

INTERACTING REINFORCED STOCHASTIC PROCESSES: STATISTICAL INFERENCE BASED ON THE WEIGHTED EMPIRICAL MEANS

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ABSTRACT. This work deals with a system of *interacting reinforced stochastic processes*, where each process $X^j = (X_{n,j})_n$ is located at a vertex j of a finite weighted directed graph, and it can be interpreted as the sequence of “actions” adopted by an agent j of the network. The interaction among the dynamics of these processes depends on the weighted adjacency matrix W associated to the underlying graph: indeed, the probability that an agent j chooses a certain action depends on its personal “inclination” $Z_{n,j}$ and on the inclinations $Z_{n,h}$, with $h \neq j$, of the other agents according to the entries of W . The best known example of reinforced stochastic process is the Pòlya urn.

The present paper characterizes the asymptotic behavior of the *weighted empirical means* $N_{n,j} = \sum_{k=1}^n q_{n,k} X_{k,j}$, proving their almost sure synchronization and some central limit theorems in the sense of stable convergence. By means of a more sophisticated decomposition of the considered processes adopted here, these findings complete and improve some asymptotic results for the personal inclinations $Z^j = (Z_{n,j})_n$ and for the empirical means $\bar{X}^j = (\sum_{k=1}^n X_{k,j}/n)_n$ given in recent papers (e.g. [1, 2, 18]). Our work is motivated by the aim to understand how the different rates of convergence of the involved stochastic processes combine and, from an applicative point of view, by the construction of confidence intervals for the common limit inclination of the agents and of a test statistics to make inference on the matrix W , based on the weighted empirical means. In particular, we answer a research question posed in [1].

Keywords: *Interacting Random Systems; Reinforced Stochastic Processes; Urn Models; Complex Networks; Preferential Attachment; Weighted Empirical Means; Synchronization; Asymptotic Normality.*

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1. FRAMEWORK, MODEL AND MOTIVATIONS

The stochastic evolution of systems composed by elements which interact among each other has always been of great interest in several scientific fields. For example, economic and social sciences deal with agents that take decisions under the influence of other agents. In social life, preferences and beliefs are partly transmitted by means of various forms of social interaction and opinions are driven by the tendency of individuals to become more similar when they interact. Hence, a collective phenomenon, that we call “synchronization”, reflects the result of the interactions among different individuals. The underlying idea is that individuals have opinions that change according to the influence of other individuals giving rise to a sort of collective behavior.

In particular, there exists a growing interest in systems of *interacting urn models* (e.g. [3, 6, 9, 11, 16, 20, 22, 26, 28, 31]) and their variants and generalizations (e.g. [1, 2, 18]). Our work is placed in the stream of this scientific literature. Specifically, it deals with the class of the so-called *interacting reinforced stochastic processes* considered in [1, 2] with a general network-based interaction and in [18] with a mean-field interaction. Generally speaking, by reinforcement in a stochastic dynamics we mean any mechanism for which the probability that a given event occurs has an increasing dependence on the number of times that the same event occurred in the past. This “*reinforcement mechanism*”, also known as “*preferential attachment rule*” or “*Rich get richer rule*” or “*Matthew effect*”, is a key feature governing the dynamics of many biological, economic and social systems (see, e.g. [32]). The best known example of reinforced stochastic process is the standard Eggenberger-Pòlya urn [21, 29], which has been widely studied and generalized (some recent variants can be found in [4, 5, 8, 10, 12, 14, 23, 24, 27]).

A *Reinforced Stochastic Process* (RSP) can be defined as a stochastic process in which, along the

time-steps, an agent performs an action chosen in the set $\{0, 1\}$ in such a way that the probability of adopting “action 1” at a certain time-step has an increasing dependence on the number of times that the agent adopted “action 1” in the previous actions. Formally, it is a stochastic process $X = \{X_n : n \geq 1\}$ taking values in $\{0, 1\}$ and such that

$$(1) \quad P(X_{n+1} = 1 \mid Z_0, X_1, \dots, X_n) = Z_n,$$

with

$$(2) \quad Z_n = (1 - r_{n-1})Z_{n-1} + r_{n-1}X_n,$$

where Z_0 is a random variable with values in $[0, 1]$, $\mathcal{F}_n := \sigma(Z_0) \vee \sigma(X_k : 1 \leq k \leq n)$ and $(r_n)_{n \geq 0}$ is a sequence of real numbers in $(0, 1)$ such that

$$(3) \quad \lim_n n^\gamma r_n = c > 0 \quad \text{with } 1/2 < \gamma \leq 1.$$

(We refer to [18] for a discussion on the case $0 < \gamma \leq 1/2$, for which there is a different asymptotic behavior of the model that is out of the scope of this research work.) The process X describes the sequence of actions along the time-steps and, if at time-step n , the “action 1” has taken place, that is $X_n = 1$, then for “action 1” the probability of occurrence at time-step $(n+1)$ increases. Therefore, the larger Z_{n-1} , the higher the probability of having $X_n = 1$ and so the higher the probability of having Z_n greater than Z_{n-1} . This means the larger the number of times in which $X_k = 1$ with $1 \leq k \leq n$, the higher the probability Z_n of observing $X_{n+1} = 1$.

As told before, the best known example of reinforced stochastic process is the standard Eggenberger-Pòlya urn, where an urn contains a red and b white balls and, at each discrete time, a ball is drawn out from the urn and then it is put again inside the urn together with one additional ball of the same color. In this case, we have

$$Z_n = \frac{a + \sum_{k=1}^n X_k}{a + b + n}.$$

It is immediate to verify that

$$Z_0 = \frac{a}{a+b} \quad \text{and} \quad Z_{n+1} = (1 - r_n)Z_n + r_n X_{n+1}$$

with $r_n = (a + b + n + 1)^{-1}$ and so $\gamma = c = 1$.

In the present work we are interested in the analysis of a system of $N \geq 2$ interacting reinforced stochastic processes $\{X^j = (X_{n,j})_{n \geq 1} : 1 \leq j \leq N\}$ positioned at the vertices of a weighted directed graph $G = (V, E, W)$, where $V := \{1, \dots, N\}$ denotes the set of vertices, $E \subseteq V \times V$ the set of edges and $W = [w_{h,j}]_{h,j \in V \times V}$ the weighted adjacency matrix with $w_{h,j} \geq 0$ for each pair of vertices. The presence of the edge $(h, j) \in E$ indicates a “direct influence” that the vertex h has on the vertex j and it corresponds to a strictly positive element $w_{h,j}$ of W , that represents a weight quantifying this influence. We assume the weights to be normalized so that $\sum_{h=1}^N w_{h,j} = 1$ for each $j \in V$. The interaction between the processes $\{X^j : j \in V\}$ is explicitly inserted in Equation (1) and it is modeled as follows: for any $n \geq 0$, the random variables $\{X_{n+1,j} : j \in V\}$ are conditionally independent given \mathcal{F}_n with

$$(4) \quad P(X_{n+1,j} = 1 \mid \mathcal{F}_n) = \sum_{h=1}^N w_{h,j} Z_{n,h} = w_{j,j} Z_{n,j} + \sum_{h \neq j} w_{h,j} Z_{n,h},$$

where $\mathcal{F}_n := \sigma(Z_{0,h} : h \in V) \vee \sigma(X_{k,j} : 1 \leq k \leq n, j \in V)$ and, for each $h \in V$, the evolution dynamics of the single process $(Z_{n,h})_{n \geq 0}$ is the same as in (2), that is

$$(5) \quad Z_{n,h} = (1 - r_{n-1})Z_{n-1,h} + r_{n-1}X_{n,h},$$

with $Z_{0,h}$ a random variable taking values in $[0, 1]$ and $(r_n)_{n \geq 0}$ a sequence of real numbers in $(0, 1)$ such that condition (3) holds true.

As an example, we can imagine that $G = (V, E)$ represents a network of N individuals that at each time-step have to make a choice between two possible actions $\{0, 1\}$. For any $n \geq 1$, the random variables $\{X_{n,j} : j \in V\}$ take values in $\{0, 1\}$ and they describe the actions adopted by the agents of the network along the time-steps; while each random variable $Z_{n,h}$ takes values in $[0, 1]$ and it can be interpreted as the “personal inclination” of the agent h of adopting “action 1”. Thus, the probability

that the agent j adopts “action 1” at time-step $(n + 1)$ is given by a convex combination of j ’s own inclination and the inclination of the other agents at time-step n , according to the “influence-weights” $w_{h,j}$ as in (4). Note that, from a mathematical point of view, we can have $w_{jj} \neq 0$ or $w_{jj} = 0$. In both cases we have a reinforcement mechanism for the personal inclinations of the agents: indeed, by (5), whenever $X_{n,h} = 1$, we have a positive increment in the personal inclination of the agent h , that is $Z_{n,h} \geq Z_{n-1,h}$. However, only in the case $w_{jj} > 0$, this fact results in a greater probability of having $X_{n+1,j} = 1$ according to (4). Therefore, if $w_{jj} > 0$, then we have a “true self-reinforcing” mechanism; while, in the opposite case, we have a reinforcement property only in the own inclination of the single agent, but this does not affect the probability (4).

The literature [2, 18, 16, 20] focus on the asymptotic behavior of the stochastic processes of the personal inclinations $\{Z^j = (Z_{n,j})_n : j \in V\}$ of the agents; while [1] studies the average of times in which the agents adopt “action 1”, i.e. the stochastic processes of the empirical means $\{\bar{X}_n^j = (\frac{1}{n} \sum_{k=1}^n X_{k,j})_n : j \in V\}$. The results given in [1], together with the resulting statistical tools, represent a great improvement in any area of application, since the “actions” $X_{n,j}$ adopted by the agents of the network are much more likely to be observed than their personal inclinations $Z_{n,j}$ of adopting these actions. More specifically, in that paper, under suitable assumptions, it is proved that all the empirical means converge almost surely to the same limit random variable (almost sure synchronization), which is also the common limit random variable of the stochastic processes $Z^j = (Z_{n,j})_n$, say Z_∞ . Moreover, some Central Limit Theorems (CLTs) for the empirical means hold true and they lead to the construction of asymptotic confidence intervals for the common limit random variable Z_∞ and of a statistical test to make inference on the weighted adjacency matrix W of the network in the case $\gamma = 1$.

In the present paper, we continue in this direction: indeed, we not only extend the results obtained in [1] for the empirical means to the “weighted empirical means”, but, using a more sophisticated decomposition, we obtain two improvements: first, we here handle the two cases, $\gamma < 1$ and $\gamma = 1$, in the same way (while in [1] we use two different arguments) and, second, we here solve a research question posed in [1] and, consequently, we succeed in constructing a test statistics to make inference on the weighted adjacency matrix W of the network for all values of the model parameters (not only in the case $\gamma = 1$). More precisely, in this paper we focus on the weighted average of times in which the agents adopt “action 1”, i.e. we study the stochastic processes of the *weighted empirical means* $\{N^j = (N_{n,j})_n : j \in V\}$ defined, for each $j \in V$, as $N_0^j := 0$ and, for any $n \geq 1$,

$$(6) \quad N_{n,j} := \sum_{k=1}^n q_{n,k} X_{k,j}, \quad \text{where} \quad q_{n,k} := \frac{a_k}{\sum_{l=1}^n a_l},$$

with $(a_k)_{k \geq 1}$ a suitable sequence of strictly positive real numbers. Since $\sum_{k=1}^n q_{n,k} = 1$, we have the relation

$$\sum_{k=1}^{n-1} q_{n,k} X_{k,j} = \frac{\sum_{l=1}^{n-1} a_l}{\sum_{l=1}^n a_l} \left(\sum_{k=1}^{n-1} q_{n-1,k} X_{k,j} \right) = (1 - q_{n,n}) N_{n-1,j}$$

and so we get

$$(7) \quad N_{n,j} = (1 - q_{n,n}) N_{n-1,j} + q_{n,n} X_{n,j}.$$

Note that this framework includes as special case the process of the standard empirical means studied in [1], which corresponds to the case $a_k = 1$ for any $k \geq 1$ (and hence $q_{n,k} = 1/n$ for any $1 \leq k \leq n$). Furthermore, the above dynamics (4), (5) and (7) can be expressed in a compact form, using the random vectors $\mathbf{X}_n := (X_{n,1}, \dots, X_{n,N})^\top$ for $n \geq 1$, $\mathbf{N}_n := (N_{n,1}, \dots, N_{n,N})^\top$ and $\mathbf{Z}_n := (Z_{n,1}, \dots, Z_{n,N})^\top$ for $n \geq 0$, as

$$(8) \quad E[\mathbf{X}_{n+1} | \mathcal{F}_n] = W^\top \mathbf{Z}_n,$$

where $W^\top \mathbf{1} = \mathbf{1}$ by the normalization of the weights, and

$$(9) \quad \begin{cases} \mathbf{Z}_n = (1 - r_{n-1}) \mathbf{Z}_{n-1} + r_{n-1} \mathbf{X}_n, \\ \mathbf{N}_n = (1 - q_{n,n}) \mathbf{N}_{n-1} + q_{n,n} \mathbf{X}_n. \end{cases}$$

Under suitable assumptions, we prove the almost sure synchronization of the stochastic processes $N^j = (N_{n,j})_n$, with $j \in V$, toward the same limit random variable Z_∞ , which is the common limit random

variable of the stochastic processes $Z^j = (Z_{n,j})_n$ and we provide some CLTs in the sense of stable convergence. In particular, we assume

$$(10) \quad \lim_n n^\nu q_{n,n} = q > 0 \quad \text{with } 1/2 < \nu \leq 1$$

and the asymptotic covariances in the provided CLTs depend on the random variable Z_∞ , on the eigen-structure of the weighted adjacency matrix W and on the parameters γ , c and ν , q governing the asymptotic behavior of the sequence $(r_n)_n$ and $(q_{n,n})_n$, respectively. We also discuss the possible statistical applications of these convergence results: asymptotic confidence intervals for the common limit random variable Z_∞ and test statistics to make inference on the weighted adjacency matrix W of the network. In particular, as said before, we obtain a statistical test on the matrix W for all values of the model parameters (not only in the case $\gamma = \nu = q = 1$ as in [1]). Moreover, our results give a hint regarding a possible ‘‘optimal choice’’ of ν and q and so point out the advantages of employing the weighted empirical means with $\nu < 1$, instead of the simple empirical means.

Finally, we point out that the existence of joint central limit theorems for the pair $(\mathbf{Z}_n, \mathbf{N}_n)$ is not obvious because the ‘‘discount factors’’ in the dynamics of the increments $(\mathbf{Z}_n - \mathbf{Z}_{n-1})_n$ and $(\mathbf{N}_n - \mathbf{N}_{n-1})_n$ are generally different. Indeed, as shown in (9), these two stochastic processes follow the dynamics

$$(11) \quad \begin{cases} \mathbf{Z}_n - \mathbf{Z}_{n-1} = r_{n-1}(\mathbf{X}_n - \mathbf{Z}_{n-1}), \\ \mathbf{N}_n - \mathbf{N}_{n-1} = q_{n,n}(\mathbf{X}_n - \mathbf{N}_{n-1}), \end{cases}$$

and so, when we assume $\nu \neq \gamma$, it could be surprising that in some cases there exists a common convergence rate for the pair $(\mathbf{Z}_n, \mathbf{N}_n)$. It is worthwhile to note that dynamics similar to (11) have already been considered in the Stochastic Approximation literature. Specifically, in [30] the authors established a CLT for a pair of recursive procedures having two different step-sizes. However, this result does not apply to our situation. Indeed, the covariance matrices Σ_μ and Σ_θ in their main result (Theorem 1) are deterministic, while the asymptotic covariance matrices in our CLTs are random (as said before, they depend on the random variable Z_∞). This is why we do not use the simple convergence in distribution, but we employ the notion of stable convergence, which is, among other things, essential for the considered statistical applications. Moreover in [30], the authors find two different convergence rates, depending on the two different step-sizes, while, as already said, we find a common convergence rate also in some cases with $\nu \neq \gamma$.

Summing up, this work complete the convergence results obtained in [1, 2] for the stochastic processes of the personal inclinations $Z^j = (Z_{n,j})_n$ and of the empirical means $\bar{X}^j = (\bar{X}_{n,j})_n$, and it extend them to the weighted empirical means $N^j = (N_{n,j})_n$. However the main focus here concerns the new decomposition employed for the analysis of the asymptotic behavior of the pair $(\mathbf{Z}_n, \mathbf{N}_n)$, that, among other things, allows us to solve the research question arisen in [1] regarding the statistical test on W in the case $\gamma < 1$. Thus, in what follows, we will go fast on the point in common with [1, 2], while we concentrate on the novelties.

The rest of the paper is organized as follows. In Section 2 we describe the notation and the assumptions used along the paper. In Section 3 and Section 4 we illustrate our main results and we discuss some possible statistical applications. An interesting example of interacting system is also provided in order to clarify the statement of the theorems and the related comments. Section 5 and Section 6 contain the proofs or the main steps of the proofs of our results, while the technical details have been gathered in the appendix. In particular, Subsection 5.2 contains the main ingredient of the proofs of the CLTs, that is a suitable decomposition of the joint stochastic process $(\mathbf{Z}_n, \mathbf{N}_n)$. Finally, for the reader’s convenience, the appendix also supplies a brief review on the notion of stable convergence and its variants (e.g. see [13, 15, 17, 25, 33]).

2. NOTATION AND ASSUMPTIONS

Throughout all the paper, we will assume $N \geq 2$ and adopt the same notation used in [1, 2]. In particular, we denote by $\text{Re}(z)$, $\text{Im}(z)$, \bar{z} and $|z|$ the real part, the imaginary part, the conjugate and the modulus of a complex number z . Then, for a matrix A with complex elements, we let \bar{A} and A^\top be its conjugate and its transpose, while we indicate by $|A|$ the sum of the modulus of its elements. The

identity matrix is denoted by I , independently of its dimension that will be clear from the context. The spectrum of A , i.e. the set of all the eigenvalues of A repeated with their multiplicity, is denoted by $Sp(A)$, while its sub-set containing the eigenvalues with maximum real part is denoted by $\lambda_{\max}(A)$, i.e. $\lambda^* \in \lambda_{\max}(A)$ whenever $\mathcal{Re}(\lambda^*) = \max\{\mathcal{Re}(\lambda) : \lambda \in Sp(A)\}$. The notation $\text{diag}(a_1, \dots, a_d)$ indicates the diagonal matrix of dimension d with diagonal elements a_1, \dots, a_d . Finally, we consider any vector \mathbf{v} as a matrix with only one column (so that all the above notations apply to \mathbf{v}) and we indicate by $\|\mathbf{v}\|$ its norm, i.e. $\|\mathbf{v}\|^2 = \bar{\mathbf{v}}^\top \mathbf{v}$. The vectors and the matrices whose elements are all ones or zeros are denoted by $\mathbf{1}$ and $\mathbf{0}$, respectively, independently of their dimension that will be clear from the context.

For the matrix W we make the following assumption:

Assumption 2.1. *The weighted adjacency matrix W is irreducible and diagonalizable.*

The irreducibility of W reflects a situation in which all the vertices are connected among each others and hence there are no sub-systems with independent dynamics (see [2, 3] for further details). The diagonalizability of W allows us to find a non-singular matrix \tilde{U} such that $\tilde{U}^\top W (\tilde{U}^\top)^{-1}$ is diagonal with complex elements $\lambda_j \in Sp(W)$. Notice that each column \mathbf{u}_j of \tilde{U} is a left eigenvector of W associated to the eigenvalue λ_j . Without loss of generality, we take $\|\mathbf{u}_j\| = 1$. Moreover, when the multiplicity of some λ_j is bigger than one, we set the corresponding eigenvectors to be orthogonal. Then, if we define $\tilde{V} = (\tilde{U}^\top)^{-1}$, we have that each column \mathbf{v}_j of \tilde{V} is a right eigenvector of W associated to the eigenvalue λ_j such that

$$(12) \quad \mathbf{u}_j^\top \mathbf{v}_j = 1, \quad \text{and} \quad \mathbf{u}_h^\top \mathbf{v}_j = 0, \quad \forall h \neq j.$$

These constraints combined with the above assumptions on W (precisely, $w_{h,j} \geq 0$, $W^\top \mathbf{1} = \mathbf{1}$ and the irreducibility) imply, by Frobenius-Perron Theorem, that $\lambda_1 := 1$ is an eigenvalue of W with multiplicity one, $\lambda_{\max}(W) = \{1\}$ and

$$(13) \quad \mathbf{u}_1 = N^{-1/2} \mathbf{1}, \quad N^{-1/2} \mathbf{1}^\top \mathbf{v}_1 = 1 \quad \text{and} \quad v_{1,j} := [\mathbf{v}_1]_j > 0 \quad \forall 1 \leq j \leq N.$$

Moreover, we recall the relation

$$(14) \quad \sum_{j=1}^N \mathbf{u}_j \mathbf{v}_j^\top = I.$$

Finally, we set $\alpha_j := 1 - \lambda_j \in \mathbb{C}$ for each $j \geq 2$, i.e. for each λ_j belonging to $Sp(W) \setminus \{1\}$, and we denote by λ^* an eigenvalue belonging to $Sp(W) \setminus \{1\}$ such that

$$\mathcal{Re}(\lambda^*) = \max\{\mathcal{Re}(\lambda_j) : \lambda_j \in Sp(W) \setminus \{1\}\}.$$

Throughout all the paper, we assume that the two sequences $(r_n)_{n \geq 0}$ and $(q_{n,n})_{n \geq 1}$, which appear in (9), satisfy the following assumption:

Assumption 2.2. *There exist real constants $\gamma, \nu \in (1/2, 1]$ and $c, q > 0$ such that*

$$(15) \quad r_{n-1} = \frac{c}{n^\gamma} + O\left(\frac{1}{n^{2\gamma}}\right) \quad \text{and} \quad q_{n,n} = \frac{q}{n^\nu} + O\left(\frac{1}{n^{2\nu}}\right).$$

In particular, it follows

$$\lim_n n^\gamma r_n = c > 0 \quad \text{and} \quad \lim_n n^\nu q_{n,n} = q > 0.$$

The following remark will be useful for a certain proof in the sequel.

Remark 2.1. Recalling that $q_{n,n} = a_n / \sum_{l=1}^n a_l$, the second relation in (15) implies that $\sum_{n=1}^{+\infty} a_n = +\infty$. Indeed, the above relation together with $\sum_{n=1}^{+\infty} a_n = \ell < +\infty$ entails $a_n = q\ell n^{-\nu} + O(n^{-2\nu})$ and so, since $\nu \leq 1$, $\sum_{n=1}^{+\infty} a_n = +\infty$, which is a contradiction.

In the special case considered in [1], where the random variables $N_{n,j}$ correspond to the standard empirical means ($a_n = 1$ for each n), we have $\nu = 1$ and $q = 1$. Other possible choices are the following:

- $\sum_{l=1}^n a_l = n^\delta$ with $\delta > 0$, which brings to

$$a_n = n^\delta - (n-1)^\delta$$

and

$$q_{n,n} = 1 - \frac{\sum_{l=1}^{n-1} a_l}{\sum_{l=1}^n a_l} = 1 - \left(1 - \frac{1}{n}\right)^\delta = \delta n^{-1} + O(n^{-2}),$$

so that we have $\nu = 1$ and $q = \delta > 0$;

- $\sum_{l=1}^n a_l = \exp(bn^\delta)$ with $b > 0$ and $\delta \in (0, 1/2)$, which brings to

$$a_n = \exp(bn^\delta) - \exp(b(n-1)^\delta)$$

and

$$\begin{aligned} q_{n,n} &= 1 - \frac{\sum_{l=1}^{n-1} a_l}{\sum_{l=1}^n a_l} = 1 - \exp\left[b\left((n-1)^\delta - n^\delta\right)\right] \\ &= bn^\delta \left(1 - (1 - n^{-1})^\delta\right) + O\left(n^{2\delta}(1 - (1 - n^{-1})^\delta)^2\right) = bn^\delta (\delta n^{-1} + O(n^{-2})) + O(n^{-(2-2\delta)}) \\ &= b\delta n^{-(1-\delta)} + O(n^{-(2-\delta)}) + O(n^{-(2-2\delta)}) = b\delta n^{-(1-\delta)} + O(n^{-2(1-\delta)}), \end{aligned}$$

so that $\nu = (1 - \delta) \in (1/2, 1)$ and $q = b\delta > 0$.

