9 A brief introduction to the mathematical work of Isaac Newton

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THE ANXIETIES OF A YOUNG STUDENT: *QUESTIONES QUAEDAM PHILOSOPHIAE* (1664–5)

When Isaac Newton entered Trinity College in 1661 the political situation in the university, and in England in general, was still fluid and unstable. The Restoration of the Stuart monarchy had occurred just a year earlier and Cambridge, a hotbed of Puritan sympathizers, was under pressure. It is easy to imagine the feeling of anxiety that the young Isaac must have experienced in such political turmoil. To Newton's generation the future looked uncertain, as no established and recognized authority that could validate truth and guarantee justice was easily discernible. Such instability also characterized the fields of natural philosophy and theology. The Aristotelian philosophy still taught at Cambridge was in disrepute. Various new philosophies were being ambitiously advanced by Bacon, Descartes, and Hobbes, amongst others, as substitutes for the old forms of knowledge. It was perceived that a choice between them would have important theological implications, and recent European and English history had shown how easily theological debates could translate into political unrest and the miseries of war.

Newton soon sought an answer to these concerns in books that he could borrow or acquire, and in notebooks he would fill in minute and legible handwriting, organizing his ideas according to the genre of theological commonplace books. As we shall see in a moment, some mathematical books polarized Newton's attention with a strength that, I feel, was not due to any choice on his part but to the fact that his mind was extraordinarily equipped for mathematical inventiveness.

His annotations after the winter of 1664 reveal the voyage of an independent mind that took the existing literature on the most advanced mathematical topics as a springboard for creating new concepts and methods. It is agreed by all commentators that within the span of a few years Newton became one of the greatest mathematicians of all ages. Perhaps less known is the fact that he was committed to carving out a new role for mathematics within a broad-ranging philosophical agenda. Indeed, Newton was convinced that mathematics was an important tool that could yield a resolution of the daunting issues that puzzled the natural philosophers of his age; most notably, the theologically laden problems concerning the heliocentric System of the World. Is there any proof that the Copernican system is true? And if so, what makes the planets revolve around the Sun? Newton was convinced that mathematics could provide definitive answers to such questions – not hypothetical conjectures but certain answers to the questions that had embroiled Galileo and Descartes in heated disputes with philosophers and theologians. But could the certainty of mathematical thought be injected into the hypothetical discourse of natural philosophy? In order to achieve this end, the relationship of mathematics to other disciples, such as optics and astronomy, had to be conceived in a new light.

Newton's early forays outside the standard Aristotelian curriculum – where he could not find answers to his anxieties concerning the nature of the soul and God, the ultimate structure of reality, the System of the World, the phenomena of life, sensation and volition – are recorded in the "Trinity College Notebook." The range of topics broached in these annotations, most notably in the "Questiones [sic] quaedam philosophiae" he probably penned in 1664–5, is impressive for the modern reader, but indeed it is typical of the broad view of philosophy entertained in late-seventeenth-century Europe.[1] In the "Questiones" one finds annotations on atoms and the "vacuum," motion, perception (especially vision), the Sun, comets, planets and the stars, the structure of the Universe and matter, electrical, magnetic and optical phenomena, heat and cold, gravity and levity, earth, water, air and fire, minerals, the faculties of memory, imagination, fantasy and invention, the soul and God, and much more. The attention of the historian of mathematics will be captured by Newton's annotations on infinity and indivisibles that open the "Questiones" with entries on first matter, atoms, the vacuum, quantity, and motion. Some of Newton's sources here are Walter Charleton's *Physiologia* (1654), Henry More's *Immortality of the Soul* (1659), and John Wallis's *De angulo contactus* (1656) and *Mathesis universalis* (1657). The young Newton was addressing the vexed question of the composition of the mathematical continuum, and the related questions concerning the composition of matter and the representation of continuously varying motion. An impact between hard atoms seemed to cause a discontinuous change of velocity. On the other hand, the speed of a body in free-fall might be conceived of as increasing continuously. These issues remained important in Newton's more mature mathematization of motion. Newton was soon to concentrate his mind on mathematical methods in which these ancient dilemmas acquired a new form, and perhaps became more tractable.

EARLY MATHEMATICAL STUDIES IN THE "WASTE BOOK" AND THE "COLLEGE NOTEBOOK" (1665)

The young Newton had very few mathematical books on his desk (see Table 9.1). His early annotations are edited in volume 1 of the

Table 9.1 *Mathematical books annotated by Newton in the 1660s*

René Descartes, *Geometria*, Amsterdam, 1659–61
François Viète, *Opera mathematica*, Leiden, 1646
Frans van Schooten, *Exercitationum mathematicarum*, Leiden, 1657
William Oughtred, *Clavis mathematicae*, 3d edn, Oxford, 1652
John Wallis, *Operum mathematicorum Pars Altera*, Oxford, 1656
John Wallis *Commercium epistolicum*, Oxford, 1658

Mathematical Papers of Isaac Newton: some of the most interesting are to be found in the "Waste Book" and the "College Notebook."[2] The new art of algebra, in which symbols for constant and variable magnitudes were manipulated, attracted his attention. Newton was introduced to this new method by Viète and Oughtred's works.[3]

The seminal text in Newton's mathematical formation was Descartes's *Géométrie*.[4] Descartes had proposed – or so he claimed in the opening sentence of the work – a novel method for the solution of all the problems of geometry. According to tradition, geometrical problems could be solved by the intersection of plane curves: in Euclid's *Elements*, for instance, via the intersection of a circle and a straight line. More advanced problems, such as the angle trisection or the duplication of the cube, could be solved by the intersection of conics (namely the circle, the parabola, the ellipse and the hyperbola). There were certain problems (such as the section of an angle into an arbitrary number of equal parts or the "squaring" of the circle) that could only be solved by using more complex curves (as the quadratrix or the spiral). In the *Géométrie* Descartes had explained how "equations" (what we would call polynomial equations in two unknowns) could be used in the process of geometrical problem solving. The curves employed in geometrical problem solving could be conceived of as loci of points (in the plane) the coordinates of which satisfy a relation expressed by an equation.[5] In the early seventeenth-century algebra and geometry were still considered two separate disciplines, the former dealing with continuous magnitudes, and the latter with discrete ones. Descartes, therefore, had to overcome conceptual obstacles that should not be underestimated. Most notably, in order to apply algebra to geometry, one had to learn how to interpret all four algebraic operations in geometric terms.

By 1665 Newton had already mastered Descartes's method. The representation and study of plane curves via (polynomial) equations was no mystery to him. While Descartes had used oblique (and sometimes orthogonal) coordinates, Newton experimented with polar (and bipolar) coordinates as well.[6] It is in this context that he began studying the properties of cubic curves (the graphs of third-degree polynomial equations in two unknowns).[7] Newton was particularly interested in a method for determining the subnormal to a curve that Descartes had developed in the *Géométrie* and applied in the *Dioptrique* to devise non-spherical lenses.[8]

Somewhat simplifying, one might say that Newton became a mathematician by studying Descartes's *Géométrie*. This short essay, which in its Latin edition was accompanied by a lengthy commentary by Frans van Schooten and other Dutch mathematicians, provided a systematic method for tackling geometrical problems. It also contained a treatment of algebraic equations: one could learn how to reduce them to a canonical form, how to determine the interval in which their roots were to be found, and how to construct the roots geometrically via the intersection of curves. The young Newton brought these results to perfection. In this context he conceived a rule concerning the number of imaginary roots of an algebraic equation, a rule inspired by Descartes's rule of signs.[9]

There were, however, open problems in the *Géométrie*. Descartes's method for drawing normals was clumsy, whereas the algorithm by Jan Hudde featured in an appendix to the Latin edition proved much more promising. Most disappointingly, Descartes had confined himself to the algebraic treatment of what he called "geometrical" curves (those we would identify as "algebraic" curves) and had quite explicitly excluded "mechanical" curves ("transcendental" curves in modern terms) as lacking "exactness."[10] The most perceptive mathematicians of Newton's generation understood that the next step in mathematical development was to devise a method for tackling the curves that Descartes had excluded from his treatment. At the middle of the century, a mechanical curve – the cycloid – attracted the attention of French, Italian and English mathematicians (including such worthies as Torricelli, Roberval, Pascal, and Wallis). Mechanical curves were interesting as objects of study for a whole series of reasons. They naturally emerged as solutions of problems concerning technology (for example, the cycloid was used by Huygens in his study of horology) and natural philosophy (for example, in his correspondence with Mersenne, Descartes identified the logarithmic spiral as the solution of a problem on motion). Mechanical curves also occurred as solutions of problems concerning the area of curvilinear surfaces and the arc-length of curves. It was often the case that the arc-length of a geometrical curve or the area bounded by it could be expressed by a mechanical curve. Most notably, it was known that the area bounded by an hyperbola is expressed by the logarithmic curve. The calculation of logarithms, a very important issue in seventeenth-century mathematics, table making, navigation, surveying, and astronomy, was thus related to the mathematical treatment of a curve that had been excluded from Descartes's canon. Thus, mechanical curves, which are not expressible via polynomial equations in two Cartesian coordinates, emerged as a promising topic for young ambitious mathematicians – and Isaac Newton certainly wasn't lacking in ambition. But how could one deal with mechanical curves?

