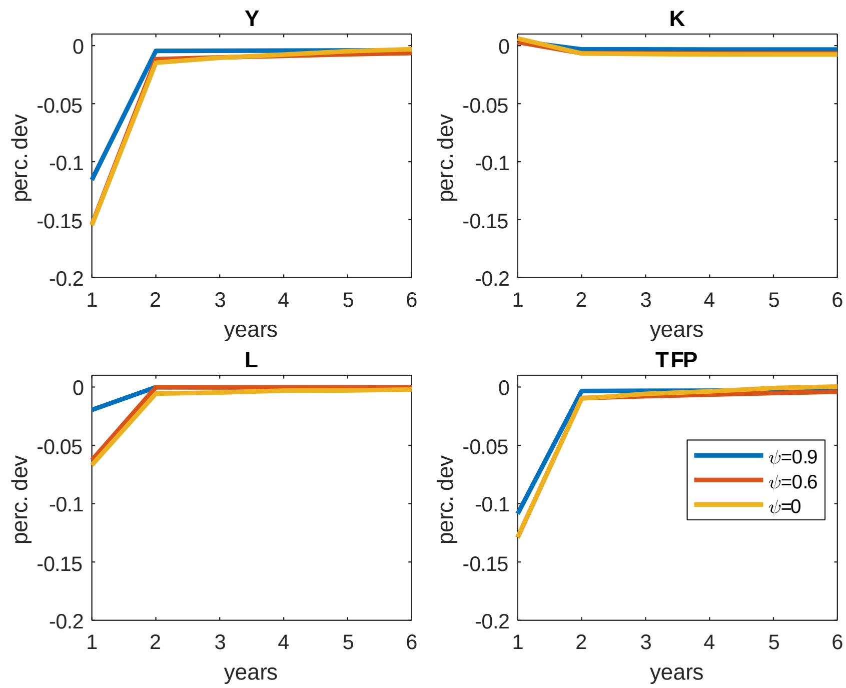


Figure 9: Aggregate dynamics different levels of persistency

3.3 Quantitative analysis: The Credit Crunch shock

Given the importance of the persistence parameter revealed in the previous simulation, we solve the dynamics with different values for the probability ψ. Figure 11 reports the result of these simulations.

For each value of ψ, the credit crunch generate the standard dynamics we observed in all the previous specification, that is a drop in output, labor, capital, and TFP

Figure 12 reports the fluctuation of prices for the three different model calibrations.

As we can see, under the current assumption concerning the evolution of individual ability, we never observe the rise in wage that we instead observed in the previous section, independently on the value of the persistence of productivity.

As discussed above, for the full understanding of firms dynamics it is important to examine the joint distribution of productivity and wealth. Figure 13

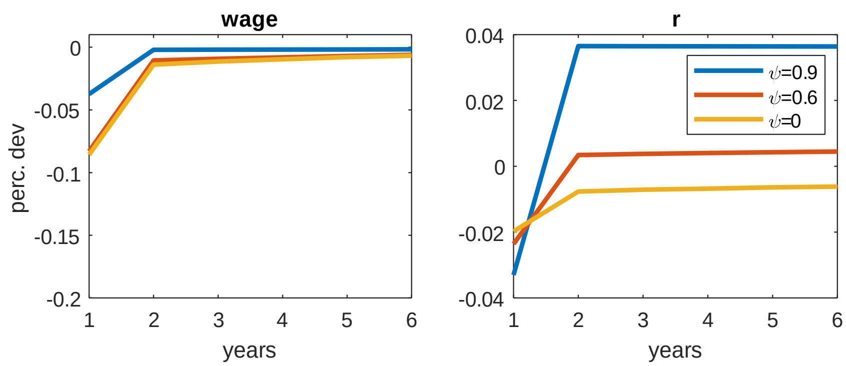


Figure 10: Equilibrium wage and interest rate for different level of persistency

reports the TFP distribution of active entrepreneurs in the economy with the lowest persistence level. We can see that the change generated by the credit crunch is very small, meaning that variations in firms entering and exit was moderate. Then, we look at the wealth distribution of wealth, reported in figure 13 on the right panel. From this graph, it is possible to see that in this specification, the distribution of wealth is very concentrated even for low values of the persistence of the idiosyncratic process. This implies that a few individuals are above the wealth threshold, and then the few individuals are able to enter during the crunch.

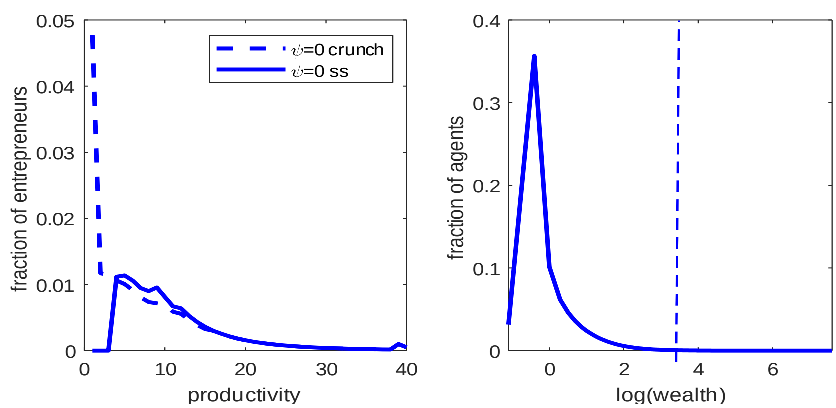


Figure 11: Productivity distributions (left panel). Wealth distribution (right panel)

Consequently, this prevents the dramatic drop in labor supply and the resulting rise of wage.

4 Conclusion

In this article, we develop a general equilibrium model to study the economic implications of different assumptions concerning the underlying idiosyncratic process of firms productivity.

To test the importance of assumptions on the productivity process we designed two aggregate shock and examined how the economy responds to these shocks for different modelization of the productivity process. We obtain that the parametrization of the individual process is crucial both in characterizing the firm level effects of the shock and in determining the magnitude of the fluctuations of the aggregate variables.

We think this study highlights the importance of modelization choice about the underlying process governing the evolution of firms' productivity. We think the argument is particularly relevant and this paper can be a good starting point for future research, considering the growing importance of heterogeneity in macroeconomics.

5 Appendix

5.1 HJB Equation

In discrete the individual value function is

$$v(a,s) = \max_c \left[u(c) + \beta \left\{ (1-\varrho) v(a', s) + \varrho \sum_{s'} p(s') v(a', s') \right\} \right]$$

Where, for simplicity we use $\varrho = (1-\psi)$. Substitute the discount factor $\frac{1}{1+\rho} = \beta$ and use the b.c. to substitute $a' = M(a,s) + (1+r)a - c(a,s)$ into the value function

$$v(a,s) = \max_c \left[u(c) + \frac{1}{1+\rho} \left\{ \begin{array}{c} (1-\varrho) v(M(a,s) + (1+r)a - c(a,s), s) \\ +\varrho \sum_{s'} p(s') v(M(a,s) + (1+r)a - c(a,s), s') \end{array} \right\} \right]$$

in a Δ fraction of time we have:

$$v(a,z) = \max_c \left[\Delta u(c) + \frac{1}{1+\Delta\rho} \left\{ \begin{array}{c} (1-\Delta\varrho) v[a + \Delta(M(a,s) + ra - c(a,s)), s] \\ +\Delta\varrho \sum_{s'} p(s') v[a + \Delta(M(a,s) + ra - c(a,s)), s'] \end{array} \right\} \right]$$

Define $sav(a,s) = M(a,s) + ra - c(a,s)$ and subtract $v(a,z)$ and manipulate a bit

$$v(a,z) - v(a,z) = \max_{c_t} \left[\Delta u(c_t) + \frac{1}{1+\Delta\rho} \left\{ \begin{array}{c} (1-\Delta\varrho) v[a + \Delta sav(a,s), s] \\ +\Delta\varrho \sum_{s'} p(s') v[a + \Delta sav(a,s), s'] \end{array} \right\} - v(a,z) \right]$$

$$0 = \max_{c_t} \left[\Delta u(c_t) + \frac{1}{1+\Delta\rho} \left\{ \begin{array}{c} (1-\Delta\varrho) v[a + \Delta sav(a,s), s] + \\ +\Delta\varrho \sum_{s'} p(s') v[a + \Delta sav(a,s), s'] + \\ -(1+\Delta\rho) v(a,z) \end{array} \right\} \right]$$

$$0 = \max_{c_t} \left[\Delta u(c_t) + \frac{1}{1+\Delta\rho} \left\{ \begin{array}{c} (1-\Delta\varrho) v[a + \Delta sav(a,s), s] + \\ +\Delta\varrho \sum_{s'} p(s') vv[a + \Delta sav(a,s), s] + \\ -v(a,z) - \Delta\rho v(a,z) \end{array} \right\} \right]$$

$$0 = \max_{c_t} \left[\Delta u(c_t) + \frac{1}{1+\Delta\rho} \left\{ \begin{array}{c} v[a + \Delta sav(a,s), s] - v(a,z) + \\ -\Delta\varrho v[a + \Delta sav(a,s), s] \\ +\Delta\varrho \sum_{s'} p(s') v[a + \Delta sav(a,s), s'] - \Delta\rho v(a,z) \end{array} \right\} \right]$$

$$0=\max_c\left[\Delta u\left(c\right)+\frac{1}{1+\Delta\rho}\left\{\begin{array}{c} v\left[a+\Delta sav\left(a,s\right),s\right]-v\left(a,s\right)+\\ -\Delta\varrho v\left[a+\Delta sav\left(a,s\right),s\right]\\ +\Delta\varrho\sum_{s'}p\left(s'\right)v\left[a+\Delta sav\left(a,s\right),s'\right]\end{array}\right\}-\frac{\Delta\rho}{1+\Delta\rho}v\left(a,s\right)\right]$$

Now divide by Δ

$$0=\max_c\left[\frac{\Delta u\left(c\right)}{\Delta}+\frac{1}{1+\Delta\rho}\left\{\begin{array}{c} \frac{v[a+\Delta sav(a,s),s]-v(a,s)}{\Delta}+\\ -\frac{\Delta}{\Delta}\varrho v\left[a+\Delta sav\left(a,s\right),s\right]\\ +\frac{\Delta}{\Delta}\varrho\sum_{s'}p\left(s'\right)v\left[a+\Delta sav\left(a,s\right),s\right]\end{array}\right\}-\frac{1}{\Delta}\frac{\Delta}{1+\Delta\rho}\rho v\left(a,s\right)\right]$$

$$0=\max_c\left[u\left(c\right)+\frac{1}{1+\Delta\rho}\left\{\begin{array}{c} \frac{v[a+\Delta sav(a,s),s]-v(a,s)}{\Delta}+\\ -\varrho v\left[a+\Delta sav\left(a,s\right),s\right]\\ +\varrho\sum_{s'}p\left(s'\right)v\left[a+\Delta sav\left(a,s\right),s'\right]\end{array}\right\}-\frac{\rho}{1+\Delta\rho}v\left(a,s\right)\right]$$

Take the limit $\Delta\rightarrow 0$

$$0=\lim_{\Delta\rightarrow 0}\max_c\left[u\left(c\right)+\frac{1}{1+\Delta\rho}\left\{\begin{array}{c} \frac{v[a+\Delta sav(a,s),s]-v(a,z)}{\Delta}+\\ -\varrho v\left[a+\Delta sav\left(a,s\right),s\right]\\ +\varrho\sum_{s'}p\left(s'\right)v\left[a+\Delta sav\left(a,s\right),s\right]\end{array}\right\}-\frac{\rho}{1+\Delta\rho}v\left(a,s\right)\right]$$

$$0=\max_c\left[u\left(c\right)+\left\{\begin{array}{c} -v'\left(a,s\right)sav\left(a,s\right)\\ +\varrho\left[\sum_{s'}p\left(s'\right)v\left(a,s'\right)-v\left(a,s\right)\right]\end{array}\right\}-\rho v\left(a,s\right)\right]$$

Note we applied $\lim_{\Delta\rightarrow 0}\frac{v[a+\Delta sav(a,s),s]-v(a,z)}{\Delta}=-\partial_a v\left(a,s\right)sav\left(a,s\right)=-v'\left(a,s\right)sav\left(a,s\right)$.