To ease the notation, we set $\widehat{r}_{n-1} := cn^{-\gamma}$ and $\widehat{q}_{n,n} := qn^{-\nu}$, so that condition (15) can be rewritten as

$$r_{n-1} = \widehat{r}_{n-1} + O\left(\frac{1}{n^{2\gamma}}\right) \quad \text{and} \quad q_{n,n} = \widehat{q}_{n,n} + O\left(\frac{1}{n^{2\nu}}\right).$$

For the CLTs provided in the sequel, we make also the following assumption:

Assumption 2.3. *When $\gamma = 1$, we assume the condition $c > 1/[2(1 - \mathcal{R}e(\lambda^*))]$, i.e. $\mathcal{R}e(\lambda^*) < 1 - (2c)^{-1}$. When $\nu = 1$, we assume $q > 1/2$.*

Note that in Assumption 2.2 condition (15) for the sequence $(r_n)_n$ is slightly more restrictive than the one assumed in [1, 2]. However, it is always verified in the applicative contexts we have in mind. The reason behind this choice is that we want to avoid some technical complications in order to focus on the differences brought by the use of the weighted empirical means, specially on the relationship between the pair (γ, ν) and the asymptotic behaviors of the considered stochastic processes. For the same reason, in the CLTs for the case $\nu = \gamma$, we add also the following assumption:

$$(16) \quad q \neq c\alpha_j \quad \forall j \geq 2.$$

We think that this condition is not necessary. Indeed, if there exists $j \geq 2$ such that $q = c\alpha_j$, we conjecture that our proofs still work (but changing the asymptotic expression adopted for a certain quantity, see the proof of Lemma 5.1) and they lead to exactly the same asymptotic covariances provided in the CLTs under the above condition (16). Our conjecture is motivated by the fact that this is what happens in [1] for the simple empirical means. Moreover, the expressions obtained for the asymptotic covariances in the following CLTs do not require condition (16). However, as told before, we do not want to make the following proofs even heavier and so, when $\nu = \gamma$, we will work under condition (16).

3. MAIN RESULTS ON THE JOINT STOCHASTIC PROCESS

The first achievement concerns the *almost sure synchronization* of all the involved stochastic processes, that is

$$(17) \quad \mathbf{Y}_n := \begin{pmatrix} \mathbf{Z}_n \\ \mathbf{N}_n \end{pmatrix} \xrightarrow{a.s.} Z_\infty \mathbf{1},$$

where Z_∞ is a random variable with values in $[0, 1]$. This fact means that all the stochastic processes $Z^j = (Z_{n,j})_n$ and $N^j = (N_{n,j})_n$ positioned at different vertices $j \in V$ of the graph converge almost surely to the same random variable Z_∞ .

The synchronization for the first component of \mathbf{Y}_n , that is

$$(18) \quad [\mathbf{Y}_n]_1 = \mathbf{Z}_n \xrightarrow{a.s.} Z_\infty \mathbf{1},$$

is the result contained in [2, Theorem 3.1], while for the second component, we prove in the present work the following result:

Theorem 3.1. *Under Assumptions 2.1 and 2.2, we have*

$$(19) \quad [\mathbf{Y}_n]_2 = \mathbf{N}_n \xrightarrow{a.s.} Z_\infty \mathbf{1}.$$

Regarding the distribution of Z_∞ , we recall that [2, Theorems 3.5 and 3.6] state the following two properties:

- (i) $P(Z_\infty = z) = 0$ for any $z \in (0, 1)$.
- (ii) If we have $P(\bigcap_{j=1}^N \{Z_{0,j} = 0\}) + P(\bigcap_{j=1}^N \{Z_{0,j} = 1\}) < 1$, then $P(0 < Z_\infty < 1) > 0$.

In particular, these facts entail that the asymptotic covariances in the following CLTs are “truly” random. Indeed, their random part $Z_\infty(1 - Z_\infty)$ is different from zero with probability greater than zero and almost surely different from a constant in $(0, 1)$.

Furthermore, it is interesting to note that the almost sure synchronization holds true without any assumptions on the initial configuration \mathbf{Z}_0 and for any choice of the weighted adjacency matrix W with the required assumptions. Finally, note that the synchronization is induced along time independently of the fixed size N of the network, and so it does not require a large-scale limit (i.e. the limit for $N \rightarrow +\infty$), which is usual in statistical mechanics for the study of interacting particle systems.

Regarding the convergence rate and the second-order asymptotic distribution of $(\mathbf{Y}_n - Z_\infty \mathbf{1})$, setting for each $\gamma \in (1/2, 1]$

$$(20) \quad \gamma_0 := \max \left\{ \frac{1}{2}, 2\gamma - 1 \right\} \in [1/2, 1],$$

$$(21) \quad \tilde{\Sigma}_\gamma := \tilde{\sigma}_\gamma^2 \mathbf{1}\mathbf{1}^\top \quad \text{with} \quad \tilde{\sigma}_\gamma^2 := \frac{\|\mathbf{v}_1\|^2 c^2}{N(2\gamma - 1)}$$

and

$$(22) \quad \tilde{U} = (\mathbf{u}_1 \quad \mathbf{u}_2 \quad \dots \quad \mathbf{u}_N) = (N^{-1/2} \mathbf{1} \quad U) \quad \text{with} \quad U := (\mathbf{u}_2 \quad \dots \quad \mathbf{u}_N),$$

we obtain the following result:

Theorem 3.2. *Under all the assumptions stated in Section 2, the following statements hold true:*

- (a) *If $1/2 < \nu < \gamma_0$, then*

$$(23) \quad n^{\nu/2}(\mathbf{Y}_n - Z_\infty \mathbf{1}) \longrightarrow \mathcal{N} \left(\mathbf{0}, Z_\infty(1 - Z_\infty) \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{U} S^{(q)} \tilde{U}^\top \end{pmatrix} \right) \quad \text{stably,}$$

where, for $1 \leq j_1, j_2 \leq N$,

$$(24) \quad [S^{(q)}]_{j_1 j_2} := \frac{q}{2} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2}.$$

- (b) *If $\gamma_0 < \nu < 1$, then*

$$(25) \quad n^{\gamma - \frac{1}{2}}(\mathbf{Y}_n - Z_\infty \mathbf{1}) \longrightarrow \mathcal{N} \left(\mathbf{0}, Z_\infty(1 - Z_\infty) \tilde{\Sigma}_\gamma \right) \quad \text{stably.}$$

- (c) *If $\nu = \gamma_0 < 1$, then*

$$(26) \quad n^{\gamma - \frac{1}{2}}(\mathbf{Y}_n - Z_\infty \mathbf{1}) \longrightarrow \mathcal{N} \left(\mathbf{0}, Z_\infty(1 - Z_\infty) \left(\tilde{\Sigma}_\gamma + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{U} S^{(q)} \tilde{U}^\top \end{pmatrix} \right) \right) \quad \text{stably,}$$

where $S^{(q)}$ is the same matrix defined in (a) by (24).

- (d) *If $\nu = \gamma_0 = 1$ (that is $\nu = \gamma = 1$), then*

$$(27) \quad \sqrt{n}(\mathbf{Y}_n - Z_\infty \mathbf{1}) \longrightarrow \mathcal{N} \left(\mathbf{0}, Z_\infty(1 - Z_\infty) \left(\tilde{\Sigma}_1 + \begin{pmatrix} \tilde{U} S^{11} \tilde{U}^\top & \tilde{U} S^{12} \tilde{U}^\top \\ \tilde{U} S^{21} \tilde{U}^\top & \tilde{U} S^{22} \tilde{U}^\top \end{pmatrix} \right) \right) \quad \text{stably,}$$

where $S^{21} = (S^{12})^\top$ and, for $2 \leq j_1, j_2, j \leq N$,

$$\begin{aligned} [S^{11}]_{11} &= [S^{11}]_{j_1 1} = [S^{11}]_{1 j_2} := 0, \\ [S^{11}]_{j_1 j_2} &:= \frac{c^2}{c(\alpha_{j_1} + \alpha_{j_2}) - 1} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2}, \\ [S^{12}]_{11} &= [S^{12}]_{1 j_2} := 0, \\ [S^{12}]_{j_1 1} &:= \frac{c(q-c)}{c\alpha_{j_1} + q - 1} \mathbf{v}_{j_1}^\top \mathbf{v}_1, \\ [S^{12}]_{j_1 j_2} &:= \frac{cq(c\alpha_{j_1} + c - 1)}{(c\alpha_{j_1} + c\alpha_{j_2} - 1)(c\alpha_{j_1} + q - 1)} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2}, \\ [S^{22}]_{11} &:= \frac{(q-c)^2}{2q-1} \|\mathbf{v}_1\|^2, \\ [S^{22}]_{j_1 1} &= [S^{22}]_{1 j_1} := \frac{q(q-c)(c+q-1)}{(c\alpha_j + q - 1)(2q-1)} \mathbf{v}_j^\top \mathbf{v}_1, \\ [S^{22}]_{j_1 j_2} &:= q^2 \frac{c^3(\alpha_{j_1} + \alpha_{j_2}) + 2c^2q(\alpha_{j_1}\alpha_{j_2} + 1) - c^2(\alpha_{j_1}\alpha_{j_2} + \alpha_{j_1} + \alpha_{j_2} + 2)}{(2q-1)(c(\alpha_{j_1} + \alpha_{j_2}) - 1)(c\alpha_{j_1} + q - 1)(c\alpha_{j_2} + q - 1)} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2} \\ &\quad + q^2 \frac{c(q-1)^2(\alpha_{j_1} + \alpha_{j_2}) - (2c+q-1)(q-1)}{(2q-1)(c(\alpha_{j_1} + \alpha_{j_2}) - 1)(c\alpha_{j_1} + q - 1)(c\alpha_{j_2} + q - 1)} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2}. \end{aligned}$$

(e) If $\gamma_0 < \nu = 1$, then

$$(28) \quad n^{\gamma-\frac{1}{2}}(\mathbf{Y}_n - Z_\infty \mathbf{1}) \longrightarrow \mathcal{N}\left(\mathbf{0}, Z_\infty(1 - Z_\infty) \left(\tilde{\Sigma}_\gamma + \frac{\|\mathbf{v}_1\|^2 c^2}{N[2q - (2\gamma - 1)]} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}\mathbf{1}^\top \end{pmatrix} \right)\right) \quad \text{stably.}$$

Remark 3.1. Looking at the asymptotic covariance matrices in the different cases of the above theorem, note that in case (a) the convergence rate of the first component is bigger than the one of the second component. Indeed, from our previous work [2], we know that it is $n^{\gamma_0/2}$. On the other hand, there are cases (see (b), (c) and (e)) in which the convergence rates of the two components are the same, although the discount factors $r_n \sim cn^{-\gamma}$ and $q_{n,n} \sim qn^{-\nu}$ in (9) have different convergence rates.

Remark 3.2. Recall that we have

$$1 \leq 1 + \|\mathbf{v}_1 - \mathbf{u}_1\|^2 = \|\mathbf{v}_1\|^2 \leq N.$$

Therefore we obtain the following lower and upper bounds (that do not depend on W) for $\tilde{\sigma}_\gamma^2$ and for the second term in the asymptotic covariance of relation (28):

$$\frac{c^2}{N(2\gamma - 1)} \leq \tilde{\sigma}_\gamma^2 \leq \frac{c^2}{2\gamma - 1}$$

and

$$\frac{c^2}{N[2q - (2\gamma - 1)]} \leq \frac{\|\mathbf{v}_1\|^2 c^2}{N[2q - (2\gamma - 1)]} \leq \frac{c^2}{2q - (2\gamma - 1)}.$$

Notice that the lower bound is achieved when $\mathbf{v}_1 = \mathbf{u}_1 = N^{-1/2}\mathbf{1}$, i.e. when W is doubly stochastic, which means $W\mathbf{1} = W^\top\mathbf{1} = \mathbf{1}$.

Remark 3.3. The results of Theorem 3.2 extend those presented in [1], since they are valid only for $q = \nu = 1$ which here corresponds to a special situation in case (d) and (e) of Theorem 3.2. Indeed, when $q = \nu = 1$ and $\gamma < 1$ we have that [1, Theorem 3.2] coincides with the result of case (e) while, when $q = \nu = 1$ and $\gamma = 1$, we have that [1, Theorem 3.4] coincides with the result of case (d), because we have

$$\begin{pmatrix} \tilde{U}S^{11}\tilde{U}^\top & \tilde{U}S^{12}\tilde{U}^\top \\ \tilde{U}S^{21}\tilde{U}^\top & \tilde{U}S^{22}\tilde{U}^\top \end{pmatrix} = \begin{pmatrix} U\hat{S}_{ZZ}U^\top & U\hat{S}_{ZN}\tilde{U}^\top \\ \tilde{U}\hat{S}_{ZN}^\top U^\top & \tilde{U}\hat{S}_{NN}\tilde{U}^\top \end{pmatrix},$$

where the matrices \hat{S}_{ZZ} , \hat{S}_{ZN} and \hat{S}_{NN} are defined in [1].

Remark 3.4. The main goal of this work is to provide results for a system of $N \geq 2$ interacting reinforced stochastic processes. However, it is worth to note that Theorem 3.1, and the consequent limit (17), hold true also for $N = 1$. Moreover, statements (d) and (e) of Theorem 3.2 with $N = 1$ are true and they correspond to [1, Theorems 3.2 and 3.3]. Finally statements (a), (b) and (c) of Theorem 3.2 with $N = 1$ (and so without the condition on λ^*) can be proven with the same proof provided in the sequel (see the following Remark 5.3).

We conclude this section with the example of the “mean-field” interaction.

Example 3.1. The mean-field interaction can be expressed in terms of a particular weighted adjacency matrix W as follows: for any $1 \leq j_1, j_2 \leq N$

$$(29) \quad w_{j_1, j_2} = \frac{\alpha}{N} + (1 - \alpha)\delta_{j_1, j_2} \quad \text{with } \alpha \in [0, 1],$$

where δ_{j_1, j_2} is equal to 1 when $j_1 = j_2$ and to 0 otherwise. Note that W in (29) is irreducible for $\alpha > 0$ and so we are going to consider this case. Since W is doubly stochastic, we have $\mathbf{v}_1 = \mathbf{u}_1 = N^{-1/2}\mathbf{1}$. Moreover, since W is also symmetric, we have $\tilde{U} = \tilde{V}$ and so $\tilde{U}\tilde{U}^\top = I$ and $\tilde{V}^\top\tilde{V} = I$. Finally, we have $\lambda_j = 1 - \alpha$ for all $j \geq 2$ and, consequently, we obtain

$$\begin{aligned} S^{(q)} &= \frac{q}{2}I, \quad \{[S^{11}]_{j_1 j_2} : 2 \leq j_1, j_2 \leq N\} = \frac{c^2}{2c\alpha - 1}I, \\ [S^{12}]_{j_1 1} &= 0 \text{ for } 2 \leq j_1 \leq N, \quad \{[S^{12}]_{j_1 j_2} : 2 \leq j_1, j_2 \leq N\} = \frac{qc(c\alpha + c - 1)}{(2c\alpha - 1)(c\alpha + q - 1)}I, \\ [S^{22}]_{11} &= \frac{(q - c)^2}{2q - 1}, \quad [S^{22}]_{j1} = [S^{22}]_{1j} = 0 \text{ for } 2 \leq j \leq N, \\ \{[S^{22}]_{j_1 j_2} : 2 \leq j_1, j_2 \leq N\} &= \\ &= \frac{(qc)^2[(\alpha^2 + 1)(2q - 1) + 2\alpha(c - 1) - 1 + (2\alpha - c^{-1})(q - 1)^2 - 2c^{-1}(q - 1)]}{(2q - 1)(2c\alpha - 1)(c\alpha + q - 1)^2}I, \end{aligned}$$

and the condition $\mathcal{R}e(\lambda^*) < 1 - (2c)^{-1}$ when $\gamma = 1$ becomes $2c\alpha > 1$.

4. USEFUL RESULTS FOR STATISTICAL APPLICATIONS

The first convergence result provided in this section can be used for the construction of asymptotic confidence intervals for the limit random variable Z_∞ , that requires to know the following quantities:

- N : the number of agents in the network;
- \mathbf{v}_1 : the right eigenvector of W associated to $\lambda_1 = 1$ (note that it is not required to know the whole weighted adjacency matrix W , e.g. we have $\mathbf{v}_1 = \mathbf{u}_1 = N^{-1/2}\mathbf{1}$ for any doubly stochastic matrix);
- γ and c : the parameters that describe the first-order asymptotic approximation of the sequence $(r_n)_n$;
- ν and q : the parameters that describe the first-order asymptotic approximation of the sequence $(q_{n,n})_n$ (recall that the weights $q_{n,k}$ are chosen and so ν and q are always known and, moreover, they can be optimally chosen).

We point out that it is not required the observation of the random variables $Z_{n,j}$, nor the knowledge of the initial random variables $\{Z_{0,j} : j \in V\}$ and nor of the exact expression of the sequence $(r_n)_n$. They are based on the weighted empirical means of the random variables $X_{n,j}$, that are typically observable.

The second result stated in this section can be employed for the construction of asymptotic critical regions for statistical tests on the weighted adjacency matrix W based on the weighted empirical means (given the values of γ , ν , c and q). In particular, we point out that in our previous work [1] we succeeded to provide a testing procedure based on the standard empirical means only for the case $\gamma = 1$; while we announced further future investigation for the case $1/2 < \gamma < 1$. In the present work we face and solve this issue, providing a test statistics for all the values of the parameters. Indeed the following Theorem 4.2 covers all the cases for the pair (γ, ν) .

Let us consider the decomposition $\mathbf{N}_n = \mathbf{1}\tilde{N}_n + \mathbf{N}'_n$, where

$$(30) \quad \mathbf{1}\tilde{N}_n := \mathbf{u}_1 \mathbf{v}_1^\top \mathbf{N}_n = N^{-1/2} \mathbf{1} \mathbf{v}_1^\top \mathbf{N}_n \quad \text{and} \quad \mathbf{N}'_n := \mathbf{N}_n - \mathbf{1}\tilde{N}_n = (I - \mathbf{u}_1 \mathbf{v}_1^\top) \mathbf{N}_n.$$

Concerning the first term, by (13) and the almost sure synchronization (17), we immediately obtain

$$\tilde{N}_n \xrightarrow{a.s.} Z_\infty.$$

Moreover, under all the assumptions stated in Section 2, setting

$$(31) \quad \tilde{\sigma}^2 := \frac{\|\mathbf{v}_1\|^2}{N} \times \begin{cases} \frac{q}{2} & \text{if } \nu < \gamma_0 \quad \text{or} \quad \nu = \gamma_0 < 1, \\ \frac{(q-c)^2}{2q-1} & \text{if } \nu = \gamma_0 = 1 \text{ (that is } \nu = \gamma = 1), \\ \frac{c^2}{2q - (2\gamma - 1)} & \text{if } \gamma_0 < \nu = 1, \end{cases}$$

where γ_0 and $\tilde{\sigma}_\gamma^2$ are defined in (20) and in (21), respectively, we have the following result:

Theorem 4.1. *Under all the assumptions stated in Section 2, the following statements hold true:*

(a) *If $\nu < \gamma_0$, then*

$$n^{\nu/2}(\tilde{N}_n - Z_\infty) \longrightarrow \mathcal{N}(0, Z_\infty(1 - Z_\infty)\tilde{\sigma}^2) \quad \text{stably.}$$

(b) *If $\gamma_0 < \nu < 1$, then*

$$n^{\gamma-1/2}(\tilde{N}_n - Z_\infty) \longrightarrow \mathcal{N}(0, Z_\infty(1 - Z_\infty)\tilde{\sigma}_\gamma^2) \quad \text{stably.}$$

(c) *If $\nu = \gamma_0$ or $\nu = 1$ (i.e. $\nu = \gamma_0 < 1$ or $\nu = \gamma_0 = 1$ or $\gamma_0 < \nu = 1$), then*

$$n^{\gamma-1/2}(\tilde{N}_n - Z_\infty) \longrightarrow \mathcal{N}(0, Z_\infty(1 - Z_\infty)(\tilde{\sigma}_\gamma^2 + \tilde{\sigma}^2)) \quad \text{stably.}$$

Note that $\tilde{\sigma}^2$ has not been defined in the case $\gamma_0 < \nu < 1$, i.e. in the case (b) of the above result, because in this case it does not appear in the asymptotic covariance matrix.

In the following remark, we point out the advantages of employing the weighted empirical means with $\nu < 1$, instead of the simple empirical means (for which we have $\nu = q = 1$), providing a brief discussion on the possible ‘‘optimal choice’’ of ν and q :

Remark 4.1. The convergence rates and the asymptotic variances expressed in the cases of the above Theorem 4.1 allows us to make some considerations on the existence of an ‘‘optimal’’ choice of the parameters ν and q in order to ‘‘maximize the convergence’’ of \tilde{N}_n towards the random limit Z_∞ . Indeed, first note that the convergence rate in case (a) is slower than the rates of the other two cases, and, moreover, the asymptotic variance in case (c) is strictly larger than the variance in case (b). Hence, the interval $\gamma_0 < \nu < 1$ in case (b) provides an ‘‘optimal’’ range of values where the parameter ν should be chosen. In addition, looking into the proof of Theorem 4.1, it is possible to investigate more deeply into the behavior of \tilde{N}_n and so derive more accurate optimality conditions on the values of ν and q (see the following Remark 6.2).

Analogously, concerning the term $\mathbf{N}'_n = (I - \mathbf{u}_1 \mathbf{v}_1^\top) \mathbf{N}_n$, from (13) and the almost sure synchronization (17), we obtain

$$\mathbf{N}'_n \xrightarrow{a.s.} \mathbf{0}.$$

Moreover, setting

$$(32) \quad \tilde{U}_{-1} := (\mathbf{0} \quad \mathbf{u}_2 \quad \dots \quad \mathbf{u}_N) = (\mathbf{0} \quad U),$$

we get the following theorem:

Theorem 4.2. *Under all the assumptions stated in Section 2, the following statements hold true:*

(a) *If $\nu < \gamma$, then*

$$n^{\frac{\nu}{2}} \mathbf{N}'_n \longrightarrow \mathcal{N}\left(\mathbf{0}, Z_\infty(1 - Z_\infty) \tilde{U}_{-1} S^{(q)} \tilde{U}_{-1}^\top\right) \quad \text{stably,}$$

where $S^{(q)}$ is defined in (24).

(b) If $\nu = \gamma$, then

$$n^{\frac{\nu}{2}} \mathbf{N}'_n \longrightarrow \mathcal{N} \left(\mathbf{0}, Z_\infty(1 - Z_\infty) \tilde{U}_{-1} S_\gamma^{22} \tilde{U}_{-1}^\top \right) \quad \text{stably,}$$

where, for any $2 \leq j_1, j_2 \leq N$, we have that $[S_\gamma^{22}]_{11}$, $[S_\gamma^{22}]_{1j_2}$ and $[S_\gamma^{22}]_{j_1 1}$ are not needed to be defined since the first column of \tilde{U}_{-1} is $\mathbf{0}$, while the remaining elements $[S_\gamma^{22}]_{j_1 j_2}$ are defined as

$$(33) \quad \begin{aligned} & q^2 \frac{c^3(\alpha_{j_1} + \alpha_{j_2}) + 2c^2q(\alpha_{j_1}\alpha_{j_2} + 1) - \mathbb{1}_{\{\gamma=1\}}c^2(\alpha_{j_1}\alpha_{j_2} + \alpha_{j_1} + \alpha_{j_2} + 2)}{(2q - \mathbb{1}_{\{\gamma=1\}})(c(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_2} + q - \mathbb{1}_{\{\gamma=1\}})} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2} \\ & + q^2 \frac{c(q - \mathbb{1}_{\{\gamma=1\}})^2(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}}(2c + q - 1)(q - 1)}{(2q - \mathbb{1}_{\{\gamma=1\}})(c(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_2} + q - \mathbb{1}_{\{\gamma=1\}})} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2}. \end{aligned}$$

(c) If $\gamma < \nu$, then

$$n^{\frac{\nu}{2}} \mathbf{N}'_n \longrightarrow \mathcal{N} \left(\mathbf{0}, Z_\infty(1 - Z_\infty) \tilde{U}_{-1} S \tilde{U}_{-1}^\top \right) \quad \text{stably,}$$

where, for any $2 \leq j_1, j_2 \leq N$, we have that $[S]_{11}$, $[S]_{1j_2}$ and $[S]_{j_1 1}$ are not needed to be defined since the first column of \tilde{U}_{-1} is $\mathbf{0}$, while the remaining elements $[S]_{j_1 j_2}$ are defined as

$$q^2 \left(\left(\frac{\lambda_{j_1} \lambda_{j_2}}{\alpha_{j_1} \alpha_{j_2}} \right) \frac{1}{2q - \mathbb{1}_{\{\nu=1\}}(2\gamma - 1)} + \left(\frac{\lambda_{j_1}}{\alpha_{j_1}} + \frac{\lambda_{j_2}}{\alpha_{j_2}} \right) \frac{1}{2q - \mathbb{1}_{\{\nu=1\}}\gamma} + \frac{1}{2q - \mathbb{1}_{\{\nu=1\}}} \right) \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2}.$$

Note that the convergence rate for (\mathbf{N}'_n) is always $n^{\nu/2}$.

