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Reducibility of 1-d Schrödinger equation with unbounded time quasiperiodic perturbations. III

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In this paper, we study the reducibility of time quasiperiodic perturbations of the quantum harmonic or anharmonic oscillator in one space dimension. We modify known algorithms obtaining a reducibility result which allows us to deal with perturbations of order strictly larger than the ones considered in all the previous papers. *Published by AIP Publishing.* https://doi.org/10.1063/1.5048726

I. INTRODUCTION

In this paper, we study the reducibility of the time dependent Schrödinger equation

$$\mathrm{i}\dot{\psi} = H(\omega t)\psi, \tag{1.1}$$

$$H(\omega t) := (-\mathrm{i}\partial_x - \epsilon W_1(x, \omega t))^2 + V(x) + \epsilon W_0(x, -\mathrm{i}\partial_x, \omega t), \qquad x \in \mathbb{R}, \tag{1.2}$$

where V is a smooth potential growing as $V(x) \simeq |x|^{2\ell}$, $\ell \geq 1$, as $x \to \infty$, and W_i are real valued C^∞ functions (symbols), depending in a quasiperiodic way on time. More precisely, we prove the existence of a unitary (in L^2) transformation depending in a quasiperiodic way on time, which conjugates the system to a diagonal time independent one. The main point is that we allow here perturbations which are of order higher than those treated in all previous papers. In particular, we include the case of a harmonic oscillator subject to a magnetic forcing.

From a physical point of view, the main consequence is that a time quasiperiodic perturbation of the kind considered here does not transfer indefinitely energy to a quantum particle. From a mathematical point of view, this is expressed by the fact that the Sobolev norms of the solutions of (1.1) stay bounded for all time. We recall that (1.2) was also studied for more theoretical reasons: it is well known that the classical Duffing oscillator, namely, the Hamiltonian system with Hamiltonian $\xi^2 + x^4 + \epsilon x^{\beta_0} \cos(\omega t)$, exhibits small chaotic islands when $\epsilon \neq 0$. The question is whether the quantum system has some behaviors which are a quantum counterpart of this nonregular behavior. Furthermore, a point of interest is whether this depends on the value of the exponent β_0 or not. As a consequence of reducibility, one gets that the quantum perturbed system qualitatively behaves forever as the unperturbed one, in sharp contrast with what happens in the classical case. Here, we prove that this is the case as far as $\beta_0 < 3$. Previously, this was known for integer values of $\beta_0 \leq 4$ or for real values $\beta_0 < 2$. We also expect our result to be the best one achievable with variants of the present technique.

We now describe more in detail our assumptions and compare the present result with the previous ones. To fix this, consider the case where W_0 is independent of $-\mathrm{i}\partial_x$ so that it is a function of x and ωt only. In this case, the precise assumption is that $\forall k \geq 0$, and there exists C_k such that the following estimates are fulfilled:

$$\left|\partial_x^k W_0(x, \omega t)\right| \leq C_k \langle x\rangle^{\beta_0 - k}, \quad \beta_0 < 2\ell - 1\,, \tag{1.3}$$

$$\left|\partial_x^k W_1(x, \omega t)\right| \leq C_k \langle x\rangle^{\beta_1 - k}, \quad \beta_1 \begin{cases} \leq \ell & \text{if} \quad \ell \in [1,2) \\ < 2(\ell - 1) & \text{if} \quad 2 \leq \ell. \end{cases} \tag{1.4}$$

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In the literature, the best known results are those of Ref. 1, which required $\beta_0 < \ell$ and $\beta_1 < \ell - 1$ for $\ell \in [1, 2]$, while, in the case $\ell > 2$, the order β_1 was subject to the limitation $\beta_1 < \ell/2$.

The problem of reducibility of equations of the form of (1.1) has a long history, and the main results have been obtained in Refs. 2–8, and 1 (see Ref. 1 for a more detailed history). We also mention that our result is limited to the one dimensional case, while some results on these problems in more than one dimension have been recently obtained.[9–13] We also recall that related techniques have been used in order to get a control on the growth of Sobolev norms in Refs. 14 and 15.

We now outline the strategy of the proof, which is the same as that of Refs. 8 and 1, which in turn is a development of the ideas of Refs. 16–18. Such ideas consist of exploiting pseudodifferential calculus in order to conjugate the Hamiltonian to a new one which is a smoothing perturbation of a time independent operator and then applying a Kolmogorov-Arnold-Moser (theory) (KAM)-reducibility scheme in order to complete the reduction to constant coefficients. More applications of these ideas can be found in several papers (see, e.g., Refs. 19–23, and 13).

In the present paper, in order to prove our reducibility result, we proceed as follows: first, by a gauge transformation, we eliminate from the perturbation the terms containing first order derivatives. Then, we develop a variant of the theory of Refs. 8 and 1 in order to reduce the perturbation to a smoothing one. The main difference is that here we do not eliminate time from the normal form that we construct. More precisely, we first use the theory of Ref. 1 (a variant of Theorem 3.19 of that paper) in order to conjugate (1.2) to a system which is a perturbation of H_0 belonging to a better class of symbols (essentially those considered in Ref. 8), and then, we apply the theory of Ref. 14 in order to conjugate the so obtained system to another one which is a smoothing perturbation of a diagonal time dependent system. Finally, we eliminate time from the latter system by an explicit transformation which is done at the quantum level. Actually, we recall that in Refs. 8 and 1, the main limitation to the order of the perturbation came from the construction of the transformation eliminating time from the perturbation.

In Sec. II, we give a precise statement of our main result and Sec. III contains its proof. Section III is split into 4 subsections: in Subsection III A, we give some preliminaries; in Subsection III B, we eliminate W_1; and in Subsection III C, we give some smoothing theorems reducing the system to a time dependent normal form. Finally, in Subsection III D, we eliminate time from the normal form and conclude the proof.

II. STATEMENT OF THE MAIN RESULT

We start by giving the precise assumptions on the potential. When $\ell > 1$, we assume that

$$V(x) = V(-x), \tag{2.1}$$

and that it admits an asymptotic expansion of the form

$$V(x) \sim |x|^{2\ell} + \sum_{j\geq 1} V_{2(\ell-j)}(x), \tag{2.2}$$

with V_a homogeneous of degree a, namely, such that $V_a(\rho x) = \rho^{aV}(x)$, $\forall \rho > 0$. We also assume that

$$V'(x) \neq 0, \quad \forall x \neq 0. \tag{2.3}$$

In the case $\ell = 1$, we assume that $V(x) = x^2$.

