THE PHILLIP ISLAND PENGUIN PARADE (A MATHEMATICAL TREATMENT)

1. Introduction

The goal of this paper is to provide a simple, but rigorous, mathematical model which describes the formation of groups of penguins on the shore at sunset.

The results that we obtain are the following. First of all, we provide the construction of a mathematical model to describe the formation of groups of penguins on the shore and their march towards their burrows; this model is based on systems of ordinary differential equations, with a number of degree of freedom which is variable in time (we show that the model admits a unique solution, which needs to be appropriately defined). Then, we give some rigorous mathematical results which provide sufficient conditions for a group of penguins to reach the burrows. In addition, we provide some numerical simulations which show that the mathematical model well predicts, at least at a qualitative level, the formation of clusters of penguins and their march towards the burrows; these simulations are easily implemented by images and videos.

It would be desirable to have empirical data about the formation of penguins clusters on the shore and their movements, in order to compare and adapt the model to experimental data and possibly give a quantitative description of concrete scenarios.

The methodology used is based on direct observations on site, strict interactions with experts in biology and penguin ecology, mathematical formulation of the problem and rigorous deductive arguments, and numerical simulations.

In this introduction, we will describe the elements which lead to the construction of the model, presenting its basic features and also its limitations. Given the interdisciplinary flavor of the subject, it is not possible to completely split the biological discussion from the mathematical formulation, but we can mention that the main mathematical equation is given in formula (1.1). Before (1.1), the main information coming from live observations are presented. After (1.1), the mathematical quantities involved in the equation are discussed and elucidated. The existence and uniqueness theory for equation (1.1) is presented in Section 2. Some rigorous mathematical results about equation (1.1) are given in Section 3. Roughly speaking, these are results which give sufficient conditions on the initial data of the system and on the external environment for the successful homecoming of the penguins, and their precise formulation requires the development of the mathematical framework in (1.1).

In Section 4 we present numerics, images and videos which support our intuition and set the mathematical model of (1.1) into a concrete framework that is easily comparable with the real-world phenomenon.

Prior to this, we think it is important to describe our experience of the penguins parade in Phillip Island, both to allow the reader who is not familiar with the event to concretely take part in it, and to describe some peculiar environmental aspects which are crucial to understand our description (for instance, the weather in Phillip Island is completely different from the Antarctic one, so many of our considerations are meant to be limited to this

particular habitat) – also, our personal experience in this bio-mathematical adventure is a crucial point, in our opinion, to describe how scientific curiosity can trigger academic activities.

1.1. **Description of the penguins parade.** An extraordinary event in the state of Victoria, Australia, consists in the march of the little penguins (whose scientific name is *Eudyptula minor*) who live in Phillip Island. At sunset, when it gets too dark for the little penguins to hunt their food in the sea, they come out to return to their homes (which are small cavities in the terrain, that are located at some dozens of meters from the water edge). What follows is the mathematical description that came out of the observations on site at Phillip Island, enriched by the scientific discussions we later had with penguin ecologists.

By watching the penguins parade in Phillip Island, it seemed to us that some simple features appeared in the very unusual pattern followed by the little penguins. First of all, they have the strong tendency to gather together in a sufficiently large number before starting their march home. They have the tendency to march on a straight line, compactly arranged in a cluster, or group. To make this group, they move back and forth, waiting for other fellows or even going back to the sea if no other mate is around.

If a little penguin remains isolated, some parameters in the model proposed may lead to a complete stop of the individual. More precisely, in the model that we propose, there is a term which makes the velocity vanish. In practice, this interruption in the penguin's movement is not due to physical impediments, but rather to the fact that there is no other penguin in a sufficiently small neighbourhood: in this sense, at a mathematical level, a quantified version of the notion of "isolation" leads the penguin to stop.

Of course, from the point of view of ethology, it would be desirable to have further non-invasive tests to measure how the situation that we describe is felt by the penguin at an emotional level (at the moment, we are not aware of experiments like this in the literature). Also, it would be highly desirable to have some precise experiments to determine how many penguins do not manage to return to their burrows within a certain time after dusk and stay either in the water or in the vicinity of the shore.

On one hand, in our opinion, it is likely that rigorous experiments on site will demonstrate that the phenomenon for which an isolated penguin stops is rather uncommon, but not completely exceptional, in nature. On the other hand, our model is general enough to take into account the possibility that a penguin stops its march, and, at a quantitative level, we emphasized this feature in the pictures of Section 4 to make the situation visible.

The reader who does not want to take into account the stopping function in the model can just set this function to be identically equal to 1 (the mathematical formulation of this remark will be given after formula (1.9)). In this particular case, our model will still exhibit the formation of groups of penguins moving together.

Though no experimental test has been run on the emotive feelings of penguins during their homecoming, in the parade that we have seen live it indeed happened that one little penguin remained isolated from the others: even though (s)he was absolutely fit and no concrete obstacle was obstructing the motion, (s)he got completely stuck for half an hour and the staff of the Nature Park had to go and provide assistance. We stress again that the fact that the penguin stopped did not seem to be caused by any physical impediment (as confirmed to us by the Ranger on site), since no extreme environmental condition was occurring, the

animal was not underweight, and was able to come out of the water and move effortlessly on the shore autonomously for about 15 meters, before suddenly stopping.

For a short video (courtesy of Phillip Island Nature Parks) of the little penguins parade, in which the formation of groups is rather evident, see e.g. the file `Penguins1.MOV`, available at the webpage
`https://youtu.be/x488k4n3ip8`

The simple features listed above are likely to be a consequence of the morphological structure of the little penguins and of the natural environment. As a matter of fact, little penguins are a marine-terrestrial species. They are highly efficient swimmers but possess a rather inefficient form of locomotion on land (indeed, flightless penguins, as the ones in Phillip Island, waddle, more than walk). At dusk, about 80 minutes after sunset according to the data collected in [22], little penguins return ashore after their fishing activity in the sea. Since their bipedal locomotion is slow and rather goofy (at least from the human subjective perception, but also in comparison with the velocity or agility that is well known to be typical of predators in nature), and the easily recognizable countershading of the penguins is likely to make them visible to predators, the transition between the marine and terrestrial environment may be particularly stressful for the penguins (see [17]) and this fact is probably related to the formation of penguins groups (see e.g. [7]). Thus, in our opinion, the rules that we have listed may be seen as the outcome of the difficulty of the little penguins to perform their transition from a more favourable environment to an habitat in which their morphology turns out to be suboptimal.

At the moment, there seems to be no complete experimental evidence measuring the subjective perceptions of the penguins with respect to the surrounding environments. Nevertheless, given the swimming ability of the penguins and the environmental conditions, one may well conjecture that an area of high potential danger for a penguin is the one adjacent to the shore-line, since this is a habitat which provides little or no shelter, and it is also in an area of reduced visibility. As a matter of fact, to protect the penguins in this critical area next to the water edge, the Rangers in Phillip Island implemented a control on the presence of the foxes in the proximity of the shore, with the aim of limiting the number of possible predators.

1.2. **Comparison with the existing literature.** We observe that, to the best of our knowledge, there is still no specific mathematical attempt to describe in a concise way the penguins parade. The mathematical literature of penguins has mostly focused on the description of the heat flow in the penguins feathers (see [9]), on the numerical analysis to mark animals for later identification (see [23]), on the statistics of the Magellanic penguins at sea (see [24]), on the hunting strategies of fishing penguins (see [13]), and on the isoperimetric arrangement of the Antarctic penguins to prevent the heat dispersion caused by the polar wind and on the crystal structures and solitary waves produced by such arrangements (see [12] and [20]). We remark that the climatic situation in Phillip Island is rather different from the Antarctic one and, given the very mild temperatures of the area, we do not think that heat considerations should affect too much the behaviour and the moving strategies of the Victorian little penguins and their tendency to cluster seems more likely to be a defensive strategy against possible predators.

Though no mathematical formulation of the little penguins parade has been given till now, a series of experimental analysis has been recently performed on the specific environment of Phillip Island. We recall, in particular, [7], in which the association of the little penguins in groups is described, by collecting data spanning over several years, [5], in which there is a description of the effect of fog on the orientation of the little penguins (which may actually not come back home in conditions of poor visibility), [18] and [21], which presents a data analysis to show the fractal structure in space and time for the foraging of the little penguins, also in relation to Lévy flights and fractional Brownian motions.

For an exhaustive list of publications focused on the behaviour of the little penguins of Phillip Island, we refer to the web page

https://www.penguins.org.au/conservation/research/publications/

This pages contains more than 160 publications related to the environment of Phillip Island, with special emphasis on the biology of little penguins.