In the following example we go on with the analysis of the mean-field interaction.

Example 4.1. If we consider again the mean-field interaction (see (29)), we have $\mathbf{N}'_n = (I - N^{-1} \mathbf{1} \mathbf{1}^\top) \mathbf{N}_n$ (because $\mathbf{v}_1 = \mathbf{u}_1 = N^{-1/2} \mathbf{1}$). Moreover, since $\tilde{U} = \tilde{V}$ and so $\tilde{V}^\top \tilde{V} = I$, we find $S^{(q)} = \frac{q}{2} I$,

$$\begin{aligned} \{[S_\gamma^{22}]_{j_1 j_2} : 2 \leq j_1, j_2 \leq N\} &= s_\gamma^{22} I \quad \text{with} \\ s_\gamma^{22} &:= \frac{q^2 [c^2(\alpha^2 + 1)(2q - \mathbb{1}_{\{\gamma=1\}}) + 2c^2\alpha(c - \mathbb{1}_{\{\gamma=1\}}) - \mathbb{1}_{\{\gamma=1\}}c^2 + 2\alpha c(q - \mathbb{1}_{\{\gamma=1\}})^2 - \mathbb{1}_{\{\gamma=1\}}(2c + q - 1)(q - 1)]}{(2q - \mathbb{1}_{\{\gamma=1\}})(2c\alpha - \mathbb{1}_{\{\gamma=1\}})(c\alpha + q - \mathbb{1}_{\{\gamma=1\}})^2} \end{aligned}$$

and

$$S = sI \quad \text{with } s := q^2 \left(\left(\frac{1 - \alpha}{\alpha} \right)^2 \frac{1}{2q - \mathbb{1}_{\{\nu=1\}}(2\gamma - 1)} + 2 \frac{(1 - \alpha)}{\alpha} \frac{1}{2q - \mathbb{1}_{\{\nu=1\}}\gamma} + \frac{1}{2q - \mathbb{1}_{\{\nu=1\}}} \right).$$

Hence, since $\tilde{U}_{-1} I \tilde{U}_{-1}^\top = U U^\top = I - N^{-1} \mathbf{1} \mathbf{1}^\top$, we get that

$$n^{\nu/2} (I - N^{-1} \mathbf{1} \mathbf{1}^\top) \mathbf{N}_n \longrightarrow \mathcal{N} \left(0, Z_\infty(1 - Z_\infty) s^* (I - N^{-1} \mathbf{1} \mathbf{1}^\top) \right) \quad \text{stably,}$$

where s^* is equal to $q/2$ or s_γ^{22} or s , according to the values of ν and γ . Finally, using the relations $U^\top U = I$ and $U U^\top = I - N^{-1} \mathbf{1} \mathbf{1}^\top$ and employing \tilde{N}_n as a strong consistent estimator of Z_∞ , we get

$$\frac{n^{\nu/2}}{[\tilde{N}_n(1 - \tilde{N}_n)s^*]^{1/2}} U^\top \mathbf{N}_n \stackrel{d}{\sim} \mathcal{N}(0, I)$$

and

$$\frac{n^\nu}{\tilde{N}_n(1 - \tilde{N}_n)s^*} \mathbf{N}_n^\top (I - N^{-1} \mathbf{1} \mathbf{1}^\top) \mathbf{N}_n \stackrel{d}{\sim} \chi_{N-1}^2.$$

Given the values of γ , ν , c and q , this result can be used in order to perform a statistical test on the parameter α in the definition of W (see (29)).

5. PROOF OF THE RESULTS ON THE JOINT STOCHASTIC PROCESS

Here we prove the convergence results stated in Section 3.

5.1. Proof of Theorem 3.1. As already recalled (see (18)), we have $\mathbf{Z}_n \xrightarrow{a.s.} Z_\infty$. Hence, since the condition $W^\top \mathbf{1} = \mathbf{1}$ and the equality (8), we get $E[\mathbf{X}_n | \mathcal{F}_{n-1}] \xrightarrow{a.s.} Z_\infty \mathbf{1}$. Therefore, the convergence $\mathbf{N}_n \xrightarrow{a.s.} Z_\infty \mathbf{1}$ follows from [1, Lemma B.1] with $c_k = k^\nu$, $v_{n,k} = c_k q_{n,k}$ and $\eta = 1$. Note that the assumptions on the weights $q_{n,k} = a_k / \sum_{l=1}^n a_l$, easily implies that c_k and $v_{n,k}$ satisfy the conditions required in the employed lemma: indeed, by definition, we have $\sum_{k=1}^n q_{n,k} = 1$ and from the second relation in (15) we get $\sum_{n=1}^{+\infty} a_n = +\infty$ and

$$n^\nu a_n = q \sum_{l=1}^n a_l + O\left(n^{-\nu} \sum_{l=1}^n a_l\right) = q \sum_{l=1}^n a_l + O\left(a_n (n^\nu q_{n,n})^{-1}\right) = q \sum_{l=1}^n a_l + O(a_n),$$

and so we obtain

$$\lim_n v_{n,k} = c_k a_k \lim_n \frac{1}{\sum_{l=1}^n a_l} = 0, \quad \lim_n v_{n,n} = \lim_n c_n q_{n,n} = q, \quad \lim_n \sum_{k=1}^n \frac{v_{n,k}}{c_k} = \lim_n \sum_{k=1}^n q_{n,k} = 1$$

and

$$\begin{aligned} \sum_{k=1}^n |v_{n,k} - v_{n,k-1}| &= \frac{1}{\sum_{l=1}^n a_l} \sum_{k=1}^n k^\nu a_k - (k-1)^\nu a_{k-1} \\ &= \frac{1}{\sum_{l=1}^n a_l} \left[\sum_{k=1}^n q \left(\sum_{l=1}^k a_l - \sum_{l=1}^{k-1} a_l \right) + O\left(\sum_{k=1}^n a_k \right) \right] \\ &= q \frac{\sum_{k=1}^n a_k}{\sum_{l=1}^n a_l} + O(1) = O(1). \end{aligned}$$

□

5.2. Decomposition of the joint stochastic process. In this section we describe the main tool used in the following proofs, that is a suitable decomposition of the joint stochastic process $\mathbf{Y} := (\mathbf{Y}_n)_n$. Indeed, in order to determine the convergence rate and the second-order asymptotic distribution of $(\mathbf{Y}_n - Z_\infty \mathbf{1})$ for any values of the parameters, we need to decompose \mathbf{Y} into a sum of “primitive” stochastic processes, and then establish the asymptotic behavior for each one of them. As we will see, they converge at different rates.

Let us express the dynamics (9) of the stochastic processes $(\mathbf{Z}_n)_n$ and $(\mathbf{N}_n)_n$ as follows:

$$(34) \quad \begin{cases} \mathbf{Z}_n - \mathbf{Z}_{n-1} = -\hat{r}_{n-1} (I - W^\top) \mathbf{Z}_{n-1} + \hat{r}_{n-1} \Delta \mathbf{M}_n + \Delta \mathbf{R}_{Z,n}, \\ \mathbf{N}_n - \mathbf{N}_{n-1} = -\hat{q}_{n,n} (\mathbf{N}_{n-1} - W^\top \mathbf{Z}_{n-1}) + \hat{q}_{n,n} \Delta \mathbf{M}_n + \Delta \mathbf{R}_{N,n}, \end{cases}$$

where $\Delta \mathbf{M}_n := (\mathbf{X}_n - W^\top \mathbf{Z}_{n-1})$ is a martingale increment with respect to the filtration $\mathcal{F} := (\mathcal{F}_n)_n$, while $\Delta \mathbf{R}_{Z,n} := (r_{n-1} - \hat{r}_{n-1})(\mathbf{X}_n - \mathbf{Z}_{n-1})$ and $\Delta \mathbf{R}_{N,n} := (q_{n,n} - \hat{q}_{n,n})(\mathbf{X}_n - \mathbf{Z}_{n-1})$ are two remainder terms. Hence, by means of (34), the dynamics of the stochastic process \mathbf{Y} can be expressed as

$$(35) \quad \mathbf{Y}_n = (I - Q_n) \mathbf{Y}_{n-1} + R_n \Delta \mathbf{M}_{Y,n} + \Delta \mathbf{R}_{Y,n},$$

where $\Delta \mathbf{M}_{Y,n} := (\Delta \mathbf{M}_n, \Delta \mathbf{M}_n)^\top$, $\Delta \mathbf{R}_{Y,n} := (\Delta \mathbf{R}_{Z,n}, \Delta \mathbf{R}_{N,n})^\top$,

$$(36) \quad Q_n := \begin{pmatrix} \hat{r}_{n-1}(I - W^\top) & \mathbf{0} \\ -\hat{q}_{n,n} W^\top & \hat{q}_{n,n} I \end{pmatrix} \quad \text{and} \quad R_n := \begin{pmatrix} \hat{r}_{n-1} I & \mathbf{0} \\ \mathbf{0} & \hat{q}_{n,n} I \end{pmatrix}.$$

Now, we want to decompose the stochastic process \mathbf{Y} in a sum of stochastic processes, whose dynamics are of the same types of (35), but more tractable. To this purpose, we set $U_j := (\mathbf{u}_{j(1)}, \mathbf{u}_{j(2)})$ for each $j = 1, \dots, N$, and we impose the following relations:

$$(37) \quad U_j = U_j^* P_j \quad \text{with} \quad U_j^* := \begin{pmatrix} \mathbf{u}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{u}_j \end{pmatrix} \quad \text{and} \quad P_j := \begin{pmatrix} 1 & 0 \\ g(\lambda_j) & 1 \end{pmatrix},$$

and, for any $n \geq 1$,

$$(38) \quad Q_n U_j = U_j D_{Q,j,n}, \quad \text{where} \quad D_{Q,j,n} := \begin{pmatrix} \hat{r}_{n-1}(1 - \lambda_j) & \mathbf{0} \\ -\lambda_j h_n(\lambda_j) & \hat{q}_{n,n} \end{pmatrix}.$$

We recall that λ_j and \mathbf{u}_j denote the eigenvalues and the left eigenvectors of W , respectively. The above functions g and h_n will be suitable defined later on. In particular, we will define h_n in such a way that the sequence $(h_n(\lambda_j))_n$ converges to zero at the biggest possible rate. In order to solve the above system of equations, we firstly observe that, by (37), we have

$$(39) \quad \mathbf{u}_{j(1)} = \begin{pmatrix} \mathbf{u}_j \\ g(\lambda_j)\mathbf{u}_j \end{pmatrix}, \quad \mathbf{u}_{j(2)} = \begin{pmatrix} \mathbf{0} \\ \mathbf{u}_j \end{pmatrix},$$

$$(40) \quad Q_n U_j = Q_n U_j^* P_j = U_j^* \begin{pmatrix} \widehat{r}_{n-1}(1-\lambda_j) & 0 \\ -\widehat{q}_{n,n}\lambda_j & \widehat{q}_{n,n} \end{pmatrix} P_j = U_j^* \begin{pmatrix} \widehat{r}_{n-1}(1-\lambda_j) & 0 \\ -\widehat{q}_{n,n}\lambda_j + \widehat{q}_{n,n}g(\lambda_j) & \widehat{q}_{n,n} \end{pmatrix}$$

and

$$(41) \quad U_j D_{Q,j,n} = U_j^* P_j D_{Q,j,n} = U_j^* \begin{pmatrix} \widehat{r}_{n-1}(1-\lambda_j) & 0 \\ \widehat{r}_{n-1}(1-\lambda_j)g(\lambda_j) - \lambda_j h_n(\lambda_j) & \widehat{q}_{n,n} \end{pmatrix}.$$

Then, combining together (40) and (41) in order to satisfy (38), we obtain

$$-\widehat{q}_{n,n}\lambda_j + \widehat{q}_{n,n}g(\lambda_j) = \widehat{r}_{n-1}(1-\lambda_j)g(\lambda_j) - \lambda_j h_n(\lambda_j),$$

from which we get the equality

$$(42) \quad \lambda_j[\widehat{q}_{n,n} - h_n(\lambda_j)] = g(\lambda_j)[\widehat{q}_{n,n} - \widehat{r}_{n-1}(1-\lambda_j)].$$

Now, for all values of γ , ν and $j \in \{1, \dots, N\}$, we want to define $g(\lambda_j)$ and $h_n(\lambda_j)$ in such a way that (42) is verified for any n and $h_n(\lambda_j)$ vanishes to zero with the biggest possible rate. To this end, we note that by (42) we have the following two facts:

- If $\lambda_j = 0$, we can set $g(\lambda_j) = g(0) = 0$ and $h_n(\lambda_j) = h_n(0)$ is not relevant.
- If $\lambda_j \neq 0$, $g(\lambda_j)$ does not depend on n only if $h_n(\lambda_j) = \widehat{r}_{n-1}(1-\lambda_j)$, which implies $g(\lambda_j) = \lambda_j$, or if $h_n(\lambda_j) = \widehat{q}_{n,n}$, which implies $g(\lambda_j) = 0$.

Hence, since \widehat{r}_{n-1} and $\widehat{q}_{n,n}$ have convergence rates n^γ and n^ν , respectively, we choose to set

$$(43) \quad h_n(x) := \begin{cases} \widehat{r}_{n-1}(1-x) & \text{if } \nu < \gamma, \\ \widehat{q}_{n,n} \mathbb{1}_{\{x \neq 1\}} & \text{if } \nu \geq \gamma \end{cases}$$

and

$$(44) \quad g(x) := \begin{cases} x & \text{if } \nu < \gamma, \\ \mathbb{1}_{\{x=1\}} & \text{if } \nu \geq \gamma. \end{cases}$$

Note that, since $\lambda_1 = 1$, we have $g(\lambda_1) = g(1) = 1$ and $h_n(\lambda_1) = h_n(1) = 0$ regardless the values of ν and γ .

Now, recalling that \mathbf{v}_j , for $j = 1, \dots, N$, denote the right eigenvectors of W , we set $V_j := (\mathbf{v}_{j(1)}, \mathbf{v}_{j(2)})$, for each $j = 1, \dots, N$, with the condition

$$V_j = V_j^* P_j^{-\top} \quad \text{where} \quad V_j^* := \begin{pmatrix} \mathbf{v}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_j \end{pmatrix} \quad \text{and} \quad P_j^{-\top} := \begin{pmatrix} 1 & -g(\lambda_j) \\ 0 & 1 \end{pmatrix},$$

so that we have

$$(45) \quad \mathbf{v}_{j(1)} = \begin{pmatrix} \mathbf{v}_j \\ \mathbf{0} \end{pmatrix} \quad \text{and} \quad \mathbf{v}_{j(2)} = \begin{pmatrix} -g(\lambda_j)\mathbf{v}_j \\ \mathbf{v}_j \end{pmatrix}.$$

Note that, we also have

$$(46) \quad V_j^\top Q_n = D_{Q,j,n} V_j^\top.$$

Moreover, by (12), we have

$$(47) \quad \mathbf{u}_{j(i)}^\top \mathbf{v}_{j(i)} = 1, \quad \text{and} \quad \mathbf{u}_{h(l)}^\top \mathbf{v}_{j(i)} = 0, \quad \forall h \neq j \text{ or } l \neq i.$$

Finally, since $\{\mathbf{u}_{j(i)} : j = 1, \dots, N; i = 1, 2\}$ and $\{\mathbf{v}_{j(i)} : j = 1, \dots, N; i = 1, 2\}$ satisfy, for any $j \in \{1, \dots, N\}$, the relation

$$(48) \quad U_j V_j^\top = \mathbf{u}_{j(1)} \mathbf{v}_{j(1)}^\top + \mathbf{u}_{j(2)} \mathbf{v}_{j(2)}^\top = \begin{pmatrix} \mathbf{u}_j \mathbf{v}_j^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{u}_j \mathbf{v}_j^\top \end{pmatrix}$$

and since (14), the stochastic process $\{\mathbf{Y}_n : n \geq 1\}$ can be decomposed as

$$(49) \quad \mathbf{Y}_n = \sum_{j=1}^N \mathbf{Y}_{j,n} \quad \text{with } \mathbf{Y}_{j,n} := U_j V_j^\top \mathbf{Y}_n.$$

The dynamics of each term $\mathbf{Y}_{j,n}$ can be deduced from (35) by multiplying this equation by $U_j V_j^\top = U_j^* V_j^{*\top}$ and using (46) and the relation $V_j^\top \mathbf{Y}_n = V_j^\top \mathbf{Y}_{j,n}$. We thus obtain

$$(50) \quad \mathbf{Y}_{j,n} = U_j (I - D_{Q,j,n}) V_j^\top \mathbf{Y}_{j,n-1} + U_j^* D_{R,n} V_j^{*\top} \Delta \mathbf{M}_{Y,n} + U_j V_j^\top \Delta \mathbf{R}_{Y,n},$$

where

$$(51) \quad D_{R,n} := \begin{pmatrix} \hat{r}_{n-1} & 0 \\ 0 & \hat{q}_{n,n} \end{pmatrix}.$$

For the sequel, it will be useful to decompose \mathbf{Y}_n further as

$$(52) \quad \mathbf{Y}_n = \sum_{j=1}^N \mathbf{Y}_{j,n} = \sum_{j=1}^N \mathbf{Y}_{j(1),n} + \sum_{j=1}^N \mathbf{Y}_{j(2),n},$$

where, for any $j \in \{1, \dots, N\}$,

$$(53) \quad \mathbf{Y}_{j,n} = \mathbf{Y}_{j(1),n} + \mathbf{Y}_{j(2),n} \quad \text{and} \quad \mathbf{Y}_{j(i),n} := \mathbf{u}_{j(i)} \mathbf{v}_{j(i)}^\top \mathbf{Y}_n = \mathbf{u}_{j(i)} \mathbf{v}_{j(i)}^\top \mathbf{Y}_{j,n}, \quad \text{for } i = 1, 2,$$

and set

$$(54) \quad \tilde{\mathbf{Y}}_n := \mathbf{Y}_{1(1),n} = \mathbf{u}_{1(1)} \mathbf{v}_{1(1)}^\top \mathbf{Y}_n = \begin{pmatrix} \mathbf{u}_1 \mathbf{v}_1^\top \mathbf{Z}_n \\ \mathbf{u}_1 \mathbf{v}_1^\top \mathbf{Z}_n \end{pmatrix} = \tilde{Z}_n \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} \quad \text{with } \tilde{Z}_n := N^{-1/2} \mathbf{v}_1^\top \mathbf{Z}_n,$$

and

$$(55) \quad \begin{aligned} \hat{\mathbf{Y}}_n &:= \mathbf{Y}_n - \tilde{\mathbf{Y}}_n = \mathbf{Y}_n - \mathbf{Y}_{1(1),n} = \sum_{j=2}^N \mathbf{Y}_{j(1),n} + \sum_{j=1}^N \mathbf{Y}_{j(2),n} \\ &= \sum_{j=2}^N \mathbf{Y}_{j(1),n} + \mathbf{Y}_{1(2),n} + \sum_{j=2}^N \mathbf{Y}_{j(2),n}. \end{aligned}$$

Remark 5.1. Note that the random vectors $\tilde{\mathbf{Y}}_n$ and $\hat{\mathbf{Y}}_n$ correspond to the random vectors $\tilde{Z}_n(\mathbf{1}, \mathbf{1})^\top$ and $(\hat{\mathbf{Z}}_n, \hat{\mathbf{N}}_n)^\top$, respectively, considered in [1, 2] in the case $a_n = 1$ for each n (so that we have $\nu = 1$ and $q = 1$): indeed, we have

$$(56) \quad \mathbf{1} \tilde{Z}_n = \mathbf{u}_1 \mathbf{v}_1^\top \mathbf{Z}_n, \quad \hat{\mathbf{Z}}_n = (I - \mathbf{u}_1 \mathbf{v}_1^\top) \mathbf{Z}_n = \mathbf{Z}_n - \mathbf{1} \tilde{Z}_n, \quad \hat{\mathbf{N}}_n = \mathbf{N}_n - \mathbf{1} \tilde{Z}_n,$$

where \tilde{Z}_n and $\hat{\mathbf{Z}}_n$ are exactly the same stochastic processes considered in [1, 2], while \mathbf{N}_n (and so $\hat{\mathbf{N}}_n$) differs from the stochastic process considered in [1] because here the random variables $N_{n,j}$ are defined in terms of a generic sequence (a_n) (see (6)) satisfying suitable assumptions.

Finally, it is worthwhile to point out that the decomposition of $\hat{\mathbf{Y}}_n$ in terms of the stochastic processes $\mathbf{Y}_{j(i),n}$ is a new element with respect to the previous works and, as we will see in the sequel, it will be the key tool in order to obtain the exact convergence rate of $\hat{\mathbf{Y}}_n$. Indeed, the convergence rate and the second-order asymptotic distribution of $\hat{\mathbf{Y}}_n$ will be the result of the different asymptotic behaviors of the three quantities in the last term of (55).

5.3. Central limit theorems for $\tilde{\mathbf{Y}}_n$ and $\hat{\mathbf{Y}}_n$. The convergence rate and the second-order asymptotic distribution of $(\mathbf{Y}_n - Z_\infty \mathbf{1})$ will be obtained by studying separately and then combining together the second-order convergence of $\tilde{\mathbf{Y}}_n$ to $Z_\infty \mathbf{1}$ and the second-order convergence of $\hat{\mathbf{Y}}_n$ to $\mathbf{0}$. To this regards, we recall that, by [2, Theorem 4.2], under Assumptions 2.1 and 2.2, we have for $1/2 < \gamma \leq 1$ that

$$(57) \quad n^{\gamma - \frac{1}{2}} (\tilde{\mathbf{Y}}_n - Z_\infty \mathbf{1}) \longrightarrow \mathcal{N} \left(\mathbf{0}, Z_\infty (1 - Z_\infty) \tilde{\Sigma}_\gamma \right) \quad \text{stably in the strong sense,}$$

where $\tilde{\Sigma}_\gamma$ is defined in (21). In this work we fully describe the second-order convergence of $\hat{\mathbf{Y}}_n$, proving the following theorem:

Theorem 5.1. *Under all the assumptions stated in Section 2, the following statements hold true:*

(a) *If $\nu < \gamma$, then*

$$n^{\nu/2} \widehat{\mathbf{Y}}_n \longrightarrow \mathcal{N} \left(\mathbf{0}, Z_\infty(1 - Z_\infty) \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{U} S^{(q)} \widetilde{U}^\top \end{pmatrix} \right) \quad \text{stably,}$$

where \widetilde{U} and $S^{(q)}$ are defined in (22) and (24), respectively.