Remark 2.1. The above assumptions on the potential are needed in order to apply the theory of Ref. 1, which in turn deeply exploits the theory of Ref. 24. We think that it should be possible to extend the results to more general potentials, for example, to noneven potentials and to potentials admitting expansions more general then (2.2). Such an extension would require a nontrivial amount of work for which we do not have a true motivation. For this reason, we limit ourselves to the considered class of potentials.

We denote by λ_j^v the sequence of the eigenvalues of

$$H_0 := -\partial_{xx} + V(x). \tag{2.4}$$

According to the results of Ref. 24, they form a sequence $\lambda_j^v \sim cj^d$, with $d = \frac{2\ell}{\ell+1}$. In what follows, *we will identify L^2 with ℓ^2 by introducing the basis of the eigenvector of H_0. We also define a reference operator $K_0 := H_0^{\frac{\ell+1}{2\ell}}$.*

Definition 2.2. For $s \geq 0$, we define the Sobolev-like spaces $\mathcal{H}^s := D(K_0^s)$ (domain of the s-power of the operator K_0) endowed by the graph norm. For negative s, the space $\mathcal{H}^s$ is the dual of $\mathcal{H}^{-s}$.

We will denote by $\mathcal{B}(\mathcal{H}^{s_1}; \mathcal{H}^{s_2})$ the space of bounded linear operators from $\mathcal{H}^{s_1}$ to $\mathcal{H}^{s_2}$.

Proposition 2.3. For $s \geq 0$, given a function $u \in \mathcal{H}^s$, one has that

$$\|u\|_{\mathcal{H}^s} \simeq \|u\|_{H^{\frac{\ell+1}{\ell}s}} + \|\langle x\rangle^{(\ell+1)s} u\|_{L^2}, \tag{2.5}$$

where $\langle x\rangle := \sqrt{1+x^2}$, H^s is the standard Sobolev space, and $\|.\|_{H^s}$ the corresponding norm.

The proof is given in Sec. III A.

Now, we recall the class of real valued symbols introduced in Ref. 1 (see also Refs. 24 and 8). We emphasize that we will never use complex valued symbols. Define

$$\lambda(x,\xi) := \left(1 + \xi^2 + |x|^{2\ell}\right)^{\frac{1}{2\ell}}. \tag{2.6}$$

Definition 2.4. The space S^{m_1,m_2} is the space of the functions (symbols) $g \in C^\infty(\mathbb{R}^2)$ such that $\forall k_1$, $k_2 \geq 0$, there exists C_{k_1,k_2} with the property that

$$\left|\partial_\xi^{k_1}\partial_x^{k_2} g(x,\xi)\right| \leq C_{k_1,k_2}\,[\lambda(x,\xi)]^{m_1-\ell k_1}\langle x\rangle^{m_2-k_2}. \tag{2.7}$$

Remark 2.5. The constants C_{k_1,k_2} form a family of seminorms which allow one to endow S^{m_1,m_2} with the structure of Fréchet space. Thus, one can consider also the space $C^\infty(\mathbb{T}^n, S^{m_1,m_2})$ which is a Fréchet space too.

We will also use the following class of symbols.

Definition 2.6. The space S_V^m is the space of the functions (symbols) $g \in C^\infty(\mathbb{R})$ such that $\forall k \geq 0$, there exists C_k with the property that

$$\left|\partial_x^k g(x)\right| \leq C_k \langle x\rangle^{m-k}. \tag{2.8}$$

Remark 2.7. Here the index V is written in order to recall that this is a space composed by "potentials," thus depending on x only. For these kinds of functions, we will systematically identify the function with the corresponding multiplication operator.

Remark 2.8. We have $S_V^m \subset S^{0,m}$, and, defining

$$[m] := \max\{m, 0\}, \tag{2.9}$$

one also has $S^{m_1,m_2} \subset S^{m_1+[m_2],0}$.

To a symbol $g \in S^{m_1,m_2}$, we associate its Weyl quantization, namely, the operator $Op^w(g)$, defined by

$$Op^w(g)\psi(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mathrm{i}(x-y)\cdot\xi} g\left(\frac{x+y}{2};\xi\right)\psi(y)dyd\xi. \tag{2.10}$$

Given a symbol g, we will often denote by the corresponding capital letter (in this case G) the corresponding Weyl operator, and vice versa.

The frequencies ω will be assumed to vary in the set

$$\Omega := [1, 2]^n,$$

or in suitable closed subsets $O \subset \Omega$. We will denote by $|O|$ the Lebesgue measure of a Borel set.

Definition 2.9. Given a family of time dependent unitary operators $U(t)$, we will say that it conjugates H_1, to H_2, if given an arbitrary solution ψ of the Schrödinger equation $i\dot{\psi} = H_1\psi$, the function φ defined by $\psi = U(t)\varphi$ satisfies $i\dot{\varphi} = H_2\varphi$.

Our main result is the following Theorem whose proof will occupy the rest of the paper.

Theorem 2.10. *Let $w_0 \in C^\infty(\mathbb{T}^n; S^{\beta_0^{(1)},\beta_0^{(2)}})$, $W_1 \in C^\infty(\mathbb{T}^n; S_V^{\beta_1})$ be real valued symbols. Consider the linear operator*

$$H(\omega t) = (-\mathrm{i}\partial_x - \epsilon W_1(x, \omega t))^2 + V(x) + \epsilon W_0(\omega t) \tag{2.11}$$

(where of course $W_0 := Op^w(w_0)$) and define

$$\beta := \max\left\{\beta_0^{(1)}, [\beta_0^{(2)}], [\beta_1 + 1]\right\}.$$

Assume $\beta < 2\ell - 1$ and $\beta_1 \leq \ell$.