We recall that there is also a wide literature from the point of view of biology and ethology focused on collective mathematical behaviours, also in terms of formation of groups and hierarchies (see e.g. [3] [19] and [10]).

The mathematical literature studying the collective behaviour of animal groups is also rather broad: we mention in particular [1], which studied the local rules of interaction of individual birds in airborne flocks, [6], which analyzed the self-organization from a microscopic to a macroscopic scale, [4], which took into account movements with a speed depending on an additional variable, and [15] for different models on opinion formation within an interacting group.

We remark that our model is specifically tailored on the Phillip Island penguins : for instance, other colonies of penguins, such as those in St Kilda, exhibit behaviours different from those in Phillip Island, due to the different environmental conditions, see e.g. the scientific report by [14] for additional information on the penguins colony on the St Kilda breakwater.

1.3. **Mathematical formulation.**

In this section we provide a mathematical description of the penguins parade, which was described in Section 1.1. The idea for providing an equation for this parade is to prescribe that the velocity of a group of penguins which travels in line is influenced by the natural environment and by the position of the other visible groups. Anytime a group is formed, the equation needs to be modified to encode the formation of this new structure. The main mathematical notation is described in Table 1.

In further details, to translate into a mathematical framework the simple observations on the penguins behaviour that we listed in Subsection 1.1, we propose the following equation:

$$\dot{p}_i(t) = \mathfrak{P}_i\big(p(t), w(t); t\big)\left(\varepsilon + \mathcal{V}_i\big(p(t), w(t); t\big)\right) + f\big(p_i(t), t\big). \tag{1.1}$$

The variable $t \geqslant 0$ represents time and $p(t)$ is a vector valued function of time, that takes into account the positions of the different groups of penguins. Roughly speaking, at time t, there are $n(t)$ groups of penguins, therefore $p(t)$ is an array with $n(t)$ components, and so we will write

$$p(t) = \big(p_1(t), \ldots, p_{n(t)}(t)\big). \tag{1.2}$$

We stress that $n(t)$ may vary in time (in fact, it will be taken to be piecewise constant), hence the spatial dimension of the image of p is also a function of time. For any $i \in \{1, \ldots, n(t)\}$,

$p_i(t)$	one-dimensional position of the ith group of penguins at time t
$w_i(t)$	number of penguins belonging to the ith group of penguins at time t
f	function describing the environment (sea, shore, presence of predators, etc.)
$\mathfrak{P}_i$	stopping function
ε	speed of a solitary penguin in a neutral condition (may be zero)
$\mathcal{V}_i$	strategic speed of the ith group of penguins (depending on the position of the penguins, on the size of the group and on time)
v	speed of "large" penguins groups
m_i	influence of the "visible" penguins ahead and behind on the speed of the ith group
$\mathfrak{s}$	eye-sight of the penguins

TABLE 1. Notation.

the ith group of penguins contains a number of penguins denoted by $w_i(t)$ (thus, the number of penguins belonging to each group is also a function of time).

In further detail, the following notation is used. The function $n : [0, +\infty) \to \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$, is piecewise constant and nonincreasing, namely there exist a (possibly finite) sequence $0 = t_0 < t_1 < \cdots < t_j < \ldots$ and integers $n_1 > \cdots > n_j > \ldots$ such that

$$n(t) = n_j \in \mathbb{N}_0 \text{ for any } t \in (t_{j-1}, t_j). \tag{1.3}$$

In this model, for simplicity, the spatial occupancy of a cluster of penguins coincide with that of a single penguin: of course, in reality, there is a small repulsion playing among the penguins, which cannot stay too close to one another. This additional complication may also be taken into account in our model, by enlarging the spatial size of the cluster in dependence of the numerousness of the penguins in the group. In any case, for practical purposes, we think it is not too inaccurate to identify a group of penguins with just a single element, since the scale at which the parade occurs (several dozens of meters) is much larger than the size of a single penguin (little penguins are only about 30 cm. tall).

We also consider the array $w(t) = \big(w_1(t), \ldots, w_{n(t)}(t)\big)$. We assume that w_i is piecewise constant, namely, $w_i(t) = \bar{w}_{i,j}$ for any $t \in (t_{j-1}, t_j)$, for some $\bar{w}_{i,j} \in \mathbb{N}_0$, namely the number of little penguins in each group remains constant, till the next penguins join the group at time t_j (if, for the sake of simplicity, one wishes to think that initially all the little penguins are separated one from the other, one may also suppose that $w_i(t) = 1$ for all $i \in \{1, \ldots, n_1\}$ and $t \in [0, t_1)$).

By possibly renaming the variables, we suppose that the initial position of the groups is increasing with respect to the index, namely

$$p_1(0) < \cdots < p_{n_1}(0). \tag{1.4}$$

The parameter $\varepsilon \geqslant 0$ represents a drift velocity of the penguins towards their house, which is located at the point $H \in (0, +\infty)$. The parameter ε, from the biological point of view, represents the fact that each penguin, in a neutral situation, has a natural tendency to move towards its burrow. We can also allow $\varepsilon = 0$ in our treatment (namely, the existence and uniqueness theory in Section 2 remains unchanged if $\varepsilon = 0$ and the rigorous results in Section 3 present cases in which they still hold true when $\varepsilon = 0$, compare in particular with assumptions (3.1) and (3.3)).

For concreteness, if $p_i(T) = H$ for some $T \geqslant 0$, we can set $p_i(t) := H$ for all $t \geqslant T$ and remove p_i from the equation of motion – that is, the penguin has safely come back home.

For any $i \in \{1, \ldots, n(t)\}$, the quantity $\mathcal{V}_i\big(p(t), w(t); t\big)$ represents the strategic velocity of the ith group of penguins and it can be considered as a function with domain varying in time

$$\mathcal{V}_i(\cdot, \cdot; t) : \mathbb{R}^{n(t)} \times \mathbb{N}^{n(t)} \to \mathbb{R},$$

i.e.

$$\mathcal{V}_i(\cdot, \cdot; t) : \mathbb{R}^{n_j} \times \mathbb{N}^{n_j} \to \mathbb{R} \quad \text{for any } t \in (t_{j-1}, t_j),$$

and, for any $(p, w) = (p_1, \ldots, p_{n(t)}, w_1, \ldots, w_{n(t)}) \in \mathbb{R}^{n(t)} \times \mathbb{N}^{n(t)}$, it is of the form

$$\mathcal{V}_i(p, w; t) := \Big(1 - \mu\big(w_i\big)\Big) m_i\big(p, w; t\big) + v\mu\big(w_i\big). \tag{1.5}$$

In this setting, for any $(p, w) = (p_1, \ldots, p_{n(t)}, w_1, \ldots, w_{n(t)}) \in \mathbb{R}^{n(t)} \times \mathbb{N}^{n(t)}$, we have that

$$m_i\big(p, w; t\big) := \sum_{j \in \{1, \ldots, n(t)\}} \operatorname{sign}(p_j - p_i)\, w_j\, \mathfrak{s}(|p_i - p_j|), \tag{1.6}$$

where $\mathfrak{s} \in \operatorname{Lip}([0, +\infty))$ is nonnegative and nonincreasing and, as usual, we denoted the "sign function" as

$$\mathbb{R} \ni r \mapsto \operatorname{sign}(r) := \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r = 0, \\ -1 & \text{if } r < 0. \end{cases}$$

Also, for any $\ell \in \mathbb{N}$, we set

$$\mu(\ell) := \begin{cases} 1 & \text{if } \ell \geqslant \kappa, \\ 0 & \text{if } \ell \leqslant \kappa - 1, \end{cases} \tag{1.7}$$

for a fixed $\kappa \in \mathbb{N}$, with $\kappa \geqslant 2$, and $v > \varepsilon$.

In our framework, the meaning of the strategic velocity of the ith group of penguins is the following. When the group of penguins is too small (i.e. it contains less than κ little penguins), then the term involving μ vanishes, thus the strategic velocity reduces to the term given by m_i; this term, in turn, takes into account the position of the other groups of penguins. That is, each penguin is endowed with a "eye-sight" (i.e., the capacity of seeing the other penguins that are "sufficiently close" to them), which is modelled by the function $\mathfrak{s}$ (for instance, if $\mathfrak{s}$ is identically equal to 1, then the penguin has a "perfect eye-sight"; if $\mathfrak{s}(r) = e^{-r^2}$, then the penguin sees close objects much better than distant ones; if $\mathfrak{s}$ is compactly supported, then the penguin does not see too far objects, etc.). Based on the position of the other mates that (s)he sees, the penguin has the tendency to move either forward or backward (the more penguins (s)he sees ahead, the more (s)he is inclined to move forward, the more penguins (s)he sees behind, the more (s)he is inclined to move backward, and nearby penguins weight more than distant ones, due to the monotonicity of $\mathfrak{s}$). This strategic tension coming from the position of the other penguins is encoded by the function m_i.