Finally, substitute again $\varrho=\left(1-\psi\right)$ and obtain

$$\rho v\left(a,s\right)=\max_c\left[u\left(c\right)+-v'\left(a,s\right)sav\left(a,s\right)+\left(1-\psi\right)\left[\sum_{s'}p\left(s'\right)v\left(a,s\right)-v\left(a,s\right)\right]\right]$$

5.2 KF Equation

In discrete time the joint distribution G evolves according to the following law of motion.

$$\begin{aligned} G_{t+1}\left(a,s\right) &= \left(1-\varrho\right)\int_{a_{-1}:g(a_{-1},s)=a}G_t\left(a_{-1},s\right)da+\varrho p\left(s\right)\sum_{s'}\int_{(a_{-1},s'):g(a_{-1},s')=a}G_t\left(a_{-1},s'\right)da\\ G_{t+1}\left(a,s\right) &= \left(1-\varrho\right)G_t\left(a-sav\left(a,s\right),s\right)+\varrho p\left(s\right)\sum_{s'}G_t\left(a-sav\left(a,s\right),s'\right)\end{aligned}$$

where we define the wealth of previous period as $a_{-1} = a - sav(a,s)$ with $sav(a,s) = M(a,s) + ra - c$. and define $\varrho = 1 - \psi$ the probability to change productivity.

In Δ fraction of time we have

$$G_{t+\Delta}(a,s) = (1 - \Delta\varrho) G_t(a - \Delta sav(a,s), s) + \Delta\varrho p(s) \sum_{s'} G_t(a - \Delta sav(a,s), s')$$

subtract $G_t(a,z)$ from both sides and manipulate a bit.

$$\begin{aligned} G_{t+\Delta}(a,s) - G_t(a,s) &= (1 - \Delta\varrho) G_t(a - \Delta sav(a,s), s) + \\ &\quad + \Delta\varrho p(s) \sum_{s'} G_t(a - \Delta sav(a,s), s') - G_t(a,s) \end{aligned}$$

$$\begin{aligned} G_{t+\Delta}(a,s) - G_t(a,s) &= G_t(a - \Delta sav(a,s), s) - \Delta\varrho G_t(a - \Delta sav(a,s), s) + \\ &\quad + \Delta\varrho p(s) \sum_{s'} G_t(a - \Delta sav(a,s), s') - G_t(a,s) \\ G_{t+\Delta}(a,s) - G_t(a,z) &= G_t(a - \Delta sav(a,s), s) - G_t(a,s) + \\ &\quad - \Delta\varrho \left[G_t(a - \Delta sav(a,s), s) - p(s) \sum_{s'} G_t(a - \Delta sav(a,s), s') \right] \end{aligned}$$

Divide by Δ

$$\begin{aligned} \frac{G_{t+\Delta}(a,s) - G_t(a,z)}{\Delta} &= \frac{G_t(a - \Delta sav(a,s), s) - G_t(a,s)}{\Delta} + \\ &\quad - \frac{\Delta\varrho}{\Delta} \left[G_t(a - \Delta sav(a,s), s) - p(s) \sum_{s'} G_t(a - \Delta sav(a,s), s') \right] \end{aligned}$$

Hence take the limit of Δ going to 0

$$\begin{aligned} \lim_{\Delta \to 0} \frac{G_{t+\Delta}(a,s) - G_t(a,z)}{\Delta} &= \lim_{\Delta \to 0} \frac{G_t(a - \Delta sav(a,s), s) - G_t(a,s)}{\Delta} + \\ &\quad - \lim_{\Delta \to 0} \varrho \left[G_t(a - \Delta sav(a,s), s) - p(s) \sum_{s'} G_t(a - \Delta sav(a,s), s') \right] \end{aligned}$$

Note that $\lim_{\Delta \to 0} \frac{G_t(a - \Delta sav) - G_t(a,z)}{\Delta} = -\partial_a G_t(a,s)\, sav(a,s) = -g_t(a,s)\, sav(a,s)$.

Then obtain:

$$\partial_t G_t(a,s) = - - g_t(a,s)\, sav(a,s) - \varrho \left[G_t(a,s) - p(s) \sum_{s'} G_t(a,s') \right]$$

note that in steady state $\partial_t G_t(a,s) = 0$ and $G_t = G$:

$$0 = -g(a,s)\, sav(a,s) - \varrho \left[G(a,s) - p(s) \sum_{s'} G(a,s') \right]$$

Finally, differentiate w.r.t. a and substitute again $\varrho = 1 - \psi$. Obtain

$$0 = -\partial_a \left[g(a,s)\, sav(a,s) \right] - (1-\psi) \left[g(a,s) - p(s) \sum_{s'} g(a,s') \right]$$

Part III

The Global Decline of Labor Share and the Rising Capital Misallocation

Francesca Crucitti and Lorenza Rossi

Abstract

We provide a general equilibrium model able to reconcile four important facts: the global decline in the relative price of investment goods, the global decline of the labor share of income, the increase in capital misallocation and the low total factor productivity growth. Starting from an exogenous fall in the relative price of investment, the model generates an endogenous decline in the labor share and in the rental rate of capital. Then, the presence of heterogenous producers and borrowing constraint, rise capital misallocation and narrow TFP growth. In this respect, our explanation for the decline of the real rental rate of capital and the consequent rising capital misallocation, is alternative to the one offered by Gopinath et al (2017). Also we claim that, though the Euro convergence process may have reinforced these facts, it cannot be used as a unique explanation

6 Introduction

Over the past decades, most of the advanced economies around the world shared similar trends in four important macroeconomic variables: declining labor share of income, rising capital misallocation, low total factor productivity growth and declining relative price of investment goods. What we aim to do in this paper is to develop a theoretical framework which can reconcile all these facts. More specifically, we try to set a connection between the decline in the price of relative price of investment goods, the decline in the labor share and the increasing capital misallocation. In this respect, we provide some element for understanding the roots of the slowdown in productivity growth. We offer a theoretical model able to better interpret the underlying facts and to recognize the links between these different economic phenomena.

To do this, we develop a model with a well-structured supply side. We defined two final goods production sectors, one for the investment good and one for the consumption good, and an intermediate good production sector. In the model, two different sectors of production for investment and consumption allows simulating an investment-specific technology change, which is the exogenous elements determining the fall in the relative price of the investment good.

Firms in both the final goods production sectors are identical and they operate in perfectly competitive markets. Besides, they use only one input, that is the intermediate good. Neither capital or labor is directly employed in the production of final goods.

The characterization of the intermediate production sector is key in the model. Firms in this sector are heterogeneous in term of productivity and they are subject to a borrowing constraint for capital. The presence of productivity heterogeneity and financial frictions is necessary for the model to generate some degree of misallocation. We assign in the sector Constant Elasticity of Substitution production technology with capital and labor being the two production inputs. Hence, capital and labor demand comes only from the intermediate good production sector. Besides, since we set the elasticity of substitution between capital and labor to be larger than one, we allow for variation in the respective income share of the two production inputs.

The design of the economy then is suited to study the long-run effects of a change in the level of technology in the investment good production sector and on its relative price. We can both do macro and micro-level analysis so that we can asses the effect of the change in the relative price of investment on misallocation. Finally, we can also quantify TFP losses directly coming from misallocation.

In a nutshell, in the model, the reduction of the relative price of investment leads agents to accumulate more wealth so that the aggregate stock of capital in the economy increases, depressing the rental rate. Consequently, firms substitute labor with capital, which became relatively cheaper. It directly lowers the labor share but it also gives rise to the increase in misallocation of capital. In fact, as the rental rate of capital fall, firms with higher collateral, that are unconstrained, increase their capital demand and then the marginal productivity of capital of these firms lower. Differently, constrained firms, despite being potentially productive cannot increase their capital so that they do not experience a decline in their marginal capital productivity. Therefore, the allocative efficiency of capital worsen.

Besides, the presence of the collateral constraint also restrains the fall in labor share, since most of the firms are not able to give up some labor for larger capital, again because they are constrained. Simulating the same change in the relative price of investment that is simulated in Karabarbounis et al (2014) we get half of the decline in the labor share that they get. We argue that precisely is because of heterogeneity and borrowing limit.

At the very macro level, the decline in the relative price of investment generates growth in consumption, output and total factor productivity. However, what is noticeable is that, relative to all the other variables, the TFP growth is significantly lower and this result is strictly connected to the increase in input misallocation. When we compute analytically the level of TFP we would observe in the economy without misallocation we obtain that productivity would grow by about 20 percentage point more if production input were efficiently allocated among firms.

Our paper is related to several strands of literature. Several economic studies documented the four mentioned trends and empirically tested their significance.

The decline in the relative price of investment itself (or in other words the decline in the relative cost of capital) is something which is also well documented and captured the attention of economists, especially in recent years. IMF (2014) has examined changes in the relative price of investment in the advanced economies since 1980. It documents a downward trend that levels off in the early twenty-first century. In explaining this movement, it points to the work of Gordon (1990), who emphasizes the role of research and development that is embodied in cheaper, more efficient investment goods. Finally, the IMF's study asserts that any induced increase in the volume of investment was insufficient to offset the negative impact of this trend on real interest rates. It then suggests that the decline in the relative price of investment is strictly connected to low levels of interest rate, another important and documented facts that many advanced economies share. Already Krusell (1998) documents a significant declining trend in the relative investment price due to positive technology change. A similar argument is in Justiniano et al. (2010) but applied in the contest of business cycle model. Eichengreen, (2015) connects the declining trend in the relative price of investment to the slowdown in productivity growth, taking part in the discussion about secular stagnation.

