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Averaging theorems for NLS: probabilistic and deterministic results

MAT07

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Contents

Introduction

In this thesis, we study the dynamics of NLS, in particular, we deal with the problem of the construction of prime integrals, either in the probabilistic or in the deterministic case.

In the first part of the thesis, we consider the non linear Schrödinger equation on the one dimensional torus with a defocusing polynomial nonlinearity and we study the dynamics corresponding to initial data in a set of a large measure with respect to the Gibbs measure. We prove that along the corresponding solutions the modulus of the Fourier coefficients is approximately constant for long time. The proof is obtained by adapting to the context of Gibbs measure for PDEs some tools of Hamiltonian perturbation theory (see [6, 31, 21, 22]).

In the second part, we consider the nonlinear Schrödinger equation on the two dimensional torus with a time-dependent nonlinearity starting with cubic terms. In this case, using perturbation theory techniques, we construct an approximate integral of motion that changes slowly for initial data with small H^1-norm, this allows to ensure long time existence of solutions in $H^1(\mathbb{T}^2)$. The main difficulty is that $H^1(\mathbb{T}^2)$ is not an algebra.

We now describe more in detail the problem we study in the first part of the thesis which is also the main result of the thesis. The system we consider is the defocousing NLS on the one dimensional torus

$$i\dot{\psi} = -\Delta\psi + F'\left(|\psi|^2\right)\psi, \quad x \in \mathbb{T}, \tag{0.0.1}$$

where F is a polynomial of degree $q \geq 2$, $F(x) := \sum_{j=2}^{q} c_j x^j$, s.t. $F(x) \geq 0$ for any $x \geq 0$ and $c_2 \neq 0$. This is a Hamiltonian system with Hamiltonian H given by

$$H = H_2 + P \tag{0.0.2}$$

where

$$H_2 := \frac{1}{2}\int_0^{2\pi} |\nabla\psi(x)|^2 dx,$$

$$P = \sum_{j=2}^{q} H_{2j}, \quad H_{2j} := \frac{c_j}{2j}\int_0^{2\pi} |\psi(x)|^{2j} dx.$$

The associated Gibbs measure is formally defined by

$$d\mu_\beta = \frac{e^{-\beta\left(H(\psi)+\frac{1}{2}\|\psi\|_{L^2}^2\right)}}{Z(\beta)} d\psi d\bar{\psi},\ \beta > 0\ , \quad Z(\beta) := \int_{H^s} e^{-\beta\left(H(\psi)+\frac{1}{2}\|\psi\|_{L^2}^2\right)} d\psi d\bar{\psi} \tag{0.0.3}$$

where β plays the role of the inverse of the temperature.

The measure is supported on H^s space with $s < \frac{1}{2}$, so using Gibbs measure one actually studies solution with low regularity. The parameter β will be very large so the measure is concentrated on "small" data namely with size of order $\beta^{-\frac{1}{2}}$ and P can be thought as a small perturbation of H_2.

First, in Chapter 2, we recall the results of [30, 16, 18, 19, 25] that show that the Gibbs measure is well defined and invariant and furthermore that the flow of (0.0.1) is almost surely globally well-posed on any one of the spaces H^s with s s.t. $\frac{1}{2} - \frac{1}{q-1} < s < \frac{1}{2}$.

In Chapter 3, we prove our main result ([10]):

Theorem 0.0.1. *There exist $\beta^*, C, C' > 0$ s.t. for any $\eta_1, \eta_2 > 0$, β fulfilling*

$$\beta > \max\left\{\beta^*, \frac{C}{\eta_1^{\frac{10}{7}}\eta_2^{\frac{5}{7}}}\right\}$$

and any $\mathtt{k} \in \mathbb{Z}$, there exists a measurable set $\mathfrak{J}_\mathtt{k} \subset H^s$ with $\mu_\beta(\mathfrak{J}_\mathtt{k}^c) < \eta_2$ s.t., if the initial datum $\psi(0) \in \mathfrak{J}_\mathtt{k}$ then the solution exists globally in H^s and one has

$$\left|\frac{|\psi_\mathtt{k}(t)|^2 - |\psi_\mathtt{k}(0)|^2}{\frac{C'}{(1+\mathtt{k}^2)\beta}}\right| < \eta_1\ ,\quad \forall |t| < C'\eta_1\sqrt{\eta_2}\beta^{2+\varsigma}\ ,\ \varsigma = \frac{1}{10}\ . \tag{0.0.4}$$

Remark 0.0.2. The quantity $|\psi_\mathtt{k}|^2$ appears since it is the action of the linearized system. Theorem 0.0.1 shows that, for general initial data, $|\psi_\mathtt{k}|^2$ moves very little compared to its typical size over a time scale of order $\beta^{2+\varsigma}$.

Remark 0.0.3. If one considers (0.0.1) as a perturbation of the cubic integrable NLS, then one has that the main term of the perturbation is (in the equation) $|\psi|^4\psi$ whose size can be thought to be of order $\beta^{-5/2}$ which is of order β^{-2} smaller then the linear part. For this reason one can think that the effective perturbation is of size β^{-2}. So one expects to obtain a control of the dynamics of the actions over a time scale at least of order β^2.

Theorem 0.0.1, not only gives a rigorous proof of this fact, but also shows that this is true over a longer time scale. We do not expect the value of ς to be optimal.

Remark 0.0.4. In order to cover times longer than β^2, we have to face the problem of small denominators. Indeed the nonlinear corrections to the frequencies become relevant and an important part of the proof consists in giving an estimate of the measure of the phase space in which the nonlinear frequencies are nonresonant.

The proof of our result is based on the generalization to the context of Gibbs measure for PDEs of Poincaré's method of construction of approximate integrals of motion ([34, 26]). The standard way of using this method consists in first using a formal algorithm giving the construction of objects which are expected to be approximate integrals of motion and then adding estimates in order to show that this actually happens. This is the way we proceed. So, first, we develop a formal scheme of construction of the approximate integrals of motion. This is delicate due to the fact that the linearized system is completely resonant and we have to find a way to use the nonlinear modulation of the frequencies in order to control each one of the actions. So we obtain a function $\Phi_{\mathtt{k}}$ which is a modification of the action $|\psi_{\mathtt{k}}|^2$ and is expected to be an approximate integral of motion. In the second part of Chapter 3, we estimate the $L^2(\mu_\beta)$-norm of $\dot{\Phi}_{\mathtt{k}}$, showing that it is small. We remark that all the estimates can be done using the Gaussian measure associated to the linearized system that is absolutely continuous respect to the Gibbs measure. The main ingredient of this section is the exploitation of the decay of Fourier modes of functions in the support of the Gibbs measure. Finally we use the invariance of the Gibbs measure and Chebyshev's theorem in order to pass from the estimate of $\dot{\Phi}_{\mathtt{k}}$ to the estimate of $|\Phi_{\mathtt{k}}(t) - \Phi_{\mathtt{k}}(0)|$. Finally, we show that this implies the control of $|\psi_{\mathtt{k}}|^2$.

In the second part of the thesis, we study the following NLS system:

$$i\psi_t = -2\Delta\psi + 2a(x, \omega t)|\psi|^2\psi, \qquad x \in \mathbb{T}^2 \tag{0.0.5}$$

where a is a smooth function quasiperiodic in time and $\omega \in \mathbb{R}^d$.
We remark that equation (0.0.5) is Hamiltonian with Hamiltonian function given by

$$H(\psi, \omega t) = H_0(\psi) + H_1(\psi, \omega t), \quad H_0(\psi) = \int_{\mathbb{T}^2} |\nabla\psi|^2 dx \tag{0.0.6}$$

$$H_1(\psi, \omega t) = \int_{\mathbb{T}^2} a(x, \omega t)|\psi(x)|^4 dx. \tag{0.0.7}$$

As anticipated above, we construct an approximate integral which is a deformation of the H^1-norm ([7]). In dimension 1, this would be a trivial problem and a control of the solution over exponentially long time would be possible. However, the situation is much more complicated in dimension 2, since H^1 is not an algebra. As in Chapter 3, first we use a formal algorithm that gives the construction of the object which is expected to be approximate integral of motion and then we add estimates in order to show that this actually happens. The formal algorithm is quite standard, the difficulty comes from the fact that averaging involves here the study of the L^p-norms of the solution of the Schrödinger equation on $\mathbb{T}^2$. Here the main tool is the Bourgain's estimate

$$\|e^{it\Delta}\psi\|_{L^4_{tx}} \leq C\|\psi\|_{H^\epsilon},\ \forall\epsilon > 0$$

and the interpolation estimate that one deduces from it. Using such estimate together with some tools coming from Hamiltonian theory, denoting for any $K \in \mathbb{N}$ in the usual way

$$\|\cdot\|_{C^K(\mathbb{T}^d)} := \begin{cases} \sup_{\mathbb{T}^d} |\cdot| & \text{if } K = 0, \\ \sup_{\mathbb{T}^d} |\cdot| + \sum_{|\alpha|=1}^{K} \sup_{\mathbb{T}^d} |D^\alpha \cdot| & \text{if } K \neq 0, \end{cases}$$

we are able to make three steps of perturbation theory and to get the following

Theorem 0.0.5. *Assume that $a \in C^\infty(\mathbb{T}^{d+2})$ and that the frequency ω is Diophantine, namely that there exist γ and τ s.t.*

$$|\omega \cdot k + k_0| \geq \frac{\gamma}{1 + |k|^\tau}\ , \quad \forall (k, k_0) \in \mathbb{Z}^{n+1} \setminus \{0\}\ , \tag{0.0.8}$$

then, given $K \in \mathbb{N}$, there exist $\epsilon_*, C > 0$ and a functional $\Phi^{(3)} \in C^\infty(\mathbb{T}^d; H^1(\mathbb{T}^2, \mathbb{C}))$ with the following properties

$$\left|\frac{d}{dt}\Phi^{(3)}(\omega t, \psi(t))\right| \leq C\,\|\psi(t)\|_{H^1}^{10}\,, \tag{0.0.9}$$

$$\sup_{\|\psi\|_{H^1} < \epsilon_*} \left\|\Phi^{(3)}(\omega t, \psi) - H_0(\psi)\right\|_{C^K(\mathbb{T}^d)} \leq C\,\|\psi\|_{H^1}^4\,. \tag{0.0.10}$$

Theorem 0.0.6. *With the same assumptions and notations, if $\epsilon := \|\psi_0\|_{H^1} < \epsilon_*$, then the solution of (0.0.5) with initial data ψ_0 exists up to times t s.t. $|t| < \epsilon^{-6}$ and fulfills*

$$\|\psi(t)\|_{H^1} < 2\epsilon\ . \tag{0.0.11}$$

Part I

Probabilistic result

Chapter 1

Measures on infinite dimensional spaces

The contents of this chapter are largely based on some lectures given by Albeverio in Milan in 2015 and on [1, 33].

In statistical mechanics, to describe a system, moving from detailed information about a single particle, to global information, one uses a probabilistic approach. In particular, in the case of a Hamiltonian system one can use the Gibbs measure, but in general, in many problems of mathematics, physics and their applications studied from a probabilistic point of view one can define different measures on suitable phase-space, so heuristic integrals of the following form can arise:

$$\text{“}\int_\Gamma e^{-s\Phi(\gamma)} f(\gamma) d\gamma\text{”} \tag{1.0.1}$$

where Φ is a real-valued function lower bounded; f is a complex-valued function, γ is thought to be a member of some space Γ “on which the integration extends”, $d\gamma$ is a heuristic “flat measure”.

If Γ is finite dimensional, say $\Gamma = \mathbb{R}^n$, then $d\gamma$ is thought of as Lebesgue measure, everything is well known and the measure is absolutely continuous respect to the Lebesgue measure, instead if Γ is infinite dimensional, $d\gamma$ has no clear meaning. In this first chapter we explain how to give a sense to such expression and to the heuristic integral (1.0.1) in the case of an infinite dimensional space, since they arise in many areas of mathematics and physics, in particular in connection with the solution of partial differential equations like the Schrödinger equation.

In particular, in this first chapter, following [1, 33], we give only some results without proofs, about the non existence of an infinite dimensional measure analogous to the Lebesgue measure and about the construction of *abstract Wiener space*, to conclude with the presentation of Kolmogorov's Theorem about measures that gives us the possibility to give meaning to the Gaussian measure on $H^s(\mathbb{T})$, starting from a sequence of finite dimensional measures satisfying a suitable condition. This will be an essential point for the results of the next chapters.

1.1 Difference between Borel measures on finite and infinite dimensional Hilbert spaces

To construct probability measures in infinite dimensional, the first difficult is that there is not an analogous of the Lebesgue measure (a σ-additive Borel measure invariant under rotations or translation).

In particular, we study the case of $\mathcal{H}$, a (separable) Hilbert space, with norm $\|\cdot\|$, scalar product $\langle,\rangle$ and Borel σ-algebra $\mathfrak{B}(\mathcal{H})$. First, we recall the definition of *regular measure.*

Definition 1. Let $\mathcal{H}$ be a separable Hilbert space. A Borel measure μ on $(\mathcal{H}, \mathfrak{B}(\mathcal{H}))$ is called *regular* if for any $B \in \mathfrak{B}(\mathcal{H})$ we have

$$\mu(B) = \inf_{\substack{B\subseteq U\\ U \text{ open}}} \mu(U)$$

and

$$\mu(B) = \sup_{\substack{K\subseteq B\\ K \text{ compact}}} \mu(K).$$

In particular the following holds

Proposition 1.1.1. *Let $\mathcal{H}$ be a separable Hilbert space. Then any positive finite measure μ on $(\mathcal{H}, \mathfrak{B}(\mathcal{H}))$ is regular.*

Proof. See Lemma 26.2 of [13]. □

If the dimension of $\mathcal{H}$ is finite, the Lebesgue measure on Borel σ-algebra of $\mathcal{H}$ can be characterized as the (unique up to multiplicative constants) regular measure which is invariant under rotations and translations in $\mathcal{H}$ while if $\mathcal{H}$ is infinite dimensional, the following result holds (see [1, 33]):

Theorem 1.1.2. *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. Then there cannot exist a σ-additive Borel measure μ which is invariant under rotations (or translations) and assignes a positive finite value to any open ball.*

Remark 1.1.3. This results highlights the impossibility of the existence of any rotations or translations invariant regular σ-additive Borel measure on an infinite dimensional Hilbert space. Hence in infinite dimensions there cannot be a direct analogue of the standard Gaussian measure on $\mathbb{R}^n$, namely of the probability measure $\mu_G(dx) = \frac{e^{-\|x\|^2}}{(2\pi)^{\frac{n}{2}}}$, $x \in \mathbb{R}^n$. In particular, in infinite dimensional Hilbert space one have to do some work due to the loosing of σ-additivity.

1.2 Abstract Wiener spaces

In the present section we give some elements of the theory of abstract Wiener spaces.

Abstract Wiener spaces are mathematical objects used to construct a "good" measure on an infinite dimensional vector-space. Roughly speaking, they are triples $(i, \mathcal{H}, \mathcal{B})$ where $\mathcal{B}$ is a Banach space with norm $|\cdot|$, $\mathcal{H}$ is a real separable infinite dimensional Hilbert space with inner product $\langle , \rangle$ and norm $\|\cdot\|$ contains in $\mathcal{B}$ and i is the inclusion of $\mathcal{H}$ in $\mathcal{B}$ and it is a function that takes a *cylinder set measure* (that we will define later) on $\mathcal{H}$ to a true measure on $\mathcal{B}$.

In particular, we shall see that, given a real separable infinite dimensional Hilbert space $(\mathcal{H}, \langle , \rangle, \|\cdot\|)$, there exists a Banach space $(\mathcal{B}, |\cdot|)$ where $\mathcal{H}$ is densely embedded and a Borel measure on $\mathcal{B}$ whose Fourier transform is $\phi(x) = e^{-\frac{1}{2}\|x\|^2}$, where $x \in \mathcal{B}^* \subset \mathcal{H}$ and $\|\cdot\|$ is the $\mathcal{H}$-norm. Let us introduce some definitions.

Definition 2. A Gaussian measure on a Banach space $(\mathcal{B}, |\cdot|)$ is a probability measure on the Borel σ-algebra on $\mathcal{B}$ such that for each $x \in \mathcal{B}^*$, the random variable $x : \mathcal{B}(\mathbb{C}) \to \mathbb{R}$ has a Gaussian distribution on $\mathbb{R}(\mathbb{C})$.

Definition 3. A cylinder set $Z \subset \mathcal{H}$ of a separable Hilbert space $\mathcal{H}$ is a set of the form

$$Z = \{x \in \mathcal{H}, \text{ s.t. } Px \in F\}$$

with $P : \mathcal{H} \to \mathcal{H}$ is a projection operator on $\mathcal{H}$ with finite dimensional range, i.e. $P\mathcal{H} \equiv \mathbb{R}^n(\mathbb{C}^n)$ for some $n \in \mathbb{N}$, and $F \in \mathfrak{B}(P\mathcal{H})$ is a Borel set in $P\mathcal{H}$. In the following we shall denote by $\sigma(\mathcal{Z})$ the σ-algebra generated by all cylinder sets.

Definition 4. A *cylinder measure* on $\mathcal{H}$ is a positive and finitely additive set function ν defined on the σ-algebra $\sigma(\mathcal{Z})$ of cylinder sets.

Let us consider the *cylinder measure* ν on $\mathcal{H}$ given on the cylindrical sets of $\mathcal{H}$ by the following formula

$$\nu(\{x \in \mathcal{H}, \text{ s.t. } Px \in F\}) = (2\pi)^{-\frac{n}{2}} \int_F e^{-\frac{1}{2}\|Px\|^2} d(Px), \ F \in \mathcal{B}(P\mathcal{H})$$

ν is called *standard Gaussian measure* associated with $\mathcal{H}$.

Remark 1.2.1. By Theorem 1.1.2, if $\mathcal{H}$ is infinite dimensional, then the *standard Gaussian measure* associated with $\mathcal{H}$ is not σ-additive on $\sigma(\mathcal{Z})$, so we cannot work on $\mathcal{H}$ but we need to enlarge the space.

Definition 5. A norm $|\cdot|$ on $\mathcal{H}$ is called measurable if for any $\epsilon > 0$, there exists $P_\epsilon : \mathcal{H} \to \mathcal{H}$ s.t.

$$\nu(\{x \in \mathcal{H} \text{ s.t. } |P(x)| > \epsilon\}) < \epsilon, \tag{1.2.1}$$

for any $P : \mathcal{H} \to \mathcal{H}$ s.t. its range is orthogonal to that of P_ϵ in $(\mathcal{H}, \langle , \rangle)$.

Given $|\cdot|$ a measurable norm, we can construct $\mathcal{B}$ the Banach space as the completion of H in the $|\cdot|$-norm and i is the inclusion of $\mathcal{H}$ in $\mathcal{B}$ and it is continuous. Analogously, the dual map $i^* : \mathcal{B}^* \to \mathcal{H}^*$, which is given by restriction, i.e. $i^*(x) = x_{|\mathcal{H}}$, is continuous. Identifying $\mathcal{H} \equiv \mathcal{H}^*$ we have the following chain of densely embedded subspaces

$$\mathcal{B}^* \subset \mathcal{H} \subset \mathcal{B}.$$

The triple $(i, \mathcal{H}, \mathcal{B})$ is called an *abstract Wiener space.*

Let us consider a particular kind of cylinder sets in $\mathcal{H}$. Given $y_1, ..., y_n \in \mathcal{B}^*$, and $F \in \mathcal{B}(\mathbb{R}^n)$, let $Z_F(y_1, ..., y_n)$ be the subset of $\mathcal{H}$

$$Z_F(y_1, ..., y_n) := \{x \in \mathcal{H} \text{ s.t. } (i^*y_1(x), ..., i^*y_n(x)) \in F\}.$$

Analogously the subset of $\mathcal{B}$ defined as

$$\{x \in \mathcal{B} \text{ s.t. } (y_1(x), ..., y_n(x)) \in F\}, \tag{1.2.2}$$

is called a cylinder set of $\mathcal{B}$.

The following holds:

Theorem 1.2.2. *The σ-algebra on $\mathcal{B}$ generated by the cylinder sets of the form* (1.2.2) *coincides with the Borel σ-algebra on $\mathcal{B}$. Moreover the Gaussian measure μ on $\mathcal{B}$ is an extension of the standard Gaussian measure ν on H in the sense that*

$$\mu(\{x \in \mathcal{B} | (y_1(x), ..., y_n(x)) \in F\}) = \nu(\{x \in H | (i^*y_1(x), ..., i^*y_n(x)) \in F\}).$$

1.3 Kolmogorov's Theorem

In this section, we present Kolmogorov's Theorem, that is one of the basic tools for the construction of probability measures on infinite dimensional spaces and that guarantees that a suitably "consistent" collection of finite-dimensional distributions will define a unique probability measure on an infinite dimensional space. The original version of this theorem was established by Kolmogorov in the case where $\Gamma = \mathbb{R}^{[0,T]}$, but it was later generalized to "projective" limit spaces. We want to present here a sufficiently powerful version of the theorem but before we need a little introduction. Let $\Omega = \mathbb{R}^{[0,T]} = \{\gamma : [0,T] \to \mathbb{R}\}$ be the set of all maps from the interval $[0,T]$ into $\mathbb{R}$ and let $\mathcal{F}([0,T])$ be the set of all finite subsets of the interval $[0,T]$. We introduce in $\mathcal{F}([0,T])$ the partial order relation $\leq$ defined by

$$J \leq K \text{ if } J \subseteq K,$$

as a consequence of definition of $\mathcal{F}([0,T])$, for any $J, K \in \mathcal{F}([0,T])$, there is an $H \in \mathcal{F}([0,T])$ such that $J \leq H$ and $K \leq H$. Given a $J \in \mathcal{F}([0,T])$, with $J = \{t_1, t_2, ..., t_n\}$, $0 \leq t_1 < t_2... < t_n \leq T$, let us consider the set $\mathbb{R}^J$ of all maps from J to $\mathbb{R}$. An element of $\mathbb{R}^J$ is an n-ple $(\gamma(t_1), \gamma(t_2), ..., \gamma(t_n))$ and clearly $\mathbb{R}^J$ is naturally isomorphic to $\mathbb{R}^n$, n being the cardinality of J. Let us consider on $\mathbb{R}^J$ the Euclidean topology and the Borel σ- algebra $\mathfrak{B}(\mathbb{R}^J)$. For any $J \in \mathcal{F}([0,T])$ let us consider the projection $\Pi_J : \Omega \to \mathbb{R}^J$ which assigns to each path $\gamma \in \Omega$ its values at the points of J:

$$\gamma \mapsto \Pi_J(\gamma) = (\gamma(t_1), \gamma(t_2), ..., \gamma(t_n)),\ \gamma \in \mathbb{R}^{[0,T]},\ J = \{t_1, t_2, ..., t_n\}.$$

Let us consider the *cylinder sets*, i e. the subsets of Ω of the form $\Pi_J^{-1}(B_J)$ for some $J \in \mathcal{F}([0,T])$ and some Borel set $B_J \in \mathfrak{B}(\mathbb{R}^J)$. Let C denote the set of all cylinder sets, and let $\mathcal{A}$ be σ-algebra generated by the cylinder sets. Given a measure μ on $(\Omega, \mathcal{A})$, for any $J \in \mathcal{F}([0,T])$ it is possible to construct a measure μ_J on $(\mathbb{R}^J, \mathfrak{B}(\mathbb{R}^J))$ as $\mu_J := \Pi_J(\mu)$, i.e.

$$\mu_J(B_J) := \mu(\Pi_J^{-1}(B_J),\ B_J \in \mathfrak{B}(\mathbb{R}^J).$$

Given two elements $J, K \in \mathcal{F}([0,T])$, with $J \leq K$, let $\Pi_J^K : \mathbb{R}^K \to \mathbb{R}^J$ the projection map, which is continuous hence Borel measurable. The measures μ_J on $(\mathbb{R}^J, \mathfrak{B}(\mathbb{R}^J))$ and μ_K on $(\mathbb{R}^K, \mathfrak{B}(\mathbb{R}^K))$ are related by the equation $\mu_J = \Pi_J^K(\mu_K)$, that means

$$\mu_J(B_J) := \mu_K((\Pi_J^K)^{-1}(B_J)),\ B_J \in \mathfrak{B}(\mathbb{R}^J), \tag{1.3.1}$$

as one can verify by means of the equation $\Pi_J = \Pi_J^K \circ \Pi_K$.

Definition 6. A family of measures $\{\mu_J\}_{J\in\mathcal{F}([0,T])}$ satisfying the compatibility condition (1.3.1) is called a *projective family of measures.*

However, we are interested in the converse problem, in fact, knowing that there is a family of measures $\{\mu_J\}_{J\in\mathcal{F}([0,T])}$ satisfying the compatibility condition (1.3.1), we want to construct a measure μ on $(\mathbb{R}^{[0,T]}, A)$ such that for any $J \in \mathcal{F}([0,T])$ one has that $\mu_J = \Pi_J(\mu)$. Kolmogorov's Theorem guarantees that it is possible and that there exists an unique measure μ on $(\mathbb{R}^{[0,T]}, A)$ such that for any $J \in \mathcal{F}([0,T])$ one has that $\mu_J = \Pi_J(\mu)$. So the theorem guarantees that it is possible to construct a measure on the (infinite dimensional) space $\Omega = \mathbb{R}^{[0,T]}$ by means of its "finite dimensional approximations".

Theorem 1.3.1 (Kolmogorov's Theorem)**.** *For any projective family $\{\mu_J\}_{J\in\mathcal{F}([0,T])}$ of probability measures on $(\mathbb{R}^J, \mathcal{B}(\mathbb{R}^J))$ there exists a unique probability measure μ on $(\mathbb{R}^{[0,T]}, A)$ such that*

$$\mu_J = \Pi_J(\mu). \tag{1.3.2}$$

The measure μ described by Kolmogorov's Theorem is said the *projective limit of the projective family $\{\mu_J\}$.*

Remark 1.3.2. The result of Kolmogorov's Theorem can be generalized in several directions. In particular, an other version of the Kolmogorov's Theorem can be formulated as

Theorem 1.3.3. *Suppose that for each $n \geq 1$, μ_n is a Borel probability measure on $\mathbb{R}^n(\mathbb{C}^n)$ s.t. for every $n, k \geq 1$ and every Borel set $E \subset \mathbb{R}^n(\mathbb{C}^n)$ one has*

$$\mu_{n+k}\left(E \times \mathbb{R}^k(\mathbb{C}^k)\right) = \mu_n(E).$$

Then there exists a unique probability measure μ on the product σ-algebra of $\mathbb{R}^\infty(\mathbb{C}^\infty)$ such that for any $n \geq 1$ and any Borel subset $E \subset \mathbb{R}^n(\mathbb{C}^n)$, the measure

$$\mu(E \times \mathbb{R}(\mathbb{C}) \times \mathbb{R}(\mathbb{C})....) = \mu_n(E).$$

Remark 1.3.4. This last formulation of Kolmogorov's Theorem highlights the point of view we will use in the next section to construct some Gaussian measures on $H^s(\mathbb{T})$.

We conclude this section with the description of an important class of functions on $\mathbb{R}^{[0,T]}$ and of their integral with respect to the measure μ described by Kolmogorov's Theorem.

Definition 7. A function $f : \mathbb{R}^{[0,T]} \to \mathbb{C}$ of the form

$$f(\gamma) = g(\gamma(t_1), ..., \gamma(t_n)),\ \gamma \in \mathbb{R}^{[0,T]}, \tag{1.3.3}$$

with $0 \leq t_1 < ... < t_n \leq T$ and $g : \mathbb{R}^n \to \mathbb{C}$ is a Borel bounded function, is said cylinder function.

In particular, if $J = \{t_1, ..., t_n\}$ the cylinder function (1.3.3) can be written as $f = g \circ \Pi_J$. This representation provides an integration formula, indeed the integral of f with respect to the measure μ, the projective limit of the family of measures $\{\mu_J\}$, is given by:

$$\int_{\mathbb{R}^{[0,T]}} f(\gamma)d\mu(\gamma) = \int_{\mathbb{R}^{[0,T]}} g \circ \Pi_J(\gamma)d\mu(\gamma) = \int_{\mathbb{R}^J} g(x_1, ..., x_n)d\mu_j(x_1, ..., x_n).$$

1.4 Gaussian measures on $H^s(\mathbb{T})$

In this section we analyze more in detail the Gaussian measure on H^s. We consider

$$d\mu_{g,\sigma} = Z^{-1}e^{-\frac{1}{2}\|\psi\|^2_{H^\sigma}}d\psi d\bar{\psi}.$$

By Kolmogorov's Theorem, this can be seen as *projective limit* of

$$\begin{aligned} d\mu_{g,\sigma,N} &= Z_N^{-1}e^{-\frac{1}{2}\|P_{\leq N}\psi\|^2_{H^\sigma}}dP_{\leq N}\psi dP_{\leq N}\bar{\psi} \\ &= Z_N^{-1}\prod_{|n|\leq N} e^{-\frac{1}{2}k^{2\sigma}|\psi_k|^2}d\psi_k d\bar{\psi}_k. \end{aligned}$$

However we can not take a limit as $N \to \infty$ in $H^\sigma(\mathbb{T})$. In fact we have the following

Lemma 1.4.1. *Let $s < \sigma - \frac{1}{2}$, $M > N \geq 0$, then*

$$\mathbb{E}\left[\|P_{\leq M}\psi - P_{\leq N}\psi\|^2_{H^s}\right] \leq CN^\alpha, \tag{1.4.1}$$

where $\alpha = 2(\sigma - s) - 1 > 0$.
Moreover, if $s \geq \sigma - \frac{1}{2}$, $\mathbb{E}[\|P_{\leq M}\psi - P_{\leq N}\psi\|^2_{H^s}]$ is infinite.

Proof. Using Fourier coordinates, we have

$$\mathbb{E}\left[\|P_{\leq M}\psi - P_{\leq N}\psi\|^2_{H^s}\right] = \frac{\int_{\mathbb{C}^\infty}\sum_{N<|k|\leq M} k^{2s}|\psi_k|^2 e^{-\sum_k k^{2\sigma}|\psi_k|^2}d\psi_k d\bar{\psi}_k}{\int_{\mathbb{C}^\infty} e^{-\sum_k k^{2\sigma}|\psi_k|^2}d\psi_k d\bar{\psi}_k}.$$

Using the substitution $\psi_k = \frac{\sqrt{2z_k}}{(1+k^2)^{\frac{\sigma}{2}}}e^{i\theta_k}$ and the independence of the variables, one has that $\mathbb{E}[\|P_{\leq M}\psi - P_{\leq N}\psi\|^2_{H^s}]$ is equal to

$$\frac{\sum_{N<|k|\leq M} k^{2(s-\sigma)}\int_{\mathbb{R}^+} z_k e^{-z_k}dz_k \prod_{j\neq k}\int_{\mathbb{R}} z_j e^{-z_j}dz_j}{\prod_j \int_{\mathbb{R}^+} z_j e^{-z_j}dz_j} = \sum_{N<|k|\leq M} k^{2(s-\sigma)} < \infty$$

if and only if $s < \sigma - \frac{1}{2}$.
Moreover, if $s < \sigma - \frac{1}{2}$, then one has $\sum_{N<|k|\leq M} k^{2(s-\sigma)} \leq CN^{\alpha}$, where $\alpha = 2(\sigma - s) - 1 > 0$. □

So, if $s < \sigma - \frac{1}{2}$, then $\mu_{g,\sigma}$ is a probability measure on $H^s(\mathbb{T})$.

Remark 1.4.2. Lemma 1.4.1 implies that

$$\|P_{\leq M}\psi - P_{\leq N}\psi\|_{H^s} = \infty$$

only on a set of measure 0, so in particular this means that the subset of $\mathbb{C}^{\infty}$ of sequences $\{\psi_k\} = \psi$ that are not Cauchy sequences has measure 0. In particular this means that

$$\mu_{g,\sigma}(H^{s_1}) = 0$$

for any $s_1 \geq \sigma - \frac{1}{2}$.

The following lemma helps to understand well what is the support of $\mu_{g,\sigma}$.

Lemma 1.4.3. *Let $s < \sigma - \frac{1}{2}$, $a < \frac{1}{2}$, then*

$$\mu_{g,\sigma}\left(\{\|\psi\|_{H^s} > K\}\right) \leq Ce^{-aK^2}$$

for all $K > 0$.

Proof.

$$\begin{aligned}
e^{aK^2}\mu_{g,\sigma}\left(\{\|\psi\|_{H^s} < K\}\right) =& e^{aK^2}\int_{\{\|\psi\|_{H^s}\geq K\}} d\mu_{g,\sigma} \leq \int_{H^s} e^{a\|\psi\|^2_{H^s}} d\mu_{g,\sigma} \\
=& \frac{\int_{\mathbb{C}^{\infty}} e^{a\|\psi\|^2_{H^s}} e^{-\|\psi\|^2_{H^\sigma}} d\psi d\bar{\psi}}{\int_{\mathbb{C}^{\infty}} e^{-\|\psi\|^2_{H^\sigma}} d\psi d\bar{\psi}} \\
=& \frac{\int_{\mathbb{C}^{\infty}} e^{-\sum_k (1+k^2)^{\sigma}\left(1-\frac{a}{(1+k^2)^{\sigma-s}}\right)|\psi_k|^2} d\psi d\bar{\psi}}{\int_{\mathbb{C}^{\infty}} e^{-\sum_k (1+k^2)^{\sigma}|\psi_k|^2} d\psi d\bar{\psi}} \\
=& \prod_k \frac{\int_{\mathbb{R}^+} e^{-\left(1-\frac{a}{(1+k^2)^{\sigma-s}}\right)z_k} dz_k}{\int_{\mathbb{R}^+} e^{z_k^2} dz_k} \\
=& \prod_k \int_{\mathbb{R}^+} e^{-\left(1-\frac{a}{(1+k^2)^{\sigma-s}}\right)z_k} dz_k \\
=& \prod_k \left(1 + \frac{2a}{(1+k^2)^{\sigma-s} - 2a}\right) = C(s,\sigma)
\end{aligned}$$

where in the fourth line we use the substitution $\psi_k = \frac{\sqrt{2z_k}}{(1+k^2)^{\frac{\sigma}{2}}} e^{i\theta_k}$, $z_k \in \mathbb{R}^+$, $\theta_k \in [2\pi, 0)$ and the fact that $\int_{\mathbb{R}^+} e^{-z} dz = 1$. □

Remark 1.4.4. From the previous lemma, if K goes to $+\infty$, we obtain that for any $s < \sigma - \frac{1}{2}$,

$$\mu_{g,\sigma}\left(\{\|\psi\|_{H^s} = +\infty\}\right) = 0.$$

In particular, we obtain that, for any $s < \sigma - \frac{1}{2}$, $\mu_{g,\sigma}\left(H^s\right) = 1$, so the support of $\mu_{g,\sigma}$ is H^s for any $s < \sigma - \frac{1}{2}$.

Remark 1.4.5. One can describe the definition of Gaussian measure on $H^s(\mathbb{T})$ in terms of Wiener space, so $\mathcal{H} = H^1(\mathbb{T})$ and $\mathcal{B} = H^s(\mathbb{T})$ with $s < \frac{1}{2}$.

Chapter 2

Invariant measures for NLS

In this chapter, following the construction of Bourgain in [16], we construct the Gibbs measure associated to (0.0.1), showing that it is invariant under the dynamics of the flow of (0.0.1).

The proof of the invariance of the measure is complicated and the first step is the introduction of a formal definition of the *invariance* of a measure.

To this goal, initially, we consider a finite dimensional Hamiltonian system on $\mathbb{R}^{2n}$ with Hamiltonian $H(p,q) = H(p_1,...,p_n,q_1,...,q_n)$. The equations of the motion are

$$\begin{cases} \dot{p}_j = \frac{\partial H}{\partial q_j}, \\ \dot{q}_j = -\frac{\partial H}{\partial p_j} \end{cases} \quad j = 1,...,n, \tag{2.0.1}$$

or in a compact way

$$\dot{x} = X(x)$$

where $x = (p,q)$ and $X(x)$ is the Hamiltonian vector field. Since X is an Hamiltonian vector field one has

$$\begin{aligned} \text{div}X &= \sum_{j=1}^{n} \left[\frac{\partial}{\partial p_j} X_j + \frac{\partial}{\partial q_j} X_{j+n} \right] \\ &= \sum_{j=1}^{n} \left[\frac{\partial}{\partial p_j} \frac{\partial H}{\partial q_j} - \frac{\partial}{\partial q_j} \frac{\partial H}{\partial p_j} \right] = 0. \end{aligned}$$

By Liouville's theorem, we know that $\frac{d}{dt} Vol = \text{div}X$, so in particular, we obtain that $\frac{d}{dt} Vol = 0$ that means that the Lebesgue measure

$$dpdq = \prod_{j=1}^{n} dp_j dq_j$$

is invariant under the dynamics of (2.0.1). However, if we consider a different measure this way of proceed fails and to study the evolution of a measure under the dynamics it is convenient to introduce the following definition.

Definition 8. Given a measure space (Y, μ), we say that the measure is *invariant* under a μ-measurable transformation $T : Y \to Y$ if $\mu = \mu \circ T^{-1}$, i.e. for any μ-measurable set $A \subseteq Y$ one has $\mu(A) = \mu(T^{-1}(A))$.

In particular, defined the Gibbs measure associated to (2.0.1) as

$$d\mu_{\beta,H} = Z^{-1} e^{-\beta H(p,q)} dpdq \tag{2.0.2}$$

where β is the inverse of the temperature, we have

Lemma 2.0.1. *$\mu_{\beta,H}$ is invariant under the dynamics of* (2.0.1).

Proof. Denoting by $\Phi(t)$ the flow of (2.0.1) at time t, since $\frac{d}{dt}H = 0$, for any μ-measurable set $A \subset \mathbb{R}^{2n}$ one has

$$\begin{aligned}
\mu_{\beta,H}(\Phi(-t)A) =& \mu_{\beta,H}\left\{(p,q) \in \Phi(-t)A\right\} \\
=& \mu_{\beta,H}\left\{\Phi(t)(p,q) \in A\right\} = Z^{-1}\int_A e^{-\beta H(p(t)q(t))} dp(t)dq(t) \\
=& Z^{-1}\int_A e^{-\beta H(p(0)q(0))} dp(0)dq(0) = \mu_{\beta,H}(A)
\end{aligned}$$

where in the last line we use the invariance of H and of Lebesgue measure. □

Remark 2.0.2. With the same reason, in finite dimension and with a reasonable F conserved under the dynamics of (2.0.1), also $d\mu_F = Z^{-1}e^{-F(p,q)}dpdq$ is *invariant.*

Since the invariance of the measure depends on the invariance of the Hamiltonian H, a natural question is if one can construct also in infinite dimension some measures that are invariant under the flow of an Hamiltonian PDE. In finite dimension this is trivial due to the invariance of the Lebesgue measure, while in infinite dimension this is not so easy since there is not an analogous of the Lebesgue measure.

2.0.1 Construction of Gibbs measure for defocusing NLS on the torus

System (0.0.1) is a Hamiltonian system with Hamiltonian given by

$$H = H_2 + P \tag{2.0.3}$$

where

$$H_2 := \frac{1}{2}\int_0^{2\pi} |\nabla\psi(x)|^2 dx,$$

$$P = \sum_{j=2}^{q} H_{2j}, \quad H_{2j} := \frac{c_j}{2j}\int_0^{2\pi} |\psi(x)|^{2j} dx.$$

Note that the L^2-norm, i.e. $\int_0^{2\pi} |\psi(x)|^2 dx$, is conserved by (0.0.1). The flow of (0.0.1) is almost surely globally well-posed on any one of the spaces H^s with s fulfilling $\frac{1}{2} - \frac{1}{q-1} < s < \frac{1}{2}$ (see e.g. [16, 18], see also [25]). *We fix s in this range once for all.* In analogy with the finite dimensional case, the Gibbs measure associated to (0.0.1) is formally defined by

$$d\mu_\beta = \frac{e^{-\beta\left(H(\psi)+\frac{1}{2}\|\psi\|^2_{L^2}\right)}}{Z(\beta)} d\psi d\bar{\psi}, \ \beta > 0 \ , \quad Z(\beta) := \int_{H^s} e^{-\beta\left(H(\psi)+\frac{1}{2}\|\psi\|^2_{L^2}\right)} d\psi d\bar{\psi} \tag{2.0.4}$$

where β plays the role of the inverse of the temperature.