The eye-sight function can be also considered as a modification of the interaction model based simply on metric distance. Another interesting feature which has been observed in several animal groups (see e.g. [1]), is the so-called "topological interaction" model, in which every agent interacts only with a fixed number of agents, among the ones which are closer. A modification of the function $\mathfrak{s}$ can also take into account this possibility. It is of course

very interesting to investigate by direct observations how much topological, quantitative and metric considerations influence the formation and the movement of little penguin clusters.

When the group of penguins is sufficiently large (i.e. it contains at least κ little penguins), then the term involving μ is equal to 1; in this case, the strategic velocity is v (that is, when the group of penguins is sufficiently rich in population, its strategy is to move forward with cruising speed equal to v).

The function $\mathfrak{P}_i\big(p(t), w(t); t\big)$ describes the case of extreme isolation of the ith individual from the rest of the herd. Here, we take $\overline{d} > \underline{d} > 0$, a nonincreasing function $\varphi \in \mathrm{Lip}(\mathbb{R}, [0,1])$, with $\varphi(r) = 1$ if $r \leqslant \underline{d}$ and $\varphi(r) = 0$ if $r \geqslant \overline{d}$, and, for any $\ell \in \mathbb{N}_0$,

$$\mathfrak{w}(\ell) := \begin{cases} 1 & \text{if } \ell \geqslant 2, \\ 0 & \text{if } \ell = 1, \end{cases} \tag{1.8}$$

and we take as stopping function the function with variable domain

$$\mathfrak{P}_i(\cdot, \cdot; t) : \mathbb{R}^{n(t)} \times \mathbb{N}^{n(t)} \to [0,1],$$

i.e.

$$\mathfrak{P}_i(\cdot, \cdot; t) : \mathbb{R}^{n_j} \times \mathbb{N}^{n_j} \to [0,1] \quad \text{for any } t \in (t_{j-1}, t_j),$$

given, for any $(p, w) = (p_1, \ldots, p_{n(t)}, w_1, \ldots, w_{n(t)}) \in \mathbb{R}^{n(t)} \times \mathbb{N}^{n(t)}$, by

$$\mathfrak{P}_i(p, w; t) := \max\left\{ \mathfrak{w}(w_i), \max_{\substack{j \in \{1, \ldots, n(t)\} \\ j \neq i}} \varphi(|p_i - p_j|) \right\}. \tag{1.9}$$

Here the notation "Lip" stands for bounded and Lipschitz continuous functions.

The case of φ identically equal to 1 can be also comprised in our setting. In this case, also $\mathfrak{P}_i$ is identically one (which corresponds to the case in which the stopping function has no effect).

The stopping function describes the fact that the group may present the tendency to suddenly stop. This happens when the group contains only one element (i.e., $\mathfrak{w}_i = 0$) and the other groups are far apart (at distance larger than $\overline{d}$).

Conversely, if the group contains at least two little penguins, or if there is at least another group sufficiently close (say at distance smaller than $\underline{d}$), then the group is self-confident, namely the function $\mathfrak{P}_i\big(p(t), w(t); t\big)$ is equal to 1 and the total intentional velocity of the group coincides with the strategic velocity.

Interestingly, the stopping function $\mathfrak{P}_i$ may be independent of the eye-sight function $\mathfrak{s}$: namely a little penguin can stop if (s)he feels too much exposed, even if (s)he can see other little penguins (for instance, if $\mathfrak{s}$ is identically equal to 1, the little penguin always sees the other members of the herd, still (s)he can stop if they are too far apart).

The function $f \in \mathrm{Lip}(\mathbb{R} \times [0, +\infty))$ takes into account the environment. For a neutral environment, one has that this term vanishes (where neutral means here that the environment does not favour or penalize the homecoming of the penguins). In practice, it may take into account the ebb and flow of the sea on the foreshore (where the little penguins parade starts), the possible ruggedness of the terrain, the presence of predators, etc. (as a variation, one can consider also a stochastic version of this term). This environment function can take into account several characteristics at the same time. For example, a possible situation that we wish to model is that in which the sea occupies the spatial region $(-\infty, 0)$, producing waves that are periodic in time, with frequency ϖ and amplitude δ; suppose also that the shore is

located in the spatial region $(-\infty, 0)$, presenting a steep hill in the region $(1, 2)$ which can slow down the motion of the penguins, whose burrows are located at the point 4. In this setting, a possible choice of the environment function f is

$$\mathbb{R} \times [0, +\infty) \ni (p, t) \longmapsto f(p, t) = \delta \sin(\varpi t + \phi)\, \chi_{(-\infty, 0)}(p) - h\, \chi_{(1,2)}(p).$$

In this notation $h > 0$ is a constant that takes into account "how steep" the hill located in the region $(1, 2)$ is, $\phi \in \mathbb{R}$ is an initial phase of the wave in the sea, and χ_E is the characteristic function of a set E, namely

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Given the interpretations above, equation (1.1) tries to comprise the pattern that we described in words and to set the scheme of motion of the little penguins into a mathematical framework.

1.4. Preliminary presentation of the mathematical results.
In this paper, three main mathematical results will be presented. First of all, in Section 2, we provide an existence and uniqueness theory for the solutions of equation (1.1).

From the mathematical viewpoint, we remark that (1.1) does not fall into the classical framework of the standard Cauchy initial value problem for ordinary differential equations (compare e.g. with formula (2.3) and Theorem 2.1 in [2]), since the right hand side of the equation is not Lipschitz continuous (and, in fact, it is not even continuous). This mathematical complication is indeed the counterpart of the real motion of the little penguins in the parade, which have the tendency to change their speed rather abruptly to maintain contact with the other elements of the herd. That is, on view, it does not seem unreasonable to model, as a simplification, the speed of the penguin as a discontinuous function, to take into account the sudden modifications of the waddling according to the position of the other penguins, with the conclusive aim of gathering together a sufficient number of penguins in a group which eventually will march concurrently in the direction of their burrows.

Then, in Section 3 we provide two rigorous results which guarantee suitable conditions under which all the penguins, or some of them, safely return to their burrows. In Theorem 3.1 we establish that if the sum of the drift velocity and the environmental function is strictly positive and if there is a time (which can be the initial time or a subsequent one) for which the group at the end of the line consists of at least two penguins, then all the penguins reach their burrows in a finite time, which can be explicitly estimated.

Also, in Theorem 3.2 we prove that if the sum of the drift and cruise velocities and of the environmental function is strictly positive and if there is a time for which one of the penguins group is sufficiently numerous, then all the penguins of this group and of the groups ahead safely return home in a finite time, which can be explicitly estimated.

Rigorous statements and proofs will be given in Sections 2 and 3.

1.5. Detailed organization of the paper.
The mathematical treatment of equation (1.1) that we provide in this paper is the following.

In Section 2, we provide a notion of solution for which (1.1) is uniquely solvable in the appropriate setting. This notion of solution will be obtained by a "stop-and-go" procedure, which is compatible with the idea that when two (or more) groups of penguins meet, they form a new, bigger group which will move coherently in the sequel of the march.

In Section 3, we discuss a couple of concrete examples in which the penguins are able to safely return home: namely, we show that there are "nice" conditions in which the strategy of the penguins allows a successful homecoming.

In Section 4, we present a series of numerical simulations to compare our mathematical model with the real-world experience. This part also contains some figures produced by the numerics.

Several possible structural generalizations of the model proposed are presented in Section 5. Furthermore, the model that we propose can be easily generalized to a multi-dimensional setting, as discussed in Section 6.

The conclusions of our work will be summarized in Section 7.

2. Existence and uniqueness theory for equation (1.1)

We stress that equation (1.1) does not lie within the setting of ordinary differential equations, since the right hand side is not Lipschitz continuous (due to the discontinuity of the functions w and m_i, and in fact the right hand side also involves functions with domain varying in time). As far as we know, the weak formulations of ordinary differential equations as the ones treated by [8] do not take into consideration the setting of equation (1.1), so we briefly discuss here a direct approach to the existence and uniqueness theory for such equation. To this end, and to clarify our direct approach, we present two illustrative examples (see e.g. [11]).