Then there exists C, $\epsilon_* > 0$, and $\forall|\epsilon| < \epsilon_*$, a closed set $\Omega(\epsilon) \subset \Omega$ and, $\forall\omega \in \Omega(\epsilon)$, there exists a unitary (in L^2) time quasiperiodic map $U_\omega(\omega t)$ conjugating (2.11) to $H_\infty := diag(\lambda_j^\infty)$, with $\lambda_j^\infty = \lambda_j^\infty(\omega, \epsilon)$ independent of time and

$$\left|\lambda_j^\infty - \lambda_j^v\right| \leq C\epsilon j^{\frac{\beta}{\ell+1}}. \tag{2.12}$$

Furthermore, one has

1. $\lim_{\epsilon\to 0}|\Omega - \Omega(\epsilon)| = 0$;
2. *$\forall s$, $r \geq 0$, $\exists\epsilon_{s,r} > 0$ and s_r such that if $|\epsilon| < \epsilon_{s,r}$, then the map $\phi \mapsto U_\omega(\phi)$ is of class $C^r(\mathbb{T}^n; \mathcal{B}(\mathcal{H}^{s+s_r}; \mathcal{H}^s))$; when $r = 0$, one has $s_0 = 0$.*
3. *$\exists b > 0$ such that $\forall s \geq 0$, $\|U_\omega(\phi) - \mathbf{1}\|_{\mathcal{B}(\mathcal{H}^{s+\beta}; \mathcal{H}^s)} \leq C_s\epsilon^b$.*

As usual, the boundedness of Sobolev norms and the pure point nature of the Floquet spectrum follow.

Consider now the following class of symbols (already used in Refs. 24 and 8):

Definition 2.11. The space S^m is the space of the symbols $g \in C^\infty(\mathbb{R}^2)$ such that $\forall k_1, k_2 \geq 0$, there exists C_{k_1,k_2} with the property that

$$\left|\partial_\xi^{k_1}\partial_x^{k_2} g(x, \xi)\right| \leq C_{k_1,k_2}\left[\lambda(x, \xi)\right]^{m-\ell k_1-k_2}. \tag{2.13}$$

Remark 2.12. If a symbol w is of class S^m ($m \geq 0$), and depends on x only, then it is a polynomial. Indeed taking m derivatives with respect to x, one gets a function which tends to zero as $\xi \to \infty$. Since it is actually independent of ξ, it must be identically zero, and thus, the original function is a polynomial.

By a small variant of the Proof of Theorem 2.10, one gets the following result.

Theorem 2.13. *Let $w \in C^\infty(\mathbb{T}^n; S^\beta)$ be a symbol and consider the time dependent operator $H(\omega t) := H_0 + \epsilon\, W(\omega t)$; assume $\beta < 2\ell$, then the same conclusion of Theorem 2.10 holds.*

The details of the proof are omitted.

Remark 2.14. In Ref. 8, the same result was obtained under the stronger limitation $\beta < \ell + 1$ in the general case, or $\beta < (3\ell + 1)/2$ in the case where $\langle w \rangle = 0$.

III. PROOF

First, remark that a quasiperiodic family of unitary transformations $U(\omega t)$ conjugates $H(\omega t)$ to

$$H_+(\phi) = U_{\omega *}H(\phi) := U(\phi)^{-1}\Big(H(\phi)U(\phi) - \mathrm{i}\omega \cdot \partial_\phi U(\phi)\Big). \tag{3.1}$$

A. A few results on pseudodifferential calculus

An operator G will be said to be pseudodifferential of class OPS^{m_1,m_2} if there exists a symbol $g \in S^{m_1,m_2}$ such that $G = Op^w(g)$. Similarly, we will say that $G \in OPS^m$ if there exists a symbol $g \in S^m$ such that $G = \mathrm{Op}^w(g)$.

Remark 3.1. If $W \in S^m_V$ is a function, then by direct computation one has

$$Op^w(\xi W(x)) = -\mathrm{i}W\partial_x - \frac{\mathrm{i}}{2}W_x = \frac{-\mathrm{i}\partial_x \circ W - W\mathrm{i}\partial_x}{2}. \tag{3.2}$$

In particular, H_0 is the Weyl quantization of the symbol

$$h_0(x,\xi) := \xi^2 + V(x), \tag{3.3}$$

and the operator H of (2.11) is the Weyl quantization

$$h(x,\xi,\omega t) = (\xi - \epsilon W_1(x,\omega t))^2 + V(x) + \epsilon w_0(x,\xi,\omega t). \tag{3.4}$$

According to the standard theory of pseudodifferential operators (see, in particular, Lemma 3.4 of Ref. 1), given two symbols $a \in S^{m_1,m_2}$ and $b \in S^{m'_1,m'_2}$, there exists a symbol $g \in S^{m_1+m'_1,m_2+m'_2}$ such that $Op^w(a)Op^w(b) = Op^w(g)$. One denotes $a\sharp b := g$, and furthermore, one has $g = ab + S^{m_1+m'_1-\ell,m_2+m'_2-1}$, and there exists a full asymptotic expansion of $a\sharp b$. Furthermore, the symbol $(a\sharp b - b\sharp a)/\mathrm{i}$ of $1/\mathrm{i}$ times the commutator of the two Weyl operators is called the Moyal Bracket of a and b and will be denoted by $\{a;b\}_M$. It turns out that

$$\{a;b\}_M \in S^{m_1+m'_1-\ell,m_2+m'_2-1}, \quad \{a;b\}_M = \{a;b\} + S^{m_1+m_2-3\ell,m_1+m_2-3}, \tag{3.5}$$

where $\{.;.\}$ denotes the Poisson Bracket.

We also recall the following two lemmas, proved in Ref. 1, which will be used in our proof.

Lemma 3.2 (Lemma 3.8 in Ref. 1). *Let $\chi \in S^{m,0}$ have the further property that $\partial_x\chi \in S^{\ell,0}$. Assume $m \le \ell + 1$, then $X := Op^w(\chi)$ is self-adjoint and $e^{-i\epsilon X}$ leaves invariant all the spaces $\mathcal{H}^s$.*

Lemma 3.3 (Lemma 3.11 in Ref. 1). *Let $\chi \in S^{\alpha,0}$ with $\alpha < \ell$, then given $g \in S^{m_1,m_2}$, one has*

$$e^{\mathrm{i}\epsilon X}Op^w(g)e^{-\mathrm{i}\epsilon X} \in OPS^{m_1,m_2},$$

and its symbol is given by

$$g + \epsilon\{g,\chi\} + \epsilon S^{m_1+2(\alpha-\ell),m_2-2}. \tag{3.6}$$

Remark 3.4. In the following, we will need to consider the case where χ is also time dependent, namely, $\chi \in C^\infty(\mathbb{R}, S^{\alpha,0})$, with $\alpha < \ell$. In such a case, according to the formula (3.1), the operator $e^{-i\epsilon X}$ conjugates $Op^w(g)$ to

$$e^{\mathrm{i}\epsilon X}Op^w(g)e^{-\mathrm{i}\epsilon X} + e^{\mathrm{i}\epsilon X}\partial_t e^{-\mathrm{i}\epsilon X}.$$

According to Remark 3.12 of Ref. 1, one has

$$e^{\mathrm{i}\epsilon X}\partial_t e^{-\mathrm{i}\epsilon X} \in OPS^{\alpha,0}.$$

The application of the Calderon Vaillancourt Theorem yields the following Lemma.