Remark 2.0.3. Instead of the Hamiltonian H, we consider the function $H(\psi) + \frac{1}{2}\|\psi\|^2_{L^2}$ to avoid the problems at frequency 0.

From now on, we shall work using the Fourier coordinates. In these coordinates, H_2 becomes

$$H_2 := \frac{1}{2}\sum_k k^2 |\psi_k|^2.$$

Define the H^1-norm:

$$\|\psi\|^2_{H^1} := \sum_k (1+k^2)|\psi_k|^2,$$

then we can express $H_2 + \frac{1}{2}\|\psi\|^2_{L^2} = \frac{1}{2}\|\psi\|^2_{H^1}$ and we formally defined the Gaussian measure by

$$d\mu_{g,\beta} := \frac{e^{-\frac{\beta}{2}\|\psi\|^2_{H^1}}}{Z_g(\beta)} d\psi d\bar{\psi}, \tag{2.0.5}$$

with

$$Z_g(\beta) := \int_{H^s} e^{-\frac{\beta}{2}\|\psi\|^2_{H^1}} d\psi d\bar{\psi}.$$

To give sense to this expression one can proceed as in Section 1.4 , seeing it as a projective limit of finite dimensional Gaussian measures

$$d\mu_{\beta,g,N} := \frac{e^{-\frac{\beta}{2}\|P_{\le N}(\psi)\|^2_{H^1}}}{Z_{g,N}(\beta)} dP_{\le N}\psi dP_{\le N}\bar{\psi} = \frac{e^{-\frac{\beta}{2}\sum_{|k|\le N}\left(1+k^2\right)|\psi_k|^2}}{Z_{g,N}(\beta)} dP_{\le N}\psi dP_{\le N}\bar{\psi},$$

$$Z_{g,N}(\beta) := \int_{P_{\leq N}(H^s)} e^{-\frac{\beta}{2}\sum_{|k|\leq N}\left(1+k^2\right)|\psi_k|^2} \prod_{|k|\leq N} d\psi_k d\bar{\psi}_k,$$

where $P_{\leq N}\left(\{\psi_k\}_{k\in\mathbb{Z}}\right) := \{\psi_k\}_{|k|\leq N}$.

Now, we can express (3.1.2) as

$$\begin{aligned} d\mu_\beta =& \frac{e^{-\beta\left(P+\frac{1}{2}\int_0^{2\pi}(|\nabla\psi(x)|^2+|\psi(x)|^2)dx\right)}}{Z(\beta)} d\psi d\bar{\psi} \\ =& \frac{e^{-\beta P}}{Z(\beta)} Z_g(\beta) \frac{e^{-\frac{\beta}{2}\|\psi\|^2_{H^1}}}{Z_g(\beta)} d\psi d\bar{\psi} = e^{-\beta P}\frac{Z_g(\beta)}{Z(\beta)}\mu_{g,\beta} d\psi d\bar{\psi}. \end{aligned} \tag{2.0.6}$$

As in section 1.4, one can prove that the support of the Gaussian measure $\mu_{g,\beta}$ is $H^s(\mathbb{T})$ for $s < \frac{1}{2}$, by Sobolev's inequality, we know that $\psi \in L^p(\mathbb{T})$ a.s. for any $p < \infty$, so in particular, due the definition of P and the fact that P is a positive function, one has

$$0 < e^{-\beta P(\psi)} \leq 1 \text{ a.s.} \tag{2.0.7}$$

Moreover, one can prove the following lemma which proof is in Appendix A.

Lemma 2.0.4. *There exist $\beta^*, \tilde{C} > 0$ s.t. for any $\beta > \beta^*$. one has*

$$1 \geq \int_{H^s} e^{-\beta P} d\mu_{g,\beta} \geq e^{-2\tilde{C}}. \tag{2.0.8}$$

In particular, since $\int_{H^s} e^{-\beta P} d\mu_{g,\beta} = \frac{Z(\beta)}{Z_g(\beta)}$, this means that if β is sufficiently large

$$1 \leq \frac{Z_g(\beta)}{Z(\beta)} \leq e^{2\tilde{C}}, \tag{2.0.9}$$

so, using (2.0.7) and (2.0.9), we can conclude that μ_β is a good probability measure on any H^s, $s < \frac{1}{2}$ for β large enough.

Remark 2.0.5. If P would not be a positive function, we could not obtain estimate (2.0.7) and (2.0.9). In that case we need to introduce an invariant cutoff in L^2-norm to ensure that $e^{-\beta P}\frac{Z_g(\beta)}{Z(\beta)} \in L^1(\mu_{g,\beta})$ and so to define the Gibbs measure.

One can get the following Lemma, which proof is in Appendix A, that shows how to control the Gibbs measure of set A with its Gaussian measure.

Lemma 2.0.6. *There exist $\beta^*, \tilde{C} > 0$ s.t. for any $\beta > \beta^*$ and for any function $\mu_{g,\beta}$-measurable set $A \subset H^s$, one has:*

$$\mu_\beta(A) \leq \mu_{g,\beta}(A)e^{\tilde{C}}.$$

We emphasize that the constant $\tilde{C}$ is independent of β and q, where q is the degree of the polynomial F (see (0.0.1)).

2.1 Truncated approximation to NLS and invariance of the Gibbs measure

After giving sense to the definition of the Gibbs measure, in this section we revisit some results of [16] to prove that μ_β is an invariant measure under the flow of (0.0.1). The main idea of this section is to use a local well posedness of the flow of (0.0.1) in H^s space with $s < \frac{1}{2}$ to construct a "finite" dimensional system that approximates (0.0.1), it will be local well posed and the associated "finite" dimensional Gibbs measure would be invariant. Using this fact, we can obtain the almost sure global well posedness of the "finite" dimensional system and finally the almost sure global well posedness of (0.0.1) and the invariance of μ_β under its dynamics.

Using Fourier coordinates, $\psi_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \psi(x) e^{-ikx} dx$, fixed $N \in \mathbb{N}$, we can denote by $P_{\leq N}$ the Dirichlet projection onto the frequencies $\{|n| \leq N\}$ and set $P_{>N} := \mathrm{Id} - P_{\leq N}$. We denote by

$$E_N = P_{\leq N} L^2(\mathbb{T}) = \mathrm{span}\{e^{inx} : |n| \leq N\},$$

$$E_N^{\perp} = P_{>N} L^2(\mathbb{T}) = \mathrm{span}\{e^{inx} : |n| > N\}.$$

We introduce now the following "truncated" system (FNLS)

$$i\dot{\psi}^N = -\Delta \psi^N + P_{\leq N}\left(F'\left(\left| P_{\leq N}\psi^N \right|^2 \right) P_{\leq N}\psi^N \right), \quad x \in \mathbb{T}, \tag{2.1.1}$$

that is an approximation of (0.0.1).

Remark 2.1.1. System (2.1.1) is not a finite dimensional system. It is an Hamiltonian system with Hamiltonian H_N given by

$$H_N = H_2 + \tilde{P} \tag{2.1.2}$$

where

$$H_2(\psi^N) := \frac{1}{2} \int_0^{2\pi} |\nabla \psi^N(x)|^2 dx,$$

$$\tilde{P}(\psi^N) = \sum_{j=2}^{q} \tilde{H}_{2j}(\psi^N), \quad \tilde{H}_{2j}(\psi^N) := \frac{c_j}{2j} \int_0^{2\pi} |P_{\leq N}\psi^N(x)|^{2j} dx$$

and the equation of motion are given by $\dot{\psi}^N = -i\frac{\partial H^N}{\partial \bar{\psi}^N}$.

One can get the following result about local existence of the flow of (0.0.1) and (2.1.1)

Proposition 2.1.2. *[Prop 3.1 of [19] (N. Burq, P.Gérard and N.Tzvetkov)] Let*

$$\frac{1}{2} - \frac{1}{q-1} < s < \frac{1}{2}.$$

Then, for any $\psi_0 \in H^s(\mathbb{T})$*, there exists* $T > 0$ *s.t.*

$$\begin{cases} i\dot{\psi} = -\Delta\psi + F'(|\psi|^2)\,\psi, & x \in \mathbb{T}, \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{2.1.3}$$

and

$$\begin{cases} i\dot{\psi}^N = -\Delta\psi^N + P_{\leq N}\left(F'\left(|P_{\leq N}\psi^N|^2\right) P_{\leq N}\psi^N\right), & x \in \mathbb{T}, \\ {\psi^N}_{|t=0} = \psi_0 \end{cases} \tag{2.1.4}$$

have respectively a unique solution

$$\psi, \psi^N \in C([-T,T], H^s(\mathbb{T})) \cap L^p([-T,T], L^\infty(\mathbb{T}))$$

for some $p > q-1$.
Moreover, there exists $\theta = \theta(q) > 0$*, s.t., given* $K > 0$*, for any* ψ_0 *s.t.* $\|\psi_0\|_{H^s} < K$*, one has that the corresponding solution* $\psi(t,x)$ *satisfies*

$$\|\psi(t,x)\|_{H^s(\mathbb{T})} < 2K$$

$$\|\psi^N(t,x)\|_{H^s(\mathbb{T})} < 2K$$

for any $|t| < T \sim \frac{1}{K^\theta}$.

In the following we will denote by Φ^t_{NLS} and by Φ^t_{FNLS} respectively the flow of (0.0.1) and of (2.1.1).

Moreover, we can see (2.1.1) as an infinite dimensional system of ODEs for the Fourier coefficients $\{\psi_k\}_{k\in\mathbb{Z}}$, where the high frequencies $\{|n| > N\}$ evolve linearly since they evolve according to the following equation:

$$\dot{\psi}^N_k = -ik^2\psi^N_k, \quad |k| > N. \tag{2.1.5}$$

One can introduce the following finite dimensional system of ODEs:

$$i\dot{\phi}^N = -\Delta\phi^N + P_{\leq N}\left(F'\left(|P_{\leq N}\phi^N|^2\right) P_{\leq N}\phi^N\right), \quad x \in \mathbb{T}, \tag{2.1.6}$$

with $\phi^N = P_{\leq N}\psi^N$, i.e. $\phi^N_k = 0$ for any $|k| > N$.

This is a Hamiltonian finite system with Hamiltonian $H_{N,low}$ given by

$$H_{N,low} = H_{2,low} + \tilde{P} \tag{2.1.7}$$

where

$$H_{2,low}(\phi^N) := \frac{1}{2}\int_0^{2\pi} |\nabla\phi^N(x)|^2 dx,$$

$$\tilde{P}(\phi^N) = \sum_{j=2}^{q} \tilde{H}_{2j}(\phi^N), \quad \tilde{H}_{2j}(\phi^N) := \frac{c_j}{2j}\int_0^{2\pi} |P_{\leq N}\phi^N(x)|^{2j} dx.$$

We denote by $\Phi_{FNLS_{low}}(t,\tau)$ the solution maps of (2.1.6) sending initial data at time τ to solutions at time t, for simplicity, we set

$$\Phi^t_{FNLS_{low}} := \Phi_{FNLS_{low}}(t,0).$$

Since $\Phi_{FNLS_{low}}(t,\tau)$ preserves the L^2-norm ($\|\phi^N\|^2_{L^2} = \sum_{|k|\leq N} |\phi^N_k|^2$) that is the Euclidean distance on $\mathbb{C}^N$, the flow of (2.1.6) is globally well posed.

Remark 2.1.3. Since $P_{>N}\psi^N$ evolves linearly, the flow of system (2.1.5) is globally well posed, so in particular, we have that the flow of (2.1.1) is globally well posed for any $N \in \mathbb{N}$ and in particular we have the following relations:

$$\Phi^t_{FNLS} = \Phi^t_{FNLS_{low}} P_{\leq N} + P_{>N} \text{ and } P_{\leq N}\Phi^t_{FNLS} = \Phi^t_{FNLS_{low}} P_{\leq N}.$$

However, given $\psi_0 \in H^s$, denote $\psi^N(t) := \Phi_{FNLS}(t)\psi_0$, there is no uniform control in N on $\|\psi^N(t)\|_{H^s}$ for any time.

The next lemma, that will be proved in Appendix B, shows how the truncated system (2.1.1) approximates system (0.0.1) as N goes to $+\infty$.

Lemma 2.1.4. *[Approximation Lemma]*
Let $K > 0, T > 0, \psi_0 \in H^s$, with $\|\psi_0\|_{H^s} \leq K$. Suppose that for any N, $\psi^N(t) = \Phi^t_{FNLS}\psi_0$ satisfies

$$\|\Phi^t_{FNLS}(\psi_0)\|_{H^s} \leq K, \ |t| \leq T.$$

Then, there exists an unique solution $\psi(t) := \Phi^t_{NLS}\psi_0$ to (0.0.1) on $[-T,T]$ with initial data ψ_0. Moreover, given $0 < s_1 < s$,

$$\|\psi(t) - \psi^N(t)\|_{H^{s_1}} = \|\Phi^t_{NLS}\psi_0 - \Phi^t_{FNLS}\psi_0\|_{H^{s_1}} \to 0, \quad \text{as } N \to \infty. \tag{2.1.8}$$

Using the definition of E_N and $E_N^\perp$, one can write the Gaussian measure $\mu_{g,\beta}$ on $L^2(\mathbb{T})$ as

$$\mu_{g,\beta} = \mu_{g,\beta,N} \otimes \mu^\perp_{g,\beta,N} \tag{2.1.9}$$

where $\mu_{g,\beta,N}$ and $\mu^{\perp}_{g,\beta,N}$ are the marginal distribution of $\mu_{g,\beta}$ restricted onto E_N and $E_N^{\perp}$ respectively. So, formally, we can write

$$d\mu_{g,\beta,N} = \frac{e^{-\frac{\beta}{2}\|P_{\leq N}\psi\|^2_{H^1}}}{Z_{g,N}(\beta)} dP_{\leq N}\psi, \qquad Z_{g,N}(\beta) := \int_{\mathbb{C}^N} e^{-\frac{\beta}{2}\|P_{\leq N}\psi\|^2_{H^1}} dP_{\leq N}\psi \tag{2.1.10}$$

$$d\mu^{\perp}_{g,\beta,N} = \frac{e^{-\frac{\beta}{2}\|P_{>N}\psi\|^2_{H^1}}}{\hat{Z}_{g,N}(\beta)} dP_{>N}\psi, \qquad \hat{Z}_{g,N}(\beta) := \int_{P_{>N}(H^s)} e^{-\frac{\beta}{2}\|P_{>N}\psi\|^2_{H^1}} dP_{>N}\psi. \tag{2.1.11}$$

We consider now the finite dimensional system (2.1.6), the associated Gibbs measure $\mu_{\beta,N,low}$ is given by

$$d\mu_{\beta,N,low} = \frac{e^{-\beta\left(\tilde{P}(\phi)+\frac{1}{2}\|\phi\|^2_{H^1}\right)}}{Z_{N,low}(\beta)} d\phi d\bar{\phi}, \ \beta > 0 \ , \tag{2.1.12}$$

$$Z_{N,low}(\beta) := \int_{\mathbb{C}^N} e^{-\beta\left(\tilde{P}(\phi)+\frac{1}{2}\|\phi\|^2_{H^1}\right)} d\phi d\bar{\phi}. \tag{2.1.13}$$

Since $\tilde{P}(\phi) + \frac{1}{2}\|\phi\|^2_{H^1} = H_{N,low}(\phi) + \frac{1}{2}\|\phi\|^2_{L^2}$ is invariant under the dynamics of (2.1.6), by Liouville's Theorem, one has that $\mu_{\beta,N,low}$ is invariant under the flow of (2.1.6). Moreover, $\mu^{\perp}_{g,\beta,N}$ is invariant under the flow of (2.1.5) since $P_{>N}\psi$ evolves linearly and in particular $|\psi^N_k(t)|^2 = |\psi^N_k(0)|^2$ for any $|k| > N$, $t > 0$.

We can now define the Gibbs measure for the truncated system (2.1.1) as

$$\mu_{\beta,N} := \mu_{\beta,N,low} \otimes \mu^{\perp}_{g,\beta,N}. \tag{2.1.14}$$

So, explicitly, we can express $d\mu_{\beta,N}$ in the following way

$$\begin{aligned} d\mu_{\beta,N} =& \frac{e^{-\beta\left(\tilde{P}(\psi)+\frac{1}{2}\int_0^{2\pi}(|\nabla\psi(x)|^2+|\psi(x)|^2)dx\right)}}{Z_N(\beta)} d\psi d\bar{\psi} \\ =& \frac{e^{-\beta\tilde{P}(\psi)}}{Z_N(\beta)} Z_g(\beta) \frac{e^{-\frac{\beta}{2}\|\psi\|^2_{H^1}}}{Z_g(\beta)} d\psi d\bar{\psi} = e^{-\beta\tilde{P}(\psi)} \frac{Z_g(\beta)}{Z_N(\beta)} \mu_{g,\beta} d\psi d\bar{\psi}, \end{aligned} \tag{2.1.15}$$

where

$$Z_N(\beta) := \int_{H^s} e^{-\beta\left(\tilde{P}(\psi)+\frac{1}{2}\|\psi\|^2_{H^1}\right)} d\psi d\bar{\psi} \tag{2.1.16}$$

$$= \int_{H^s} e^{-\beta\left(\sum_{j=2}^q \frac{c_j}{2j}\int_0^{2\pi}|P_{\leq N}\psi(x)|^{2j}dx+\frac{1}{2}\|\psi\|^2_{H^1}\right)} d\psi d\bar{\psi}. \tag{2.1.17}$$

As in the case of μ_β, with the same reasoning, we can obtain the following

Lemma 2.1.5. *There exists constants $\beta^*, \tilde{C} > 0$ s.t. for any $\beta > \beta^*$ and for any $N \in \mathbb{N}$, one has*

$$1 \geq \int_{H^s} e^{-\beta \tilde{P}} d\mu_{g,\beta} \geq e^{-2\tilde{C}}. \tag{2.1.18}$$

In particular, this means that for any $N \in \mathbb{N}$

$$1 \leq \frac{Z_g(\beta)}{Z_N(\beta)} \leq e^{2\tilde{C}}, \tag{2.1.19}$$

so we can conclude that $\mu_{\beta,N}$ is a good probability measure on any H^s for any $N \in \mathbb{N}$.

The proof of this Lemma and of the next Lemmas of this section are contained in Appendix A.

Remark 2.1.6. Using the definition of $\mu_{\beta,N}$, the invariance of $\mu_{\beta,N,low}$ under the flow of (2.1.6) and the invariance of $\mu^{\perp}_{g,\beta,N}$ under the flow of (2.1.5), we conclude that $\mu_{\beta,N}$ is invariant under the flow of the truncated system (2.1.1).

Lemma 2.1.7. *There exist $\beta^*, \tilde{C} > 0$ s.t. for any $\beta > \beta^*$, for any $N \in \mathbb{N}$ and for any $\mu_{g,\beta}$-measurable set $A \subset H^s$, one has:*

$$\mu_{\beta,N}(A) \leq \mu_{g,\beta}(A) e^{\tilde{C}}.$$

We emphasize that the constant $\tilde{C}$ is independent of β, N and q, where q is the degree of the polynomial F (see (0.0.1)).

Lemma 2.1.8. *There exists $C > 0$ s.t. for any $\epsilon, \beta > 0$, there exists $N_0 \in \mathbb{N}$ s.t. for any $N > N_0$ and any μ_β-measurable set $A \in H^s(\mathbb{T})$, one has*

$$|\mu_{\beta,N}(A) - \mu_\beta(A)| < \epsilon e^{C\left(1+\frac{1}{\beta}\right)}. \tag{2.1.20}$$

2.1.1 Almost sure global well posedness for NLS

First, in this section we present a result due to Bourgain in [16] that shows that the flow of (2.1.1) is well posed except for a set of small measure, using this result we get the μ_β-almost sure global well posedness for the flow of (0.0.1).

For any $M > 0$, we denote by

$$B_M := \{\psi \in H^s(\mathbb{T}) : \|\psi\|_{H^s} \leq M\}.$$

and we have the following Lemma, which proof is in Appendix A

Lemma 2.1.9. *For any $s_1 < \frac{1}{2}$ there exists a constant $C > 0$ s.t. for any $\beta > 0$, $a < \frac{1}{2}$, $M > 0$, one has*

$$\mu_\beta\left(\{\|\psi\|_{H^{s_1}} > M\}\right) \leq C(s_1)e^{-a\beta M^2}.$$

So, we get the following Lemma about the well posedness of the flow of (2.1.1) except for a set of small measure.

Lemma 2.1.10. *For any $T < \infty, \epsilon > 0, N \in \mathbb{N}$, $\beta \geq 1$, there exists $\Omega_N = \Omega_N(T, \epsilon)$ s.t.*

- $\mu_{\beta,N}(\Omega_N^c) < \epsilon$.
- *For $\psi_0 \in \Omega_N$, there exists an unique solution $\Phi^t_{FNLS}(\psi_0)$ to (2.1.1) s.t.*

$$\|\Phi^t_{FNLS}(\psi_0)\|_{H^s} \lesssim \left(\frac{\log\frac{T}{\epsilon}}{\beta}\right)^{\frac{1}{2}}, \ |t| \leq T. \tag{2.1.21}$$

We emphasize that estimate (2.1.21) is independent of N.

Proof. From local theory, there exists $\theta = \theta(q) > 0$ (q is the degree of the polynomial F (see (0.0.1))) s.t. for any $M > 0$ and for any initial data $\psi_0 \in B_M$ one has that $\Phi^t_{NLS}(\psi_0)$ and $\Phi^t_{FNLS}(\psi_0)$ are locally well-posed on $[-\delta, \delta]$, $\delta \sim (1+M)^{-\theta}$ uniformly in N (see Prop. 2.1.2 and Prop. 3.1 of [19]). In particular we know that for any $N \in \mathbb{N}$ and for any $t \in [-\delta, \delta]$, $\Phi^t_{NLS}(\psi_0), \Phi^t_{FNLS}(\psi_0) \in B_{2M}$.
We define the set

$$\Omega_N := \cap_{j=-[\frac{T}{\delta}]}^{[\frac{T}{\delta}]}\Phi^j_{FNLS}\left(B_M\right). \tag{2.1.22}$$

So

$$\mu_{\beta,N}\left(\Omega_N^c\right) \leq \sum_{j=-[\frac{T}{\delta}]}^{[\frac{T}{\delta}]} \mu_{\beta,N}\left(\Phi^j_{FNLS}\left(B_M^c\right)\right),$$

using the invariance of the measure, one gets

$$\mu_{\beta,N}\left(\Phi^t_{FNLS}\left(B_M^c\right)\right) = \mu_{\beta,N}\left(\left(B_M^c\right)\right),$$

so in particular

$$\begin{aligned}\mu_{\beta,N}\left(\Omega_N^c\right) \leq& 2\left[\frac{T}{\delta}\right]\mu_{\beta,N}\left(B_M^c\right) \leq 2e^{\tilde{C}}\left[\frac{T}{\delta}\right]\mu_{g,\beta}\left(B_M^c\right)\\ \leq& 2e^{\tilde{C}}\left[\frac{T}{\delta}\right]e^{-aM^2\beta} \lesssim TM^\theta e^{-aM^2\beta},\end{aligned} \tag{2.1.23}$$

where in the first line we use Lemma 2.1.7 and in the last line we use Lemma 2.1.9.

Choosing

$$M \sim \left(\frac{\log \frac{T}{\epsilon}}{\beta} \right)^{\frac{1}{2}},$$

one obtains that

$$\mu_{\beta,N}\left(\Omega_N^c\right) < \epsilon.$$

By construction, we have that for any $\psi_0 \in \Omega_N$,

$$\|\Phi^{j\delta}_{FNLS}(\psi_0)\|_{H^s} \lesssim M,\ j = 0, \pm 1, ..., \pm \left[\frac{T}{\delta}\right],$$

so, by local theory, we have that for any $\psi_0 \in \Omega_N$,

$$\|\Phi^{t}_{FNLS}(\psi_0)\|_{H^s} \lesssim 2M \sim 2\left(\frac{\log \frac{T}{\epsilon}}{\beta} \right)^{\frac{1}{2}},\ |t| \leq T.$$

□

Using this result and the Lemma 2.1.4, one can obtain the following result that gives the well posedness existence of the flow of (0.0.1) except for a set of small measure.

Lemma 2.1.11. *For any $T < \infty, \epsilon > 0$, $\beta \geq 1$, there exists $\Omega = \Omega(T, \epsilon)$ and $C > 0$, independent of ϵ, β, T. s.t.*

- $\mu_\beta(\Omega^c) < \epsilon$.
- *For $\psi_0 \in \Omega$, there exists an unique solution ψ to (0.0.1) on $[-T;T]$ s.t.*

$$\|\Phi^{t}_{NLS}(\psi_0)\|_{H^s} \lesssim \left(\frac{\log \frac{T}{\epsilon}}{\beta} \right)^{\frac{1}{2}},\ |t| \leq T. \tag{2.1.24}$$

Proof. Let $\Omega_N(T, \epsilon)$ as in Lemma 2.1.10. By Lemma 2.1.10, one has

$$\|\Phi^{t}_{FNLS}(\psi_0)\|_{H^s} \leq 2M$$

for any $|t| \leq T$ and any $\psi_0 \in \Omega_N(T, \epsilon)$. By Lemma 2.1.4, given $0 < s_1 < s$, there exists $N_1 \in \mathbb{N}$ s.t.

$$\|\Phi^{t}_{NLS}(\psi_0) - \Phi^{t}_{NLS}(\psi_0)\|_{H^{s_1}} \ll 1,\ |t| \leq T,$$

for any $N \geq N_1$. So, in particular, one has

$$\|\Phi^t_{NLS}(\psi_0)\|_{H^{s_1}} \lesssim M \sim \left(\frac{\log \frac{T}{\epsilon}}{\beta}\right)^{\frac{1}{2}}, \ |t| \leq T. \tag{2.1.25}$$

Moreover, by Lemma 2.1.8 and using the fact that $\beta \geq 1$

$$\mu_\beta \left(\Omega^c_N(T,\epsilon)\right) \leq \mu_{\beta,N} \left(\Omega^c_N(T,\epsilon)\right) + \epsilon e^C \leq 2\epsilon e^C. \tag{2.1.26}$$

□

Using this result, we obtain the following theorem that gives the μ_β-almost sure global well posedness for the flow of (0.0.1).

Theorem 2.1.12. *For $\beta \geq 1$, system* (0.0.1) *is μ_β-almost sure global well posed.*

Proof. Given $\epsilon > 0$, let $T_j = 2^j$, $\epsilon_j = \frac{\epsilon}{2^j}$.
We define

$$\Omega_j := \Omega_{T_j,\epsilon_j}, \ \ \Omega_\epsilon := \cap_{j=1}^\infty \Omega_j. \tag{2.1.27}$$

Then one has

- $\mu_\beta \left(\Omega^c_\epsilon\right) \leq \sum_{j=1}^\infty \mu_\beta \left(\Omega^c_j\right) \leq \sum_{j=1}^\infty \frac{\epsilon}{2^j} = \epsilon.$
- If $\psi_0 \in \Omega_\epsilon$, then there exists an unique solution ψ on $[-T_j, T_j]$ for any $j \in \mathbb{N}$, so in particular we have global solution for any $\psi_0 \in \Omega_\epsilon$.

In particular, denoting by

$$\Sigma = \cup_{\epsilon>0}\Omega_\epsilon,$$

one has

- $\mu_\beta \left(\Sigma^c\right) = \inf_{\epsilon>0} \epsilon = 0.$
- If $\psi_0 \in \Sigma$, then $\psi_0 \in \Omega_\epsilon$ for some $\epsilon > 0$, so, in particular, there exists a unique global solution $\psi(t)$ with $\psi_{|t=0} = \psi_0$.

□

2.1.2 Invariance of Gibbs measure under NLS

Finally, always following [16] and some lectures by Ho, using the μ_β-almost global well posedness of the flow of (0.0.1), we obtain the main result of this section about the invariance of μ_β under the flow of (0.0.1).

Theorem 2.1.13. *The Gibbs measure μ_β is invariant under the flow of* (0.0.1).

Proof. Since $\mu_\beta(\Sigma^c) = 1$ and the flow of (0.0.1) is reversible, to obtain the invariance of the Gibbs measure under the flow of (0.0.1), it is sufficient to prove that for any μ_β-measurable set $A \subset \Sigma$ and for any $t \in \mathbb{R}$, one has

$$\mu_\beta(A) \le \mu_\beta\left(\Phi^t_{NLS}(A)\right). \tag{2.1.28}$$

We denote by $\mathcal{F} := \{F \subset H^s, F \text{ closed }\}$.

We consider a μ_β-measurable set $A \subseteq \Sigma$, by inner regularity, there exists a sequence $\{F_n\} \in \mathcal{F}$ s.t. $F_n \subseteq A$ and $\mu_\beta(A) = \lim_{n\to\infty} \mu_\beta(F_n)$, namely

$$\mu_\beta(A) = \sup_{\substack{F\subseteq A\\ F\in\mathcal{F}}} \mu_\beta(F).$$

This implies that to obtain the invariance of the measure it is sufficient to prove (2.1.28) for closed sets. In fact if (2.1.28) holds for closed sets, then

$$\begin{aligned}\mu_\beta(A) &= \lim_{n\to\infty} \mu_\beta(F_n)\\ &\le \limsup_{n\to\infty} \mu_\beta(F_n)\\ &\le \limsup_{n\to\infty} \mu_\beta\left(\Phi^t_{NLS}(F_n)\right)\\ &\le \mu_\beta\left(\Phi^t_{NLS}(A)\right)\end{aligned}$$

where the last line is true since $F_n \subset A$. Given a closed set $F \subset H^s$ and $s < \sigma < \frac{1}{2}$, we denote by

$$K_n := \{\psi \in F : \|\psi\|_{H^\sigma} \le n\}.$$

Then K_n is a compact set in H^s. We have

$$\begin{aligned}\mu_\beta(F) &= \lim_{n\to\infty} \mu_\beta(K_n)\\ &\le \limsup_{n\to\infty} \mu_\beta(K_n)\\ &\le \limsup_{n\to\infty} \mu_\beta\left(\Phi^t_{NLS}(K_n)\right)\\ &\le \mu_\beta\left(\Phi^t_{NLS}(F)\right)\end{aligned}$$

where the last line is true since $K_n \subseteq F$, so to prove the invariance of the Gibbs measure, it is sufficient to prove (2.1.28) for compact sets.

Let K be a compact set in H^s, bounded in H^σ, $\sigma < s$. By Lemma 2.1.8, we know that $\mu_{\beta,N} \rightharpoonup \mu_\beta$ but, by Portmanteau's theorem, this implies that, for any $\epsilon > 0$, one has

$$\mu_\beta\left(\Phi^t_{NLS}(K) + \bar{B}_\epsilon\right) \geq \limsup \mu_{\beta,N}\left(\Phi^t_{NLS}(K) + \bar{B}_\epsilon\right). \tag{2.1.29}$$

So, using local theory and Lemma 2.1.4, we get that for $|t| \ll 1$ and any $0 < \epsilon_1 \ll 1$, there exists $0 < \epsilon \ll 1$ and $N_0 \in \mathbb{N}$ s.t. for any $N > N_0$, one has

$$\begin{aligned}\Phi^t_{FNLS}\left(K + B_{\epsilon_1}\right) \subset & \Phi^t_{FNLS}\left(K\right) + B_{\frac{\epsilon}{2}} \\ \subset & \Phi^t_{NLS}\left(K\right) + B_\epsilon,\end{aligned} \tag{2.1.30}$$

where the first inclusion is true by local theory, while the second is true by Lemma 2.1.4. So, in particular we get

$$\mu_{\beta,N}\left(\Phi^t_{FNLS}\left(K + B_{\epsilon_1}\right)\right) \leq \mu_{\beta,N}\left(\Phi^t_{NLS}\left(K\right) + B_\epsilon\right),$$

but using the invariance of $\mu_{\beta,N}$ respect to Φ^t_{FNLS}, we get also

$$\mu_{\beta,N}\left(K + B_{\epsilon_1}\right) = \mu_{\beta,N}\left(\Phi^t_{FNLS}\left(K + B_{\epsilon_1}\right)\right) \leq \mu_{\beta,N}\left(\Phi^t_{NLS}\left(K\right) + B_\epsilon\right). \tag{2.1.31}$$

Hence,

$$\begin{aligned}\mu_\beta\left(K\right) \leq \mu_\beta\left(K + B_{\epsilon_1}\right) \leq & \liminf_N \mu_{\beta,N}\left(K + B_{\epsilon_1}\right) \\ \leq & \liminf_N \mu_{\beta,N}\left(\Phi^t_{FNLS}(K) + B_\epsilon\right) \\ \leq & \limsup_N \mu_{\beta,N}\left(\Phi^t_{FNLS}(K) + \bar{B}_\epsilon\right) \\ \leq & \mu_\beta\left(\Phi^t_{NLS}(K) + \bar{B}_\epsilon\right),\end{aligned}$$

where we get the third inequality by (2.1.31) and the last inequality is true for (2.1.29). So, finally, sending ϵ to 0, we get

$$\mu_\beta\left(K\right) \leq \mu_\beta\left(\Phi^t_{NLS}(K)\right). \tag{2.1.32}$$

So, we obtain the thesis for compact sets, so, due to the previous observations, we get the thesis for closed sets and then for any measurable sets in H^s. $\square$

Chapter 3

A large probability averaging Theorem for the defocusing NLS

Introduction to Chapter 3

This chapter is devoted to the proof of Theorem 0.0.1 and we will follow [10]. For completeness, we report Theorem 0.0.1.

Theorem 3.0.1. *There exist $\beta^* > 1, C, C' > 0$ s.t. for any $\eta_1, \eta_2 > 0$, β fulfilling*

$$\beta > \max\left\{\beta^*, \frac{C}{\eta_1^{\frac{10}{7}}\eta_2^{\frac{5}{7}}}\right\}$$

and any $\mathtt{k} \in \mathbb{Z}$, there exists a measurable set $\mathfrak{J}_{\mathtt{k}} \subset H^s$ with $\mu_\beta(\mathfrak{J}_{\mathtt{k}}^c) < \eta_2$ s.t., if the initial datum $\psi(0) \in \mathfrak{J}_{\mathtt{k}}$ then the solution exists globally in H^s and one has

$$\left|\frac{|\psi_{\mathtt{k}}(t)|^2 - |\psi_{\mathtt{k}}(0)|^2}{\frac{C'}{(1+\mathtt{k}^2)\beta}}\right| < \eta_1 \ , \quad \forall |t| < C'\eta_1\sqrt{\eta_2}\beta^{2+\varsigma} \ , \ \varsigma = \frac{1}{10} \ . \tag{3.0.1}$$

Remark 3.0.2. The expectation value of $\psi_{\mathtt{k}}$ is $C_1/\sqrt{(1+\mathtt{k}^2)\beta}$, with a suitable constant C_1.

Remark 3.0.3. This results shows that in Gibbs measure, for large β, with high probability, the single k-action changes very little during the motion respect to its expectation value and for large time.
In fact, for example, if we consider

$$\eta_1 = \eta_2^{1/2} \text{ and } \beta = (C+1)\eta_2^{-20} \gg \beta^*,$$

we get that, for all initial datum $\psi(0) \in \mathcal{J}_{\mathtt{k}}$, with $\mu_\beta(\mathcal{J}^c_{\mathtt{k}}) < \frac{(C+1)^{1/20}}{\beta^{1/20}}$, one has

$$\left| \frac{|\psi_{\mathtt{k}}(t)|^2 - |\psi_{\mathtt{k}}(0)|^2}{\frac{C'}{(1+\mathtt{k}^2)\beta}} \right| < \frac{(C+1)^{1/40}}{\beta^{1/40}} \ , \ \ \forall |t| < C' \beta^{2+\frac{1}{20}}. \tag{3.0.2}$$

Corollary 3.0.4. *Under the same assumption of Theorem 0.0.1 and for any $\alpha < 1/2$, there exists a measurable set $\mathfrak{I}_\alpha \subset H^s$ with $\mu_\beta(\mathfrak{I}^c_\alpha) < \eta_2$ s.t., if the initial datum $\psi(0) \in \mathfrak{I}_\alpha$ then the solution exists globally in H^s and one has*

$$\left| \frac{|\psi_{\mathtt{k}}(t)|^2 - |\psi_{\mathtt{k}}(0)|^2}{[(1+\mathtt{k}^2)^\alpha \beta]^{-1}} \right| < \eta_1 \quad , \ \ \forall |t| < C' \eta_1 \sqrt{\eta_2} \beta^{2+\varsigma} \quad , \quad \forall \mathtt{k} \in \mathbb{Z}, \quad \varsigma = \frac{1}{10}. \tag{3.0.3}$$

Corollary 3.0.4 controls all the actions at the same time at the prize of giving a slightly worst control on the actions with large index.

Theorem 0.0.1 is essentially an averaging theorem for perturbations of a linear resonant system.

We recall that previous results giving long time stability of the actions in (0.0.1) have been obtained in [3] and [17]. The first two results allow to control the dynamics for exponentially long times, but only for initial data close in energy norm to some finite dimensional manifold, so essentially for a very particular set of initial data. Bourgain [17] was able to exploit the nonlinear modulation of the frequencies in order to show that for most (in a suitable sense, not related to Gibbs measure) initial data in H^s with $s \gg 1$ the Sobolev norm of the solution is controlled for times longer then any inverse power of the small parameter.

Nothing is known for solutions with low regularity as those dealt with in the present thesis and in [10].

Our result can be compared also to the result of Huang Guan [28], who proved a large probability averaging theorem for perturbations of KdV equation. We emphasize that the result of [28] deals with the quite artificial case in which the perturbation is smoothing, namely it maps functions with some regularity into functions with higher regularity. In our case we deal with the natural local perturbation given by a polynomial in ψ. Furthermore [28] only deals with smooth solution. We also recall [29] in which a weaker version of averaging theorem is obtained for solutions of some NLS-type equations. In that paper the initial datum is required to be more regular that in Theorem 0.0.1 and the times covered are shorter.

Finally we mention the papers [8, 9, 4] which deal with very smooth initial data and perturbations of nonresonant linear system. These results are clearly in a context very different from ours.

As anticipated in the introduction, first, we develop a formal scheme of construction of the approximate integrals of motion which is slightly different from the standard one. This is due to the fact that the linearized system is completely resonant and we have to find a way to use the nonlinear modulation of the frequencies in order to control each one of the actions. We have also to restrict our construction to the region of the phase space in which the frequencies are nonresonant. This is obtained by eliminating (through cutoff functions) the regions of the phase space where the linear combinations of the frequencies that are met along the construction are smaller than δ, where δ is a parameter that will be determined at the end of the construction.

Once we obtained a function $\Phi_{\mathtt{k}}(\psi)$ close to $|\psi_{\mathtt{k}}|^2$ which is expected to be an approximate integral of motion, we need to estimate its derivative in $L^2(\mu_\beta)$ showing that it is small. To this end, we first recall that all the estimates can be done by working with the Gaussian measure associated to the linearized system, then we introduce the class of functions which will be needed for the construction. Then we show how to control the $L^2(\mu_\beta)$ norm of such functions. Essentially using the decay of the Fourier mods of functions in the support of Gibbs measure, we show that the integral of a function of our class on the resonant region is small with δ. Then we choose δ to minimize the $L^2(\mu_\beta)$ norm of $\dot{\Phi}_{\mathtt{k}}$. Finally, using the invariance of Gibbs measure, we prove Theorem 0.0.1.