Example 2.1. Setting $x : [0, +\infty) \to \mathbb{R}$, the ordinary differential equation

$$\dot{x}(t) = \begin{cases} -1 & \text{if } x(t) \geqslant 0, \\ 1 & \text{if } x(t) < 0 \end{cases} \tag{2.1}$$

is not well posed. Indeed, taking an initial datum $x(0) < 0$, it will evolve with the formula $x(t) = t + x(0)$ for any $t \in [0, -x(0)]$ till it hits the zero value. At that point, equation (2.1) would prescribe a negative velocity, which becomes contradictory with the positive velocity prescribed to the negative coordinates.

Example 2.2. The ordinary differential equation

$$\dot{x}(t) = \begin{cases} -1 & \text{if } x(t) > 0, \\ 0 & \text{if } x(t) = 0, \\ 1 & \text{if } x(t) < 0 \end{cases} \tag{2.2}$$

is similar to the one in (2.1), in the sense that it does not fit into the standard theory of ordinary differential equations, due to the lack of continuity of the right hand side. But, differently from the one in (2.1), it can be set into an existence and uniqueness theory by a simple "reset" algorithm.

Namely, taking an initial datum $x(0) < 0$, the solution evolves with the formula $x(t) = t + x(0)$ for any $t \in [0, -x(0)]$ till it hits the zero value. At that point, equation (2.2) would prescribe a zero velocity, thus a natural way to continue the solution is to take $x(t) = 0$ for any $t \in [-x(0), +\infty)$ (similarly, in the case of positive initial datum $x(0) > 0$, a natural way to continue the solution is $x(t) = -t + x(0)$ for any $t \in [0, x(0)]$ and $x(t) = 0$ for any $t \in [x(0), +\infty)$). The basic idea for this continuation method is to flow the equation according to the standard Cauchy theory of ordinary differential equations for as long as possible, and then, when the classical theory breaks, "reset" the equation with respect of the

datum at the break time (this method is not universal and indeed it does not work for (2.1), but it produces a natural global solution for (2.2)).

In the light of Example 2.2, we now present a framework in which equation (1.1) possesses a unique solution (in a suitable "reset" setting). To this aim, we first notice that the initial number of groups of penguins is fixed to be equal to n_1 and each group is given by a fixed number of little penguins packed together (that is, the number of little penguins in the ith initial group being equal to $\bar{w}_{i,1}$ and i ranges from 1 to n_1). So, we set $\bar{w}_1 := (\bar{w}_{1,1}, \ldots, \bar{w}_{n_1,1})$ and $\bar{\mathfrak{w}}_{i,1} = \mathfrak{w}(\bar{w}_{i,1})$, where $\mathfrak{w}$ was defined in (1.8). For any $p = (p_1, \ldots, p_{n_1}) \in \mathbb{R}^{n_1}$, let also

$$\mathfrak{P}_{i,1}(p) := \max \left\{ \bar{\mathfrak{w}}_{i,1}, \max_{\substack{j \in \{1,\ldots,n_1\} \\ j \neq i}} \varphi(|p_i - p_j|) \right\}. \tag{2.3}$$

The reader may compare this definition with the one in (1.9). For any $i \in \{1, \ldots, n_1\}$ we also set

$$\bar{\mu}_{i,1} := \mu(\bar{w}_{i,1}),$$

where μ is the function defined in (1.7), and, for any $p = (p_1, \ldots, p_{n_1}) \in \mathbb{R}^{n_1}$,

$$\bar{m}_{i,1}(p) := \sum_{j \in \{1,\ldots,n_1\}} \text{sign}\,(p_j - p_i)\, \bar{w}_{j,1}\, \mathfrak{s}(|p_i - p_j|).$$

This definition has to be compared with (1.6). Recalling (1.4) we also set

$$\mathcal{D}_1 := \{p = (p_1, \ldots, p_{n_1}) \in \mathbb{R}^{n_1} \text{ s.t. } p_1 < \cdots < p_{n_1}\}.$$

We remark that if $p \in \mathcal{D}_1$ then

$$\bar{m}_{i,1}(p) = \sum_{j \in \{i+1,\ldots,n_1\}} \bar{w}_{j,1}\, \mathfrak{s}(|p_i - p_j|) - \sum_{j \in \{1,\ldots,i-1\}} \bar{w}_{j,1}\, \mathfrak{s}(|p_i - p_j|)$$

and therefore

$$\bar{m}_{i,1}(p) \text{ is bounded and Lipschitz for any } p \in \mathcal{D}_1. \tag{2.4}$$

Then, we set

$$\mathcal{V}_{i,1}(p) := (1 - \bar{\mu}_{i,1})\, \bar{m}_{i,1}(p) + v\bar{\mu}_{i,1}.$$

This definition has to be compared with the one in (1.5). Notice that, in view of (2.4), we have that

$$\mathcal{V}_{i,1}(p) \text{ is bounded and Lipschitz for any } p \in \mathcal{D}_1. \tag{2.5}$$

So, we set

$$G_{i,1}(p, t) := \mathfrak{P}_{i,1}(p)\, (\varepsilon + \mathcal{V}_{i,1}(p)) + f(p_i, t).$$

From (2.3) and (2.5), we have that $G_{i,1}$ is bounded and Lipschitz in $\mathcal{D}_1 \times [0, +\infty)$. Consequently, from the global existence and uniqueness of solutions of ordinary differential equations, we have that there exist $t_1 \in (0, +\infty]$ and a solution $p^{(1)}(t) = (p_1^{(1)}(t), \ldots, p_{n_1}^{(1)}(t)) \in \mathcal{D}_1$ of the Cauchy problem

$$\begin{cases} \dot{p}_i^{(1)}(t) = G_{i,1}\big(p^{(1)}(t),\, t\big) & \text{for } t \in (0, t_1), \\ p^{(1)}(0) & \text{given in } \mathcal{D}_1 \end{cases}$$

and

$$p^{(1)}(t_1) \in \partial \mathcal{D}_1, \tag{2.6}$$

see e.g. Theorem 1.4.1 in the book [16].

Notice that, as customary in the mathematical literature, we denoted by ∂ the "topological boundary" of a set. In particular,

$$\partial \mathcal{D}_1 \;=\; \{p = (p_1, \ldots, p_{n_1}) \in \mathbb{R}^{n_1} \text{ s.t. } p_1 \leqslant \ldots \leqslant p_{n_1}$$
$$\text{and there exists } i \in \{1, \ldots, n_1 - 1\} \text{ s.t. } p_i = p_{i+1}\}.$$

The idea for studying the Cauchy problem in our framework is thus that, as long as the trajectory of the system stays in the interior of the domain $\mathcal{D}_1$, the forcing term remains uniformly Lipschitz, thus the flow does not develop any singularity. Hence the trajectory exists and it is defined up to the time (if any) in which it meets the boundary of the domain $\mathcal{D}_1$, that, in the biological framework, corresponds to the situation in which two (or more) penguins meet (i.e., they occupy the same position at the same time). In this case, the standard flow procedure of the ordinary differential equation is stopped, we will merge the joint penguins into a common cluster, and then repeat the argument.

In further detail, the solution of (1.1) will be taken to be $p^{(1)}$ in $[0, t_1)$, that is, we set $p(t) := p^{(1)}(t)$ for any $t \in [0, t_1)$. We also set that $n(t) := n_1$ and $w(t) := (\bar{w}_{1,1}, \ldots, \bar{w}_{n_1,1})$. With this setting, we have that p is a solution of equation (1.1) in the time range $t \in (0, t_1)$ with prescribed initial datum $p(0)$. Condition (2.6) allows us to perform our "stop-and-go" reset procedure as follows: we denote by n_2 the number of distinct points in the set $\{p_1^{(1)}(t_1), \ldots, p_{n_1}^{(1)}(t_1)\}$. Notice that (2.6) says that if t_1 is finite then $n_2 \leqslant n_1 - 1$ (namely, at least two penguins have reached the same position). In this way, the set of points $\{p_1^{(1)}(t_1), \ldots, p_{n_1}^{(1)}(t_1)\}$ can be identified by the set of n_2 distinct points, that we denote by $\{p_1^{(2)}(t_1), \ldots, p_{n_2}^{(2)}(t_1)\}$, with the convention that

$$p_1^{(2)}(t_1) < \cdots < p_{n_2}^{(2)}(t_1).$$

For any $i \in \{1, \ldots, n_2\}$, we also set

$$\bar{w}_{i,2} := \sum_{\substack{j \in \{1, \ldots, n_1\} \\ p_j^{(1)}(t_1) = p_i^{(2)}(t_1)}} \bar{w}_{j,1}.$$

This says that the new group of penguins indexed by i contains all the penguins that have reached that position at time t_1.