Lemma 3.5. Let $g \in S^{m_1,m_2}$, then one has

$$Op^w(g) \in \mathcal{B}(\mathcal{H}^{s_1+s}; \mathcal{H}^s), \quad \forall s, \quad \forall s_1 \geq m_1 + [m_2]. \tag{3.7}$$

In order to deal with functions p which depend on (x, ξ) through h_0 only, namely, such that there exist a $\tilde{p}$ with the property that

$$p(x, \xi) = \tilde{p}(h_0(x, \xi)),$$

we introduce the following class of symbols.

Definition 3.6. A function $\tilde{p} \in C^\infty$ will be said to be of class $\widetilde{S}^m$ if there exist constants C_k such that

$$\left| \frac{\partial^k \tilde{p}}{\partial E^k}(E) \right| \leq C_k \langle E \rangle^{\frac{m}{2\ell} - k}. \tag{3.8}$$

Sometimes symbols of this class are also called classical symbols.

By abuse of notation, we will say that $p \in \widetilde{S}^m$ if there exists $\tilde{p} \in \widetilde{S}^m$ such that $p(x, \xi) = \tilde{p}(h_0(x, \xi))$. We say that the corresponding Weyl operator $\mathrm{Op}^w(p)$ belongs to the class $\widetilde{OPS}^m$.

Proof of Proposition 2.3. Proposition 2.3 is a direct consequence of the fact that, according to the results of Ref. 25, for any real s, the operator K_0^s (recall that $K_0 = H_0^{\frac{\ell+1}{2\ell}}$) is a pseudodifferential operator whose symbol has the form

$$(\xi^2 + |x|^{2\ell})^{\frac{s(\ell+1)}{2\ell}} + \text{lower order terms}.$$

In turn, for, $s \geq 0$, this is bounded by a constant times $\langle \xi \rangle^{s\frac{\ell+1}{\ell}} + \langle x \rangle^{s(\ell+1)}$. As a consequence, one deduces the equivalence (2.5). □

Let $p \in S^{m_1,m_2}$ be a symbol and define its average by

$$\langle p \rangle(x, \xi) := \frac{1}{T(E)} \int_0^{T(E)} p(\Phi_{h_0}^\tau(x, \xi)) d\tau \bigg|_{E = h_0(x,\xi)}, \tag{3.9}$$

where $\Phi_{h_0}^\tau(x, \xi)$ is the classical flow of the Hamiltonian h_0 and $T(E)$ is the period of the classical orbits of h_0 at energy E. Consider the homological equation

$$p + \{h_0; \chi\} = \langle p \rangle. \tag{3.10}$$

Then the following lemma summarizes the results of Lemmas 3.13 and 3.14 of Ref. 1.

Lemma 3.7. Assume that $p \in S^{m_1,m_2}$, then one has $\langle p \rangle \in \tilde{S}^{m_1+[m_2]}$, and furthermore, the homological equation (3.10) has a solution $\chi \in S^{m_1+[m_2]-\ell+1,0}$.

B. Reduction of W_1

In order to eliminate the magnetic term, we will perform a gauge transformation; first of all, we study the properties of such a transformation.

Lemma 3.8. Let $b \in S_V^\alpha$ with $\alpha \leq \ell + 1$, then the unitary transformation $e^{-ib(x)}$ maps the spaces $\mathcal{H}^s$ into themselves and leaves invariant the space of the pseudodifferential operators of class OPS^{m_1,m_2}.

Proof. In order to show that the spaces $\mathcal{H}^s$ are left invariant by the transformation $e^{-\mathrm{i}b}$ generated by b, we apply Theorem 1.2 of Ref. 26 according to which it is enough to verify that $[b, K_0]K_0^{-1}$ is a bounded operator. This is easily verified by remarking that

$$[b, K_0]K_0^{-1} = Op^w(\{b, h_0^{\frac{\ell+1}{2\ell}}\}h_0^{-\frac{\ell+1}{2\ell}}) + \text{lower order terms},$$

and by explicit computation of the Poisson bracket, the function $\{b, h_0^{\frac{\ell+1}{2\ell}}\}h_0^{-\frac{\ell+1}{2\ell}}$ is bounded together with all its derivatives, provided $\alpha - 1 \le \ell$. Thus, by the Calderon Vaillancourt Theorem, cf. Lemma 3.5, the corresponding operator is bounded. We come to the transformation of pseudodifferential operators.

According to the standard approach to Egorov's theorem, it is enough to prove that, denoting by $\Phi_b^t(x,\xi) = (x, \xi - tb'(x))$ the Hamiltonian flow of the Hamiltonian function b, one has $g \circ \Phi_b^t \in S^{n_1,n_2}$ for any $g \in S^{n_1,n_2}$ for any n_1, n_2. First, remark that one has

$$|g(x, \xi - tb'(x))| \le C\langle x\rangle^{n_2}\lambda(x, \xi - tb'(x))^{n_1}.$$

But, by the definition of λ,

$$\lambda(x, \xi - tb'(x)) \le C(1 + \xi^2 + (tb'(x))^2 + |x|^{2\ell}) \le C_1\lambda(x,\xi)$$

since, by assumption, $|tb'(x)|^2 \le \langle x\rangle^{2(\alpha-1)} \le \langle x\rangle^{2\ell}$, for $|t| \le 1$. For the derivatives of g, one proceeds similarly. □

Lemma 3.9. There exists $b \in C^\infty(\mathbb{T}^n; S_V^{[\beta_1+1]})$ such that the transformation

$$U^{(1)}(\phi) : \psi(x) \mapsto e^{-\mathrm{i}\epsilon b(\phi,x)}\psi(x) \tag{3.11}$$

conjugates (2.11) to

$$H^{(1)}(\phi) := -\partial_{xx} + V(x) + \epsilon W_0^{(1)}(\phi), \quad w_0^{(1)} \in C^\infty(\mathbb{T}^n, S^{\beta,0}), \tag{3.12}$$

where $W_0^{(1)} = Op^w\left(w_0^{(1)}\right)$.