3.1 Preliminaries

We recall that the system (0.0.1) is a Hamiltonian system with Hamiltonian H given by

$$H = H_2 + P \tag{3.1.1}$$

where

$$H_2 := \frac{1}{2}\int_0^{2\pi} |\nabla\psi(x)|^2 dx,$$

$$P = \sum_{j=2}^{q} H_{2j}, \quad H_{2j} := \frac{c_j}{2j}\int_0^{2\pi} |\psi(x)|^{2j} dx.$$

We consider the Gibbs measure μ_β associated to this Hamiltonian, which is known to be invariant with respect to Φ^t_{NLS} ([16, 30, 39, 38]) and that is formally defined as

$$d\mu_\beta = \frac{e^{-\beta\left(H(\psi)+\frac{1}{2}\|\psi\|^2_{L^2}\right)}}{Z(\beta)} d\psi d\bar{\psi}, \ \beta > 0 \ , \quad Z(\beta) := \int_{H^s} e^{-\beta\left(H(\psi)+\frac{1}{2}\|\psi\|^2_{L^2}\right)} d\psi d\bar{\psi} \tag{3.1.2}$$

where β plays the role of the inverse of the temperature.

Given a function $f : H^s \to \mathbb{C}$, $f \in L^2(H^s, \mu_\beta)$, we define its average and its L^2-norm with respect to the measure μ_β as:

$$\langle f \rangle := \int_{H^s} f d\mu_\beta,$$

$$\|f\|^2_{\mu_\beta} := \int_{H^s} |f|^2 d\mu_\beta.$$

Remark 3.1.1. From the invariance of μ_β, one has that the average $\langle f \rangle$ and the L^2-norm $\|f\|_{\mu_\beta}$ of the functions are preserved along the flow, namely $\langle f \circ \Phi^t_{NLS} \rangle = \langle f \rangle$, $\|f \circ \Phi^t_{NLS}\|_{\mu_\beta} = \|f\|_{\mu_\beta}$ for any t.

Given a function $f : H^s \to \mathbb{C}$, we denote by

$$\|f\|^2_{g,\beta} := \int_{H^s} |f|^2 d\mu_{g,\beta}$$

its L^2-norm respect to $\mu_{g,\beta}$. From now on, we shall work using the Fourier coordinates. The following lemmas will be proved in Appendix A.

Lemma 3.1.2. *There exist $\beta^*, \tilde{C} > 0$ s.t. for any $\beta > \beta^*$ and for any function $f \in L^2(H^s, \mu_{g,\beta})$, one has:*

$$\|f\|_{\mu_\beta} \leq \|f\|_{g,\beta} e^{\tilde{C}}.$$

We emphasize that the constant $\tilde{C}$ is independent of β and q.

Lemma 3.1.3. *For any $\frac{q-1}{2q} < s_1 < \frac{1}{2}$, there exists $C_{sob}, D' > 0$ s.t. for any $\beta > 0$ and any function $f \in L^2(H^s, \mu_{g,\beta})$, one has*

$$\|f\|_{\mu_\beta} \geq e^{-\frac{C_{sob}}{2\beta} q \max_j c_j D'^j} \left\| f \chi_{\left\{\|\psi\|_{H^{s_1}} < \frac{D'}{\beta}\right\}} \right\|_{g,\beta}$$

where $\chi_{\{U\}}(\psi)$ is the characteristic function of the set U.

The next lemma shows that every moment of μ_β is well defined.

Lemma 3.1.4. *There exists $\beta^* > 0$ s.t., for any $s_1 < \frac{1}{2}$, $n \in \mathbb{N}$, $\beta > \beta^*$, one has*

$$\|\psi\|^n_{H^{s_1}} \in L^1(H^s, \mu_\beta) \cap L^1(H^s, \mu_{g,\beta}).$$

Finally, for the special case of the function $|\psi_{\mathtt{k}}|^2$, that is the $k-$action of the linearized system, we have the following lemma.

Lemma 3.1.5. *There exists $\beta^* > 0, C > 0$ s.t. for any $\beta > \beta^*$ s.t.*

$$\left\| |\psi_{\mathtt{k}}|^2 \right\|_{\mu_\beta} \geq \frac{C}{\beta\,(1 + \mathtt{k}^2)}.$$

3.2 Polynomials with frequency dependent coefficients

In this section we introduce a class of function on H^s which will be stable under the perturbative construction and we prove some results needed for the rest of the proof.

Definition 9. Let B_1, B_2 be two Banach spaces, we say that $F(y) : B_1 \to B_2$ is a polynomial of degree n if there exists a n-multilinear form $\tilde{F}$ s.t. for any $y \in B_1$, one has $F(y) = \tilde{F}(\underbrace{y, y, ..., y}_{n})$.

Remark 3.2.1. In particular a polynomial $f : H^s \to \mathbb{C}$ of degree n has the form:

$$f(\psi) = \sum_{l,m} \psi^l \bar{\psi}^m f_{l,m} \tag{3.2.1}$$

where $l = \{l_k\}$, $m = \{m_k\}$, $l_k, m_k \in \mathbb{N}$, $\sum_k l_k + m_k = n$, $f_{l,m} \in \mathbb{C}$, $\psi^l = ...\psi_{-k}^{l_{-k}}...\psi_k^{l_k}...$ and the same for $\bar{\psi}^m$.

Definition 10. We say that a polynomial f of the form (3.2.1) of degree $2n$ is of class P_{2n} if it fulfills the *null momentum* condition, i.e.

$$f_{l,m} \neq 0 \text{ only if } \sum_{k \in Supp(l)} k = \sum_{k \in Supp(m)} k \text{ and } \sum_k l_k = \sum_k m_k = n. \tag{3.2.2}$$

On P_{2n}, we introduce the following norm

$$|||f||| := \sup_{l,m} |f_{l,m}| \,. \tag{3.2.3}$$

Remark 3.2.2. In the following, due to (3.2.2), we will write a polynomial $f \in P_{2n}$ also in the equivalent following form, more convenient in a lot of situations

$$f(\psi) = \sum_{\substack{k=(k_1,\ldots,k_{2n}) \\ \sum_{i=1}^n k_i = \sum_{i=n+1}^{2n} k_i}} f_k \prod_{i=1}^n \psi_{k_i} \bar{\psi}_{k_{i+n}}. \tag{3.2.4}$$

The next lemma shows that the polynomials of class P_{2n} are smooth polynomials on H^{s_1}, $\frac{1}{2} - \frac{1}{n} < s_1 < \frac{1}{2}$.

Lemma 3.2.3. *Let n be a positive integer and s_1 s.t. $\frac{1}{2} - \frac{1}{2n} < s_1 < \frac{1}{2}$, $f \in P_{2n}$, then there exists $C(s_1, n) > 0$ s.t.*

$$|f(\psi)| \leq C(s_1, n) \|\psi\|_{H^{s_1}}^{2n} |||f|||. \tag{3.2.5}$$

Proof.

$$\begin{aligned}|f(\psi)| &\leq \sum_{\substack{k_1,\ldots,k_{2n}\\ \sum_{i=1}^n k_i=\sum_{i=n+1}^{2n} k_i}} |f_{k_1,\ldots,k_{2n}}| \prod_{i=1}^{2n} |\psi_{k_i}| \\ &\leq |||f||| \sum_{\substack{k_1,\ldots,k_{2n}\\ \sum_{i=1}^n k_i=\sum_{i=n+1}^{2n} k_i}} \prod_{i=1}^{2n} |\psi_{k_i}|.\end{aligned}$$

We define $\varphi := \{\varphi_k\} := \{|\psi_k|\}$, $\tilde{\varphi} := \sum_k \varphi_k e^{ikx}$, so, using Sobolev's embedding $H^{s_1} \subset L^{2n}$ for $\frac{1}{2} - \frac{1}{2n} < s_1 < \frac{1}{2}$, one has:

$$\begin{aligned}|f(\psi)| \leq |||f||| \sum_{\substack{k_1,\ldots,k_{2n}\\ \sum_{i=1}^n k_i=\sum_{i=n+1}^{2n} k_i}} \prod_{i=1}^{2n} \varphi_{k_i} &= \|\tilde{\varphi}\|_{L^{2n}}^{2n} |||f||| \\ \leq C(s_1,n)\|\tilde{\varphi}\|_{H^{s_1}}^{2n} |||f||| &= C(s_1,n)\|\psi\|_{H^{s_1}}^{2n} |||f|||.\end{aligned}$$

□

We will also consider the functions $f \in C^r(\ell^1, P_{2n})$, $f : \ell^1 \ni \omega = \{\omega_j\} \to f(\psi,\omega) = \sum_{\substack{k=(k_1,\ldots,k_{2n})\\ \sum_{i=1}^n k_i=\sum_{i=n+1}^{2n} k_i}} f_k(\omega) \prod_{i=1}^n \psi_{k_i}\bar{\psi}_{k_{i+n}}$. In the following ω_j will be the nonlinear modulation of the j-th frequency.
Actually we need to keep the information of the size of the different derivative of f. So, we give the following definition.

Definition 11. We will say that $f \in P^r(2n, \{A_i\}_{i=0}^r)$ if $f \in C^r(\ell^1, P_{2n})$ and

$$\sup_{\substack{\omega,k\\ |j|=i}} \left| \frac{\partial^{|j|} f_k(\omega)}{\partial \omega^j} \right| < A_i, \quad \forall i = 0, ..., r.$$

Remark 3.2.4. $\mathrm{Max}_i A_i$ is a norm for $C^r(\ell^1, P_{2n})$.

Given a function $f \in C^r(\ell^1, P_{2n})$, we also consider

$$f_{ph}(\psi) := f(\psi, |\psi|^2),$$

conversely, we will say that $\tilde{f} : H^s \to \mathbb{C}$ is of class $P^r(2n, \{A_i\}_{i=0}^r)$ if there exists a function $F(\psi,\omega) \in P^r(2n, \{A_i\}_{i=0}^r)$ s.t. $F(\psi,\omega)_{|_{\omega=\{|\psi_k|^2\}}} = \tilde{f}(\psi)$.

Remark 3.2.5. If $f \in P_{2n}$ with $|||f||| < \infty$, then $f \in P^\infty(2n, \{A_i\}_{i=0}^\infty)$ with $A_0 = |||f|||$ and $A_i = 0$ for any $i \geq 0$. For simplicity, we will write $f \in P^\infty(2n, |||f|||)$.

Remark 3.2.6. From Lemma 3.2.3, for any $n \in \mathbb{N}$ and for any s_1 s.t. $\frac{1}{2} - \frac{1}{2n} < s_1 < \frac{1}{2}$, for any $r \geq 0$ and for any $f \in P^r(2n, \{A_i\}_{i=0}^r)$, one has

$$|f(\psi)| \leq A_0 C(s_1, n) \|\psi\|_{H^{s_1}}^{2n}. \tag{3.2.6}$$

The connection of the norm of $P^0(2n, A_0)$ and the L^2-norm is given by

Lemma 3.2.7. *Let n be an integer, denoted by $C_g(n) := 2^{n+2}[(2n)!]^{\frac{3}{2}}(2n-1)^2 \left(\sum_l \frac{1}{1+l^2}\right)^n$, for any $\beta > 0$, and $f_{ph} \in P^0(2n, A_0)$, one has*

$$\|f_{ph}\|_{g,\beta} \leq \frac{A_0 C_g(n)}{\beta^n}. \tag{3.2.7}$$

Proof. Writing $f_{ph} = \sum_{k=(k_1,\dots,k_{2n})} f_k(\psi) \prod_{i=1}^n \psi_{k_i} \bar{\psi}_{k_{n+i}}$, one has

$$\|f_{ph}\|_{g,\beta}^2 = \int_{H^s} |f_{ph}|^2 d\mu_{g,\beta} = \int_{H^s} \sum_{k,j} f_k(\psi) \bar{f}_j(\psi) \prod_{i=1}^n \psi_{k_i} \psi_{j_{n+i}} \bar{\psi}_{j_i} \bar{\psi}_{k_{n+i}} d\mu_{g,\beta}. \tag{3.2.8}$$

Let s_1 be s.t. $\max\left\{s, \frac{n-1}{2n}\right\} < s_1 < \frac{1}{2}$, by Lemma 3.2.3, there exists a constant C s.t. $|f|^2 \leq C A_0^2 \|\psi\|_{H^{s_1}}^{4n}$, moreover by Lemma 3.1.4, $\|\psi\|_{H^{s_1}}^{4n} \in L^1(H^s, \mu_{g,\beta})$. So we can exchange the order between the integral and the series and (3.2.8) becomes

$$\sum_{k,j} \int_{H^s} f_k(\psi) \bar{f}_j(\psi) \prod_{i=1}^n \psi_{k_i} \psi_{j_{n+i}} \bar{\psi}_{j_i} \bar{\psi}_{k_{n+i}} d\mu_{g,\beta} =$$

$$\sum_{k,j} \frac{\int_{H^s} f_k(\psi) \bar{f}_j(\psi) \prod_{i=1}^n \psi_{k_i} \psi_{j_{n+i}} \bar{\psi}_{j_i} \bar{\psi}_{k_{n+i}} e^{-\frac{\beta}{2} \sum_{S_{kj}} (1+l^2)|\psi_l|^2} \prod_{S_{kj}} d\psi_l d\bar{\psi}_l}{\prod_{S_{kj}} \int_{H^s} e^{-\frac{\beta}{2}(1+l^2)|\psi_l|^2} d\psi_l d\bar{\psi}_l} \tag{3.2.9}$$

where $S_{kj} := Supp(k, j)$. It is useful to use the following notation: given a set K of indices $(k_1, \dots, k_{2n})$ with an even number of components, we denote

$$K_1 := \{k_1, \dots, k_n\}, \quad K_2 := \{k_{n+1}, \dots, k_{2n}\}.$$

Using the substitution $\psi_l = \frac{\sqrt{2 z_l}}{\sqrt{\beta(1+l^2)}} e^{i\theta_l}$, $z_l \in \mathbb{R}^+$, $\theta_l \in [0, 2\pi)$, one has that the only integrals different from 0 are the terms in which $K_1 \cup J_2 = K_2 \cup J_1$.

We denote by $\mathcal{T}$ the set of (k, j) s.t. $K_1 \cup J_2 = K_2 \cup J_1$ and with both k and j fulfilling the zero momentum condition, namely $\sum_{i=1}^n k_i = \sum_{i=n+1}^{2n} k_i$, $\sum_{i=1}^n j_i = \sum_{i=n+1}^{2n} j_i$. Thus (3.2.9) is bounded by

$$A_0^2 \sum_{k,j \in \mathcal{T}} \frac{2^{2n}}{\beta^{2n} \prod_{i=1}^n (1+k_i^2)(1+j_{n+i}^2)} \int \prod_{i=1}^n z_{k_i} z_{j_{i+n}} e^{-\sum_{S_{kj}} z_l} \prod_{S_{kj}} dz_l$$

$$\leq A_0^2 \frac{2^{2n}(2n)!}{\beta^{2n}} \sum_{k,j\in\mathcal{T}} \frac{1}{\prod_{i=1}^n \left(1+k_i^2\right)\left(1+j_{n+i}^2\right)}.$$

So,

$$\|f_{ph}\|_{g,\beta}^2 \leq \frac{A_0^2 2^{2n}(2n)!}{\beta^{2n}} \sum_{(k,j)\in\mathcal{T}} \frac{1}{\prod_{i=1}^n \left(1+k_i^2\right)\left(1+j_{n+i}^2\right)}. \tag{3.2.10}$$

Since we sum on $(k,j)\in\mathcal{T}$, we have that, having fixed $K_1 \cup J_2 = K_2 \cup J_1$ we have $(2n)!$ way to rearrange $K_1 \cup J_2$ and $(2n)!$ way to rearrange $K_2 \cup J_1$, so

$$\sum_{(k,j)\in\mathcal{T}} \frac{1}{\prod_{i=1}^n \left(1+k_i^2\right)\left(1+j_{n+i}^2\right)} \leq [(2n)!]^2 \sum_{\substack{k_1,\dots,k_n,\\ j_{n+1},\dots,j_{2n}}} \frac{1}{\prod_{i=1}^n \left(1+k_i^2\right)\left(1+j_{n+i}^2\right)}$$

$$= [(2n)!]^2 \left(\sum_l \frac{1}{1+l^2}\right)^{2n}.$$

So, finally,

$$\|f_{ph}\|_{g,\beta}^2 \leq \frac{A_0^2 2^{2n}[(2n)!]^3 \left(\sum_i \frac{1}{1+i^2}\right)^{2n}}{\beta^{2n}} \leq \frac{A_0^2 C_g^2(n)}{\beta^{2n}}$$

with $C_g(n)^2 := 2^{2n+4}[(2n)!]^3(2n-1)^4 \left(\sum_l \frac{1}{1+l^2}\right)^{2n}$.

□

Remark 3.2.8. According to Lemma 3.1.2, one also has

$$\|f_{ph}\|_{\mu_\beta} \leq \frac{A_0 C_g(n)}{\beta^n}. \tag{3.2.11}$$

The Poisson brackets of two functions f, g with $f \in P_{2n}$ and $g \in P^r\left(2m, \{A_i\}_{i=0}^r\right)$ is formally, given by

$$\{f,g\} := L_f(g) := -i\sum_k \left(\frac{\partial f}{\partial \psi_k}\frac{\partial g}{\partial \bar{\psi}_k} - \frac{\partial g}{\partial \psi_k}\frac{\partial f}{\partial \bar{\psi}_k}\right).$$

Remark 3.2.9. If $f \in P_n$, $g \in P_m$, then

$$\{f,g\} \in P_{n+m-2}.$$

Lemma 3.2.10. *Consider $f \in P_{2n}$, $|||f||| < D$, $g_{ph} \in P^r\left(2m, \{A_i\}_{i=0}^r\right)$. Then*

$$\{f, g_{ph}\} = F_1 + F_2, \tag{3.2.12}$$

where

$$F_1 \in P^r\left(2n+2m-2, 2nmD\{A_i\}_{i=0}^r\right), \tag{3.2.13}$$

$$F_2 \in P^{r-1}(2n+2m, 2nD\{A_{i+1}\}_{i=0}^{r-1}). \tag{3.2.14}$$

Proof. Writing $g_{ph} = \sum_{k=(k_1,...k_{2m})} g_k\left(\{|\psi_k|^2\}\right)\psi_{k_1}...\psi_{k_m}\bar{\psi}_{k_{m+1}}...\bar{\psi}_{k_{2m}}$, then it is immediate to verify that (3.2.12) holds with

$$F_1 = \sum_{k=(k_1,...k_{2m})} g_k(\{|\psi_j|^2\})\left\{f, \psi_{k_1}...\psi_{k_m}\bar{\psi}_{k_{m+1}}...\bar{\psi}_{k_{2m}}\right\}$$

$$F_2 = \sum_{k=(k_1,...k_{2m})} \psi_{k_1}...\psi_{k_m}\bar{\psi}_{k_{m+1}}...\bar{\psi}_{k_{2m}}\left\{f, g_k(\{|\psi_j|^2\})\right\} =$$

$$= \sum_{k=(k_1,...k_{2m})} \left(\sum_l \frac{\partial g_k(\{|\psi_j|^2\})}{\partial \omega_l}\right)\psi_{k_1}...\psi_{k_m}\bar{\psi}_{k_{m+1}}...\bar{\psi}_{k_{2m}}\left\{f, |\psi_l|^2\right\}$$

and, by Remark 3.2.9, $F_1 \in P^r\left(2n+2m-2, 2nmD\{A_i\}_{i=0}^r\right)$ and $F_2 \in P^{r-1}(2n+2m, 2nD\{A_{i+1}\}_{i=0}^{r-1})$ hold.

□

Actually, we shall use a more particular class of functions in which the range of the indices is subject to a further restriction. This is related to the fact that in our construction we shall fix an index $\mathtt{k}$ corresponding to the action we want to conserve. To this end, we introduce the following definition:

Definition 12. Given $M > 0$, $\mathtt{k} \in \mathbb{Z}$, a linear combination

$$G(k_1, ..., k_{2n}) := \sum_{i=1}^{2n} a_i k_i$$

with $a_i \in \mathbb{Z}$, $|a_i| \leq M$, we will say that the relation

$$G(k_1, ..., k_{2n}) = \mathtt{k}$$

is $(M, \mathtt{k})$-admissible.

Lemma 3.2.11. *Given $D > 0$, let be $f \in P_{2n}$, $|||f||| < D$, $g_{ph}(\psi, \bar{\psi}) \in P^r(2m, \{A_i\}_{i=0}^r)$, $M > 0$, $\mathtt{k} \in \mathbb{Z}$.*
Assume that

$$g_{ph} = \sum_{\substack{k=(k_1,...,k_{2m}) \ s.t. \\ G_k(k_1,...,k_{2m})=\mathtt{k}}} g_k\left(\{|\psi_k|^2\}\right)\psi_{k_1}...\psi_{k_m}\bar{\psi}_{k_{m+1}}...\bar{\psi}_{k_{2m}},$$

where, for any k, $G_k = \mathtt{k}$ is $(M, \mathtt{k})$-admissible. Then

$$\{f, g_{ph}\} = F_1 + F_2$$

where

$$F_1 = \sum_{\substack{k'=(k'_1,\dots,k'_{2n+2m-2}) \\ \tilde{G}_{k'}(k'_1,\dots,k'_{2n+2m-2})=\mathtt{k}}} F_{1,k'}\psi_{k'_1}\dots\psi_{k'_{n+m-1}}\bar{\psi}_{k'_{n+m}}\dots\bar{\psi}_{k'_{2m+2n-2}}, \qquad (3.2.15)$$

$$F_2 = \sum_{\substack{k''=(k''_1,\dots,k''_{2n+2m}) \\ \hat{G}_{k''}(k''_1,\dots,k''_{2n+2m})=\mathtt{k}}} F_{2,k''}\psi_{k''_1}\dots\psi_{k''_{m+n}}\bar{\psi}_{k''_{m+n+1}}\dots\bar{\psi}_{k''_{2m+2n}} \qquad (3.2.16)$$

where for any k', k'', *the relations* $\tilde{G}_{k'} = \mathtt{k}$, $\hat{G}_{k''} = \mathtt{k}$ *are* $(2M, \mathtt{k})$-*admissible.*

Proof. Writing $f = \sum_{l=(l_1,\dots l_{2n})} f_l\psi_{l_1}\dots\psi_{l_n}\bar{\psi}_{l_{n+1}}\dots\bar{\psi}_{l_{2n}}$, by Lemma 3.2.10, we have $F_1 \in P^r\left(2n+2m-2, 2nmD\{A_i\}_{i=0}^r\right)$, $F_2 \in P^{r-1}(2n+2m, 2nD\{A_i\}_{i=1}^r)$.

Moreover, each term of F_1 is originated by two terms that depend respectively on $l = (l_1,\dots l_{2n})$ and $k = (k_1,\dots k_{2m})$ s.t. $\sum_{i=1}^n l_i = \sum_{i=n+1}^{2n} l_i$, $\sum_{i=1}^m k_i = \sum_{i=m+1}^{2m} k_i$ and $\{l_1,\dots l_n\} \cap \{k_{m+1},\dots k_{2m}\} \neq \varnothing$ or $\{l_{n+1},\dots l_{2n}\} \cap \{k_1,\dots k_m\} \neq \varnothing$. Without losing generality, we can suppose $l_1 = k_{m+1}$.
We form a vector of indices $k' = (l_2,\dots l_n, k_1,\dots,k_m, l_{n+1},\dots l_{2n}, k_{m+2},\dots,k_{2m})$ s.t. $\sum_{i=2}^n l_i + \sum_{i=1}^m k_i = \sum_{i=n+1}^{2n} k_i + \sum_{i=m+2}^{2m} k_i$. Moreover, $k_{m+1} = \sum_{i=1}^m k_i - \sum_{i=m+2}^{2m} k_i$. By hypothesis, we can write $G_k(k_1,\dots,k_{2m}) = \sum_{i=1}^{2m} a_i k_i$ with $a_i \in \mathbb{N}, |a_i| < M$, so

$$\mathtt{k} = G_k(k_1,\dots,k_{2m}) = \sum_{i=1}^{2m} a_i k_i = \sum_{i=1}^{m}(a_i + a_{m+1})k_i + \sum_{i=m+2}^{2m}(a_i - a_{m+1})k_i =$$

$$= \sum_{i=1}^{m} b_i k_i + \sum_{i=m+2}^{2m} b_i k_i = \tilde{G}_k(k_1,\dots,k_m,k_{m+2},\dots,k_{2m})$$

$$= \tilde{G}_{k'}(l_2,\dots,l_n,k_1,\dots,k_m,l_{n+1},\dots,l_{2n},k_{m+2},\dots,k_{2m}).$$

We note that $|b_i| < 2M$ and $\tilde{G}_k$ is a linear combination only of $\{k_1,\dots,k_m,k_{m+2},\dots,k_{2m}\}$ so it is independent of the *null-momentum* condition related to $(l_2,\dots,l_n,k_1,\dots,k_m,l_{n+1},\dots,l_{2n},k_{m+2},\dots,k_{2m})$, so we obtain the thesis for F_1.
For F_2 the situation is simpler. Again each term of F_2 is originated by two terms that depend respectively on l and k s.t. $\sum_{i=1}^n l_i = \sum_{i=n+1}^{2n} l_i$, $\sum_{i=1}^m k_i = \sum_{i=m+1}^{2m} k_i$ and $\{l_1,\dots l_n\}\cap\{k_{m+1},\dots k_{2m}\} \neq \emptyset$ or $\{l_{n+1},\dots l_{2n}\}\cap\{k_1,\dots k_m\} \neq \emptyset$.
We obtain a vector of indices $k'' = (l_1,\dots,l_n,k_1,\dots,k_m,l_{n+1},\dots,l_{2n},k_{m+1},\dots,k_{2m})$ s.t. $\sum_{i=1}^n l_i + \sum_{i=1}^m k_i = \sum_{i=n+1}^{2n} k_i + \sum_{i=m+1}^{2m} k_i$ and

$$\mathtt{k} = G_k(k_1,\dots k_{2m}) = \tilde{G}_{k''}(l_1,\dots,l_n,k_1,\dots,k_m,l_{n+1},\dots,l_{2n},k_{m+1},\dots,k_{2m}).$$

$\square$

Remark 3.2.12. This result holds also in the particular case in which g_k is a constant independent of $\{|\psi_j|^2\}$.

In particular, one can obtain the following improvement of Lemma 3.2.7:

Lemma 3.2.13. *Let n be an integer, $M > 0$, $\mathtt{k} \in \mathbb{Z}$, let*

$$f_{ph} = \sum_{\substack{k=(k_1,...,k_{2n}) \\ G_k(k_1,...,k_{2n})=\mathtt{k}}} f_k\left(\{|\psi_k|^2\}\right) \psi_{k_1}...\psi_{k_n}\bar{\psi}_{k_{n+1}}...\bar{\psi}_{k_{2n}},$$

s.t. $f_{ph} \in P^0(2n, A_0)$ and for any k, $G_k(k_1, ..., k_{2n}) = \mathtt{k}$ is $(M, \mathtt{k})$-admissible. Then, for any $\beta > 0$, one has

$$\|f_{ph}\|_{g,\beta} \leq \frac{A_0 C_g(n) M^2}{(1+\mathtt{k}^2)\,\beta^n}. \tag{3.2.17}$$

The proof of this lemma is very technical and it is deferred to Appendix C.0.1.

3.3 Formal construction of perturbed actions

In this section we look for a formal integral of motion which is a higher order perturbation of $\Phi_{\mathtt{k},2} := |\psi_{\mathtt{k}}|^2$. Thus we fix once for all the value of $\mathtt{k}$.

To present the construction, we describe first an equivalent one, which however is difficult to manage directly. Since H_2 is completely resonant, it is well known that one can construct, formally a canonical transformation T which transforms the Hamiltonian into

$$H_2 + Z_4 + Z_6 + R_8 \tag{3.3.1}$$

with Z_4 and Z_6 which Poisson commute with H_2. In particular Z_4 has been computed in many papers (see e.g. [3]) and is given by

$$Z_4(\psi) := \frac{c_2}{2}\left(\sum_k |\psi_k|^2\right)^2 - \frac{c_2}{2}\sum_k |\psi_k|^4 \ . \tag{3.3.2}$$

Then, following the ideas by Poincaré, we look for $\tilde{\Phi}_{\mathtt{k},6}$, Poisson commuting with H_2, s.t. $\tilde{\Phi}^{(6)}_{\mathtt{k}} := \Phi_{\mathtt{k},2} + \tilde{\Phi}_{\mathtt{k},6}$ is an approximate integral of motion of (3.3.1). Computing the Poisson bracket of this quantity with (3.3.1), one has that this is a quantity of order at least 8 if

$$\left\{Z_4, \tilde{\Phi}_{\mathtt{k},6}\right\} = \{\Phi_{\mathtt{k},2}, Z_6\} =: \mathcal{R}_6 \ , \tag{3.3.3}$$

which is clearly impossible since the l.h.s. is of order 8 and the r.h.s. of order 6, so we will modify it. Since Z_4 depends on the actions only, one has

$$\{Z_4, \cdot\} = i\sum_j \omega_j \left(\psi_j \frac{\partial}{\partial\psi_j} - \bar{\psi}_j \frac{\partial}{\partial\bar{\psi}_j}\right) ,$$

with $\omega_j := c_2\left(|\psi_j|^2 + \sum_k |\psi_k|^2\right)$. So one is led to separate the regions where the ω_j's are resonant and those in which they are non resonant. The resonant regions and the nonresonant regions will be defined precisely in the following. Denote $\mathcal{R}_6^{NR}$ the restriction of Z_6 to the nonresonant regions, we will solve the equation

$$\left\{Z_4, \tilde{\Phi}_{\mathbf{k},6}\right\} = \mathcal{R}_6^{NR} . \tag{3.3.4}$$

Looking for $\tilde{\Phi}_{\mathbf{k}}^{(6)}$ in the class of polynomials with frequency dependent coefficients, the approximate integral of motion that we are going to construct is given by the sixth order truncation of $T^{-1}\tilde{\Phi}_{\mathbf{k}}^{(6)}$. We proceed now to the construction of the integral of motion. Define the operator $L_{H_2} := \{H_2, \cdot\}$, we have that for any $f \in P_{2n}$

$$L_{H_2} f = \{H_2, f\} \equiv -i\sum_{l,m} f_{l,m} \left\langle \mathbf{k^2}, (l-m)\right\rangle \psi^l \bar{\psi}^m$$

where $\langle \mathbf{k^2}, (l-m)\rangle := \sum_j k_j^2 (l_j - m_j)$.
Equivalently, for any for any $f \in P_{2n}$, we can write

$$L_{H_2} f = -i\sum_k f_k \left(\sum_k k^2 \left(\sum_{i=1}^{n} \delta_{k_i,k} - \sum_{i=n+1}^{2n} \delta_{k_i,k}\right)\right) \prod_{i=1}^{n} \psi_{k_i} \bar{\psi}_{k_{i+n}},$$

where $\delta_{x,y}$ is kronecker's delta.

Definition 13. We denote by

$$N_{H_2} := \ker L_{H_2} = \left\{f \in \cup_{n\in\mathbb{N}} P_{2n} : f_{l,m} \neq 0 \Leftrightarrow \left\langle \mathbf{k^2}, (l-m)\right\rangle = 0\right\},$$

$$R_{H_2} := \left\{f \in \cup_{n\in\mathbb{N}} P_{2n} : f_{l,m} \neq 0 \Leftrightarrow \left\langle \mathbf{k^2}, (l-m)\right\rangle \neq 0\right\}.$$

Remark 3.3.1. $L_{H_2} : R_{H_2} \to R_{H_2}$ is formally invertible.

Given a polynomial f, we indicate the projection of f on N_{H_2} by $f^{N_{H_2}}$ and the projection on R_{H_2} by $f^{R_{H_2}}$.

In particular, we have

$$H_4^{R_{H_2}} := \frac{c_2}{4} \sum_{\substack{k_1+k_2=k_3+k_4 \\ k_1^2+k_2^2 \neq k_3^2+k_4^2}} \psi_{k_1}\psi_{k_2}\bar{\psi}_{k_3}\bar{\psi}_{k_4},$$

$$Z_4 = H_4^{N_{H_2}}.$$

Define now

$$\chi_4 := -L_{H_2}^{-1} H_4^{R_{H_2}},\ \chi_6 := -L_{H_2}^{-1}\left(\frac{1}{2}\left\{\chi_4, H_4^{R_{H_2}}\right\} + \{\chi_4, Z_4\} + H_6\right)^{R_{H_2}},$$

$$\Phi_{k,4} := L_{\chi_4}|\psi_k|^2,\ \Phi_{k,6} := \frac{1}{2}L_{\chi_4}^2|\psi_k|^2 + L_{\chi_6}|\psi_k|^2$$

and

$$Z_6 := H_6^{N_{H_2}} + \left(\frac{1}{2}\left\{\chi_4, H_4^{R_{H_2}}\right\} + \{\chi_4, Z_4\}\right)^{N_{H_2}},$$

to proceed, we have to define the resonant/nonresonant decomposition of the phase-space.

Definition 14. For any $n > 0$, we denote by

$$\mathcal{M}_{2n} := \left\{k = \{k_j\} \in \mathbb{Z}^{2n} \text{ s.t. } \sum_{j=1}^{n} k_j = \sum_{j=n+1}^{2n} k_j,\ \sum_{j=1}^{n} k_j^2 = \sum_{j=n+1}^{2n} k_j^2\right\}$$

Write

$$Z_6 = \sum_{k\in\mathcal{M}_6} \tilde{Z}_{6,k}\psi_{k_1}\psi_{k_2}\psi_{k_3}\bar{\psi}_{k_4}\bar{\psi}_{k_5}\bar{\psi}_{k_6},$$

computing

$$\mathcal{R}_6 = \{\Phi_{\mathtt{k},2}, Z_6\},$$

one gets

$$\mathcal{R}_6 = \sum_{k\in\mathcal{M}_6} Z_{6,k,\mathtt{k}} \tag{3.3.5}$$

with

$$Z_{6,k,\mathtt{k}} := -i\tilde{Z}_{6,k}\left(\delta_{k_1,\mathtt{k}} + \delta_{k_2,\mathtt{k}} + \delta_{k_3,\mathtt{k}} - \delta_{k_4,\mathtt{k}} - \delta_{k_5,\mathtt{k}} - \delta_{k_6,\mathtt{k}}\right)\psi_{k_1}\psi_{k_2}\psi_{k_3}\bar{\psi}_{k_4}\bar{\psi}_{k_5}\bar{\psi}_{k_6},$$

where $\delta_{j,\mathtt{k}}$ is Kronecker's delta.

We introduce a function $\rho \in \mathcal{C}_0^\infty$, s.t.

$$\rho(x) = \begin{cases} 1 \text{ if } |x| > 2 \\ 0 \text{ if } |x| < 1 \end{cases}. \tag{3.3.6}$$

Recalling that $\omega_j := c_2\left(|\psi_j|^2 + \sum_k |\psi_k|^2\right)$, we denote by

$$\begin{aligned} a_k(\psi) :=& \frac{1}{c_2}\left(\omega_{k_1} + \omega_{k_2} + \omega_{k_3} - \omega_{k_4} - \omega_{k_5} - \omega_{k_6}\right) \\ =& \left(|\psi_{k_1}|^2 + |\psi_{k_2}|^2 + |\psi_{k_3}|^2 - |\psi_{k_4}|^2 - |\psi_{k_5}|^2 - |\psi_{k_6}|^2\right) \end{aligned} \tag{3.3.7}$$

and, given $0 < \delta < 1$, we define the decomposition $\mathcal{R}_6 := \mathcal{R}_6^{NR} + \mathcal{R}_6^R$ with

$$\mathcal{R}_6^{NR} := \sum_k Z_{6,k,\mathtt{k}} \rho \left(\frac{a_k(\psi)}{\delta} \right)$$

and

$$\mathcal{R}_6^R := \sum_k Z_{6,k,\mathtt{k}} \left(1 - \rho \left(\frac{a_k(\psi)}{\delta} \right) \right).$$

We define $\tilde{\Phi}_{\mathtt{k},6}$ to be the solution of equation (3.3.4), which is explicitely given by

$$\tilde{\Phi}_{\mathtt{k},6} := i \sum_{k \in \mathcal{M}_6} \frac{Z_{6,k,\mathtt{k}}}{c_2 a_k(\psi)} \rho \left(\frac{a_k(\psi)}{\delta} \right).$$

Remark 3.3.2. $\tilde{\Phi}_{\mathtt{k},6}(\psi) \in P^2 \left(6, \left\{ \frac{A_i}{\delta^{i+1}} \right\}_{i=0}^2 \right) \subset P^2 \left(6, \left\{ \frac{A}{\delta^{i+1}} \right\}_{i=0}^2 \right)$ with $A := \max_i A_i$.

Finally we define the approximate integral of motion is given by

$$\Phi_{\mathtt{k}}^{(6)} := \Phi_{\mathtt{k},2} + \Phi_{\mathtt{k},4} + \Phi_{\mathtt{k},6} + \tilde{\Phi}_{\mathtt{k},6} + L_{\chi_4} \tilde{\Phi}_{\mathtt{k},6}. \tag{3.3.8}$$

The following lemma gives the structure of its time derivative.