EARLY DISCOVERIES IN ORGANIC GEOMETRY: *HOW TO DRAW TANGENTS TO MECHANICALL LINES* (1665)[11]

A promising intake came not from the symbolism of algebra but from a research field located at the intersection between pure geometry and mixed mathematics, the so-called "geometria organica," which studied the tracing of curves by instruments. This was an important topic, since in order to determine the point of intersection of curves in the construction of geometrical solutions, it was convenient to think of the curves as generated by a continuous motion driven by some instrument (an οργανυου), such as the compass and a straightedge in Euclid's *Elements*. It is the continuity of the motion generating the curves by means of a tracing mechanism that guarantees that a point of intersection can be located exactly, under the assumption that the mechanism is handled in an idealized situation in which any imprecision can be avoided.[12] Descartes had devised several mechanisms for generating curves. In *De organica conicarum sectionum* *in plano descriptione tractatus* (1646) van Schooten had presented several mechanisms for generating conic sections. This research field was connected with practical applications, for instance, lens grinding and sundial design. In the late 1660s Newton was able to devise a mechanism for generating conics that he later extended to higher-order curves.[13]

In 1665 in manuscripts entitled "How to Draw Tangents to Mechanicall Lines," Newton deployed organic descriptions in order to determine tangents to mechanical curves, that is, plane curves such as the spiral, the cycloid, the quadratrix, and the logarithmic curve that were not acceptable according to the criteria stated by Descartes in the *Géométrie*.[14] Indeed, Descartes had stated that "mechanical" curves (what nowadays we would call "transcendental" curves) could not be accepted in geometry: he developed a complex argument according to which they were not "exact." Descartes accepted only curves that could be expressed by algebraic equations, and he provided a method for finding the normal (and, therefore, the tangent) to a point on one of these curves. Newton was able to determine the tangent to mechanical curves by applying a kinematical technique known to Gilles Personne de Roberval. Such a method for determining tangents "without calculation" pleased Newton. By 1665, the young master in algebraic analyses was already experimenting with non-algebraic approaches to geometrical problems.

EARLY DISCOVERIES IN THE METHOD OF SERIES AND FLUXIONS: THE *OCTOBER TRACT ON FLUXIONS* (1666)

It was by reading van Heureat and Hudde's annotations to the *Géométrie*, and Wallis's *Arithmetica infinitorum* (1656), and possibly by exchanging ideas and books with Isaac Barrow – who in 1663 had been appointed Lucasian Professor of Mathematics in Cambridge – that Newton began mastering a research field that he named "new analysis." In early modern Europe, "analysis" was a term that had a complex semantic stratification, as it was used in medicine, chemistry, philosophy, and mathematics. To mathematicians, "analysis" meant the "art of mathematical discovery," as outlined in Pappos's *Mathematical Collections*, a fourth-century CE work whose Latin edition had appeared in 1588. What was "new" in the analysis Newton was interested in; that is, the use of the infinite and infinitesimal? Descartes's method of problem solving was confined to the use of finite magnitudes (such as finite segments) expressed by "finite equations" (i.e., polynomial equations with a finite number of terms). Wallis, instead, had conceived curvilinear surfaces as composed of an infinite number of infinitesimal components, and had calculated their areas by means of sums (or products) with an infinite number of terms (or factors).

Newton's "new analysis," the method of series and fluxions, is certainly the most celebrated among Newton's discoveries. In this section, this fundamental turning point in the history of mathematics will be very briefly sketched. Our treatment of the subject, of course, will hardly do justice to the fascinating and enthralling complexity of this intellectual adventure. For brevity's sake, we shall subdivide the discovery process into three steps.

In the first step Newton generalized the results contained in Wallis's *Arithmetica infinitorum* and made his first mathematical discovery: the binomial theorem (winter 1664–5). How to raise a binomial to a positive integer exponent was something already known.[15] Newton was instead interested in calculating binomials raised to a negative integer or even a fractional exponent.[16] What Newton did was the following: he identified a general form for the coefficients that occur when a binomial is raised to an positive integer exponent;[17] he tabulated the coefficients, and then, by an act of faith in the universality of these forms, he extrapolated and interpolated the tables, thus finding coefficients for negative and fractional exponents. Thus Newton discovered the binomial theorem.[18] He tested its validity by verifying that it led to correct result for known cases (achievable by methods such as long division and root extraction). Newton seems to have been aware that the binomial theorem lacked a conclusive demonstration: he found it by a heuristic method and he convinced himself of its validity by successful applications. Its rigorous proof remained beyond the power of the best mathematical minds until the nineteenth century.

The second step consisted in the development of a notation and algorithm for calculating tangents to plane curves. Probably following Barrow, Newton conceived curves as generated by the motion of a point. He later called "fluents" such magnitudes generated by continuous motion, and named "fluxions" their instantaneous rate of flow. Newton claimed that during an infinitesimal "moment" of time, the fluxion can be considered as constant. The infinitesimal increment of a fluent quantity (for example, a point moving along a straight line with a variable speed) will be equal to its instantaneous speed (or fluxion) multiplied by a moment of time. Such infinitesimal increments were called by Newton "moments" of the fluent quantity. Last, Newton calculated the slope of a plane curve by the ratio of the moment of the ordinate to the moment of the abscissa.[19] It was via the above conception and method that the notion of infinitesimal magnitude entered into Newton's mathematical practice. The algorithm that Newton devised for the calculation of the slope of a plane curve is basically the one still used in schools, with the crucial difference that today we think in terms of functions rather than curves, and justify the calculation of the derivative through limiting procedures, rather than through infinitesimals. It must be added that such an algorithm had already been sketched in Barrow's lectures, and it is fair to say that it is highly probable that Newton had drawn inspiration from them.[20]

The third step in Newton's discovery of the method of series and fluxions is the so-called "fundamental theorem of the calculus." This is somewhat of a misnomer, as today we understand it as an inverse relation between operators acting on functions. In Newton's day, however, it was understood as a relation between curves. Torricelli, James Gregory, van Heureat, and Barrow had proved theorems concerning curves that showed that the operation for calculating the slope is the inverse of the operation for calculating the area subtended to a curve. Indeed, when a kinematical conception of magnitudes is accepted – that is, if one conceives of curves as traced by the motion of a point – such a relation is somewhat intuitively given as follows. It was customary, for example after Galileo and his followers, to represent the varying speed of a falling body as a graph whose ordinate is the speed and abscissa is the time. It was also understood that the area of the surface bounded by that graph is proportional to the space travelled by the body.[21] Newton's proof of the "fundamental theorem" that he penned in 1665 was probably influenced by Barrow's lectures.[22]

What was new about Newton's (and Leibniz's) approach to the fundamental theorem was the fact that they immediately seized the opportunity to use it in order to facilitate the calculation of areas of curvilinear surfaces and the arc-lengths of curves. Once an algorithm for calculating tangents is given, one can construct what Newton called "tables of curves" (namely, integral tables) by applying the algorithm to increasingly difficult curves. Newton tabulated equations of curves and of the slopes of their tangents as early as 1665–6.[23] It was in this context that he developed techniques for what we would call integration by variable substitution and by parts.

The range of problems that Newton's new art of discovery – the new analysis that he was soon to term the "method of series and fluxions" – could broach successfully is really impressive. Newton could calculate the area and arc-length of curves important for astronomy such as the ellipse, or even of mechanical curves such as the cycloid by expanding, via the binomial theorem, the ordinate as a power series and then integrating it term by term. He was able to express trigonometric magnitudes and logarithms in terms of power series. He could calculate tangents and radii of curvature to all known curves. He could systematize what we call the integration of irrational functions by the use of (integral) tables. The generality of this new method brought mathematics up to a level that only a handful of mathematicians in Europe could dream of, most notably the Scotsman James Gregory, who was developing similar results.

Newton systematized his notes into a short treatise that is known as the *1666 Tract on Fluxions*.[24] By the end of the 1660s, the Lucasian Professor, Isaac Barrow, realized that his younger fellow in Trinity deserved to be known outside the walls of the University.