Proof. One has

$$\begin{aligned} [U^{(1)}]^{-1} \circ (\mathrm{i}\partial_x) \circ U^{(1)} &= \mathrm{i}\partial_x - \epsilon b_x, \\ -[U^{(1)}]^{-1} \circ \left(\mathrm{i}\omega \cdot \partial_\phi U^{(1)}\right) &= -\epsilon\omega \cdot \partial_\phi b, \end{aligned} \tag{3.13}$$

while the operators of multiplication are invariant under the transformation (with ϕ considered as a parameter). Thus, if we define b by

$$b(\phi, x) = \int_0^x W_1(\phi, y)dy, \tag{3.14}$$

we get $b \in S^{[\beta_1+1]}$, and $(\mathrm{i}\partial_x + W_1(x,\phi))^2$ is conjugated to the differential operator $(\mathrm{i}\partial_x)^2 - \epsilon\omega \cdot \partial_\phi b$.

The transformation of W_0 is an immediate corollary of Lemma 3.8. Finally, by summing up, one gets the thesis. □

C. Smoothing theorems

Thanks to Lemma 3.9, in the case $\ell = 1$, Theorem 2.10 follows from Theorem 2.4 of Ref. 1, so **we concentrate on the case $\ell > 1$.**

In this case, the conjugation of $H^{(1)}$ to a Hamiltonian with a smoothing perturbation is obtained through the combination of a few smoothing theorems which essentially have already been proved in previous papers, but are here combined in a new way. Precisely, in Theorem 3.10, we conjugate $H^{(1)}$ to a new Hamiltonian $H^{(2)}$ which is a perturbation of H_0, with perturbation which, up to a smoothing remainder, belongs to the class $C^\infty(\mathbb{T}^n, S^\beta)$. In Theorem 3.11, we conjugate (again up to a smoothing remainder) $H^{(2)}$ to a new Hamiltonian which is time dependent and diagonal (with respect to the basis of the eigenfunctions of H_0). In Lemma 3.13, we show that the (still time dependent) eigenvalues of the diagonal Hamiltonian are symbols of class $\tilde{S}^\beta$ as functions of λ_j^v. In Lemma 3.15, we eliminate time from the above eigenvalues. Finally, in Lemma 3.16, we summarize the results and draw the last conclusions in a form suitable for the application of the KAM theory of Ref. 8.

The first result that we need is a smoothing theorem which is a variant of Theorem 3.19 of Ref. 1.

Theorem 3.10 [*Smoothing Theorem 1*]. *Consider the Hamiltonian* (3.12)*, and assume*

$$\beta < 2\ell - 1\,;$$

fix an arbitrary $\kappa > 0$. Then there exists a time dependent family of unitary transformations $U^{(2)}(\phi)$ which conjugates the Hamiltonian (3.12) *to a pseudo-differential operator $H^{(2)}$ with the symbol $h^{(2)}$ given by*

$$h^{(2)}(\phi, x, \xi) = h_0(x, \xi) + \epsilon z^{(2)}(h_0(x, \xi), \phi) + \epsilon r^{(2)}(x, \xi, \phi), \tag{3.15}$$

where $z^{(2)} \in C^\infty(\mathbb{T}^n; \tilde{S}^\beta)$ is a function of (x, ξ) through h_0 only, while the remainder fulfills

$$r^{(2)} \in C^\infty(\mathbb{T}^n; S^{-\kappa,0}). \tag{3.16}$$

Furthermore, one has

1. *$\forall r \geq 0$, $\exists s_r$ such that the map $\phi \mapsto U^{(2)}(\phi)$ is of class $C^r(\mathbb{T}^n; \mathcal{B}(\mathcal{H}^{s+s_r}; \mathcal{H}^s))$; when $r = 0$, one has $s_0 = 0$.*
2. $\left\| U^{(2)}(\phi) - \mathbf{1} \right\|_{\mathcal{B}(\mathcal{H}^{s+\beta}; \mathcal{H}^s)} \leq C_s \epsilon.$

Proof. The proof is essentially identical to the Proof of Theorem 3.19 of Ref. 1, the difference is that one makes the first transformation reducing (3.12) to the form (3.41) of Ref. 1, and then, instead of eliminating the time dependence from the average of $w_0^{(1)}$, one iterates the previous step (as in Ref. 14), in order to get a normal form which is a function of time, but depending on the space variables through h_0 only.

Precisely, fix

$$\delta := \min\{\ell - 1; \beta - (2\ell - 1)\},$$

which is strictly positive due to the assumption on β, and define the sequence

$$\alpha_0 := \beta, \quad \alpha_{j+1} = \alpha_j - \delta.$$

Then, we are going to prove that there exist sequences of (time dependent) symbols $z_j \in C^\infty(\mathbb{T}^n; \tilde{S}^\beta)$, $w_j^{(1)} \in C^\infty(\mathbb{T}^n; S^{\alpha_j,0})$, and $\chi_j \in C^\infty(\mathbb{T}^n; S^{\alpha_{j-1}-\ell+1,0})$ with the property that $e^{-\mathrm{i}\epsilon Op^w(\chi_{j+1})}$ conjugates the Hamiltonian operator with symbol

$$h_j^{(1)} := h_0 + \epsilon z_j + \epsilon w_j^{(1)} \tag{3.17}$$

to a Hamiltonian operator with the same symbol, but with j replaced by $j + 1$. We proceed by induction. The result is true for $j = 0$ by taking $z_j = 0$. Assume that the result true for some j. Next, we consider the homological equation (3.10) with p replaced by $w_j^{(1)}$. Let χ_{j+1} be its solution, and use the operator $e^{-\mathrm{i}\epsilon Op^w(\chi_{j+1})}$ to conjugate the Hamiltonian to a new Hamiltonian with a symbol which, according to Lemma 3.3, is given by

$$h_0 + \epsilon \left\{ h_0; \chi_{j+1} \right\} + \epsilon S^{\alpha_j - (2\ell - 1 - \alpha_j),0} \tag{3.18}$$

$$+ \epsilon z_j + \epsilon^2 S^{\alpha_0 - (2\ell - 1 - \alpha_j),0} \tag{3.19}$$

$$+ \epsilon w_j^{(1)} + \epsilon^2 S^{\alpha_j - (2\ell - 1 - \alpha_j),0} \tag{3.20}$$

$$+ \epsilon S^{\alpha_j - \ell + 1,0}. \tag{3.21}$$

Define $z_{j+1} = z_j + \epsilon \langle w_j^{(1)} \rangle$ and $w_{j+1}^{(1)}$ to be the sum of all the remainder terms above. Then, one deduces the claimed statement with $j + 1$ in place of j. □

We now apply Theorem 3.8 of Ref. 14 which gives the following.