(b) *If $\nu = \gamma$, then*

$$n^{\gamma/2} \widehat{\mathbf{Y}}_n \longrightarrow \mathcal{N} \left(\mathbf{0}, Z_\infty(1 - Z_\infty) \begin{pmatrix} \widetilde{U} S_\gamma^{11} \widetilde{U}^\top & \widetilde{U} S_\gamma^{12} \widetilde{U}^\top \\ \widetilde{U} S_\gamma^{21} \widetilde{U}^\top & \widetilde{U} S_\gamma^{22} \widetilde{U}^\top \end{pmatrix} \right) \quad \text{stably,}$$

where $S_\gamma^{21} = (S_\gamma^{12})^\top$ and, for $2 \leq j_1, j_2 \leq N$,

$$[S_\gamma^{11}]_{11} = [S^{11}]_{j_1 1} = [S^{11}]_{1 j_2} := 0,$$

$$[S_\gamma^{11}]_{j_1 j_2} := \frac{c^2}{c(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}}} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2},$$

$$[S_\gamma^{12}]_{11} = [S^{12}]_{1 j_2} := 0,$$

$$[S_\gamma^{12}]_{j_1 1} := \frac{c(q - c)}{c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}}} \mathbf{v}_{j_1}^\top \mathbf{v}_1,$$

$$[S_\gamma^{12}]_{j_1 j_2} := \frac{cq(c\alpha_{j_1} + c - \mathbb{1}_{\{\gamma=1\}})}{(c\alpha_{j_1} + c\alpha_{j_2} - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2},$$

$$[S_\gamma^{22}]_{11} := \frac{(q - c)^2}{2q - \mathbb{1}_{\{\gamma=1\}}} \|\mathbf{v}_1\|^2,$$

$$[S_\gamma^{22}]_{j_1} = [S_\gamma^{22}]_{1 j} := \frac{q(q - c)(c + q - \mathbb{1}_{\{\gamma=1\}})}{(c\alpha_j + q - \mathbb{1}_{\{\gamma=1\}})(2q - \mathbb{1}_{\{\gamma=1\}})} \mathbf{v}_j^\top \mathbf{v}_1,$$

$$[S_\gamma^{22}]_{j_1 j_2} := q^2 \frac{c^3(\alpha_{j_1} + \alpha_{j_2}) + 2c^2q(\alpha_{j_1}\alpha_{j_2} + 1) - \mathbb{1}_{\{\gamma=1\}}c^2(\alpha_{j_1}\alpha_{j_2} + \alpha_{j_1} + \alpha_{j_2} + 2)}{(2q - \mathbb{1}_{\{\gamma=1\}})(c(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_2} + q - \mathbb{1}_{\{\gamma=1\}})} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2}$$

$$+ q^2 \frac{c(q - \mathbb{1}_{\{\gamma=1\}})^2(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}}(2c + q - 1)(q - 1)}{(2q - \mathbb{1}_{\{\gamma=1\}})(c(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_2} + q - \mathbb{1}_{\{\gamma=1\}})} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2}.$$

(c) *If $\gamma < \nu$, then*

$$n^{\gamma - \frac{\nu}{2}} \widehat{\mathbf{Y}}_n \longrightarrow \mathcal{N} \left(\mathbf{0}, Z_\infty(1 - Z_\infty) \frac{c^2}{N[2q - \mathbb{1}_{\{\nu=1\}}(2\gamma - 1)]} \|\mathbf{v}_1\|^2 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{11}^\top \end{pmatrix} \right) \quad \text{stably.}$$

Remark 5.2. Note that, when $\nu \neq \gamma$ the convergence rates of the first and the second component of $\widehat{\mathbf{Y}}_n$ are always different: indeed from [2], we know that, under our assumptions, the convergence rate of $\widehat{\mathbf{Z}}_n$ is always $n^{\gamma/2}$, while the above theorem shows that the convergence rate of $\widehat{\mathbf{N}}_n$ changes according to the pair (γ, ν) .

Regarding the proof of Theorem 5.1, we note that, using the definition (55) of $\widehat{\mathbf{Y}}_n$ given in Section 5.2, we can say that this random variable can be decomposed in a sum of suitable random variables that have the form

$$(58) \quad \sum_{j \in J} \sum_{i \in I_j} \mathbf{Y}_{j(i), n},$$

where $J \subseteq \{1, \dots, N\}$, $I_j \subseteq \{1, 2\}$ for any $j \in J$ and $Y_{j(i), n}$ is defined in (53). Hence, in order to characterize the asymptotic behavior of $\widehat{\mathbf{Y}}_n$, we first establish the second-order asymptotic behavior of the above general sum (58) under certain specifications of the sets J and I_j (see Lemma 5.1) and then we combine them together appropriately according to their convergence rates.

Lemma 5.1. *Under all the assumptions stated in Section 2, consider the general sum (58) in the following cases:*

- (i) $\nu < \gamma$, $J = \{2, \dots, N\}$ and $I_j = \{1\}$ for all $j \in J$;
- (ii) $\nu < \gamma$, $J = \{1, \dots, N\}$ and $I_j = \{2\}$ for all $j \in J$;
- (iii) $\nu = \gamma$, $J = \{1, \dots, N\}$, $I_1 = \{2\}$ and $I_j = \{1, 2\}$ for all $j \in J \setminus \{1\}$;
- (iv) $\nu > \gamma$, $J = \{2, \dots, N\}$ and $I_j = \{1\}$ for all $j \in J$;
- (v) $\nu > \gamma$, $J = \{1\}$ and $I_1 = \{2\}$;
- (vi) $\nu > \gamma$, $J = \{2, \dots, N\}$ and $I_j = \{2\}$ for all $j \in J$.

Then, in all the above listed cases, we have

(59)

$$t_n(J(I)) \sum_{j \in J} \sum_{i \in I_j} \mathbf{Y}_{j(i),n} \xrightarrow{\text{stably}} \mathcal{N} \left(\mathbf{0}, Z_\infty(1 - Z_\infty) \sum_{j_1 \in J} \sum_{j_2 \in J} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2} \sum_{i_1 \in I_{j_1}} \sum_{i_2 \in I_{j_2}} d^{j_1(i_1), j_2(i_2)} \mathbf{u}_{j_1(i_1)} \mathbf{u}_{j_2(i_2)}^\top \right),$$

where

$$(60) \quad t_n(J(I)) := \begin{cases} n^{\gamma/2} & \text{for cases (i), (iii) and (iv)} \\ n^{\nu/2} & \text{for cases (ii) and (vi)} \\ n^{\gamma - \frac{\nu}{2}} & \text{for case (v)}. \end{cases}$$

and $d^{j_1(i_1), j_2(i_2)}$ are constants corresponding to the result of suitable limits computed in Section A.4 of the appendix.

Proof of Theorem 5.1. From the above lemma, we immediately get the proof of Theorem 5.1. Indeed, in case (a) we get

$$n^{\nu/2} \widehat{\mathbf{Y}}_n = \frac{1}{n^{(\gamma-\nu)/2}} n^{\gamma/2} \sum_{j=2}^N \mathbf{Y}_{j(1),n} + n^{\nu/2} \sum_{j=1}^N \mathbf{Y}_{j(2),n}$$

where, considering the above cases (i) and (ii), the first term in the sum converges in probability to zero, while the second term converges stably to the desired Gaussian kernel, that is the Gaussian kernel with zero mean and random covariance matrix

$$Z_\infty(1 - Z_\infty) \sum_{j_1=1}^N \sum_{j_2=1}^N \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2} d^{j_1(2), j_2(2)} \mathbf{u}_{j_1(2)} \mathbf{u}_{j_2(2)}^\top,$$

where

$$\mathbf{u}_{j_1(2)} \mathbf{u}_{j_2(2)}^\top = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{u}_{j_1} \mathbf{u}_{j_2}^\top \end{pmatrix}.$$

In case (b) we simply have

$$n^{\gamma/2} \widehat{\mathbf{Y}}_n = n^{\gamma/2} \left(\sum_{j=2}^N \mathbf{Y}_{j(1),n} + \sum_{j=1}^N \mathbf{Y}_{j(2),n} \right),$$

where the right-hand term converges stably to the desired Gaussian kernel (see the above case (iii)), that is the Gaussian kernel with zero mean and random covariance matrix

$$Z_\infty(1 - Z_\infty) \sum_{j_1=1}^N \sum_{j_2=1}^N \sum_{i_1=1}^2 \sum_{i_2=1}^2 (1 - \mathbb{1}_{\{j_1=i_1=1\}} \mathbb{1}_{\{j_2=i_2=1\}}) \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2} d^{j_1(i_1), j_2(i_2)} \mathbf{u}_{j_1(i_1)} \mathbf{u}_{j_2(i_2)}^\top,$$

where

$$\begin{aligned} \mathbf{u}_{j_1(1)} \mathbf{u}_{j_2(1)}^\top &= \begin{pmatrix} \mathbf{u}_{j_1} \mathbf{u}_{j_2}^\top & \mathbb{1}_{\{j_2=1\}} \mathbf{u}_{j_1} \mathbf{u}_{j_2}^\top \\ \mathbb{1}_{\{j_1=1\}} \mathbf{u}_{j_1} \mathbf{u}_{j_2}^\top & \mathbb{1}_{\{j_1=1\}} \mathbb{1}_{\{j_2=1\}} \mathbf{u}_{j_1} \mathbf{u}_{j_2}^\top \end{pmatrix}, \\ \mathbf{u}_{j_1(1)} \mathbf{u}_{j_2(2)}^\top &= \begin{pmatrix} \mathbf{0} & \mathbf{u}_{j_1} \mathbf{u}_{j_2}^\top \\ \mathbf{0} & \mathbb{1}_{\{j_1=1\}} \mathbf{u}_{j_1} \mathbf{u}_{j_2}^\top \end{pmatrix}, \\ \mathbf{u}_{j_1(2)} \mathbf{u}_{j_2(1)}^\top &= \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{u}_{j_1} \mathbf{u}_{j_2}^\top & \mathbb{1}_{\{j_2=1\}} \mathbf{u}_{j_1} \mathbf{u}_{j_2}^\top \end{pmatrix}, \\ \mathbf{u}_{j_1(2)} \mathbf{u}_{j_2(2)}^\top &= \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{u}_{j_1} \mathbf{u}_{j_2}^\top \end{pmatrix}. \end{aligned}$$

Finally, in case (c), we obtain

$$n^{\gamma-\frac{\nu}{2}}\widehat{\mathbf{Y}}_n = \frac{1}{n^{(\nu-\gamma)/2}}n^{\frac{\gamma}{2}}\sum_{j=2}^N\mathbf{Y}_{j(1),n} + n^{\gamma-\frac{\nu}{2}}\mathbf{Y}_{1(2),n} + \frac{1}{n^{(\nu-\gamma)}}n^{\frac{\nu}{2}}\sum_{j=2}^N\mathbf{Y}_{j(2),n},$$

where, considering the above cases (iv), (v) and (vi), we have that the first and the third terms in the sum converge in probability to zero, while the second term converges stably to the desired Gaussian kernel, that is the Gaussian kernel with zero mean and random covariance matrix

$$Z_\infty(1-Z_\infty)\|\mathbf{v}_1\|^2d^{1(2),1(2)}\mathbf{u}_{1(2)}\mathbf{u}_{1(2)}^\top,$$

where

$$(61) \quad \mathbf{u}_{1(2)}\mathbf{u}_{1(2)}^\top = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{u}_1\mathbf{u}_1^\top \end{pmatrix} = \frac{1}{N} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}\mathbf{1}^\top \end{pmatrix}.$$

□

We now go on with the proof of Lemma 5.1.

Proof of Lemma 5.1. Since this proof is quite long, we split it into various steps and the technical computations and details are collected in the appendix.

First step: decomposition of the general sum (58).

First of all, we observe that, for any set $J \subseteq \{1, \dots, N\}$, the dynamics of $\sum_{j \in J} \mathbf{Y}_{j,n}$ can be obtained by summing up equation (50) for $j \in J$:

$$\sum_{j \in J} \mathbf{Y}_{j,n} = \left(\sum_{j \in J} U_j (I - D_{Q,j,n}) V_j^\top \right) \sum_{j \in J} \mathbf{Y}_{j,n-1} + \left(\sum_{j \in J} U_j^* D_{R,n} V_j^{*\top} \right) \Delta \mathbf{M}_{Y,n} + \left(\sum_{j \in J} U_j V_j^\top \right) \Delta \mathbf{R}_{Y,n}.$$

Then, recalling that $\operatorname{Re}(\alpha_j) > 0$ for each $j \geq 2$ because $\operatorname{Re}(\lambda_j) < 1$ for each $j \geq 2$, and taking an integer $m_0 \geq 2$ large enough such that for $n \geq m_0$ we have $\operatorname{Re}(\alpha_j)cn^{-\gamma} < 1$ for each $j \geq 2$ and $qn^{-\nu} < 1$, we can write

$$(62) \quad \begin{aligned} \sum_{j \in J} \mathbf{Y}_{j,n} &= \left(\sum_{j \in J} U_j A_{m_0, n-1}^j V_j^\top \right) \sum_{j \in J} \mathbf{Y}_{j, m_0} \\ &+ \sum_{k=m_0}^{n-1} \left(\sum_{j \in J} U_j A_{k+1, n-1}^j V_j^\top U_j^* D_{R,k} V_j^{*\top} \right) \Delta \mathbf{M}_{Y, k+1} \\ &+ \sum_{k=m_0}^{n-1} \left(\sum_{j \in J} U_j A_{k+1, n-1}^j V_j^\top \right) \Delta \mathbf{R}_{Y, k+1} \quad \text{for } n \geq m_0, \end{aligned}$$

where, for any $j \in J$,

$$(63) \quad A_{k+1, n-1}^j = \begin{cases} \prod_{m=k+1}^{n-1} (I - D_{Q,j,m}) & \text{for } m_0 \leq k \leq n-2 \\ I & \text{for } k = n-1. \end{cases}$$

Setting for any $x = a_x + ib_x \in \mathbb{C}$ with $a_x > 0$ and $1/2 < \delta \leq 1$,

$$p_k^\delta(x) := \prod_{m=m_0}^k \left(1 - \frac{x}{m^\delta} \right) \text{ for } k \geq m_0$$

and

$$F_{k+1, n-1}^\gamma(x) := \frac{p_{n-1}^\gamma(x)}{p_k^\gamma(x)} \text{ for } m_0 \leq k \leq n-1,$$

It is easy to see that, for $j = 1$, we have

$$(64) \quad A_{k+1, n-1}^1 = \begin{pmatrix} 1 & 0 \\ 0 & F_{k+1, n-1}^\nu(q) \end{pmatrix} \text{ for } m_0 \leq k \leq n-1$$

and, for $j \geq 2$, after some calculations reported in Section A.2 of the appendix, we obtain

$$(65) \quad A_{k+1,n-1}^j = \begin{pmatrix} F_{k+1,n-1}^\gamma(c\alpha_j) & 0 \\ \lambda_j G_{k+1,n-1}(c\alpha_j, q) & F_{k+1,n-1}^\nu(q) \end{pmatrix} \quad \text{for } m_0 \leq k \leq n-1,$$

where

$$(66) \quad G_{k+1,n-1}(x, q) := \sum_{l=k+1}^{n-1} F_{l+1,n-1}^\gamma(x) h_l(1-c^{-1}x) F_{k+1,l-1}^\nu(q).$$

Then, since $V_j^\top U_j^* = P_j^{-1}$, equation (62) can be rewritten as

$$(67) \quad \sum_{j \in J} \mathbf{Y}_{j,n} = \left(\sum_{j \in J} U_j A_{m_0,n-1}^j V_j^\top \right) \sum_{j \in J} \mathbf{Y}_{j,m_0} + \sum_{k=m_0}^{n-1} \mathbf{T}_{k+1,n-1}^J + \sum_{k=m_0}^{n-1} \boldsymbol{\rho}_{k+1,n-1}^J \quad \text{for } n \geq m_0,$$

with

$$\begin{aligned} \mathbf{T}_{k+1,n-1}^J &= \left(\sum_{j \in J} U_j A_{k+1,n-1}^j P_j^{-1} D_{R,k} V_j^{*\top} \right) \Delta \mathbf{M}_{Y,k+1}, \\ \boldsymbol{\rho}_{k+1,n-1}^J &= \left(\sum_{j \in J} U_j A_{k+1,n-1}^j V_j^\top \right) \Delta \mathbf{R}_{Y,k+1}. \end{aligned}$$

In order to get a similar decomposition for the general sum (58), we set, for any $j \in J$,

$$(68) \quad U_{j(1)}^0 := (\mathbf{u}_{j(1)} \quad \mathbf{0}) = \begin{pmatrix} \mathbf{u}_j & \mathbf{0} \\ g(\lambda_j) \mathbf{u}_j & \mathbf{0} \end{pmatrix} \quad \text{and} \quad U_{j(2)}^0 := (\mathbf{0} \quad \mathbf{u}_{j(2)}) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{u}_j \end{pmatrix}$$

and taking into account the last relation in (53), we get

$$(69) \quad \sum_{j \in J} \sum_{i \in I_j} \mathbf{Y}_{j(i),n} = C_{m_0,n-1}^{J(I)} \sum_{j \in J} \mathbf{Y}_{j,m_0} + \sum_{k=m_0}^{n-1} \mathbf{T}_{k+1,n-1}^{J(I)} + \sum_{k=m_0}^{n-1} \boldsymbol{\rho}_{k+1,n-1}^{J(I)} \quad \text{for } n \geq m_0,$$

with

$$\begin{aligned} C_{m_0,n-1}^{J(I)} &= \sum_{j \in J} \sum_{i \in I_j} U_{j(i)}^0 A_{m_0,n-1}^j V_j^\top, \\ \mathbf{T}_{k+1,n-1}^{J(I)} &= \left(\sum_{j \in J} \sum_{i \in I_j} U_{j(i)}^0 A_{k+1,n-1}^j P_j^{-1} D_{R,k} V_j^{*\top} \right) \Delta \mathbf{M}_{Y,k+1}, \\ \boldsymbol{\rho}_{k+1,n-1}^{J(I)} &= \left(\sum_{j \in J} \sum_{i \in I_j} U_{j(i)}^0 A_{k+1,n-1}^j V_j^\top \right) \Delta \mathbf{R}_{Y,k+1}. \end{aligned}$$

In the sequel of the proof, we will establish the asymptotic behavior of the general sum (58) by studying separately the three terms $C_{m_0,n-1}^{J(I)} \sum_{j \in J} \mathbf{Y}_{j,m_0}$, $\sum_{k=m_0}^{n-1} \mathbf{T}_{k+1,n-1}^{J(I)}$ and $\sum_{k=m_0}^{n-1} \boldsymbol{\rho}_{k+1,n-1}^{J(I)}$ in the six cases (i)-(vi) specified in the statement of the considered lemma.

Second step: asymptotic behavior of $C_{m_0,n-1}^{J(I)} \sum_{j \in J} \mathbf{Y}_{j,m_0}$.

From (64), (65), (66) and (68), taking into account the fact that in all the considered cases with $1 \in J$, i.e. (ii), (iii) and (v), we have $1 \notin I_1$, we get

$$\left| C_{m_0,n}^{J(I)} \right| = O(C_n^{11}) + O(C_n^{21}) + O(C_n^{22}),$$

where

$$\begin{aligned} C_n^{11} &:= \sum_{j \in J, j \neq 1} \mathbb{1}_{\{1 \in I_j\}} |F_{m_0, n-1}^\gamma(c\alpha_j)|, \\ C_n^{21} &:= \sum_{j \in J, j \neq 1} \mathbb{1}_{\{2 \in I_j\}} |G_{m_0, n-1}(c\alpha_j, q)|, \\ C_n^{22} &:= \sum_{j \in J} \mathbb{1}_{\{2 \in I_j\}} |F_{m_0, n-1}^\nu(q)|. \end{aligned}$$

Using (81) in the appendix and denoting by a^* the real part of $\alpha^* := 1 - \lambda^*$, it is immediate to see that

$$C_n^{11} = \sum_{j \in J, j \neq 1} \mathbb{1}_{\{1 \in I_j\}} \begin{cases} O\left(\exp\left(-ca^* \frac{n^{1-\gamma}}{1-\gamma}\right)\right) & \text{if } 1/2 < \gamma < 1 \\ O(n^{-ca^*}) & \text{if } \gamma = 1 \end{cases}$$

and

$$C_n^{22} = \sum_{j \in J} \mathbb{1}_{\{2 \in I_j\}} \begin{cases} O\left(\exp\left(-q \frac{n^{1-\nu}}{1-\nu}\right)\right) & \text{if } 1/2 < \nu < 1 \\ O(n^{-q}) & \text{if } \nu = 1. \end{cases}$$

For the term C_n^{21} , we apply Lemma A.4 so that we get:

Case $\nu < \gamma$: We have $G_{m_0, n-1}(c\alpha_j, q) = O(n^{-(\gamma-\nu)} |F_{m_0, n-1}^\nu(q)| + |F_{m_0, n-1}^\gamma(c\alpha_j)|)$ by means of Lemma A.4 and so

$$C_n^{21} = \sum_{j \in J, j \neq 1} \mathbb{1}_{\{2 \in I_j\}} O\left(n^{-(\gamma-\nu)} |F_{m_0, n-1}^\nu(q)| + |F_{m_0, n-1}^\gamma(c\alpha_j)|\right),$$

where, as above, by (81), we have $|F_{m_0, n-1}^\nu(q)| = O\left(\exp\left(-q \frac{n^{1-\nu}}{1-\nu}\right)\right)$ and

$$|F_{m_0, n-1}^\gamma(c\alpha_j)| = \begin{cases} O\left(\exp\left(-ca^* \frac{n^{1-\gamma}}{1-\gamma}\right)\right) & \text{if } 1/2 < \gamma < 1 \\ O(n^{-ca^*}) & \text{if } \gamma = 1. \end{cases}$$

Case $\nu > \gamma$: We have $G_{m_0, n-1}(c\alpha_j, q) = O(n^{-(\nu-\gamma)} |F_{m_0, n-1}^\nu(q)| + |F_{m_0, n-1}^\gamma(c\alpha_j)|)$ by means of Lemma A.4 and so

$$C_n^{21} = \sum_{j \in J, j \neq 1} \mathbb{1}_{\{2 \in I_j\}} O\left(n^{-(\nu-\gamma)} |F_{m_0, n-1}^\nu(q)| + |F_{m_0, n-1}^\gamma(c\alpha_j)|\right),$$

where, as above, by (81), we have $|F_{m_0, n-1}^\gamma(c\alpha_j)| = O\left(\exp\left(-ca^* \frac{n^{1-\gamma}}{1-\gamma}\right)\right)$ and

$$|F_{m_0, n-1}^\nu(q)| = \begin{cases} O\left(\exp\left(-q \frac{n^{1-\nu}}{1-\nu}\right)\right) & \text{if } 1/2 < \nu < 1 \\ O(n^{-q}) & \text{if } \nu = 1. \end{cases}$$

Case $\nu = \gamma$: By assumption (16)¹ and Lemma A.4, we have $G_{m_0, n-1}(c\alpha_j, q) = O(|F_{m_0, n-1}^\gamma(q)| + |F_{m_0, n-1}^\gamma(c\alpha_j)|)$ and so

$$C_n^{21} = \sum_{j \in J, j \neq 1} \mathbb{1}_{\{2 \in I_j\}} O\left(|F_{m_0, n-1}^\gamma(q)| + |F_{m_0, n-1}^\gamma(c\alpha_j)|\right),$$

where, as above, by (81), we have for $x = q$ or $x \in \{c\alpha_j : j \in J, j \neq 1\}$

$$|F_{m_0, n-1}^\gamma(x)| = \begin{cases} O\left(\exp\left(-a_x \frac{n^{1-\gamma}}{1-\gamma}\right)\right) & \text{if } 1/2 < \nu = \gamma < 1 \\ O(n^{-a_x}) & \text{if } \nu = \gamma = 1 \end{cases}$$

¹If there exists j such that $q = c\alpha_j$, we have to consider the other asymptotic expression given in Lemma A.4.

and so, setting $x^* := \min\{q, ca^*\}$, we can write

$$C_n^{21} = \sum_{j \in J, j \neq 1} \mathbb{1}_{\{2 \in I_j\}} \begin{cases} O\left(\exp\left(-x^* \frac{n^{1-\gamma}}{1-\gamma}\right)\right) & \text{if } 1/2 < \nu = \gamma < 1 \\ O(n^{-x^*}) & \text{if } \nu = \gamma = 1. \end{cases}$$

Summing up, taking into account the conditions $ca^* > 1/2$ when $\gamma = 1$ and $q > 1/2$ when $\nu = 1$, we can conclude that in all the six cases (i)-(vi) we have $t_n(J(I)) \left| C_{m_0, n-1}^{J(I)} \right| \rightarrow 0$ and so

$$t_n(J(I)) C_{m_0, n-1}^{J(I)} \sum_{j \in J} \mathbf{Y}_{j, m_0} \xrightarrow{a.s.} \mathbf{0}.$$

Third step: asymptotic behavior of $\sum_{k=m_0}^{n-1} \rho_{k+1, n-1}^{J(I)}$.