There is a large literature that indicates among the main causes of low TFP growth the rising input misallocation among heterogeneous firms. Gopinath et. al (2017) establish a causal relationship between the decline in the real interest rate and the rise in capital misallocation. They report that these dynamics lead to a declining TFP. Also, Gopinath et al. (2017) focus on southern European countries claiming that in that circumstance, the main responsible for the increase in misallocation was the decline in the real interest rate induced by the euro convergence process. Their model, however, does not investigate the decline in the labor share of income and that in the relative price of investment goods. Similarly, Calligaris et al. (2017) find that resource misallocation has played a sizeable role in slowing down Italian productivity growth. Restuccia et al. (2013) using a simple model explains how aggregate TFP can be lower by inputs misallocation across heterogeneous production units. Dias et al (2016), using firm-level data, investigate whether changes in resource misallocation may have contributed to the poor economic performance of some southern and peripheral European countries. Furthermore, our model offers an explanation for the decline of the real rental rate of capital based by the relative decline price of investment goods, which is thus alternative to the one offered by Gopinath et al (2017), among others. To our opinion, this can be seen as a valid explanation since it may contribute not only to explain the long-run dynamics of the EU economies but also that of the US and more in general of all the OECD countries. The global decline in the labor share of income has been indeed accompanied by a global stagnation in TFP growth together with a decline of the relative price of investment goods and increased capital misallocation. Also, we claim that while the euro convergence process can be dated as starting in the early '90s, the long-run facts we aim to explain are well documented already in the early eighties in most of the countries. Thus, though the Euro convergence process may have reinforced these facts, it cannot be used as a unique explanation of these facts. Finally, all these papers do not investigate the decline of the labor share of income and its relationship with capital misallocation and TFP stagnation.

According to the World Economic Outlook 2017 by the International Monetary Fund, in advanced economies, labor income shares began trending down in the 1980s, reached their lowest level just prior to the global financial crisis of 2008, and have not recovered materially since then. A similar result is described in Dao et al. (2017). Also, Rodriguez et al. (2018) after showing that in most regions of the world labor shares have fallen starting from about 1980, they argue that this decrease is driven by a decrease in intra-sector labor shares as opposed to movements in activity towards sectors with lower labor shares. In the same direction goes the paper by Karabarbounis et al. (2014), they provide further evidence that the global labor share decline reflects declines within industries. These pieces of evidence suggest that the cause is something different than changes in industrial composition, as it was instead claimed in older literature. Indeed, Karabarbounis et al. (2014) argument that the decline of the labor share can be explained by efficiency gains in capital-producing which, by lowering the relative price of investment goods, induced firms to shift away from labor input and toward capital input in production. However, in their model firms are homogeneous and the TFP is exogenous so that their model is not able to replicate the increased capital misallocation and the low TFP growth.

Finally, looking at the modelization and the design of the economy we set, this paper is connected to the large literature on occupational choice and financial frictions. The binary occupational choice between being worker or entrepreneurs and the modelization of collateral constraint for firms closely follow Buera (2009), Buera et al (2013).

Moving the attention toward the research question, our paper builds on empirical studies which document the decline in the labor share over the medium run as Krueger et al (1999) Dorn et al. (2017), Jones (2003), and Bentolila et al (2003), Rodriguez et al. (2010) use UN data and are the broadest studies of trends in labor shares. Karabarbounis et al (2013) who establish a causal relationship between the trend in labor share and in the relative price of investment.

The theoretical attempts to analyze the dynamics of labor share are less numerous, Acemoglu (2003). Blanchard and Giavazzi (2003), through a general equilibrium model, argument that a potential explanation for the observed decline in labor share is the decrease in the bargaining power of unions. Choi et al (2009) and Colciago and Rossi (2015) study the main drivers of the dynamics of labor share in a business cycle model.

Our main contribution is in the ability of our model to explain all the four facts mentioned above using a simple model characterized by heterogeneity among firms and financial frictions. The model provided allows connecting the dynamics of labor share of income with the secular stagnation phenomenon, through the dynamics of input misallocation caused by the decline in the relative price of investment and the subsequent decline of the real rental rate of capital.

At the best of our knowledge, we are the first attempt to reconcile the increase in misallocation with the labor share decline.

The rest of the paper is organized as follows. The model is presented in section 2. Section 3 reports calibration, solution strategy and the results of quantitative analysis. Finally, section 4 concludes

7 The Model

We consider an economic environment in which final consumption and investment goods are produced by using an intermediate good y which production function is characterized by CES technology. There is no aggregate uncertainty and all economic agents have perfect foresight. All payments in this economy are made in terms of the final consumption good, which is the numeraire. Individual are heterogeneous with respect to entrepreneurial ability z and individual wealth a.

Individual wealth evolves endogenously according to individual optimal saving decisions, while the entrepreneurial ability follows a stochastic process. We assume that each individual retains her productivity with probability ψ while with probability $(1-\psi)$ she loses the current productivity and has to draw a new one. The new draw is from a time-invariant distribution with a cumulative density $\mu(z) = 1 - z^{-\eta}$ and it is independent of her previous productivity level.

Agents face a discrete choice relative to their occupation. According to their individual state, i.e. productivity and wealth level, individuals choose whether to be a worker or to be an entrepreneur and set up their own firms. The Individual who decides to be worker offers the unit of labor whose she is endowed to the labor market and receives as compensation the equilibrium wage. Differently, the individual who decides to be entrepreneurs do not access to the labor market and she only receives profits from running the firm.

All the individuals have the same utility function:

$$U = E_t \int_{t=0}^{\infty} e^{-\rho t} u\left(c\right) dt \tag{29}$$

where ρ is the intertemporal discount factor and c is the level of consumption.

Individuals can purchases consumption and investment goods x, from final good producers at their relative price. They use investment good to accumulate wealth such that $\dot{a} = x - \delta a$.

7.1 Supply side

7.1.1 Final consumption good producers

Identical competitive producers assemble the final consumption good from intermediate inputs y and sell it to the household at a price P^C. They produce final consumption with the technology:

$$C = Y^c \tag{30}$$

where Y^c is the quantity of input Y used in production of the final consumption good. The consumption good producers purchase these inputs from perfectly competitive intermediate producers.

Consumption good is the numeraire in the economy and it has a price of $P^C = 1$.

7.1.2 Final investment good producers

Identical competitive producers assemble the final investment good from intermediate inputs y and sell it to the household at a price P_t^X. They produce final investment with the technology:

$$X = \frac{1}{\xi} Y^X \tag{31}$$

where Y^X is the quantity of input y used in production of the final investment good. The exogenous variable ξ denotes the technology level in the production of the consumption good relative to the investment good. A decline of ξ implies an improvement in the technology of producing the investment good relative to the consumption good. The price of final investment good P^X is then ξ which is also equal to the relative price of investment to consumption $\xi = \frac{P^X}{P^C}$

7.1.3 Intermediate goods producers

Firms produce good y which can be used both for consumption and for investment purpose. Production function of firm in the sector is

$$y = z \left[\varkappa \left(Ak \right)^{\theta} + \left(1 - \varkappa \right) \left(Bl \right)^{\theta} \right]^{\frac{\upsilon}{\theta}} \tag{32}$$

The production technology is characterized by decreasing return to scale $\upsilon < 1$. This assumption allows for a richer characterization of the occupational choice. Assuming constant return to scale would simplify the analysis since it would become a function of productivity only. With decreasing return to scale, instead, both the productivity and the initial wealth matter. The parameter θ is the input elasticity of substitution, in the limiting case of $\sigma \to 1$ and $\theta \to 0$ the function collapses to a Cobb-Douglas. A and B are input specific technology.

Finally, $\varkappa \in (0,1)$ is the distribution parameter, reflects capital intensity in production.

Firm's profit maximization problem

Producers of intermediate good are subject to collateral constraint. More specifically, we assume that entrepreneurs' capital rental k is limited by a collateral constraint $k \leq \lambda a$, where λ measures the degree of credit frictions, with $\lambda \to \infty$ corresponding to perfect credit markets and $\lambda = 1$ to financial autarky.

The profit-maximization problem of the producer of intermediate input y_i is:

$$\begin{aligned} \Pi &= \max_{k,l} y - (r+\delta) k - wl \\ s.t \quad k &\leq \lambda a \end{aligned} \tag{33}$$

From cost minimization, obtain the following demands for capital and labor:

$$k = \min \left\{ \lambda a, \left(\frac{A^{\theta} \upsilon z \varkappa}{(r+\delta)} \right)^{\frac{1}{(1-\upsilon)}} \left[\frac{\varkappa A^{\theta} + (1-\varkappa)}{B^{\theta} \left(\frac{\varkappa}{1-\varkappa} \left(\frac{A}{B}\right)^{\theta} \frac{w_t}{r_t+\delta} \right)^{\frac{\theta}{\theta-1}}} \right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}} \right\} \tag{34}$$

$$l = k \left(\frac{1-\varkappa}{\varkappa} \left(\frac{B}{A} \right)^{\theta} \frac{r+\delta}{w} \right)^{\frac{1}{1-\theta}} \tag{35}$$

7.2 Individual problem

7.2.1 Occupational choice

Agents $i(a,z)$ who decide to be entrepreneur obtain as income the realized profit $M(a,z) = \Pi(a,z)$. The occupational choice of the agent is then defined as $oc(a,z) = 1$ and labor and capital demand is $l(a,z), k(a,z) > 0$. Differently, the income of an agent $i(a,z)$ who decide to be worker is given by the wage $M(a,z) = w$. Her occupational choice, capital and labor demand are $oc(a,z), l(a,z), k(a,z) = 0$.

7.2.2 Utility maximization

The agent chooses consumption c and investment x in order to maximize her utility, subject to the period budget constraint.

$$M(a,z) = \max\left[w, \Pi(a,z)\right] \tag{36}$$

Utility maximization problem

$$\max_{c_{it}} E_t \int_{t=0}^{\infty} e^{-\rho t} u(c)\, dt \tag{37}$$

s.t. the budget constraint:

$$c(a,z) + \xi \dot{a}(a,z) = M(a,z) + ar + \delta(1-\xi)a \tag{38}$$

Or, equivalently, recalling the law of motion for wealth accumulation, we can write the budget constraint as

$$\begin{aligned} c(a,z) + \xi x(a,z) &= M(a,z) + (r+\delta)a \\ x_t &= \dot{a} + \delta a \end{aligned} \tag{39}$$

Writing the problem recursively, the first order condition is:

$$u'(c) = \frac{1}{\xi} v'(a,z) \tag{40}$$

7.3 Aggregate TFP and misallocation

In this section, we explain how we measure misallocation in the model and how we compute the loss coming from misallocation in term of total factor productivity.

As a measure for misallocation, we use the covariance between the marginal productivity and the amount of input used in the firms. We distinguish then between labor misallocation and capital misallocation.

We start by defining the marginal productivity of capital and labor given the CES production technology we defined above.

The marginal productivity of capital is

$$mpk(a,z) = z\upsilon\varkappa A^{\theta} k^{\theta-1}\left[\varkappa (Ak)^{\theta} + (1-\varkappa)(Bl)^{\theta}\right]^{\frac{\upsilon}{\theta}-1}$$

As with standard production technology, the marginal productivity is increasing in individual ability z and it is decreasing in the amount of capital k. Optimal condition imposes that the entrepreneurs choose the level of capital such mpk equalized $(r+d)$, so that the capital demand is increasing in individual ability. However, in the model, the presence of borrowing constraint limits the possibility to invest of some entrepreneurs and it leads to variability in firms marginal productivity of capital. In particular, constrained firms will have higher marginal productivity.