PUBLICATION PROPOSALS: *DE ANALYSI* (1669) AND *DE METHODIS* (1671)

Whereas we are uncertain about the nature of Barrow's relationships with Newton in the early years of Barrow's tenure of the Lucasian Chair, it is certain that in 1669 the two were in deep contact on matters related to optics and mathematics.[25] When Barrow decided to quit the Chair in 1669, it was not chance that he was succeeded by Newton. The importance of this event for Newton's intellectual life cannot be overestimated, and now with a modicum of teaching duties, he could devote himself to research in a relatively safe environment. Before passing the Chair to Newton, Barrow took an equally important decision. He asked his young colleague to write about his new analysis in order to communicate this discovery to other mathematicians.

Thus, through Barrow's intermediation, a short manuscript tract entitled *De analysi per aequationes numero terminorum infinitas* (*On the analysis by means of equations with an infinite number of terms*) was dispatched in July 1669 to London.[26] The addressee was John Collins, an amateur mathematician who made a living – if only a modest one – out of his entrepreneurial activities in the field of mathematical book publishing. This sector was in crisis because of the depression in the print business caused by the Great Fire, but Collins managed to supervise the printing of several books, mostly related to algebra. In this discipline there was great need to update what was available on the English market.

Newton devoted *De analysi* to his techniques for calculating arc-lengths and curvilinear areas via infinite series. Newton was convinced that infinite power series, which he called "infinite equations," were the means for solving some of the most advanced open problems high up on the agenda of the mathematicians of his age. When he summarized its contents for Leibniz in 1676 he stated that the "limits of analysis are enlarged" by the use of "infinite equations" in such a way that "by their help analysis reaches . . . to all problems."[27] Yet, in *De analysi* Newton avoided including his most powerful method for obtaining series expansions, the binomial theorem. Instead, he relied on the safer methods of long division and root extraction. "De analysi" included power series expansions for the natural logarithm function and trigonometric functions (such as the sine, cosine, and arctangent), the quadrature of the cycloid, and the quadrature (and rectification) of the quadratrix. It ended with a proof of the fundamental theorem of the calculus and, most interestingly, with an attempt to determine the interval of convergence for power series.[28]

When Collins received Newton's tract he was thrilled, although it is debatable whether he really understood what he had in his hands. For Newton, getting in touch with Collins meant having free access to a network of mathematical correspondents, both British and Continental, and to the bustling world of printers and booksellers active in the capital. Newton could not have been offered the option to print his method of series in a more conspicuous and attractive way, although nothing came out if it. The extant correspondence between Newton and Collins reveals much of Newton's changing approach to publishing mathematics in the period from 1669 – the year in which he was elected Lucasian Professor in succession to Barrow – to late 1670. Collins had several proposals for Newton: for example, to issue *De analysi* together with some of Barrow's works he was expecting to publish. While waiting for Newton's permission, Collins made copies of *De analysi*, a short tract that would not have been so expensive to print, and we have good reasons to think that he circulated information about this youthful work by correspondence with British and Continental mathematicians. Reading the epistolary exchange between Newton and Collins leads the historian to follow a zigzag path: at first, Newton seems close to accepting

Collins's invitations to print *De analysi*, or even to dispatch him a more extensive treatise in which he had systematized his discoveries; but then – within a matter of weeks – we find him withdrawing his promise, much to Collins's frustration.

The new treatise was the so-called *De methodis serierum et fluxionum* (*On the methods of series and fluxions*) he had completed in 1671.[29] While *De analysi* would have been suitable for print publication, *De methodis*, a much longer treatise, would have changed the history of mathematics if it had been printed in the 1670s. In the opening lines Newton brought to perfection his method of series expansion,[30] and employed it for what we would call the integration of first order differential equations. In the second part of the treatise Newton reworked some of the results of the "1666 Tract on fluxions." He introduced a notation for fluents and fluxions (but not the familiar dotted one yet), presented his improved Hudde algorithm, which he applied to the calculation of maxima and minima, tangents and radii of curvature (a fluxional measure for the radius of curvature of a plane curve was provided). The treatise included two long tables of curves (two integral tables, to use Leibnizian terminology) that allowed Newton to solve very advanced problems related to the rectification and quadrature of curves. This was a masterpiece that was only published in an English translation more than sixty years (1736) after its composition, a delay caused primarily by Newton's idiosyncratic attitude towards publication (an attitude that was fuelled by the polemics surrounding the validity of the *experimentum crucis* in the years 1672–6).[31]

Newton had achieved results that would have made him famous all over Europe as the most creative mathematician alive. Yet in letters sent to Collins he stated his increasing reluctance to print them. By the mid 1670s, Newton was quite adamant in not allowing his mathematical jewels to escape from his hands. To the few lucky ones who had corresponded with him on mathematical subjects and who had had access to his manuscripts he ordered silence and secrecy. What was the origin of Newton's anxieties over printing his mathematical discoveries?

NEWTON'S PHILOSOPHICAL AGENDA: THE *LUCASIAN LECTURES ON OPTICS* (1670–1672)

To answer the above question it is necessary to broaden one's historical perspective a little in order to take into consideration the philosophical agenda that Newton set himself in the 1670s. We can learn about it by looking briefly at his dealings with the Royal Society and by reading some passionate annotations that he jotted down when comparing the methods of the ancient mathematicians to those of the moderns, as epitomized by Descartes.

Newton became a member of the Royal Society in 1672 after having presented his reflection telescope to the society. This innovation fitted in well with the desiderata of the newly established institution: which assigned great importance to microscopy, and the improvement of telescopic observations. As is well known, in 1672 Newton was to submit his famous paper on the *experimentum crucis*, in which he claimed to have proved a new theory concerning light and colors. For most of the Royal Society's members, Newton's confidence in having "proved" a new physical theory could only sound provocatively arrogant: no such statements were expected. The theory had already received a thorough treatment in the Lucasian lectures on optics that Newton deposited in 1672 and dated retrospectively from 1670. In the third lecture, he stated that by the use of "geometry" the science of colors, and natural philosophy in general, could achieve the "highest evidence."[32] He also expressed his annoyance towards those natural philosophers who were confining themselves to "conjectures and probabilities." Newton might have had in mind Robert Hooke who in the *Micrographia* (1665) had warned readers to consider any discourse concerning the "causes of things" contained in the book simply a "small conjecture," a "doubtful problem," and an "uncertain guess."[33] Newton's discourse against probabilism offered a view of natural philosophy at odds with what an influent member of the Royal Society such as Boyle was promoting. It is a discourse that intertwined with Newton's well-known rejection of what is now known as hypothetico-deductive method, a method that was championed by Descartes.

Hooke was expressing values deeply felt in the Royal Society. One should bear in mind that, just after the restoration of the Stuarts, many natural philosophers belonging to the Royal Society wished to make it clear that no "unquestionable" or "dogmatic conclusions" should be feared from them. Politically opinionated philosophers or dogmatic theologians were not admitted in the society, which instead promoted an innocuous mitigated skepticism. That is why any discourse aimed at reaching certainty was looked upon with suspicion, while skepticism and probabilism were approved of in some of the most influential Royal Society manifestos, such as Hooke's masterpiece on microscopy and Glanvill's *Scepsis scientifica* (1665). In his *Lucasian Lectures on Optics* and his 1672 paper, Newton broke with this code of behavior by stating that the theory of colors he was proposing – a topic that he knew was regarded as "belonging to physics"[34] – was not "an hypothesis but [of the] most rigid consequence."[35]

Newton's claim that he was reaching "the highest evidence" in the theory of colors was targeted by Hooke in the heated debate that poisoned Newton's life in the years following the publication of the "New Theory about Light and Colors" (1672). The effect of the dispute concerning the *experimentum crucis* on Newton's reluctance to print his mathematical results cannot be overestimated. Newton's great paper of 1672 was fiercely attacked, and this frustrating experience was to lead him – possibly out of spite – to avoid publishing his results in other fields of enquiry. In a letter concerning the project of printing his lectures on optics (dated May 25, 1672), Newton wrote to Collins: "I have now determined otherwise of them; finding already by that little use I have made of the Presse, that I shall not enjoy my former serene liberty till I have done with it."[36]

Newton claimed that his natural philosophy was certain because it was mathematical. However, in order to profile himself as the philosopher who via the use of mathematics could transcend the kind of probabilism defended in texts such as Hooke's *Micrographia*, Newton had to avoid becoming embroiled in a further polemic concerning the certainty of mathematical methods. Newton was keenly aware that his method of series and fluxions was open to debate, because of the guesswork surrounding his theory of series (for example, when the coefficients of all the terms of the series are determined by discerning a pattern in the first terms) and the use of rather problematic concepts – such as that of "infinitesimal" or "moment" – in the method of fluxions. He knew that mathematicians who had published on the new analysis – such as Wallis, whose methods really stood at the root of his use of infinite series – had to withstand the criticisms of the defenders of the rigor and certainty of ancient geometry. Such debate would have been lethal for Newton, a philosopher who had claimed to be able to bring evidence into natural philosophy via the use of geometry: for if mathematics is to endow philosophy with evidence, it must be practiced according to criteria that guarantee the certainty of its methods. Newton showed annoyance with the qualitative models of the mechanical philosophy: Descartes, and in general the followers of the corpuscular philosophy, had attempted to explain natural phenomena in terms of impacts of invisible, hypothetical, corpuscles. Newton instead wished to "deduce" the laws of nature (most eminently, the laws of optics and the laws of gravitation) from the phenomena. This deduction led to a much greater certainty compared to the "elegant perhaps and charming" romances (as Cotes would say in his preface to the second edition of the *Principia* (1713)) of the corpuscularists, and was the province of the mathematician.