We recall that, by Assumption 2.2, we have $|\Delta \mathbf{R}_{Z, k+1}| = O(k^{-2\gamma})$ and $|\Delta \mathbf{R}_{N, k+1}| = O(k^{-2\nu})$. Then, from (64), (65), (66) and (68), taking into account the fact that in all the considered cases with $1 \in J$, i.e. (ii), (iii) and (v), we have $1 \notin I_1$, we get

$$\left| \sum_{k=m_0}^{n-1} \rho_{k+1, n-1}^{J(I)} \right| = O(\rho_n^{11}) + O(\rho_n^{21}) + O(\rho_n^{22}),$$

where

$$\begin{aligned} \rho_n^{11} &:= \sum_{j \in J, j \neq 1} \mathbb{1}_{\{1 \in I_j\}} \sum_{k=m_0}^{n-1} k^{-2\gamma} |F_{k+1, n-1}^\gamma(c\alpha_j)|, \\ \rho_n^{21} &:= \sum_{j \in J, j \neq 1} \mathbb{1}_{\{2 \in I_j\}} \sum_{k=m_0}^{n-1} k^{-2\gamma} |G_{k+1, n-1}(c\alpha_j, q)|, \\ \rho_n^{22} &:= \sum_{j \in J} \mathbb{1}_{\{2 \in I_j\}} \sum_{k=m_0}^{n-1} (k^{-2\gamma} + k^{-2\nu}) |F_{k+1, n-1}^\nu(q)|. \end{aligned}$$

Using Lemma A.2 (with $\beta = 2\gamma > 1$, $e = 1$ and $\delta = \gamma$), we get

$$\rho_n^{11} = \sum_{j \in J, j \neq 1} \mathbb{1}_{\{1 \in I_j\}} \begin{cases} O(n^{-\gamma}) & \text{if } 1/2 < \gamma < 1, \\ O(n^{-ca^*}) & \text{if } \gamma = 1 \text{ and } 1/2 < ca^* < 1, \\ O(n^{-1} \ln(n)) & \text{if } \gamma = 1 \text{ and } ca^* = 1, \\ O(n^{-1}) & \text{if } \gamma = 1 \text{ and } ca^* > 1. \end{cases}$$

For ρ_n^{22} , we observe that we have $k^{-2\gamma} = O(k^{-2\nu})$ when $\nu \leq \gamma$ and $k^{-2\nu} = O(k^{-2\gamma})$ when $\nu > \gamma$. Therefore, using Lemma A.2 (with $e = 1$ and $\delta = \nu$ and $\beta = 2\nu > 1$ if $\nu \leq \gamma$ and $\beta = 2\gamma > 1$ if $\nu > \gamma$), we obtain for the case $\nu \leq \gamma$

$$\begin{aligned} \rho_n^{22} &= \sum_{j \in J} \mathbb{1}_{\{2 \in I_j\}} O\left(\sum_{k=m_0}^{n-1} k^{-2\nu} |F_{k+1, n-1}^\nu(q)|\right) \\ &= \sum_{j \in J} \mathbb{1}_{\{2 \in I_j\}} \begin{cases} O(n^{-\nu}) & \text{if } 1/2 < \nu < 1, \\ O(n^{-q}) & \text{if } \nu = 1 \text{ and } 1/2 < q < 1, \\ O(n^{-1} \ln(n)) & \text{if } \nu = 1 \text{ and } q = 1, \\ O(n^{-1}) & \text{if } \nu = 1 \text{ and } q > 1, \end{cases} \end{aligned}$$

and for the case $\nu > \gamma$

$$(70) \quad \begin{aligned} \rho_n^{22} &= \sum_{j \in J} \mathbb{1}_{\{2 \in I_j\}} O \left(\sum_{k=m_0}^{n-1} k^{-2\gamma} |F_{k+1, n-1}^\nu(q)| \right) \\ &= \sum_{j \in J} \mathbb{1}_{\{2 \in I_j\}} \begin{cases} O(n^{-2\gamma+\nu}) & \text{if } 1/2 < \nu < 1, \\ O(n^{-q}) & \text{if } \nu = 1 \text{ and } 1/2 < q < 2\gamma - 1, \\ O(n^{-q} \ln(n)) & \text{if } \nu = 1 \text{ and } q = 2\gamma - 1 > 1/2, \\ O(n^{-2\gamma+1}) & \text{if } \nu = 1 \text{ and } q > \max\{1/2, 2\gamma - 1\}. \end{cases} \end{aligned}$$

For the term ρ_n^{21} , we apply Lemma A.2 and Lemma A.4 so that we get:

Case $\nu < \gamma$: We have $G_{k+1, n-1}(c\alpha_j, q) = O(n^{-(\gamma-\nu)} |F_{k+1, n-1}^\nu(q)| + k^{-(\gamma-\nu)} |F_{k+1, n-1}^\gamma(c\alpha_j)|)$ by means of Lemma A.4, and so we get

$$\rho_n^{21} = \sum_{j \in J, j \neq 1} \mathbb{1}_{\{2 \in I_j\}} O \left(n^{-(\gamma-\nu)} \sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma}} |F_{k+1, n-1}^\nu(q)| + \sum_{k=m_0}^{n-1} \frac{1}{k^{3\gamma-\nu}} |F_{k+1, n-1}^\gamma(c\alpha_j)| \right),$$

where, by Lemma A.2, the first term is $O(n^{-3\gamma+2\nu})$, while for the second term we have

$$\sum_{k=m_0}^{n-1} \frac{1}{k^{3\gamma-\nu}} |F_{k+1, n-1}^\gamma(c\alpha_j)| = \begin{cases} O(n^{-2\gamma+\nu}) & \text{if } 1/2 < \gamma < 1, \\ O(n^{-ca^*}) & \text{if } \gamma = 1 \text{ and } 1/2 < ca^* < 2 - \nu, \\ O(n^{-2+\nu} \ln(n)) & \text{if } \gamma = 1 \text{ and } ca^* = 2 - \nu, \\ O(n^{-2+\nu}) & \text{if } \gamma = 1 \text{ and } ca^* > 2 - \nu. \end{cases}$$

Case $\nu > \gamma$: We have $G_{k+1, n-1}(c\alpha_j, q) = O(n^{-(\nu-\gamma)} |F_{k+1, n-1}^\nu(q)| + k^{-(\nu-\gamma)} |F_{k+1, n-1}^\gamma(c\alpha_j)|)$ by means of Lemma A.4, and so we get

$$\rho_n^{21} = \sum_{j \in J, j \neq 1} \mathbb{1}_{\{2 \in I_j\}} O \left(n^{-(\nu-\gamma)} \sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma}} |F_{k+1, n-1}^\nu(q)| + \sum_{k=m_0}^{n-1} \frac{1}{k^{\gamma+\nu}} |F_{k+1, n-1}^\gamma(c\alpha_j)| \right),$$

where, by Lemma A.2, the second term is $O(n^{-\nu})$, while the sum in the first term has the asymptotic behavior given in (70).

Case $\nu = \gamma$: By assumption (16)² and Lemma A.4, we have $G_{k+1, n-1}(c\alpha_j, q) = O(|F_{k+1, n-1}^\gamma(q)| + |F_{k+1, n-1}^\gamma(c\alpha_j)|)$, and so we get

$$\rho_n^{21} = \sum_{j \in J, j \neq 1} \mathbb{1}_{\{2 \in I_j\}} O \left(\sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma}} |F_{k+1, n-1}^\gamma(q)| + \sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma}} |F_{k+1, n-1}^\gamma(c\alpha_j)| \right),$$

where, by Lemma A.2, we have for $x = q$ or $x \in \{c\alpha_j : j \in J, j \neq 1\}$

$$\sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma}} |F_{k+1, n-1}^\gamma(x)| = \begin{cases} O(n^{-\gamma}) & \text{if } 1/2 < \nu = \gamma < 1, \\ O(n^{-a_x}) & \text{if } \nu = \gamma = 1 \text{ and } 1/2 < a_x < 1, \\ O(n^{-1} \ln(n)) & \text{if } \nu = \gamma = 1 \text{ and } a_x = 1, \\ O(n^{-1}) & \text{if } \nu = \gamma = 1 \text{ and } a_x > 1 \end{cases}$$

and so, setting $x^* := \min\{q, ca^*\}$, we can write

$$\rho_n^{21} = \sum_{j \in J, j \neq 1} \mathbb{1}_{\{2 \in I_j\}} \begin{cases} O(n^{-\gamma}) & \text{if } 1/2 < \nu = \gamma < 1, \\ O(n^{-x^*}) & \text{if } \nu = \gamma = 1 \text{ and } 1/2 < x^* < 1, \\ O(n^{-1} \ln(n)) & \text{if } \nu = \gamma = 1 \text{ and } x^* = 1, \\ O(n^{-1}) & \text{if } \nu = \gamma = 1 \text{ and } x^* > 1. \end{cases}$$

²If there exists j such that $q = c\alpha_j$, we have to consider the other asymptotic expression given in Lemma A.4.

Summing up, taking into account the conditions $ca^* > 1/2$ when $\gamma = 1$ and $q > 1/2$ when $\nu = 1$, from the asymptotic behavior given above we easily obtain that in all the cases (i)-(v) we have

$$(71) \quad t_n(J(I)) \left| \sum_{k=m_0}^{n-1} \rho_{n,k}^{J(I)} \right| \xrightarrow{\text{a.s.}} \mathbf{0}.$$

In the case (vi), the evaluation of the asymptotic behavior given in (70) for the term ρ_n^{22} is not enough in order to conclude that $t_n(J(I))\rho_n^{22} \rightarrow 0$ a.s. Therefore, we need a better evaluation, that we can get applying Lemma A.2 in a different way. Indeed, in the case (vi), taking $u > 1$ and applying Lemma A.2 with $e = u$, $\delta = \nu$ and $\beta = 2\gamma u > 1$, we find

$$\begin{aligned} (t_n(J(I))\rho_n^{22})^u &= n^{u\nu/2} O \left(\sum_{k=m_0}^{n-1} k^{-2\gamma u} |F_{k+1,n-1}^\nu(q)|^u \right) \\ &= n^{u\nu/2} \begin{cases} O(n^{-2\gamma u + \nu}) & \text{if } 1/2 < \nu < 1, \\ O(n^{-qu}) & \text{if } \nu = 1 \text{ and } 1/2 < q < 2\gamma - u^{-1}, \\ O(n^{-qu} \ln(n)) & \text{if } \nu = 1 \text{ and } q = 2\gamma - u^{-1} > 1/2, \\ O(n^{-2\gamma u + 1}) & \text{if } \nu = 1 \text{ and } q > \max\{1/2, 2\gamma - u^{-1}\}. \end{cases} \end{aligned}$$

Hence, from the above relations we get that it is possible to find $u > 1$ large enough such that $(t_n(J(I))\rho_n^{22})^u \rightarrow 0$ a.s, that trivially implies $t_n(J(I))\rho_n^{22} \rightarrow 0$ a.s. Therefore also in the case (vi), we can conclude that (71) holds true.

Fourth step: asymptotic behavior of $\sum_{k=m_0}^{n-1} \mathbf{T}_{k+1,n-1}^{J(I)}$.

We aim at proving that, for each of the cases (i)–(vi), the quantity $t_n(J(I)) \sum_{k=m_0}^{n-1} \mathbf{T}_{k+1,n-1}^{J(I)}$ converges stably to the desired Gaussian kernel. For this purpose, we apply Theorem B.1. More precisely, we set $\mathcal{G}_{k,n} = \mathcal{F}_{k+1}$ and, given the fact that condition (c1) required in this theorem is obviously satisfied, we check only conditions (c2) and (c3).

For condition (c2), we have to study the convergence of

$$t_n(J(I))^2 \sum_{k=m_0}^{n-1} \mathbf{T}_{k+1,n-1}^{J(I)} (\mathbf{T}_{k+1,n-1}^{J(I)})^\top.$$

To this end, we note that

$$\begin{aligned} \sum_{k=m_0}^{n-1} \mathbf{T}_{k+1,n-1}^{J(I)} (\mathbf{T}_{k+1,n-1}^{J(I)})^\top &= \sum_{j_1 \in J} \sum_{j_2 \in J} U_{j_1(i_1)}^0 \left(\sum_{k=m_0}^{n-1} \mathbf{T}_{k+1,n-1}^{j_1} (\mathbf{T}_{k+1,n-1}^{j_2})^\top \right) U_{j_2(i_2)}^{0,\top} \\ &= \sum_{j_1 \in J} \sum_{j_2 \in J} \left[\sum_{k=m_0}^{n-1} \mathbf{T}_{k+1,n-1}^{j_1} (\mathbf{T}_{k+1,n-1}^{j_2})^\top \right]_{(i_1, i_2)} \mathbf{u}_{j_1(i_1)} \mathbf{u}_{j_2(i_2)}^\top, \end{aligned}$$

where

$$\mathbf{T}_{k+1,n-1}^j := A_{k+1,n-1}^j P_j^{-1} D_{R,k} V_j^{*\top} \Delta \mathbf{M}_{Y,k+1}.$$

Thus, we can focus on the convergence of $t_n(J(I))^2 \sum_{k=m_0}^{n-1} \mathbf{T}_{k+1,n-1}^{j_1} (\mathbf{T}_{k+1,n-1}^{j_2})^\top$. Regarding to this, we observe that

$$\mathbf{T}_{k+1,n-1}^{j_1} (\mathbf{T}_{k+1,n-1}^{j_2})^\top = A_{k+1,n-1}^{j_1} H_{k+1}^{j_1, j_2} (A_{k+1,n-1}^{j_2})^\top,$$

where

$$\begin{aligned}
H_{k+1}^{j_1, j_2} &:= P_{j_1}^{-1} D_{R,k} V_{j_1}^{*\top} \Delta \mathbf{M}_{Y, k+1} \Delta \mathbf{M}_{Y, k+1}^\top V_{j_2}^* D_{R,k} P_{j_2}^{-\top} \\
&= P_{j_1}^{-1} D_{R,k} V_{j_1}^{*\top} \begin{pmatrix} I \\ I \end{pmatrix} \Delta \mathbf{M}_{k+1} \Delta \mathbf{M}_{k+1}^\top \begin{pmatrix} I & I \end{pmatrix} V_{j_2}^* D_{R,k} P_{j_2}^{-\top} \\
&= P_{j_1}^{-1} D_{R,k} \mathbf{1} \mathbf{v}_{j_1}^\top \Delta \mathbf{M}_{k+1} \Delta \mathbf{M}_{k+1}^\top \mathbf{v}_{j_2} \mathbf{1}^\top D_{R,k} P_{j_2}^{-\top}, \\
&= \mathbf{h}_k^{j_1} \mathbf{v}_{j_1}^\top \Delta \mathbf{M}_{k+1} \Delta \mathbf{M}_{k+1}^\top \mathbf{v}_{j_2} (\mathbf{h}_k^{j_2})^\top \\
&= \beta_{k+1}^{j_1, j_2} \mathbf{h}_k^{j_1} (\mathbf{h}_k^{j_2})^\top,
\end{aligned}$$

with

$$\beta_{k+1}^{j_1, j_2} := \mathbf{v}_{j_1}^\top \Delta \mathbf{M}_{k+1} \Delta \mathbf{M}_{k+1}^\top \mathbf{v}_{j_2} \quad \text{and}$$

$$\mathbf{h}_k^j := P_j^{-1} D_{R,k} \mathbf{1} = \begin{pmatrix} \hat{r}_{k-1} \\ \hat{q}_{k,k} - \hat{r}_{k-1} g(\lambda_j) \end{pmatrix}.$$

Now, we set $\mathbf{d}_{k,n}^j := A_{k+1, n-1}^j \mathbf{h}_k^j$, so that we can write

$$(72) \quad \sum_{k=m_0}^{n-1} \mathbf{T}_{k+1, n-1}^{j_1} (\mathbf{T}_{k+1, n-1}^{j_2})^\top = \sum_{k=m_0}^{n-1} \beta_{k+1}^{j_1, j_2} \mathbf{d}_{k,n}^{j_1} (\mathbf{d}_{k,n}^{j_2})^\top.$$

Hence, in order to obtain the almost sure convergence of $t_n(J(I))^2 \sum_{k=m_0}^{n-1} \mathbf{T}_{k+1, n-1}^{j_1} (\mathbf{T}_{k+1, n-1}^{j_2})^\top$, by means of the usual martingale arguments (see [1, Lemma B.1]) and the technical results collected in Section A of the appendix, it is enough to prove the convergence of $t_n(J(I))^2 \sum_{k=m_0}^{n-1} \mathbf{d}_{k,n}^{j_1} (\mathbf{d}_{k,n}^{j_2})^\top$. Indeed, since $\{X_{n,j} : j = 1, \dots, N\}$ are conditionally independent given \mathcal{F}_n , we have

$$E[\Delta M_{n,h} \Delta M_{n,j} | \mathcal{F}_{n-1}] = 0 \quad \text{for } h \neq j;$$

while, for each j , using the normalization $W^\top \mathbf{1} = \mathbf{1}$, we have

$$E[(\Delta M_{n,j})^2 | \mathcal{F}_{n-1}] = \left(\sum_{h=1}^N w_{h,j} Z_{n-1,h} \right) \left(1 - \sum_{h=1}^N w_{h,j} Z_{n-1,h} \right) \xrightarrow{a.s.} Z_\infty (1 - Z_\infty).$$

Therefore, we get

$$E[(\Delta \mathbf{M}_n)(\Delta \mathbf{M}_n)^\top | \mathcal{F}_{n-1}] \xrightarrow{a.s.} Z_\infty (1 - Z_\infty) I$$

and so

$$E[\beta_{n+1}^{j_1, j_2} | \mathcal{F}_n] = \mathbf{v}_{j_1}^\top E[\Delta \mathbf{M}_{n+1} (\Delta \mathbf{M}_{n+1})^\top | \mathcal{F}_n] \mathbf{v}_{j_2} \xrightarrow{a.s.} Z_\infty (1 - Z_\infty) \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2},$$

from which we finally obtain

$$a.s. - \lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} \mathbf{T}_{k+1, n-1}^{j_1} (\mathbf{T}_{k+1, n-1}^{j_2})^\top = Z_\infty (1 - Z_\infty) \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2} \lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} \mathbf{d}_{k,n}^{j_1} (\mathbf{d}_{k,n}^{j_2})^\top.$$

In order to compute the limits in the last term of the above relation, we observe that, by means of (64) and (65), we have the following analytic expression of $\mathbf{d}_{k,n}^j$:

$$(73) \quad \mathbf{d}_{k,n}^1 = A_{k+1, n-1}^1 \mathbf{h}_{1,k} = \begin{pmatrix} \hat{r}_{k-1} \\ (\hat{q}_{k,k} - \hat{r}_{k-1}) F_{k+1, n-1}^\nu(q) \end{pmatrix}$$

and, for $j \geq 2$,

$$(74) \quad \mathbf{d}_{k,n}^j = A_{k+1, n-1}^j \mathbf{h}_{j,k} = \begin{pmatrix} \hat{r}_{k-1} F_{k+1, n-1}^\gamma(c\alpha_j) \\ \lambda_j \hat{r}_{k-1} G_{k+1, n-1}(c\alpha_j, q) + (\hat{q}_{k,k} - \hat{r}_{k-1} g(\lambda_j)) F_{k+1, n-1}^\nu(q) \end{pmatrix}.$$

Using these equalities, in Section A.4 of the appendix, for all the considered cases (i) – (vi), we find the limit of each component of $t_n(J(I))^2 \sum_{k=m_0}^{n-1} \mathbf{d}_{k,n}^{j_1} (\mathbf{d}_{k,n}^{j_2})^\top$, that is we compute

$$d^{j_1(i_1), j_2(i_2)} := \lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)},$$

where $d_{k,n}^{j(1)}$ and $d_{k,n}^{j(2)}$ are, respectively, the first and the second component of $\mathbf{d}_{k,n}^j$ given in (73) and (74). Summing up, we have

$$\sum_{k=m_0}^{n-1} \mathbf{T}_{k+1,n-1}^{J(I)} (\mathbf{T}_{k+1,n-1}^{J(I)})^\top \xrightarrow{a.s.} Z_\infty (1 - Z_\infty) \sum_{j_1 \in J, j_2 \in J} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2} \sum_{i_1 \in I_{j_1}, i_2 \in I_{j_2}} d_{k,n}^{j_1(i_1), j_2(i_2)} \mathbf{u}_{j_1(i_1)} \mathbf{u}_{j_2(i_2)}^\top.$$

For the check of condition (c3) of Theorem B.1, we observe that, by (64), (65), (66) and (68), taking into account the fact that in all the considered cases with $1 \in J$, i.e. (ii), (iii) and (v), we have $1 \notin I_1$, we can write

$$|\mathbf{T}_{k+1,n-1}^{J(I)}| = O(\Gamma_{k+1,n-1}^{11}) + O(\Gamma_{k+1,n-1}^{21}) + O(\Gamma_{k+1,n-1}^{22}),$$

where $\Gamma_{k+1,n-1}^{11}$, $\Gamma_{k+1,n-1}^{21}$ and $\Gamma_{k+1,n-1}^{22}$ are the following deterministic quantities:

$$\begin{aligned} \Gamma_{k+1,n-1}^{11} &:= \sum_{j \in J, j \neq 1} \mathbb{1}_{\{1 \in I_j\}} \widehat{r}_{k-1} |F_{k+1,n-1}^\gamma(c\alpha_j)|, \\ \Gamma_{k+1,n-1}^{21} &:= \sum_{j \in J, j \neq 1} \mathbb{1}_{\{2 \in I_j\}} \widehat{r}_{k-1} |G_{k+1,n-1}(c\alpha_j, q)| \\ \Gamma_{k+1,n-1}^{22} &:= \sum_{j \in J} \mathbb{1}_{\{2 \in I_j\}} (\widehat{r}_{k-1} + \widehat{q}_{k,k}) |F_{k+1,n-1}^\nu(q)|. \end{aligned}$$

Therefore, we find for any $u > 1$

$$\begin{aligned} \left(\sup_{m_0 \leq k \leq n-1} |t_n(J(I)) \mathbf{T}_{k+1,n-1}^{J(I)}| \right)^{2u} &\leq t_n(J(I))^{2u} \sum_{k=m_0}^{n-1} |\mathbf{T}_{k+1,n-1}^{J(I)}|^{2u} = \\ &t_n(J(I))^{2u} \left\{ \sum_{k=m_0}^{n-1} O((\Gamma_{k+1,n-1}^{11})^{2u}) + \sum_{k=m_0}^{n-1} O((\Gamma_{k+1,n-1}^{21})^{2u}) + \sum_{k=m_0}^{n-1} O((\Gamma_{k+1,n-1}^{22})^{2u}) \right\}. \end{aligned}$$

We now analyze the last three terms. For the first one, by Lemma A.2 with $\beta = 2\gamma u$, $e = 2u$ and $\delta = \gamma$, we have

$$\begin{aligned} \sum_{k=m_0}^{n-1} O((\Gamma_{k+1,n-1}^{11})^{2u}) &= \sum_{j \in J, j \neq 1} I_{\{1 \in I_j\}} O \left(\sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma u}} |F_{k+1,n-1}^\gamma(c\alpha_j)|^{2u} \right) \\ &= \sum_{j \in J, j \neq 1} I_{\{1 \in I_j\}} \begin{cases} O(n^{-\gamma(2u-1)}) & \text{if } 1/2 < \gamma < 1, \\ O(n^{-2ca^*u}) & \text{if } \gamma = 1 \text{ and } 1/2 < ca^* < 1 - (2u)^{-1}, \\ O(n^{-2u+1} \ln(n)) & \text{if } \gamma = 1 \text{ and } ca^* = 1 - (2u)^{-1}, \\ O(n^{-2u+1}) & \text{if } \gamma = 1 \text{ and } ca^* > 1 - (2u)^{-1}. \end{cases} \end{aligned}$$