Lemma 3.3.3. *Write*

$$\left\{ H, \Phi_{\mathtt{k}}^{(6)} \right\} = -\mathcal{R}_6^R + R$$

then

$$R = \sum_{j=4}^{q+1} R_{2j} + \sum_{j=5}^{q+2} R_{2j,1} + \sum_{j=6}^{q+3} R_{2j,2} + \sum_{j=7}^{q+5} R_{2j,3}, \tag{3.3.9}$$

with $R_{2j} \in P_{2j}$, and there exists $C > 0$ s.t.

$$R_{2j,l} \in P^{3-l} \left(2j, \left\{ \frac{C}{\delta^{m+l}} \right\}_{m=0}^{3-l} \right).$$

Proof. One has

$$\left\{H, \Phi_{\mathtt{k}}^{(6)}\right\} = \{H_2, \Phi_{\mathtt{k},2}\}$$

$$+ \{H_2, \Phi_{\mathtt{k},4}\} + \{H_4, \Phi_{\mathtt{k},2}\} + \left\{H_2, \tilde{\Phi}_{\mathtt{k},6}\right\} \tag{3.3.10}$$

$$+ \{Z_6, \Phi_{k,2}\} + \left\{Z_4, \tilde{\Phi}_{\mathtt{k},6}\right\} + \left\{H_4^{R_{H_2}}, \tilde{\Phi}_{\mathtt{k},6}\right\} + \left\{H_2, L_{\chi_4}\tilde{\Phi}_{\mathtt{k},6}\right\} \tag{3.3.11}$$

$$+ \sum_{j=2}^{n-2} \left(\{H_{2j}, \Phi_{k,6}\} + \left\{H_{2j}, L_{\chi_4}\tilde{\Phi}_{\mathtt{k},6}\right\} + \left\{H_{2(j+1)}, \Phi_{k,4}\right\} + \left\{H_{2(j+1)}, \tilde{\Phi}_{\mathtt{k},6}\right\} + \left\{H_{2(j+2)}, \Phi_{k,2}\right\}\right) \tag{3.3.12}$$

$$+ \left\{H_{2(n-1)}, \Phi_{k,6}\right\} + \left\{H_{2(n-1)}, L_{\chi_4}\tilde{\Phi}_{\mathtt{k},6}\right\} + \{H_{2n}, \Phi_{k,6}\} + \left\{H_{2n}, \tilde{\Phi}_{\mathtt{k},6}\right\} \tag{3.3.13}$$

$$+ \{H_{2n}, \Phi_{k,6}\} + \left\{H_{2n}, L_{\chi_4}\tilde{\Phi}_{\mathtt{k},6}\right\}. \tag{3.3.14}$$

Due to the construction, we have that $\{H_2, \Phi_{\mathtt{k},2}\} = 0$ and $\{H_2, \Phi_{\mathtt{k},4}\} = -\left\{H_4, \tilde{\Phi}_{\mathtt{k},2}\right\}$. Due to the fact that a_k and ρ depend on the actions only and $\{Z_{6,k,\mathtt{k}}, H_2\} = 0$, one has $\left\{H_2, \tilde{\Phi}_{\mathtt{k},6}\right\} = 0$ so that (3.3.10) vanishes.
Since Z_4 is a function of the actions only, we have also

$$\left\{Z_4, \tilde{\Phi}_{\mathtt{k},6}\right\} = i\sum_k \{Z_4, Z_{6,k,\mathtt{k}}\} \frac{\rho\left(\frac{a_k(\psi)}{\delta}\right)}{c_2 a_k(\psi)} = \sum_k Z_{6,k,\mathtt{k}}\rho\left(\frac{a_k(\psi)}{\delta}\right) = \mathcal{R}_6^{NR}.$$

We note that $\left\{H_4^{R_{H_2}}, \tilde{\Phi}_{\mathtt{k},6}\right\} = -\left\{H_2, L_{\chi_4}\tilde{\Phi}_{\mathtt{k},6}\right\}$ in fact, by the definition of χ_4 and $\left\{H_2, \tilde{\Phi}_{\mathtt{k},6}\right\} = 0$, one has

$$\left\{H_2, L_{\chi_4}\tilde{\Phi}_{\mathtt{k},6}\right\} = -\left\{H_2, \left\{L_{H_2}^{-1}H_4^{R_{H_2}}, \tilde{\Phi}_{\mathtt{k},6}\right\}\right\} =$$

$$= \left\{L_{H_2}^{-1}H_4^{R_{H_2}}, \left\{\tilde{\Phi}_{\mathtt{k},6}, H_2\right\}\right\} + \left\{\tilde{\Phi}_{\mathtt{k},6}, L_{H_2}L_{H_2}^{-1}H_4^{R_{H_2}}\right\} = \left\{\tilde{\Phi}_{\mathtt{k},6}, H_4^{R_{H_2}}\right\}.$$

So, by (3.3.5), line (3.3.11) reduces to $\sum_k Z_{6,k,\mathtt{k}}\left(\rho\left(\frac{a_k(\psi)}{\delta}\right) - 1\right) = -\mathcal{R}_6^R$.
It remains to study now (3.3.12), (3.3.13) and (3.3.14). Using Lemma 3.2.10, we have

$$\left\{H_{2j}, \tilde{\Phi}_{\mathtt{k},6}\right\} = F_{1,j} + F_{2,j},$$

$$F_{1,j} \in P^2\left(2j+4, \left\{\frac{C}{\delta^{i+1}}\right\}_{i=0}^2\right), \; F_{2,j} \in P^1\left(2j+6, \left\{\frac{C}{\delta^{i+2}}\right\}_{i=0}^1\right),$$

$$L_{\chi_4}\tilde{\Phi}_{\mathtt{k},6} = E_1 + E_2, \; E_1 \in P^2\left(8, \left\{\frac{C}{\delta^{i+1}}\right\}_{i=0}^2\right), \; E_2 \in P^1\left(10, \left\{\frac{C}{\delta^{i+2}}\right\}_{i=0}^1\right),$$

so

$$\left\{H_{2j}, L_{\chi_4}\tilde{\Phi}_{\mathtt{k},6}\right\} = F_{3,j} + F_{4,j} + F_{5,j},$$

$$F_{3,j} \in P^2\left(2j+6, \left\{\frac{C}{\delta^{i+1}}\right\}_{i=0}^{2}\right),\ F_{4,j} \in P^1\left(2j+8, \left\{\frac{C}{\delta^{i+2}}\right\}_{i=0}^{1}\right),$$

$$F_{5,j} \in P^0\left(2j+10, \frac{C}{\delta^3}\right),$$

$$\{H_{2j}, \Phi_{k,2}\} \in P_{2j},$$

$$\{H_{2j}, \Phi_{k,4}\} \in P_{2j+2},$$

$$\{H_{2j}, \Phi_{k,6}\} \in P_{2j+4}.$$

□

3.4 Measure estimates

In this section we estimate $\|\Phi_{\mathtt{k}}^{(6)} - |\psi_{\mathtt{k}}|^2\|_{\mu_\beta}^2$ and $\left\|\left\{H, \Phi_{\mathtt{k}}^{(6)}\right\}\right\|_{\mu_\beta}$.

Lemma 3.4.1. *There exists a constant $C > 0$ s.t. for any $\beta > 1$, $\delta \in (0,1)$ s.t. $0 < \delta\beta < 1$, one has*

$$\|\Phi_{\mathtt{k}}^{(6)} - |\psi_{\mathtt{k}}|^2\|_{g,\beta}^2 \leq \frac{C}{(1+\mathtt{k}^2)^2 \min\{\delta^2\beta^6, \delta^4\beta^{10}\}}, \tag{3.4.1}$$

$$\|R\|_{g,\beta}^2 \leq \frac{C}{(1+\mathtt{k}^2)^2\,\delta^6\beta^{14}}, \tag{3.4.2}$$

where R is defined by (3.3.9).

Proof. We recall that

$$\Phi_{\mathtt{k}}^{(6)} - |\psi_{\mathtt{k}}|^2 = \Phi_{\mathtt{k},4} + \Phi_{\mathtt{k},6} + \tilde{\Phi}_{\mathtt{k},6} + L_{\chi_4}\tilde{\Phi}_{\mathtt{k},6}.$$

By construction, $\Phi_{\mathtt{k},4} \in P_4$, $\Phi_{\mathtt{k},6} \in P_6$ and there exists $C_1 > 0$ s.t. $\Phi_{\mathtt{k},6} \in P^2\left(6, \left\{\frac{C_1}{\delta^{i+1}}\right\}_{i=0}^{2}\right)$ and, using Lemma 3.2.11, there exists $C_2 > 0$ s.t. $L_{\chi_4}\tilde{\Phi}_{\mathtt{k},6} = E_1 + E_2$, $E_1 \in P^2\left(8, \left\{\frac{C_2}{\delta^{i+1}}\right\}_{i=0}^{2}\right)$, $E_2 \in P^1\left(10, \left\{\frac{C_2}{\delta^{i+2}}\right\}_{i=0}^{1}\right)$.
Moreover, $P^2\left(6, \left\{\frac{C_1}{\delta^{i+1}}\right\}_{i=0}^{2}\right) \subset P^0\left(6, \frac{C_1}{\delta}\right)$, $P^2\left(8, \left\{\frac{C_2}{\delta^{i+1}}\right\}_{i=0}^{2}\right) \subset P^0\left(8, \frac{C_2}{\delta}\right)$

and $P^1\left(10, \{\frac{C_2}{\delta^{i+2}}\}_{i=0}^1\right) \subset P^0\left(10, \frac{C_2}{\delta^2}\right)$. So, using Lemma 3.2.13 with $M = 2$, we obtain

$$\|\Phi_{\mathtt{k}}^{(6)} - |\psi_{\mathtt{k}}|^2\|_{g,\beta}^2 \leq \frac{C}{(1+\mathtt{k}^2)^2}\left(\frac{1}{\beta^4} + \frac{1}{\beta^6} + \frac{1}{\delta^2\beta^6} + \frac{1}{\delta^2\beta^8} + \frac{1}{\delta^4\beta^{10}}\right) \leq$$

$$\leq \frac{5C}{(1+\mathtt{k}^2)^2 \min\{\delta^2\beta^6, \delta^4\beta^{10}\}}$$

where we used $0 < \delta\beta < 1$. Using (3.3.9), Lemma 3.3.3, Lemma 3.2.11 and Lemma 3.2.13 with $M = 4$, we get

$$\|R\|_{g,\beta}^2 \leq \frac{C}{(1+\mathtt{k}^2)^2}\left(\sum_{j=4}^{n+1}\frac{1}{\beta^{2j}} + \sum_{j=5}^{n+2}\frac{1}{\delta^2\beta^{2j}} + \sum_{j=6}^{n+3}\frac{1}{\delta^4\beta^{2j}} + \sum_{j=7}^{n+5}\frac{1}{\delta^6\beta^{2j}}\right)$$

so

$$\|R\|_{g,\beta}^2 \leq \frac{C}{(1+\mathtt{k}^2)^2\,\delta^6\beta^{14}}.$$

□

It remains to estimate the resonant part, namely $\left\|\mathcal{R}_6^R\right\|_{g,\beta}^2$.

Lemma 3.4.2. *There exists a constant $\tilde{C} > 0$ s.t. for any $\beta > 0$ and $\delta > 0$ s.t. $0 < \delta\beta < 1$, one has*

$$\left\|\mathcal{R}_6^R\right\|_{g,\beta}^2 \leq \tilde{C}\frac{(\delta\beta)^{\frac{2}{3}}}{\beta^6\,(1+\mathtt{k}^2)^2}. \tag{3.4.3}$$

The very technical proof is deferred to Appendix C. We remark that the difficult part consists in showing the presence of $(1+\mathtt{k}^2)^2$ at the denominators.

Finally, we obtain the following

Lemma 3.4.3. *There exists a constant $C > 0$ s.t. for any $\beta > 0$, one has*

$$\left\|\dot{\Phi}_{\mathtt{k}}^{(6)}\right\|_{g,\beta} = \left\|\left\{H, \Phi_{\mathtt{k}}^{(6)}\right\}\right\|_{g,\beta} \leq \frac{C}{(1+\mathtt{k}^2)\,\beta^{3+\frac{1}{10}}}.$$

Proof. By Lemma 3.3.3, we know that

$$\left\{H, \Phi_{\mathtt{k}}^{(6)}\right\} = -\mathcal{R}_6^R + R.$$

Using Lemmas 3.4.3 and 3.4.1, we can choose δ in such a way that (3.4.2) and (3.4.3) have the same size:

$$\frac{1}{\delta^6\beta^{14}} = \frac{(\delta\beta)^{\frac{2}{3}}}{\beta^6}.$$

It follows that $\delta = \frac{1}{\beta^{\frac{13}{10}}}$ and the thesis. □

Finally, using these results and Lemma 3.1.2, we obtain

Lemma 3.4.4. *There exists $\beta^*, C > 0$ s.t. for any $\beta > \beta^*$, one has*

$$\left\|\dot{\Phi}_{\mathrm{k}}^{(6)}\right\|_{\mu_\beta} \leq \frac{C}{(1+\mathrm{k}^2)\,\beta^{3+\frac{1}{10}}}.$$

Proof. This results is a simple consequence of Lemma 3.4.3 and Lemma 3.1.2. □

3.5 Proof of Theorem 0.0.1

Proof of Theorem 0.0.1 Using Chebyshev's inequality, one has

$$\mu_\beta\left\{\psi : |\Phi_{\mathrm{k}}^{(6)}(\psi(t)) - \Phi_{\mathrm{k}}^{(6)}(\psi(0))| > \eta_1\||\psi_{\mathrm{k}}|^2\|_{\mu_\beta}\right\} \leq \frac{\left\|\Phi_{\mathrm{k}}^{(6)}(\psi(t)) - \Phi_{\mathrm{k}}^{(6)}(\psi(0))\right\|_{\mu_\beta}^2}{\eta_1^2\||\psi_{\mathrm{k}}|^2\|_{\mu_\beta}^2}. \tag{3.5.1}$$

But $\Phi_{\mathrm{k}}^{(6)}(\psi(t)) - \Phi_{\mathrm{k}}^{(6)}(\psi(0)) = \int_0^t \dot{\Phi}_{\mathrm{k}}^{(6)}(\psi(s))ds$, so

$$\left\|\Phi_{\mathrm{k}}^{(6)}(\psi(t)) - \Phi_{\mathrm{k}}^{(6)}(\psi(0))\right\|_{\mu_\beta} \leq \int_0^t \left\|\dot{\Phi}_{\mathrm{k}}^{(6)}(\psi(s))\right\|_{\mu_\beta} ds.$$

Thanks to the invariance of the measure, the $L^2(\mu_\beta)$-norm is conserved under the dynamics, so for any $t \in \mathbb{R}$, we have

$$\left\|\dot{\Phi}_{\mathrm{k}}^{(6)}(\psi(t))\right\|_{\mu_\beta} = \left\|\dot{\Phi}_{\mathrm{k}}^{(6)}(\psi(0))\right\|_{\mu_\beta} = \left\|\dot{\Phi}_{\mathrm{k}}^{(6)}\right\|_{\mu_\beta},$$

and in particular we obtain

$$\left\|\Phi_{\mathrm{k}}^{(6)}(\psi(t)) - \Phi_{\mathrm{k}}^{(6)}(\psi(0))\right\|_{\mu_\beta} \leq t\left\|\dot{\Phi}_{\mathrm{k}}^{(6)}\right\|_{\mu_\beta}.$$

So,

$$\mu_\beta\left\{\psi : |\Phi^{(6)}_{\mathtt{k}}(\psi(t)) - \Phi^{(6)}_{\mathtt{k}}(\psi(0))| > \eta_1 \||\psi_{\mathtt{k}}|^2\|_{\mu_\beta}\right\} \leq t^2 \frac{\left\|\dot{\Phi}^{(6)}_{\mathtt{k}}\right\|^2_{\mu_\beta}}{\eta_1^2 \||\psi_{\mathtt{k}}|^2\|^2_{\mu_\beta}} \leq \eta_2 \tag{3.5.2}$$

for any $|t| < \frac{\eta_1\sqrt{\eta_2}\beta^{2+\frac{1}{10}}}{C}$, where we used Lemmas 3.1.5 and 3.4.4. Using this result, we can study the variation of the $\mathtt{k}$-action. In fact

$$\mu_\beta\left\{\psi : \left||\psi_{\mathtt{k}}(t)|^2 - |\psi_{\mathtt{k}}(0)|^2\right| > \eta_1 \||\psi_{\mathtt{k}}|^2\|_{\mu_\beta}\right\} \leq \tag{3.5.3}$$

$$\leq \mu_\beta\left\{\psi : \left|\Phi^{(6)}_{\mathtt{k}}(\psi(t)) - \Phi^{(6)}_{\mathtt{k}}(\psi(0))\right| > \frac{\eta_1}{3} \||\psi_{\mathtt{k}}|^2\|_{\mu_\beta}\right\}$$
$$+\mu_\beta\left\{\psi : \left|\Phi^{(6)}_{\mathtt{k}} - |\psi_{\mathtt{k}}|^2\right|(t) > \frac{\eta_1}{3} \||\psi_{\mathtt{k}}|^2\|_{\mu_\beta}\right\}$$
$$+\mu_\beta\left\{\psi : \left|\Phi^{(6)}_{\mathtt{k}} - |\psi_{\mathtt{k}}|^2\right|(0) > \frac{\eta_1}{3} \||\psi_{\mathtt{k}}|^2\|_{\mu_\beta}\right\}$$
$$\leq \frac{\eta_2}{2} + 18 \frac{\left\|\Phi^{(6)}_{\mathtt{k}} - |\psi_{\mathtt{k}}|^2\right\|^2_{\mu_\beta}}{\eta_1^2 \, \||\psi_{\mathtt{k}}|^2\|^2_{\mu_\beta}} \leq \eta_2$$

for any $\beta > \frac{C}{\eta_1^{\frac{10}{7}}\eta_2^{\frac{5}{7}}}$, $|t| < \frac{\eta_1\sqrt{\eta_2}\beta^{2+\frac{1}{10}}}{C}$, where we used Chebyshev's inequality, the conservation of the Gibbs measure, (3.4.1) with $\delta = \frac{1}{\beta^{\frac{13}{10}}}$ and Lemma 3.1.2 to estimate the second and the third term. Then Theorem 0.0.1 is obtained by reformulating this inequality.

□

Proof of Corollary 3.0.4 We consider two sequences $\eta_{1,k} := \eta_1(1+k^2)^{\frac{1}{2}}$, $\eta_{2,k} := \frac{\eta_2}{(1+k^2)}\left(\sum_j \frac{1}{1+j^2}\right)^{-1}$.
For any $k \in \mathbb{Z}$ and any $\alpha < 1/2$, we define

$$\mathcal{I}_{\alpha,k} := \left\{\psi : \left||\psi_k(t)|^2 - |\psi_k(0)|^2\right| \leq \frac{\eta_1}{(1+k^2)^\alpha \beta}\right\}.$$

Using Theorem 0.0.1, one has

$$\mu_\beta(\mathcal{I}^c_{\alpha,k}) \leq \mu_\beta\left\{\psi : \left||\psi_k(t)|^2 - |\psi_k(0)|^2\right| > \frac{\eta_1}{(1+k^2)^{\frac{1}{2}}\beta}\right\} =$$

$$\mu_\beta\left\{\psi : \left||\psi_k(t)|^2 - |\psi_k(0)|^2\right| > \frac{\eta_{1,k}}{(1+k^2)\beta}\right\} \leq \eta_{2,k}$$

for any $|t| < C'\eta_1\sqrt{\eta_2}\beta^{2+\varsigma}$.
Denote $\mathcal{I}_\alpha := \cup_k \mathcal{I}_{\alpha,k}$, one has that

$$\mu_\beta\left(\mathcal{I}_\alpha^c\right) \leq \sum_k \mu_\beta\left(\mathcal{I}_{\alpha,k}^c\right) \leq \eta_2. \tag{3.5.4}$$

□

Part II

Deterministic result

Chapter 4

Long time existence in H^1 for time-dependent NLS on the 2-d torus

Introduction to Chapter 4

In this Chapter we follow [7].

We study the NLS equation (0.0.5), namely

$$\mathrm{i}\psi_t = -2\Delta\psi + 2a(x,\omega t)|\psi|^2\psi, \qquad x \in \mathbb{T}^2.$$

with $\omega \in \mathbb{R}^d$ and $a \in C^\infty(\mathbb{T}^{d+2})$. We study the possibility of using Hamiltonian perturbation theory for the study of the dynamics.

To explain the situation, remark first that equation (0.0.5) is Hamiltonian with Hamiltonian function given by

$$H(\psi) = H_0(\psi) + H_1(\psi,\omega t), \quad H_0(\psi) = \int_{\mathbb{T}^2} |\nabla\psi|^2 dx \tag{4.0.1}$$

$$H_1(\psi,\omega t) = \int_{\mathbb{T}^2} a(x,\omega t)|\psi(x)|^4 dx. \tag{4.0.2}$$

which, for small initial data is a perturbation of H_0. When written in terms of the Fourier coefficients $\widehat{\psi}_k$ of ψ, H_0 takes the very simple form

$$H_0 = \sum_{k\in\mathbb{Z}^2} |k|^2 \left|\widehat{\psi}_k\right|^2 ,$$

which is the sum of infinitely many harmonic oscillators with integer frequency. It is thus natural to study the completely resonant normal form of the system. Consider the standard Sobolev space H^s and denote by $B_s(R)$ the open ball *in* H^s of radius R centered at the origin, then Theorem 6.2 of [12] (which is a development of [11]) gives the following result

Theorem 4.0.1. *Assume that the function a is analytic over $\mathbb{T}^{d+2}$, that the frequency is Diophantine, namely that there exist γ and τ s.t.*

$$|\omega \cdot k + k_0| \geq \frac{\gamma}{1+|k|^\tau} \ , \quad \forall (k, k_0) \in \mathbb{Z}^{d+1} \setminus \{0\} \ . \tag{4.0.3}$$

Fix $s > 1$, then there exists $C, \epsilon > 0$ and a canonical transformation $\mathcal{T} : B_s(\epsilon) \to H^s$ s.t.

$$H \circ \mathcal{T} = H_0 + Z + R$$

with Z independent of time and fulfilling $\{Z, H_0\} = 0$ and the following estimates hold

$$\begin{aligned}
&\sup_{\psi \in B_s(\epsilon)} \|\psi - \mathcal{T}(\psi)\|_{H^s} \leq C\epsilon^2, \\
&\sup_{\psi \in B_s(\epsilon)} |Z(\psi)| \leq C\epsilon^4, \\
&\sup_{\psi \in B_s(\epsilon)} |R(\psi)| \leq C \exp\left[-\left(\frac{C}{\epsilon}\right)^{2/(\tau+1)}\right], \\
&\sup_{\psi \in B_s(\epsilon)} \|X_R(\psi)\|_{H^s} \leq C \exp\left[-\left(\frac{C}{\epsilon}\right)^{2/(\tau+1)}\right]
\end{aligned}$$

where X_R is the Hamiltonian vector field, namely $X_R := \mathrm{i}\left(\frac{\partial \mathrm{w}}{\partial \psi}, -\frac{\partial \mathrm{w}}{\partial \bar{\psi}}\right)$.

Now, the question is *what are the dynamical consequences of Theorem 4.0.1?* Since Z is a resonant normal form, one can conclude that H_0, namely the square of the H^1 norm is an integral of motion for $H_0 + Z$, and therefore it is almost conserved in the complete system *provided some H^s norm, with s strictly larger than 1, remains smaller than ϵ for such times.* The problem is that it is impossible to see whether this happens or not, so the above theorem is useless, unless it is combined with a deeper analysis of the dynamics. For example in [24, 36, 35] the authors study in detail the form of Z and construct some particular very interesting solutions which are of interest, but nothing is known for general solutions.

On the other hand, in dimension 1 the resonant normal form has proved to be useful for the understanding of remarkable stability properties of the dynamics [3] in the energy space, so it is natural to try to use Hamiltonian perturbation to study the 2-d NLS in the energy space.

The problem is that $H^1(\mathbb{T}^2)$ is not an algebra, and therefore an analogue of Theorem 4.0.1 is not known in this space.

In this chapter, we use a variant of the normal form theory in order to construct a function on H^1 which is a deformation of H_0 and is an approximate integral of motion for initial data which have small H^1 norm. We deduce existence of solutions in H^1 for times of order ϵ^{-6}. For completeness, we report the main Theorems of this Chapter.

Theorem 4.0.2. *Assume that $a \in C^\infty(\mathbb{T}^{d+2})$ and that the frequency ω is Diophantine (namely it fulfills* (4.0.3)*), then there exist $\epsilon_*, C > 0$ and a functional $\Phi^{(3)} \in C^\infty(\mathbb{T}^d; H^1(\mathbb{T}^2, \mathbb{C}))$ with the following properties*

$$\left| \frac{d}{dt} \Phi^{(3)}(\omega t, \psi(t)) \right| \le C \left\| \psi(t) \right\|_{H^1}^{10} , \tag{4.0.4}$$

$$\sup_{\|\psi\|_{H^1} < \epsilon_*} \left\| \Phi^{(3)}(\omega t, \psi) - H_0(\psi) \right\|_{C^K(\mathbb{T}^d)} \le C \left\| \psi \right\|_{H^1}^4 . \tag{4.0.5}$$

Theorem 4.0.3. *With the same assumptions and notations, if $\epsilon := \|\psi_0\|_{H^1} < \epsilon_*$, then the solution of* (0.0.5) *with initial data ψ_0 exists up to times t s.t. $|t| < \epsilon^{-6}$ and fulfills*

$$\|\psi(t)\|_{H^1} < 2\epsilon \ . \tag{4.0.6}$$

The idea of the proof is to use an algorithm of direct construction of integrals of motion which originates from celestial mechanics [26] and then to exploit the explicit expression of the so obtained quantities in order to estimate their time derivative. The naif idea is that, since H_1 is the integral of a polynomial in $\psi(x)$, one can expect the approximate integral of motion and its time derivative to have the same structure. If this were true then one could use the fact that H^1 is embedded in L^p, $\forall p$, in order to get a control of such a quantity.

This naif idea turns out to be wrong, since the construction of the approximate integral of motion involves a procedure of averaging with respect to the flow of the linearized equation (namely with respect to $e^{-\mathrm{i}\Delta t}$) and the Bessel spaces H^s_p, namely the spaces of functions f s.t. $(-\Delta)^{s/2} f \in L^p$, are not invariant under such a flow, if $p \neq 2$. To overcome this problem, to estimate the average of the perturbation and to obtain our result, the main tool we need is the famous L^4 estimate by Bourgain, namely

$$\forall \epsilon > 0 \ \exists\, C \text{ such that } \|e^{-\mathrm{i}\Delta \mathrm{t}} \psi\|_{L^4_{tx}} \le C \|\psi\|_{H^\epsilon} \ . \tag{4.0.7}$$

We conclude this introduction by mentioning that we expect the present result to be the fundamental tool for the extension to dimension 2 of results of the kind of [5, 3]. We also recall that a preliminary interesting result in this direction has been recently proved in [32].

4.1 Formal scheme

4.1.1 Preliminaries

To define precisely the phase space, we first consider $H^1(\mathbb{T}^2;\mathbb{C})$ as a *real* Hilbert space endowed by the (weak) scalar product

$$\langle \psi_1; \psi_2 \rangle_{L^2} := 2\Re \int_{\mathbb{T}^2} \psi_1(x)\bar{\psi}_2(x)dx \ , \tag{4.1.1}$$

that we use in order to define the L^2 gradient of a function $W \in C^1(H^1(\mathbb{T}^2;\mathbb{C}))$ by

$$\langle \nabla_{L^2} W; h \rangle_{L^2} = dWh \ . \tag{4.1.2}$$

Furthermore, in order to give H^1 a symplectic structure, we define the Poisson operator as the operator of multiplication by i, which in a real Hilbert space is actually an operator, so that the Hamiltonian vector field of a function W is $\mathrm{i}\nabla_{L^2} W$.

We now extend the phase-space by adding the angles $\alpha \in \mathbb{T}^d$ and their conjugated variables $I \in \mathbb{R}^d$, so that the phase space turns out to be $\mathcal{P} := H^1(\mathbb{T}^2,\mathbb{C}) \oplus \mathbb{R}^d \oplus \mathbb{T}^d$. The extended Hamiltonian is

$$H^{ext}(I,\psi) = H_0^{ext}(I,\alpha,\psi) + H_1(\alpha,\psi), \tag{4.1.3}$$

where

$$H_0^{ext}(I,\psi) = \sum_{j=1}^{d} \omega_j I_j + \int_{\mathbb{T}^2} |\nabla\psi|^2 dx$$

and

$$H_1(\alpha,\psi) = \int_{\mathbb{T}^2} a(x,\alpha)|\psi(x)|^4 dx. \tag{4.1.4}$$

Given a function $W \in C^\infty(\mathcal{P})$ we will denote by X_W its Hamiltonian vector field. Remark that

$$X_W = \left(\mathrm{i}\nabla_{L^2} W, \frac{\partial W}{\partial \alpha}, -\frac{\partial W}{\partial I} \right) \ .$$

In general the vector field is a map from $\mathcal{P}$ to its dual $\mathcal{P}^* = H^{-1} \oplus \mathbb{R}^d \oplus \mathbb{T}^d$.

Definition 15. Given two functions, $F, G \in C^\infty(\mathcal{P})$, we define their Poisson brackets by

$$\{F; G\} := dF X_G \ . \tag{4.1.5}$$

In general the Poisson brackets of a couple of smooth functions can fail to exist. In the following we will use the explicit form of the functions in order to show that in the cases we meet such a quantity is well defined.

We will often denote $L_F := \{F, \cdot\}$.

Definition 16. We say that a function $G : H^1 \to \mathbb{C}$ is in *normal form* if

$$\{G, H_0\} = 0.$$

4.1.2 The algorithm

In this subsection we describe an algorithm due to Giorgilli to construct approximate integral of motion.

Given a sequence $\{\chi_n\}_{n\geq 1}$, we define recursively

$$E_0 = 1,$$

and for $n \geq 1$,

$$E_n = \sum_{j=1}^{n} \frac{j}{n} L_{\chi_j} E_{n-j}.$$

Given a function $W(\alpha, \psi)$ we define its average by

$$\langle W \rangle(\psi) := \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{T}^d} d\alpha \int_0^{2\pi} W(\alpha, e^{-\mathrm{i}\Delta \mathrm{t}}\psi) dt,$$

and (again recursively) the sequence $\{\Psi_n\}_{n\geq 1}$ by:

$$\Psi_1 = H_1, \tag{4.1.6}$$

$$\Psi_n = -\frac{n-1}{n} L_{\chi_{n-1}} H_1 - \sum_{j=1}^{n-1} \frac{j}{n} E_{n-j} \langle \Psi_j \rangle, \quad n \geq 2. \tag{4.1.7}$$

We have now the following theorem due to Giorgilli (for the proof see [26]).

Theorem 4.1.1. *Let χ_n be a solution of the homological equation*

$$L_{H_0^{ext}} \chi_n + \Psi_n = \langle \Psi_n \rangle \quad n \geq 1, \tag{4.1.8}$$

Define $\Phi_n := E_n H_0$, fix N and define

$$\Phi^{(N)} := H_0 + \sum_{j=1}^{N} \Phi_j.$$

Then one has

$$\left\{ H^{ext}, \Phi^{(N)} \right\} = \{H_1, \Phi_N\}.$$

Actually we are able to develop this construction only for three steps.

4.1.3 The solutions of the homological equation (4.1.8)

Given a function $W(\alpha,\psi)$, expand it in Fourier series in α, i.e.

$$W(\alpha,\psi)=\sum_{k\in\mathbb{Z}^d}\widehat{W}_k(\psi)e^{\mathrm{i}k\cdot\alpha}\ . \tag{4.1.9}$$

Lemma 4.1.2 (Lemma 6.4 of [12]). *Given a function W on $\mathcal{P}$, the solution of the homological equation*

$$L_{H_0^{ext}}\chi+W=\langle W\rangle \tag{4.1.10}$$

is given by

$$\chi(\alpha,\psi)=\sum_{k\in\mathbb{Z}^d}\widehat{\chi}_k(\psi)e^{\mathrm{i}k\cdot\alpha}$$

where

$$\chi_0(\psi)\ =\ \frac{1}{2\pi}\int_0^{2\pi}t(W_0-\langle W\rangle)(e^{-\mathrm{i}\Delta t}\psi)dt, \tag{4.1.11}$$

and for $k\neq 0$

$$\widehat{\chi}_k(\psi)\ =\ \frac{e^{-\mathrm{i}2\pi k\cdot\omega}}{1-e^{-\mathrm{i}2\pi k\cdot\omega}}\int_0^{2\pi}e^{\mathrm{i}k\cdot\omega t}\widehat{W}_k(e^{-\mathrm{i}\Delta t}\psi)dt. \tag{4.1.12}$$

For the proof see [12].

4.2 Estimate of χ_i, $i=1,2,3$

From now on we will use the notation $a\preceq b$ to mean “there exists a positive constant C s.t. $a\le Cb$”.

We associate to a polynomial W in H^1 homogeneous of degree k the unique symmetric multilinear form $\widetilde{W}$ s.t.

$$\widetilde{W}(\psi,...,\psi)=W(\psi)\ . \tag{4.2.1}$$

The same notation will be used for polynomials taking values in Banach spaces.

For example, one has

$$\widetilde{H_1}(\psi_1,\psi_2,\psi_3,\psi_4)=\frac{1}{4!}\sum_{\varsigma}\int_{\mathbb{T}^2}\psi_{\varsigma(1)}\overline{\psi_{\varsigma(2)}}\psi_{\varsigma(3)}\overline{\psi_{\varsigma(4)}}dx, \tag{4.2.2}$$

where the sum is over the permutations of $\{1, 2, 3, 4\}$, and

$$\widetilde{\nabla_{L^2} H_1}(\psi_1, \psi_2, \psi_3) = \frac{1}{3!} \sum_{\varsigma} \psi_{\varsigma(1)} \overline{\psi_{\varsigma(2)}} \psi_{\varsigma(3)}, \tag{4.2.3}$$

where now ς are permutations of $\{1, 2, 3\}$.

We remark that in particular one has

$$dW(\psi)h = k\widetilde{W}(\psi, ..., \psi, h) = \langle \nabla_{L^2} W; h \rangle_{L^2} \tag{4.2.4}$$

which is a formula useful for the study of the property of the gradient of functions.

Before to present the estimate of χ_i and of its gradient, we introduce some spaces of functions that we use in the following and the main technical tools we need.
First we introduce the Bessel Spaces $H^s_p(\mathbb{T}^2)$ ([23, 37]), namely the space of functions $\psi \in L^p(\mathbb{T}^2)$ s.t. $(-\Delta)^{s/2}\psi \in L^p(\mathbb{T}^2)$, with norm

$$\|\psi\|_{H^s_p(\mathbb{T}^2)} = \|\psi\|_{L^p(\mathbb{T}^2)} + \|(-\Delta)^{s/2}\psi\|_{L^p(\mathbb{T}^2)}. \tag{4.2.5}$$

We introduce now the main technical tools we need.

Lemma 4.2.1. *[Bourgain's estimate] $\forall \epsilon > 0$ and for any $\psi \in H^\epsilon(\mathbb{T}^2)$, one has*

$$\|e^{\mathrm{i}\Delta \mathrm{t}}\psi\|_{L^4_{tx}} \preceq \|\psi\|_{H^\epsilon(\mathbb{T}^2)}. \tag{4.2.6}$$

Proof. See [14, 15, 20]. □

Lemma 4.2.2. *[Interpolation lemma] Let $p \geq 4$, $\epsilon > 0$, $\psi \in H^{1-\frac{4}{p}+\epsilon}(\mathbb{T}^2)$, then*

$$\|e^{\mathrm{i}\Delta \mathrm{t}}\psi\|_{L^p_{tx}} \preceq \|\psi\|_{H^{1-\frac{4}{p}+\epsilon}(\mathbb{T}^2)}.$$

The proof is postponed to Appendix D.2

Remark 4.2.3.
Let $s \in \mathbb{R}$, $n \in \mathbb{N}$, then, for any $\psi \in H^s_p(\mathbb{T}^n)$, one has $e^{i\Delta t}((-\Delta)^{s/2}\psi) = (-\Delta)^{s/2}(e^{i\Delta t}\psi)$.

From the above Lemmas and by the definition of the norm in Bessel space $H^s_p(\mathbb{T}^2)$ one immediately gets the following Corollary.

Corollary 4.2.4. *$\forall \epsilon > 0$, $p \geq 4$ and for any $\psi \in H^{s+\epsilon}(\mathbb{T}^2)$, $s \in \mathbb{R}$, one has*

$$\|e^{\mathrm{i}\Delta \mathrm{t}}\psi\|_{L^p_t H^s_{p,x}} \preceq \|\psi\|_{H^{s+1-\frac{4}{p}+\epsilon}(\mathbb{T}^2)}. \tag{4.2.7}$$

Furthermore, recalling that $\|\psi\|_{H^s_p(\mathbb{T}^2)}$ and $\|(I-\Delta)^{s/2}\psi\|_{L^p(\mathbb{T}^2)}$ are equivalent norms, we will use the following Lemma, which proof is postponed to Appendix D.2.

Lemma 4.2.5. *[Leibniz rule for fractional Laplacian on torus, Grafakos-2018] Let $s>0, n\in\mathbb{N}$, $1<p,p_1,p_2,p_3,p_4<\infty$, $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{p}$, then there exists $C(s,n,p,p_1,p_2,q_1,q_2)>0$ s.t. for any f,g smooth 2π-periodic functions (in each variable) on $\mathbb{R}^n$, denoting $(I-\Delta)^{s/2}=J_s$, one has*

$$\|J_s(fg)\|_{L^p(\mathbb{T}^n)} \le C\|J_s(f)\|_{L^{p_1}(\mathbb{T}^n)}\|g\|_{L^{p_2}(\mathbb{T}^n)} + \|f\|_{L^{q_1}(\mathbb{T}^n)}\|J_s(g)\|_{L^{q_2}(\mathbb{T}^n)}. \tag{4.2.8}$$

Using all these results, we have the following results about the estimate of $\chi_i,\ \Psi_i\ i=1,2,3$ and their gradient (for the proof, see Appendix D.1).

Lemma 4.2.6. *For any $s\ge 0$, $\epsilon>0$ one has*

$$\left\|\widetilde{\langle\Psi_1\rangle}(\alpha,\psi_1,\psi_2,\psi_3,\psi_4)\right\|_{C^K(\mathbb{T}^d,\mathbb{R})} \preceq \|\psi_1\|_{H^{-s}}\prod_{j=2}^{4}\|\psi_j\|_{H^{s+2\epsilon}}, \tag{4.2.9}$$

$$\|\widetilde{\chi_1}(\alpha,\psi_1,\psi_2,\psi_3,\psi_4)\|_{C^K(\mathbb{T}^d,\mathbb{R})} \preceq \|\psi_1\|_{H^{-s}}\prod_{j=2}^{4}\|\psi_j\|_{H^{s+2\epsilon}}. \tag{4.2.10}$$

Lemma 4.2.7. *For any $s\ge 0$, $\epsilon>0$ one has*

$$\left\|\widetilde{\langle\Psi_2\rangle}(\alpha,\psi_1,\dots,\psi_6)\right\|_{C^K(\mathbb{T}^d,\mathbb{R})} \preceq \|\psi_1\|_{H^{\varepsilon}}\prod_{2\le i\le 6}\|\psi_i\|_{H^{s+\frac{1}{2}+\varepsilon}}, \tag{4.2.11}$$

$$\|\widetilde{\chi_2}(\alpha,\psi_1,\dots,\psi_6)\|_{C^K(\mathbb{T}^d,\mathbb{R})} \preceq \|\psi_1\|_{H^{\epsilon}}\prod_{2\le i\le 6}\|\psi_i\|_{H^{s+\frac{1}{2}+\epsilon}}. \tag{4.2.12}$$

Lemma 4.2.8. *For any $s\ge 0$, $\epsilon>0$, one has*

$$\left\|\widetilde{\langle\Psi_3\rangle}(\alpha,\psi_1,\dots,\psi_8)\right\|_{C^K(\mathbb{T}^d,\mathbb{R})} \preceq \|\psi_1\|_{H^{\epsilon}}\prod_{j=2}^{8}\|\psi_j\|_{H^{s+\epsilon+\frac{2}{3}}}, \tag{4.2.13}$$

$$\|\widetilde{\chi_3}(\alpha,\psi_1,\dots,\psi_8)\|_{C^K(\mathbb{T}^d,\mathbb{R})} \preceq \|\psi_1\|_{H^{\epsilon}}\prod_{j=2}^{8}\|\psi_j\|_{H^{s+\epsilon+\frac{2}{3}}}. \tag{4.2.14}$$

The proofs of these lemmas are very technical and are in Appendix D.1.