The mathematics that Newton was using in his optical work was not particularly advanced, except in a few cases, such as the study of atmospheric refraction (1685–95).[37] Yet mathematics played a fundamental role in Newton's optical work, especially in the study of interference phenomena, and in the deduction of the properties of matter that exact measurements concerning these phenomena allowed. It is true that Newton's optical work is largely independent from the methods of series and fluxions. Still, what I wish to suggest is that Newton felt compelled to defend an image of himself as the natural philosopher who, because of mathematics, could go beyond the hypothetical discourse "blazoned about everywhere."[38] Newton cherished the idea that he was able to define a level of discourse where natural philosophy could be practiced "with highest evidence" and hypotheses avoided: for all his life he stood by this position. For Newton, printing *De analysi* or *De methodis* would have meant tying his name as an author to conjectural and heuristic mathematical methods, something which might have led to a deflation of the high status assigned to mathematics in his philosophical agenda.

There is another aspect of Newton's philosophy that we should consider in order to appreciate the reasons behind his reluctance to grant Collins permission to print the method of series and fluxions. In the 1670s Newton began to develop a profound distaste for Cartesianism, mechanical philosophy, and "modern philosophers" in general. Descartes epitomized the *hubris* of the moderns. The mechanical philosophy was the invention of a presumptuous Frenchman, who instead of looking with reverence to the distant past had dared to rebuild philosophy from scratch starting from an act of denial, a hyperbolic doubt cast on all past knowledge. Descartes the mathematician, at least the one revealed in the *Géométrie*, was as aggressive and innovative as the philosopher of the *Discours*. In the *Géométrie* one could read that the ancient geometers had not possessed any systematic method for solving geometrical problems: the length of their books and the disorderly presentation of their results were positive proof that they were just gathering together those propositions "on which they had happened by accident."[39]

Descartes's disparaging attitude towards the Ancients, in philosophy and mathematics, was anathema for Newton, who looked at the distant past with reverence and admiration. Of course, one should bear in mind that reverently referring to an antediluvian Hebrew sage such as Noah is something that served quite a different function for Newton from the citing of a Greek Alexandrian mathematician such as Euclid. A certain resonance between Newton's philosophical and mathematical classicism can be discerned, and actually proved by textual evidence. Among Newton's contemporaries, Huygens was the one to win his admiration. In the *Horologium oscillatorium* (1673) the Dutch polymath showed him how one could carry out cutting-edge research in pure and mixed mathematics by means that were purely geometrical, without the help of any equations, infinite series, or infinitesimals. The cycloid, a daunting transcendental curve for mid-seventeenth-century mathematicians, was tamed with elegance in the *Horologium*, and put to good use in the study of pendulum motion. This example exerted a lasting impression on Newton's mind: the lesson was that one could use geometry rather than algebra, in imitation of the Ancients.

As we shall see in the next sections, Newton embarked on a research program aimed at refuting Descartes's claims about the superiority of modern algebra over ancient geometry. Newton concluded that algebra, as well as the method of series and fluxions, were just heuristic tools: they were useful in the art of discovery, but lacked the certainty and elegance of geometry. As Newton was to tell David Gregory in 1694, "algebra is fit enough to find out, but entirely unfit to consign to writing and commit to posterity."[40] Algebra should not be printed. But Newton was a great algebraist, and indeed was proud of his results in the field of algebra and calculus. From the mid 1670s to the early 1690s, he resolved this tension between his mathematical practices and methodological agendas by using the register of print publication for demonstrative geometry, and that of scribal publication (i.e., manuscript circulation and correspondence) for heuristic algebra. Indeed, in 1676 he wrote two well thought-out letters for Leibniz on his mathematical methods (including the binomial theorem).[41] What is more, in the 1680s and 1690s he allowed some of this correspondence on algebra and calculus to be printed in Wallis's works (see p. XXX). Newton was thus able to circulate some of his symbolical results without having to commit them to "posterity" in the way he would have through printed books.

Studying the publishing strategies of Newton the mathematician is a challenging task. It might be too much to say that the above-mentioned philosophical factors *caused* Newton to reject Collins's publication proposals in the 1670s. Rather, one might say that it was a number of philosophical ideas, and political and religious concerns that propelled Newton's polemical reading of Descartes and the "modern philosophers;" that these concerns together with Newton's tense dealings with the Royal Society – like force vectors – pointed his mind away from the prospect of committing to print his symbolical, rather Cartesian, modern and uncertain mathematical discoveries. We should avoid describing Newton's thought and behavior as governed by causal laws. Newton's approach to publication was far from coherent, and the historical record does not afford us any simplistic description. Yet his dealings with Collins and Wallis reveal something of his authorial strategies, of the way in which he wished to profile himself vis-à-vis his contemporaries, of the role he attributed to himself as a rediscoverer of ancient exemplars and the defender of an anti-Baconian way of envisaging the relationship between mathematics and natural philosophy. Ultimately, we must accept that the late publication of Newton's early mathematical writings is also the result of contingencies such as the depression in printing caused by the Great Fire of London (1666), Collins's death (1683), and the publication conventions in England under the Restoration, an age in which scribal publication flourished.

It is often believed that after the creative outburst of the *anni mirabiles* of his youth Newton abandoned mathematics for other interests like alchemy, theology, and natural philosophy. However, the extant manuscripts distributed across the eight magnificent volumes edited by Whiteside, disprove such a view. Newton continued to be productive as a mathematician until the mid 1690s, when his move from Cambridge to London as Master of the Mint brought a real change to his lifestyle. In the following sections, I shall attempt to provide a survey of Newton's mature mathematical work.

CARTESIAN ALGEBRA: THE *ARITHMETICA UNIVERSALIS* (1673–1684)

Sometime between the autumn of 1683 and early winter of 1684, Newton, according to the statutes of the Lucasian Chair, deposited a set of lectures that were printed much later with the title *Arithmetica Universalis* (1707).[42] The lectures bear dates ranging from 1673 to 1683, but these were added in retrospect and it is highly unlikely that they were ever delivered to Cambridge students. In the *Lucasian Lectures on Algebra* Newton drew on results he had obtained in the 1660s and observations he had recorded in 1670 while working on the project of publishing a treatise by Gerard Kinckhuysen.[43] From several points of view, Newton's professed anti-Cartesianism notwithstanding, these lectures can be described as a fulfillment of Descartes's program, since algebra is here extensively presented as the tool to be used in the resolution of geometrical problems.

Several lectures were devoted to a systematic treatment of the roots of algebraic equations. Descartes's "rule of signs" already gave an upper bound for the number of positive roots of an equation by examining changes of sign. Newton added a new rule for determining the number of "impossible" or imaginary roots. The rules for expressing the coefficients of equations as symmetric functions of the roots were well known. Newton used these functions to find formulas for sums of powers of the roots. He also expressed rules for finding bounds between which the roots of an equation must lie.[44]

This is an impressive list that reveal Newton's prowess as an algebraist. The *Lucasian Lectures on Algebra*, however, contain some critical comments on the use of algebra that are worth considering. When, in 1707, the lectures appeared under the title of *Arithmetica universalis*, readers were somewhat puzzled. On the one hand, Newton's lectures can be seen as a fulfillment of the program outlined by Descartes in the *Géométrie* because of the extraordinary results on the theory of equations listed above. On the other hand, the lectures contain criticisms directed at Cartesian "construction" or "synthesis."[45] In the last section Newton argues that the demarcation between acceptable and unacceptable means of construction (or synthesis), as well as the characterization of the relative simplicity of such means proposed by Descartes, are far too dependent upon algebraic criteria.