For the third term, we observe that $\widehat{r}_{k-1} = O(\widehat{q}_{k,k})$ when $\nu \leq \gamma$ and $\widehat{q}_{k,k} = O(\widehat{r}_{k-1})$ when $\nu > \gamma$. Hence, by Lemma A.2 with $e = 2u$, $\delta = \nu$ and $\beta = 2\nu u$ if $\nu \leq \gamma$ and $\beta = 2\gamma u$ if $\nu > \gamma$, we get for the case $\nu \leq \gamma$

$$\begin{aligned} \sum_{k=m_0}^{n-1} O((\Gamma_{k+1,n-1}^{22})^{2u}) &= \sum_{j \in J} I_{\{2 \in I_j\}} O \left(\sum_{k=m_0}^{n-1} \frac{1}{k^{2\nu u}} |F_{k+1,n-1}^\nu(q)|^{2u} \right) \\ &= \sum_{j \in J} I_{\{2 \in I_j\}} \begin{cases} O(n^{-\nu(2u-1)}) & \text{if } 1/2 < \nu < 1, \\ O(n^{-2qu}) & \text{if } \nu = 1 \text{ and } 1/2 < q < 1 - (2u)^{-1}, \\ O(n^{-2u+1} \ln(n)) & \text{if } \nu = 1 \text{ and } q = 1 - (2u)^{-1}, \\ O(n^{-2u+1}) & \text{if } \nu = 1 \text{ and } q > 1 - (2u)^{-1}, \end{cases} \end{aligned}$$

and for the case $\nu > \gamma$

(75)

$$\begin{aligned} \sum_{k=m_0}^{n-1} O((\Gamma_{k+1,n-1}^{22})^{2u}) &= \sum_{j \in J} I_{\{2 \in I_j\}} O \left(\sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma u}} |F_{k+1,n-1}^\nu(q)|^{2u} \right) \\ &= \sum_{j \in J} I_{\{2 \in I_j\}} \begin{cases} O(n^{-2\gamma u + \nu}) & \text{if } 1/2 < \nu < 1, \\ O(n^{-2qu}) & \text{if } \nu = 1 \text{ and } 1/2 < q < \gamma - (2u)^{-1}, \\ O(n^{-2\gamma u + 1} \ln(n)) & \text{if } \nu = 1 \text{ and } q = \gamma - (2u)^{-1} > 1/2, \\ O(n^{-2\gamma u + 1}) & \text{if } \nu = 1 \text{ and } q > \max\{1/2, \gamma - (2u)^{-1}\}. \end{cases} \end{aligned}$$

For the second term, we apply Lemma A.2 together with Lemma A.4 so that we get:

Case $\nu < \gamma$: We have $G_{k+1,n-1}(c\alpha_j, q) = O(n^{-(\gamma-\nu)} |F_{k+1,n-1}^\nu(q)| + k^{-(\gamma-\nu)} |F_{k+1,n-1}^\gamma(c\alpha_j)|)$ by means of Lemma A.4, and so we find

$$\begin{aligned} \sum_{k=m_0}^{n-1} O((\Gamma_{k+1,n-1}^{21})^{2u}) &= \sum_{j \in J, j \neq 1} I_{\{2 \in I_j\}} O \left(\sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma u}} |G_{k+1,n-1}(c\alpha_j, q)|^{2u} \right) = \\ &= \sum_{j \in J, j \neq 1} I_{\{2 \in I_j\}} O \left(n^{-2(\gamma-\nu)u} \sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma u}} |F_{k+1,n-1}^\nu(q)|^{2u} + \sum_{k=m_0}^{n-1} \frac{1}{k^{4\gamma u - 2\nu u}} |F_{k+1,n-1}^\gamma(c\alpha_j)|^{2u} \right), \end{aligned}$$

where, by Lemma A.2, the first term is $O(n^{-4\gamma u + 2\nu u + \nu})$, while for the second term we have

$$\sum_{k=m_0}^{n-1} \frac{1}{k^{4\gamma u - 2\nu u}} |F_{k+1,n-1}^\gamma(c\alpha_j)|^{2u} = \begin{cases} O(n^{-4\gamma u + 2\nu u + \gamma}) & \text{if } 1/2 < \gamma < 1, \\ O(n^{-2ca^*u}) & \text{if } \gamma = 1 \text{ and } 1/2 < ca^* < 2 - \nu - (2u)^{-1}, \\ O(n^{-2ca^*u} \ln(n)) & \text{if } \gamma = 1 \text{ and } ca^* = 2 - \nu - (2u)^{-1}, \\ O(n^{-4u + 2\nu u + 1}) & \text{if } \gamma = 1 \text{ and } ca^* > 2 - \nu - (2u)^{-1}. \end{cases}$$

Case $\nu > \gamma$: We have $G_{k+1,n-1}(c\alpha_j, q) = O(n^{-(\nu-\gamma)} |F_{k+1,n-1}^\nu(q)| + k^{-(\nu-\gamma)} |F_{k+1,n-1}^\gamma(c\alpha_j)|)$ by means of Lemma A.4, and so we find

$$\begin{aligned} \sum_{k=m_0}^{n-1} O((\Gamma_{k+1,n-1}^{21})^{2u}) &= \sum_{j \in J, j \neq 1} I_{\{2 \in I_j\}} O \left(\sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma u}} |G_{k+1,n-1}(c\alpha_j, q)|^{2u} \right) = \\ &= \sum_{j \in J, j \neq 1} I_{\{2 \in I_j\}} O \left(n^{-2(\nu-\gamma)u} \sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma u}} |F_{k+1,n-1}^\nu(q)|^{2u} + \sum_{k=m_0}^{n-1} \frac{1}{k^{2\nu u}} |F_{k+1,n-1}^\gamma(c\alpha_j)|^{2u} \right), \end{aligned}$$

where, by Lemma A.2, the second term is $O(n^{-2\nu u + \gamma})$, while the sum in the first term has the asymptotic behavior given in (75).

Case $\nu = \gamma$: By assumption (16)³ and Lemma A.4, we have $G_{k+1,n-1}(c\alpha_j, q) = O(|F_{k+1,n-1}^\gamma(q)| + |F_{k+1,n-1}^\gamma(c\alpha_j)|)$, and so we find

$$\sum_{k=m_0}^{n-1} O((\Gamma_{k+1,n-1}^{21})^{2u}) = \sum_{j \in J, j \neq 1} I_{\{2 \in I_j\}} O \left(\sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma u}} |F_{k+1,n-1}^\gamma(q)|^{2u} + \sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma u}} |F_{k+1,n-1}^\gamma(c\alpha_j)|^{2u} \right),$$

where, by Lemma A.2, we have for $x = q$ or $x \in \{c\alpha_j : j \in J, j \neq 1\}$

$$\sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma u}} |F_{k+1,n-1}^\gamma(x)|^{2u} = \begin{cases} O(n^{-\gamma(2u-1)}) & \text{if } 1/2 < \nu = \gamma < 1, \\ O(n^{-2a_x u}) & \text{if } \nu = \gamma = 1 \text{ and } 1/2 < a_x < 1 - (2u)^{-1}, \\ O(n^{-2u+1} \ln(n)) & \text{if } \nu = \gamma = 1 \text{ and } a_x = 1 - (2u)^{-1}, \\ O(n^{-2u+1}) & \text{if } \nu = \gamma = 1 \text{ and } a_x > 1 - (2u)^{-1} \end{cases}$$

³If there exists j such that $q = c\alpha_j$, we have to consider the other asymptotic expression given in Lemma A.4.

and so, setting $x^* := \min\{q, ca^*\}$, we can write

$$\sum_{k=m_0}^{n-1} O\left((\Gamma_{k+1, n-1}^{21})^{2u}\right) = \sum_{j \in J, j \neq 1} I_{\{2 \in I_j\}} \begin{cases} O\left(n^{-\gamma(2u-1)}\right) & \text{if } 1/2 < \nu = \gamma < 1, \\ O\left(n^{-2x^*u}\right) & \text{if } \nu = \gamma = 1 \text{ and } 1/2 < x^* < 1 - (2u)^{-1}, \\ O\left(n^{-2u+1} \ln(n)\right) & \text{if } \nu = \gamma = 1 \text{ and } x^* = 1 - (2u)^{-1}, \\ O\left(n^{-2u+1}\right) & \text{if } \nu = \gamma = 1 \text{ and } x^* > 1 - (2u)^{-1}. \end{cases}$$

Summing up, taking into account the conditions $ca^* > 1/2$ when $\gamma = 1$ and $q > 1/2$ when $\nu = 1$, we can conclude that in all the six cases (i)-(vi), there exists a suitable $u > 1$ such that

$$\left(\sup_{m_0 \leq k \leq n-1} |t_n(J(I)) \mathbf{T}_{k+1, n-1}^{J(I)}| \right)^{2u} \xrightarrow{L^1} 0.$$

This convergence trivially implies condition (c3) of Theorem B.1. \square

5.4. Proof of Theorem 3.2. The proof of Theorem 3.2 follows by recalling that

$$(\tilde{\mathbf{Y}}_n - Z_\infty \mathbf{1}) = (\tilde{\mathbf{Y}}_n - Z_\infty \mathbf{1}) + \hat{\mathbf{Y}}_n,$$

where the convergence rate for the first term is $n^{\gamma-\frac{1}{2}}$ for any parameters (see (57)), while the convergence rate of the second term is n^e , with e specified in Theorem 5.1 according to the values of the parameters. Therefore, we can have three different cases:

- If $e < \gamma - \frac{1}{2}$, then we have

$$n^e(\mathbf{Y}_n - Z_\infty \mathbf{1}) = \frac{n^e}{n^{\gamma-\frac{1}{2}}} n^{\gamma-\frac{1}{2}}(\tilde{\mathbf{Y}}_n - Z_\infty \mathbf{1}) + n^e \hat{\mathbf{Y}}_n,$$

where the first term converges in probability to zero and the second term converges stably to a certain Gaussian kernel. This occurs only in case (a) with $e = \nu/2$ and $\nu < \gamma_0$.

- If $e > \gamma - \frac{1}{2}$, then we have

$$n^{\gamma-\frac{1}{2}}(\mathbf{Y}_n - Z_\infty \mathbf{1}) = n^{\gamma-\frac{1}{2}}(\tilde{\mathbf{Y}}_n - Z_\infty \mathbf{1}) + \frac{n^{\gamma-\frac{1}{2}}}{n^e} n^e \hat{\mathbf{Y}}_n,$$

where the first term converges stably (in the strong sense) to the Gaussian kernel given in (57) and the second term converges in probability to zero. This occurs in case (a) with $e = \nu/2$ and $\gamma_0 < \nu < \gamma$, in case (b) with $e = \gamma/2$ and $\nu = \gamma < 1$ and in case (c) with $e = \gamma - \nu/2$ and $\gamma < \nu < 1$.

- If $e = \gamma - \frac{1}{2}$, then we have

$$n^{\gamma-\frac{1}{2}}(\mathbf{Y}_n - Z_\infty \mathbf{1}) = n^{\gamma-\frac{1}{2}}(\tilde{\mathbf{Y}}_n - Z_\infty \mathbf{1}) + n^{\gamma-\frac{1}{2}} \hat{\mathbf{Y}}_n,$$

where the first term converges stably in the strong sense to the Gaussian kernel given in (57) and the second term is \mathcal{F}_n -measurable and it converges stably to a certain Gaussian kernel. Thus, in this case, we can apply Theorem B.2 in Appendix. This occurs in case (a) with $e = \nu/2$ and $\nu = \gamma_0 < 1$, in case (b) with $e = \gamma/2$ and $\nu = \gamma = 1$ (i.e. $\nu = \gamma_0 = 1$) and in case (c) with $e = \gamma - \nu/2$ and $\gamma < \nu = 1$ (i.e. $\gamma_0 < \nu = 1$). \square

Remark 5.3. As told in Remark 3.4, statements (a), (b) and (c) of Theorem 3.2 with $N = 1$ (and so without the condition on λ^*) can be proven with the same proof. Specifically, it is enough to take into account that when $N = 1$, we have $\hat{\mathbf{Y}}_n = Y_{1(2)}$ and $\tilde{Z}_n = Z_n$.

6. PROOF OF THE RESULTS FOR STATISTICAL APPLICATIONS

Here we prove the convergence results stated in Section 4. As we will see, the decomposition of \mathbf{Y}_n given in Section 5.2 is a fundamental tool also for the proof of these results.

6.1. Proof of Theorem 4.1. For the proof of this result, we need the following lemma:

Lemma 6.1. *Let us set*

$$(76) \quad \beta := \frac{\nu}{2} \mathbb{1}_{\{\nu \leq \gamma\}} + \left(\gamma - \frac{\nu}{2} \right) \mathbb{1}_{\{\gamma < \nu\}}.$$

Then, under all the assumptions stated in Section 2, we have

$$n^\beta \mathbf{Y}_{1(2)} \xrightarrow{a.s.} \mathcal{N} \left(\mathbf{0}, Z_\infty(1 - Z_\infty) \frac{\|\mathbf{v}_1\|^2}{N} d^{1(2),1(2)} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{11}^\top \end{pmatrix} \right),$$

where

$$d^{1(2),1(2)} = \begin{cases} \frac{q}{2} & \text{for } \nu < \gamma, \\ \frac{(q-c)^2}{2q^{-1}\{\nu=1\}} & \text{for } \nu = \gamma, \\ \frac{c^2}{2q^{-1}\{\nu=1\}(2\gamma-1)} & \text{for } \gamma < \nu. \end{cases}$$

Proof. We observe that $\mathbf{Y}_{1(2)}$ can be written as the general sum (58) with $J = \{1\}$ and $I_1 = \{2\}$. Therefore case $\nu > \gamma$ coincides with the case (v) of Lemma 5.1, taking into account the value $d^{1(2),1(2)}$ computed in Section A.4 for this case and equality (61). The cases $\nu < \gamma$ and $\nu = \gamma$ follows from the same arguments employed for the proof of Lemma 5.1, setting $t_n(J(I)) = n^{\nu/2}$ and using the value $d^{1(2),1(2)}$ obtained in Section A.4 when $\nu \leq \gamma$. \square

Remark 6.1. Note that, when $\nu = \gamma$ and $q = c$, we have $d^{1(2),1(2)} = 0$ and so we obtain that $n^\beta \mathbf{Y}_{1(2)}$ converges to $\mathbf{0}$ in probability. This means that in this case the convergence of $\mathbf{Y}_{1(2)}$ to $\mathbf{0}$ is faster than $n^{-\beta} = n^{-\gamma/2}$.

Proof of Theorem 4.1. The convergence rate and the second-order asymptotic distribution of \tilde{N}_n can be obtained by combining the second-order convergences of the two stochastic processes \tilde{Z}_n and $(\tilde{N}_n - \tilde{Z}_n)$. In order to get the convergence results for these two last processes, we observe that

$$\begin{aligned} N^{-1/2} \mathbf{u}_1^\top (\mathbf{0} \ I) \tilde{\mathbf{Y}}_n &= \tilde{Z}_n N^{-1/2} \mathbf{u}_1^\top \mathbf{1} = \tilde{Z}_n \quad \text{and} \\ N^{-1/2} \mathbf{u}_1^\top (\mathbf{0} \ I) \mathbf{Y}_{1(2),n} &= N^{-1/2} \mathbf{u}_1^\top (\mathbf{0} \ I) \mathbf{u}_{1(2)} \mathbf{v}_{1(2)}^\top \mathbf{Y}_n = N^{-1/2} \mathbf{u}_1^\top (-\mathbf{u}_1 \mathbf{v}_1^\top \quad \mathbf{u}_1 \mathbf{v}_1^\top) \mathbf{Y}_n = \\ &(\tilde{N}_n - \tilde{Z}_n) N^{-1/2} \mathbf{u}_1^\top \mathbf{1} = \tilde{N}_n - \tilde{Z}_n \end{aligned}$$

(where we have used (54) for the first equality and relations (53), (39), (45), (30) and (13) for the other equalities). Hence, from the convergence result stated in (57) and Lemma 6.1, together with Remark 6.1, we obtain that \tilde{Z}_n converges in probability to the random variable Z_∞ with rate $n^{\gamma-1/2}$ and $(\tilde{N}_n - \tilde{Z}_n)$ converges in probability to zero with at least rate n^β defined in (76). Then, since $\tilde{N}_n = \tilde{Z}_n + (\tilde{N}_n - \tilde{Z}_n)$, it is possible to follow analogous arguments to those used in the proof of Theorem 3.2 to combine the asymptotic behaviors of \tilde{Z}_n and $(\tilde{N}_n - \tilde{Z}_n)$. More precisely:

- (a) in the case $\nu < \gamma_0$, we necessarily have $\gamma_0 = 2\gamma - 1 \leq \gamma$ (since $\gamma \leq 1$) and so we have $\beta = \nu/2 < (\gamma - 1/2)$. Thus \tilde{N}_n has the same convergence rate and the same asymptotic variance as $(\tilde{N}_n - \tilde{Z}_n) = N^{-1/2} \mathbf{u}_1^\top (\mathbf{0} \ I) \mathbf{Y}_{1(2),n}$, that is (see Lemma 6.1) we get

$$n^{\nu/2} (\tilde{N}_n - Z_\infty) \longrightarrow \mathcal{N} (0, Z_\infty(1 - Z_\infty) \tilde{\sigma}^2) \quad \text{stably}$$

with $\tilde{\sigma}^2 = q/2$;

- (b) in the case $\gamma_0 < \nu < 1$, we have $\beta > (\gamma - 1/2)$ and hence \tilde{N}_n has the same asymptotic behavior as $\tilde{Z}_n = N^{-1/2} \mathbf{u}_1^\top (\mathbf{0} \ I) \tilde{\mathbf{Y}}_n$, that is (see (57))

$$n^{\gamma-\frac{1}{2}} (\tilde{N}_n - Z_\infty) \longrightarrow \mathcal{N} (0, Z_\infty(1 - Z_\infty) \tilde{\sigma}_\gamma^2) \quad \text{stably};$$

- (c) If $\nu = \gamma_0$ (i.e. $\nu = 2\gamma - 1 \leq \gamma$) or $\nu = 1$, we have $\beta = (\gamma - 1/2)$ and hence the asymptotic behavior of \tilde{N}_n follows by combining the convergence results for $(\tilde{N}_n - \tilde{Z}_n)$ and \tilde{Z}_n as done in the proof of Theorem 3.2, and so we get

$$n^{\gamma-\frac{1}{2}} (\tilde{N}_n - Z_\infty) \longrightarrow \mathcal{N} (0, Z_\infty(1 - Z_\infty) (\tilde{\sigma}_\gamma^2 + \tilde{\sigma}^2)) \quad \text{stably},$$

where $\tilde{\sigma}^2$ is defined in (31).

□

Remark 6.2. Returning to Remark 4.1, we observe that in the proof of Theorem 4.1 the asymptotic behavior of \tilde{N}_n is obtained as the combination of the asymptotic behaviors of $\tilde{N}_n - \tilde{Z}_n$ and \tilde{Z}_n . In case (b), \tilde{Z}_n converges slower than $\tilde{N}_n - \tilde{Z}_n$, and so only the rate and the asymptotic variance of \tilde{Z}_n appear in the statement of the result. However, if we look at an higher level of approximation, we should also consider the process $\tilde{N}_n - \tilde{Z}_n$, that converges to zero with at least rate n^β . Then, we can note that β as a function of ν has its maximum in $\nu = \gamma$, which hence provides the “optimal value” of ν . In addition, in this case the quantity $d^{1(2)1(2)}$ as a function of q has its minimum in $q = c$, which hence gives the “optimal value” of q . Note that, as told in the previous Remark 6.1, when $\nu = \gamma$ and $q = c$, we have $n^\beta \mathbf{Y}_{1(2)} \rightarrow \mathbf{0}$ in probability and so also $n^\beta(\tilde{N}_n - \tilde{Z}_n) \rightarrow 0$ in probability. This means that in this case the convergence of $\tilde{N}_n - \tilde{Z}_n$ to zero is faster than $n^{-\beta} = n^{-\gamma/2}$.

6.2. Proof of Theorem 4.2. Recalling (30), together with (14) and the fact that

$$U_j^* V_j^{*\top} = \begin{pmatrix} \mathbf{u}_j \mathbf{v}_j^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{u}_j \mathbf{v}_j^\top \end{pmatrix},$$

we can write $\mathbf{N}'_n = \sum_{j=2}^N \mathbf{u}_j \mathbf{v}_j^\top \mathbf{N}_n = (\mathbf{0} \ I) \sum_{j=2}^N U_j^* V_j^{*\top} \mathbf{Y}_n$. Now we can use the decomposition $\mathbf{Y}_n = (\tilde{\mathbf{Y}}_n + \hat{\mathbf{Y}}_n)$ and the fact that $U_j^* V_j^{*\top} \tilde{\mathbf{Y}}_n = \mathbf{0}$ for any $2 \leq j \leq N$ (by (12) and (54)) in order to obtain the equality

$$\mathbf{N}'_n = (\mathbf{0} \ I) \sum_{j=2}^N U_j^* V_j^{*\top} \hat{\mathbf{Y}}_n.$$

Hence, the convergence rate and the second-order asymptotic distribution of \mathbf{N}'_n can be obtained by using the convergences stated in Theorem 5.1 or in Lemma 5.1. Specifically, case (a) follows from Theorem 5.1(a), observing that (by (12)) we have

$$(\mathbf{0} \ I) \sum_{j=2}^N U_j^* V_j^{*\top} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{U} \end{pmatrix} = (\mathbf{0} \ I) \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{U}_{-1} \end{pmatrix} = (\mathbf{0} \ \tilde{U}_{-1}).$$

Case (b) follows from Theorem 5.1(b), observing that (by (12)) we have

$$(\mathbf{0} \ I) \sum_{j=2}^N U_j^* V_j^{*\top} \begin{pmatrix} \tilde{U} & \mathbf{0} \\ \mathbf{0} & \tilde{U} \end{pmatrix} = (\mathbf{0} \ I) \begin{pmatrix} \tilde{U}_{-1} & \mathbf{0} \\ \mathbf{0} & \tilde{U}_{-1} \end{pmatrix} = (\mathbf{0} \ \tilde{U}_{-1}).$$

Finally, case (c) cannot be obtained directly by using the convergences stated in Theorem 5.1 since in this case we have (by (12))

$$(\mathbf{0} \ I) \sum_{j=2}^N U_j^* V_j^{*\top} \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} = N^{1/2} (\mathbf{0} \ I) \sum_{j=2}^N U_j^* V_j^{*\top} \begin{pmatrix} \mathbf{0} \\ \mathbf{u}_1 \end{pmatrix} = (\mathbf{0} \ I) \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} = \mathbf{0}.$$

Therefore, we need to express \mathbf{N}'_n in the following equivalent way:

$$\mathbf{N}'_n = (\mathbf{0} \ I) \sum_{j=2}^N U_j^* V_j^{*\top} \hat{\mathbf{Y}}_n = (\mathbf{0} \ I) \left(\sum_{j=2}^N \mathbf{Y}_{j(1),n} + \sum_{j=2}^N \mathbf{Y}_{j(2),n} \right),$$

where for the last equality we have used the decomposition (55) of $\hat{\mathbf{Y}}_n$ and the fact that $U_j^* V_j^{*\top} \mathbf{Y}_{1(2),n} = U_j^* V_j^{*\top} \mathbf{u}_{1(2)} \mathbf{v}_{1(2)}^\top \mathbf{Y}_n = \mathbf{0}$ for $2 \leq j \leq N$. Now, we recall that, in case (c), that is $\nu > \gamma$, we have $g(\lambda_1) = g(1) = 1$ and $g(\lambda_j) = 0$ for $2 \leq j \leq N$ and so we get $(\mathbf{0} \ I) \mathbf{u}_{j(1)} = \mathbf{0}$ for $2 \leq j \leq N$. As a consequence, since $\mathbf{Y}_{j(1),n} = \mathbf{u}_{j(1)} \mathbf{v}_{j(1)}^\top \mathbf{Y}_n$, we have that $(\mathbf{0} \ I) \sum_{j=2}^N \mathbf{Y}_{j(1),n} = \mathbf{0}$, and the desired convergence result follows from case (vi) of Lemma 5.1. □

APPENDIX A. COMPUTATIONS FOR THE PROOF OF LEMMA 5.1

In all the sequel, given $(z_n)_n, (z'_n)_n$ two sequences of complex numbers, the notation $z_n = O(z'_n)$ means $|z_n| \leq C|z'_n|$ for a suitable constant $C > 0$ and n large enough. Moreover, if $z'_n \neq 0$, the notation $z_n \sim z'_n$ with $z \in \mathbb{C} \setminus \{0\}$ means $\lim_n z_n/z'_n = z$ and, finally, the notation $z_n = o(z'_n)$ means $\lim_n z_n/z'_n = 0$.