Similarly, for labor we have

$$mpl\,(a,z) = zv\,(1-\varkappa)\,B^{\theta}l^{\theta-1}\left[\varkappa\,(Ak)^{\theta}+(1-\varkappa)\,(Bl)^{\theta}\right]^{\frac{v}{\theta}-1}$$

Labor misallocation is then computed as:

$$misL = \int \left(mpl\,(a,z) - MPL\right)\left(l\,(a,z) - \frac{L}{1-L}\right) g^{e}\,(da,dz) \tag{41}$$

where $mpl\,(a,z)$ is the marginal labor productivity of the individual firm, $MPL = \int mpl\,(a,z)\,g^{e}\,(da,dz)$ is the average labor productivity $\frac{L}{1-L}$ and is average labor employed and $g^{e}\,(a,z) = \frac{g(a,z)_{|oc(a,z)=1}}{1-L}$ is the stationary distribution conditional on being entrepreneur.

Similarly for capital we have:

$$misK = \int \left(mpk\,(a,z) - MPK\right)\left(k\,(a,z) - \frac{K}{1-L}\right) g^{e}\,(da,dz) \tag{42}$$

where $mpk(a,z)$ is firm's capital productivity, $MPK = \int mpk\,(a,z)\,g^{e}\,(da,dz)$ is the average capital productivity and $\frac{K}{1-L}$ is average capital employed.

In an economy where capital and labor are perfectly allocated $misL = misK = 0$ because the marginal productivity of both the input is equalized across producers. Thus, for labor we would observe $mpl\,(a,z) = MPL = \int mpl\,(a,z)\,g^{e}\,(da,dz)$ and then $(mpl\,(a,z) - MPL)$. The same would hold for capital.

Instead, if there is misallocation the marginal productivity of inputs vary among firms. In particular, we would register firms with higher than average input productivity but lower than average input employed,so

$$\begin{aligned}\left(mpk\left(a,z\right)-MPK\right)\left(k\left(a,z\right)-\frac{K}{1-L}\right) &< 0\\ \left(mpl\left(a,z\right)-MPL\right)\left(l\left(a,z\right)-\frac{L}{1-L}\right) &< 0\end{aligned}$$

Concluding, we can say that the lower the value of $misK$, $misL$ the worse the allocation of production factors.

In doing the quantitative evaluation of the model in the next section we want to quantify the TFP loss deriving from misallocation.

Take as given the stationary level of aggregate capital K, labor L and the measure of entrepreneurs that characterizes the equilibrium of the economy g^e, we allocate capital $k\left(a,z\right)$ and labor $l\left(a,z\right)$ among entrepreneurs in order to maximize the aggregate output:

$$\max_{k(a,z)l(a,z)}\int z\left[\varkappa\left(Ak\left(a,z\right)\right)^{\theta}+\left(1-\varkappa\right)\left(Bl\left(a,z\right)\right)^{\theta}\right]^{\frac{\upsilon}{\theta}}g^{e}\left(da,dz\right) \tag{43}$$

s.t.

$$\int k\left(a,z\right)g^{e}\left(da,dz\right) = K \tag{44}$$

$$\int l\left(a,z\right)g^{e}\left(da,dz\right) = L \tag{45}$$

As discussed, an efficient allocation implies that the marginal product of labor and capital is equalized across produces. Thus, rearranging the first order conditions of the problem we obtain that the efficient level of TFP is:

$$TFP^{ef}=\frac{Y^{ef}}{\left[\varkappa\left(AK\right)^{\theta}+\left(1-\varkappa\right)\left(BL\right)^{\theta}\right]^{\frac{\upsilon}{\theta}}}=\left[\int_{(a,z):oc(a,z)=1}z^{\frac{1}{(1-\upsilon)}}g\left(da,dz\right)\right]^{1-\upsilon} \tag{46}$$

Consequently the efficient level of output is

$$Y^{ef}=\left[\varkappa\left(AK\right)^{\theta}+\left(1-\varkappa\right)\left(BL\right)^{\theta}\right]^{\frac{\upsilon}{\theta}}\left[\int_{(a,z):oc(a,z)=1}z^{\frac{1}{(1-\upsilon)}}g\left(da,dz\right)\right]^{1-\upsilon} \tag{47}$$

In the appendix we provide a full derivation of the solution of the problem.

7.4 Equilibrium

As in standard Ayagari model, individuals' consumption-saving decision and the evolution of the joint distribution of their income and wealth can be summarized with two differential equations:

- Hamilton-Jacobi-Bellman (HJB) equation

$$\rho v(a,z) = \max_c \left[u(c(a,z)) + \left\{ \begin{array}{c} v'(a,z)\left(\frac{1}{\xi}(M(a,z)+(r+\delta)a - c(a,z)) - \delta a\right) \\ +(1-\psi)\left[p(z)\int_z v(a,z') - v(a,z)\right] \end{array} \right\} \right] \tag{48}$$

- Kolmogorov Forward (or Fokker-Planck). Define $s(a,z) = \dot{a} - a = \frac{1}{\xi_t}(y - c + (r_t + (1-\xi)\delta)a)$, then

$$0 = -\partial_a [g(a,z)s(a,z)] + (1-\psi)\left[\int_z p(z)g(a,z) - g(a,z)\right] \tag{49}$$

  we provide in the apendix a full derivation of the HJB and the Fokker-Planck equations

Finally capital and labor market clearing condition and the aggregate resource constraint are:

$$K = \int_{(a,z):oc(a,z)=1} k(a,z)g(a,z)\,dadz = \int_{(a,z)} ag(a,z)\,dadz \tag{50}$$

$$\begin{aligned} L &= \int_{(a,z):oc(a,z)=1} l(a,z)g(a,z)\,dadz \\ &= 1 - \int_{(a,z):oc(a,z)=1} g(a,z)\,dadz \end{aligned} \tag{51}$$

$$\begin{aligned} Y &= \int_{(a,z):oc(a,z)=1} y(a,z)g(a,z)\,dadz \\ &= \int_{(a,z)} c(a,z)g(a,z)\,dadz + \xi\int_{(a,z)} x(a,z)g(a,z)\,dadz \end{aligned} \tag{52}$$

8 Long run quantitative exploration

We compare here the long run stationary equilibria for different model specifications and different shocks.

8.1 Calibration

Calibration is at yearly bases**.** We calibrate the parameter $\eta = 14.6$ in the Pareto distribution to target the wealth share of the top 10% equal to 0.65;. the parameter of the collateral constraint $\lambda = 3.95$ targets the external credit ratio equal to 60% of total capital; the parameter $\varkappa = 1.17$ in the production function targets an initial labor share equal to 0.65; $\rho = 0.067$ is set to obtain an initial interest rate equal to 0.045. The value for θ, the elasticity of substitution in the production function, is 1.3 as in Karabarbounis et al. 2013, while we follow Buera and Shin (2013) to set $\nu = 0.8$ and $\psi = 0.894$. The utility function is assumed to be a standard CRRA $u(c) = \frac{c^{1-\mu}}{1-\mu}$ with parameter $\mu = 1.5$. Finally, the depreciation rate is $\delta = 0.06$. In the initial state, the price of investment ξ is set equal to one.

8.2 Solution strategy

To solve for the stationary equilibrium of the model we follow the solution strategy proposed by Moll et. al (2017).

The objective is to calculate stationary equilibria functions $v(a, z)$ and $g(a, z)$ and the equilibrium prices r, w, which satisfy the equilibrium conditions 48 49 50 51

The algorithm that we use to find the equilibrium prices of the model is the algorithms typically used to solve discrete-time heterogeneous agent models in the contest of Aiyagari-Hugget economy. The main steps are:

1. Begin iteration with initial guess r^{guess},w^{guess}
2. Given $r = r^{guess}$ and $w = w^{guess}$ solve the HJB equation and calculate saving policy function
3. Given the saving function, solve the KF for g
4. Given functions of individual choices and the distribution, compute aggregates

5. Check market clearing condition. If not satisfied, update the guess $r^{guess} = r^{new}$ and $w^{guess} = w^{new}$

The main differences with respect to the discrete-time version of the solution are the methodologies used to calculate the two equilibrium equation: $v(a,z)$ and $g(a,z)$.

we solve the HJB equation (7) using a finite difference method

8.3 Decline in investment price

In this section we analyze the long-run effects of a decline in the relative price of investment. In the benchmark model the relative price of investment is $\xi = 1$. Then, following Karabarbounis et al (2014) we assume a change of -0.25% in ξ. We compute the stationary equilibrium of the economy when $\xi = 0.75$. A part from the change in the relative price of investment everything else is identical to the benchmark model. We collect in Table 1 below the variations of the main variables generated by the variation in ξ.

The first column presents the percentage change of the variables when ξ drops

Table 1

Variable name	Mod (perc var)	Kn 2013 (perc var)
ξ	-25	-25
Consumption	6.13	20.1
Output	6.68	22.8
Nominal investment	13.2	30.8
Labor share	-1.07	-2.6
TFP	0.26	–
OP gap L	0	–
OP gap K	-15.6	–
Capital input	51	67.8
Labor input	-0.34	1.4
Wage	5.28	19.2
Rental Rate	-56.2	22.1

In the model, a positive technology shock in the investment goods production sector and the consequent decline in the relative price of investment leads to a lower rental rate of capital and lower labor share. As investment goods become cheaper than consumption goods, agents buy more investment and then, over time, they accumulate more wealth. In the long run, this wealth accumulation generates an important rise in the stock of capital in the economy, (aggregate capital increases by more than 50%). Consequently, the rental rate falls, by 56.2%.

On the supply side, directly because of the decline of capital rental rate and the increase in capital supply, entrepreneurs modify the optimal set of input in production. Both capital and labor employed in production rise but, capital increases dramatically more. Indeed, though both wage and labor input increases, their growth is so modest that the total labor share decreases. In other words, the growth of labor income is so modest that it is not able to catch up with the growth of total income mainly driven by the increase in K. The fall in investment price induced growth to all the main aggregate variables, and the positive correlation between consumption, output and nominal investment is maintained. However, in the model the growth of these aggregates, although all positive, are relatively lower than expected. Comparing our results with the one obtained by Karabarbounis et al. (2014) we can immediately observe that the relative change reported in their paper are significantly different.

For instance, while in this model the observed growth in consumption is around 6% in Karabarbounis et al. (2014) is larger than 20% A similar difference emerges for output.

As we said at the beginning of the section, we follow their calibration to set the value of ξ in the second economy. Besides, the supply side structure of our paper is very similar to the one that they have. The main difference is that here there is heterogeneity among producers and the presence of financial frictions. These ingredients are the fundamental drivers of the differences in the result of the two models.

We can argue that the lack of growth in this model with heterogeneous producers and incomplete markets is caused by a certain degree of misallocation. Indeed, what is noticeable is that, relative to all the others variable, the TFP growth is significantly lower. This aspect cannot be analyzed in Karabarbounis et al. (2014) since there TFP is completely exogenous.

As shown in the table, the drop of ξ worsened the allocative efficiency of production input in the economy. Entrepreneurs who were constrained before the change on ξ are still constrained and then they cannot change the level of capital employed in production. Thus, the marginal productivity of this set of entrepreneurs remained unchanged.