Thus, having the "new number of groups", that is n_2, the "new number of little penguins in each group", that is $\bar{w}_2 = (\bar{w}_{1,2}, \ldots, \bar{w}_{n_2,2})$, and the "new initial datum", that is $p^{(2)}(t_1) = \big(p_1^{(2)}(t_1), \ldots, p_{n_2}^{(2)}(t_1)\big)$, we can solve a new differential equation with these new parameters, exactly in the same way as before, and keep iterating this process.

Indeed, recursively, we suppose that we have found $t_1 < t_2 < \cdots < t_k$, $p^{(1)} : [0, t_1] \to \mathbb{R}^{n_1}$, $\ldots$, $p^{(k)} : [0, t_k] \to \mathbb{R}^{n_k}$ and $\bar{w}_1 \in \mathbb{N}_0^{n_1}$, $\ldots$, $\bar{w}_k \in \mathbb{N}_0^{n_k}$ such that, setting

$$p(t) := p^{(j)}(t) \in \mathcal{D}_j, \qquad n(t) := n_j$$
$$\text{and} \qquad w(t) := \bar{w}_j \qquad \text{for } t \in [t_{j-1}, t_j) \text{ and } j \in \{1, \ldots, k\},$$

one has that p solves (1.1) in each interval (t_{j-1}, t_j) for $j \in \{1, \ldots, k\}$, with the "stop condition"

$$p^{(j)}(t_j) \in \partial \mathcal{D}_j,$$

where

$$\mathcal{D}_j := \{p = (p_1, \ldots, p_{n_j}) \in \mathbb{R}^{n_j} \text{ s.t. } p_1 < \cdots < p_{n_j}\}.$$

Then, since $p^{(k)}(t_k) \in \partial \mathcal{D}_k$, if t_k is finite, we find $n_{k+1} \leqslant n_k - 1$ such that the set of points $\{p_1^{(k)}(t_k), \ldots, p_{n_k}^{(k)}(t_k)\}$ coincides with a set of n_{k+1} distinct points, that we denote by $\{p_1^{(k+1)}(t_k), \ldots, p_{n_k}^{(k+1)}(t_k)\}$, with the convention that

$$p_1^{(k+1)}(t_k) < \cdots < p_{n_k}^{(k+1)}(t_k).$$

For any $i \in \{1, \ldots, n_{k+1}\}$, we set

$$\bar{w}_{i,k+1} := \sum_{\substack{j \in \{1, \ldots, n_k\} \\ p_j^{(k)}(t_k) = p_i^{(k+1)}(t_k)}} \bar{w}_{j,k}. \tag{2.7}$$

It is useful to observe that, in light of (2.7),

$$\sum_{i \in \{1, \ldots, n_{k+1}\}} \bar{w}_{i,k+1} = \sum_{i \in \{1, \ldots, n_k\}} \bar{w}_{i,k},$$

which says that the total number of little penguins remains always the same (more precisely, the sum of all the little penguins in all groups is constant in time).

Let also $\bar{\mathfrak{w}}_{i,k+1} = \mathfrak{w}(\bar{w}_{i,k+1})$. Then, for any $i \in \{1, \ldots, n_{k+1}\}$ and any $p = (p_1, \ldots, p_{n_{k+1}}) \in \mathbb{R}^{n_{k+1}}$, we set

$$\mathfrak{P}_{i,k+1}(p) := \max \left\{ \bar{\mathfrak{w}}_{i,k+1}, \max_{\substack{j \in \{1, \ldots, n_{k+1}\} \\ j \neq i}} \varphi(|p_i - p_j|) \right\}.$$

For any $i \in \{1, \ldots, n_{k+1}\}$ we also define

$$\bar{\mu}_{i,k+1} := \mu(\bar{w}_{i,k+1}),$$

where μ is the function defined in (1.7) and, for any $p \in \mathbb{R}^{n_{k+1}}$,

$$\bar{m}_{i,k+1}(p) := \sum_{j \in \{1, \ldots, n_{k+1}\}} \operatorname{sign}(p_j - p_i)\, \bar{w}_{j,k+1}\, \mathfrak{s}(|p_i - p_j|).$$

We notice that $\bar{m}_{i,k+1}(p)$ is bounded and Lipschitz for any $p \in \mathcal{D}_{k+1} := \{p = (p_1, \ldots, p_{n_{k+1}}) \in \mathbb{R}^{n_{k+1}}$ s.t. $p_1 < \cdots < p_{n_{k+1}}\}$.

We also define

$$\mathcal{V}_{i,k+1}(p) := (1 - \bar{\mu}_{i,k+1})\, \bar{m}_{i,k+1}(p) + v\bar{\mu}_{i,k+1}$$

and

$$G_{i,k+1}(p,t) := \mathfrak{P}_{i,k+1}(p)\left(\varepsilon + \mathcal{V}_{i,k+1}(p)\right) + f(p_i, t).$$

In this way, we have that $G_{i,k+1}$ is bounded and Lipschitz in $\mathcal{D}_{k+1} \times [0, +\infty)$ and so we find the next solution $p^{(k+1)}(t) = (p_1^{(k+1)}(t), \ldots, p_{n_{k+1}}^{(k+1)}(t)) \in \mathcal{D}_{k+1}$ in the interval (t_k, t_{k+1}), with $p^{(k+1)}(t_{k+1}) \in \partial \mathcal{D}_{k+1}$, by solving the ordinary differential equation

$$\dot{p}_i^{(k+1)}(t) = G_{i,k+1}\left(p^{(k+1)}(t), t\right).$$

This completes the iteration argument and provides the desired notion of solution for equation (1.1).

3. Examples of safe return home

Here, we provide some sufficient conditions for the penguins to reach their home, located at the point H, which is taken to be "far away with respect to the initial position of the penguins", namely we suppose that

$$H > \max_{i\in\{1,\ldots,n(0)\}} p_i(0),$$

and ε has to be thought sufficiently small. Let us mention that, in the parade that we saw live, one little penguin remained stuck and did not manage to return home – so, giving a mathematical treatment of the case in which the strategy of the penguins turns out to be successful somehow reassured us on the fate of the species.

To give a mathematical framework of the notion of homecoming, we introduce the function

$$[0,+\infty) \ni t \mapsto \mathcal{N}(t) := \sum_{\substack{j\in\{1,\ldots,n(t)\}\\ p_j(t)=H}} w_j(t).$$

In the setting of Subsection 1.3, the function $\mathcal{N}(t)$ represents the number of penguins that have safely returned home at time t.

For counting reasons, we also point out that the total number of penguins is constant and given by

$$\mathcal{M} := \sum_{j\in\{1,\ldots,n(0)\}} w_j(0) = \sum_{j\in\{1,\ldots,n(t)\}} w_j(t),$$

for any $t \geqslant 0$.

The first result that we present says that if at some time the group of penguins that stay further behind gathers into a group of at least two elements, then all the penguins will manage to eventually return home. The mathematical setting goes as follows:

Theorem 3.1. *Let $t_o \geqslant 0$ and assume that*

$$\varepsilon + \inf_{(r,t)\in\mathbb{R}\times[t_o,+\infty)} f(r,t) \geqslant \iota \tag{3.1}$$

for some $\iota > 0$, and

$$w_1(t_o) \geqslant 2. \tag{3.2}$$

Then, there exists $T \in \left[t_o, \, t_o + \frac{H-p_1(t_o)}{\iota}\right]$ such that

$$\mathcal{N}(T) = \mathcal{M}.$$

Proof. We observe that $w_1(t)$ is nondecreasing in t, by (2.7), and therefore (3.2) implies that $w_1(t) \geqslant 2$ for any $t \geqslant t_o$. Consequently, from (1.8), we obtain that $\mathfrak{w}(w_1(t)) = 1$ for any $t \geqslant t_o$. This and (1.9) give that $\mathfrak{P}_1\big(p, w(t); t\big) = 1$ for any $t \geqslant t_o$ and any $p \in \mathbb{R}^{n(t)}$. Accordingly, the equation of motions in (1.1) gives that, for any $t \geqslant t_o$,

$$\dot{p}_1(t) = \varepsilon + \mathcal{V}_1\big(p(t), w(t); t\big) + f\big(p_1(t), t\big) \geqslant \varepsilon + f\big(p_1(t), t\big) \geqslant \iota,$$

by (3.1). That is, for any $j \in \{1, \ldots, n(t)\}$,

$$p_j(t) \geqslant p_1(t) \geqslant \min\{H, \, p_1(t_o) + \iota\,(t - t_o)\},$$

which gives the desired result. $\qquad\square$

A simple variation of Theorem 3.1 says that if, at some time, a group of little penguins reaches a sufficiently large size, then all the penguins in this group (as well as the ones ahead) safely reach their home. The precise statement (whose proof is similar to the one of Theorem 3.1, up to technical modifications, and is therefore omitted) goes as follows:

Theorem 3.2. *Let $t_o \geqslant 0$ and assume that*

$$\varepsilon + v + \inf_{(r,t)\in\mathbb{R}\times[t_o,+\infty)} f(r,t) \geqslant \iota \tag{3.3}$$

for some $\iota > 0$, and

$$w_{j_o}(t_o) \geqslant \kappa,$$

for some $j_o \in \{1,\dots,n(t_o)\}$, where κ is defined in (1.7).