Given $1/2 < \delta \leq 1$, $x = a_x + i b_x \in \mathbb{C}$ with $a_x > 0$ and an integer $m_0 \geq 2$ such that $a_x m^{-\delta} < 1$ for all $m \geq m_0$, let us set

$$(77) \quad p_n^\delta(x) := \prod_{m=m_0}^n \left(1 - \frac{x}{m^\delta}\right) \quad \text{for } n \geq m_0.$$

A.1. Some technical results. We first recall the following result, which has been proved in [2].

Lemma A.1. [2, Lemma A.4] *We have*

$$(78) \quad |p_n^\delta(x)| = \begin{cases} O\left(\exp\left(-a_x \frac{n^{1-\delta}}{1-\delta}\right)\right) & \text{for } 1/2 < \delta < 1 \\ O(n^{-a_x}) & \text{for } \delta = 1 \end{cases}$$

and

$$(79) \quad |p_n^\delta(x)^{-1}| = \begin{cases} O\left(\exp\left(a_x \frac{n^{1-\delta}}{1-\delta}\right)\right) & \text{for } 1/2 < \delta < 1 \\ O(n^{a_x}) & \text{for } \delta = 1. \end{cases}$$

Therefore, if we set

$$(80) \quad F_{k+1,n}^\delta(x) := \frac{p_n^\delta(x)}{p_k^\delta(x)} \quad \text{for } m_0 \leq k \leq n,$$

we have

$$(81) \quad |F_{k+1,n}^\delta(x)| = \begin{cases} O\left(\exp\left(\frac{a_x}{1-\delta}(k^{1-\delta} - n^{1-\delta})\right)\right) & \text{for } 1/2 < \delta < 1 \\ O\left(\left(\frac{k}{n}\right)^{a_x}\right) & \text{for } \delta = 1. \end{cases}$$

Now, we prove two other results.

Lemma A.2. *Given $\beta > 1$ and $e > 0$, we have*

$$(82) \quad \sum_{k=m_0}^n \frac{1}{k^\beta} |F_{k+1,n}^\delta(x)|^e = \begin{cases} O(n^{-(\beta-\delta)}) & \text{if } 1/2 < \delta < 1, \\ O(n^{-ea_x}) & \text{if } \delta = 1 \text{ and } ea_x < \beta - 1, \\ O(n^{-(\beta-1)} \ln(n)) & \text{if } \delta = 1 \text{ and } ea_x = \beta - 1, \\ O(n^{-(\beta-1)}) & \text{if } \delta = 1 \text{ and } ea_x > \beta - 1. \end{cases}$$

Proof. The desired relations immediately follows from (81) using the well-known relation

$$(83) \quad \sum_{k=1}^n \frac{1}{k^{1-a}} = \begin{cases} O(1) & \text{for } a < 0, \\ \ln(n) + d + O(n^{-1}) = \ln(n) + O(1) & \text{for } a = 0, \\ a^{-1} n^a + O(1) & \text{for } 0 < a \leq 1, \\ a^{-1} n^a + O(n^{a-1}) & \text{for } a > 1, \end{cases}$$

where d is the Euler-Mascheroni constant, and the relation

$$(84) \quad \sum_{k=1}^n \frac{\exp(ak^b/b)}{k^\beta} = O\left(\int_1^n \frac{\exp(at^b/b)}{t^\beta} dt\right) = O\left(\left[\frac{\exp(at^b/b)}{at^{b+\beta-1}}\right]_1^n + \frac{(b+\beta-1)}{a} \int_1^n \frac{\exp(at^b/b)}{t^{b+\beta}} dt\right) \\ = O\left(\frac{\exp(an^b/b)}{n^{b+\beta-1}}\right) \quad \text{for } a > 0, b > 0, \beta > 1.$$

Indeed, for the case $\delta = 1$, it is enough to apply (83) with $a = ea_x - (\beta - 1)$; while, for the case $1/2 < \delta < 1$, it is enough to apply (84) with $a = ea_x$, $b = 1 - \delta$ and β . \square

The following lemma extends [2, Lemma A.5].

Lemma A.3. *Given $1/2 < \delta_1 \leq \delta_2 \leq 1$, $\beta > \delta_1$ and $x_1, x_2 \in \mathbb{C}$ with $\operatorname{Re}(x_1) > 0$, $\operatorname{Re}(x_2) > 0$, let $m_0 \geq 2$ be an integer such that $\max\{\operatorname{Re}(x_1), \operatorname{Re}(x_2)\}m^{-\delta_1} < 1$ for all $m \geq m_0$. Then we have*

$$(85) \quad \lim_n n^{\beta-\delta_1} \sum_{k=m_0}^n k^{-\beta} F_{k+1,n}^{\delta_1}(x_1) F_{k+1,n}^{\delta_2}(x_2) = \begin{cases} \frac{1}{x_1+x_2} & \text{if } 1/2 < \delta_1 = \delta_2 < 1, \\ \frac{1}{x_1+x_2-\beta+1} & \text{if } \delta_1 = \delta_2 = 1 \text{ and } \operatorname{Re}(x_1+x_2) > \beta-1, \\ \frac{1}{x_1} & \text{if } 1/2 < \delta_1 < \delta_2 \leq 1. \end{cases}$$

Proof. Let us start with observing that, in each considered case, relation (78) implies

$$(86) \quad \lim_n n^{\beta-\delta_1} |p_n^{\delta_1}(x_1)| |p_n^{\delta_2}(x_2)| = 0.$$

Indeed, in particular, when $\delta_1 = \delta_2 = 1$ we have the additional condition $\operatorname{Re}(x_1+x_2) > \beta-1$.

Now, fix $k \geq 2$ and let us set $\eta := \beta - \delta_1$ and $\ell_n^\delta(x) := 1/p_n^\delta(x)$ and define the following quantity

$$\begin{aligned} D_k &:= \frac{1}{k^\eta} \ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2) - \frac{1}{(k-1)^\eta} \ell_{k-1}^{\delta_1}(x_1) \ell_{k-1}^{\delta_2}(x_2) \\ &= \left(\frac{1}{k^\eta} - \frac{1}{(k-1)^\eta} \right) \ell_{k-1}^{\delta_1}(x_1) \ell_{k-1}^{\delta_2}(x_2) + \frac{1}{k^\eta} \left(\ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2) - \ell_{k-1}^{\delta_1}(x_1) \ell_{k-1}^{\delta_2}(x_2) \right) \\ &= \ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2) \left[\left(\frac{1}{k^\eta} - \frac{1}{(k-1)^\eta} \right) \frac{\ell_{k-1}^{\delta_1}(x_1) \ell_{k-1}^{\delta_2}(x_2)}{\ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2)} + \frac{1}{k^\eta} \left(1 - \frac{\ell_{k-1}^{\delta_1}(x_1) \ell_{k-1}^{\delta_2}(x_2)}{\ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2)} \right) \right]. \end{aligned}$$

Then, we observe the following:

$$(87) \quad \left(\frac{1}{k^\eta} - \frac{1}{(k-1)^\eta} \right) = -\frac{\eta}{k^{1+\eta}} + O\left(\frac{1}{k^{2+\eta}}\right) \quad \text{for } k \rightarrow +\infty$$

and

$$(88) \quad \frac{\ell_{k-1}^{\delta_1}(x_1) \ell_{k-1}^{\delta_2}(x_2)}{\ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2)} = \left(1 - \frac{x_1}{k^{\delta_1}}\right) \left(1 - \frac{x_2}{k^{\delta_2}}\right) = 1 + \frac{x_1 x_2}{k^{(\delta_1+\delta_2)}} - \frac{x_1}{k^{\delta_1}} - \frac{x_2}{k^{\delta_2}}.$$

Now, by using (87) and (88) in the above expression of D_k , we have for $k \rightarrow +\infty$

$$\begin{aligned} D_k &= \ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2) \left[\left(-\frac{\eta}{k^{\eta+1}} + O(1/k^{\eta+2}) \right) \left(1 + \frac{x_1 x_2}{k^{(\delta_1+\delta_2)}} - \frac{x_1}{k^{\delta_1}} - \frac{x_2}{k^{\delta_2}} \right) + \frac{1}{k^\eta} \left(-\frac{x_1 x_2}{k^{(\delta_1+\delta_2)}} + \frac{x_1}{k^{\delta_1}} + \frac{x_2}{k^{\delta_2}} \right) \right] \\ &= \begin{cases} \ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2) \left[\frac{x_1+x_2}{k^{\eta+\delta}} - \frac{\eta}{k^{\eta+1}} + o(1/k^{\eta+\delta}) \right] & \text{if } \delta_1 = \delta_2 = \delta, \\ \ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2) \left[\frac{x_1}{k^{\eta+\delta_1}} + o(1/k^{\eta+\delta_1}) \right] & \text{if } \delta_1 < \delta_2 \end{cases} \\ &= \begin{cases} \ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2) \frac{x_1+x_2}{k^{\eta+\delta}} + o(\ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2) / k^{\eta+\delta}) & \text{if } \delta_1 = \delta_2 = \delta < 1, \\ \ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2) \frac{x_1+x_2-\eta}{k^{\eta+1}} + o(\ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2) / k^{\eta+1}) & \text{if } \delta_1 = \delta_2 = 1 \text{ and } \operatorname{Re}(x_1+x_2) > \eta, \\ \ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2) \frac{x_1}{k^{\eta+\delta_1}} + o(\ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2) / k^{\eta+\delta_1}) & \text{if } \delta_1 < \delta_2. \end{cases} \end{aligned}$$

that is

$$(89) \quad D_k \sim \begin{cases} \frac{x_1+x_2}{k^{\eta+\delta}} \ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2) & \text{if } 1/2 < \delta_1 = \delta_2 = \delta < 1, \\ \frac{x_1+x_2-\eta}{k^{\eta+1}} \ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2) & \text{if } \delta_1 = \delta_2 = 1 \text{ and } \operatorname{Re}(x_1+x_2) > \eta, \\ \frac{x_1}{k^{\eta+\delta_1}} \ell_k^{\delta_1}(x_1) \ell_k^{\delta_2}(x_2) & \text{if } 1/2 < \delta_1 < \delta_2 \leq 1. \end{cases}$$

Now, following the same arguments used in the proof of [2, Lemma A.5], in order to conclude, we apply [2, Corollary A.2] with

$$z_n = D_n, \quad v_n = n^\eta p_n^{\delta_1}(x_1) p_n^{\delta_2}(x_2), \quad w_n = \frac{\ell_{n,1} \ell_{n,2}}{n^{\eta+\delta_1} D_n}, \quad w = \begin{cases} \frac{1}{x_1+x_2} & \text{if } 1/2 < \delta_1 = \delta_2 = \delta < 1 \\ \frac{1}{x_1+x_2-\eta} & \text{if } \delta_1 = \delta_2 = 1 \text{ and } \operatorname{Re}(x_1+x_2) > \eta \\ \frac{1}{x_1} & \text{if } 1/2 < \delta_1 < \delta_2 \leq 1. \end{cases}$$

Indeed, $\lim_n v_n = 0$ by (86), $\lim_n w_n = w \neq 0$ by (89),

$$\begin{aligned} \lim_n v_n \sum_{k=m_0}^n z_k &= \lim_n n^\eta p_n^{\delta_1}(x_1) p_n^{\delta_2}(x_2) \sum_{k=m_0}^n D_k \\ &= \lim_n n^\eta p_n^{\delta_1}(x_1) p_n^{\delta_2}(x_2) \left(\frac{\ell_n^{\delta_1}(x_1) \ell_n^{\delta_2}(x_2)}{n^\eta} - \frac{\ell_{m_0-1}^{\delta_1}(x_1) \ell_{m_0-1}^{\delta_2}(x_2)}{(m_0-1)^\eta} \right) = 1 \end{aligned}$$

by (86) and $z'_n = z_n w_n = r_n^2 \ell_{n,1} \ell_{n,2}$. \square

A.2. Analytic expression of $A_{k+1,n-1}^j$ with $j \geq 2$. Let us recall the definition of the following quantities for $j \geq 2$:

$$A_{k+1,n-1}^j = \prod_{m=k+1}^{n-1} (I - D_{Q,j,m}), \quad \text{where} \quad D_{Q,j,n} = \begin{pmatrix} \widehat{r}_{n-1}(1 - \lambda_j) & 0 \\ -\lambda_j h_n(\lambda_j) & \widehat{q}_{n,n} \end{pmatrix}$$

$$\text{with } h_n \text{ defined in (43), that is } h_n(x) = \begin{cases} \widehat{r}_{n-1}(1 - x) & \text{if } \nu < \gamma, \\ \widehat{q}_{n,n} \mathbb{1}_{\{x \neq 1\}} & \text{if } \nu \geq \gamma. \end{cases}$$

The aim of this section is to compute the product above and so finding the useful expression of $A_{k+1,n-1}^j$ presented in (65), i.e.

$$A_{k+1,n-1}^j = \begin{pmatrix} F_{k+1,n-1}^\gamma(c\alpha_j) & 0 \\ \lambda_j G_{k+1,n-1}(c\alpha_j, q) & F_{k+1,n-1}^\nu(q) \end{pmatrix},$$

where

$$G_{k+1,n-1}(c\alpha_j, q) = \sum_{l=k+1}^{n-1} F_{l+1,n-1}^\gamma(c\alpha_j) h_l(\lambda_j) F_{k+1,l-1}^\nu(q).$$

It is straightforward to see that $[A_{k+1,n-1}^j]_{21} = 0$,

$$[A_{k+1,n-1}^j]_{11} = \prod_{m=k+1}^{n-1} (1 - \widehat{r}_{m-1}(1 - \lambda_j)) = F_{k+1,n-1}^\gamma(c\alpha_j),$$

$$[A_{k+1,n-1}^j]_{22} = \prod_{m=k+1}^{n-1} (1 - \widehat{q}_{m,m}) = F_{k+1,n-1}^\nu(q),$$

while it is not immediate to determine $[A_{k+1,n-1}^j]_{12}$. To this end, let us set $x_{n-1} := [A_{k+1,n-1}^j]_{21}$ and observe that, since $A_{k+1,n-1}^j = A_{k+1,n-2}^j (I - D_{Q,j,n-1})$ and $x_{k+1} = \lambda_j h_{k+1}(\lambda_j)$, we have that

$$\begin{aligned} x_{n-1} &= x_{n-2} (1 - \widehat{r}_{n-1}(1 - \lambda_j)) + [A_{k+1,n-2}^j]_{22} \lambda_j h_{n-1}(\lambda_j) \\ &= x_{n-2} F_{n-1,n-1}^\gamma(c\alpha_j) + F_{k+1,n-2}^\nu(q) \lambda_j h_{n-1}(\lambda_j) \\ &= x_{n-3} F_{n-2,n-1}^\gamma(c\alpha_j) + F_{k+1,n-3}^\nu(q) \lambda_j h_{n-2}(\lambda_j) F_{n-1,n-1}^\gamma(c\alpha_j) + F_{k+1,n-2}^\nu(q) \lambda_j h_{n-1}(\lambda_j) \\ &= \dots \\ &= x_{k+1} F_{k+2,n-1}^\gamma(c\alpha_j) + \sum_{l=k+2}^{n-2} F_{k+1,l-1}^\nu(q) \lambda_j h_l(\lambda_j) F_{l+1,n-1}^\gamma(c\alpha_j) + F_{k+1,n-2}^\nu(q) \lambda_j h_{n-1}(\lambda_j) \\ &= \sum_{l=k+1}^{n-1} F_{k+1,l-1}^\nu(q) \lambda_j h_l(\lambda_j) F_{l+1,n-1}^\gamma(c\alpha_j) \\ &= \lambda_j G_{k+1,n-1}(c\alpha_j, q). \end{aligned}$$

A.3. Asymptotic behavior of $G_{k+1,n-1}(x, q)$. Let us recall the definition

$$G_{k+1,n-1}(x, q) := \sum_{l=k+1}^{n-1} F_{l+1,n-1}^\gamma(x) h_l(1 - c^{-1}x) F_{k+1,l-1}^\nu(q).$$

Here we prove the following result:

Lemma A.4. *When $\nu = \gamma$, we have for $x \in \mathbb{C} \setminus \{0\}$*

(90)

$$G_{k+1,n-1}(x, q) = \begin{cases} \frac{q}{x-q} \left(F_{k+1,n-1}^\gamma(q) - F_{k+1,n-1}^\gamma(x) \right) & \text{if } x \neq q, \\ \frac{q}{1-\gamma} F_{k+1,n-1}^\gamma(q) \left[(n-1)^{1-\gamma} - (k+1)^{1-\gamma} \right] + O(F_{k+1,n-1}^\gamma(q)) & \text{if } x = q \text{ and } 1/2 < \gamma < 1, \\ q F_{k+1,n-1}^\gamma(q) \ln \left(\frac{n-1}{k+1} \right) + O(k^{-1} F_{k+1,n-1}^\gamma(q)) & \text{if } x = q \text{ and } \gamma = 1. \end{cases}$$

When $\nu \neq \gamma$, we have for $x \in \mathbb{C} \setminus \{0\}$

$$(91) \quad G_{k+1,n-1}(x, q) = C(x, q) \left(\frac{F_{k+1,n-1}^\nu(q)}{(n-1)^\mu} - \frac{F_{k+1,n-1}^\nu(x)}{k^\mu} \right) + O \left(\frac{|F_{k+1,n-1}^\nu(q)|}{n^{2\mu}} + \frac{|F_{k+1,n-1}^\nu(x)|}{k^{2\mu}} \right),$$

where $\mu := |\gamma - \nu|$ and

$$C(x, q) := \begin{cases} -\frac{x}{q} & \text{if } \nu < \gamma, \\ \frac{q}{x} & \text{if } \nu > \gamma. \end{cases}$$

Proof. Recalling the definition (80), we can write

$$(92) \quad \begin{aligned} G_{k+1,n-1}(x, q) &= \sum_{l=k+1}^{n-1} F_{l+1,n-1}^\gamma(x) h_l(1 - c^{-1}x) F_{k+1,l-1}^\nu(q) = \sum_{l=k+1}^{n-1} \frac{p_{n-1}^\gamma(x)}{p_l^\gamma(x)} h_l(1 - c^{-1}x) \frac{p_{l-1}^\nu(q)}{p_k^\nu(q)} \\ &= \frac{p_{n-1}^\gamma(x)}{p_k^\nu(q)} \sum_{l=k+1}^{n-1} \frac{h_l(1 - c^{-1}x)}{(1 - \hat{q}_{l,l})} X_l, \quad \text{where} \quad X_l := \frac{p_l^\nu(q)}{p_l^\gamma(x)}. \end{aligned}$$

Moreover, recalling the definition (43), we have for $x \neq 0$

$$h_l(1 - c^{-1}x) = \begin{cases} \hat{r}_{l-1} c^{-1}x = x l^{-\gamma} & \text{if } \nu < \gamma, \\ \hat{q}_{l,l} = q l^{-\nu} & \text{if } \nu \geq \gamma. \end{cases}$$

Let us start with the case $\nu = \gamma$. In this case, we have

$$\begin{aligned} \Delta X_l &:= X_l - X_{l-1} = \left(1 - \frac{X_{l-1}}{X_l} \right) X_l = \left(\frac{x-q}{q} \frac{q}{l^\gamma(1 - q l^{-\gamma})} \right) X_l \\ &= \frac{x-q}{q} \frac{\hat{q}_{l,l}}{1 - \hat{q}_{l,l}} X_l = \frac{x-q}{q} \frac{h_l(1 - c^{-1}x)}{1 - \hat{q}_{l,l}} X_l. \end{aligned}$$

It follows that

$$\frac{x-q}{q} \sum_{l=k+1}^{n-1} \frac{h_l(1 - c^{-1}x)}{1 - \hat{q}_{l,l}} X_l = X_{n-1} - X_k.$$

Since

$$(93) \quad \begin{aligned} \frac{p_{n-1}^\gamma(x)}{p_k^\nu(q)} X_{n-1} &= \frac{p_{n-1}^\gamma(x)}{p_k^\nu(q)} \frac{p_{n-1}^\nu(q)}{p_{n-1}^\gamma(x)} = F_{k+1,n-1}^\nu(q) \text{ and} \\ \frac{p_{n-1}^\gamma(x)}{p_k^\nu(q)} X_k &= \frac{p_{n-1}^\gamma(x)}{p_k^\nu(q)} \frac{p_k^\nu(q)}{p_k^\gamma(x)} = F_{k+1,n-1}^\gamma(x), \end{aligned}$$

we find by (92)

$$\frac{x-q}{q} G_{k+1,n-1}(x, q) = \left(F_{k+1,n-1}^\gamma(q) - F_{k+1,n-1}^\gamma(x) \right)$$

and so for $x \neq q$ we get

$$G_{k+1,n-1}(x, q) = \frac{q}{x-q} \left(F_{k+1,n-1}^\gamma(q) - F_{k+1,n-1}^\gamma(x) \right).$$

When $\nu = \gamma$ and $x = q$, we have $X_l = 1$ and so we obtain (by (92) together with (83))

$$\begin{aligned} G_{k+1,n-1}(x, q) &= qF_{k+1,n-1}^\gamma(q) \sum_{l=k+1}^{n-1} \frac{1}{l^\gamma(1-ql^{-\gamma})} = qF_{k+1,n-1}^\gamma(q) \sum_{l=k+1}^{n-1} \frac{1}{l^\gamma} + O\left(\sum_{l \geq k+1} l^{-2\gamma}\right) \\ &= qF_{k+1,n-1}^\gamma(q) \begin{cases} \frac{(n-1)^{1-\gamma}}{1-\gamma} - \frac{(k+1)^{1-\gamma}}{1-\gamma} + O(1) + O(k^{-(2\gamma-1)}) & \text{if } 1/2 < \gamma < 1 \\ \ln(n-1) - \ln(k+1) + O(n^{-1}) + O(k^{-1}) & \text{if } \gamma = 1, \end{cases} \end{aligned}$$

which implies the two different asymptotic behavior in (90) according to the value of γ .