At the same time, due to the dramatic fall in the rental rate, the marginal productivity of the unconstrained entrepreneurs significantly decreased while they were increasing their capital and labor input. Finally, an additional consequence of the large drop in the rental rate is that many of the entrepreneurs who were unconstrained before the change become constrained when ξ fall then, of course, they exhibit larger marginal productivity than the unconstrained ones but a lower level of capital employed.

All these facts lead to larger productivity and size differences among firms and then more misallocation.

Losses for misallocation Above we have just documented that the exogenous change in the relative price of investment generates a significant misallocation of production factors. In this paragraph, we want to quantify the losses connected to this misallocation in term of aggregate productivity, output, and consumption. Specifically, we calculate the level of this economic aggregates when capital and labor are efficiently allocated among existing entrepreneurs.

Taking as given the equilibrium stock of capital and labor input, we use the equation 46, which defines the efficient level of aggregate total factor productivity. We then compare the growth of TFP we have reported in the previous section when ξ fall to the growth we would observe if capital and labor inputs were efficiently allocated. We finally compute the corresponding growth of consumption and output.

In this experiments, an efficient allocation of production factors leads to a TFP growth which is 20.15% higher than the one obtained in the specification with misallocation. This directly reflects on output and consumption. Clearly, also output reports the identical difference, it is 20.15% higher in this specification. Finally, consumption growth is 18.97% larger.

8.3.1 Direct versus General Equilibrium effects

In this section we disentangle the direct effects of a change in the relative price of investment from the general equilibrium effects, meaning the effects generated by the consequent variation in the prices of capital and labor.

In the first experiment, we solve the model with $\xi = 0.9$, keeping both the rental rate and the wage fixed to their values in the benchmark economy. This would isolate the effect of the change in ξ from the effect generated by the change in prices. Then, we run the complementary experiment, i.e. we maintained fixed

$\xi = 1$ but we set $r = 0.038$ and $w = 0.61$, which are the equilibrium interest rate and wage in the economy with $\xi = 0.9$.

Ideally, in this exercise we should set $\xi = 0.75$, however, allowing the investment price to fall by so much while keeping the rental rate particularly high we obtain an equilibrium where capital supply is extremely large. In fact, very cheap investment opportunity together with a very high interest rate induce agents to accumulate wealth to such an extent that none would be constrained.

Table 2 reports in the first column how the variables in the first experiment change (in percentage) with respect to the baseline economy. The second column collects the percentage change of the variable in the second experiment. The third column presents the variation in general equilibrium.

Table 2			
Variable Name	$\Delta\xi$ effect	$\Delta\,(r, w)$ effect	total
r	0	-22.2	-22.2
Wage	0	1.86	1.86
ξ	-0.1	0	-0.1
Consumption	14.7	-2.36	2.13
Output	27.1	-15.3	2.32
Nominal investment	22	-8.59	4.47
Labor share (percentage point)	-0.21	-0.22	-0.38
TFP	$+4.1$	-3	0.11
OP gap L	0	0	0
OP gap K	$+69.2$	-46.2	-5.5

In the third column, we can see the same pattern we analyzed above, with variations clearly smaller in magnitude.

More interesting is the analysis of the results reported in the first and second column. Indeed, as it is possible to see, the decline in the relative price of investment per sè is the driver in the increase in output, consumption and, of course, in investment. Besides, it has an overall positive effect on allocative efficiency. Instead, it is the decline in the rental rate the responsible for the worsened allocation.

As already mentioned at the beginning of this section, the lower ξ induces agents to accumulate wealth and then, if the rental rate is kept fixed, to move away from the borrowing constrained. Consumption and output increase along with capital and, since less individual are constrained, allocative efficiency improves and aggregate total factor productivity grows.

Differently from the decrease in ξ, the decrease in the rental rate does not lead agents to save more and then to move away from the constraint but, rather, it generates the opposite effects, being a disincentive to save. Thus, in the long-run, aggregate investment is lower and more individuals are subject to the borrowing limit. Furthermore, the low rental rate makes it profitable for relatively less productive agents to employ more capital and, at the margin, to leave the labor market and become entrepreneurs. The final result is a poorer aggregate economic outcome and more misallocation of factors among the active firms. This result is in line with the explanation of Gopinath et al (2017) who directly connect the dynamics of interest rate to the dynamics of capital misallocation. Remarkably, our model offers an explanation for the decline of the real rental rate of capital, which is alternative to the one offered by these authors based by the euro convergence process. We offer an explanation for the decline of the real rental rate of capital based on the relative decline price of investment goods, which is thus alternative to the one offered by Gopinath et al (2017), among others. We believe this can be seen as a valid explanation since it may contribute not only to explain the long-run dynamics of the EU economies but also that of the US and more in general of all the OECD countries. The global decline in the labor share of income has been indeed accompanied by a global stagnation in TFP growth together with a decline of the relative price of investment goods and increased capital misallocation. Also, we claim that while the euro convergence process can be dated as starting in the early '90s, the long-run facts we aim to explain are well documented already in the early '80 in most of the countries. Thus, though the Euro convergence process may have reinforced these facts, it cannot be used as a unique explanation of these facts

9 Conclusion

In this article, we develop a general equilibrium model which can reconcile four important long-run economic phenomena: the global decline in the relative price of investment goods, the global decline the labor share of income, the increase in capital misallocation and the low total factor productivity growth that is still plaguing many advanced economies.

The model is characterized by the presence of heterogeneous firms and financial frictions. Both are crucial to derive our theoretical results and they largely differentiate this analysis from the existing literature. Another fundamental ingredient is the CES production technology, which allows for variable factor shares.

After defining the theoretical model, we conduct a quantitative exploration to compute the long run implications of a drop in the relative price of investment goods. We documented that this change generates a rise in output and consumption but also it increases misallocation of capital. Moreover, we also obtain that the growth in TFP is significantly lower compared to the growth of the other aggregates. This fact is in accordance with data and very well documented in the economic literature. We argue that the low total factor productivity growth is directly connected to the increase input misallocation. However, we offer an alternative explanation to the one presented in the literature.

To further test the importance of misallocation, or more broadly, to test the importance of heterogeneity and credit frictions in this contest, we provide an analytical expression for aggregate TFP in case production factors were perfectly allocated among active firms. Thus, we compare this "efficient" level of TFP with the one that is instead obtained in stationary equilibrium. We register a significant difference between the two values, providing support to the fact that there a strong connection between misallocation and aggregate productivity.

Concluding, our results highlight the importance of heterogeneity in macro models and stress how incomplete markets can be crucial.

We think this model can be a good starting point for future research. We are aware that the assumption on the elasticity of substitution being greater than one can be controversial. Along this line, we are going to modify the production function.

One way to move in this direction could be to introduce a differentiation between ICT and no-ICT capital. In this regard, Carbonero et al. (2017) estimate an elasticity of substitution between labor and ICT capital statistically larger than one (their estimate is close to 1.18). Besides, they provide further empirical evidence on the relationship between the decline in the relative price of ICT capital and the decline in the labor share.

Alternatively, we could also allow for substitutability between different types of workers, as it was proposed by Saumik (2018). By doing this, we can obtain variability of labor share even when the elasticity of substitution between capital and labor is smaller than unity.

Moreover, adding workers heterogeneity or simply a distinction between skilled and unskilled, would allow for interesting welfare analysis and inequality considerations. Taking the model to the data is also in our research agenda

10 Appendix

10.1 Optimal Capital and labor demand

Find the expression of optimal capital demand taking into account the borrowing constraint

FOCs:

$$\begin{aligned}
y(k,l) &= z\left[\varkappa(Ak)^{\theta}+(1-\varkappa)(Bl)^{\theta}\right]^{\frac{\upsilon}{\theta}} \\
\frac{\partial y}{\partial k} &= r+\delta : \frac{\upsilon}{\theta} z\left[\varkappa(Ak)^{\theta}+(1-\varkappa)(Bl)^{\theta}\right]^{\upsilon/\theta-1}\theta\varkappa A^{\theta}k^{\theta-1}=r+\delta \quad (53)\\
\frac{\partial y}{\partial l} &= w : \frac{\upsilon}{\theta} z\left[\varkappa(Ak)^{\theta}+(1-\varkappa)(Bl)^{\theta}\right]^{\upsilon/\theta-1}\theta(1-\varkappa)B^{\theta}l^{\theta-1}=w \quad (54)
\end{aligned}$$

Divide 53, by 54 get

$$\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\left(\frac{k}{l}\right)^{\theta-1}=\frac{r+\delta}{w}$$

Solving for l

$$\begin{aligned}
l^{\theta-1} &= k^{\theta-1}\frac{w}{r+\delta}\left(\frac{A}{B}\right)^{\theta}\frac{\varkappa}{1-\varkappa} \\
l &= k\left(\frac{w}{r+\delta}\left(\frac{A}{B}\right)^{\theta}\frac{\varkappa}{1-\varkappa}\right)^{\frac{1}{\theta-1}} \qquad (55)
\end{aligned}$$

Plug 55, into 54 and solve for k

$$\begin{aligned}
\upsilon z\left[\varkappa(Ak)^{\theta}+(1-\varkappa)(Bl)^{\theta}\right]^{\frac{\upsilon-\theta}{\theta}}(1-\varkappa)B^{\theta}k^{\theta-1}\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{w}{r+\delta}\right)^{\frac{\theta-1}{\theta-1}} &= w \\
\upsilon z\left[\varkappa(Ak)^{\theta}+(1-\varkappa)(Bl)^{\theta}\right]^{\frac{\upsilon-\theta}{\theta}}(1-\varkappa)B^{\theta}k^{\theta-1}\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{w}{r+\delta} &= w \\
\frac{A^{\theta}\upsilon z\varkappa}{r+\delta}\left[\varkappa(Ak)^{\theta}+(1-\varkappa)(Bl)^{\theta}\right]^{\frac{\upsilon-\theta}{\theta}}k^{\theta-1} &= 1 \\
\left[\varkappa(Ak)^{\theta}+(1-\varkappa)(Bl)^{\theta}\right]^{\frac{\upsilon-\theta}{\theta}} &= \frac{r+\delta}{A^{\theta}\upsilon z\varkappa}k^{1-\theta} \\
\varkappa(Ak)^{\theta}+(1-\varkappa)(Bl)^{\theta} &= \left(\frac{r+\delta}{A^{\theta}\upsilon z\varkappa}\right)^{\frac{\theta}{\upsilon-\theta}}k^{\frac{(1-\theta)\theta}{\upsilon-\theta}}
\end{aligned}$$