   Then, there exists $T \in \left[t_o,\, t_o + \frac{H - p_{j_o}(t_o)}{\iota}\right]$ such that

$$\mathcal{N}(T) \geqslant \sum_{j\in\{j_o,\dots,n(t_o)\}} w_j(t_o).$$

4. Pictures, videos and numerics

In this section, we present some simple numerical experiments to facilitate the intuition at the base of the model presented in (1.1). These simulations may actually show some of the typical treats of the little penguins parade, such as the oscillations and sudden change of direction, the gathering of the penguins into clusters and the possibility that some elements of the herd remain isolated, either on the land or in the sea.

The possibility that a penguin remains isolated also in the sea may actually occur in the real-world experience, as demonstrated by the last penguin in the herd on the video (courtesy of Phillip Island Nature Parks) named `Penguins2.MOV` available online at the webpage `https://youtu.be/dVk1uYbH_Xc`

In our simulations, for the sake of simplicity, we considered 20 penguins returning to their burrows from the shore – some of the penguins may start their trip from the sea (that occupies the region below level 0 in the simulations) in which waves and currents may affect the movements of the animals. The pictures that we produce (see Section 9) have the time variable on the horizontal axis and the space variable on the vertical axis (with the burrow of the penguins community set at level 4 for definiteness). The pictures are, somehow, self-explanatory. For instance, in Figure 1, we present a case in which, fortunately, all the little penguins manage to safely return home, after having gathered into groups: as a matter of fact, in the first of these pictures all the penguins safely reach home together at the same time (after having rescued the first penguin, who stayed still for a long period due to isolation); on the other hand, the second of these pictures shows that a first group of penguins, which was originated by the animals that were on the land at the initial time, reaches home slightly before the second group of penguins, which was originated by the animals that were in the sea at the initial time (notice also that the motion of the penguins in the sea appears to be affected by waves and currents).

We also observe a different scenario depicted in Figure 4 (with two different functions to represent the currents in the sea): in this situation, a big group of 18 penguins gathers together (collecting also penguins who were initially in the water) and safely returns home. Two penguins remain isolated in the water, and they keep slowly moving towards their final destination (that they eventually reach after a longer time).

Similarly, in Figure 2, almost all the penguins gather into a single group and reach home, while two penguins get together in the sea, they come to the shore and slowly waddle towards their final destination, and one single penguin remains isolated in the water, moved by the currents.

The situation in Figure 3 is slightly different, since the last penguin at the beginning moves towards the others, but (s)he does not manage to join the forming group by the time the other penguins decide to move consistently towards their burrows – so, unfortunately this last penguin, in spite of the initial effort, finally remains in the water.

With simple modifications of the function f, one can also consider the case in which the waves of the sea change with time and their influence may become more (or less) relevant for the swimming of the little penguins: as an example of this feature, see Figures 5 and 6.

In Figures 8 and 7 we give some examples of what happens when varying the parameters that we used in the numerics of the other figures. For example we consider different values of κ, the parameter which encodes when a group of penguins is big enough to be self confident and waddle home without being influenced by the other groups of penguins in sight.

By considering small values of κ we can represent a strong preference of the penguins to go straight towards their homes, instead of first trying to form a large group. This situation is depicted in the second picture of Figure 8 where we see that after a few time the penguins form two distinct small groups and go towards home without trying to form a unique large group together.

On the contrary, considering a large value of κ represents the preference of the penguins to gather in a very large group before starting their march towards home, like in the first picture of Figure 8. This situation could represent for example the penguins being timorous because of the presence of predators.

We think that the case in which one penguin, or a small number of penguins, remain(s) in water even after the return of the main group is worth of further investigation also by means of concrete experiments. One possible scenario is that the penguins in the water will just wait long enough for other penguins to get close to the shore and join them to form a new group; on the other hand, if all the other penguins have already returned, the few ones remained in the water will have to accept the risk of returning home isolated from the other conspecifics and in an unprotected situation, and we think that interesting biological features could be detected in this case.

Finally, we recall that once a group of little penguins is created, then it moves consistently altogether. This is of course a simplifying assumption, and it might happen in reality that one or a few penguins leave a large group after its formation – perhaps because one penguin is slower than the other penguins of the group, perhaps because (s)he gets distracted by other events on the beach, or simply because (s)he feels too exposed being at the side of the group and may prefer to form a new group in which (s)he finds a more central and protected position. We plan to describe this case in detail in a forthcoming project (also possibly in light of morphological and social considerations and taking into account a possible randomness in the system).

The situation in which one little penguin seems to think about leaving an already formed group can be observed in the video (courtesy of Phillip Island Nature Parks) named `Penguins2.MOV` and available online at
`https://youtu.be/dVk1uYbH_Xc`

(see in particular the behaviour of the second penguin from the bottom, i.e. the last penguin of the already formed large cluster).

We point out that all these pictures have been easily obtained by short programs in MATLAB.

We describe here the algorithm of the basic program, with waves of constant size and standard behaviour of all the little penguins. The modified versions (periodic strong waves, tired little penguins and so on) can be easily inferred from it.

We take into account N little penguins, we set their house at $H = 4$ and the sea below the location 0. Strong waves can go beyond the location 0 in some cases, but in the standard program we just consider normal ones. We take a small ε to represent the natural predisposition to go home of the little penguins, and we define a constant $\delta = (N + 1)\varepsilon$ that we need to define the velocity of the little penguins. We define the waves as $\mathrm{WAVE} = \delta \sin(T)$, where T is the array of times. The speed of the animals is related to the one of waves in such a way that it becomes the strongest just when the little penguins form a group that is big enough.

The program starts with a "for" loop that counts all the animals in a range near the chosen little penguin. This "for" loop gives us two values: the indicator of the parameter PAN (short for "panic") and the function W, that represents the number of animals in the same position of the one we are considering. We needed this function since we have seen that when the little penguins form a group that is big enough, they proceed towards their home with a cruise speed that is bigger than it was before. We define this cruise speed as vc (short for "velocity") in the program.

Then we start computing the speed V of the little penguin. If PAN is equal to zero, the little penguin freezes. His velocity is zero if he is on the shore (namely his position is greater or equal than zero), or it is given by the waves if he is in the water. It is worth noting that at each value of time the "for" loop counts the value of PAN, hence a little penguin can leave the stopping condition if he sees some mates and start moving again.

If PAN is not zero we have mainly two cases, according to the fact that a big group is formed or not. If this has happened, namely $W > \frac{N}{2}$, then the little penguin we are considering is in the group, so he goes towards home with a cruise speed vc, possibly modified by the presence of waves. If the group is not formed yet, the animal we are considering is surrounded by some mates, but they are not enough to proceed straight home. His speed is positive or negative, namely he moves forward or backward, in dependence of the amount of little penguins that he has ahead of him or behind him. Its speed is given by:

$$V = \varepsilon + M$$

where M is the number of penguins ahead of him minus the number of animals behind him multiplied by $\frac{\delta}{N}$, and ε has been defined before. As in the other cases, the speed can be modified by the presence of waves if the position is less than zero.

Now that we have computed the speed of the animal, we can obtain his position P after a discrete time interval t by considering $P(k + 1) = P(k) + Vt$.

The last "for" loop is done in order to put in the same position two animals that are closed enough. Then we reset the counting variables PAN, W and M and we restart the loop.

For completeness, we made the source codes of all the programs available on the webpage
`https://www.dropbox.com/sh/odgic3a0ke5qp0q/AABIMaasAcTwZQ3qKRoB--xra?dl=0`

An example of the code is given in Section 8. The simplicity of these programs shows that the model in (1.1) is indeed very simple to implement numerically, still producing sufficiently "realistic" results in terms of cluster formation and cruising speed of the groups. The parameters in the code are chosen as examples, producing simulations that show some features similar to those observed on site and in the videos. From one picture to another, what is varying is the initial conditions and the environment function (minor modifications in the code would allow also to change the number of penguins, their eye-sight, the drift and cruise velocities, the stopping function, and also to take into account multi-dimensional cases).