Theorem 3.11 [*Smoothing Theorem 2*]. *Under the assumptions of Theorem 3.10. There exists a unitary (time-dependent) operator $U^{(3)}(\phi)$ in $L^2(\mathbb{R})$ which conjugates $H^{(2)}$ [and thus the Hamiltonian (2.11)] to the Hamiltonian*

$$H^{(3)}(\phi) := H_0 + \epsilon Z^{(3)}(\phi) + \epsilon R^{(3)}(\phi), \tag{3.22}$$

where $Z^{(3)}(\phi) \in C^\infty(\mathbb{T}^n, \widetilde{OPS}^\beta)$ commutes with H_0, i.e., $[Z^{(3)}(\omega t), H_0] = 0$, while $R^{(3)} \in C^\infty(\mathbb{T}^n, OPS^{-\kappa,0})$. Furthermore, one has

1. *$\forall r \geq 0$, $\exists s_r$ such that the map $\phi \mapsto U^{(3)}(\phi)$ is of class $C^r(\mathbb{T}^n; \mathcal{B}(\mathcal{H}^{s+s_r}; \mathcal{H}^s))$; when $r = 0$, one has $s_0 = 0$.*
2. $\left\| U^{(3)}(\phi) - \mathbf{1} \right\|_{\mathcal{B}(\mathcal{H}^{s+\beta};\mathcal{H}^s)} \leq C_s \epsilon$.

Proof. First, we recall that according to Theorems 7 and 8 of Ref. 24, there exists a pseudodifferential operator $Q \in OPS^{-(\ell+1)}$ such that

$$H_0 = K_1^{\frac{2\ell}{\ell+1}} + Q, \text{ and } [K_1; Q] = 0, \tag{3.23}$$

and the spectrum of K_1 is $\{j + \sigma\}_{j\geq 0}$ with[27] $\sigma > 0$. Remark that K_1 and Q are diagonal on the basis of the eigenfunctions of H_0. Then, Theorem 3.8 of Ref. 14 applies and gives the result with $Z^{(3)}$ which commutes with K_1; however, since the eigenvalues of K_1 are simple, $Z^{(3)}$ commutes also with H_0. □

Remark 3.12. By the previous theorem, the matrix of the operator $Z^{(3)}(\omega t)$ is diagonal on the basis of the eigenfunctions of H_0. Thus, on this basis

$$Z^{(3)}(\phi) = \text{diag}_{j\geq 0}\mu_j(\phi) \tag{3.24}$$

with suitable smooth functions $\mu_j(\phi)$ which satisfy for any $m \in \mathbb{N}$, the estimate $\|\mu_j\|_{C^m(\mathbb{T}^n)} \leq C_m j^{\frac{\beta}{\ell+1}}$ for a suitable constant $C_m > 0$.

We are now going to show that, due to the property that $Z^{(3)}(\omega t)$ is a pseudodifferential operator, the $\mu_j{}'s$ are essentially smooth functions of the eigenvalues of H_0, i.e., of λ_j^v.

Lemma 3.13. For any κ there exists a smooth function $z_\mu \in C^\infty(\mathbb{T}^n; \tilde{S}^\beta)$ and a sequence of functions $\delta_j(\phi)$ such that

$$\mu_j(\phi) = z_\mu(\lambda_j^v, \phi) + \delta_j(\phi), \tag{3.25}$$

and for any $m \geq 0$, there exist C_m such that

$$\|\delta_j\|_{C^m(\mathbb{T}^n)} \leq C_m \langle j \rangle^{-\kappa}. \tag{3.26}$$

Proof. The Lemma follows by the following inductive claim: For any $k \in \mathbb{N}$, there exists a smooth function $z_\mu^{\{k\}}(H_0)$ of the operator H_0 (spectrally defined as a function of H_0) with $z_\mu^{\{k\}} \in C^\infty(\mathbb{T}^n, OPS^\beta)$ such that

$$Z^{(3)} = z_\mu^{\{k\}}(H_0) + OPS^{\beta-(k+1)(\ell+1)}.$$

The claim is proved by arguing by induction on k.

Proof for $k = 0$

Denote by $z^{(3)}$, the symbol of $Z^{(3)}$ (where we drop the dependence on ϕ). Let $\eta(E)$ be a smooth compactly supported function and write

$$z^{(3)} = z_0^{(3)} + z_R^{(3)},$$

where $z_R^{(3)}(x, \xi, \omega t) := z^{(3)}(x, \xi, \omega t)\eta(h_0(x, \xi))$ and $z_0^{(3)} := z^{(3)} - z_R^{(3)}$.

By the commutation property, one has

$$\left\{z^{(3)}; h_0\right\}_M = 0 \implies \left\{z_0^{(3)}; h_0\right\} =: \delta \in S^{\beta-\ell-3}. \tag{3.27}$$

Denote by $\Phi^t_{h_0}$, the flow of the Hamiltonian system with Hamiltonian h_0, and define the average of $z_0^{(3)}$ according to (3.9). By Lemma 4.16 of Ref. 8, one has $\langle z^{(3)}\rangle \in C^\infty(\mathbb{T}^n, \tilde{S}^\beta)$.

Define now

$$\breve{z}_0^{(3)} := z_0^{(3)} - \langle z_0^{(3)}\rangle,$$

and remark that the average of δ vanishes so that $\breve{z}_0^{(3)}$ is the only solution with zero average of the equation

$$\left\{h_0; \breve{z}^{(3)}\right\} = \delta.$$

Now, according to Lemma 4.17 of Ref. 8, such a solution is of class $S^{\beta-2(\ell+1)}$. It follows that $\breve{z}_0^{(3)} \in S^{\beta-2(\ell+1)}$. Furthermore, by the standard argument, one has that $\langle z_0^{(3)}\rangle(x,\xi) = \langle z_0^{(3)}\rangle(h_0(x,\xi),\phi)$ depending on (x,ξ) through h_0 only. Finally, by functional calculus, one has that the Weyl operator of $\langle z_0^{(3)}\rangle(h_0,\phi)$ is given by

$$\langle z_0^{(3)}\rangle(H_0) + OPS^{\beta-(\ell+1)}. \tag{3.28}$$

Thus, one has

$$Z^{(3)} = \langle z_0^{(3)}\rangle(H_0) + OPS^{\beta-(\ell+1)}.$$

Thus, the claimed statement for $k = 0$ follows by defining $z_\mu^{\{0\}} := \langle z_0^{(3)}\rangle$.