On the LHS substitute again the expression for l obtained in 55

$$\begin{aligned}
\varkappa\left(Ak\right)^{\theta}+\left(1-\varkappa\right)B^{\theta}k^{\theta}\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{w}{r+\delta}\right)^{\frac{\theta}{\theta-1}} &= \left(\frac{r+\delta}{A^{\theta}\upsilon z\varkappa}\right)^{\frac{\theta}{\upsilon-\theta}}k^{\frac{(1-\theta)\theta}{\upsilon-\theta}}\\
k^{\theta}\left[\varkappa A^{\theta}+\left(1-\varkappa\right)B^{\theta}\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{w}{r+\delta}\right)^{\frac{\theta}{\theta-1}}\right] &= \left(\frac{r+\delta}{A^{\theta}\upsilon z\varkappa}\right)^{\frac{\theta}{\upsilon-\theta}}k^{\frac{(1-\theta)\theta}{\upsilon-\theta}}\\
\left(\frac{r+\delta}{A^{\theta}\upsilon z\varkappa}\right)^{-\frac{\theta}{\upsilon-\theta}}\left[\varkappa A^{\theta}+\left(1-\varkappa\right)B^{\theta}\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{w}{r+\delta}\right)^{\frac{\theta}{\theta-1}}\right] &= k^{\frac{(1-\theta)\theta}{\upsilon-\theta}-\theta}\\
\left(\frac{r+\delta}{A^{\theta}\upsilon z\varkappa}\right)^{-\frac{\theta}{\upsilon-\theta}}\left[\varkappa A^{\theta}+\left(1-\varkappa\right)B^{\theta}\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{w}{r+\delta}\right)^{\frac{\theta}{\theta-1}}\right] &= k^{\frac{\theta-\theta^{2}-\upsilon\theta+\theta^{2}}{\upsilon-\theta}}\\
\left(\frac{r+\delta}{A^{\theta}\upsilon z\varkappa}\right)^{-\frac{\theta}{\upsilon-\theta}}\left[\varkappa A^{\theta}+\left(1-\varkappa\right)B^{\theta}\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{w}{r+\delta}\right)^{\frac{\theta}{\theta-1}}\right] &= k^{\frac{\theta(1-\upsilon)}{\upsilon-\theta}}\\
\left(\frac{r+\delta}{A^{\theta}\upsilon z\varkappa}\right)^{-\frac{\theta}{\upsilon-\theta}\frac{\upsilon-\theta}{\theta(1-\upsilon)}}\left[\varkappa A^{\theta}+\left(1-\varkappa\right)B^{\theta}\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{w}{r+\delta}\right)^{\frac{\theta}{\theta-1}}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}} &= k\\
\left(\frac{r+\delta}{A^{\theta}\upsilon z\varkappa}\right)^{\frac{-1}{(1-\upsilon)}}\left[\varkappa A^{\theta}+\left(1-\varkappa\right)B^{\theta}\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{w}{r+\delta}\right)^{\frac{\theta}{\theta-1}}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}} &= k \qquad (56)
\end{aligned}$$

56 defines the optimal level of capital. However, given the collateral constraint and then the limited possibility to borrow, the feasible investment k^{*} will be:

$$k^{*}=\min\left(k,\lambda a\right) \tag{57}$$

Now that we obtained the actual level of investment, labor demand will be simply given by plugging k^{*} into 54. Note that we need to solve numerically for l

$$\upsilon z\left[\varkappa\left(Ak^{*}\right)^{\theta}+\left(1-\varkappa\right)\left(Bl^{*}\right)^{\theta}\right]^{\upsilon/\theta-1}\left(1-\varkappa\right)B^{\theta}\left(l^{*}\right)^{\theta-1}=w \tag{58}$$

10.2 HJB Equation and the FOC

Define $\varrho=(1-\psi)$. Then, in discrete time the indivudal problem is

$$v\left(a,z\right)=\max_{c}\left[u\left(c\right)+\beta\left\{\left(1-\varrho\right)v\left(a',z\right)+\varrho\sum_{z'}p\left(z\right)v\left(a',z'\right)\right\}\right]$$

s.t

$$\xi a' = y + (r + \delta + (1-\delta)\xi) a - c$$

After some manipulation the budget constraint can be re-write as

$$\begin{aligned}
a' &= (y + (r + \delta + (1-\delta)\xi) a - c)\frac{1}{\xi} \\
a' &= \left(\frac{y}{\xi} + \frac{(r+\delta)a}{\xi} + (1-\delta)a - \frac{c}{\xi}\right)
\end{aligned}$$

Substitute the discount factor β and using the b.c. substitute a' into the value function

$$v(a,z) = \max_c \left[u(c) + \frac{1}{1+\rho} \left\{ \begin{array}{c} (1-\varrho)v\left((y + (r+\delta+(1-\delta)\xi)a - c, z)\frac{1}{\xi}, z\right) \\ +\varrho \sum_{z'} p(z) v\left((y + (r+\delta+(1-\delta)\xi)a - c, z)\frac{1}{\xi}, z'\right) \end{array} \right\} \right]$$

in a Δ fraction of time we have:

$$v(a,z) = \max_{c_t} \left[\Delta u(c) + \frac{1}{1+\Delta\rho} \left\{ \begin{array}{c} (1-\Delta\varrho)v\left((\Delta y + (\Delta r + \Delta\delta + (1-\Delta\delta)\xi)a - \Delta c, z)\frac{1}{\xi}\right) \\ +\Delta\varrho \sum_{z'} p(z) v\left((\Delta y + (\Delta r + \Delta\delta + (1-\Delta\delta)\xi)a - \Delta c, z')\frac{1}{\xi}\right) \end{array} \right\} \right]$$

Now subtract $v(a,z)$ from both sides and manipulate a bit.

$$\begin{aligned}
0 &= \max_c \left[\Delta u(c) + \frac{1}{1+\Delta\rho} \left\{ \begin{array}{c} (1-\Delta\varrho)v\left(\Delta\frac{y}{\xi} + \Delta\frac{(r+\delta)a}{\xi} + \frac{(1-\Delta\delta)\xi a}{\xi} - \Delta\frac{c}{\xi}, z\right) \\ +\Delta\varrho \sum_{z'} p(z') v\left(\Delta\frac{y}{\xi} + \Delta\frac{(r+\delta)a}{\xi} + \frac{(1-\Delta\delta)\xi a}{\xi} - \Delta\frac{c}{\xi}, z'\right) \end{array} \right\} - v(a,z) \right] \\
0 &= \max_c \left[\Delta u(c) + \frac{1}{1+\Delta\rho} \left\{ \begin{array}{c} (1-\Delta\varrho)v\left(a + \Delta\left(-\delta a + \frac{y}{\xi} + \frac{r+\delta}{\xi}a + -\frac{c}{\xi}\right), z\right) + \\ +\Delta\varrho \sum_{z'} p(z') v\left(a + \Delta\left(-\delta a + \frac{y}{\xi} + \frac{r+\delta}{\xi}a + -\frac{c}{\xi}\right), z'\right) + \\ -(1+\Delta\rho)v(a,z) \end{array} \right\} \right] \\
0 &= \max_c \left[\Delta u(c) + \frac{1}{1+\Delta\rho} \left\{ \begin{array}{c} (1-\Delta\varrho)v\left(a + \Delta\left(-\delta a + \frac{y}{\xi} + \frac{r+\delta}{\xi}a + -\frac{c}{\xi}\right), z\right) - v(a,z) + \\ +\Delta\varrho \sum_{z'} p(z') v\left(a + \Delta\left(-\delta a + \frac{y}{\xi} + \frac{r+\delta}{\xi}a + -\frac{c}{\xi}\right), z'\right) + \\ -\Delta\rho v(a,z) \end{array} \right\} \right] \\
0 &= \max_c \left[\begin{array}{c} \Delta u(c) + \frac{1}{1+\Delta\rho} \left\{ \begin{array}{c} v\left(a + \Delta\left(-\delta a + \frac{y}{\xi} + \frac{r+\delta}{\xi}a + -\frac{c}{\xi}\right), z\right) - v(a,z) + \\ +\Delta\varrho \sum_{z'} p(z') v\left(a + \Delta\left(-\delta a + \frac{y}{\xi} + \frac{r+\delta}{\xi}a + -\frac{c}{\xi}\right), z'\right) + \\ \Delta\varrho v\left(a + \Delta\left(-\delta a + \frac{y}{\xi} + \frac{r+\delta}{\xi}a + -\frac{c}{\xi}\right), z\right) \end{array} \right\} \\ -\frac{\Delta}{1+\Delta\rho}\rho v(a,z) \end{array} \right]
\end{aligned}$$

divide by Δ

$$0=\max_c\left[u(c)+\frac{1}{1+\Delta\rho}\left\{\begin{array}{c}\frac{v\left(a+\Delta\left(-\delta a+\frac{y}{\xi}+\frac{r+\delta}{\xi}a+-\frac{c}{\xi}\right),z\right)-v(a,z)}{\Delta}+\\+\varrho\sum_{z'}p(z')\,v\left(a+\Delta\left(-\delta a+\frac{y}{\xi}+\frac{r+\delta}{\xi}a+-\frac{c}{\xi}\right),z'\right)+\\\varrho v\left(a+\Delta\left(-\delta a+\frac{y}{\xi}+\frac{r+\delta}{\xi}a+-\frac{c}{\xi}\right),z\right)\end{array}\right\}-\frac{1}{1+\Delta\rho}\rho v(a,z)\right]$$

Take the limit $\Delta\to 0$

$$0=\lim_{\Delta\to 0}\left\{\max_c\left[\begin{array}{c}u(c)+\frac{1}{1+\Delta\rho}\left\{\begin{array}{c}\frac{v\left(a+\Delta\left(-\delta a+\frac{y}{\xi}+\frac{r+\delta}{\xi}a+-\frac{c}{\xi}\right),z\right)-v(a,z)}{\Delta}+\\+\varrho\sum_{z'}p(z')\,v\left(a+\Delta\left(-\delta a+\frac{y}{\xi}+\frac{r+\delta}{\xi}a+-\frac{c}{\xi}\right),z'\right)+\\\varrho v\left(a+\Delta\left(-\delta a+\frac{y}{\xi}+\frac{r+\delta}{\xi}a+-\frac{c}{\xi}\right),z\right)\end{array}\right\}\\-\frac{1}{1+\Delta\rho}\rho v(a,z)\end{array}\right]\right\}$$

Note that here apply:

$$\lim_{x\to 0}\frac{f(y+x)-f(y)}{x}=f'(y)$$

In this case:

$$\lim_{\Delta\to 0}\frac{v\left(a+\Delta\left(-\delta a+\frac{y}{\xi}+\frac{r+\delta}{\xi}a+-\frac{c}{\xi}\right),z\right)-v(a,z)}{\Delta}=-v\prime(a,z)\left(-\delta a+\frac{y}{\xi}+\frac{r+\delta}{\xi}a+-\frac{c}{\xi}\right)$$

Finally, the value function is:

$$\rho v(a,z)=\max_c\left[u(c)+\left\{-v\prime(a,z)\left(-\delta a_t+\frac{y_t}{\xi}+\frac{(r_t+\delta)\,a_t}{\xi}-\frac{c_t}{\xi_t}\right)+\varrho\left[\sum_{z'}p(z')\,v(a,z')-v(a,z)\right]\right\}\right]$$

and then the FOC:

$$u'(c)=\frac{1}{\xi}v'(a,z)$$

$$\begin{aligned}u'(c)&=c^{-\mu}\\c^{-\mu}&=\frac{1}{\xi}v'(a,z)\\c&=\left(\frac{1}{\xi}v'(a,z)\right)^{\frac{1}{-\mu}}\end{aligned}$$