One should bear in mind the canon that Descartes had adopted in his *Géométrie*. As Henk Bos has explained, Descartes's method of problem solving was divided, according to the Pappusian canon, into an analytical part (resolution) and a synthetic one (construction).[46]

The analytical part is algebraic: it consists in reducing the problem to a polynomial equation. If the equation is in one unknown, the problem is determinate. The equation's real roots would correspond to the solutions of the problem. Methods for the calculation of the roots of algebraic equations up to the fourth degree had already been developed in the sixteenth century. But even when formulas were available, they did not provide indications about how one could achieve what was sought for the solution of a geometric problem: namely, a geometrical construction (or synthesis). Algebra could do only half of the business required by early-modern mathematicians; a geometrical construction was needed.

Descartes accepted the traditional idea that such constructions had to be performed through the intersection of curves. One had to choose two curves such that their intersections determine segments whose lengths geometrically represent the real roots of the equation.

The synthetic part of the process of problem solving, known as the "construction of the equation," opened up a series of questions. Which curves were admissible in the solution of problems? Which curves, among the admissible ones, were to be preferred in terms of simplicity? In asking himself these questions Descartes was continuing – albeit on a different plane of abstraction and generality – a long debate concerning the role and classification of curves in the solution of problems. Descartes prescribed that in the "construction of the equation" one had to use "geometric" (i.e., algebraic) curves of the lowest possible degree, whereas Euclid – it might be recalled – accepted in the geometrical constructions of the *Elements* only the use of the circle and the straight line.

In the final part of his lectures on algebra, devoted to the construction of third-degree algebraic equations (i.e., to the geometric construction of segments representing the real roots of third-degree algebraic equations), Newton fiercely disagreed with Descartes. The message that Newton wished to deliver was that in geometrical constructions algebraic criteria are misleading. Descartes had admitted all geometric (algebraic) curves as means of construction. Most of these, however, were, hopelessly complex according to Newton. On the other hand, Newton claimed that simple means of construction could be found in some "mechanical" (transcendental) curves, such as the cycloid. Newton maintained that it would be wrong to think that a curve can be accepted or rejected on the basis of its defining equation: "[I]t is not the equation but its description which produces a geometrical curve," he argued.[47] A circle is a simple and admissible geometrical curve not because it is expressible by means of an equation, but because its description can be carried out by means of a compass, one of the most fundamental constructions postulated in geometry. Further, expressing of a hierarchy of simplicity in terms of the degree of algebraic equations is something foreign to geometry. The circle and ellipse are of the same degree, but the former is simpler: "It is not the simplicity of its equation, but the ease of its description, which primarily indicates that a curve is to be admitted into the construction of problems . . . On the simplicity, indeed, of a construction the algebraic representation has no bearing. Here the descriptions of curves alone come into the reckoning."[48] When practicing geometry, Newton insisted, curves must be seen as being traced by motion, hence their defining equation is irrelevant.

Newton's passionate insistence on the idea that curves must be primarily seen as traced by motion rather than as loci of equations, has deep roots in his conception of the relationship between geometry and mechanical practice, a conception whose importance in Newton's mathematized natural philosophy can hardly be overestimated. In the 1690s, in writings devoted to projective geometry, Newton reconsidered this issue. Here we read that the "species" of a curve is not revealed by its equation but by the "reason for its genesis."[49] The geometer who has learned about the mechanical genesis of curves has an epistemological advantage over the algebraist: he knows the nature of curves because he masters their construction. Newton seems to suggest that we know what we can construct, not what we can calculate. Furthermore, Descartes's algebraic distinction between geometrical (algebraic) and mechanical (transcendental) curves, and his rejection of the latter, would have imposed unacceptable limitations on the geometrized natural philosophy that Newton was promoting. The non-algebraic integrability of ovals that Newton proved in Lemma 28, Book 1, of the *Principia* implies that the Kepler equation cannot be solved by the means admitted in the *Géométrie*.[50] Mechanical (transcendental) curves are basic elements of the geometrical structure of natural philosophy. In Newton's opinion, Descartes's rejection of these curves as lacking exactness (because they are not loci of polynomial equations in Cartesian coordinates) depends upon an unwarranted privilege given to algebra. Newton declared in the preface to the *Principia* that as these curves belong to Nature regardless of the complexity of their algebraic expression, their tracing is perfectly "executed by the most perfect mechanics of all." Indeed, as Newton wrote in the 1690s contra Descartes, "any plane figures executed by God, nature or any technician you will are measured by geometry in the hypothesis that they are exactly constructed."[51]

Newton's fiery invectives against the Cartesian algebraic method, which abound in the final section of the *Lucasian Lectures on Algebra*, are not at all paradoxical, as is often claimed. They are rather the expression of some of his most deeply felt philosophical convictions. Newton made it clear that in the section on the "construction of equations" he was talking about the synthetic, constructive phase of the problem-solving process. The analytical stage, discussed in the section devoted to the reduction of geometrical questions to equations, can be carried on in algebraic terms.[52] Indeed, in the Lucasian lectures algebra is proposed as one of the admissible analytical tools. In the synthetic, constructive stage, however, algebra must not play any role. Descartes had claimed that in the synthesis or construction of a problem, only intersections between geometrical (algebraic) curves could be used. For Newton, what was relevant in geometrical construction was not the equation of the curves, but the fact that an elegant and simple tracing mechanism was deployed.

IN SEARCH OF A GEOMETRICAL ANALYSIS: THE *SOLUTIO PROBLEMATIS VETERUM DE LOCO SOLIDO* (LATE 1670S)

Newton distanced himself from Descartes's analysis as well. "Analysis," we should bear in mind, meant a method of discovery. Newton considered his method of series and fluxions a kind of new analysis, a new method of discovery that could be used but was unworthy of publication. In writings penned in the late 1670s and early 1690s, Newton searched for a geometric analysis, a geometric method of discovery alternative to the symbolic, algebraic one.

One of the reasons why Newton distanced himself from algebra as a tool of discovery was the fact that geometry is aesthetically more pleasing. The Ancients' geometrical method – Newton often affirmed – is "more elegant by far than the Cartesian one." The enthusiastic acknowledgement of the elegance and conciseness of geometry compared to the "tediousness" of the "algebraic calculus" is a topos that recurs frequently in Newton's mathematical manuscripts.[53] The importance in Newton's mindset of such aesthetic evaluations can hardly be overestimated.

Moreover, according to Newton, algebraic analysis does not reveal how the geometrical synthesis can be performed. After the geometrical analysis of a problem it is often possible to reach a construction (a synthesis) by simply reversing the steps of the analysis, but after an algebraic analysis one is left with an additional and artificial problem: Descartes's problem of the construction of the roots of the equation. Newton concluded that such constructions were largely a Cartesian contrivance extraneous to the ancient geometrical tradition.[54]

Newton's search for a geometric analysis led him to read the compilation by Pappos entitled *Mathematicae collectiones*, published in Urbino in 1588. Newton's attention was particularly focused on a branch of the lost Euclidean corpus: the books on *Porisms*. In the early 1690s, Newton gave voice to his hopes and idiosyncrasies as follows:

Whence it happens that a resolution which proceeds by means of appropriate porisms is more suited to composing demonstrations than is common algebra. Through algebra you easily arrive at equations, but always to pass therefrom to the elegant constructions and demonstrations which usually result by means of the method of porisms is not so easy, nor is one's ingenuity and power of invention so greatly exercised and refined in this analysis.[55]

"Porisms" are elliptically referred to in the seventh book of the *Collectiones*, where Pappos tells his readers that the Ancients possessed a method of discovery, a "method of analysis," that allowed them to reach their extraordinary results. This method had been illustrated in several works, of which Euclid's three books on porisms were the most advanced. Early modern mathematicians were tantalized and tried to reconstruct this method from Pappos, who provided some lemmas as an introduction to the reading of Euclid's work. For Pappos's fourth-century CE readers everything was quite clear, since they had Euclid's work, but for early modern mathematicians the situation was really frustrating, since that work was lost (as it still is). Newton came to the conclusion that porisms consisted in the kind of results that nowadays we would classify as pertaining to projective geometry.[56] Particularly notable, in this respect, is a small treatise entited *Solutio Problematis Veterum de Loco Solido* that he penned in the late 1670s.[57]

A problem on which Newton had much to say was the so-called Pappos problem of 3 or 4 lines. This problem was central to the *Géométrie*, where its algebraic solution was presented as a paradigm for the superiority of Descartes's method over that of the Ancients. Indeed, according to Descartes, neither Euclid nor Apollonius had been able to thoroughly tackle the generalization of the Pappos problem to *n* lines. Newton was of a different opinion. In the late 1670s, commenting upon Descartes's solution, he stated with vehemence:

> To be sure, their [the Ancients'] method is more elegant by far than the Cartesian one. For he [Descartes] achieved the result [the solution of the Pappos problem] by an algebraic calculus which, when transposed into words (following the practice of the Ancients in their writings), would prove to be so tedious and entangled as to provoke nausea, nor might it be understood. But they accomplished it by certain simple proportions, judging that nothing written in a different style was worthy to be read, and in consequence they were concealing the analysis by which they found their constructions.[58]

With the benefit of hindsight, we might consider this Newtonian statement a misunderstanding of the role and strength of Cartesian algebra. Of course, when algebraic symbols are translated into connected prose, they often lead to rather opaque mathematical demonstrations. It might be claimed that the introduction of symbolism at the beginning of the seventeenth century was proposed by its defenders as a vehicle for freeing mathematical demonstrations from cumbersome verbal formulations. Further, only algebra could enable generalizations unthinkable in geometry: in the case at hand, a streamlined generalization of the Pappos problem from 4 to *n* lines.