4.3 Proof of Theorems 0.0.5 and 0.0.6

Lemma 4.3.1. *Let W be a homogeneous polynomial of degree n and $\widetilde{W}$ the unique symmetric multilinear form s.t.*

$$\widetilde{W}(\psi,...,\psi) = W(\psi). \tag{4.3.1}$$

If there exist $\alpha, \beta \in \mathbb{R}$ s.t.

$$|\widetilde{W}(\psi_1,...,\psi_n)| \preceq \|\psi_1\|_{H^\alpha} \prod_{j=2}^{n} \|\psi_j\|_{H^\beta}, \tag{4.3.2}$$

then one has

$$\left\| \widetilde{\nabla_{L^2} W}(\psi_1,...\psi_{n-1}) \right\|_{H^{-\alpha}} \preceq \prod_{j=1}^{n-1} \|\psi_j\|_{H^\beta}, \tag{4.3.3}$$

Proof. For any test function $h \in H^\alpha$ with $\|h\|_{H^\alpha} = 1$, we need to consider

$$\langle \widetilde{\nabla_{L^2} W}(\psi_1,...,\psi_{n-1}); h \rangle_{L^2}.$$

For any $i = 1,..,n$, we denote $\varphi_i = \psi_i$ and $\varphi_n = h$, so we have

$$\langle \widetilde{\nabla_{L^2} W}(\psi_1,...,\psi_{n-1}); h \rangle_{L^2} = n\widetilde{W}(\psi_1,...,\psi_{n-1},h) = n\widetilde{W}(\varphi_1,...,\varphi_n).$$

Using (4.3.2), we obtain

$$\left| \langle \widetilde{\nabla_{L^2} W}(\psi_1,...,\psi_{n-1}); h \rangle_{L^2} \right| \preceq \|h\|_{H^\alpha} \prod_{j=2}^{n} \|\psi_j\|_{H^\beta},$$

so we get the thesis. □

Remark 4.3.2. By Lemma 4.3.1 and Lemma 4.2.6, we obtain that for any $s \geq 0$, $\epsilon > 0$, one has

$$\|\widetilde{\nabla_{L^2} \langle \Psi_1 \rangle}(\alpha, \psi_1, \psi_2, \psi_3)\|_{C^K(\mathbb{T}^d, H^s)} \preceq \prod_{i=1}^{3} \|\psi_i\|_{H^{s+\epsilon}}, \tag{4.3.4}$$

$$\|\widetilde{\nabla_{L^2} \chi_1}(\alpha, \psi_1, \psi_2, \psi_3)\|_{C^K(\mathbb{T}^d, H^s)} \preceq \prod_{i=1}^{3} \|\psi_i\|_{H^{s+\epsilon}} \tag{4.3.5}$$

and for any $s > 0$, $0 < \epsilon \leq s$,

$$\|\widetilde{\nabla_{L^2} \langle \Psi_1 \rangle}(\alpha, \psi_1, \psi_2, \psi_3)\|_{C^K(\mathbb{T}^d, H^{-s})} \preceq \|\psi_1\|_{H^{-s+\epsilon}} \prod_{2 \leq i \leq 3} \|\psi_i\|_{H^s}, \tag{4.3.6}$$

$$\|\widetilde{\nabla_{L^2} \chi_1}(\alpha, \psi_1, \psi_2, \psi_3)\|_{C^K(\mathbb{T}^d, H^{-s})} \preceq \|\psi_1\|_{H^{-s+\epsilon}} \prod_{2 \leq i \leq 3} \|\psi_i\|_{H^s}. \tag{4.3.7}$$

Remark 4.3.3. By Lemma 4.3.1, Lemma 4.2.7 and Lemma 4.2.8 we get that for any $s \geq 0, \epsilon > 0$, one has

$$\left\| \widetilde{\nabla_{L^2} \langle \Psi_2 \rangle}(\alpha, \psi_1, \ldots, \psi_5) \right\|_{C^K(\mathbb{T}^d, H^{-\epsilon})} \preceq \prod_{1 \leq i \leq 5} \|\psi_i\|_{H^{s+\frac{1}{2}+\epsilon}}, \tag{4.3.8}$$

$$\left\| \widetilde{\nabla_{L^2} \chi_2}(\alpha, \psi_1, \ldots, \psi_5) \right\|_{C^K(\mathbb{T}^d, H^{-\epsilon})} \preceq \prod_{1 \leq i \leq 5} \|\psi_i\|_{H^{s+\frac{1}{2}+\epsilon}}, \tag{4.3.9}$$

$$\left\| \widetilde{\nabla_{L^2} \langle \Psi_3 \rangle}(\alpha, \psi_1, \ldots, \psi_7) \right\|_{C^K(\mathbb{T}^d, H^{-\epsilon})} \preceq \prod_{j=1}^{7} \|\psi_j\|_{H^{s+\epsilon+\frac{2}{3}}}, \tag{4.3.10}$$

$$\left\| \widetilde{\nabla_{L^2} \chi_3}(\alpha, \psi_1, \ldots, \psi_7) \right\|_{C^K(\mathbb{T}^d, H^{-\epsilon})} \preceq \prod_{j=1}^{7} \|\psi_j\|_{H^{s+\epsilon+\frac{2}{3}}}. \tag{4.3.11}$$

Lemma 4.3.4. *For any $s \in (0,1)$, $\epsilon > 0$, one has*

$$\|\psi_j\|_{H^s_6(\mathbb{T}^2)} \preceq \|\psi_j\|_{H^{s+\frac{2}{3}+\epsilon}(\mathbb{T}^2)}. \tag{4.3.12}$$

Proof. See Appendix D.2. □

Lemma 4.3.5. *For any $s \in (0,1)$, $\epsilon > 0$, we have*

$$\left\| \widetilde{H_1}(\alpha, \psi_1, \psi_2, \psi_3, \psi_4) \right\|_{C^K(\mathbb{T}^d, \mathbb{R})} \preceq \|\psi_1\|_{H^{-s}} \prod_{j=2}^{4} \|\psi_j\|_{H^{s+\frac{2}{3}+\epsilon}},$$

$$\left\| \widetilde{\nabla_{L^2} H_1}\,(\alpha, \psi_1, \psi_2, \psi_3) \right\|_{C^K(\mathbb{T}^d, H^s)} \preceq \prod_{i=1}^{3} \|\psi_j\|_{H^{s+\frac{2}{3}+\epsilon}}.$$

Proof.

$$\begin{aligned}
\left| \widetilde{H_1}(\alpha, \psi_1, \psi_2, \psi_3, \psi_4) \right| &\preceq \left| \int_{\mathbb{T}^2} a(x, \alpha) \psi_1 \psi_2 \psi_3 \psi_4 dx \right| \\
&\preceq \|a(x, \alpha)\psi_1\|_{H^{-s}} \cdot \left\| \prod_{j=2}^{4} \psi_j \right\|_{H^s} \\
&\preceq \|\psi_1\|_{H^{-s}} \cdot \left\| \prod_{j=2}^{4} \psi_j \right\|_{H^s} \\
&\preceq \|\psi_1\|_{H^{-s}} \cdot \prod_{j=2}^{4} \|\psi_j\|_{H^s_6} \\
&\preceq \|\psi_1\|_{H^{-s}} \cdot \prod_{j=2}^{4} \|\psi_j\|_{H^{s+\frac{2}{3}+\epsilon}}.
\end{aligned}$$

where in the third line we use Lemma 4.2.5, knowing that $a \in C^\infty(\mathbb{T}^{d+2})$, and in the last line we use Lemma 4.3.4.
Using Lemma 4.3.1, we get also that

$$\left\| \widetilde{\nabla_{L^2} H_1} (\alpha, \psi_1, \psi_2, \psi_3) \right\|_{C^K(\mathbb{T}^d, H^s)} \preceq \prod_{i=1}^{3} \|\psi_j\|_{H^{s+\frac{2}{3}+\epsilon}}.$$

□

Lemma 4.3.6. *For any $\epsilon > 0$, one has*

$$\left\| \widetilde{\Phi_1}(\alpha, \psi_1, \psi_2, \psi_3, \psi_4) \right\|_{C^K(\mathbb{T}^d, \mathbb{R})} \preceq \prod_{i=1}^{4} \|\psi_i\|_{H^{\frac{2}{3}+\epsilon}}, \tag{4.3.13}$$

$$\left\| \widetilde{\Phi_2}(\alpha, \psi_1, ..., \psi_6) \right\|_{C^K(\mathbb{T}^d, \mathbb{R})} \preceq \prod_{i=1}^{6} \|\psi_i\|_{H^{\frac{1}{2}+\epsilon}}, \tag{4.3.14}$$

$$\left\| \widetilde{\Phi_3}(\alpha, \psi_1, ..., \psi_8) \right\|_{C^K(\mathbb{T}^d, \mathbb{R})} \preceq \prod_{i=1}^{8} \|\psi_i\|_{H^{\frac{2}{3}+\epsilon}}, \tag{4.3.15}$$

$$\left\| \widetilde{\nabla_{L^2} \Phi_3}(\alpha, \psi_1, ..., \psi_7) \right\|_{C^K(\mathbb{T}^d, H^{-\epsilon})} \preceq \prod_{i=1}^{7} \|\psi_i\|_{H^{\frac{2}{3}+\epsilon}}. \tag{4.3.16}$$

Proof. By definition, we have

$$\Phi_1 = E_1 H_0, \ \Phi_2 = E_2 H_0, \ \Phi_3 = E_3 H_0.$$

Using the definition of E_i and of Ψ_i for $i = 1, 2, 3$ and the homological equation (4.1.8), denoting $h = \sum \omega_j I_j$, one has

$$\Phi_1 = \Psi_1 - \langle \Psi_1 \rangle + \{h, \chi_1\}, \tag{4.3.17}$$

$$\Phi_2 = - L_{\chi_1} \langle \Psi_1 \rangle - \langle \Psi_2 \rangle + \frac{1}{2} L_{\chi_1} \{h, \chi_1\} + \{h, \chi_2\}, \tag{4.3.18}$$

$$\begin{aligned} \Phi_3 = & - \frac{1}{2} L_{\chi_1}^2 \langle \Psi_1 \rangle + \frac{1}{6} L_{\chi_1}^2 \{h, \chi_1\} - L_{\chi_1} \langle \Psi_2 \rangle \\ & - L_{\chi_2} \langle \Psi_1 \rangle + \frac{2}{3} L_{\chi_2} \{h, \chi_1\} - \langle \Psi_3 \rangle + \{h, \chi_3\}. \end{aligned} \tag{4.3.19}$$

To get (4.3.13), we need to estimate $\left\| \widetilde{\Psi_1}(\alpha, \psi_1, \psi_2, \psi_3, \psi_4) \right\|_{C^K(\mathbb{T}^d, \mathbb{R})}$, $\left\| \widetilde{\langle \Psi_1 \rangle}(\alpha, \psi_1, \psi_2, \psi_3, \psi_4) \right\|_{C^K(\mathbb{T}^d, \mathbb{R})}$ and $\left\| \widetilde{\{h, \chi_1\}}(\alpha, \psi_1, \psi_2, \psi_3, \psi_4) \right\|_{C^K(\mathbb{T}^d, \mathbb{R})}$. Using the fact that $\Psi_1 = H_1$, $\alpha \in C^\infty(\mathbb{T}^d)$, $h = \sum \omega_j I_j$ and that $L_{\chi_{i_k}} h = \sum \omega_j \frac{\partial}{\partial \alpha_j}$, by Lemma 4.3.5 and Lemma 4.2.6, one has (4.3.13).

To prove (4.3.14), it is sufficient to note that

$$\begin{aligned}
&\left\|\widetilde{\Phi_2}(\alpha,\psi_1,...,\psi_6)\right\|_{C^K(\mathbb{T}^d,\mathbb{R})}\\
&\preceq \left\|\widetilde{\nabla_{L^2}\chi_1}(\alpha,\psi_1,...,\psi_3)\right\|_{C^K(\mathbb{T}^d,H^{\epsilon})}\left\|\widetilde{\nabla_{L^2}\langle\Psi_1\rangle}(\alpha,\psi_4,...,\psi_6)\right\|_{C^K(\mathbb{T}^d,H^{\epsilon})}\\
&+\|\widetilde{\chi_2}(\alpha,\psi_1,...,\psi_6)\|_{C^K(\mathbb{T}^d,\mathbb{R})}+\left\|\widetilde{\langle\Psi_2\rangle}(\alpha,\psi_1,...,\psi_6)\right\|_{C^K(\mathbb{T}^d,\mathbb{R})}.
\end{aligned}$$

So, by Remark 4.3.2 and Lemma 4.2.7, one gets (4.3.14). In a similar way, we know that

$$\begin{aligned}
&\left\|\widetilde{\Phi_3}(\alpha,\psi_1,...,\psi_8)\right\|_{C^K(\mathbb{T}^d,\mathbb{R})}\\
&\preceq \left\|\widetilde{\nabla_{L^2}\chi_1}(\alpha,\psi_1,...,\psi_3)\right\|_{C^K(\mathbb{T}^d,H^{\epsilon})}\left\|\nabla_{L^2}\widetilde{L_{\chi_1}\langle}\Psi_1\rangle(\alpha,\psi_4,...,\psi_8)\right\|_{C^K(\mathbb{T}^d,H^{-\epsilon})}\\
&+\left\|\widetilde{\nabla_{L^2}\chi_1}(\alpha,\psi_1,...,\psi_3)\right\|_{C^K(\mathbb{T}^d,H^{\epsilon})}\left\|\widetilde{\nabla_{L^2}L_{\chi_1}}\chi_1(\alpha,\psi_4,...,\psi_8)\right\|_{C^K(\mathbb{T}^d,H^{-\epsilon})}\\
&+\left\|\widetilde{\nabla_{L^2}\chi_1}(\alpha,\psi_1,...,\psi_3)\right\|_{C^K(\mathbb{T}^d,H^{\epsilon})}\left\|\widetilde{\nabla_{L^2}\langle\Psi_2\rangle}(\alpha,\psi_4,...,\psi_8)\right\|_{C^K(\mathbb{T}^d,H^{-\epsilon})}\\
&+\left\|\widetilde{\nabla_{L^2}\langle\Psi_1\rangle}(\alpha,\psi_1,...,\psi_3)\right\|_{C^K(\mathbb{T}^d,H^{\epsilon})}\left\|\widetilde{\nabla_{L^2}\chi_2}(\alpha,\psi_4,...,\psi_8)\right\|_{C^K(\mathbb{T}^d,H^{-\epsilon})}\\
&+\left\|\widetilde{\nabla_{L^2}\chi_1}(\alpha,\psi_1,...,\psi_3)\right\|_{C^K(\mathbb{T}^d,H^{\epsilon})}\left\|\widetilde{\nabla_{L^2}\chi_2}(\alpha,\psi_4,...,\psi_8)\right\|_{C^K(\mathbb{T}^d,H^{-\epsilon})}\\
&+\|\widetilde{\chi_3}(\alpha,\psi_1,...,\psi_8)\|_{C^K(\mathbb{T}^d,\mathbb{R})}+\left\|\widetilde{\langle\Psi_3\rangle}(\alpha,\psi_1,...,\psi_8)\right\|_{C^K(\mathbb{T}^d,\mathbb{R})}.
\end{aligned}$$

By Lemmas 4.3.1, 4.2.8, Remarks 4.3.2, 4.3.3 and proceeding as in Lemma 4.2.7, we get (4.3.15).
The proof of (4.3.16) is a little more complicated. By the definition of Φ_3, we have

$$\begin{aligned}
&\left\|\widetilde{\nabla_{L^2}\Phi_3}(\alpha,\psi_1,...,\psi_7)\right\|_{C^K(\mathbb{T}^d,H^{-\epsilon})}\\
&\preceq \left\|\widetilde{\nabla_{L^2}\chi_1}\left(\alpha,\psi_1,\psi_2,\nabla_{L^2}\widetilde{L_{\chi_1}\langle}\Psi_1\rangle(\alpha,\psi_3,...,\psi_7)\right)\right\|_{C^K(\mathbb{T}^d,H^{-\epsilon})} && (4.3.20)\\
&+\left\|\widetilde{\nabla_{L^2}\chi_1}\left(\alpha,\psi_1,\psi_2,\widetilde{\nabla_{L^2}L_{\chi_1}}\chi_1(\alpha,\psi_3,...,\psi_7)\right)\right\|_{C^K(\mathbb{T}^d,H^{-\epsilon})} && (4.3.21)\\
&+\left\|\widetilde{\nabla_{L^2}\chi_1}\left(\alpha,\psi_1,\psi_2,\widetilde{\nabla_{L^2}\langle\Psi_2\rangle}(\psi_3,...,\psi_7)\right)\right\|_{C^K(\mathbb{T}^d,H^{-\epsilon})} && (4.3.22)\\
&+\left\|\widetilde{\nabla_{L^2}\langle\Psi_1\rangle}\left(\psi_1,\psi_2,\widetilde{\nabla_{L^2}\chi_2}(\alpha,\psi_3,...,\psi_7)\right)\right\|_{C^K(\mathbb{T}^d,H^{-\epsilon})} && (4.3.23)\\
&+\left\|\widetilde{\nabla_{L^2}\chi_3}\left(\alpha,\psi_1,\psi_2,\psi_3,...,\psi_7\right)\right\|_{C^K(\mathbb{T}^d,H^{-\epsilon})} && (4.3.24)\\
&+\left\|\widetilde{\nabla_{L^2}\langle\Psi_3\rangle}\left(\alpha,\psi_1,\psi_2,\psi_3,...,\psi_7\right)\right\|_{C^K(\mathbb{T}^d,H^{-\epsilon})}. && (4.3.25)
\end{aligned}$$

By Remark 4.3.3, we get the estimate of (4.3.24) and (4.3.25). To obtain the thesis we estimate explicitly only (4.3.22), since the other terms are similar. By Remarks 4.3.2, 4.3.3, we have

$$\begin{aligned}
&\left\|\widetilde{\nabla_{L^2}\chi_1}\left(\alpha,\psi_1,\psi_2,\widetilde{\nabla_{L^2}\langle\Psi_2\rangle}(\alpha,\psi_3,...,\psi_7)\right)\right\|_{C^K(\mathbb{T}^d,H^{-\epsilon})}\\
\preceq&\|\psi_1\|_{H^\epsilon}\|\psi_2\|_{H^\epsilon}\left\|\widetilde{\nabla_{L^2}\langle\Psi_2\rangle}(\alpha,\psi_3,...,\psi_7)\right\|_{C^K(\mathbb{T}^d,H^{-\epsilon})}\\
\preceq&\prod_{i=1}^{7}\|\psi_i\|_{H^{\frac{1}{2}+\epsilon}}\leq\prod_{i=1}^{7}\|\psi_i\|_{H^{\frac{2}{3}+\epsilon}}.
\end{aligned}$$

□

Proof of Theorem 0.0.5. We have

$$\begin{aligned}
&\left|\frac{d}{dt}\Phi^{(3)}\right|=|\{\Phi_3,H_1\}|\\
\preceq&\left\|\widetilde{\nabla_{L^2}\Phi_3}\left(\alpha,\psi_1,\psi_2,\psi_3\right)\right\|_{C^K(\mathbb{T}^d,H^{-\epsilon})}\left\|\widetilde{\nabla_{L^2}H_1}\left(\alpha,\psi_1,\psi_2,\psi_3\right)\right\|_{C^K(\mathbb{T}^d,H^{\epsilon})}\\
\preceq&\|\psi\|^{10}_{H^{\frac{2}{3}+\epsilon}}
\end{aligned}$$

where we use Lemma 4.3.6 and Lemma 4.3.5. Using Lemma 4.3.6, we get (4.0.5). □

Proof of Theorem 0.0.6. First remark that local existence in H^1 is standard and that

$$\|\psi\|^2_{H^1}=\|\psi\|^2_{L^2}+H_0(\psi)\ ,$$

thus, exploiting the conservation of the L^2 norm and Theorem 0.0.5 one can bound the H^1 norm of the solution for the considered times. □

Appendix A

Lemmas on Gaussian and Gibbs measure

First, we recall that both Gibbs and Gaussian measures are constructed with a limit procedure starting from the "finite dimensional" measure which, in the Gaussian case, is defined by

$$\mu_{\beta,g,N} := \frac{e^{-\frac{\beta}{2}\|P_{\leq N}(\psi)\|^2_{H^1}}}{Z_{g,N}(\beta)} = \frac{e^{-\frac{\beta}{2}\sum_{|k|\leq N}\left(1+k^2\right)|\psi_k|^2}}{Z_{g,N}(\beta)},$$

$$Z_{g,N}(\beta) := \int_{P_{\leq N}(H^s)} e^{-\frac{\beta}{2}\sum_{|k|\leq N}\left(1+k^2\right)|\psi_k|^2} \prod_{|k|\leq N} d\psi_k d\bar{\psi}_k,$$

where $P_{\leq N}\left(\{\psi_k\}_{k\in\mathbb{Z}}\right) := \{\psi_k\}_{|k|\leq N}$. (See [16]).

Lemma A.0.1. *Let N be an integer, $1 > \gamma > 0$, then there exists $\tilde{C}(\gamma) > 0$ s.t. for any $\beta > 0$ one has*

$$\frac{\int_{P_{\leq N}(H^s)} \prod_{|k|\leq N} \chi_{\left\{|\psi_k| < \frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}} e^{-\frac{\beta}{2}\left(1+k^2\right)|\psi_k|^2} d\psi_k d\bar{\psi}_k}{Z_{g,N}(\beta)} \geq e^{-\tilde{C}(\gamma)}.$$

Moreover $\tilde{C}$ is independent of N.

Proof. Using the independence of all the variables, one gets

$$\frac{\int_{P_{\leq N}(H^s)} \prod_{|k|\leq N} \chi_{\left\{|\psi_k| < \frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}} e^{-\frac{\beta}{2}\left(1+k^2\right)|\psi_k|^2} d\psi_k d\bar{\psi}_k}{Z_{g,N}(\beta)} =$$

$$= \prod_{|k|\le N} \frac{2\pi \int_0^\infty \chi_{\left\{\rho_k < \frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}} e^{-\frac{\beta}{2}\left(1+k^2\right)\rho_k^2} \rho_k d\rho_k}{2\pi \int_0^\infty e^{-\frac{\beta}{2}(1+k^2)\rho_k^2} \rho_k d\rho_k} = \prod_{|k|\le N} \frac{\int_0^{\frac{\left(1+k^2\right)^{1-\gamma}}{2}} e^{-z_k} dz_k}{\int_0^\infty e^{-z_k} dz_k} =$$

$$= \prod_{|k|\le N} \left(1 - e^{-\frac{\left(1+k^2\right)^{1-\gamma}}{2}}\right) \ge \prod_{k\in\mathbb{Z}} \left(1 - e^{-\frac{\left(1+k^2\right)^{1-\gamma}}{2}}\right)$$

$$= e^{\sum_{|k|\in\mathbb{Z}} \log\left(1-e^{-\frac{\left(1+k^2\right)^{1-\gamma}}{2}}\right)} = e^{-\tilde{C}(\gamma)}.$$

□

As $N \to \infty$, we get the following lemma

Lemma A.0.2. *Let γ be $1 > \gamma > 0$. Then, for any $\beta > 0$, one has*

$$\lim_{N\to\infty} \frac{\int_{P_{\le N}(H^s)} \prod_{|k|\le N} \chi_{\left\{|\psi_k| < \frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}} e^{-\frac{\beta}{2}\left(1+k^2\right)|\psi_k|^2} d\psi_k d\bar{\psi}_k}{Z_{g,N}(\beta)}$$

$$= \int_{H^s} \left(\prod_{k\in\mathbb{Z}}^{\infty} \chi_{\left\{|\psi_k| < \frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}} \right) d\mu_{g,\beta}.$$

Proof. For any $M > N$, $M \in \mathbb{N}$, one has

$$\int_{P_{\le N}(H^s)} \prod_{|k|\le N} \chi_{\left\{|\psi_k| < \frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}} \frac{e^{-\frac{\beta}{2}\sum_{|k|\le N}\left(1+k^2\right)|\psi_k|^2} \prod_{|k|\le N} d\psi_k d\bar{\psi}_k}{Z_{g,N}(\beta)}$$

$$= \int_{P_{\le M}(H^s)} \prod_{|k|\le N} \chi_{\left\{|\psi_k| < \frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}} \frac{e^{-\frac{\beta}{2}\sum_{|k|<M}\left(1+k^2\right)|\psi_k|^2} \prod_{|k|<M} d\psi_k d\bar{\psi}_k}{Z_{g,M}(\beta)}.$$

So, one has

$$\lim_{M\to\infty} \int_{P_{\le M}(H^s)} \prod_{|k|\le N} \chi_{\left\{|\psi_k| < \frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}} \frac{e^{-\frac{\beta}{2}\sum_{|k|<M}\left(1+k^2\right)|\psi_k|^2} \prod_{|k|<M} d\psi_k d\bar{\psi}_k}{Z_{g,M}(\beta)} =$$

$$= \int_{H^s} \prod_{|k|\le N} \chi_{\left\{|\psi_k| < \frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}} d\mu_{g,\beta}.$$

But $\prod_{|k|\le N}\chi_{\left\{|\psi_k|<\frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}} \to \prod_{k\in\mathbb{Z}}\chi_{\left\{|\psi_k|<\frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}}$ a.e. on H^s as $N\to\infty$. Since $1\in L^1(H^s,\mu_{g,\beta})$ and $\prod_{|k|\le N}\chi_{\left\{|\psi_k|<\frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}}\le 1$, by Lebesgue's dominated convergence Theorem,

$$\lim_{N\to\infty}\int_{H^s}\prod_{|k|\le N}\chi_{\left\{|\psi_k|<\frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}}d\mu_{g,\beta}=\int_{H^s}\lim_{N\to\infty}\prod_{|k|\le N}\chi_{\left\{|\psi_k|<\frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}}d\mu_{g,\beta}=$$

$$=\int_{H^s}\prod_{k\in\mathbb{Z}}\chi_{\left\{|\psi_k|<\frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}}d\mu_{g,\beta}.$$

□

Remark A.0.3. From Lemma A.0.1 and Lemma A.0.2, we know that, if $1>\gamma>0$ and $\beta>0$, one has

$$\int_{H^s}\prod_{k\in\mathbb{Z}}\chi_{\left\{|\psi_k|<\frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}}d\mu_{g,\beta}\ge e^{-\tilde{C}(\gamma)}. \tag{A.0.1}$$

Proof of Lemma 2.0.4 We remark that $P=\sum_{j=2}^{q}H_{2j}=\sum_{j=2}^{q}\frac{c_j}{2j}\|\psi\|_{L^{2j}}^{2j}$. The first inequality is obvious.
We analyze now the second inequality. By the definition of P, if we fix s_1, by Sobolev's inequality $H^{s_1}(\mathbb{T})\subset L^r(\mathbb{T})$ if $r\in[1,\frac{2}{1-2s_1}]$. Therefore, choosing $\frac{q-1}{2q}<s_1<\frac{1}{2}$, there exists a constant C_{sob} s.t.

$$\|\psi\|_{L^{2j}}<C_{sob}^{\frac{1}{2j}}\|\psi\|_{H^{s_1}},\ j=2,...,q. \tag{A.0.2}$$

We fix $\frac{1}{2}+s_1<\gamma<1$, denote $D':=\sum_{j\in\mathbb{Z}}\frac{1}{(1+j^2)^{\gamma-s_1}}$, then we have:

$$\int_{H^s}e^{-\beta P}d\mu_{g,\beta}\ge\int_{H^s}\chi_{\left\{\|\psi\|_{H^{s_1}}^2\le\frac{D'}{\beta}\right\}}e^{-\beta P}d\mu_{g,\beta}\ge$$

$$\int_{H^s}\chi_{\left\{\|\psi\|_{H^{s_1}}^2\le\frac{D'}{\beta}\right\}}e^{-\frac{C_{sob}}{\beta}\left(\sum_{\substack{j=2,\dots,q\\ c_j\ge0}}\frac{c_jD'^j}{\beta^{j-1}}\right)}d\mu_{g,\beta}\ge\int_{H^s}\chi_{\left\{\|\psi\|_{H^{s_1}}^2\le\frac{D'}{\beta}\right\}}e^{-\frac{C_{sob}}{\beta}q\max_j c_jD'^j}d\mu_{g,\beta}$$

$$\ge e^{-\frac{C_{sob}}{\beta}q\max_j c_jD'^j}\int_{H^s}\prod_{k\in\mathbb{Z}}\chi_{\left\{|\psi_k|<\frac{1}{(1+k^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}}d\mu_{g,\beta}$$

$$\ge e^{-\frac{C_{sob}}{\beta}q\max c_jD'^j}e^{-\tilde{C}(\gamma)}\ge e^{-2\tilde{C}(\gamma)},$$

where the inequalities in the last line are true thanks to Lemma A.0.2 and for β sufficiently large. □

Remark A.0.4. μ_β is a good probability measure on H^s since $\mu_\beta < \mu_{g,\beta}$ and $e^{-2\tilde{C}(\gamma)} \leq \frac{Z(\beta)}{Z_g(\beta)} \leq 1$.

For the proof is sufficient to note that

$$\int_{H^s} e^{-\beta\left(\sum_{i=4}^{n} \frac{c_i}{i}\|\psi\|_{L^i}^i\right)} d\mu_{g,\beta} = \frac{Z(\beta)}{Z_g(\beta)}.$$

Using this result, we can obtain Lemma 3.1.2 to estimate the L^2-norm in the Gibbs measure with the norm in Gaussian measure.
Proof of Lemma 3.1.2 We have

$$\|f\|_{\mu_\beta}^2 = \int_{H^s} |f|^2 d\mu_\beta \leq \frac{\int_{H^s} |f|^2 d\mu_{g,\beta}}{\int_{H^s} e^{-\beta P} d\mu_{g,\beta}}$$

and, from Lemma 2.0.4,

$$\|f\|_{\mu_\beta}^2 \leq \|f\|_{g,\beta}^2 e^{2\tilde{C}(\gamma)}.$$

□

Proof of Lemma 2.0.6 It is a simple application of Lemma 3.1.2 with $f = \chi(A)$.

□

Proof of Lemma 2.1.5 The proof is the same of Lemma 2.0.4. The only difference is that instead of $\|\psi\|_{L^{2j}}$, we have to work with $\|P_{\leq N}\psi\|_{L^{2j}}$, but again, by Sobolev's inequality, we have

$$\|P_{\leq N}\psi\|_{L^{2j}} < C_{sob}^{\frac{1}{2j}} \|P_{\leq N}\psi\|_{H^{s_1}} \leq C_{sob}^{\frac{1}{2j}} \|\psi\|_{H^{s_1}}, \ j = 2, ..., q. \tag{A.0.3}$$

□

Remark A.0.5. The constant $\tilde{C}$ is independent of N and is the same constant of Lemma 3.1.2.

Proof of Lemma 2.1.7 It is a simple application of Lemma 2.1.5 with $f = \chi(A)$. □

Proof of Lemma 3.1.3 As above we fix $\frac{q-1}{2q} < s_1 < \frac{1}{2}$ and $\frac{1}{2} + s_1 < \gamma < 1$, we denote $D' := \sum_{j\in\mathbb{Z}} \frac{1}{(1+j^2)^{\gamma - s_1}}$, so we have:

$$\|f\|_{\mu_\beta}^2 = \int_{H^s} |f|^2 d\mu_\beta \geq \int_{H^s} |f|^2 e^{-\beta P} d\mu_{g,\beta} \geq$$

$$\geq \int_{H^s} |f|^2 \chi_{\left\{\|\psi\|_{H^{s_1}}^2 \leq \frac{D'}{\beta}\right\}} e^{-\beta P} d\mu_{g,\beta} \geq$$

$$\geq e^{-\frac{C_{sob}}{\beta} q \max_j c_j D'^j} \int_{H^s} |f|^2 \chi_{\left\{\|\psi\|^2_{H^{s_1}} \leq \frac{D'}{\beta}\right\}} d\mu_{g,\beta}$$

$$= e^{-\frac{C_{sob}}{\beta} q \max_j c_j D'^j} \left\| f \chi_{\left\{\|\psi\|^2_{H^{s_1}} \leq \frac{D'}{\beta}\right\}} \right\|^2_{g,\beta}.$$

□

We are now ready to give the proof of Lemma 3.1.5, namely the estimate from below of the L^2-norm of the actions in Gibbs measure.

Proof of Lemma 3.1.5 We fix $\frac{q-1}{2q} < s_1 < \frac{1}{2}$ and $\frac{1}{2} + s_1 < \gamma < 1$, we denote $D' := \sum_{j \in \mathbb{Z}} \frac{1}{(1+j^2)^{\gamma - s_1}}$, so

$$\left\| |\psi_{\mathtt{k}}|^2 \chi_{\left\{\|\psi\|^2_{H^{s_1}} \leq \frac{D}{\beta}\right\}} \right\|^2_{g,\beta} \geq \int_{H^s} |\psi_{\mathtt{k}}|^4 \prod_{k \in \mathbb{Z}} \chi_{\left\{|\psi_j| < \frac{1}{(1+j^2)^{\frac{\gamma}{2}} \sqrt{\beta}}\right\}} d\mu_{g,\beta} =$$

$$\lim_{N \to \infty} \frac{\int_{P_{\leq N}(H^s)} |\psi_{\mathtt{k}}|^4 \prod_{j \in \mathbb{Z}} \chi_{\left\{|\psi_j| < \frac{1}{(1+j^2)^{\frac{\gamma}{2}} \sqrt{\beta}}\right\}} e^{-\frac{\beta}{2} \sum_{|j|<N} \left(1+j^2\right) |\psi_j|^2} \prod_{|j|<N} d\psi_j d\bar{\psi}_j}{\int_{P_{\leq N}(H^s)} e^{-\frac{\beta}{2} \sum_{j<N} (1+j^2) |\psi_j|^2} \prod_{|j|<N} d\psi_j d\bar{\psi}_j}. \tag{A.0.4}$$

Using the independence of the variables, we have that (A.0.4) is equal to

$$\frac{\int_{\mathbb{C}} |\psi_{\mathtt{k}}|^4 \chi_{\left\{|\psi_{\mathtt{k}}| < \frac{1}{(1+\mathtt{k}^2)^{\frac{\gamma}{2}} \sqrt{\beta}}\right\}} e^{-\frac{\beta}{2}\left(1+\mathtt{k}^2\right)|\psi_{\mathtt{k}}|^2} d\psi_{\mathtt{k}} d\bar{\psi}_{\mathtt{k}}}{\int_{\mathbb{C}} e^{-\frac{\beta}{2}(1+\mathtt{k}^2)|\psi_{\mathtt{k}}|^2} d\psi_{\mathtt{k}} d\bar{\psi}_{\mathtt{k}}} \times$$

$$\times \lim_{N \to \infty} \frac{\int_{P^{-\mathtt{k}}_{\leq N}(H^s)} \prod_{\substack{j \in \mathbb{Z} \\ j \neq \mathtt{k}}} \chi_{\left\{|\psi_j| < \frac{1}{(1+j^2)^{\frac{\gamma}{2}} \sqrt{\beta}}\right\}} e^{-\frac{\beta}{2} \sum_{\substack{|j|<N \\ j \neq \mathtt{k}}} \left(1+j^2\right)|\psi_j|^2} \prod_{\substack{|j|<N \\ j \neq \mathtt{k}}} d\psi_j d\bar{\psi}_j}{\int_{P^{-\mathtt{k}}_{\leq N}(H^s)} e^{-\frac{\beta}{2} \sum_{\substack{|j|<N \\ j \neq \mathtt{k}}} (1+j^2)|\psi_j|^2} \prod_{\substack{|j|<N \\ j \neq \mathtt{k}}} d\psi_j d\bar{\psi}_j}, \tag{A.0.5}$$

where $P^{-\mathtt{k}}_{\leq N}(H^s)$ the Dirichlet projection onto the frequencies $\{|n| \leq N,\ n \neq \mathtt{k}\}$. Furthermore, since

$$\frac{\int_{\mathbb{C}} \chi_{\left\{|\psi_{\mathtt{k}}| < \frac{1}{(1+\mathtt{k}^2)^{\frac{\gamma}{2}} \sqrt{\beta}}\right\}} e^{-\frac{\beta}{2}\left(1+\mathtt{k}^2\right)|\psi_{\mathtt{k}}|^2} d\psi_{\mathtt{k}} d\bar{\psi}_{\mathtt{k}}}{\int_{\mathbb{C}} e^{-\frac{\beta}{2}(1+\mathtt{k}^2)|\psi_{\mathtt{k}}|^2} d\psi_{\mathtt{k}} d\bar{\psi}_{\mathtt{k}}} < 1,$$

one has that (A.0.5) is lower than

$$\frac{\int_{\mathbb{C}} |\psi_{\mathtt{k}}|^4 \chi_{\left\{|\psi_{\mathtt{k}}|<\frac{1}{(1+\mathtt{k}^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}} e^{-\frac{\beta}{2}\left(1+\mathtt{k}^2\right)|\psi_{\mathtt{k}}|^2} d\psi_{\mathtt{k}} d\bar{\psi}_{\mathtt{k}}}{\int_{\mathbb{C}} e^{-\frac{\beta}{2}(1+\mathtt{k}^2)|\psi_{\mathtt{k}}|^2} d\psi_{\mathtt{k}} d\bar{\psi}_{\mathtt{k}}} \times$$

$$\times \lim_{N\to\infty} \frac{\int_{P_{\leq N}(H^s)} \prod_{j\in\mathbb{Z}} \chi_{\left\{|\psi_j|<\frac{1}{(1+j^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}} e^{-\frac{\beta}{2}\sum_{|j|<N}\left(1+j^2\right)|\psi_j|^2} \prod_{|j|<N} d\psi_j d\bar{\psi}_j}{\int_{P_{\leq N}(H^s)} e^{-\frac{\beta}{2}\sum_{|j|<N}(1+j^2)|\psi_j|^2} \prod_{|j|<N} d\psi_j d\bar{\psi}_j}$$

$$\geq \frac{\int_0^{\frac{1}{(1+\mathtt{k}^2)^{\frac{\gamma}{2}}\sqrt{\beta}}} \rho_{\mathtt{k}}^5 e^{-\frac{\beta}{2}\left(1+\mathtt{k}^2\right)\rho_{\mathtt{k}}^2} d\rho_{\mathtt{k}}}{\int_0^\infty \rho_{\mathtt{k}} e^{-\frac{\beta}{2}(1+\mathtt{k}^2)\rho_{\mathtt{k}}^2} d\rho_{\mathtt{k}}} \int_{H^s} \prod_{k\in\mathbb{Z}} \chi_{\left\{|\psi_j|<\frac{1}{(1+j^2)^{\frac{\gamma}{2}}\sqrt{\beta}}\right\}} d\mu_{g,\beta}$$

$$\geq \frac{4}{\beta^2\left(1+\mathtt{k}^2\right)^2} \int_0^{\frac{\left(1+\mathtt{k}^2\right)^{1-\gamma}}{2}} z_{\mathtt{k}}^2 e^{-z_{\mathtt{k}}} dz_k e^{-2\tilde{C}(\gamma)}$$

$$\geq \frac{e^{-\tilde{C}(\gamma)}}{\beta^2\left(1+\mathtt{k}^2\right)^2} \int_0^{\frac{\left(1+\mathtt{k}^2\right)^{1-\gamma}}{2}} z_{\mathtt{k}}^2 e^{-z_{\mathtt{k}}} dz_{\mathtt{k}} \geq \frac{e^{-\tilde{C}(\gamma)}}{\beta^2\left(1+\mathtt{k}^2\right)^2} \int_0^{\frac{1}{2}} x^2 e^{-x} dx,$$

where in the last line we use Lemma *A*.0.2. So, for β large enough, using Lemma 3.1.3, one has

$$\||\psi_{\mathtt{k}}|^2\|_{\mu_\beta}^2 \geq e^{-\frac{C_{sob}}{\beta} q \max_j c_j D'^q} \left\| |\psi_{\mathtt{k}}|^2 \chi_{\left\{\|\psi\|_{H^{s_1}}^2 \leq \frac{D'}{\beta}\right\}} \right\|_{g,\beta}^2$$

$$\geq e^{-\frac{C_{sob}}{\beta} q \max_j c_j D'^q} \frac{e^{-\tilde{C}(\gamma)}}{\beta^2\left(1+\mathtt{k}^2\right)^2} \int_0^{\frac{1}{2}} x^2 e^{-x} dz_{\mathtt{k}} = \frac{C_1^2(\gamma)}{\beta^2\left(1+\mathtt{k}^2\right)^2}.$$

□

The support of the Gaussian measure is described in the following lemma in which the main part is that we specify the dependence on β of the r.h.s. *Proof of Lemma* 2.1.9 We consider

$$e^{a\beta M^2} \mu_\beta\left(\{\|\psi\|_{H^{s_1}} > M\}\right) \leq e^{2\tilde{C}} e^{a\beta M^2} \mu_{g,\beta}\left(\{\|\psi\|_{H^{s_1}} > M\}\right)$$

$$= e^{2\tilde{C}} \int_{\{\|\psi\|_{H^{s_1}} > M\}\cap H^s} e^{a\beta M^2} d\mu_{g,\beta} \leq e^{2\tilde{C}} \int_{\{\|\psi\|_{H^{s_1}} > M\}\cap H^s} e^{a\|\psi\|_{H^{s_1}}^2} d\mu_{g,\beta}$$

$$\leq e^{2\tilde{C}} \int_{H^s} e^{a\beta\|\psi\|_{H^{s_1}}^2} d\mu_{g,\beta} = e^{2\tilde{C}} \int_{H^s} e^{a\beta\sum_j\left(1+j^2\right)^{s_1}|\psi_j|^2} d\mu_{g,\beta}$$

$$= e^{2\tilde{C}} \frac{\int_{H^s} e^{a\beta \sum_j \left(1+j^2\right)^{s_1}|\psi_j|^2 - \frac{\beta}{2}\sum_j \left(1+j^2\right)|\psi_j|^2} \prod_j d\psi_j d\bar{\psi}_j}{\int_{H^s} e^{-\frac{\beta}{2}\sum_j (1+j^2)|\psi_j|^2} \prod_j d\psi_j d\bar{\psi}_j}$$

$$= e^{2\tilde{C}} \prod_j \frac{\int_{\mathbb{C}} e^{a\beta \left(1+j^2\right)^{s_1}|\psi_j|^2 - \frac{\beta}{2}\left(1+j^2\right)|\psi_j|^2} d\psi_j d\bar{\psi}_j}{\int_{\mathbb{C}} e^{-\frac{\beta}{2}(1+j^2)|\psi_j|^2} d\psi_j d\bar{\psi}_j}. \tag{A.0.6}$$

Using the substitution $\psi_j = \frac{\sqrt{2z_j}}{\sqrt{\beta(1+j^2)}} e^{i\theta_j}$, $z_j \in \mathbb{R}^+$, $\theta_j \in [0, 2\pi)$ and the fact that $\int_{\mathbb{R}^+} e^{-z} dz = 1$, one has that (A.0.6) is equal to

$$e^{2\tilde{C}} \prod_j \int_0^\infty e^{-\left(1-2a\left(1+j^2\right)^{s_1-1}\right)z_k} dz_k$$

$$= e^{2\tilde{C}} \prod_j \left(1 + \frac{2a}{\left(1+j^2\right)^{1-s_1} - 2a}\right) = C.$$

□

Remark A.0.6. From the previous lemma, if M goes to $+\infty$, we obtain that for any $s_1 < \frac{1}{2}$,

$$\mu_\beta \left(\{\|\psi\|_{H^{s_1}} = +\infty\}\right) = 0.$$

In particular, we obtain that, for any $s_1 > s$, $\mu_\beta \left(H^s \setminus H^{s_1}\right) = 0$.