10.3 Aggregate TFP

$$Y = \int_{(a,z):oc(a,z)=1} z \left[\varkappa \left(Ak\left(a,z\right) \right)^{\theta} + \left(1-\varkappa\right) \left(Bl\left(a,z\right) \right)^{\theta} \right]^{\frac{\upsilon}{\theta}} g\left(da,dz\right)$$

10.4 Efficient Allocation

Problem: Social planner maximize output given taking as given K and L. It solves

$$\max_{k,l} \int_{(a,z):oc(a,z)=1} z \left[\varkappa \left(Ak\left(a,z\right) \right)^{\theta} + \left(1-\varkappa\right) \left(Bl\left(a,z\right) \right)^{\theta} \right]^{\frac{\upsilon}{\theta}} g\left(da,dz\right)$$

s.t.

$$\begin{aligned} \int_{(a,z):oc(a,z)=1} k\left(a,z\right) g\left(da,dz\right) &= K \\ \int_{(a,z):oc(a,z)=1} l\left(a,z\right) g\left(da,dz\right) &= L \end{aligned}$$

F.o.c.: An efficient allocation implies that the marginal product of labor and capital is equalized across produces:

$$\begin{aligned} mpk &= \upsilon z \left[\varkappa \left(Ak\left(a,z\right) \right)^{\theta} + \left(1-\varkappa\right) \left(Bl\left(a,z\right) \right)^{\theta} \right]^{\frac{\upsilon}{\theta}-1} \varkappa A^{\theta} k^{\theta-1}\left(a,z\right) \\ mpl &= \upsilon z \left[\varkappa \left(Ak\left(a,z\right) \right)^{\theta} + \left(1-\varkappa\right) \left(Bl\left(a,z\right) \right)^{\theta} \right]^{\frac{\upsilon}{\theta}-1} \left(1-\varkappa\right) B^{\theta} l^{\theta-1}\left(a,z\right) \end{aligned}$$

Consequently

$$\begin{aligned} mpk &= \int_{(a,z):oc(a,z)=1} \upsilon z \left[\varkappa \left(Ak\left(a,z\right) \right)^{\theta} + \left(1-\varkappa\right) \left(Bl\left(a,z\right) \right)^{\theta} \right]^{\frac{\upsilon}{\theta}-1} \varkappa A^{\theta} k^{\theta-1}\left(a,z\right) g\left(da,dz\right) \\ mpl &= \int_{(a,z):oc(a,z)=1} \upsilon z \left[\varkappa \left(Ak\left(a,z\right) \right)^{\theta} + \left(1-\varkappa\right) \left(Bl\left(a,z\right) \right)^{\theta} \right]^{\frac{\upsilon}{\theta}-1} \left(1-\varkappa\right) B^{\theta} l^{\theta-1}\left(a,z\right) g\left(da,dz\right) \end{aligned}$$

Rearranging the f.o.cs., we can write the optimal demand of capital as:

$$k = \left(\frac{A^{\theta} \upsilon z \varkappa}{mpk} \right)^{\frac{1}{(1-\upsilon)}} \left[\varkappa A^{\theta} + \left(1-\varkappa\right) B^{\theta} \left(\frac{\varkappa}{1-\varkappa} \left(\frac{A}{B} \right)^{\theta} \frac{mpl}{mpk} \right)^{\frac{\theta}{\theta-1}} \right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}}$$

then, aggregating

$$
\begin{aligned}
K &= \int_{(a,z):oc(a,z)=1} k(a,z)\, g(da,dz) \\
&= \int_{(a,z):oc(a,z)=1} \left(\frac{A^{\theta}\upsilon z \varkappa}{mpk}\right)^{\frac{1}{(1-\upsilon)}} \left[\varkappa A^{\theta} + (1-\varkappa) B^{\theta} \left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta} \frac{mpl}{mpk}\right)^{\frac{\theta}{\theta-1}}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}} g(da,dz)
\end{aligned}
$$

take out of the integral the constant terms and obtain:

$$
\left[\varkappa A^{\theta} + (1-\varkappa) B^{\theta} \left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta} \frac{mpl}{mpk}\right)^{\frac{\theta}{\theta-1}}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}} \left(\frac{A^{\theta}\upsilon\varkappa}{mpk}\right)^{\frac{1}{(1-\upsilon)}} \int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g(da,dz) = K
$$

Similarly, solving for the optimal demand of labor and aggregating we have:

$$
\begin{aligned}
L &= \int_{(a,z):oc(a,z)=1} l(a,z)\, g(da,dz) \\
&= \int_{(a,z):oc(a,z)=1} k(a,z) \left(\frac{1-\varkappa}{\varkappa}\left(\frac{B}{A}\right)^{\theta} \frac{mpk}{mpl}\right)^{\frac{1}{1-\theta}} g(da,dz) \\
&= \left(\frac{1-\varkappa}{\varkappa}\left(\frac{B}{A}\right)^{\theta} \frac{mpk}{mpl}\right)^{\frac{1}{1-\theta}} \int_{(a,z):oc(a,z)=1} k(a,z)\, g(da,dz)
\end{aligned}
$$

Substitute $\int_a \int_z k(a,z)\, g(da,dz)$ with the expression above

$$
\begin{aligned}
L = {} & \left(\frac{1-\varkappa}{\varkappa}\left(\frac{B}{A}\right)^{\theta} \frac{mpk}{mpl}\right)^{\frac{1}{1-\theta}} \left[\varkappa A^{\theta} + (1-\varkappa) B^{\theta} \left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta} \frac{mpl}{mpk}\right)^{\frac{\theta}{\theta-1}}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}} \\
& \left(\frac{A^{\theta}\upsilon\varkappa}{mpk}\right)^{\frac{1}{(1-\upsilon)}} \int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g(da,dz)
\end{aligned}
$$

Isolate the integral term:

$$\begin{aligned}
\frac{1}{\int_a \int_z z^{\frac{1}{(1-\upsilon)}} g\left(da, dz\right)} &= \frac{1}{L}\left(\frac{1-\varkappa}{\varkappa}\left(\frac{B}{A}\right)^{\theta}\frac{mpk}{mpl}\right)^{\frac{1}{1-\theta}} \\
& \left[\varkappa A^{\theta} + \left(1-\varkappa\right)B^{\theta}\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{mpl}{mpk}\right)^{\frac{\theta}{\theta-1}}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}}\left(\frac{A^{\theta}\upsilon\varkappa}{mpk}\right)^{\frac{1}{(1-\upsilon)}}
\end{aligned}$$

Do the same for capital

$$\frac{1}{\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g\left(da, dz\right)} = \frac{1}{K}\left[\varkappa A^{\theta} + \left(1-\varkappa\right)B^{\theta}\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{mpl}{mpk}\right)^{\frac{\theta}{\theta-1}}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}}\left(\frac{A^{\theta}\upsilon\varkappa}{mpk}\right)^{\frac{1}{(1-\upsilon)}}$$

Since the right and side of the two equations are the same, then

$$\begin{aligned}
&\frac{1}{K}\left[\varkappa A^{\theta} + \left(1-\varkappa\right)B^{\theta}\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{mpl}{mpk}\right)^{\frac{\theta}{\theta-1}}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}}\left(\frac{A^{\theta}\upsilon\varkappa}{mpk}\right)^{\frac{1}{(1-\upsilon)}} \\
=\ &\frac{1}{L}\left(\frac{1-\varkappa}{\varkappa}\left(\frac{B}{A}\right)^{\theta}\frac{mpk}{mpl}\right)^{\frac{1}{1-\theta}}\left[\varkappa A^{\theta} + \left(1-\varkappa\right)B^{\theta}\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{mpl}{mpk}\right)^{\frac{\theta}{\theta-1}}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}}\left(\frac{A^{\theta}\upsilon\varkappa}{mpk}\right)^{\frac{1}{(1-\upsilon)}}
\end{aligned}$$

Solving for $\frac{mpl}{mpk}$ obtain

$$\begin{aligned}
\frac{1}{K} &= \frac{1}{L}\left(\frac{1-\varkappa}{\varkappa}\left(\frac{B}{A}\right)^{\theta}\frac{mpk}{mpl}\right)^{\frac{1}{1-\theta}} \\
\frac{L}{K} &= \left(\frac{1-\varkappa}{\varkappa}\left(\frac{B}{A}\right)^{\theta}\frac{mpk}{mpl}\right)^{\frac{1}{1-\theta}} \\
\left(\frac{L}{K}\right)^{1-\theta} &= \frac{1-\varkappa}{\varkappa}\left(\frac{B}{A}\right)^{\theta}\frac{mpk}{mpl} \\
\frac{mpl}{mpk} &= \frac{1-\varkappa}{\varkappa}\left(\frac{B}{A}\right)^{\theta}\left(\frac{L}{K}\right)^{\theta-1}
\end{aligned}$$

We found then an expression for the ratio of the two marginal productivity.

Recall the equation defining aggregate capital

$$\frac{1}{K}\left[\varkappa A^{\theta}+(1-\varkappa)B^{\theta}\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{mpl}{mpk}\right)^{\frac{\theta}{\theta-1}}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}}\left(\frac{A^{\theta}\upsilon\varkappa}{mpk}\right)^{\frac{1}{(1-\upsilon)}}=\frac{1}{\int_{(a,z):oc(a,z)=1}z^{\frac{1}{(1-\upsilon)}}g\left(da,dz\right)}$$

Substitute expression for $\frac{mpl}{mpk}$ and compute mpk

$$\begin{aligned}\frac{1}{\int_{(a,z):oc(a,z)=1}z^{\frac{1}{(1-\upsilon)}}g\left(da,dz\right)} &= \frac{1}{K}\left[\frac{\varkappa A^{\theta}+(1-\varkappa)B^{\theta}}{\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{1-\varkappa}{\varkappa}\left(\frac{B}{A}\right)^{\theta}\left(\frac{L}{K}\right)^{\theta-1}\right)^{\frac{\theta}{\theta-1}}}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}}\left(\frac{A^{\theta}\upsilon\varkappa}{mpk}\right)^{\frac{1}{(1-\upsilon)}}\\
\frac{1}{\int_{(a,z):oc(a,z)=1}z^{\frac{1}{(1-\upsilon)}}g\left(da,dz\right)} &= \frac{1}{K}\left[\varkappa A^{\theta}+(1-\varkappa)B^{\theta}\left(\frac{L}{K}\right)^{\theta-1\frac{\theta}{\theta-1}}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}}\left(\frac{A^{\theta}\upsilon\varkappa}{mpk}\right)^{\frac{1}{(1-\upsilon)}}\end{aligned}$$

$$\begin{aligned}\frac{K}{\int_{(a,z):oc(a,z)=1}z^{\frac{1}{(1-\upsilon)}}g\left(da,dz\right)} &= \left[\varkappa A^{\theta}+(1-\varkappa)B^{\theta}\left(\frac{L}{K}\right)^{\theta}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}}\left(\frac{A^{\theta}\upsilon\varkappa}{mpk}\right)^{\frac{1}{(1-\upsilon)}}\\
\left(\frac{A^{\theta}\upsilon\varkappa}{mpk}\right)^{\frac{1}{(1-\upsilon)}} &= \frac{K}{\int_{(a,z):oc(a,z)=1}z^{\frac{1}{(1-\upsilon)}}g\left(da,dz\right)}\left[\varkappa A^{\theta}+(1-\varkappa)B^{\theta}\left(\frac{L}{K}\right)^{\theta}\right]^{-\frac{\upsilon-\theta}{\theta(1-\upsilon)}}\end{aligned}$$

$$\begin{aligned}\frac{A^{\theta}\upsilon\varkappa}{mpk} &= \left(\frac{K}{\int_{(a,z):oc(a,z)=1}z^{\frac{1}{(1-\upsilon)}}g\left(da,dz\right)}\right)^{(1-\upsilon)}\left[\varkappa A^{\theta}+(1-\varkappa)B^{\theta}\left(\frac{L}{K}\right)^{\theta}\right]^{-\frac{\upsilon-\theta}{\theta}}\\
mpk &= A^{\theta}\upsilon\varkappa\left(\frac{\int_{(a,z):oc(a,z)=1}z^{\frac{1}{(1-\upsilon)}}g\left(da,dz\right)}{K}\right)^{(1-\upsilon)}\left[\varkappa A^{\theta}+(1-\varkappa)B^{\theta}\left(\frac{L}{K}\right)^{\theta}\right]^{\frac{\upsilon-\theta}{\theta}}\end{aligned}$$