Now, let us consider the case $\nu \neq \gamma$ and introduce the sequence $\{y_l; l \geq 1\}$ defined as $y_l := l^{-\mu}$, with $\mu = |\gamma - \nu|$. Then, we have

$$\begin{aligned} y_l X_l - y_{l-1} X_{l-1} &= \Delta y_l X_l + y_{l-1} \Delta X_l = \left(\frac{1}{l^\mu} - \frac{1}{(l-1)^\mu}\right) X_l + \left(\frac{1}{l^\mu} + O\left(\frac{1}{l^{1+\mu}}\right)\right) \Delta X_l \\ &= \left(\frac{-\mu}{l^{1+\mu}} + O\left(\frac{1}{l^{2+\mu}}\right)\right) X_l + \left(\frac{1}{l^\mu} + O\left(\frac{1}{l^{1+\mu}}\right)\right) \Delta X_l, \end{aligned}$$

where

$$\Delta X_l := X_l - X_{l-1} = \left(1 - \frac{X_{l-1}}{X_l}\right) X_l = R_l X_l$$

with

$$R_l := \left(1 - \frac{X_{l-1}}{X_l}\right) = \frac{x l^{-\gamma} - q l^{-\nu}}{1 - q l^{-\nu}} = \frac{\widehat{r}_{l-1} c^{-1} x - \widehat{q}_{l,l}}{1 - \widehat{q}_{l,l}} = O\left(\frac{1}{l^{\min\{\gamma, \nu\}}}\right).$$

Taking into account that $\mu + \min\{\gamma, \nu\} < 1 + \mu$ for $\nu \neq \gamma$, we obtain that

$$y_l X_l - y_{l-1} X_{l-1} = \left[\frac{R_l}{l^\mu} + O\left(\frac{1}{l^{1+\mu}}\right)\right] X_l = K(x, q) \frac{h_l(1 - c^{-1}x)}{(1 - \widehat{q}_{l,l})} X_l + Q_l X_l,$$

where

$$K(x, q) := \left(-\frac{q}{x}\right) \mathbb{1}_{\{\nu < \gamma\}} + \left(\frac{x}{q}\right) \mathbb{1}_{\{\nu > \gamma\}} = C(x, q)^{-1}$$

and

$$Q_l := \begin{cases} \frac{x l^{-(2\gamma-\nu)}}{1 - \widehat{q}_{l,l}} + O(l^{-(1+\mu)}) & \text{if } \nu < \gamma, \\ -\frac{q l^{-(2\nu-\gamma)}}{1 - \widehat{q}_{l,l}} + O(l^{-(1+\mu)}) & \text{if } \nu > \gamma. \end{cases}$$

Note that $Q_l \sim \kappa l^{-(2\mu + \min\{\gamma, \nu\})}$ with a suitable $\kappa \neq 0$. The above expression implies that

$$(94) \quad \frac{X_{n-1}}{(n-1)^\mu} - \frac{X_k}{k^\mu} = \sum_{l=k+1}^{n-1} (y_l X_l - y_{l-1} X_{l-1}) = K(x, q) \sum_{l=k+1}^{n-1} \frac{h_l(1 - c^{-1}x)}{(1 - \widehat{q}_{l,l})} X_l + \sum_{l=k+1}^{n-1} Q_l X_l.$$

With similar computations, setting

$$R_l^* := 1 - \frac{|X_{l-1}|}{|X_l|} = \frac{|1 - q l^{-\nu}| - |1 - x l^{-\gamma}|}{|1 - q l^{-\nu}|}$$

and taking into account that $R_l^* l^{-2\mu} \sim \kappa' l^{-(2\mu + \min\{\gamma, \nu\})}$ with a suitable $\kappa' \neq 0$ and $\min\{\gamma, \nu\} < 1$ for $\nu \neq \gamma$, we find

$$\frac{|X_l|}{l^{2\mu}} - \frac{|X_{l-1}|}{(l-1)^{2\mu}} = \left[\frac{R_l^*}{l^{2\mu}} + O\left(\frac{1}{l^{1+2\mu}}\right)\right] |X_l| = Q_l^* |X_l|.$$

Then, since $Q_l \sim \kappa'' Q_l^*$ with a suitable $\kappa'' \neq 0$,

$$\sum_{l=k+1}^{n-1} Q_l X_l = O\left(\sum_{l=k+1}^{n-1} Q_l^* |X_l|\right) = O\left(\frac{|X_{n-1}|}{(n-1)^{2\mu}} - \frac{|X_k|}{k^{2\mu}}\right).$$

Finally, by (92), (93), (94) and the last above relations, we obtain for $x \neq 0$

$$G_{k+1,n-1}(x, q) = C(x, q) \left(\frac{F_{k+1,n-1}^\nu(q)}{(n-1)^\mu} - \frac{F_{k+1,n-1}^\gamma(x)}{k^\mu}\right) + O\left(\frac{|F_{k+1,n-1}^\nu(q)|}{n^{2\mu}} + \frac{|F_{k+1,n-1}^\gamma(x)|}{k^{2\mu}}\right).$$

□

A.4. Computation of the limit $d^{j_1(i_1), j_2(i_2)}$. Recall that we have

$$d_{k,n}^{j(1)} = \begin{cases} \widehat{r}_{k-1} & \text{for } j = 1 \\ \widehat{r}_{k-1} F_{k+1, n-1}^\gamma(c\alpha_j) & \text{for } j \geq 2 \end{cases}$$

and

$$d_{k,n}^{j(2)} = \begin{cases} (\widehat{q}_{k,k} - \widehat{r}_{k-1}) F_{k+1, n-1}^\nu(q) & \text{for } j = 1 \\ \lambda_j \widehat{r}_{k-1} G_{k+1, n-1}(c\alpha_j, q) + (\widehat{q}_{k,k} - \widehat{r}_{k-1} g(\lambda_j)) F_{k+1, n-1}^\nu(q) & \text{for } j \geq 2, \end{cases}$$

where g is defined in (44), and so, for each $j \geq 2$, we have $g(\lambda_j) = \lambda_j$ when $\nu < \gamma$, while $g(\lambda_j) = 0$ when $\nu \geq \gamma$.

Here, for each of the six cases (i) – (vi) listed in Lemma 5.1, we compute the limit

$$d^{j_1(i_1), j_2(i_2)} = \lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)}.$$

For all the computations, we make the assumptions stated in Section 2 and we use Lemma A.3 and Lemma A.4.

Case (i): Take $\nu < \gamma$, $j_1, j_2 \in \{2, \dots, N\}$ and $i_1 = i_2 = 1$. We have

$$\begin{aligned} \lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} &= \lim_n n^\gamma \sum_{k=m_0}^{n-1} \widehat{r}_{k-1}^2 F_{k+1, n-1}^\gamma(c\alpha_{j_1}) F_{k+1, n-1}^\gamma(c\alpha_{j_2}) \\ &= c^2 \lim_n n^\gamma \sum_{k=m_0}^{n-1} k^{-2\gamma} F_{k+1, n-1}^\gamma(c\alpha_{j_1}) F_{k+1, n-1}^\gamma(c\alpha_{j_2}) \\ &= \frac{c^2}{c(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}}}. \end{aligned}$$

Case (ii): Take $\nu < \gamma$, $j_1, j_2 \in \{1, \dots, N\}$ and $i_1 = i_2 = 2$. For $j_1 = j_2 = 1$, we have

$$\begin{aligned} \lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} &= \lim_n n^\nu \sum_{k=m_0}^{n-1} (\widehat{q}_{k,k} - \widehat{r}_{k-1})^2 F_{k+1, n-1}^\nu(q)^2 \\ &= \lim_n n^\nu \sum_{k=m_0}^{n-1} \widehat{q}_{k,k}^2 F_{k+1, n-1}^\nu(q)^2 \\ &= q^2 \lim_n n^\nu \sum_{k=m_0}^{n-1} k^{-2\nu} F_{k+1, n-1}^\nu(q)^2 = \frac{q}{2}. \end{aligned}$$

(Note that the above second equality is due to the fact that some terms are $o(n^{-\nu})$ and so we can cancel them.) Similarly, for the cases $j_1 \geq 2$, $j_2 \geq 2$ and $j_1 = 1$, $j_2 \geq 2$ and $j_1 \geq 2$, $j_2 = 1$, using Lemma A.4, which allows us to replace in the computation of the desired limit the quantity $G_{k+1, n-1}(c\alpha_j, q)$ by

$$-\frac{c\alpha_j}{q} \left(\frac{F_{k+1, n-1}^\nu(q)}{(n-1)^{\gamma-\nu}} - \frac{F_{k+1, n-1}^\gamma(c\alpha_j)}{k^{\gamma-\nu}} \right),$$

and removing the terms which are $o(n^{-\nu})$, we obtain

$$\begin{aligned} \lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} &= \lim_n n^\nu \sum_{k=m_0}^{n-1} \widehat{q}_{k,k}^2 F_{k+1,n-1}^\nu(q)^2 \\ &= q^2 \lim_n n^\nu \sum_{k=m_0}^{n-1} k^{-2\nu} F_{k+1,n-1}^\nu(q)^2 = \frac{q}{2}. \end{aligned}$$

Case (iii): Take $\nu = \gamma$, $j_1, j_2 \in \{1, \dots, N\}$ and $i_1, i_2 \in \{1, 2\}$ with $i_h \neq 1$ if $j_h = 1$. Recall assumption (16)⁴. Therefore, for $j_1 = j_2 = 1$ and $i_1 = i_2 = 2$, we have

$$\begin{aligned} \lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} &= \lim_n n^\gamma \sum_{k=m_0}^{n-1} (\widehat{q}_{k,k} - \widehat{r}_{k-1})^2 F_{k+1,n-1}^\gamma(q)^2 \\ &= \lim_n (q-c)^2 n^\gamma \sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma}} F_{k+1,n-1}^\gamma(q)^2 = \frac{(q-c)^2}{2q - \mathbb{1}_{\{\gamma=1\}}}. \end{aligned}$$

For $j_1 = 1$, $j_2 \geq 2$, $i_1 = 2$ and $i_2 = 1$, we have

$$\begin{aligned} \lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} &= \\ \lim_n n^\gamma \sum_{k=m_0}^{n-1} (\widehat{q}_{k,k} - \widehat{r}_{k-1}) F_{k+1,n-1}^\gamma(q) \widehat{r}_{k-1} F_{k+1,n-1}^\gamma(c\alpha_{j_2}) &= \\ (q-c)c \lim_n n^\gamma \sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma}} F_{k+1,n-1}^\gamma(q) F_{k+1,n-1}^\gamma(c\alpha_{j_2}) &= \frac{c(q-c)}{c\alpha_{j_2} + q - \mathbb{1}_{\{\gamma=1\}}}. \end{aligned}$$

By symmetry, for $j_1 \geq 2$, $j_2 = 1$, $i_1 = 1$ and $i_2 = 2$, we have

$$d^{j_1(i_1), j_2(i_2)} = \frac{c(q-c)}{c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}}}.$$

For $j_1 = 1$, $j_2 \geq 2$ and $i_1 = i_2 = 2$, we observe that, by means of Lemma A.4, in the computation of the considered limit, we can replace $G_{k+1,n-1}(c\alpha_j, q)$ by

$$q(c\alpha_j - q)^{-1} (F_{k+1,n-1}^\gamma(q) - F_{k+1,n-1}^\gamma(c\alpha_j)),$$

that is we can replace $d_{k,n}^{j(2)}$, with $j \geq 2$ by

$$\widehat{q}_{k,k} \left(\frac{(c\alpha_j - c)F_{k+1,n-1}^\gamma(c\alpha_j) - (q-c)F_{k+1,n-1}^\gamma(q)}{c\alpha_j - q} \right).$$

⁴In the case $q = c\alpha_j$ for some $j \geq 2$ the computations are similar, but we have to consider the other asymptotic expression given in Lemma A.4.

Therefore, we have

$$\begin{aligned}
& \lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} = \\
& \lim_n n^\gamma \sum_{k=m_0}^{n-1} (\widehat{q}_{k,k} - \widehat{r}_{k-1}) F_{k+1,n-1}^\gamma(q) \widehat{q}_{k,k} \left(\frac{(c\alpha_{j_2} - c)F_{k+1,n-1}^\gamma(c\alpha_{j_2}) - (q - c)F_{k+1,n-1}^\gamma(q)}{c\alpha_{j_2} - q} \right) = \\
& \frac{q(q - c)(c\alpha_{j_2} - c)}{c\alpha_{j_2} - q} \lim_n n^\gamma \sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma}} F_{k+1,n-1}^\gamma(c\alpha_{j_2}) F_{k+1,n-1}^\gamma(q) \\
& \quad + \frac{q(q - c)^2}{c\alpha_{j_2} - q} \lim_n n^\gamma \sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma}} F_{k+1,n-1}^\gamma(q)^2 = \\
& \frac{q(q - c)(c + q - \mathbb{1}_{\{\gamma=1\}})}{(c\alpha_{j_2} + q - \mathbb{1}_{\{\gamma=1\}})(2q - \mathbb{1}_{\{\gamma=1\}})}.
\end{aligned}$$

By symmetry, for $j_1 \geq 2$, $j_2 = 1$ and $i_1 = i_2 = 2$, we get

$$d^{j_1(i_1), j_2(i_2)} = \frac{q(q - c)(c + q - \mathbb{1}_{\{\gamma=1\}})}{(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})(2q - \mathbb{1}_{\{\gamma=1\}})}.$$

Similarly, for $j_1 \geq 2$, $j_2 \geq 2$, $i_1 = i_2 = 1$, we have

$$\begin{aligned}
\lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} &= \lim_n n^\gamma \sum_{k=m_0}^{n-1} \widehat{r}_{k-1}^2 F_{k+1,n-1}^\gamma(c\alpha_{j_1}) F_{k+1,n-1}^\gamma(c\alpha_{j_2}) \\
&= \frac{c^2}{c(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}}}.
\end{aligned}$$

For $j_1 \geq 2$, $j_2 \geq 2$, $i_1 = 1$ and $i_2 = 2$, we have

$$\begin{aligned}
& \lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} = \\
& \lim_n n^\gamma \sum_{k=m_0}^{n-1} \widehat{r}_{k-1} F_{k+1,n-1}^\gamma(c\alpha_{j_1}) \widehat{q}_{k,k} \left(\frac{(c\alpha_{j_2} - c)F_{k+1,n-1}^\gamma(c\alpha_{j_2}) - (q - c)F_{k+1,n-1}^\gamma(q)}{c\alpha_{j_2} - q} \right) = \\
& \frac{cq(c\alpha_{j_1} + c - \mathbb{1}_{\{\gamma=1\}})}{(c\alpha_{j_1} + c\alpha_{j_2} - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})}.
\end{aligned}$$

By symmetry, for $j_1 \geq 2$, $j_2 \geq 2$, $i_1 = 2$ and $i_2 = 1$, we get

$$d^{j_1(i_1), j_2(i_2)} = \frac{cq(c\alpha_{j_2} + c - \mathbb{1}_{\{\gamma=1\}})}{(c\alpha_{j_1} + c\alpha_{j_2} - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_2} + q - \mathbb{1}_{\{\gamma=1\}})}.$$

Finally, for $j_1 \geq 2$, $j_2 \geq 2$ and $i_1 = i_2 = 2$, we have

$$\begin{aligned} & \lim_n t_n (J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} = \\ & \lim_n n^\gamma \sum_{k=m_0}^{n-1} \widehat{q}_{k,k}^2 \left(\frac{(c\alpha_{j_1} - c)F_{k+1,n-1}^\gamma(c\alpha_{j_1}) - (q-c)F_{k+1,n-1}^\gamma(q)}{c\alpha_{j_1} - q} \right) \\ & \quad \left(\frac{(c\alpha_{j_2} - c)F_{k+1,n-1}^\gamma(c\alpha_{j_2}) - (q-c)F_{k+1,n-1}^\gamma(q)}{c\alpha_{j_2} - q} \right) = \\ & q^2 \frac{c^3(\alpha_{j_1} + \alpha_{j_2}) + 2c^2q(\alpha_{j_1}\alpha_{j_2} + 1) - \mathbb{1}_{\{\gamma=1\}}c^2(\alpha_{j_1}\alpha_{j_2} + \alpha_{j_1} + \alpha_{j_2} + 2)}{(2q - \mathbb{1}_{\{\gamma=1\}})(c(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_2} + q - \mathbb{1}_{\{\gamma=1\}})} \\ & \quad + q^2 \frac{c(q - \mathbb{1}_{\{\gamma=1\}})^2(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}}(2c + q - 1)(q - 1)}{(2q - \mathbb{1}_{\{\gamma=1\}})(c(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_2} + q - \mathbb{1}_{\{\gamma=1\}})}. \end{aligned}$$

Case (iv): Take $\gamma < \nu$, $j_1, j_2 \in \{2, \dots, N\}$ and $i_1 = i_2 = 1$. We have

$$\begin{aligned} \lim_n t_n (J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} &= \lim_n n^\gamma \sum_{k=m_0}^{n-1} \widehat{r}_{k-1}^2 F_{k+1,n-1}^\gamma(c\alpha_{j_1}) F_{k+1,n-1}^\gamma(c\alpha_{j_2}) \\ &= \lim_n n^\gamma \sum_{k=m_0}^{n-1} k^{-2\gamma} F_{k+1,n-1}^\gamma(c\alpha_{j_1}) F_{k+1,n-1}^\gamma(c\alpha_{j_2}) = \frac{c}{\alpha_{j_1} + \alpha_{j_2}}. \end{aligned}$$

The difference with the computations in the case $\nu < \gamma$ concerns only the fact that here it is not possible that $\gamma = 1$ since $\gamma < \nu \leq 1$.

Case (v): Take $\gamma < \nu$, $j_1, j_2 = 1$ and $i_1 = i_2 = 2$. We have

$$\begin{aligned} \lim_n t_n (J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} &= \lim_n n^{2\gamma-\nu} \sum_{k=m_0}^{n-1} (\widehat{q}_{k,k} - \widehat{r}_{k-1})^2 F_{k+1,n-1}^\nu(q)^2 \\ &= \lim_n n^{2\gamma-\nu} \sum_{k=m_0}^{n-1} \widehat{r}_{k-1}^2 F_{k+1,n-1}^\nu(q)^2 = \frac{c^2}{2q - \mathbb{1}_{\{\nu=1\}}(2\gamma - 1)}. \end{aligned}$$

(Note that the above second equality is due to the fact that some terms are $o(n^{-(2\gamma-\nu)})$ and so we can cancel them.)

Case (vi): Take $\gamma < \nu$, $j_1, j_2 \in \{2, \dots, N\}$ and $i_1 = i_2 = 2$. Using Lemma A.4, which allows us to replace in the computation of the desired limit the quantity $G_{k+1,n-1}(c\alpha_j, q)$ by

$$\frac{q}{c\alpha_j} \left(\frac{F_{k+1,n-1}^\nu(q)}{(n-1)^{\nu-\gamma}} - \frac{F_{k+1,n-1}^\gamma(c\alpha_j)}{k^{\nu-\gamma}} \right),$$

and removing the terms which are $o(n^{-\nu})$, we have

$$\begin{aligned}
& \lim_n t_n (J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} = \\
& \lim_n n^\nu \sum_{k=m_0}^{n-1} (\lambda_{j_1} \widehat{r}_{k-1} G_{k+1,n-1}(c\alpha_{j_1}, q) + \widehat{q}_{k,k} F_{k+1,n-1}^\nu(q)) (\lambda_{j_2} \widehat{r}_{k-1} G_{k+1,n-1}(c\alpha_{j_2}, q) + \widehat{q}_{k,k} F_{k+1,n-1}^\nu(q)) = \\
& \frac{\lambda_{j_1} \lambda_{j_2}}{\alpha_{j_1} \alpha_{j_2}} q^2 \lim_n n^{2\gamma-\nu} \sum_{k=m_0}^{n-1} k^{-2\gamma} F_{k+1,n-1}^\nu(q)^2 + \left(\frac{\lambda_{j_1}}{\alpha_{j_1}} + \frac{\lambda_{j_2}}{\alpha_{j_2}} \right) q^2 \lim_n n^\gamma \sum_{k=m_0}^{n-1} k^{-(\gamma+\nu)} F_{k+1,n-1}^\nu(q)^2 \\
& + q^2 \lim_n n^\nu \sum_{k=m_0}^{n-1} k^{-2\nu} F_{k+1,n-1}^\nu(q)^2 = \\
& \left(\frac{\lambda_{j_1} \lambda_{j_2}}{\alpha_{j_1} \alpha_{j_2}} \right) \frac{q^2}{2q - \mathbb{1}_{\{\nu=1\}}(2\gamma-1)} + \left(\frac{\lambda_{j_1}}{\alpha_{j_1}} + \frac{\lambda_{j_2}}{\alpha_{j_2}} \right) \frac{q^2}{2q - \mathbb{1}_{\{\nu=1\}}\gamma} + \frac{q^2}{2q - \mathbb{1}_{\{\nu=1\}}}.
\end{aligned}$$

APPENDIX B. STABLE CONVERGENCE AND ITS VARIANTS

This brief appendix contains some basic definitions and results concerning stable convergence and its variants. For more details, we refer the reader to [13, 15, 17, 25] and the references therein.

Let (Ω, \mathcal{A}, P) be a probability space, and let S be a Polish space, endowed with its Borel σ -field. A *kernel* on S , or a random probability measure on S , is a collection $K = \{K(\omega) : \omega \in \Omega\}$ of probability measures on the Borel σ -field of S such that, for each bounded Borel real function f on S , the map

$$\omega \mapsto Kf(\omega) = \int f(x) K(\omega)(dx)$$

is \mathcal{A} -measurable. Given a sub- σ -field \mathcal{H} of \mathcal{A} , a kernel K is said \mathcal{H} -measurable if all the above random variables Kf are \mathcal{H} -measurable.

On (Ω, \mathcal{A}, P) , let $(Y_n)_n$ be a sequence of S -valued random variables, let \mathcal{H} be a sub- σ -field of \mathcal{A} , and let K be a \mathcal{H} -measurable kernel on S . Then we say that Y_n converges \mathcal{H} -stably to K , and we write $Y_n \longrightarrow K$ \mathcal{H} -stably, if

$$P(Y_n \in \cdot | H) \xrightarrow{\text{weakly}} E[K(\cdot) | H] \quad \text{for all } H \in \mathcal{H} \text{ with } P(H) > 0,$$

where $K(\cdot)$ denotes the random variable defined, for each Borel set B of S , as $\omega \mapsto KI_B(\omega) = K(\omega)(B)$. In the case when $\mathcal{H} = \mathcal{A}$, we simply say that Y_n converges *stably* to K and we write $Y_n \longrightarrow K$ stably. Clearly, if $Y_n \longrightarrow K$ \mathcal{H} -stably, then Y_n converges in distribution to the probability distribution $E[K(\cdot)]$. Moreover, the \mathcal{H} -stable convergence of Y_n to K can be stated in terms of the following convergence of conditional expectations:

$$(95) \quad E[f(Y_n) | \mathcal{H}] \xrightarrow{\sigma(L^1, L^\infty)} Kf$$

for each bounded continuous real function f on S .

In [17] the notion of \mathcal{H} -stable convergence is firstly generalized in a natural way replacing in (95) the single sub- σ -field \mathcal{H} by a collection $\mathcal{G} = (\mathcal{G}_n)_n$ (called conditioning system) of sub- σ -fields of \mathcal{A} and then it is strengthened by substituting the convergence in $\sigma(L^1, L^\infty)$ by the one in probability (i.e. in L^1 , since f is bounded). Hence, according to [17], we say that Y_n converges to K *stably in the strong sense*, with respect to $\mathcal{G} = (\mathcal{G}_n)_n$, if

$$(96) \quad E[f(Y_n) | \mathcal{G}_n] \xrightarrow{P} Kf$$

for each bounded continuous real function f on S .

Finally, a strengthening of the stable convergence in the strong sense can be naturally obtained if in (96) we replace the convergence in probability by the almost sure convergence: given a conditioning system $\mathcal{G} = (\mathcal{G}_n)_n$, we say that Y_n converges to K in the sense of the *almost sure conditional convergence*, with respect to \mathcal{G} , if

$$E[f(Y_n) | \mathcal{G}_n] \xrightarrow{a.s.} Kf$$

for each bounded continuous real function f on S . The almost sure conditional convergence has been introduced in [13] and, subsequently, employed by others in the urn model literature.

We now conclude this section recalling two convergence results that we need in our proofs.

From [19, Proposition 3.1], we can get the following result.

Theorem B.1. *Let $(\mathbf{T}_{k,n})_{1 \leq k \leq k_n, n \geq 1}$ be a triangular array of d -dimensional real random vectors, such that, for each fixed n , the finite sequence $(\mathbf{T}_{k,n})_{1 \leq k \leq k_n}$ is a martingale difference array with respect to a given filtration $(\mathcal{G}_{k,n})_{k \geq 0}$. Moreover, let $(t_n)_n$ be a sequence of real numbers and assume that the following conditions hold:*

- (c1) $\mathcal{G}_{k,n} \subseteq \mathcal{G}_{k,n+1}$ for each n and $1 \leq k \leq k_n$;
- (c2) $\sum_{k=1}^{k_n} (t_n \mathbf{T}_{k,n})(t_n \mathbf{T}_{k,n})^\top = t_n^2 \sum_{k=1}^{k_n} \mathbf{T}_{k,n} \mathbf{T}_{k,n}^\top \xrightarrow{P} \Sigma$, where Σ is a random positive semidefinite matrix;
- (c3) $\sup_{1 \leq k \leq k_n} |t_n \mathbf{T}_{k,n}| \xrightarrow{L^1} 0$.

Then $t_n \sum_{k=1}^{k_n} \mathbf{T}_{k,n}$ converges stably to the Gaussian kernel $\mathcal{N}(\mathbf{0}, \Sigma)$.

The following result combines together a stable convergence and a stable convergence in the strong sense.

Theorem B.2. [7, Lemma 1] *Suppose that C_n and D_n are S -valued random variables, that M and N are kernels on S , and that $\mathcal{G} = (\mathcal{G}_n)_n$ is a filtration satisfying for all n*

$$\sigma(C_n) \subseteq \mathcal{G}_n \quad \text{and} \quad \sigma(D_n) \subseteq \sigma(\bigcup_n \mathcal{G}_n)$$

If C_n stably converges to M and D_n converges to N stably in the strong sense, with respect to \mathcal{G} , then

$$(C_n, D_n) \longrightarrow M \otimes N \quad \text{stably.}$$

(Here, $M \otimes N$ is the kernel on $S \times S$ such that $(M \otimes N)(\omega) = M(\omega) \otimes N(\omega)$ for all ω .)

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