Descartes solved the problem via algebra. He proved that for 3 or 4 lines the locus was expressed by a second-degree algebraic equation in *x* and *y*, and hence concluded that it was a conic section. To the contrary, Newton approached the problem in geometric terms. In the 1670s he developed many interesting ideas on projective geometry, reaching a result equivalent to Steiner's theorem. Indeed, Newton's geometric solution of the Pappos problem for 3 and 4 lines is grounded on an understanding of the projective definition of conics.

The solution of the Pappos problem could be achieved without algebra, in purely geometric terms. For Newton this meant a victory over the impudence of Descartes, who had dared to challenge the Ancients. In Section 5, Book 1, of his *Principia*, Newton presented his geometrical solution of the Pappos problem as having been achieved "as the ancients required." This result was understood by Newton as a victory over Descartes, a vindication of the Ancients against the Moderns. It is easy to see, however, that – *pace* Newton – his projective solution is quite modern (it is a child of the seventeenth century), and further that it cannot be generalized to *n* lines, whereas Cartesian algebra offers precisely this great advantage.

Nevertheless, one should not underestimate the values that informed Newton's opposition to Cartesian algebra. That is, with the benefit of hindsight, we should avoid judging Newton's mathematical practices and methods as dead-ends in the development of mathematics. The invectives against the use of algebraic symbols that characterize Newton's critique of Descartes's analytic geometry, and that later also informed Newton's critique of the Leibnizian calculus, must be viewed as part of a larger philosophical project that he had in mind. Reading Newton's defense of geometry as a backward move, and identifying algebraization as a progressive element in seventeenth-century mathematics, means failing to grasp the values that underlie the confrontation between mathematicians such as Huygens, Barrow, and Newton on the one side and Descartes, Wallis, and Leibniz on the other. First, Huygens, Barrow, and Newton defended visualization over algorithmic efficiency. Second, they defended geometry over algebra as better anchored to physical reality. Finally, aesthetic criteria, such as conciseness and elegance, played a role in their choice to opt for the geometrical methods of the Ancients.

MATHEMATIZING MOTION: THE *PRINCIPIA* (1684–87)

The positions concerning mathematical methods that Newton defended in the 1670s and early 1680s, as well as the strategies of mathematical publication that characterized his correspondence in that period, may well have contributed to shape the mathematical style of his *Principia*. As is well known, in this masterpiece geometry is given pride of place. Newton's alleged triumph over Cartesian algebra, his geometrical solving of the Pappos problem – hardly a problem related to the mathematization of the System of the World – is presented in two long sections (4 and 5, Book 1) devoted to the geometry of conics. In the first section of Book 1, it is stated that infinitesimal magnitudes should be avoided. Newton opted for geometrical limit procedures, the so-called "method of the first and last ratios of vanishing quantities" that he had developed around 1680 in a treatise entitled *Geometria curvilinea*.[59] In the *Principia* Newton has much to say to his readers on his methodological preference for geometry. Yet, the *Principia* is a panoply of mathematical methods. Newton, like all great scientists, was an opportunist who deployed the methods best suited to his purposes, not only geometrical ones but also rather advanced symbolical methods of integration and power series expansion.[60]

It is thanks to a vast array of methods that Newton was able to raise the mathematization of natural philosophy to levels of generality and complexity that elicited admiration even from his harshest critics. One should bear in mind that before the *Principia* mathematicians knew how to deal with rather elementary cases such as projectile (vertical and parabolic) and pendular (circular and cycloidal) motion in a constant gravitational field, and that merely basic steps had been taken in the study of resisted motion. In the *Principia* Newton dealt with topics of unthinkable difficulty for his age, such as the three-body problem, the attraction of extended bodies, motion in resisting media, the speed of sound, the precession of equinoxes, the shape of equilibrium of a rotating fluid mass, tidal motion, the irregularities in the Moon's motion, and planetary perturbations.[61] His treatment of these subjects, however, stood in need of new mechanical concepts and mathematical improvements that were only to be provided in the eighteenth century. Indeed, the *Principia* became a repertoire of open problems that polarized the attention of mathematicians for more than a century, even those who were skeptical about gravitation.

The *Principia* would often puzzle the most competent mathematical readers. In some cases, Newton does not provide all the details necessary to grasp his demonstrations. Most notably, he often proves how to reduce a problem to a quadrature, and then proceeds to state that "granting the quadrature of curvilinear figures" certain results follow. No detail, however, is given about the quadrature techniques (integrations) just alluded to in the text. We know that at least in some cases, Newton was able to send his acolytes details about rather advanced quadratures necessary to achieve some of the results of the *Principia*. This occurred for Cor. 3. Prop. 41, Book 1, where he identifies some of the spiral trajectories traversed by a body in an inverse-cube force field; and for Cor. 2, Prop. 91, Book 1, he determines the attraction exerted by a homogeneous ellipsoid of revolution on a point mass located on the prolongation of its axis of revolution.[62]

The richness of the mathematical methods of the *Principia*, Newton's pronouncements in favor of geometry, the contiguity of his geometrical limit procedures with techniques typical of the infinitesimal calculus, his employment of Taylor series, and his elliptical reference to quadrature techniques the details of which do not appear in the printed text, have given rise to the vexed question concerning Newton's use of calculus in the *Principia*: a question that played a prominent role in the controversy between Newtonians and Leibnizians. Did Newton use his calculus in the *Principia*? This is a question that cannot be broached here. It is a difficult question both because different readers of Newton's masterpiece have taken the term "calculus" to mean different things, and because the application of calculus techniques to the science of motion was far from being an established practice in Newton's times.[63] The contributions to the analytical treatment of mechanics made in the *Principia*, their magnitude notwithstanding, were soon to be superseded by the analytical dynamics promoted by Continental mathematicians such as Pierre Varignon, Johann Bernoulli, and Leonhard Euler.

LATER YEARS: *DE QUADRATURA*, *GEOMETRIAE LIBRI*, *ENUMERATIO LINEARUM TERTII ORDINIS*, *OF QUADRATURE BY ORDINATES*, THE BRACHISTOCHRONE PROBLEM, AND *COMMERCIUM EPISTOLICUM*

In the 1690s Newton worked on a lengthy treatise entitled *Geometriae libri*, achieving results that would nowadays be expressed in terms of the theory of birational correspondences of second degree.[64] He also penned his masterpiece on the theory of integration, *De quadratura curvarum*, where he introduced his dotted notation for first order and higher-order fluxions and further systematized his quadrature techniques.[65] In a draft of this work, which unfortunately was never published, Newton proved that a fluent can be expanded into a Taylor power series, the terms of which are higher-order fluxions of the given fluent multiplied by the appropriate coefficients.[66] Newton also systematized his results on the classification of cubic curves that he had achieved in the 1670s,[67] reaching an (incomplete) enumeration of 72 species.[68] Perhaps the most striking result was the statement that all cubic curves can be subdivided into five projective classes: that is, that the five diverging parabolas[69] can generate all other cubic curves by central projection.[70] Newton also wrote a short treatise entitled *Of Quadrature by Ordinates* in the context of his studies on interpolation.[71] The Newton–Cotes formula originates from this research. Newton's work on interpolation dates from 1676 and, as we have seen, was partly published in Lemma 5, Book 3, of the *Principia*.