Also, these pictures can be easily translated into animations. Simple videos that we have obtained by these numerics are available from the webpage
`https://www.youtube.com/playlist?list=PLASZVsOA5ReZgEinpnJFat66lo2kIkWTS`

The source codes of the animations are available online at
`https://www.dropbox.com/s/l1z5riqtc8jzxbs/scatter.txt?dl=0`

5. Discussion on the model proposed: simplifications, generalizations and further directions of investigation

We stress that the model proposed in (1.1) is of course a dramatic simplification of "reality". As often happens in science indeed, several simplifications have been adopted in order to allow a rigorous mathematical treatment and handy numerical computations: nevertheless the model is already rich enough to detect some specific features of the little penguins parade, such as the formation of groups, the oscillatory waddling of the penguins and the possibility of isolated and exposed individuals. Moreover, our model is flexible enough to allow specific distinctions between the single penguins (for instance, with minor modifications, one can take into account the possibility that different penguins have a different eye-sight, that they have a different reaction to isolation, or that they exhibit some specific social behaviour that favours the formation of clusters selected by specific characteristics); similarly, the modeling of the habitat may also encode different possibilities (such as the burrows of the penguins being located in different places), and multi-dimensional models can be also constructed using similar ideas (see Section 6 for details).

We observe that one can replace the quantities $v, \mathfrak{s}, \mu, \kappa, \varphi$ with $v_i, \mathfrak{s}_i, \mu_i, \kappa_i, \varphi_i$ if one wants to customize these features for every group.

Furthermore, natural modifications lead to the possibility that one or a few penguins may leave an already formed group: for instance, rather than forming one single group, the model can still consider the penguins of the cluster as separate elements, each one with its own peculiar behaviour. At the moment, for simplicity, we considered here the basic model in which, once a cluster is made up, it keeps moving without losing any of its elements – we plan to address in a future project in detail the case of groups which may also decrease the number of components, possibly in dependence of random fluctuations or social considerations among the members of the group.

In addition, for simplicity, in this paper we modelled each group to be located at a precise point: though this is not a completely unrealistic assumption (given that the scale of the individual penguin is much smaller than that of the beach), one can also easily modify this feature by locating a cluster in a region comparable to its size.

In future projects, we plan to introduce other more sophisticated models, also taking into account stochastic oscillations and optimization methods, and, in the long run, to use these models in a detailed experimental confrontation taking advantage of the automated monitoring systems which is under development in Phillip Island.

The model that we propose here is also flexible enough to allow quantitative modifications of all the parameters involved. This is quite important, since these parameters may vary due to different conditions of the environment. For instance, the eye-sight of the penguins can be reduced by the fog (see [5]), and by the effect of moonlight and artificial light (see [22]).

Similarly, the number of penguins in each group and the velocity of the herd may vary due to structural changes of the beach: roughly speaking, from the empirical data, penguins typically gather into groups of 5–10 individuals (but we have also observed much larger groups forming on the beach) within 40 second intervals, see [7], but the way these groups are built varies year by year and, for instance, the number of individuals which always gather into the same group changes year by year in strong dependence with the breeding success of the season, see again [7]. Also, tidal phenomena may change the number of little penguins in each group and the velocity of the group, since the change of the beach width alters the perception of the risk of the penguins. For instance, a low tide produces a larger beach, with higher potential risk of predators, thus making the penguins gather in groups of larger size, see [17].

6. Multi-dimensional models

It is interesting to remark that the model in (1.1) can be easily generalized to the multi-dimensional case. That is, for any $i \in \{1, \ldots, n(t)\}$ the ith coordinate p_i can be taken to have image in some $\mathbb{R}^d$. More generally, the dimension of the target space can also vary in time, by allowing for any $i \in \{1, \ldots, n(t)\}$ the ith coordinate p_i to range in some $\mathbb{R}^{d_i(t)}$, with $d_i(t)$ piecewise constant, namely $d_i(t) = d_{i,j} \in \mathbb{N}_0$ for any $t \in (t_{j-1}, t_j)$ (compare with (1.3)).

This modification just causes a small notational complication in (1.2), since each $p_i(t)$ would now be a vector in $\mathbb{R}^{d_i(t)}$ and the array $p(t)$ would now be of dimension $d_1(t) + \cdots + d_{n(t)}$. While we do not indulge here in this generalization, we observe that such mathematical extension can be useful, in practice, to consider the specific location of the burrows and describe for instance the movements of the penguins on the beach (say, a two-dimensional surface) which, as time flows, gather together in a single queue and move in the end on a one-dimensional line.

Of course, the rigorous results in Section 3 need to be structurally modified in higher dimension, since several notions of "proximity" of groups, "direction of march" and "orientation of the eye-sight" can be considered.

7. Conclusions

As a result of our direct observation at Phillip Island and a series of scientific discussions with penguin ecologists, we provide a simple, but rigorous, mathematical model which aims to describe the formation of groups of penguins on the shore at sunset and the return to their burrows.

The model is proved to possess existence and uniqueness of solutions and quantitative results on the homecoming of the penguins are given.

The framework is general enough to show the formation of groups of penguins marching together – as well as the possibility that some penguins remain isolated from the rest of the herd.

The model is also numerically implemented in simple and explicit simulations.

We believe that the method proposed can be suitably compared with the real penguins parade, thus triggering a specific field work on this rather peculiar topic. Indeed, at the moment, a precise collection of data focused on the penguins parade seems to be still missing in the literature, and we think that a mathematical formulation provides the necessary setting for describing specific behaviours in ethology, such as the formation of groups and the possible isolation of penguins, in a rigorous and quantitative way.

Given the simple and quantitative mathematical setting that we introduced here, we also believe that our formulation can be easily modified and improved to capture possible additional details of the penguins march provided by the biological data which may be collected in future specialized field work.

We hope that this problem will also take advantage of statistically sound observations by ecologists, possibly taking into account the speed of the penguins in different environments, the formation of groups of different size, the velocity of each group depending on its size and the links between group formations motivated by homecoming and the social structures of the penguin population.

Due to the lack of available biological theories and precise experimental data, the form of some of the functions considered in this article should just be considered as an example. This applies in particular to the strategic velocity function, to the eye-sight function and to the stopping function, and it would be desirable to run experiments to provide a better quantification of these notions.

It would be also interesting to detect how changes in the environment, such as modified visibility or presence of predators, influence the formation of groups, their size and their speed. In general, we think that it would be very important to provide precise conditions for clustering and to explore these conditions systematically.

In addition, it would be interesting to adapt models of this type to social studies, politics and evolutionary biology, in order to describe and quantify the phenomenon of "front runners" which "wait for the formation of groups of considerable size" in order to "more safely proceed towards their goal".