Proof of Lemma 3.1.4 Having fixed β large enough, $n > 0$, and $a < \frac{\beta}{2}$, there exists a constant $C > 0$ s.t. for any $x > C$, $x^n < e^{ax^2}$, so, one has

$$\int_{H^s} \|\psi\|^n_{H^{s_1}} d\mu_{g,\beta} < \int_{\{\|\psi\|_{H^{s_1}} < C\} \cap H^s} \|\psi\|^n_{H^{s_1}} d\mu_{g,\beta} + \int_{\{\|\psi\|_{H^{s_1}} > C\} \cap H^s} e^{a\|\psi\|^2_{H^{s_1}}} d\mu_{g,\beta}$$

$$\leq C^n + \int_{H^s} e^{a\|\psi\|^2_{H^{s_1}}} d\mu_{g,\beta} = C^n + \prod_j \left(1 + \frac{2a}{\beta\left(1+j^2\right)^{1-s_1} - 2a}\right) < \infty,$$

where in the last line we proceed as in Lemma 2.1.9. So we proved that $\|\psi\|^n_{H^{s_1}} \in L^1(H^s, d\mu_{g,\beta})$. By Lemma 3.1.2 we have that $\|\psi\|^n_{H^{s_1}} \in L^1(H^s, d\mu_\beta)$.

□

Proof of Lemma 2.1.8 We know that

$$e^{-\beta\left(\sum_{j=2}^q \frac{c_j}{2j}\int_0^{2\pi} |P_{\leq N}\psi(x)|^{2j} dx\right)} \longrightarrow e^{-\beta\left(\sum_{j=2}^q \frac{c_j}{2j}\int_0^{2\pi} |\psi(x)|^{2j} dx\right)}$$

a.s. respect to $\mu_{g,\beta}$ for $N \to \infty$.

So, by Egorov's Theorem, $e^{-\beta\left(\sum_{j=2}^q \frac{c_j}{2j}\int_0^{2\pi} |P_{\leq N}\psi(x)|^{2j} dx\right)} \longrightarrow e^{-\beta\left(\sum_{j=2}^q \frac{c_j}{2j}\int_0^{2\pi} |\psi(x)|^{2j} dx\right)}$ almost uniformly.

So in particular $e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|P_{\leq N}\psi(x)|^{2j}dx\right)} \longrightarrow e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|\psi(x)|^{2j}dx\right)}$ in measure.
For any $\epsilon > 0, N \in \mathbb{N}$, let

$$A_{N,\epsilon} = \left\{\psi \in H^s(\mathbb{T}) : \left|e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|P_{\leq N}\psi(x)|^{2j}dx\right)} - e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|\psi(x)|^{2j}dx\right)}\right| \leq \frac{1}{2}\epsilon\right\}.$$

By the convergence in measure, for any $q \geq 1$, there exists $N_{0,q} \in \mathbb{N}$ s.t. for any $N > N_{0,q}$ we have $\mu_{g,\beta}\left(A^c_{N,\epsilon}\right) < \left(\frac{\epsilon}{4}\right)^{2q}$. So, in particular, we have

$$\begin{aligned}
&\left\|e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|P_{\leq N}\psi(x)|^{2j}dx\right)} - e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|\psi(x)|^{2j}dx\right)}\right\|_{L^q(\mu_{g,\beta})}\\
\leq&\left\|\left(e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|P_{\leq N}\psi(x)|^{2j}dx\right)} - e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|\psi(x)|^{2j}dx\right)}\right)\chi_{A_{N,\epsilon}}\right\|_{L^q(\mu_{g,\beta})}\\
+&\left\|\left(e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|P_{\leq N}\psi(x)|^{2j}dx\right)} - e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|\psi(x)|^{2j}dx\right)}\right)\chi_{A^c_{N,\epsilon}}\right\|_{L^q(\mu_{g,\beta})}\\
\leq&\frac{1}{2}\epsilon\mu_{g,\beta}\left(A_{N,\epsilon}\right)^{\frac{1}{q}}\\
+&\left\|e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|P_{\leq N}\psi(x)|^{2j}dx\right)} - e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|\psi(x)|^{2j}dx\right)}\right\|_{L^{2q}(\mu_{g,\beta})}\mu_{g,\beta}\left(A^c_{N,\epsilon}\right)^{\frac{1}{2q}}\\
\leq&\frac{1}{2}\epsilon + 2\left(\mu_{g,\beta}\left(A^c_{N,\epsilon}\right)\right)^{\frac{1}{2q}} < \frac{1}{2}\epsilon + 2\left(\frac{\epsilon}{4}\right)^{2q} = \epsilon
\end{aligned} \tag{A.0.7}$$

where in the fourth line we use the definition of $A_{N,\epsilon}$, in the fifth line we use Holder inequality and in the last line we use the fact that

$$\mu_{g,\beta}\left(A_{N,\epsilon}\right) < 1,$$

$$\left\|e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|P_{\leq N}\psi(x)|^{2j}dx\right)}\right\|_{L^q(\mu_{g,\beta})} \leq 1$$

and

$$\left\|e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|\psi(x)|^{2j}dx\right)}\right\|_{L^q(\mu_{g,\beta})} \leq 1.$$

So in particular,

$$e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|P_{\leq N}\psi(x)|^{2j}dx\right)} \longrightarrow e^{-\beta\left(\sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|\psi(x)|^{2j}dx\right)}$$

in $L^q(\mu_{g,\beta})$-norm for any $q \geq 1$.
This implies the thesis.
In fact, remembering that $\tilde{P} = \sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|P_{\leq N}\psi(x)|^{2j}dx$ and $P = \sum_{j=2}^{q}\frac{c_j}{2j}\int_0^{2\pi}|\psi(x)|^{2j}dx$, for any $\epsilon > 0$, there exists $N_0 = \max\{N_{0,1}, N_{0,2}\}$,

s.t. for any $N > N_0$ and for any μ_β-measurable set $A \subset H^s(\mathbb{T})$, one has

$$
\begin{aligned}
&|\mu_{\beta,N}(A) - \mu_\beta(A)| \\
=&\left|\frac{Z_{g,\beta}}{Z_N(\beta)}\int_A e^{-\beta\tilde{P}}d\mu_{g,\beta} - \frac{Z_{g,\beta}}{Z(\beta)}\int_A e^{-\beta P}d\mu_{g,\beta}\right| \\
\leq&\left|\frac{Z_{g,\beta}}{Z_N(\beta)}\int_A \left(e^{-\beta\tilde{P}} - e^{-\beta P}\right)d\mu_{g,\beta}\right| \\
+&\left|\int_A e^{-\beta P}d\mu_{g,\beta}\left(\frac{Z_{g,\beta}}{Z_N(\beta)} - \frac{Z_{g,\beta}}{Z(\beta)}\right)\right| \\
=&\left|\frac{Z_{g,\beta}}{Z_N(\beta)}\int_{H^s} \chi_A\left(e^{-\beta\tilde{P}} - e^{-\beta P}\right)d\mu_{g,\beta}\right| \\
+&\left|\left(\frac{Z_{g,\beta}}{Z_N(\beta)} - \frac{Z_{g,\beta}}{Z(\beta)}\right)\int_{H^s} e^{-\beta P}\chi_A d\mu_{g,\beta}\right| \\
\leq&\frac{Z_{g,\beta}}{Z_N(\beta)}\mu_{g,\beta}(A)^{\frac{1}{2}}\left(\int_{H^s}\left|e^{-\beta\tilde{P}} - e^{-\beta P}\right|^2 d\mu_{g,\beta}\right)^{\frac{1}{2}} \qquad \text{(A.0.8)} \\
+&\left|\frac{Z_{g,\beta}}{Z_N(\beta)} - \frac{Z_{g,\beta}}{Z(\beta)}\right|\int_{H^s} e^{-\beta P}d\mu_{g,\beta} \\
\leq&\frac{Z_{g,\beta}}{Z_N(\beta)}\epsilon + \left|\frac{Z_{g,\beta}}{Z_N(\beta)} - \frac{Z_{g,\beta}}{Z(\beta)}\right| \qquad \text{(A.0.9)}
\end{aligned}
$$

where in line (A.0.8) we use Holder inequality and in line (A.0.9) we use the fact that $\mu_{g,\beta}(A) \leq 1$, that $\left(\int_{H^s}\left|e^{-\beta\tilde{P}} - e^{-\beta P}\right|^2 d\mu_{g,\beta}\right)^{\frac{1}{2}} < \epsilon$ for any $N > N_{0,2}$ and that $0 \leq e^{-\beta P} \leq 1$ so $\int_{H^s} e^{-\beta P}d\mu_{g,\beta} \leq 1$.
Moreover, since

$$\frac{Z_{g,\beta}}{Z_N(\beta)} = \frac{1}{\int_{H^s} e^{-\beta\tilde{P}}d\mu_g}$$

and

$$\frac{Z_{g,\beta}}{Z(\beta)} = \frac{1}{\int_{H^s} e^{-\beta P}d\mu_g},$$

proceeding as in Lemma 2.0.4 and Lemma 2.1.5, we have that there exists $C > 0$ independent of N, N_0, ϵ and β s.t.

$$\frac{Z_{g,\beta}}{Z_N(\beta)} \leq e^{C\left(1+\frac{1}{\beta}\right)} \qquad \text{(A.0.10)}$$

and

$$\begin{aligned}\left|\frac{Z_{g,\beta}}{Z_N(\beta)} - \frac{Z_{g,\beta}}{Z(\beta)}\right| \leq & \frac{\|e^{-\beta\tilde{P}} - e^{-\beta P}\|_{L^1(\mu_{g,\beta})}}{\left(\int_{H^s} e^{-\beta\tilde{P}} d\mu_g\right)\left(\int_{H^s} e^{-\beta P} d\mu_g\right)} \\ \leq & \|e^{-\beta\tilde{P}} - e^{-\beta P}\|_{L^1(\mu_{g,\beta})} e^{2C\left(1+\frac{1}{\beta}\right)}. \end{aligned} \tag{A.0.11}$$

Finally, since $\|e^{-\beta\tilde{P}} - e^{-\beta P}\|_{L^1(\mu_{g,\beta})} < \epsilon$ for any $N > N_{0,1}$, we have that, for any $N > \max\{N_{0,1}, N_{0,2}\}$

$$|\mu_{\beta,N}(A) - \mu_\beta(A)| \leq \frac{Z_{g,\beta}}{Z_N(\beta)}\epsilon + \left|\frac{Z_{g,\beta}}{Z_N(\beta)} - \frac{Z_{g,\beta}}{Z(\beta)}\right| \leq 2\epsilon e^{2C\left(1+\frac{1}{\beta}\right)}. \tag{A.0.12}$$

□

Remark A.0.7. For any $\epsilon, \beta \geq 1$, there exists $N_0 \in \mathbb{N}$ s.t. for any $N > N_0$ and any μ_β-measurable set $A \in H^s(\mathbb{T})$, one has

$$|\mu_{\beta,N}(A) - \mu_\beta(A)| < \epsilon. \tag{A.0.13}$$

Proof. We can repeat the same proof of Lemma 2.1.8 but in this case the last term of (A.0.12) is smaller than $2\epsilon e^{4C}$, loosing the dependence on β. □

Appendix B

Proof of Lemma 2.1.4

We start this section giving two results that are the key point of the proof of Proposition 2.1.2 and of Lemma 2.1.4 that were proved by Burq, Gérard and Tzvetkov in [19, 18].

Theorem B.0.1. *Given* p_1, p_2 *s.t.* $\frac{2}{p_1} + \frac{1}{p_2} = \frac{1}{2}$, $p_1 \geq 2$, $p_2 < \infty$, *the solution* ψ *of* (0.0.1) *satisfies for any finite time interval* I,

$$\|\psi\|_{L^{p_1}(I, L^{p_2}(\mathbb{T}))} \leq C|I|^{\frac{1}{p_1}} \|\psi\|_{H^{\frac{1}{p_1}}(\mathbb{T})}. \tag{B.0.1}$$

Corollary B.0.2. *Given* p_1, p_2 *s.t.* $\frac{2}{p_1} + \frac{1}{p_2} = \frac{1}{2}$, $p_1 \geq 2$, $p_2 < \infty$, *then for any* $f \in L^1([0,T], H^{\frac{1}{p_1}}(\mathbb{T}))$, *one has*

$$\left\| \int_0^t e^{i(t-\tau)\Delta} f(\tau) d\tau \right\|_{L^{p_1}([0,T], L^{p_2}(\mathbb{T}))} \leq CT^{\frac{1}{p_1}} \|f\|_{L^1([0,T], H^{\frac{1}{p_1}}(\mathbb{T}))}. \tag{B.0.2}$$

We present now the proof of the *approximation Lemma* 2.1.4 that is a little modification of the proof by Bourgain in [16].
Proof of Lemma 2.1.4 We fix $s_1 < s < \frac{1}{2}$, we choose $p_1 > q-1$ s.t. $\frac{1}{2} - \frac{1}{p_1} < s_1$. From local theory, we know that the solution of (0.0.1) and of (2.1.1) corresponding to initial data $\psi^N_{|t=0} = \psi_{|t=0} = \psi_0$ are locally well-posed on $[-t,t]$, $t \sim (1+K)^{-\theta}$ uniformly in N, in particular we know that for any $N \in \mathbb{N}$ and for any $\tau \in [-t,t]$, $\|\psi(\tau)\|_{H^{s_1}}$, $\|\psi^N(\tau)\|_{H^{s_1}} \leq 2K$. We fix $0 < \delta \ll K^{1-q}$ and we consider

$$Y_\delta := C([-\delta,\delta], H^{s_1}) \cap L^{p_1}([-\delta,\delta], W^{\sigma,p_2}) \tag{B.0.3}$$

where p_2 is given by $\frac{2}{p_1} + \frac{1}{p_2} = \frac{1}{2}$ and $\sigma = s_1 - \frac{1}{p_1} > \frac{1}{p_2}$. We set

$$\|\psi\|_{Y_\delta} := \max_{|t|\leq\delta} \|\psi(t)\|_{H^{s_1}} + \|(1-\Delta)^{\frac{\sigma}{2}}\psi\|_{L^{p_1}([-\delta,\delta], L^{p_2})}. \tag{B.0.4}$$

By the Duhamel formula, we have

$$\psi(t) = e^{i\Delta t}\psi_0 - i\int_0^t e^{i(t-\tau)\Delta}(F(\psi(\tau)))d\tau,$$

$$\psi^N(t) = e^{i\Delta t}\psi_0 - i\int_0^t e^{i(t-\tau)\Delta}(P_{\leq N}F(P_{\leq N}\psi^N(\tau)))d\tau.$$

In particular, using Strichartz estimate, we have that, for any $t \in (0,\delta)$, $\|\psi(t) - P_{\leq N}\psi^N(t)\|_{Y_\delta}$ is bounded from above by

$$C\|P_{>N}\psi_0\|_{H^{s_1}} + \left\|\int_0^t e^{i(t-\tau)\Delta}\left(F(\psi(\tau)) - P_{\leq N}F(P_{\leq N}\psi^N(\tau))\right)d\tau\right\|_{Y_\delta}. \tag{B.0.5}$$

The first term of (B.0.5) is bounded by $CN^{s_1-s}\|\psi_0\|_{H^s} \leq CN^{s_1-s}K$.

We study the second term of (B.0.5), in particular it is lower than

$$\begin{aligned}
&C\int_{-\delta}^{\delta}\left\|e^{i(t-\tau)\Delta}\left(F(\psi(\tau)) - P_{\leq N}F(P_{\leq N}\psi^N(\tau))\right)\right\|_{H^{s_1}}d\tau\\
\leq& C\int_{-\delta}^{\delta}\left\|\left(F(\psi(\tau)) - P_{\leq N}F(P_{\leq N}\psi^N(\tau))\right)\right\|_{H^{s_1}}d\tau\\
\leq& C\int_{-\delta}^{\delta}\left\|(F(\psi(\tau)) - P_{\leq N}F(\psi(\tau)))\right\|_{H^{s_1}}d\tau\\
+& C\int_{-\delta}^{\delta}\left\|P_{\leq N}\left(F(\psi(\tau)) - F(P_{\leq N}\psi^N(\tau))\right)\right\|_{H^{s_1}}d\tau\\
\leq& CN^{s_1-s}2\delta\sup_\tau\|F(\psi(\tau))\|_{H^{s1}}\\
+& C\int_{-\delta}^{\delta}\left(1 + \|\psi(\tau)\|_{L^\infty}^{q-1} + \|\psi^N(\tau)\|_{L^\infty}^{q-1}\right)\left\|\psi(\tau) - P_{\leq N}\psi^N(\tau)\right\|_{H^{s_1}}d\tau \qquad \text{(B.0.6)}\\
\leq& CN^{s_1-s}2\delta(1+K^q)\\
+& 2C\delta^\gamma\left(1 + \|\psi(\tau)\|_{L^{p_1}(L^\infty)}^{q-1} + \|\psi^N(\tau)\|_{L^{p_1}(L^\infty)}^{q-1}\right)\left\|\psi(\tau) - P_{\leq N}\psi^N(\tau)\right\|_{L^\infty(H^{s_1})} \qquad \text{(B.0.7)}\\
\leq& CN^{s_1-s}2\delta(1+K^q) \qquad \text{(B.0.8)}\\
+& 2C\delta^\gamma\left(1 + \|\psi(\tau)\|_{Y_\delta} + \|\psi^N(\tau)\|_{Y_\delta}\right)^{q-1}\left\|\psi(\tau) - P_{\leq N}\psi^N(\tau)\right\|_{Y_\delta}.
\end{aligned}$$

with $\gamma = 1 - \frac{q-1}{p_1}$.

So, we have

$$\|\psi(t) - P_{\leq N}\psi^N(t)\|_{Y_\delta} \leq CKN^{s_1-s} + 2C\delta^\gamma(1+K^{q-1})\left\|\psi(\tau) - P_{\leq N}\psi^N(\tau)\right\|_{Y_\delta}, \tag{B.0.9}$$

and in particular, if we choose $\delta \le \left(\frac{1}{4C(1+K^{q-1})}\right)^{\frac{1}{\gamma}}$, we get

$$\|\psi(t) - P_{\le N}\psi^N(t)\|_{Y_\delta} \le 2CKN^{s_1-s}. \tag{B.0.10}$$

So, for any N, we define $\alpha_{0,N} := \sup_{|t|<\delta} \|\psi(t) - P_{\le N}\psi^N(t)\|_{H^{s_1}}$ and we conclude that

$$\sup_{|t|<\delta} \|\psi(t) - P_{\le N}\psi^N(t)\|_{H^{s_1}} = \alpha_{0,N} \le 2CKN^{s_1-s}, \tag{B.0.11}$$

$$\|\psi(t)\|_{H^{s_1}} \le \|P_{\le N}\psi^N(t)\|_{H^{s_1}} + \alpha_{0,N} \le K + \alpha_{0,N} \tag{B.0.12}$$

We denote by $\psi'(t)$ the solution of (0.0.1) corresponding to the initial data $P_{\le N}\psi^N(\delta)$. By regularity, we have

$$\sup_{0<t<\delta} \|\psi(t+\delta) - \psi'(t)\|_{H^{s_1}} \le 2\|\psi(\delta) - P_{\le N}\psi^N(\delta)\|_{H^{s_1}} \le 2\alpha_{0,N}. \tag{B.0.13}$$

So, one has

$$\begin{aligned}
&\sup_{\delta<t<2\delta} \|\psi(t) - P_{\le N}\psi^N(t)\|_{H^{s_1}} \\
&\le \sup_{0<t<\delta} \|\psi(t+\delta) - \psi'(t)\|_{H^{s_1}} + \sup_{0<t<\delta} \|P_{\le N}\psi^N(t+\delta) - \psi'(t)\|_{H^{s_1}} \\
&\le 2\alpha_{0,N} + 2CKN^{s_1-s}
\end{aligned} \tag{B.0.14}$$

where we use (B.0.13) and (B.0.11) to get the last inequality.

So, one has

$$\begin{aligned}
\alpha_{1,N} := &\sup_{0<t<2\delta} \|\psi(t) - P_{\le N}\psi^N(t)\|_{H^{s_1}} \\
&\le \sup_{0<t<\delta} \|\psi(t) - P_{\le N}\psi^N(t)\|_{H^{s_1}} + \sup_{\delta<t<2\delta} \|\psi(t) - P_{\le N}\psi^N(t)\|_{H^{s_1}} \\
&\le 2\alpha_{0,N} + 4CKN^{s_1-s}
\end{aligned} \tag{B.0.15}$$

where we used (B.0.11) and (B.0.14).

We consider now T, we divide $[0,T]$ in δ interval $[t_j, t_{j+1}]$ of length $\frac{T}{\delta}$ where $t_j = \delta j,\ j = 0, ..., \frac{T}{\delta}$. Repeating the reasoning above, we obtain

$$\begin{cases} \alpha_{j,N} := \sup_{0<t<j\delta} \|\psi(t) - P_{\le N}\psi^N(t)\|_{H^{s_1}} \le 2\alpha_{j,N} + 4CKN^{s_1-s} \\ \alpha_{0,N} \le C^{j+1}KN^{s_1-s} \end{cases}. \tag{B.0.16}$$

So, we obtain

$$\sup_{0<t<T} \|\psi(t) - P_{\le N}\psi^N(t)\|_{H^{s_1}} \le C^{\frac{T}{\delta}}KN^{s_1-s}. \tag{B.0.17}$$

We study now $\|\psi(t) - \psi^N(t)\|_{H^{s_1}}$, in particular, using (B.0.17) and recalling that $P_{\leq N}\psi^N(t) - \psi^N(t) = P_{>N}\psi^N(t) = P_{>N}\psi_0$, one has

$$\begin{aligned}
&\sup_{0<t<T} \|\psi(t) - \psi^N(t)\|_{H^{s_1}} \\
&\leq \sup_{0<t<T} \|\psi(t) - P_{\leq N}\psi^N(t)\|_{H^{s_1}} + \sup_{0<t<T} \|P_{\leq N}\psi^N(t) - \psi^N(t)\|_{H^{s_1}} \\
&\leq C^{\frac{T}{\delta}} K N^{s_1 - s} + K N^{s_1 - s}.
\end{aligned} \tag{B.0.18}$$

Since $0 < s_1 < s$, we get the thesis. □

Appendix C

Technical lemmas for Chapter 3

C.0.1 Proof of Lemma 3.2.13

We recall that, given a set K of indices $(k_1, ..., k_{2n})$ with an even number of components, we denote

$$K_1 := \{k_1, ..., k_n\} \ , \quad K_2 := \{k_{n+1}, ..., k_{2n}\} \ .$$

Lemma C.0.1. *Let $k \in \mathbb{Z}^{2n}$ and $j \in \mathbb{Z}^{2m}$ be 2 integer vectors, each one fulfilling the zero momentum condition and an $(M, \mathtt{k})$ admissible condition.*

Assume that $K_1 \cup J_2 = K_2 \cup J_1$, then there exist $x, y \in K_1 \cup J_2$ and a constant C, s.t. $|x|, |y| \geq |\mathtt{k}|/C$. Furthermore $\{x, y\}$ is uniquely determined by $K_1 \cup J_2 \setminus \{x, y\}$.

Proof. For future reference we write the $(M, \mathtt{k})$ admissible conditions for the two vectors:

$$\sum_{i=1}^{2n} a_i k_i = \mathtt{k} \ , \tag{C.0.1}$$

$$\sum_{i=1}^{2n} b_i j_i = \mathtt{k} \ . \tag{C.0.2}$$

We give now a recoursive procedure in order to determine the elements x, y in the statement.

From (C.0.1) there exists l_1 s.t. $|k_{l_1}| \geq |\mathtt{k}|/2nM$. By possibily interchanging $K_1 \cup J_2$ with $K_2 \cup J_1$ and reordering the indexes, we can always assume that $l_1 = 1$. So we have

$$|k_1| \geq \frac{|\mathtt{k}|}{2nM} \ , \quad a_1 \neq 0 \ .$$

In the following we will make several cases.

We look for the “companion” of k_1 in $K_2 \cup J_1$. We have two possibilities:

(A) It belongs to J_1 and therefore, by possibly reordering the indexes it is given by j_1 (thus we have $k_1 = j_1$).

(B) It belongs to K_2 and therefore, by possibly reordering the indexes it is given by k_{n+1} (thus we have $k_1 = k_{n+1}$).

We begin by analyzing the case (A). We use the zero momentum condition on k in order to compute k_1 as a function of the other components and we substitute in (C.0.1), which takes the form

$$\sum_{i=2}^{n}(a_i - a_1)k_i + \sum_{i=1}^{n}(a_{i+n} + a_1)k_{i+n} = \mathtt{k} \ . \tag{C.0.3}$$

Then there exists at least one of the k_i's which has modulus larger then a constant times $|\mathtt{k}|$. There are two possibilities

(A.1) It belongs to K_1, thus (up to reordering) it is given by k_n:

$$|k_n| \geq \frac{|\mathtt{k}|}{2(n-1)M} \quad \& \quad a_1 \neq a_n. \tag{C.0.4}$$

(A.2) It belongs to K_2, thus (up to reordering) it is given by k_{2n}:

$$|k_{2n}| \geq \frac{|\mathtt{k}|}{2(n-1)M} \quad \& \quad a_1 \neq -a_{2n} \ . \tag{C.0.5}$$

We analyze first (A.1). Consider the companion of k_n, there are two further possibilities:

(A.1.1) It belongs to J_1, call it j_m (thus $k_n = j_m$).

(A.1.2) It belongs to K_2, call it k_{2n} (thus $k_n = k_{2n}$).

We analyze (A.1.1). In this case, given $K_1 \cup J_2 \setminus \{k_1, k_n\}$ also $K_2 \cup J_1 \setminus \{j_1, j_m\}$ is fixed. Then (C.0.3) determines k_n and then (C.0.1) determines k_1. This concludes the case (A.1.1).

We analyze now (A.1.2). Given $K_1 \cup J_2 \setminus \{k_1, k_n\}$ also $K_2 \cup J_1 \setminus \{j_1, k_{2n}\}$ is fixed. So, also $J_1 \cup J_2 \setminus \{j_1\}$ is determined. Then, by the zero momentum condition on j one determines $j_1 = k_1$. Still one has to determine $k_n = k_{2n}$. To this end one would like to use (C.0.3). This is possible if the coefficients of k_n and k_{2n} do not cancel out. If this happens, then consider

$k' := (k_1, ..., k_{n-1}, k_{n+1}, ..., k_{2n-1})$ and iterate the argument of situation (A) with it (which also fulfills the zero momentum condition). Iterating n possibly decreases by one at each step. Since k' (and its iterates) has to fulfill an $(M, \mathtt{k})$ relation, which in particular is inhomogeneous, the procedure terminates with a nontrivial k' of dimension at least 2. This concludes this case.

This concludes the analysis of (A.1).

We now analyze the case (A.2). We have two cases according to the position of the companinon of k_{2n}.

(A.2.1) It is $k_n \in K_1$ (thus $k_n = k_{2n}$).

(A.2.2) It is $j_{2m} \in J_2$ (thus $j_{2m} = k_{2n}$).

The situation of the case (A.2.1) is identical to that of (A.1.2) and has already been analyzed.

We study now (A.2.2). Given $K_1 \cup J_2 \setminus \{k_1, j_{2m}\}$ also $K_1 \cup K_2 \setminus \{k_1, k_{2n}\}$ is determined. But, by the second of (C.0.5), (C.0.3) determines k_{2n}. Then k_1 is determined by (C.0.1).

This concludes the analysis of (A).

We come to (B). Substituting $k_1 = k_{n+1}$ in (C.0.1) we get

$$(a_1 + a_{n+1})k_1 + \sum_{i=2}^{n}(a_i k_i + a_{i+n} k_{i+n}) = \mathtt{k} \ . \tag{C.0.6}$$

We have two possibilities

(B.1) $-a_1 \neq a_{n+1}$,

(B.2) $-a_1 = a_{n+1}$.

We analyze (B.1). We concentrate on j. By (C.0.2) there exists one of the j_i's which is "big". There are two cases

(B.1.1) It belongs to J_1 and thus it is $|j_1| \geq |\mathtt{k}|/2mM$.

(B.1.2) It belongs to J_2 and thus it is $|j_{2m}| \geq |\mathtt{k}|/2mM$.

Analyze (B.1.1). There are again two cases according to the companion of j_1

(B.1.1.1) It belongs to K_1, thus it is $k_n = j_1$.

(B.1.1.2) It belongs to J_2, thus it is $j_{m+1} = j_1$.

Analyze (B.1.1.1). Given $K_1 \cup J_2 \setminus \{k_1, k_n\}$ also $K_2 \cup J_1 \setminus \{k_{n+1}, j_1\}$ is determined. Thus also $J_1 \cup J_2 \setminus \{j_1\}$ is determined. So, from the zero momentum condition also $j_1 = k_n$ is determined. From (C.0.6) also k_1 is determined.

We analyze (B.1.1.2). First we remark that given $K_1 \cup J_2 \setminus \{k_1, j_{2n}\}$ also $K_2 \cup J_1 \setminus \{k_{n+1}, j_n\}$ is determined, thus $K_1 \cup K_2 \setminus \{k_1, k_{n+1}\}$ is determined, and then, by (C.0.6) also $k_1 = k_{n+1}$ is determined. Then we have to determine one further large component.

Substituting $j_1 = j_{m+1}$ in (C.0.2) one gets

$$\sum_{i=2}^{m} (b_i j_i + b_{i+m} j_{i+m}) + (b_1 + b_{m+1}) j_1 = \mathtt{k} \ . \tag{C.0.7}$$

We have two cases

(B.1.1.2.1) $b_1 + b_{m+1} \neq 0$,

(B.1.1.2.2) $b_1 + b_{m+1} = 0$.

Case (B.1.1.2.1). Given $K_1 \cup J_2 \setminus \{k_1, j_{m+1}\}$ also $K_2 \cup J_1 \setminus \{k_{n+1}, j_1\}$ is determined. Thus also $J_1 \cup J_2 \setminus \{j_1, j_{m+1}\}$ is determined, but then one can use (C.0.7) to compute j_1. This concludes the analysis of this case.

Case (B.1.1.2.2). In this case (C.0.7) becomes a $(2M, \mathtt{k})$ admissible condition for $j' := (j_2, ..., j_m, j_{m+2}, ..., j_{2m})$, which also fulfills the zero momentum condition. Thus one is again in the situation (B.1) but with j' in place of j. Iterating the construction one decreases m at each step, and therefore the procedure terminates in a finite number of steps.

We come to the case (B.1.2). We distinguish two cases according to the position of the companion of j_{2m}.

(B.1.2.1) It belongs to K_2, thus it is k_{2n}.

(B.1.2.2) It belongs to J_1, thus it is j_{2m}.

Case (B.1.2.1). Given $K_1 \cup J_2 \setminus \{k_1, j_{2m}\}$ also $K_2 \cup J_1 \setminus \{k_{n+1}, k_{2n}\}$ is determined. Thus also $J_1 \cup J_2 \setminus \{j_{2m}\}$ is determined. Then by the zero momentum condition on j also $j_{2m} = k_{2n}$ is determined and one can use (C.0.6) to determine k_1.

Case (B.1.2.2). By reasoning in a similar way one determines $k_1 = k_{n+1}$. Still one has to determine $j_m = j_{2m}$ and this can be done exactly (up to a relabellin of the indexes) as in the case (B.1.1.2). It means that if $b_1 + b_{m+1} \neq 0$ the argument is complete, otherwise we have to start a recoursion as above in the case (B.1.1.2.2).

In the case (B.2), (C.0.6) becomes an $(M, \mathtt{k})$ admissible condition for $k' := (k_2, ..., k_n, k_{n+2}, ..., k_{2n})$ which also fulfills the zero momentum condition. Thus the construction is repeated with k' in place of k and after a finite number of steps the construction stops.

□

We can now prove Lemma 3.2.13.

Proof of Lemma 3.2.13 The proof is similar to that of Lemma 3.2.7. In the same way, we get an estimate analogous to (3.2.10), the only difference is that the sum is not on $\mathfrak{T}$ but on the set of (k, j) fulfilling the assumptions of Lemma C.0.1. We denote this set by $\tilde{\mathfrak{T}}$.

So, we estimate

$$\sum_{(k,j)\in\tilde{\mathfrak{T}}} \frac{1}{\prod_{i=1}^{n}\left(1+k_i^2\right)\left(1+j_{n+i}^2\right)}. \tag{C.0.8}$$

If $\mathtt{k} = 0$, then we can proceed exactly as in Lemma 3.2.7.

If $\mathtt{k} \neq 0$, we note that at most $[(2n!)]^2$ couples (k, j) give the same set $K_1 \cup J_2 = K_2 \cup J_1$. So using Lemma C.0.1, we obtain

$$\sum_{(k,j)\in\tilde{\mathfrak{T}}} \frac{1}{\prod_{i=1}^{n}\left(1+k_i^2\right)\left(1+j_{n+i}^2\right)} \tag{C.0.9}$$

$$\leq \frac{[(2n)!]^2}{\left(1+\left(\frac{\mathtt{k}}{C}\right)^2\right)^2} \sum_{l_1,\dots,l_{2n-2}} \frac{1}{\prod_{t=1}^{2m-2}(1+l_t^2)} \tag{C.0.10}$$

$$\leq \frac{C}{\left(1+\mathtt{k}^2\right)^2}\left(\sum_{l}\frac{1}{(1+l^2)}\right)^{n-2}. \tag{C.0.11}$$

□

C.0.2 Estimate of the resonant part

First, we introduce a lemma useful to estimate the measure of the resonant region.