The above texts were Newton's last creative mathematical works. In 1696 a momentous change occurred in his life. He moved to London as Warden of the Mint, becoming a well-paid public servant. Newton had to face new challenges, as he grew involved in the political and theological debates raging in the aftermath of the Glorious Revolution. Challenges were also coming from mathematicians, since both in England and on the Continent the progress of mathematics was eroding Newton's advantage over his contemporaries. David Gregory had already threatened Newton's superiority in 1685 with the publication of his Uncle James's results on infinite series. This event prompted Newton to compile a defense, the *Matheseos universalis specimina* and *De computo serierum*,[72] texts that most probably were not meant for publication but rather for private circulation. John Craig, Ehrenfried Walther von Tschirnhaus, and David Gregory were also working on quadratures, albeit still on a lower level than the one the author of *De quadratura* could achieve.

The greatest challenge came from Leibniz, who began printing his differential (1684) and integral (1686) calculi in the *Acta Eruditorum*. He promoted the new method in France, Switzerland, and Italy. A small group of aggressively innovative mathematicians began treading in his footsteps and a steady flow of papers spread knowledge about the Leibnizian calculus, which indeed started influencing even British mathematicians. Leibniz was able to establish a form of cooperation and competition with these younger mathematicians that differed from the reverent submission of Newton's acolytes towards their master. Leibniz and his followers also brought about considerable innovations in publication practices – not only were Continentals now publishing their results in newly founded journals, but more emphasis was being laid on methods than results.

In 1685 and 1693 Wallis was able to obtain permission from Newton to print extracts from the two letters to Leibniz he had penned in 1676, together with a brief presentation (1693) of the method of series and fluxions (see Table 9.2). Wallis was inflamed by nationalism. He would often complain about the machinations of the Continentals, whom he accused of stealing English discoveries. He warned Newton that his method of fluxions was circulating on the Continent "by the name of Leibniz's calculus differentialis."[73]

In 1697 Johann Bernoulli circulated the brachistochrone problem as a challenge "to the sharpest mathematicians in the whole world." Newton's solution soon appeared anonymously in the *Philosophical Transactions*. He had probably achieved this solution through a fluxional equation similar to the (unpublished) one he had employed in the *Principia* for the solid of least resistance problem.[74]

Table 9.2 *The publication of Newton's mathematical works*

1685	Wallis, *Algebra*	paraphrased translation of letters to Leibniz (1676)
1693	Wallis, *Opera*, vol. 2	paraphrase of letters to Leibniz (1676) and additions on fluxional notation and quadratures
1697	*Philosophical Transactions*	Anonymous solution of Johann Bernoulli's two challenge problem (one is the brachistochrone)
1699	Wallis, *Opera*, vol. 3	letters to Leibniz (1676) *verbatim*
1702	David Gregory, *Astronomiae Physicae & Geometricae Elementa*	*Theoria Lunae* (in appendix). English version as a pamphlet (1702) and in English transl. of Gregory (1715)
1704	*Opticks*	*De quadratura* and *Enumeratio linearum tertii ordinis*
1706	*Optice*	*De quadratura* and *Enumeratio linearum tertii ordinis*
1707	*Arithmetica universalis*	*Lucasian lectures on algebra* (1684) edited by William Whiston. English trans., 1720, second Latin edn, 1722
1710	Harris, *Lexicon Technicum*, vol. 2	English translation of *De quadratura* and *Enumeratio linearum tertii ordinis*
1711	*Analysis per quantitatum series, fluxiones, ac differentias*	*De analysi, Methodus differentialis, De quadratura, Enumeratio linearum tertii ordinis,* some correspondence, edited by William Jones
1713	*Commercium epistolicum*	*De analysi*, mathematical correspondence
1715	*Philosophical Transactions*	Anonymous *Account of the Commercium Epistolicum*
1716	*Philosophical Transactions*	Anonymous solution of Leibniz's challenge problem (on orthogonal trajectories)
1723	*Principia* (Amsterdam)	William Jones's edition of Newton's mathematical tracts (1711) printed in *Appendix*

Newton's paper contained a geometrical construction of the curve required (a cycloid) but no fluxional analysis.[75] In 1699, Fatio de Duillier, one of Newton's *protégés*, in a work devoted to the brachistochrone, *Lineae brevissimi descensus investigatio geometrica duplex*, accused Leibniz of having plagiarized Newton's method of fluxions. This episode was dealt with diplomatically, and the case was soon brought to rest.

Many in Newton's entourage, particularly those who had enjoyed privileged access to his mathematical manuscripts by visiting him in his private quarters, were convinced that – via correspondence with Collins, Oldenburg and Newton himself – Leibniz had gained information on the calculus, which he had then printed as his own discovery after changing the notation. After studying Leibniz's manuscripts, twentieth-century historians were able to disprove this accusation. It is an established fact that by 1675 Leibniz had come up with the calculus notation and algorithm that is still in use today.

The situation degenerated when, in 1708, in the journal of the Royal Society a mathematician of minor stature, John Keill, claimed that Leibniz was a plagiarist. The latter demanded that the Royal Society protect him from the "empty and unjust braying" of such an "upstart." Consequently, a committee of the Royal Society, secretly led by its president, Isaac Newton, produced a detailed report, the *Commercium epistolicum* (1713). The committee maintained that Newton had been the "first inventor" and that "Leibniz's Differential Method was one and the same with the Method of Fluxions, excepting the Name and Mode of Notation." It was also strongly suggested that Keill's offending statements were justified.

The ensuing controversy between Newton and Leibniz involved a number of Continental and British mathematicians, theologians and pamphleteers, thus causing a complex splintering of the European mathematical community. It would be simplistic to describe the controversy as a fight between two well defined opposing groups divided by the Channel. On the Continent, philo-Newtonian outposts were especially prominent in the Low Countries and in France, where Varignon, for instance, enjoyed cordial relationships with both Newton and Leibniz. In Basel, one of the main defenders of Leibniz, Johann Bernoulli, was far from confining his role to that of a follower of the German mathematician; rather, he played on the controversy to aggrandize himself. In a paper published anonymously in 1716, after recognizing Leibniz's claim to the invention of the differential calculus, Bernoulli claimed the discovery of the much more difficult integral calculus for himself.[76]

The controversy offered the two rivals a chance to make their views on mathematical method explicit. The two of them proved to have different views as to what the fundamental contribution made by the discovery of the calculus actually was. Leibniz was much more interested in defending the importance of notation and algorithm compared to Newton, who rather praised geometry. For Leibniz what was at stake in the dispute was the invention of an efficient algorithm. He also stressed the idea that the power of the algebraic method consisted in the fact that one could free the mind from the "burden of imagination" and manipulate symbols without worrying about their meaning. Newton despised those who practiced algebra like Leibniz, dismissing them as "bunglers of mathematics."[77] He instead praised mathematical procedures in which concepts "visible to the eye" are organized in the mind of the mathematician without his loosing a firm grasp of the meaning of symbols. For Newton, mathematical concepts had to represent existing phenomena of which we have a clear intuition, such as motion and velocity. Leibniz rather conceived the rules of the calculus as symbolic manipulations well-grounded on metaphysical principles, such as that of continuity. One might say that Leibniz was a logician who gave pride of place to the basic notation and rules of the calculus, and who praised the generality of symbolical methods, whereas Newton was a geometer who sought conceptually profound solutions to advanced problems. The two mathematicians approached the dispute from very different angles.

The political arena, in which Leibniz and Newton each took on a leading role, also fueled the dispute, and this in part explains why the abstractions of mathematics could ignite so much fury. Newton had been a member of the Convention Parliament in 1689: he belonged to the entourage of Lord Halifax, and was very close to sectors of the Church of England that had promoted the Glorious Revolution. Contemporaneously, Leibniz had become a counselor of the Emperor and of the Czar, and was in the service of the Duke of Hanover. It was in part thanks to Leibniz's diplomatic efforts that his patron acceded to the throne of Great Britain and Ireland in 1714 as George I. The prospect of having Leibniz – a towering diplomat and metaphysician who actively pursued an ecumenical policy of reconciliation between the Christian Churches – as Royal Historian in London must have been a daunting one for Newton's party, which favored anti-Catholicism and a Protestant interpretation of Anglicanism. The priority dispute served the purpose of discrediting Leibniz at the Royal Court well.[78]

It might be contended that the anxieties and passions surrounding Newton's thought on mathematical method that we have reviewed in this chapter were determined by the fact that mathematics played such a prominent role in his broad-ranging philosophical agenda – one polemically oriented against the theological heresies of the mechanism promoted by the Cartesians and by Hobbes, the probabilism in vogue at the Royal Society, and the irenicism defended by the diplomatic endeavors of Leibniz. The greatest mathematician since Archimedes's time, like the Syracusan geometer and mechanic himself, used mathematics for belligerent purposes.

NOTES

1 Ms. Add. 3996, fols. 88r–135r edited in *Certain Philosophical Questions: Newton's Trinity Notebook*, ed. James E. McGuire and Martin Tamny (Cambridge: Cambridge University Press), 1983.