Proof of the inductive step

Assume that

$$Z^{(3)} = z_\mu^{\{k\}}(H_0) + OPS^{\beta-(k+1)(\ell+1)}, \quad z_\mu^{[k]} \in OPS^\beta.$$

We define $A_\mu^{\{k\}} := Z^{(3)} - z_\mu^{\{k\}}(H_0) \in OPS^{\beta-(k+1)(\ell+1)}$, and let $a_\mu^{\{k\}} \in S^{\beta-(k+1)(\ell+1)}$ be the symbol of $A_\mu^{[\{k\}}$. The following splitting holds:

$$a_\mu^{\{k\}} = a_{\mu,0}^{\{k\}} + a_{\mu,R}^{\{k\}}, \quad a_{\mu,0}^{\{k\}} := (1-\eta(h_0(x,\xi)))a_\mu^{\{k\}}, \quad a_{\mu,R}^{\{k\}} := \eta(h_0(x,\xi))a_\mu^{\{k\}}.$$

Since $Z^{(3)}$ and $z_\mu^{\{k\}}(H_0)$ commute with H_0 then $A_\mu^{\{k\}}$ also commutes with H_0, and hence, one obtains that

$$\left\{a_\mu^{\{k\}}; h_0\right\}_M = 0 \implies \left\{a_{\mu,0}^{\{k\}}; h_0\right\} =: \delta_k \in S^{\beta-(k+1)(\ell+1)-\ell-3}.$$

We now define $\breve{a}_{\mu,0}^{\{k\}} := a_{\mu,0}^{\{k\}} - \langle a_{\mu,0}^{\{k\}}\rangle$, where $\langle a_{\mu,0}^{\{k\}}\rangle$ is the average of the symbol $a_{\mu,0}^{\{k\}}$ with respect to the Hamiltonian flow of h_0. Note that the average of δ_k vanishes, and by Lemma 4.16 of Ref. 8, one has that $\langle a_{\mu,0}^{\{k\}}\rangle \equiv \langle a_{\mu,0}^{\{k\}}\rangle(h_0) \in \widetilde{S}^{\beta-(k+1)(\ell+1)}$. Hence, $\breve{a}_{\mu,0}^{\{k\}}$ is the unique solution of the homological equation

$$\left\{\breve{a}_{\mu,0}^{\{k\}}; h_0\right\} =: \delta_k,$$

and by Lemma 4.17 of Ref. 8, one has that $\breve{a}_{\mu,0}^{\{k\}} \in S^{\beta-(k+3)(\ell+1)}$, and hence,

$$a_\mu^{\{k\}} = a_{\mu,0}^{\{k\}} + a_{\mu,R}^{\{k\}} = \langle a_{\mu,0}^{\{k\}}\rangle(h_0) + \breve{a}_{\mu,0}^{\{k\}} + a_{\mu,R}^{\{k\}} = \langle a_{\mu,0}^{\{k\}}\rangle(h_0) + S^{\beta-(k+3)(\ell+1)}. \tag{3.29}$$

Finally, by functional calculus, one has that

$$\mathrm{Op}^w\left(\langle a_{\mu,0}^{\{k\}}\rangle(h_0)\right) = \langle a_{\mu,0}^{\{k\}}\rangle(H_0) + OPS^{\beta-(k+2)(\ell+1)}. \tag{3.30}$$

Therefore, (3.29) and (3.30) imply that

$$\begin{aligned} Z^{(3)} &= z_\mu^{\{k\}}(H_0) + A_\mu^{\{k\}} \overset{(3.29)}{=} z_\mu^{\{k\}}(H_0) + \mathrm{Op}^w\left(\langle a_{\mu,0}^{\{k\}}\rangle(h_0)\right) + OPS^{\beta-(k+3)(\ell+1)} \\ &\overset{(3.30)}{=} z_\mu^{\{k\}}(H_0) + \langle a_{\mu,0}^{\{k\}}\rangle(H_0) + OPS^{\beta-(k+2)(\ell+1)}. \end{aligned}$$

The claimed statement at the step $k + 1$ then follows by defining $z_\mu^{\{k+1\}}(H_0) := z_\mu^{\{k\}}(H_0) + \langle a_{\mu,0}^{\{k\}} \rangle (H_0)$. ☐

D. Elimination of time from $Z^{(3)}$ and preparation for KAM theory

In this section, we eliminate time from $Z^{(3)}$, and we get a system suitable for the application of the KAM Theorem 7.3 of Ref. 8.

First, we fix a $\tau > n - 1$, and for any $\gamma \in (0, 1)$, we define the set Ω_γ of Diophantine frequencies with constant γ by the following.

Definition 3.14. The frequencies ω belonging to the set

$$\Omega_\gamma := \left\{ \omega \in [1,2]^n \ : \ |\omega \cdot k| \geq \frac{\gamma}{|k|^\tau}, \ \forall k \in \mathbb{Z}^n \setminus \{0\} \right\} \tag{3.31}$$

are called Diophantine.

It is well known that $\left|\Omega - \Omega_\gamma\right| \leq C\gamma$ for a suitable positive constant C.

In the following, we will denote by $Lip\left(\Omega_\gamma; C^r(\mathbb{T}^n; \mathcal{B}(\mathcal{H}^s; \mathcal{H}^{s'}))\right)$ the space of Lipschitz functions from Ω_γ to $C^r(\mathbb{T}^n; \mathcal{B}(\mathcal{H}^s; \mathcal{H}^{s'}))$. Furthermore, we define

$$\langle z_\mu \rangle_\phi(E) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} z_\mu(E, \phi)\, d\phi, \tag{3.32}$$

where z_μ is given in Lemma 3.13. Note that $\langle z_\mu \rangle \in \widetilde{S}^\beta$. We now prove the following lemma.