Given $n \in \mathbb{N}$ and $k = (k_1, ..., k_n) \in \mathbb{Z}^n$, we denote by $\mathtt{M}$ the cardinality of $Supp(k)$ and for any $\epsilon > 0$, we define the non smooth cutoff function

$$\chi(x) = \begin{cases} 0 \text{ if } |x| \geq 1 \\ 1 \text{ if } |x| < 1 \end{cases}, \quad \chi_\epsilon(x) := \chi\left(\frac{x}{\epsilon}\right).$$

Lemma C.0.2. *Let $0<\epsilon$, $n\in\mathbb{N}$, $k=(k_1,...,k_n)\in\mathbb{Z}^n$, $\{a_i\}_{i=1}^n\in\mathbb{Z}^n\setminus\{0\}$. Then there exists a constant $C(n)>0$ s.t., denoting $\tilde{k}:=\min_{l\in Supp(k),a_l\neq 0}k_l$ and $\tilde{a}$ the correspondent coefficient in $\{a_i\}_{i=1}^n$,*

$$\int_{\mathbb{R}_+^{\mathtt{M}}}\left(\prod_{i=1}^n z_{k_i}\right)\chi\left(\sum_{i=1}^n a_i\frac{z_{k_i}}{k_i^2}\right)e^{-\sum_{l\in Supp(k)}z_l}\prod_{l\in Supp(k)}dz_l\leq 4\tilde{a}C(n)\tilde{k}^2\epsilon. \tag{C.0.12}$$

Proof. We have that $z^le^{-z}<(2l)^le^{-l}e^{-\frac{z}{2}}<(2n)^ne^{-\frac{z}{2}}$, so, denoting by I the left side of (C.0.12) and using the substitution $\frac{z_l}{2}=x_l$, we have

$$I\leq C_1(n)\int_{\mathbb{R}_+^{\mathtt{M}}}\chi\left(\sum_{i=1}^n 2a_i\frac{x_{k_i}}{k_i^2}\right)e^{-\sum_{l\in Supp(k)}x_l}\prod_{l\in Supp(k)}dx_l.$$

We denote $A(x):=\sum_{k_i\neq\tilde{k}}2a_i\frac{x_{k_i}}{k_i^2}$. So I is bounded from above by

$$C(n)\int_{\mathbb{R}_+^{\mathtt{M}-1}}\prod_{\substack{l\in Supp(k)\\ l\neq\tilde{k}}}dx_le^{-\sum_{l\in Supp(k)\ l\neq\tilde{k}}x_l}\int_{(-\epsilon-A(x))\frac{\tilde{k}^2}{2\tilde{a}}}^{(\epsilon-A(x))\frac{\tilde{k}^2}{2\tilde{a}}}e^{-x_{\tilde{k}}}dx_{\tilde{k}}$$

$$<C(n)\int_{\mathbb{R}_+^{\mathtt{M}-1}}\prod_{\substack{l\in Supp(k)\\ l\neq\tilde{k}}}dx_le^{-\sum_{l\in Supp(k)\ l\neq\tilde{k}}x_l}\int_{(-\epsilon-A(x))\frac{\tilde{k}^2}{2\tilde{a}}}^{(\epsilon-A(x))\frac{\tilde{k}^2}{2\tilde{a}}}dx_{\tilde{k}}=4\tilde{a}C(n)\tilde{k}^2\epsilon.$$

□

Proof of Lemma 3.4.2

$$\left\|\mathcal{R}_6^R\right\|_g^2=\left\|\sum_{k\in\mathcal{M}_6}Z_{6,k,\mathtt{k}}(\psi)\left(1-\rho\left(\frac{a_k(\psi)}{\delta}\right)\right)\right\|_g^2,$$

so

$$\left\|\mathcal{R}_6^R\right\|_g^2=$$

$$=\int_{H^s}\left(\sum_{k\in\mathcal{M}_6}Z_{6,k,\mathtt{k}}(\psi)\left(1-\rho\left(\frac{a_k(\psi)}{\delta}\right)\right)\right)\left(\sum_{j\in\mathcal{M}_6}\bar{Z}_{6,j,\mathtt{k}}(\psi)\left(1-\rho\left(\frac{a_j(\psi)}{\delta}\right)\right)\right)d\mu_\beta$$

$$=\int_{H^s}\sum_{k,j\in\mathcal{M}_6}Z_{6,k,\mathtt{k}}(\psi)\bar{Z}_{6,j,\mathtt{k}}(\psi)\left(1-\rho\left(\frac{a_j(\psi)}{\delta}\right)\right)\left(1-\rho\left(\frac{a_k(\psi)}{\delta}\right)\right)d\mu_\beta. \tag{C.0.13}$$

As in Lemma 3.2.13, for Lemmas 3.2.3 and 3.1.4, we can exchange the order between the integral and the series.
So (C.0.13) is equal to

$$\sum_{k,j\in\mathcal{M}_6}\int_{H^s} Z_{6,k,\mathbf{k}}(\psi)\bar{Z}_{6,j,\mathbf{k}}(\psi)\left(1-\rho\left(\frac{a_j(\psi)}{\delta}\right)\right)\left(1-\rho\left(\frac{a_k(\psi)}{\delta}\right)\right)d\mu_\beta.$$

We analyze, now one single term of the series, namely:

$$\tilde{Z}_{6,k}\left(\delta_{k_1,\mathbf{k}}+\delta_{k_2,\mathbf{k}}+\delta_{k_3,\mathbf{k}}-\delta_{k_4,\mathbf{k}}-\delta_{k_5,\mathbf{k}}-\delta_{k_6,\mathbf{k}}\right) \tag{C.0.14}$$

$$\times\bar{\tilde{Z}}_{6,j}\left(\delta_{k_1,\mathbf{k}}+\delta_{k_2,\mathbf{k}}+\delta_{k_3,\mathbf{k}}-\delta_{k_4,\mathbf{k}}-\delta_{k_5,\mathbf{k}}-\delta_{k_6,\mathbf{k}}\right) \tag{C.0.15}$$

$$\times\int\prod_{i=1}^{3}\psi_{j_i}\psi_{k_{3+i}}\bar{\psi}_{j_{3+i}}\bar{\psi}_{k_i}\left(1-\rho\left(\frac{a_j(\psi)}{\delta}\right)\right)\left(1-\rho\left(\frac{a_k(\psi)}{\delta}\right)\right)d\mu_\beta. \tag{C.0.16}$$

We remark that:

$$a_k(\psi):=(|\psi_{k_1}|^2+|\psi_{k_2}|^2+|\psi_{k_3}|^2-|\psi_{k_4}|^2-|\psi_{k_5}|^2-|\psi_{k_6}|^2).$$

With the transformation $\psi=re^{i\theta}$, denoted by $S_{k,j}:=Supp(k,j)$, the integral becomes

$$\frac{\int_{r_k\in\mathbb{R}_+}\prod_{i=1}^6 r_{j_i}r_{k_i}\left(1-\rho\left(\frac{\tilde{a}_j(r)}{\delta}\right)\right)\left(1-\rho\left(\frac{\tilde{a}_k(r)}{\delta}\right)\right)e^{-\beta\sum_{l\in S_{k,j}}\left(1+l^2\right)r_l^2}\prod_{k\in S_{k,j}}r_l dr_l}{\prod_{l\in S_{k,j}}\int_{\mathbb{R}_+}e^{-\beta(1+l^2)r_l^2}l_k dr_l}$$

$$\times\frac{\int_{\theta_k\in[0,2\pi]}e^{i(\theta_{j_1}+\theta_{j_2}+\theta_{j_3}+\theta_{k_4}+\theta_{k_5}+\theta_{k_6}-\theta_{j_4}-\theta_{j_5}-\theta_{j_6}-\theta_{k_1}-\theta_{k_2}-\theta_{k_3})}\prod_{l\in S_{k,j}}d\theta_l}{\prod_{kl\in S_{k,j}}\int_{\theta_l\in[0,2\pi]}d\theta_l}$$

where

$$\tilde{a}_k(r):=(r_{k_1}^2+r_{k_2}^2+r_{k_3}^2-r_{k_4}^2-r_{k_5}^2-r_{k_6}^2).$$

The only terms different from 0 are the terms where

$$\theta_{j_1}+\theta_{j_2}+\theta_{j_3}+\theta_{k_4}+\theta_{k_5}+\theta_{k_6}=\theta_{j_4}+\theta_{j_5}+\theta_{j_6}+\theta_{k_1}+\theta_{k_2}+\theta_{k_3}$$

or equivalently

$$\{j_1,j_2,j_3,k_4,k_5,k_6\}=\{j_4,j_5,j_6,k_1,k_2,k_3\}.$$

This implies that the integrals that survive have this form:

$$\frac{\int_{r_k\in\mathbb{R}_+}r_{j_1}^2r_{j_2}^2r_{j_3}^2r_{k_4}^2r_{k_5}^2r_{k_6}^2\left(1-\rho\left(\frac{\tilde{a}_j(r)}{\delta}\right)\right)\left(1-\rho\left(\frac{\tilde{a}_k(r)}{\delta}\right)\right)e^{-\beta\sum_{l\in S_{k,j}}\left(1+l^2\right)r_l^2}\prod_{l\in S_{k,j}}r_l dr_l}{\prod_{l\in S_{k,j}}\int_{\mathbb{R}_+}e^{-\beta(1+l^2)r_l^2}r_l dr_l}=$$

$$\frac{\int_{z_k\in\mathbb{R}_+} z_{j_1}z_{j_2}z_{j_3}z_{k_4}z_{k_5}z_{k_6}\left(1-\rho\left(\frac{\tilde{b}_j(z)}{\beta\delta}\right)\right)\left(1-\rho\left(\frac{\tilde{b}_k(z)}{\beta\delta}\right)\right)e^{-\sum_{l\in S_{k,j}} z_l}\prod_{l\in S_{k,j}} dz_l}{\beta^6(1+j_1)^2(1+j_2)^2(1+j_3)^2(1+k_4)^2(1+k_5)^2(1+k_6)^2\prod_{l\in S_{k,j}}\int_{\mathbb{R}_+} e^{-\sum_l z_l}dz_l}$$

where

$$\tilde{b}_k(z) := \left(\frac{z_{k_1}}{1+k_1^2}+\frac{z_{k_2}}{1+k_2^2}+\frac{z_{k_3}}{1+k_3^2}-\frac{z_{k_4}}{1+k_4^2}-\frac{z_{k_5}}{1+k_5^2}-\frac{z_{k_6}}{1+k_6^2}\right).$$

We define the non smooth cutoff function $\chi(x) = \begin{cases} 0 \text{ if } |x| \geq \delta\beta \\ 1 \text{ if } |x| \leq \delta\beta \end{cases}$

So we can increase the integral with the following integral:

$$\frac{1}{\beta^6(1+j_1)^2(1+j_2)^2(1+j_3)^2(1+k_4)^2(1+k_5)^2(1+k_6)^2}\times$$
$$\int\prod_{i=1}^{3} z_{j_i}\prod_{l=4}^{6} z_{k_l}\chi\left(\tilde{b}_j(z)\right)\chi\left(\tilde{b}_k(z)\right)e^{-\sum_{l\in S_{k,j}} z_l}\prod_{l\in S_{k,j}} dz_l. \tag{C.0.17}$$

We would to know more information on the arguments of the cutoff function that depend on the form of $Z_{6,k,\mathbf{k}}$ and $Z_{6,j,\mathbf{k}}$.
Since in $\mathcal{R}_6^R$ there are only terms in which $\{k_1,k_2,k_3\} \neq \{k_4,k_5,k_6\}$, this implies also that there are only terms in which $k_i \neq k_l$ for $i = 1,2,3$ $l = 4,5,6$, since if there exists at least an index $i \in \{1,2,3\}$, and index $l \in \{4,5,6\}$ s.t. $k_i = k_l$ this implies that $\{k_1,k_2,k_3\} = \{k_4,k_5,k_6\}$ and it is absurd.

In fact, without losing generality we can suppose that $k_1 = k_4$, this means that $k_2+k_3 = k_5+k_6$ and $k_2^2+k_3^2 = k_5^2+k_6^2$, so $k_2 = k_5$ and $k_3 = k_6$ or $k_2 = k_6$ and $k_3 = k_5$, so $\{k_1,k_2,k_3\} = \{k_4,k_5,k_6\}$.

So one has $j_i \neq j_l$ and $k_i \neq k_l$ $j = 1,2,3$, $l = 4,5,6$. Moreover we know that $\{j_1,j_2,j_3,k_4,k_5,k_6\} = \{j_4,j_5,j_6,k_1,k_2,k_3\}$ this means $\{j_1,j_2,j_3\} = \{k_1,k_2,k_3\}$ and $\{k_4,k_5,k_6\} = \{j_4,j_5,j_6\}$ and $\{j_1,j_2,j_3,j_4,j_5,j_6\} = \{k_1,k_2,k_3,k_4,k_5,k_6\} = \{j_1,j_2,j_3,k_4,k_5,k_6\}$

So, up to any permutation of the indices, we have 9 cases:

- if $j_i \neq j_l$, $k_i \neq k_l$, $\tilde{b}_k(z) = \tilde{b}_j(z) = \left(\frac{z_{j_1}}{1+j_1^2}+\frac{z_{j_2}}{1+j_2^2}+\frac{z_{j_3}}{1+j_3^2}-\frac{z_{k_4}}{1+k_4^2}-\frac{z_{k_5}}{1+k_5^2}-\frac{z_{k_6}}{1+k_6^2}\right)$,
- if $j_i \neq j_l$, $k_4 = k_5$, $\tilde{b}_k(z) = \tilde{b}_j(z) = \left(\frac{z_{j_1}}{1+j_1^2}+\frac{z_{j_2}}{1+j_2^2}+\frac{z_{j_3}}{1+j_3^2}-2\frac{z_{k_4}}{1+k_4^2}-\frac{z_{k_6}}{1+k_6^2}\right)$,
- if $j_i \neq j_l$, $k_4 = k_5 = k_6$, $\tilde{b}_k(z) = \tilde{b}_j(z) = \left(\frac{z_{j_1}}{1+j_1^2}+\frac{z_{j_2}}{1+j_2^2}+\frac{z_{j_3}}{1+j_3^2}-3\frac{z_{k_4}}{1+k_4^2}\right)$,
- if $j_1 = j_2$, $k_i \neq k_l$, $\tilde{b}_k(z) = \tilde{b}_j(z) = \left(\frac{2z_{j_1}}{1+j_1^2}+\frac{z_{j_3}}{1+j_3^2}-\frac{z_{k_4}}{1+k_4^2}-\frac{z_{k_5}}{1+k_5^2}-\frac{z_{k_6}}{1+k_6^2}\right)$,

- if $j_1 = j_2$, $k_4 = k_5$, $\tilde{b}_k(z) = \tilde{b}_j(z) = \left(\frac{2z_{j_1}}{1+j_1^2} + \frac{z_{j_3}}{1+j_3^2} - 2\frac{z_{k_4}}{1+k_4^2} - \frac{z_{k_6}}{1+k_6^2} \right)$,
- if $j_1 = j_2$, $k_4 = k_5 = k_6$, $\tilde{b}_k(z) = \tilde{b}_j(z) = \left(\frac{2z_{j_1}}{1+j_1^2} + \frac{z_{j_3}}{1+j_3^2} - 3\frac{z_{k_4}}{1+k_4^2} \right)$,
- if $j_1 = j_2 = j_3$, $k_i \neq k_l$, $\tilde{b}_k(z) = \tilde{b}_j(z) = \left(3\frac{z_{j_1}}{1+j_1^2} - \frac{z_{k_4}}{1+k_4^2} - \frac{z_{k_5}}{1+k_5^2} - \frac{z_{k_6}}{1+k_6^2} \right)$,
- if $j_1 = j_2 = j_3$, $k_4 = k_5$, $\tilde{b}_k(z) = \tilde{b}_j(z) = \left(3\frac{z_{j_1}}{1+j_1^2} - 2\frac{z_{k_4}}{1+k_4^2} - \frac{z_{k_6}}{1+k_6^2} \right)$,
- if $j_1 = j_2 = j_3$, $k_4 = k_5 = k_6$, $\tilde{b}_k(z) = \tilde{b}_j(z) = \left(3\frac{z_{j_1}}{1+j_1^2} - 3\frac{z_{k_4}}{1+k_4^2} \right)$.

We can resume all this cases writing

$$\tilde{b}_k(z) = \tilde{b}_j(z) = \tilde{b}_{kj}(z) =$$

$$= \left(a_1 \frac{z_{j_1}}{j_1^2} + a_2 \frac{z_{j_2}}{1+j_2^2} + a_3 \frac{z_{j_3}}{1+j_3^2} - a_4 \frac{z_{k_4}}{1+k_4^2} - a_5 \frac{z_{k_5}}{1+k_5^2} - a_6 \frac{z_{k_6}}{1+k_6^2} \right)$$

where $a_i \in \{0,1,2,3\}$, $\sum_{i=1}^{6} a_i = 6$, and $\{a_i\}_{i=1}^{6}$ s.t. if there exists $i \in \{1,2,3\}$ s.t. $a_i \neq 1$, for any $l \in \{1,2,3\}$, $l \neq i$ s.t. $a_l = 0$, $j_i = j_l$ and if there exists $i' \in \{4,5,6\}$ s.t $a_{i'} \neq 1$, for any $l' \in \{4,5,6\}$, $l' \neq i'$ s.t. $a_{l'} = 0$, $k_{i'} = k_{l'}$. In this way we can write (C.0.17) as

$$\frac{1}{\beta^6 \prod_{i=1}^{3} \left(1 + j_i^2\right) \left(1 + k_{3+i}^2\right)} \int \prod_{i=1}^{3} z_{j_i} z_{k_{3+i}} \chi \left(\tilde{b}_{kj}(z) \right) e^{-\sum_{l \in S_{k,j}} z_l} \prod_{l \in S_{k,j}} dz_l \tag{C.0.18}$$

where $z_i \in \mathbb{R}_+$.

To obtain the norm of the resonant part, after studying the form of any terms of the series, we have to estimate the norm of every single term.

Let N be an integer, then Lemma $C.0.2$ shows that if there exists at least an index $i = 1,2,3$, $a_i \neq 0$ s.t. $|j_i| < N$ or an index $l = 4,5,6$, $a_l \neq 0$ s.t. $|k_l| < N$, then there exists $C_1 > 0$ s.t. (C.0.18) is bounded by

$$C_1 \frac{\delta \beta N^2}{\prod_{i=1}^{3} \left(1 + j_i^2\right) \left(1 + k_{3+i}^2\right)}.$$

If every j_i and k_l really present in the argument of the cutoff is bigger than N, we adopt an other strategy, because the distance between the two hyper-planes becomes bigger and non comparable with $\delta\beta$, so the presence of the cutoff isn't so essential, because the integral isn't so different from the integral over all the space. However, if all the indices in the argument of the cutoff are bigger than N, the denominators $\beta^6 \prod_{i=1}^{3} \left(1 + j_i^2\right) \left(1 + k_{3+i}^2\right)$ is small and this helps the convergence. Obviously, since there exists at least an index j_i or k_i equal to $\mathtt{k}$, this situation is possible only if $|\mathtt{k}| \geq N$.

We denote by $T_{\mathtt{k}}$ the set of $(k, j) \in \mathbb{Z}^{12}$ s.t. $\{j_1, j_2, j_3, k_4, k_5, k_6\} = \{k_1, k_2, k_3, j_4, j_5, j_6\}$, $\sum_{i=1}^{n} k_i = \sum_{i=n+1}^{2n} k_i$, $\sum_{i=1}^{n} j_i = \sum_{i=n+1}^{2n} j_i$, and s.t. there exists at least an index $i \in \{1,2,3,4,5,6\}$ s.t. $k_i = \mathtt{k}$ and at least an index $l \in \{1,2,3,4,5,6\}$ s.t. $j_l = \mathtt{k}$.

So, if $\mathtt{k} < N$, we have

$$\left\|\mathcal{R}_6^R\right\|_g^2 \leq 9C_1 \frac{\delta\beta N^2}{\beta^6} \sum_{j,k \in T_{\mathtt{k}}} \frac{|\tilde{Z}_{6,j}||\tilde{Z}_{6,k}|}{\prod_{i=1}^{3}\left(1+j_i^2\right)\left(1+k_{3+i}^2\right)}.$$

Instead, if $\mathtt{k} \geq N$, we have that $\left\|\mathcal{R}_6^R\right\|_g^2$ is bounded by

$$9C_1 \frac{\delta\beta N^2}{\beta^6} \sum_{j,k \in T_{\mathtt{k}}} \frac{|\tilde{Z}_{6,j}||\tilde{Z}_{6,k}|}{\prod_{i=1}^{3}\left(1+j_i^2\right)\left(1+k_{3+i}^2\right)}$$

$$+\frac{9}{\beta^6} \sum_{\substack{j,k \in T_{\mathtt{k}} \text{ s.t} \\ \forall i\ |j_i|,|k_i| \geq N}} \frac{|\tilde{Z}_{6,j}||\tilde{Z}_{6,k}|}{\prod_{i=1}^{3}\left(1+j_i^2\right)\left(1+k_{3+i}^2\right)}.$$

We know also that for every j in the sum there is an index i s.t. $j_i = \mathtt{k}$ but, due to the *null momentum condition*, there must be at least an other index l s.t. $|j_l| \geq \frac{|\mathtt{k}|}{5}$ and the same holds also for any k. Moreover, from Lemma 3.2.10, $|\tilde{Z}_{6,j}|$ are uniformly limited by a constant. So, in both the cases, as in Theorem 3.2.13, we have

$$\sum_{j,k \in T_{\mathtt{k}}} \frac{|\tilde{Z}_{6,j}||\tilde{Z}_{6,k}|}{\prod_{i=1}^{3}\left(1+j_i^2\right)\left(1+k_{3+i}^2\right)} \leq \frac{C}{\left(1+k^2\right)^2} \sum_{l_1,l_2,l_3,l_4} \frac{1}{\prod_{i=1}^{4}\left(1+l_i^2\right)}$$

and, choosing $0 < \epsilon \ll 1$,

$$\sum_{\substack{j,k \in T_{\mathtt{k}} \text{ s.t} \\ \forall i\ |j_i|,|k_i| \geq N}} \frac{|\tilde{Z}_{6,j}||\tilde{Z}_{6,k}|}{\prod_{i=1}^{3}\left(1+j_i^2\right)\left(1+k_{3+i}^2\right)} \leq \frac{C}{\left(1+k^2\right)^2} \sum_{\substack{l_1,l_2,l_3,l_4 \\ \forall i,\ |l_i| > N}} \frac{1}{\prod_{i=1}^{4}\left(1+l_i^2\right)}$$

$$\leq \frac{C}{\left(1+k^2\right)^2 N^{4-4\epsilon}} \sum_{\substack{l_1,l_2,l_3,l_4 \\ \forall i,\ |l_i| > N}} \frac{1}{\prod_{i=1}^{4}\left(1+l_i^2\right)^{\frac{1+\epsilon}{2}}}.$$

One has $\sum_{\substack{l_1,l_2,l_3,l_4 \\ \forall i,\ |l_i| > N}} \frac{1}{\prod_{i=1}^{4}\left(1+l_i^2\right)^{\frac{1+\epsilon}{2}}} \sim \frac{1}{N^{4\epsilon}}$, so, we can take

$$\delta\beta N^2 = \frac{1}{N^4},$$

one has $N = \frac{1}{(\delta\beta)^{\frac{1}{6}}}$ and finally

$$\delta\beta N^2 = \frac{1}{N^4} = (\delta\beta)^{\frac{2}{3}}.$$

This implies that

$$\left\|\mathcal{R}_6^R\right\|_g^2 \leq \tilde{C}\frac{(\delta\beta)^{\frac{2}{3}}}{\beta^6\left(1+k^2\right)^2}.$$

□

Appendix D

Technical Lemmas for Chapter 4

D.1 Proof of Lemma 4.2.6, 4.2.7, 4.2.8

Proof of Lemma 4.2.6. Since $\Psi_1 = H_1$, one has

$$\langle \Psi_1 \rangle(\alpha, \psi) \;=\; \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{T}^2} \widehat{a}_0(x) \left| e^{-\mathrm{i}\Delta t} \psi \right|^4 dx dt. \tag{D.1.1}$$

The corresponding symmetric multilinear form $\langle \Psi_1 \rangle(\alpha, \psi_1, \psi_2, \psi_3, \psi_4)$ is defined by

$$\widetilde{\langle \Psi_1 \rangle}(\alpha, \psi_1, \psi_2, \psi_3, \psi_4) = \sum_{\varsigma} \widetilde{\langle \Psi_1 \rangle}_{\varsigma}(\alpha, \psi_1, \psi_2, \psi_3, \psi_4), \tag{D.1.2}$$

$$\widetilde{\langle \Psi_1 \rangle}_{\varsigma} := \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{T}^2} \widehat{a}_0(x) e^{-\mathrm{i}\Delta t} \psi_{\varsigma(1)} e^{\mathrm{i}\Delta t} \overline{\psi}_{\varsigma(2)} e^{-\mathrm{i}\Delta t} \psi_{\varsigma(3)} e^{\mathrm{i}\Delta t} \bar{\psi}_{\varsigma(4)} dx dt. \tag{D.1.3}$$

The estimate of each term of the sum is equal so we just consider the identical permutation. For any $s \geq 0$, $\epsilon > 0$, one has

$$\begin{aligned}
&\left| \widetilde{\langle \Psi_1 \rangle}_{Id}(\alpha, \psi_1, \psi_2, \psi_3, \psi_4) \right| \\
\preceq\;& \int_0^{2\pi} \| e^{-\mathrm{i}\Delta t} \psi_1 \|_{H^{-s-\epsilon}_{4,x}} \cdot \| a_0(x) \prod_{j=2}^{4} (e^{-\mathrm{i}\Delta t} \psi_j) \|_{H^{s+\epsilon}_{4/3,x}} dt & \text{(D.1.4)} \\
\preceq\;& \int_0^{2\pi} \| e^{-\mathrm{i}\Delta t} \psi_1 \|_{H^{-s-\epsilon}_{4,x}} \cdot \| a_0 \|_{C^{s+\epsilon}(\mathbb{T}^2)} \prod_{j=2}^{4} \| e^{-\mathrm{i}\Delta t} \psi_j \|_{H^{s+\epsilon}_{4,x}} dt & \text{(D.1.5)} \\
\preceq\;& \| e^{-\mathrm{i}\Delta t} \psi_1 \|_{L^4_t H^{-s-\epsilon}_{4,x}} \cdot \prod_{j=2}^{4} \| e^{-\mathrm{i}\Delta t} \psi_j \|_{L^4_t H^{s+\epsilon}_{4,x}} & \text{(D.1.6)} \\
\preceq\;& \| \psi_1 \|_{H^{-s}} \cdot \prod_{j=2}^{4} \| \psi_j \|_{H^{s+2\epsilon}}, & \text{(D.1.7)}
\end{aligned}$$

where in line (D.1.4) we use Hölder inequality in space, in line (D.1.5) we use the product estimate (4.2.8), in line (D.1.6) we use Hölder inequality in time and in (D.1.7) we use Corollary 4.2.4.
So we conclude the estimate of (4.2.9).
To prove (4.2.10), we start from $\widehat{\chi}_{10}$.

$$
\begin{aligned}
&\left|\widetilde{\widehat{\chi}_{10}}(\alpha,\psi_1,\psi_2,\psi_3,\psi_4)\right| \\
\preceq\ & \sum_{\varsigma}\int_0^{2\pi}\left|\int_{\mathbb{T}^2}\widehat{a}_0(x)e^{-\mathrm{i}\Delta t}\psi_{\varsigma(1)}e^{-\mathrm{i}\Delta t}\psi_{\varsigma(2)}e^{-\mathrm{i}\Delta t}\psi_{\varsigma(3)}e^{-\mathrm{i}\Delta t}\psi_{\varsigma(4)}dx\right|dt \\
\preceq\ & \|\psi_1\|_{H^{-s}}\|\psi_2\|_{H^{s+\epsilon}}\|\psi_3\|_{H^{s+\epsilon}}\|\psi_4\|_{H^{s+\epsilon}}\,, \qquad \text{(D.1.8)}
\end{aligned}
$$

where in the last line we proceed as in in the proof of (4.2.9).
Consider now $\widehat{\chi}_{1k}$ with $k\neq 0$. One has

$$
\begin{aligned}
&\left|\widetilde{\widehat{\chi}_{1k}}(\alpha,\psi_1,\psi_2,\psi_3,\psi_4)\right| \\
=\ & \left|\sum_{\varsigma}\frac{e^{-\mathrm{i}2\pi k\cdot\omega}}{1-e^{-\mathrm{i}2\pi k\cdot\omega}}\int_0^{2\pi}e^{\mathrm{i}k\cdot\omega t}\int_{\mathbb{T}^2}\widehat{a}_k(x)\prod_{1\le i\le 4}e^{-\mathrm{i}\Delta t}\psi_{\varsigma(i)}dxdt\right| \\
\preceq\ & \left|\frac{e^{-\mathrm{i}2\pi k\cdot\omega}}{1-e^{-\mathrm{i}2\pi k\cdot\omega}}\right|\int_0^{2\pi}\left|e^{\mathrm{i}k\cdot\omega t}\right|\left|\int_{\mathbb{T}^2}\widehat{a}_k(x)\prod_{1\le i\le 4}e^{-\mathrm{i}\Delta t}\psi_i dxdt\right| \\
\preceq\ & \frac{1+|k|^\tau}{\gamma}\cdot\int_0^{2\pi}\left|e^{\mathrm{i}k\cdot\omega t}\right|\left|\int_{\mathbb{T}^2}\widehat{a}_k(x)\prod_{1\le i\le 4}e^{-\mathrm{i}\Delta t}\psi_i dxdt\right| \\
\preceq\ & \frac{1+|k|^\tau}{\gamma}\cdot\int_0^{2\pi}\left|\int_{\mathbb{T}^2}\widehat{a}_k(x)\prod_{1\le i\le 4}e^{-\mathrm{i}\Delta t}\psi_i dxdt\right| \\
\preceq\ & \frac{1+|k|^\tau}{\gamma}\|\widehat{a}_k\|_{C^{s+2\epsilon}}\|\psi_1\|_{H^{-s}}\|\psi_2\|_{H^{s+\epsilon}}\|\psi_3\|_{H^{s+\epsilon}}\|\psi_4\|_{H^{s+\epsilon}} \\
\preceq\ & \frac{1}{1+|k|^T}\|\psi_1\|_{H^{-s}}\|\psi_2\|_{H^{s+\epsilon}}\|\psi_3\|_{H^{s+\epsilon}}\|\psi_4\|_{H^{s+\epsilon}}\ ,
\end{aligned}
$$

where the last estimate holds $\forall\, T$ and is obtained using the standard decay properties of Fourier coefficients.
Hence one has

$$
\left|\widetilde{\chi_1}(\alpha,\psi_1,\psi_2,\psi_3,\psi_4)\right|\le\sum_{k\in\mathbb{Z}^d}\left|\widehat{\chi}_{1k}(\alpha,\psi_1,\psi_2,\psi_3,\psi_4)\right|. \qquad \text{(D.1.9)}
$$

□

Proof of Lemma 4.2.7. Note that

$$
\Psi_2=-\frac{1}{2}L_{\chi_1}\Psi_1-\frac{1}{2}L_{\chi_1}\langle\Psi_1\rangle=-\frac{1}{2}L_{\chi_1}H_1-\frac{1}{2}L_{\chi_1}\langle H_1\rangle. \qquad \text{(D.1.10)}
$$

To prove the Lemma, it is sufficient to estimate the average of $L_{\chi_1} H_1$.
For $k \neq 0$, one has

$$L_{\widehat{\chi}_{1k}} H_1(\alpha, x, \psi, \overline{\psi}) = d\widehat{\chi}_{1k} X_{H_1} \qquad \text{(D.1.11)}$$
$$= \mathrm{i}\langle \nabla_{L^2} \widehat{\chi}_{1k}; \nabla_{L^2} H_1 \rangle$$

and note

$$d\widehat{\chi}_{1k} X_{H_1}$$
$$= \frac{4e^{-\mathrm{i}2\pi k\cdot\omega}}{1 - e^{-\mathrm{i}2\pi k\cdot\omega}} \int_0^{2\pi} e^{\mathrm{i}k\cdot\omega t} \int_{\mathbb{T}^2} \widehat{a}_k(x) |e^{-\mathrm{i}\Delta t}\psi|^2 \cdot \overline{e^{-\mathrm{i}\Delta t}\psi} \cdot e^{-\mathrm{i}\Delta t} \left(a(\alpha, x) \left|\psi\right|^2 \psi \right) dxdt$$

and

$$(\widehat{d\widehat{\chi}_{1k} X_{H_1}})_{k_1}$$
$$= \frac{4e^{-\mathrm{i}2\pi k\cdot\omega}}{1 - e^{-\mathrm{i}2\pi k\cdot\omega}} \int_0^{2\pi} e^{\mathrm{i}k\cdot\omega t} \int_{\mathbb{T}^2} \widehat{a}_k(x) |e^{-\mathrm{i}\Delta t}\psi|^2 \cdot \overline{e^{-\mathrm{i}\Delta t}\psi} \cdot e^{-\mathrm{i}\Delta t} \left(\widehat{a}_{k_1 - k}(x) \left|\psi\right|^2 \psi \right) dxdt.$$

The corresponding multilinear form is given by

$$\widetilde{d\widehat{\chi}_{1k} X_{H_1}}(\alpha, x, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6)$$
$$= \frac{4e^{-\mathrm{i}2\pi k\cdot\omega}}{1 - e^{-\mathrm{i}2\pi k\cdot\omega}} \int_0^{2\pi} e^{\mathrm{i}k\cdot\omega t} \int_{\mathbb{T}^2} \widehat{a}_k(x) \mathcal{A}(\alpha, x, \psi_1, \dots, \psi_6, t) dxdt,$$

where

$$\mathcal{A} = \frac{1}{6!} \sum_{\varsigma} \mathcal{A}_{\varsigma} \qquad \text{(D.1.12)}$$

with

$$\mathcal{A}_{\varsigma} := e^{-\mathrm{i}\Delta t}\psi_{\varsigma(1)} \cdot \overline{e^{-\mathrm{i}\Delta t}\psi_{\varsigma(2)}} \cdot \overline{e^{-\mathrm{i}\Delta t}\psi_{\varsigma(3)}} \cdot e^{-\mathrm{i}\Delta t} \left(a(\alpha, x) \psi_{\varsigma(4)} \overline{\psi_{\varsigma(5)}} \psi_{\varsigma(6)} \right), \qquad \text{(D.1.13)}$$

and

$$\widetilde{(\widehat{d\widehat{\chi}_{1k} X_{H_1}})}_{k_1}(\alpha, x, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6)$$
$$= \frac{4e^{-\mathrm{i}2\pi k\cdot\omega}}{1 - e^{-\mathrm{i}2\pi k\cdot\omega}} \int_0^{2\pi} e^{\mathrm{i}k\cdot\omega t} \int_{\mathbb{T}^2} \widehat{a}_k(x) \widehat{\mathcal{A}}_{k_1 - k}(x, \psi_1, \dots, \psi_6, t) dxdt.$$

Hence, the multilinear form of $\langle (\widehat{d\widehat{\chi}_{1k} X_{H_1}})_{k_1} \rangle$ is given by

$$\langle \widetilde{(\widehat{d\widehat{\chi}_{1k} X_{H_1}})}_{k_1} \rangle (\alpha, x, \psi_1, \dots, \psi_6)$$
$$= \int_0^{2\pi} \frac{4e^{-\mathrm{i}2\pi k\cdot\omega}}{1 - e^{-\mathrm{i}2\pi k\cdot\omega}} \int_0^{2\pi} e^{\mathrm{i}k\cdot\omega t_2} \int_{\mathbb{T}^2} \widehat{a}_k(x) \widehat{\mathcal{A}}_{k_1 - k}(x, e^{-\mathrm{i}\Delta t_2}\psi_1, \dots, e^{-\mathrm{i}\Delta t_2}\psi_6, t_1) dxdt_1 dt_2.$$

We have a similar expression for the case $k=0$.
To prove the Lemma it is sufficient to prove that for any $s \geq 0$ and $\epsilon > 0$, one has

$$\left|\widetilde{\langle L_{\chi_1}\Psi_1\rangle}(\alpha,\psi_1,\ldots,\psi_6)\right| \preceq \|\psi_1\|_{H^\epsilon} \prod_{2\leq i\leq 6} \|\psi_i\|_{H^{s+\frac{1}{2}+\epsilon}}. \tag{D.1.14}$$

To prove estimate (D.1.14), it suffices to show that for any k it holds

$$\left|\widetilde{\langle L_{\widehat{\chi}_{1k}}\Psi_1\rangle}(\alpha,\psi_1,\ldots,\psi_6)\right| \preceq \frac{1}{(1+|k|)^{T_1}} \|\psi_1\|_{H^\varepsilon} \prod_{2\leq i\leq 6} \|\psi_i\|_{H^{s+\frac{1}{2}+\varepsilon}}, \tag{D.1.15}$$

for some large $T_1 > 0$. Note that for any $k \neq 0$ one has,

$$\left|\frac{4e^{-\mathrm{i}2\pi k\cdot\omega}}{1-e^{-\mathrm{i}2\pi k\cdot\omega}}\right| \preceq \frac{|k|^\tau}{\gamma}, \tag{D.1.16}$$

$$\left|e^{\mathrm{i}k\cdot\omega t_2}\right| \leq 1, \tag{D.1.17}$$

and for any k one has

$$|\widehat{a}_k(x)| \preceq \frac{1}{(1+|k|)^T} \sup_{x\in\mathbb{T}^2,\alpha\in\mathbb{T}^d} |a(x,\alpha)|. \tag{D.1.18}$$

Then one has

$$\begin{aligned} &\left|\langle(\widetilde{d\widehat{\chi_{1k}X_{H_1}}})_{k_1}\rangle\right| \\ \preceq\ & \frac{(1+|k|)^{-T+\tau}}{\gamma} \int_0^{2\pi}\int_0^{2\pi}\int_{\mathbb{T}^2} \left|\widehat{\mathcal{A}}_{k_1-k}(x, e^{-\mathrm{i}\Delta t_2}\psi_1,\ldots,e^{-\mathrm{i}\Delta t_2}\psi_6,t_1)\right| dx dt_1 dt_2. \end{aligned}$$

As in D.1.12, we have

$$\widehat{\mathcal{A}}_{k_1-k} := \sum_{\varsigma} \widehat{(\mathcal{A}_\varsigma)}_{k_1-k} \tag{D.1.19}$$

so, we only consider the following term

$$\widehat{(\mathcal{A}_\varsigma)}_{k_1-k} := e^{-\mathrm{i}\Delta t}\psi_{\varsigma(1)} \cdot \overline{e^{-\mathrm{i}\Delta t}\psi_{\varsigma(2)}} \cdot \overline{e^{-\mathrm{i}\Delta t}\psi_{\varsigma(3)}} \cdot e^{-\mathrm{i}\Delta t}\left(\widehat{a}_{k_1-k}(x))\psi_{\varsigma(4)}\overline{\psi_{\varsigma(5)}}\psi_{\varsigma(6)}\right) \tag{D.1.20}$$

and its corresponding Fourier term.
Since $(\varsigma(1),\varsigma(2),\varsigma(3),\varsigma(4),\varsigma(5),\varsigma(6))$ is a permutation of $(1,2,3,4,5,6)$ then we can have two different cases.
Case. 1. $1 \in \{\varsigma(1),\varsigma(2),\varsigma(3)\}$.

Without loss of generality, we assume $1 = \varsigma(1)$ and using Hölder inequality in space, we have

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi}\int_{\mathbb{T}^2}\left|(\widehat{\mathcal{A}_\varsigma})_{k_1-k}(x, e^{-\mathrm{i}\Delta t_2}\psi_1, \ldots, e^{-\mathrm{i}\Delta t_2}\psi_6, t_1)\right| dx dt_1 dt_2 \\
\preceq\ &\int_0^{2\pi}\int_0^{2\pi}\|e^{-\mathrm{i}\Delta(t_1+t_2)}\psi_1\|_{H^{-s-\epsilon}_{4,x}}\cdot\|B_1\|_{H^{s+\epsilon}_{4/3,x}} dt_1 dt_2 \qquad \text{(D.1.21)}
\end{aligned}$$

where

$$B_1 = \overline{e^{-\mathrm{i}\Delta(t_1+t_2)}\psi_{\varsigma(2)}}\cdot\overline{e^{-\mathrm{i}\Delta(t_1+t_2)}\psi_{\varsigma(3)}}\cdot e^{-\mathrm{i}\Delta t_1}\left(\widehat{a}_{k_1-k}(x)e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(4)}\overline{e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(5)}}e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(6)}\right).$$

Denoting

$$B_2 = \left\|\overline{e^{-\mathrm{i}\Delta(t_1+t_2)}\psi_{\varsigma(2)}}\right\|_{H^{s+\epsilon}_{4,x}}\cdot\left\|\overline{e^{-\mathrm{i}\Delta(t_1+t_2)}\psi_{\varsigma(3)}}\right\|_{H^{s+\epsilon}_{4,x}}\cdot B_3,$$

and

$$B_3 = \left\|e^{-\mathrm{i}\Delta t_1}\left(\widehat{a}_{k_1-k}(x)e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(4)}\overline{e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(5)}}e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(6)}\right)\right\|_{H^{s+\epsilon}_{4,x}}$$

and using the product estimate (4.2.8), we have that (D.1.21) is controlled by

$$\int_0^{2\pi}\int_0^{2\pi}\|e^{-\mathrm{i}\Delta(t_1+t_2)}\psi_1\|_{H^{-s-\epsilon}_{4,x}}\cdot B_2 dt_1 dt_2. \qquad \text{(D.1.22)}$$

Denoting

$$B_4 = \left\|\overline{e^{-\mathrm{i}\Delta(t_1+t_2)}\psi_{\varsigma(2)}}\right\|_{L^4_{t_1}H^{s+\epsilon}_{4,x}}\cdot\left\|\overline{e^{-\mathrm{i}\Delta(t_1+t_2)}\psi_{\varsigma(3)}}\right\|_{L^4_{t_1}H^{s+\epsilon}_{4,x}}\cdot B_5,$$

$$B_5 = \left\|e^{-\mathrm{i}\Delta t_1}\left(\widehat{a}_{k_1-k}(x)e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(4)}\overline{e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(5)}}e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(6)}\right)\right\|_{L^4_{t_1}H^{s+\epsilon}_{4,x}}$$

and using Hölder inequality in time t_1, we get that (D.1.22) is controlled by

$$\int_0^{2\pi}\|e^{-\mathrm{i}\Delta(t_1+t_2)}\psi_1\|_{L^4_{t_1}H^{-s-\epsilon}_{4,x}}\cdot B_4 dt_2. \qquad \text{(D.1.23)}$$

Finally, denoting

$$B_6 = \left\|\widehat{a}_{k_1-k}(x)e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(4)}\overline{e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(5)}}e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(6)}\right\|_{H^{s+2\epsilon}_x},$$

we get that (D.1.23) is controlled by

$$
\begin{aligned}
& \int_0^{2\pi} \|\psi_1\|_{H^{-s}}\|\psi_2\|_{H^{s+2\epsilon}}\|\psi_3\|_{H^{s+3\epsilon}} \cdot B_6 dt_2 && \text{(D.1.24)}\\
\preceq\ & \frac{1}{(1+|k_1-k|)^T}\|\psi_1\|_{H^{-s}}\|\psi_2\|_{H^{s+2\epsilon}}\|\psi_3\|_{H^{s+2\epsilon}}\prod_{i=4}^{6}\|\psi_i\|_{H^{s+\frac{1}{3}+3\epsilon}} && \text{(D.1.25)}\\
\preceq\ & \frac{1}{(1+|k_1-k|)^T}\|\psi_1\|_{H^{-s}}\prod_{i=2}^{6}\|\psi_i\|_{H^{s+\frac{1}{3}+3\epsilon}}, && \text{(D.1.26)}
\end{aligned}
$$

where in (D.1.24) we use Corollary 4.2.4 with $p=4$ and in line (D.1.25) we use the product estimate (4.2.8) and 4.2.4 with $p=6$.
Case. 2. $1 \in \{\varsigma(4), \varsigma(5), \varsigma(6)\}$.
Without loss of generality, we assume $1=\varsigma(4)$ and denoting

$$
B_7 = \left(e^{-\mathrm{i}\Delta(t_1+t_2)}\psi_{\varsigma(1)}\right)\left(\overline{e^{-\mathrm{i}\Delta(t_1+t_2)}\psi_{\varsigma(2)}}\right)\left(\overline{e^{-\mathrm{i}\Delta(t_1+t_2)}\psi_{\varsigma(3)}}\right),
$$

$$
B_8 = \left(\widehat{a}_{k_1-k}(x)e^{-\mathrm{i}\Delta t_2}\psi_1\overline{e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(5)}}e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(6)}\right)
$$

and using Hölder inequality in space, we have

$$
\begin{aligned}
& \int_0^{2\pi}\int_0^{2\pi}\int_{\mathbb{T}^2}\left|(\widehat{\mathcal{A}_\varsigma})_{k_1-k}(x, e^{-\mathrm{i}\Delta t_2}\psi_1,\dots,e^{-\mathrm{i}\Delta t_2}\psi_6, t_1)\right| dx dt_1 dt_2\\
\preceq\ & \int_0^{2\pi}\int_0^{2\pi}\|B_7\|_{H_x^s}\cdot\|e^{-\mathrm{i}\Delta t_1}B_8\|_{H_x^{-s}} dt_1 dt_2\\
=\ & \frac{1}{(1+|k_1-k|)^T}\int_0^{2\pi}\int_0^{2\pi}\|B_7\|_{H_x^s}\cdot\|B_8\|_{H_x^{-s}} dt_1 dt_2. && \text{(D.1.27)}
\end{aligned}
$$

Denoting

$$
B_9 = \|e^{-\mathrm{i}\Delta t_2}\psi_1\|_{H^\epsilon_{4,x}}\|e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(5)}\|_{H^\epsilon_{8,x}}\|e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(6)}\|_{H^\epsilon_{8,x}},
$$

$$
B_{10} = \|e^{-\mathrm{i}\Delta t_2}\psi_1\|_{L^4_{t_2}H^\epsilon_{4,x}}\|e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(5)}\|_{L^8_{t_2}H^\epsilon_{8,x}}\|e^{-\mathrm{i}\Delta t_2}\psi_{\varsigma(6)}\|_{L^8_{t_2}H^\epsilon_{8,x}}
$$

and using use the product estimate (4.2.8), we get that (D.1.27) is controlled by

$$
\begin{aligned}
& \frac{1}{1+|k_1-k|^T}\int_0^{2\pi}\int_0^{2\pi}\left(\prod_{i=1}^{3}\|e^{-\mathrm{i}\Delta(t_1+t_2)}\psi_{\varsigma(i)}\|_{H^{s+\epsilon}_{6,x}}\right)\cdot B_9 dt_1 dt_2 & \text{(D.1.28)}\\
\preceq\ & \frac{1}{1+|k_1-k|^T}\int_0^{2\pi}\prod_{i=1}^{3}\|e^{-\mathrm{i}\Delta(t_1+t_2)}\psi_i\|_{L^6_{t_1}H^{s+\epsilon}_{6,x}} B_9 dt_2 & \text{(D.1.29)}\\
\preceq\ & \frac{1}{1+|k_1-k|^T}\left(\prod_{i=1}^{3}\|\psi_{\varsigma(i)}\|_{H_x^{s+\frac{1}{3}+2\epsilon}}\right) B_{10} & \text{(D.1.30)}\\
\preceq\ & \frac{1}{1+|k_1-k|^T}\|\psi_1\|_{H^{2\epsilon}}\left(\prod_{i=1}^{3}\|\psi_{\varsigma(i)}\|_{H^{s+\frac{1}{3}+2\epsilon}}\right) & \text{(D.1.31)}\\
\times\ & \|\psi_{\varsigma(5)}\|_{H^{\frac{1}{2}+2\epsilon}}\|\psi_{\varsigma(6)}\|_{H^{\frac{1}{2}+2\epsilon}},
\end{aligned}
$$

where in line (D.1.29) we use Hölder inequality in t_1, in line (D.1.30) we use first Corollary 4.2.4 with $p=6$ and then Hölder inequality in t_2 and in the last line we use two times Corollary 4.2.4, once with $p=4$ and the second with $p=8$.