Finally we have and expression for the individual capital and labor demand. Just substitute in the optimal demand defined above mpk and $\frac{mpl}{mpk}$. Recall the optimal demand for capital $k\left(a,z\right)=\left(\frac{A^{\theta}\upsilon z\varkappa}{mpk}\right)^{\frac{1}{(1-\upsilon)}}\left[\varkappa A^{\theta}+(1-\varkappa)B^{\theta}\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{mpl}{mpk}\right)^{\frac{\theta}{\theta-1}}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}}$.

Then do the substitutions $mpk=A^{\theta}\upsilon\varkappa\left(\frac{\int_{(a,z):oc(a,z)=1}z^{\frac{1}{(1-\upsilon)}}g(da,dz)}{K}\right)^{(1-\upsilon)}\left[\varkappa A^{\theta}+(1-\varkappa)B^{\theta}\left(\frac{L}{K}\right)^{\theta}\right]^{\frac{\upsilon-\theta}{\theta}}$

and $\frac{mpl}{mpk}=\frac{1-\varkappa}{\varkappa}\left(\frac{B}{A}\right)^{\theta}\left(\frac{L}{K}\right)^{\theta-1}$. Obtain

$$\begin{aligned}
k(a,z) &= \left(\frac{A^{\theta}\upsilon z\varkappa}{mpk}\right)^{\frac{1}{(1-\upsilon)}} \\
&\quad \left[\varkappa A^{\theta}+(1-\varkappa)B^{\theta}\left(\frac{\varkappa}{1-\varkappa}\left(\frac{A}{B}\right)^{\theta}\frac{1-\varkappa}{\varkappa}\left(\frac{B}{A}\right)^{\theta}\left(\frac{L}{K}\right)^{\theta-1}\right)^{\frac{\theta}{\theta-1}}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}} \\
k(a,z) &= \left(\frac{A^{\theta}\upsilon z\varkappa}{mpk}\right)^{\frac{1}{(1-\upsilon)}}\left[\varkappa A^{\theta}+(1-\varkappa)B^{\theta}\left(\frac{L}{K}\right)^{\theta}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}} \\
k(a,z) &= \left(A^{\theta}\upsilon z\varkappa\right)^{\frac{1}{(1-\upsilon)}}\left[\varkappa A^{\theta}+(1-\varkappa)B^{\theta}\left(\frac{L}{K}\right)^{\theta}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}} mpk^{\frac{-1}{(1-\upsilon)}} \\
k(a,z) &= \left(A^{\theta}\upsilon z\varkappa\right)^{\frac{1}{(1-\upsilon)}}\left[\varkappa A^{\theta}+(1-\varkappa)B^{\theta}\left(\frac{L}{K}\right)^{\theta}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}} \\
&\quad \left(A^{\theta}\upsilon\varkappa\left(\frac{\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}}g(da,dz)}{K}\right)^{(1-\upsilon)}\left[\varkappa A^{\theta}+(1-\varkappa)B^{\theta}\left(\frac{L}{K}\right)^{\theta}\right]^{\frac{\upsilon-\theta}{\theta}}\right)^{\frac{-1}{(1-\upsilon)}} \\
k(a,z) &= z^{\frac{1}{(1-\upsilon)}}\left(A^{\theta}\upsilon\varkappa\right)^{\frac{1}{(1-\upsilon)}}\left[\varkappa A^{\theta}+(1-\varkappa)B^{\theta}\left(\frac{L}{K}\right)^{\theta}\right]^{\frac{\upsilon-\theta}{\theta(1-\upsilon)}}\left(A^{\theta}\upsilon\varkappa\right)^{\frac{-1}{(1-\upsilon)}} \\
&\quad \left(\frac{\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}}g(da,dz)}{K}\right)^{(1-\upsilon)\frac{-1}{(1-\upsilon)}}\left[\varkappa A^{\theta}+(1-\varkappa)B^{\theta}\left(\frac{L}{K}\right)^{\theta}\right]^{\frac{\upsilon-\theta}{\theta}\frac{-1}{(1-\upsilon)}}
\end{aligned}$$

manipulate and simplify a bit

$$\begin{aligned}
k(a,z) &= z^{\frac{1}{(1-\upsilon)}}\left(\frac{\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}}g(da,dz)}{K}\right)^{-1} \\
k^{ef}(a,z) &= \frac{z^{\frac{1}{(1-\upsilon)}}}{\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}}g(da,dz)}K
\end{aligned}$$

Use this equation and do the same for labor

$$\begin{aligned}
l(a,z) &= k(a,z)\left(\frac{1-\varkappa}{\varkappa}\left(\frac{B}{A}\right)^{\theta}\frac{mpk}{mpl}\right)^{\frac{1}{1-\theta}} \\
l(a,z) &= \frac{z^{\frac{1}{(1-\upsilon)}}}{\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g(da,dz)} K\left(\frac{1-\varkappa}{\varkappa}\left(\frac{B}{A}\right)^{\theta}\left(\frac{1-\varkappa}{\varkappa}\left(\frac{B}{A}\right)^{\theta}\left(\frac{L}{K}\right)^{\theta-1}\right)^{-1}\right)^{\frac{1}{1-\theta}} \\
l(a,z) &= \frac{z^{\frac{1}{(1-\upsilon)}}}{\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g(da,dz)} K\left(\frac{L}{K}\right)^{1-\theta\frac{1}{1-\theta}} \\
l^{ef}(a,z) &= \frac{z^{\frac{1}{(1-\upsilon)}}}{\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g(da,dz)} L
\end{aligned}$$

We can compute now the efficient level of output, i.e. the level output the economy would produce is input were efficiently allocated

$$\begin{aligned}
Y^{ef} &= \int_{(a,z):oc(a,z)=1} z\left[\varkappa\left(Ak^{ef}(a,z)\right)^{\theta}+(1-\varkappa)\left(Bl^{ef}(a,z)\right)^{\theta}\right]^{\frac{\upsilon}{\theta}} g(da,dz) \\
Y^{ef} &= \int_{(a,z):oc(a,z)=1} z\left[\begin{array}{c}\varkappa\left(A\frac{z^{\frac{1}{(1-\upsilon)}}}{\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g(da,dz)}K\right)^{\theta} \\ +(1-\varkappa)\left(B\frac{z^{\frac{1}{(1-\upsilon)}}}{\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g(da,dz)}L\right)^{\theta}\end{array}\right]^{\frac{\upsilon}{\theta}} g(da,dz) \\
Y^{ef} &= \int_{(a,z):oc(a,z)=1} z\left[\begin{array}{c}\varkappa(AK)^{\theta}\frac{z^{\frac{\theta}{(1-\upsilon)}}}{\left[\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g(da,dz)\right]^{\theta}} \\ +(1-\varkappa)(BL)^{\theta}\frac{z^{\frac{\theta}{(1-\upsilon)}}}{\left[\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g(da,dz)\right]^{\theta}}\end{array}\right]^{\frac{\upsilon}{\theta}} g(da,dz) \\
Y^{ef} &= \int_{(a,z):oc(a,z)=1} z\left[\left[\varkappa(AK)^{\theta}+(1-\varkappa)(BL)^{\theta}\right]\frac{z^{\frac{\theta}{(1-\upsilon)}}}{\left[\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g(da,dz)\right]^{\theta}}\right]^{\frac{\upsilon}{\theta}} g(da,dz) \\
Y^{ef} &= \int_{(a,z):oc(a,z)=1} z\left[\left[\varkappa(AK)^{\theta}+(1-\varkappa)(BL)^{\theta}\right]^{\frac{\upsilon}{\theta}}\frac{z^{\frac{\theta}{(1-\upsilon)}\frac{\upsilon}{\theta}}}{\left[\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g(da,dz)\right]^{\theta\frac{\upsilon}{\theta}}}\right] g(da,dz)
\end{aligned}$$

Take out of the integral the constant term

$$
\begin{aligned}
Y^{ef} &= \left[\varkappa\left(AK\right)^{\theta}+\left(1-\varkappa\right)\left(BL\right)^{\theta}\right]^{\frac{\upsilon}{\theta}} \frac{1}{\left[\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g\left(da,dz\right)\right]^{\upsilon}} \int_{(a,z):oc(a,z)=1} z z^{\frac{\theta}{(1-\upsilon)}\frac{\upsilon}{\theta}} g\left(da,dz\right) \\
Y^{ef} &= \left[\varkappa\left(AK\right)^{\theta}+\left(1-\varkappa\right)\left(BL\right)^{\theta}\right]^{\frac{\upsilon}{\theta}} \frac{\int_{(a,z):oc(a,z)=1} z^{\frac{\theta}{(1-\upsilon)}\frac{\upsilon}{\theta}+1} g\left(da,dz\right)}{\left[\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g\left(da,dz\right)\right]^{\upsilon}} \\
Y^{ef} &= \left[\varkappa\left(AK\right)^{\theta}+\left(1-\varkappa\right)\left(BL\right)^{\theta}\right]^{\frac{\upsilon}{\theta}} \frac{\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g\left(da,dz\right)}{\left[\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g\left(da,dz\right)\right]^{\upsilon}} \\
Y^{ef} &= \left[\varkappa\left(AK\right)^{\theta}+\left(1-\varkappa\right)\left(BL\right)^{\theta}\right]^{\frac{\upsilon}{\theta}} \left[\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g\left(da,dz\right)\right]^{1-\upsilon}
\end{aligned}
$$

And the efficient TFP

$$
TFP^{ef} = \frac{Y^{ef}}{\left[\varkappa\left(AK\right)^{\theta}+\left(1-\varkappa\right)\left(BL\right)^{\theta}\right]^{\frac{\upsilon}{\theta}}} = \left[\int_{(a,z):oc(a,z)=1} z^{\frac{1}{(1-\upsilon)}} g\left(da,dz\right)\right]^{1-\upsilon}
$$

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