Lemma 3.15. For $\omega \in \Omega_\gamma$, there exists a unitary (time-dependent) operator $U^{(4)}(\omega t)$ in L^2 which conjugates (3.22) into

$$H^{(4)}(\phi) := A_0 + \epsilon R_0(\phi), \tag{3.33}$$

where

$$A_0 := \mathrm{diag}_{j \geq 0}(\lambda_j^{(0)}(\omega)), \tag{3.34}$$

$$\lambda_j^{(0)}(\omega) = \lambda_j^v + \langle z_\mu \rangle_\phi(\lambda_j^v), \quad \forall j \geq 0. \tag{3.35}$$

Furthermore, one has

1. *$\forall r \geq 0$, the map $\phi \mapsto U^{(4)}(\phi)$ is of class $C^r(\mathbb{T}^n; B(\mathcal{H}^s; \mathcal{H}^{s-\beta r}))$.*
2. $\left\| U^{(4)}(\phi) - \mathbf{1} \right\|_{\mathcal{B}(\mathcal{H}^{s+\beta}; \mathcal{H}^s)} \leq C_s \epsilon \gamma^{-1}$.
3. *For all r, one has $R_0 := Lip\left(\Omega_\gamma; C^r(\mathbb{T}^n; \mathcal{B}(\mathcal{H}^s; \mathcal{H}^{s+\kappa-\beta r-1}))\right)$.*

Proof. The transformation is obtained by eliminating time from the operator $H_0 + \epsilon z_\mu(H_0)$. Note that this latter linear operator is diagonal. According to Lemma 3.13, we write the operator $H^{(3)}(\phi)$ in (3.22) as $H^{(3)}(\phi) = D(\phi) + \epsilon Q(\phi)$, where $D(\phi) := \mathrm{diag}_{j\geq 0}(\lambda_j^v + \epsilon z_\mu(\lambda_j^v, \phi))$ and $Q(\phi) := R^{(3)}(\phi) + \mathrm{diag}_{j \geq 0}\delta_j(\phi)$. We then conjugate $H^{(4)}$ by means of a transformation of the form $U^{(4)}(\phi) = \mathrm{diag}_{j\geq 0} e^{-\epsilon c_j(\phi)}$. By (3.1), the conjugated vector field is then given by

$$H^{(4)}(\phi) = \mathrm{diag}_{j\geq 0}\left(\lambda_j^v + \epsilon z_\mu(\lambda_j^v, \phi) - \epsilon\omega \cdot \partial_\phi c_j(\phi)\right) + \epsilon U^{(4)}(\phi)^{-1} Q(\phi) U^{(4)}(\phi).$$

In order to eliminate time dependence from the diagonal part of the operator $H^{(4)}(\phi)$, we consider the Fourier expansion

$$z_\mu(\lambda_j^v, \phi)(\phi) = \sum_{k \in \mathbb{Z}^n} \widehat{z}_\mu(\lambda_j^v, k) e^{\mathrm{i}k\cdot\phi}$$

and define

$$c_j(\phi) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{\widehat{z}_\mu(\lambda_j^v, k)}{\mathrm{i}\omega \cdot k} e^{\mathrm{i}k\cdot\phi}.$$

Then the operator $H^{(4)}(\phi)$ takes the form

$$H^{(4)}(\phi) = \mathrm{diag}_{j\geq 0}\Big(\lambda_j^v + \epsilon\langle z_\mu\rangle_\phi(\lambda_j^v)\Big) + \epsilon U^{(4)}(\phi)^{-1}Q(\phi)U^{(4)}(\phi)\,,$$

where we recall the definition (3.32). A simple analysis of the transformation $U^{(4)}(\phi)$ and of the remainder $R_0(\phi) := U^{(4)}(\phi)^{-1}Q(\phi)U^{(4)}(\phi)$ shows that the properties 1-3 are fulfilled. □

As a final step, we have a Lemma which shows that $H^{(4)}$ fulfills the assumptions of Theorem 7.3 of Ref. 8 which thus gives the result.

Lemma 3.16. For any positive γ, r, κ, there exists a set $\Omega_\gamma^{(0)} \subset \Omega_\gamma$ and positive constants a, C, ϵ_* such that, if $|\epsilon| < \epsilon_*$, then for any $\omega \in \Omega_\gamma^{(0)}$, the unitary (in L^2) operator $U_1 := U^{(1)}\circ U^{(2)}\circ U^{(3)}\circ U^{(4)}$ conjugates (1.2) to (3.33), and furthermore, the following properties hold:

$$\left|\Omega_\gamma \setminus \Omega_\gamma^{(0)}\right| \leq C\gamma^a, \tag{3.36}$$

$$A_0 := \mathrm{diag}(\lambda_j^{(0)}), \tag{3.37}$$

with $\lambda_j^{(0)} = \lambda_j^{(0)}(\omega)$ Lipschitz dependent on $\omega \in \Omega_\gamma$ and fulfilling the following inequalities [with $d = 2\ell/(\ell+1)$]:

$$\left|\lambda_j^{(0)} - \lambda_j^v\right| \leq C\epsilon j^{\frac{\beta}{l+1}}, \tag{3.38}$$

$$\left|\lambda_i^{(0)} - \lambda_j^{(0)}\right| \geq \frac{1}{C}\left|i^d - j^d\right|, \tag{3.39}$$

$$\left|\frac{\Delta(\lambda_i^{(0)} - \lambda_j^{(0)})}{\Delta\omega}\right| \leq C\epsilon|i^d - j^d|, \tag{3.40}$$

$$\left|\lambda_i^{(0)} - \lambda_j^{(0)} + \omega\cdot k\right| \geq \frac{\gamma(1+|i^d - j^d|)}{1+|k|^\tau}, \quad |i-j| + |k| \neq 0, \tag{3.41}$$

where, as usual, for any Lipschitz function f, we denoted $\Delta f = f(\omega) - f(\omega')$.

Furthermore, one has

1. *the map $\phi \mapsto U_1(\phi)$ is of class $C^r(\mathbb{T}^n; \mathcal{B}(\mathcal{H}^s; \mathcal{H}^{s-\beta r}))$.*
2. $\|U_1(\phi) - \mathbf{1}\|_{\mathcal{B}(\mathcal{H}^{s+\beta};\mathcal{H}^s)} \leq C_s\epsilon\gamma^{-1}$.
3. *For all r, one has $R_0 := Lip\Big(\Omega_\gamma; C^r(\mathbb{T}^n; \mathcal{B}(\mathcal{H}^s; \mathcal{H}^{s+\kappa-\beta r-1}))\Big)$.*

The proof is exactly as the proof of Lemma 5.2 of Ref. 8 to which we refer for the details.

We end the section by remarking that Theorem 2.10 is now an immediate consequence of the Theorem 7.3 of Ref. 8.

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