So we finish the proof of (D.1.15) and the proof of (4.2.11). In a similar way we get also (4.2.12). □

Proof of Lemma 4.2.8. Note that

$$\Psi_3 = -\frac{1}{6}L^2_{\chi_1}Z_1 - \frac{1}{3}L_{\chi_2}Z_1 - \frac{2}{3}L_{\chi_2}H_1 - \frac{2}{3}L_{\chi_1}Z_2, \tag{D.1.32}$$

The worst term is $\frac{2}{3}L_{\chi_2}H_1$, so to obtain the thesis it is sufficient to estimate its average.

We denote

$$A(\psi_1,\dots,\psi_4) = \int_{\mathbb{T}^2}|\psi_1\cdots\psi_4|\,dx, \tag{D.1.33}$$

$$B(\psi_1,\psi_2,\psi_3) = \psi_1\cdot\psi_2\cdot\psi_3, \tag{D.1.34}$$

$$B_1 = B(e^{-\mathrm{i}\Delta t_3}\psi_3, e^{-\mathrm{i}\Delta t_3}\psi_4, e^{-\mathrm{i}\Delta t_3}\psi_5) \tag{D.1.35}$$

and

$$B_2 = B(e^{-\mathrm{i}\Delta(t_2+t_3)}\psi_6, e^{-\mathrm{i}\Delta(t_2+t_3)}\psi_7, e^{-\mathrm{i}\Delta(t_2+t_3)}\psi_8)). \tag{D.1.36}$$

Then we define

$$
\begin{aligned}
& \mathcal{A}_{31}(\psi_1,\dots,\psi_8,t_1,t_2,t_3) & \text{(D.1.37)}\\
=\ & A(e^{-\mathrm{i}\Delta(t_1+t_2+t_3)}\psi_1, e^{-\mathrm{i}\Delta(t_1+t_2+t_3)}\psi_2, e^{-\mathrm{i}\Delta(t_1+t_2)}B_1, e^{-\mathrm{i}\Delta t_1}B_2).
\end{aligned}
$$

In a similar way, we define

$$
\begin{aligned}
& \mathcal{A}_{32}(\psi_1,\dots,\psi_8,t_1,t_2,t_3) & \text{(D.1.38)}\\
=\ & A(e^{-\mathrm{i}\Delta(t_1+t_2+t_3)}\psi_1, e^{-\mathrm{i}\Delta(t_1+t_2+t_3)}\psi_2, e^{-\mathrm{i}\Delta(t_1+t_2+t_3)}\psi_3, e^{-\mathrm{i}\Delta t_1}B_3)
\end{aligned}
$$

where

$$B_3 = B(e^{-i\Delta(t_2+t_3)}\psi_4, e^{-i\Delta(t_2+t_3)}\psi_5, e^{-i\Delta t_2}B_4) \tag{D.1.39}$$

and

$$B_4 = B(e^{-i\Delta t_3}\psi_6, e^{-i\Delta t_3}\psi_7, e^{-i\Delta t_3}\psi_8). \tag{D.1.40}$$

We denote by $\widetilde{\mathcal{A}_{31}}$ and $\widetilde{\mathcal{A}_{32}}$ the symmetric multilinear form associated respectively to $\mathcal{A}_{31}$ and $\mathcal{A}_{32}$. Forgetting about the coefficients due to Fourier expansion, $\langle \widetilde{L_{\chi_2}H_1}\rangle$ is composed by terms of the form $\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\widetilde{\mathcal{A}_{31}}dt_1dt_2dt_3$ and $\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\widetilde{\mathcal{A}_{32}}dt_1dt_2dt_3$, so to show the Lemma it is sufficient to estimate them.
We start from $\widetilde{\mathcal{A}_{32}} = \frac{1}{8!}\sum_\varsigma \widetilde{\mathcal{A}_{32\varsigma}}$ and as in Lemma 4.2.7, we estimate $\widetilde{\mathcal{A}_{32\varsigma}}$, showing that one has

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left|\widetilde{\mathcal{A}_{32\varsigma}}\right| dt_1dt_2dt_3 \preceq \|\psi_1\|_{H^\epsilon}\left(\prod_{j=2}^{8}\|\psi_j\|_{H^{s+\epsilon+\frac{2}{3}}}\right). \tag{D.1.41}$$

As in Lemma 4.2.7, here we have more cases to study.
Case 1. $1\in\{\varsigma(1),\varsigma(2),\varsigma(3)\}$. Without loosing generality, we consider $1 = \varsigma(1)$.
Proceeding as in Lemma 4.2.7, for any $\epsilon > 0$, by product estimate (4.2.8) and Bourgain's estimate (4.2.6), one has

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left|\widetilde{\mathcal{A}_{32\varsigma}}\right| dt_1dt_2dt_3 \tag{D.1.42}$$

$$\preceq \int_0^{2\pi}\int_0^{2\pi}\|\psi_1\|_{H^\epsilon}\left\|\psi_{\varsigma(2)}\right\|_{H^\epsilon}\left\|\psi_{\varsigma(3)}\right\|_{H^\epsilon}\|B_3\|_{H^\epsilon}dt_2dt_3. \tag{D.1.43}$$

By product estimate (4.2.8) and Corollary 4.2.4, one has

$$\|B_3\|_{H^\epsilon} \preceq \left\|\psi_{\varsigma(4)}\right\|_{H^{2\epsilon+\frac{1}{3}}}\left\|\psi_{\varsigma(5)}\right\|_{H^{2\epsilon+\frac{1}{3}}}\|B_4\|_{H^{s+2\epsilon+\frac{1}{3}}} \tag{D.1.44}$$

and

$$\|B_4\|_{H^\epsilon} \preceq \left\|\psi_{\varsigma(6)}\right\|_{H^{3\epsilon+\frac{2}{3}}}\left\|\psi_{\varsigma(7)}\right\|_{H^{3\epsilon+\frac{2}{3}}}\left\|\psi_{\varsigma(8)}\right\|_{H^{3\epsilon+\frac{2}{3}}}. \tag{D.1.45}$$

Hence, one has (D.1.41) in this case.
Case 2. $1\in\{\varsigma(4),\varsigma(5)\}$ Without loosing of generality, we consider $1 = \varsigma(4)$. For any $\epsilon > 0$, by product estimate (4.2.8) and Bourgain's estimate (4.2.6), one has

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left|\widetilde{\mathcal{A}_{32\varsigma}}\right| dt_1dt_2dt_3 \tag{D.1.46}$$

$$\preceq \int_0^{2\pi}\int_0^{2\pi}\left\|\psi_{\varsigma(1)}\right\|_{H^\epsilon}\left\|\psi_{\varsigma(2)}\right\|_{H^{s+\epsilon}}\left\|\psi_{\varsigma(3)}\right\|_{H^\epsilon}\|B_3\|_{H^\epsilon}dt_2dt_3. \tag{D.1.47}$$

By product estimate (4.2.8) and Corollary 4.2.4, choosing p_1, p_2 s.t. $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{4}$, one has

$$\|B_3\|_{H^\epsilon} \preceq \|\psi_1\|_{H^{2\epsilon}} \left\|\psi_{\varsigma(5)}\right\|_{H^{2\epsilon+1-\frac{4}{p_1}}} \|B_4\|_{H^{2\epsilon+1-\frac{4}{p_2}}} \tag{D.1.48}$$

and

$$\|B_4\|_{H^\epsilon} \preceq \left\|\psi_{\varsigma(6)}\right\|_{H^{3\epsilon+1-\frac{4}{p_2}+\frac{1}{3}}} \left\|\psi_{\varsigma(7)}\right\|_{H^{s+3\epsilon+1-\frac{4}{p_2}+\frac{1}{3}}} \|\psi_8\|_{H^{3\epsilon+1-\frac{4}{p_2}+\frac{1}{3}}} . \tag{D.1.49}$$

Moreover, choosing p_1, p_2 s.t.

$$1-\frac{4}{p_1}=1-\frac{4}{p_2}+\frac{1}{3}, \tag{D.1.50}$$

one has $p_1 = 12$ and $p_2 = 6$, which implies

$$1-\frac{4}{p_1}=1-\frac{4}{p_2}+\frac{1}{3}=\frac{2}{3}. \tag{D.1.51}$$

So, one has (D.1.41) in this case.
Case 3. $1 \in \{\varsigma(6), \varsigma(7), \varsigma(8)\}$. Without loosing of generality, we consider $1 = \varsigma(6)$.
For any $\epsilon > 0$, by product estimate (4.2.8) and Bourgain's estimate (4.2.6), one has

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \left|\widetilde{\mathcal{A}_{32\varsigma}}\right| dt_1 dt_2 dt_3 \tag{D.1.52}$$

$$\preceq \int_0^{2\pi}\int_0^{2\pi} \left\|\psi_{\varsigma(1)}\right\|_{H^\epsilon} \left\|\psi_{\varsigma(2)}\right\|_{H^\epsilon} \left\|\psi_{\varsigma(3)}\right\|_{H^\epsilon} \|B_3\|_{H^\epsilon} dt_2 dt_3. \tag{D.1.53}$$

By product estimate (4.2.8) and Corollary 4.2.4, one has

$$\|B_3\|_{H^\epsilon} \preceq \left\|\psi_{\varsigma(4)}\right\|_{H^{2\epsilon+\frac{1}{2}}} \left\|\psi_{\varsigma(5)}\right\|_{H^{2\epsilon+\frac{1}{2}}} \|B_4\|_{H^{2\epsilon}} \tag{D.1.54}$$

and

$$\|B_4\|_{H^\epsilon} \preceq \|\psi_1\|_{H^{3\epsilon}} \left\|\psi_{\varsigma(7)}\right\|_{H^{3\epsilon+\frac{1}{2}}} \left\|\psi_{\varsigma(8)}\right\|_{H^{3\epsilon+\frac{1}{2}}} . \tag{D.1.55}$$

Hence, one has (D.1.41)also in this case.
In a similar way we obtain the same estimate also in the case of $\widetilde{\mathcal{A}_{31}}$. So we get (4.2.13), in a similar way one gets (4.2.14). □

D.2 Technical Lemmas

Proof of Lemma 4.2.2. By (4.2.6), we know that for every $\epsilon > 0$ one has

$$\|e^{it\Delta}\psi\|_{L^4_{tx}} \leq C_1\|\psi\|_{H^\epsilon(\mathbb{T}^2)}.$$

Moreover, using Sobolev embeddings in 2-dimension, one has

$$\|e^{it\Delta}\psi\|_{L^\infty_{tx}} \leq C_2\|e^{it\Delta}\psi\|_{L^\infty_t(0,2\pi)H^{1+\epsilon}(\mathbb{T}^2)} \leq C_3\|\psi\|_{H^{1+\epsilon}(\mathbb{T}^2)}.$$

We denote by $p_0 = \infty$, $p_1 = 4$, $s_0 = 1+\epsilon$, $s_1 = \epsilon$. Using interpolation theorem, for every $\theta \in (0,1)$, we obtain

$$\|e^{it\Delta}\psi\|_{L^{p_\theta}_{tx}} \leq C\|\psi\|_{H^{s_\theta}(\mathbb{T}^2)}$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $s_\theta = (1-\theta)s_0 + \theta s_1$.
In particular, for any $p > 4$, one has

$$\theta = \frac{4}{p},\ \theta \in (0,1)$$

and

$$s = \left(1 - \frac{4}{p}\right)(1+\epsilon) + \frac{4}{p}\epsilon = 1 - \frac{4}{p} + \epsilon.$$

So, for any $p > 4, \epsilon > 0$, we get

$$\|e^{it\Delta}\psi\|_{L^p_{tx}} \leq C\|\psi\|_{H^{1-\frac{4}{p}+\epsilon}(\mathbb{T}^2)}.$$

□

Proof of Lemma 4.2.5. Let $\tilde{\eta}$ be smooth, supported in $B(0,2)$ and equal to 1 on $B(0,1)$. Also let $\tilde{\eta}_R(x) = \tilde{\eta}(x/R)$ for $R > 0$. We define

$$\eta_R(x_1, ..., x_n) := \prod_{i=1}^n \tilde{\eta}_R(x_i).$$

Denoted by G_s the Bessel Kernel

$$G_s(x) := \frac{e^{-|x|}}{(2\pi)^{\frac{n-1}{2}} 2^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{n-s+1}{2}\right)} \int_0^\infty e^{-|x|t}\left(t + \frac{t^2}{2}\right)^{\frac{n-s-1}{2}} dt$$

for $x \neq 0$, we can express $(I-\Delta)^{s/2}f = G_s * f$, $(I-\Delta)^{s/2}f\eta_R = G_s * f\eta_R$ ([2]).

Since f is periodic, $f \in L^\infty(\mathbb{R}^n)$ and

$$f\eta_R \to f$$

pointwise as $R \to \infty$, by Dominated Convergence Theorem, one has that

$$(I-\Delta)^{s/2}(f\eta_R) \to (I-\Delta)^{s/2}(f)$$

pointwise. Moreover the function $(I-\Delta)^{s/2}(f)$ is periodic as it is given as a convolution of a periodic function with a tempered distribution. By the Kato-Ponce inequality on $\mathbb{R}^n$ [27], recalling that $(I-\Delta)^{s/2} := J_s$, for $s > 0$ we have:

$$\begin{aligned} &\|J_s(fg\eta_R^2)\|_{L^p(\mathbb{R}^n)} \\ \preceq &\|J_s(f\eta_R)\|_{L^{p_1}(\mathbb{R}^n)}\|g\eta_R\|_{L^{p_2}(\mathbb{R}^n)} + \|f\eta_R\|_{L^{q_1}(\mathbb{R}^n)}\|J_s(g\eta_R)\|_{L^{q_2}(\mathbb{R}^n)} \end{aligned} \tag{D.2.1}$$

where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p}$, $1 < p_1, p_2, q_1, q_2 < \infty$.
If $p_1 = q_1 = \infty$, this can be modified as follows:

$$\begin{aligned} &\|J_s(fg\eta_R^2)\|_{L^p(\mathbb{R}^n)} \\ \preceq &\|J_s(f\eta_R)\|_{L^\infty(\mathbb{R}^n)}\|g\eta_R\|_{L^p(\mathbb{R}^n)} + \|f\eta_R\|_{L^\infty(\mathbb{R}^n)}\|J_s(g\eta_R)\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{D.2.2}$$

Let $v_n = |B(0,1)|$. Notice that for $0 < p_2 \leq \infty$

$$\frac{\|f\eta_R\|_{L^{p_2}(\mathbb{R}^n)}}{(v_n R^n)^{\frac{1}{p_2}}} \to \|f\|_{L^{p_2}([0,2\pi]^n)} \tag{D.2.3}$$

as $R \to \infty$.
Next we show that for $1 < p_1 < \infty$

$$\frac{\|J_s(f\eta_R)\|_{L^{p_1}(\mathbb{R}^n)}}{(v_n R^n)^{\frac{1}{p_1}}} \to \|J_s(f)\|_{L^{p_1}([0,2\pi]^n)} \tag{D.2.4}$$

as $R \to \infty$.
First, we notice that for $1 < p_1 < \infty$

$$\begin{aligned} &\frac{\|J_s(f\eta_R) - J_s(f)\eta_R\|_{L^{p_1}(\mathbb{R}^n)}}{(v_n R^n)^{\frac{1}{p_1}}} \\ &\leq C\frac{2\|J_s(f)\|_{L^\infty(\mathbb{R}^n)}\|\eta_R\|_{L^{p_1}(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}\|J_s(\eta_R)\|_{L^{p_1}(\mathbb{R}^n)}}{(v_n R^n)^{\frac{1}{p_1}}} \end{aligned} \tag{D.2.5}$$

by the Kato-Ponce and Hölder inequalities.
By (D.2.3), (D.2.5) is bounded in R, so by the Dominated Convergence Theorem, letting $R \to \infty$, we obtain that

$$\frac{\|J_s(f\eta_R) - J_s(f)\eta_R\|_{L^{p_1}(\mathbb{R}^n)}}{(v_n R^n)^{\frac{1}{p_1}}} \to 0. \tag{D.2.6}$$

Using (D.2.3), (D.2.6) and

$$\left| \frac{\|J_s(f\eta_R)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)}}{v_n R^n} - \frac{\|J_s(f)\eta_R\|^{p_1}_{L^{p_1}(\mathbb{R}^n)}}{v_n R^n} \right| \leq \frac{\|J_s(f\eta_R) - J_s(f)\eta_R\|^{p_1}_{L^{p_1}(\mathbb{R}^n)}}{v_n R^n}$$

we deduce (D.2.4). To obtain the periodic Kato-Ponce, in the case where $1 < p_1, p_2, q_1, q_2 < \infty$, we divide (D.2.1) by

$$(v_n R^n)^{\frac{1}{p}} = (v_n R^n)^{\frac{1}{p_1}} (v_n R^n)^{\frac{1}{p_2}} = (v_n R^n)^{\frac{1}{q_1}} (v_n R^n)^{\frac{1}{q_2}}$$

and we use (D.2.4) and (D.2.3). □
The rest of the Appendix is devoted to prove Lemma 4.3.4.
We recall that, for any $s \in (0,1)$ and for any $n \in \mathbb{N}$ the fractional laplacian is defined also in the following way:

$$(-\Delta)^s \psi(x) := C_{n,s} \int_{\mathbb{R}^n} \frac{\psi(x) - \psi(y)}{|x-y|^{n+2s}} dy \tag{D.2.7}$$

where

$$C_{n,s} := \frac{4^s \Gamma(n/2 + s)}{\pi^{d/2} |\Gamma(-s)|}$$

and that

$$\|\psi\|_{H^s_p(\mathbb{T}^n)} := \|\psi\|_{L^p(\mathbb{T}^n)} + \|(-\Delta)^{\frac{s}{2}} \psi\|_{L^p(\mathbb{T}^n)}.$$

Moreover, for periodic function ψ on the torus, also $(-\Delta)^s \psi(x)$ is periodic, in fact:

$$\begin{aligned} (-\Delta)^s \psi(x + 2\pi) &= C_{n,s} \int_{\mathbb{R}^n} \frac{\psi(x+2\pi) - \psi(y)}{|x + 2\pi - y|^{n+2s}} dy \\ =& C_{n,s} \int_{\mathbb{R}^n} \frac{\psi(x) - \psi(y)}{|x + 2\pi - y|^{n+2s}} dy = C_{n,s} \int_{\mathbb{R}^n} \frac{\psi(x) - \psi(\tilde{y} + 2\pi)}{|x - \tilde{y}|^{n+2s}} d\tilde{y} \\ =& C_{n,s} \int_{\mathbb{R}^n} \frac{\psi(x) - \psi(\tilde{y})}{|x - \tilde{y}|^{n+2s}} d\tilde{y} = (-\Delta)^s \psi(x). \end{aligned}$$

So

$$\begin{aligned}\|\psi\|_{H^s_p(\mathbb{T}^n)} =&\|\psi\|_{L^p([0,2\pi]^n)} + \|(-\Delta)^{\frac{s}{2}}\psi\|_{L^p([0,2\pi]^n)}\\ =&\|\psi\|_{L^p([0,2\pi]^n)} + \left(\int_{[0,2\pi]^n}\left|C_{n,s}\int_{\mathbb{R}^n}\frac{\psi(x)-\psi(y)}{|x-y|^{n+s}}dy\right|^p dx\right)^{\frac{1}{p}}.\end{aligned}$$

In particular, for $n = 2$ and for periodic functions we have the following lemma which gives us an other equivalent norm on the torus.

Lemma D.2.1. *For any $s \in (0,1)$, $1 < p < \infty$ there exists two constants $C_1(s,p), C_2(s,p) > 0$, s.t. for any periodic function $\psi \in H^s_p(\mathbb{T}^2)$, one has*

$$\|\psi\|^p_{H^s_p(\mathbb{T}^2)} \le C_1\left(\|\psi\|^p_{L^p([0,2\pi]^2)} + \int_{[0,2\pi]^2}\left|\int_{[-2\pi,4\pi]^2}\frac{\psi(x)-\psi(y)}{|x-y|^{2+s}}dy\right|^p dx\right) \tag{D.2.8}$$

and

$$C_2\left(\|\psi\|^p_{L^p([0,2\pi]^2)} + \int_{[0,2\pi]^2}\left|\int_{[-2\pi,4\pi]^2}\frac{\psi(x)-\psi(y)}{|x-y|^{2+s}}dy\right|^p dx\right) \le \|\psi\|_{H^s_p(\mathbb{T}^2)}. \tag{D.2.9}$$

Proof. We start from (D.2.8). By definition, one has

$$\begin{aligned}\|\psi\|^p_{H^s_p(\mathbb{T}^2)} =& \left(\|\psi\|_{L^p([0,2\pi]^2)} + \left(\int_{[0,2\pi]^2}\left|C_{2,s}\int_{\mathbb{R}^2}\frac{\psi(x)-\psi(y)}{|x-y|^{2+s}}dy\right|^p dx\right)^{\frac{1}{p}}\right)^p\\ \le& C(p)\left(\|\psi\|^p_{L^p([0,2\pi]^2)} + \int_{[0,2\pi]^2}\left|C_{2,s}\left(\int_{[-2\pi,4\pi]^2}\frac{\psi(x)-\psi(y)}{|x-y|^{2+s}}dy + A\right)\right|^p dx\right)\end{aligned} \tag{D.2.10}$$

where

$$A := \int_{([-2\pi,4\pi]^2)^c}\frac{\psi(x)-\psi(y)}{|x-y|^{2+s}}dy.$$

So we have

$$\|\psi\|^p_{H^s_p(\mathbb{T}^2)} \le C(p)\left(\|\psi\|^p_{L^p([0,2\pi]^2)} + \int_{[0,2\pi]^2}\left|\int_{[-2\pi,4\pi]^2}\frac{\psi(x)-\psi(y)}{|x-y|^{2+s}}dy\right|^p dx + \int_{[0,2\pi]^2}|A|^p\, dx\right). \tag{D.2.11}$$

We study now A.
Given $K = (k_1,k_2) \in \mathbb{Z}^2$, denoting for any $k_1,k_2 \in \mathbb{Z}, k_1 \ge 2$,

$$B^{>}_{k_1,k_2} := [2k_1\pi, 2(k_1+1)\pi] \times [2k_2\pi, 2(k_2+1)\pi],$$

for any $k_1, k_2 \in \mathbb{Z}, k_1 \leq -1$,

$$B^{<}_{k_1,k_2} := [2(k_1-1)\pi, 2k_1\pi] \times [2k_2\pi, 2(k_2+1)\pi],$$

for any $k_1 \in \{-1,0,1\}, k_2 \in \mathbb{Z}, k_2 \geq 2$,

$$B^{\wedge}_{k_1,k_2} := [2k_1\pi, 2(k_1+1)\pi] \times [2k_2\pi, 2(k_2+1)\pi]$$

and for any $k_1 \in \{-1,0,1\}, k_2 \in \mathbb{Z}, k_2 \leq -1$,

$$B^{\vee}_{k_1,k_2} := [2k_1\pi, 2(k_1+1)\pi] \times [2(k_2-1)\pi, 2k_2\pi],$$

one has

$$|A| \leq \sum_{k_1 \geq 2, k_2} \int_{B^{>}_{k_1,k_2}} \frac{|\psi(x)| + |\psi(y)|}{|x-y|^{2+s}} dy \tag{D.2.12}$$

$$+ \sum_{k_1 \leq -1, k_2} \int_{B^{<}_{k_1,k_2}} \frac{|\psi(x)| + |\psi(y)|}{|x-y|^{2+s}} dy \tag{D.2.13}$$

$$+ \sum_{k_1 \in \{-1,0,1\}, k_2 \geq 2} \int_{B^{\wedge}_{k_1,k_2}} \frac{|\psi(x)| + |\psi(y)|}{|x-y|^{2+s}} dy \tag{D.2.14}$$

$$+ \sum_{k_1 \in \{-1,0,1\}, k_2 \leq -1} \int_{B^{\vee}_{k_1,k_2}} \frac{|\psi(x)| + |\psi(y)|}{|x-y|^{2+s}} dy. \tag{D.2.15}$$

We study explicitly the right side of (D.2.12) but one can estimate all the other terms in a similar way.

$$\begin{aligned}
&\sum_{k_1 \geq 2, k_2} \int_{B^{>}_{k_1,k_2}} \frac{|\psi(x)| + |\psi(y)|}{|x-y|^{2+s}} dy \\
&= \sum_{k_1 \geq 2, k_2 = 0} \int_{B^{>}_{k_1,k_2}} \frac{|\psi(x)| + |\psi(y)|}{|x-y|^{2+s}} dy + \sum_{k_1 \geq 2, k_2 > 0} \int_{B^{>}_{k_1,k_2}} \frac{|\psi(x)| + |\psi(y)|}{|x-y|^{2+s}} dy \\
&+ \sum_{k_1 \geq 2, k_2 < 0} \int_{B^{>}_{k_1,k_2}} \frac{|\psi(x)| + |\psi(y)|}{|x-y|^{2+s}} dy \\
&\leq \sum_{k_1 \geq 2, k_2 = 0} \frac{1}{[2\pi|k_1-1|]^{2+s}} \int_{B^{>}_{k_1,k_2}} |\psi(x)| + |\psi(y)| dy \\
&+ \sum_{k_1 \geq 2, k_2 > 0} \frac{1}{\left[2\pi\sqrt{(k_1-1)^2 + (k_2-1)^2}\right]^{2+s}} \int_{B^{>}_{k_1,k_2}} |\psi(x)| + |\psi(y)| dy \\
&+ \sum_{k_1 \geq 2, k_2 < 0} \frac{1}{\left[2\pi\sqrt{(k_1-1)^2 + (k_2)^2}\right]^{2+s}} \int_{B^{>}_{k_1,k_2}} |\psi(x)| + |\psi(y)| dy.
\end{aligned}$$

Since ψ is a periodic function with period 2π in each variable, for any k_1, k_2, one has

$$\int_{B^{>}_{k_1,k_2}} |\psi(x)| + |\psi(y)| dy = \int_{[0,2\pi]^2} |\psi(x)| + |\psi(y)| dy.$$

Moreover, since all the series are convergent, we obtain that

$$\begin{aligned}
&\sum_{k_1 \geq 2, k_2} \int_{B^{>}_{k_1,k_2}} \frac{|\psi(x)| + |\psi(y)|}{|x-y|^{2+s}} dy \\
\leq& \tilde{C}(s,p) \int_{[0,2\pi]^2} |\psi(x)| + |\psi(y)| dy = \tilde{C}(s,p) \left((2\pi)^2 |\psi(x)| + \|\psi\|_{L^1(\mathbb{T}^2)} \right).
\end{aligned}$$

In a similar way, we obtain that

$$|A| \leq \bar{C}(s,p) \int_{[0,2\pi]^2} |\psi(x)| + |\psi(y)| dy = \bar{C}(s,p) \left(|\psi(x)| + \|\psi\|_{L^1(\mathbb{T}^2)} \right).$$

So, we have

$$\int_{[0,2\pi]^2} |A|^p dx \leq \tilde{\bar{C}}(s,p) \left(\int_{[0,2\pi]^2} |\psi(x)|^p dx + \|\psi\|^p_{L^1(\mathbb{T}^2)} \right) \leq C_1 \|\psi\|^p_{L^p(\mathbb{T}^2)}. \tag{D.2.16}$$

Using (D.2.16) in (D.2.11), we get (D.2.8).
To prove (D.2.9), it is sufficient to consider

$$\begin{aligned}
&\int_{[0,2\pi]^2} \left| \int_{[-2\pi,4\pi]^2} \frac{\psi(x) - \psi(y)}{|x-y|^{2+s}} dy \right|^p dx \\
=& \int_{[0,2\pi]^2} \left| \int_{\mathbb{R}^2} \frac{\psi(x) - \psi(y)}{|x-y|^{2+s}} dy - A \right|^p dx \\
\leq& C(p,s) \left(\int_{[0,2\pi]^2} \left| C_{2,s} \int_{\mathbb{R}^2} \frac{\psi(x) - \psi(y)}{|x-y|^{2+s}} dy \right|^p dx + \|\psi\|^p_{L^p(\mathbb{T}^2)} \right) = C \|\psi\|^p_{H^s_p(\mathbb{T}^2)}
\end{aligned} \tag{D.2.17}$$

where in the last line we use (D.2.16), so we get the thesis. □

We use the previous result to show that is equivalent to take the Bessel-norm of periodic functions on the torus and to take the Bessel-norm of a suitable non periodic function with compact support. For any periodic ψ with period 2π, we define

$$\psi_{ext}(x) := \begin{cases} \psi(x) \text{ for } x \in [-2\pi, 4\pi]^2 \\ 0 \text{ otherwise} \end{cases}.$$

Lemma D.2.2. *Let $s \in (0,1)$, $1 < p < \infty$ then $\|\psi\|^p_{H^s_p(\mathbb{T}^2)}$ and $\|\psi_{ext}\|^p_{H^s_p([0,2\pi]^2)}$ are equivalent, i.e. there exist two constant $C_1, C_2 > 0$ s.t. for any $\psi \in H^s_p(\mathbb{T}^2)$, one has*

$$C_1\|\psi\|^p_{H^s_p(\mathbb{T}^2)} \le \|\psi_{ext}\|^p_{H^s_p([0,2\pi]^2)} \le C_2\|\psi\|^p_{H^s_p(\mathbb{T}^2)}. \tag{D.2.18}$$

Proof. By definition, one has

$$\begin{aligned}
&\|\psi_{ext}\|^p_{H^s_p([0,2\pi]^2)} = \left(\|\psi_{ext}\|_{L^p([0,2\pi]^2)} + \|(-\Delta)^{\frac{s}{2}}\psi_{ext}\|_{L^p([0,2\pi]^2)}\right)^p \\
\le& C\left(\|\psi_{ext}\|^p_{L^p([0,2\pi]^2)} + \|(-\Delta)^{\frac{s}{2}}\psi_{ext}\|^p_{L^p([0,2\pi]^2)}\right) \\
\le& \tilde{C}\left(\|\psi_{ext}\|^p_{L^p([0,2\pi]^2)} + \int_{[0,2\pi]^2}\left|\int_{\mathbb{R}^2}\frac{\psi_{ext}(x)-\psi_{ext}(y)}{|x-y|^{2+s}}dy\right|^p dx\right) \\
=& \tilde{C}\left(\|\psi_{ext}\|^p_{L^p([0,2\pi]^2)}\right. \qquad\qquad (\text{D.2.19}) \\
&\left.+ \int_{[0,2\pi]^2}\left|\int_{[-2\pi,4\pi]^2}\frac{\psi_{ext}(x)-\psi_{ext}(y)}{|x-y|^{2+s}}dy + \int_{([-2\pi,4\pi]^2)^c}\frac{\psi_{ext}(x)-\psi_{ext}(y)}{|x-y|^{2+s}}dy\right|^p dx\right).
\end{aligned}$$

Using the definition of ψ_{ext}, one has that (D.2.19) is equal to

$$\begin{aligned}
=& \tilde{C}\left(\|\psi\|^p_{L^p(\mathbb{T}^2)}\right. \\
&\left.+ \int_{[0,2\pi]^2}\left|\int_{[-2\pi,4\pi]^2}\frac{\psi(x)-\psi(y)}{|x-y|^{2+s}}dy + \int_{([-2\pi,4\pi]^2)^c}\frac{\psi(x)}{|x-y|^{2+s}}dy\right|^p dx\right) \\
\le& C_1\left(\|\psi\|^p_{L^p(\mathbb{T}^2)} + \int_{[0,2\pi]^2}|\psi(x)|^p\left(\int_{([-2\pi,4\pi]^2)^c}\frac{1}{|x-y|^{2+s}}dy\right)^p dx\right. \\
&\left.+ \int_{[0,2\pi]^2}\left|\int_{[-2\pi,4\pi]^2}\frac{\psi(x)-\psi(y)}{|x-y|^{2+s}}dy\right|^p dx\right) \\
\le& C_2(s,p)\left(\|\psi\|^p_{L^p(\mathbb{T}^2)} + \int_{[0,2\pi]^2}\left|\int_{[-2\pi,4\pi]^2}\frac{\psi(x)-\psi(y)}{|x-y|^{2+s}}dy\right|^p dx\right) \qquad (\text{D.2.20})
\end{aligned}$$

where in the last line we use the fact that

$$\int_{([-2\pi,4\pi]^2)^c}\frac{1}{|x-y|^{2+s}}dy < \infty.$$

So, using Lemma D.2.1, we get

$$\|\psi_{ext}\|^p_{H^s_p([0,2\pi]^2)} \le \tilde{C}\|\psi\|^p_{H^s_p(\mathbb{T}^2)}.$$

Conversely, by Lemma D.2.1

$$\begin{aligned}
\|\psi\|^p_{H^s_p(\mathbb{T}^2)} \leq & C_3 \left(\|\psi\|^p_{L^p(\mathbb{T}^2)} + \int_{[0,2\pi]^2} \left| \int_{[-2\pi,4\pi]^2} \frac{\psi(x)-\psi(y)}{|x-y|^{2+s}} dy \right|^p dx \right) \\
= & C_3 \left(\|\psi_{ext}\|^p_{L^p(\mathbb{T}^2)} + \int_{[0,2\pi]^2} \left| \int_{[-2\pi,4\pi]^2} \frac{\psi_{ext}(x)-\psi_{ext}(y)}{|x-y|^{2+s}} dy \right|^p dx \right) \\
\leq & C_4 \left(\|\psi_{ext}\|^p_{L^p(\mathbb{T}^2)} + \int_{[0,2\pi]^2} \left| \int_{\mathbb{R}^2} \frac{\psi_{ext}(x)-\psi_{ext}(y)}{|x-y|^{2+s}} dy \right|^p dx \right. \\
& \left. + \int_{[0,2\pi]^2} \left| \int_{([-2\pi,4\pi]^2)^c} \frac{\psi_{ext}(x)}{|x-y|^{2+s}} dy \right|^p dx \right) \\
\leq & C_4 \left(\|\psi_{ext}\|^p_{L^p(\mathbb{T}^2)} + \int_{[0,2\pi]^2} \left| \int_{\mathbb{R}^2} \frac{\psi_{ext}(x)-\psi_{ext}(y)}{|x-y|^{2+s}} dy \right|^p dx \right. \\
& \left. + \int_{[0,2\pi]^2} |\psi_{ext}(x)|^p \left(\int_{([-2\pi,4\pi]^2)^c} \frac{1}{|x-y|^{2+s}} dy \right)^p dx \right) \\
\leq & C_5 (\left(\|\psi_{ext}\|^p_{L^p(\mathbb{T}^2)} + \int_{[0,2\pi]^2} \left| \int_{\mathbb{R}^2} \frac{\psi_{ext}(x)-\psi_{ext}(y)}{|x-y|^{2+s}} dy \right|^p dx \right) \\
\leq & C_6 \|\psi_{ext}\|^p_{H^s_p([0,2\pi]^2)}.
\end{aligned}$$

□

Lemma D.2.3. *[Embeddings] For every $s \in (0,1)$, $1 < p < \infty$ and $\epsilon > 0$ there exist $\{C_i\}_{i=1}^4$, $C_i > 0$ for any $i = 1, ..., 4$, s.t. for any periodic function $\psi \in H^{s+2\epsilon}_p(\mathbb{T}^2)$, one has*

$$\begin{aligned}
\|\psi\|^p_{H^s_p(\mathbb{T}^2)} \leq & C_1 \|\psi_{ext}\|^p_{H^s_p([0,2\pi]^2)} \leq C_2 \|\psi_{ext}\|^p_{W^{s+\epsilon,p}([0,2\pi]^2)} \\
\leq & C_3 \|\psi_{ext}\|^p_{H^{s+2\epsilon}_p([0,2\pi]^2)} \leq C_4 \|\psi\|^p_{H^{s+2\epsilon}_p(\mathbb{T}^2)}.
\end{aligned} \tag{D.2.21}$$

Proof. The proof of this Lemma is a simple consequence of Lemma D.2.2 and of the embeddings showed in Lemma 2.1 of [23] on open domain with regular boundary. □

Proof of Lemma 4.3.4 The proof is a simple consequence of Lemma D.2.3 and of Sobolev embeddings in Sobolev space $W^{s,p}(\Omega)$ where Ω is an open subset of $\mathbb{R}^2$, and for any $s \in (0,1), p \geq 0$,

$$W^{s,p}(\Omega) := \left\{ \psi \in L^p(\Omega) : \frac{|\psi(x)-\psi(y)|}{|x-y|^{\frac{d}{p}+s}} \in L^p(\Omega \times \Omega \right\}.$$

□

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