2 Ms. Add. 4004 and Ms. Add. 4000 (Cambridge University Library), respectively.

3 D. T. Whiteside (ed.) *The Mathematical Papers of Isaac Newton*, 8 vols. (Cambridge: Cambridge University Press), 1967–81, vol. 1, pp. 25–88 (hereafter cited as MP, 1: 25–88).

4 Newton worked with the second Latin edition (1659–1661), but may also have come across the smaller first Latin edition (1649). See MP, 1: 21.

5 To avoid anachronism, Descartes did not have "Cartesian axes," but he used what we identify as "Cartesian coordinates." For example, given an ellipse, he defined y as the abscissa measured from the vertex, and x as the corresponding ordinate, so that the ellipse's equation was $x^2 = ry - (r/q)y^2$, r and q constants. *The Geometry of René Descartes with a Facsimile of the First Edition*, ed. and trans. D. E. Smith and M. L. Latham (New York: Dover, 1954), pp. 95–6.

6 MP, 1: 155–212.

7 For Newton's early (1667–8) studies on cubics, see MP, 2: 10–89. He reconsidered this topic in the late 1670 (MP, 4: 346–401) and then in the mid 1690s (MP, 7: 410–653).

8 MP, 1: 213–33. The subnormal is defined as the segment of the x-axis lying between the x-coordinate of the point at which a normal is drawn to a curve and the intercept of the normal with the x-axis.

9 MP, 1: 520–7.

10 Henk Bos, *Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction* (New York: Springer, 2001).

11 In this subsection I draw freely from my *Isaac Newton on Mathematical Certainty and Method* (Cambridge, MA: MIT Press), pp. 6–7.

12 As Newton stated in the Preface to the *Principia*, the errors are not due to the imperfections of geometry and mechanics but rather to the "artificer" who applies them.

13 He obtained this method probably inspired by de Witt and van Schooten. See MP, 2: 106–56, esp. pp. 134–51, and MP, 4: 298–303. See also Newton to Collins (August 20, 1672) in *Correspondence*, vol. 2, pp. 230–1 and MP, 2: 156–9.

14 MP, 1: 369–82.

15 For example: $(1 + x)^2 = 1 + 2x + 1x^2$.

16 The reason for this is that Newton sought to calculate the area of the surface bounded by an hyperbola and the area of the circle segment. Indeed, the equation of the hyperbola can be written via a binomial raised to a negative exponent ($y = (a + x)^{-1}$), and the equation of the circle can be written via a binomial raised to a fractional exponent ($y = (R^2 - x^2)^{1/2}$).

17 For example, the coefficients (1, 2, 1) for $(1 + x)^2 = 1 + 2x + 1x^2$.

18 For Newton's discovery of the binomial theorem, see MP, 1: 89–142.

19 In familiar Leibnizian terms this ratio is expressed as dy/dx.

20 For Newton's early discovery (1665) of the algorithm of the fluxional calculus, see MP, 1: 382–9.

21 In Leibnizian notation: $s = \int v dt = \int \frac{ds}{dt} dt = \int ds$, where s is the space, v the speed, and t the time.

22 MP, 1: 298–305, 313–15.

23 MP, 1: 305–13, 316–17, 342–3, 348–63.

24 Ms. Add. 3958.3, fols. 48v–63v in MP, 1: 400–48.

25 In this section I am drawing from my "'Specious algebra is fit enough to find out, but entirely unfit to consign to writing and commit to posterity': Newton's publication strategies as a mathematical author," *Sartoniana* 25 (2012) 161–78.

26 MP, 2: 206–47.

27 *Correspondence*, vol. 2, p. 29.

28 The reader will excuse me for some anachronistic terminology in this paragraph.

29 MP, 3: 32–328.

30 Newton included a method (MP, 3: 50 *passim*), which was later improved and systematized by Victor-Alexandre Puiseux, for developing an algebraic equation in two variables, $f(x, y) = 0$, into a fractional power series $y = \Sigma\xi^{\theta}$.

31 *The Method of Fluxions and Infinite Series*, translated and annotated by John Colson (London: H. Woodfall for J. Nourse, 1736).

32 *Optical Papers*, pp. 86, 88, and 436, 438.

33 Hooke, *Micrographia*, Preface.

34 *Optical Papers*, pp. 87 and 439.

35 *Correspondence*, vol. 1, pp. 96–7. This passage was censored by Henry Oldenburg, the secretary of the Royal Society.

36 *Correspondence*, vol. 1, p. 161.

37 MP, 7: 422–34.

38 *Optical Papers*, pp. 89 and 439.

39 *The Geometry of René Descartes with a Facsimile of the First Edition*, ed. and trans. D. E. Smith and M. L. Latham (New York: Dover, 1954), p. 17.

40 *Correspondence*, vol. 3, p. 385.

41 *Correspondence*, vol. 2, pp. 20–31 and pp. 110–29.

42 The manuscript deposited by Newton (probably written in 1683–4) bears no title: it is here referred to as *Lucasian Lectures on Algebra*. See MP, 5: 54–491.

43 The notes on Kinckhuysen's *Algebra* are edited in MP, 2: 277–447.

44 Jacqueline Stedall, *From Cardano's Great Art to Lagrange's Reflections: Filling a Gap In the History of Algebra* (European Mathematical Society: Heritage of European Mathematics, 2011).

45 MP, 5: 420–91.

46 Henk Bos, *Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction* (New York: Springer, 2001), pp. 287–9.

47 MP, 5: 424.

48 MP, 5: 425, 427.

49 MP, 7: 291.

50 The Kepler equation is $m = x - \varepsilon \sin x$, where m is the mean anomaly, ε is the eccentricity and x is the eccentric anomaly.

51 MP, 7: 287, 289.

52 MP, 5: 128–337.

53 See, for example, MP, 4: 277.

54 See, for example, MP, 7: 251.

55 MP, 7: 261.

56 Newton's studies on projective geometry are included in MP, 4: 230–5.

57 MP, 4: 282–321.

58 MP, 4: 277.

59 MP, 4: 420–519.

60 The attention of the historian of mathematics will be captured by several mathematical methods employed in the *Principia*. What follows is only a partial list. In Book 1: (i) a foundation for the determination of curvilinear areas, tangents and curvatures in terms of limits that was considered until Cauchy the most rigorous presentation of concepts and procedures essential for the calculus (Section 1); (ii) the application of limit procedures to the treatment of central force motion (Sections 2 and 3); (iii) the study of projective transformations in the plane (Section 5); (iv) a demonstration of the non-algebraic integrability of ovals and the application of the Newton-Raphson method to the Kepler equation (Section 6); (v) the reduction of the problem of central force motion to quadratures (Sections 7 and 8); (vi) the study of precessing orbits in terms of power series expansions (Sections 9); (vii) a qualitative study of the three-body problem (Section 11); (viii) multiple integrations applied to the study of the attraction of extended bodies (Sections 12 and 13);

(ix) the binomial theorem (Section 14). In Book 2: (i) Taylor series expansions (Section 3); (ii) a solution of the problem of the solid of least resistance (section 7). In Book 3: (i) interpolation techniques equivalent to the Newton–Cotes formula (Lemma 5); (ii) an application of Lambert's theorem to the study of comets.

61 With respect to this last topic, he obtained a geometric method that, when translated into symbols, paved the way for the method of variations of constants. MP, 6: 508–37.

62 See Guicciardini, *Isaac Newton on Mathematical Certainty and Method*, pp. 267–90.

63 *Ibid.*, pp. 252–4.

64 MP, 7: 200–561.

65 MP, 7: 24–182.

66 MP, 7: 96–9.

67 MP, 4: 346–401.

68 MP, 7: 579–671.

69 Equation $y^2 = ax^3 + bx^2 + cx + d$.

70 MP, 7: 411–35, 634–5.

71 MP, 7: 690–702.

72 MP, 4: 526–89, 590–653.

73 *Correspondence*, vol. 4, p. 100.

74 MP, 6: 456–80. The "brachistochrone" is the curve connecting two given points along which a body slides, without friction and under the action of constant gravity, in the least possible time.

75 Newton's fluxional analysis of the brachistochrone problem is found in MP, 8: 86–91.

76 Johann Bernoulli, "Epistola pro eminente Mathematico, Dn. Johanne Bernoullio, contra quendam ex Anglia antogonistam [sic] scripta," *Acta Eruditorum*, Julii 1716, 296–315.

77 Walter George Hiscock, *David Gregory, Isaac Newton and Their Circle* (Oxford: Oxford University Press, 1937), p. 42.

78 See Bertoloni Meli's chapter in this volume.