8. Example of a program list

```
H=4; % Position of the burrow of the penguins community
S=-2; % The sea lies in the region (-\infty,0]. For simplicity we assume
      that penguins start near the shore, that is, the initial position
      of each penguin is at least S
eps=0.005; % Drift velocity of the penguins
vc=0.05; % Cruising speed of a big enough raft of penguins
N=20; % Number of penguins
delta=(N+1)*eps; % This parameter is used to compute the strategic
                              velocity of a penguin.

% These parameters define the time interval
TMAX=(H-S)/(2*eps);
t=0.01;
```

```
T=(0:t:TMAX);
TG=T(1:1,1:12000);

P=zeros(N,length(T));
% The following is the array of the initial positions of the N penguins
P(:,1)=[-1.95 -1.5 -1.05 -0.6 -0.55 -0.4 -0.2 0.1 0.2 0.4 0.8 0.85 0.9
                          1 1.1 1.15 1.2 1.65 3 3.4];

s=(H-S)/3; % The parameter encoding the eye-sight of the penguins
pgot=(H-S)/12; % The parameter representing the stopping function
M=zeros(1,N);
V=M;
PAN=-1;
W=0;

WAVE=sin(T)*delta;  % The "environment function". In this case only
                      waves are taken into account

for k=1 : length(T)-1
  for i=1 : N
    if P(i,k)<H
      for j=1: N  % This cycle checks if the ith penguin is in panic
        if -pgot<P(i,k)-P(j,k) & P(i,k)-P(j,k)<pgot
          PAN=PAN+1;
          if P(i,k)==P(j,k)
            W=W+1;  % This counts the number of penguins in the same
                      position of the ith penguin, that is the dimension
                      of the raft
          end
        end
      end
      if PAN==0  % The ith penguin is stuck because of panic
        if -3.5<P(i,k) & P(i,k)<0
          V(i)=-WAVE(k);
          P(i,k+1)=P(i,k)+V(i)*t;
        else
          P(i,k+1)=P(i,k);
        end
      else
        if W>N/2  % The ith penguin is a member of a big enough raft,
                so it tends to go home, forgetful of the other penguins
          if -3.5<P(i,k) & P(i,k)<0  % The environment can still affect
                                        the movement of the raft
            V(i)=vc-WAVE(k);
          else
            V(i)=vc;  % If the environment does not affect the movement,
                      the penguin moves at cruise velocity
          end
        else  % The raft is not big enough, so the strategic velocity
        of the ith penguin is influenced by the other penguins in sight
```

```
        for j=1 : N
          if -s<P(i,k)-P(j,k) & P(i,k)-P(j,k)<0
            M(i)=M(i)+delta/N;  % Each penguin in sight ahead adds a
                                  delta/N to the strategic velocity of the
                                  ith penguin
          else
            if 0<P(i,k)-P(j,k) & P(i,k)-P(j,k)<s
              M(i)=M(i)-delta/N;  % Each penguin in sight behind
                                    subtracts a delta/N from the strategic
                                    velocity of the ith penguin
            end
          end
        end
        if -3.5<P(i,k) & P(i,k)<0
          V(i)=eps+M(i)-WAVE(k);
        else
          V(i)=eps+M(i);
        end
      end
      P(i,k+1)=P(i,k)+V(i)*t;
    end
  else
    P(i,k+1)=H;
  end
  PAN=-1;
  W=0;
  end
  M=zeros(1,N);
  for i=2 : N
    for j=1 : i-1
      if -0.011<P(j,k+1)-P(i,k+1) & P(j,k+1)-P(i,k+1)<0.011
        P(j,k+1)=P(i,k+1);  % For simplicity, we assume that penguins
                  close enough occupy the same position, forming a raft
                  and moving together
      end
    end
  end
end
Q=P(1:N,1:length(TG));
plot(TG,Q)
```

9. Figures
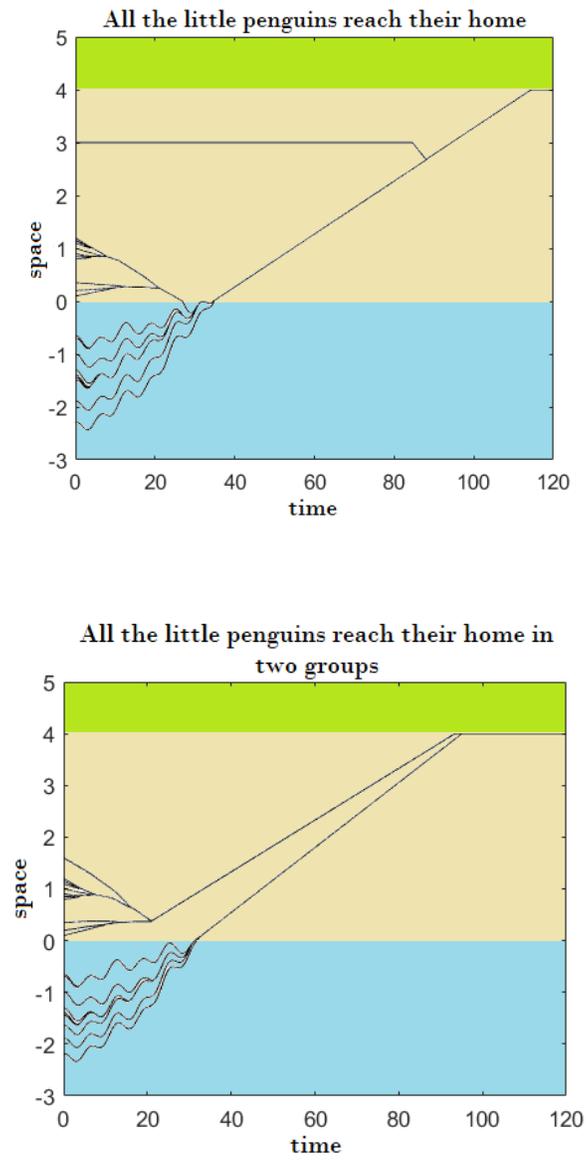
FIGURE 1. All the little penguins safely return home.
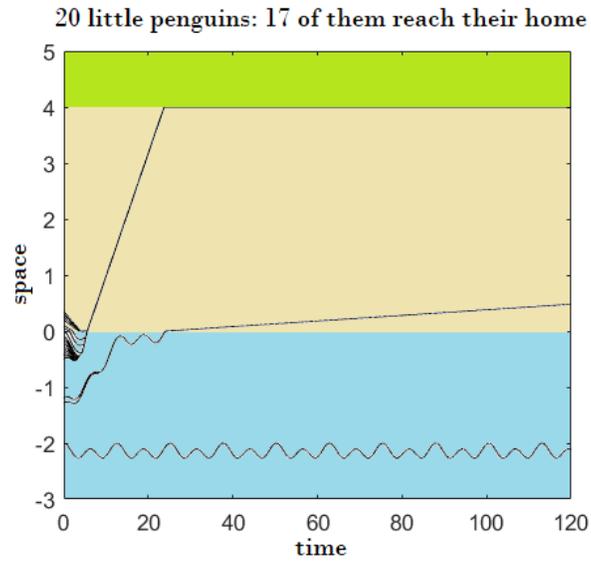
FIGURE 2. One penguin remains in the water.
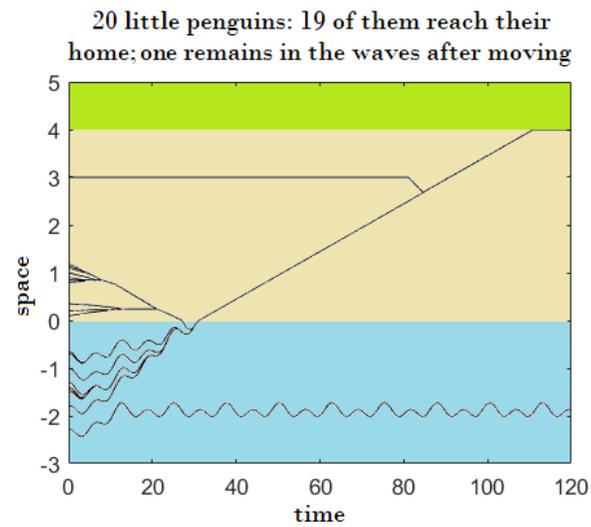
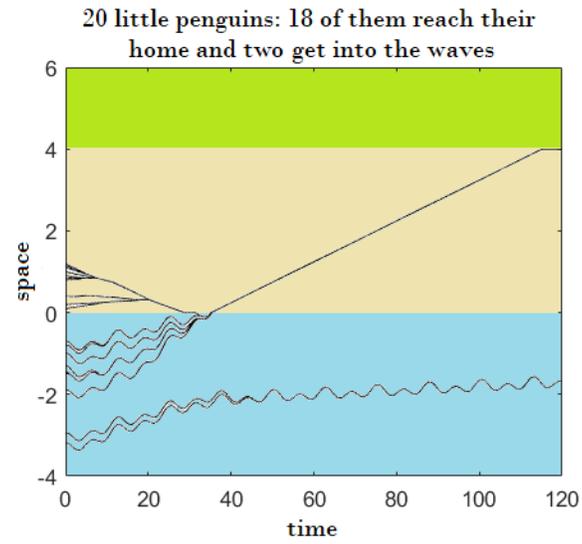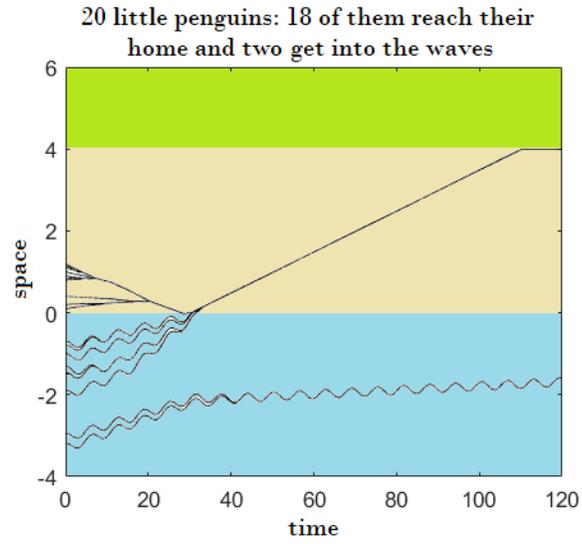FIGURE 3. One penguin moves towards the others but remains in the water.

FIGURE 4. Two penguins are still in the water after a long time.
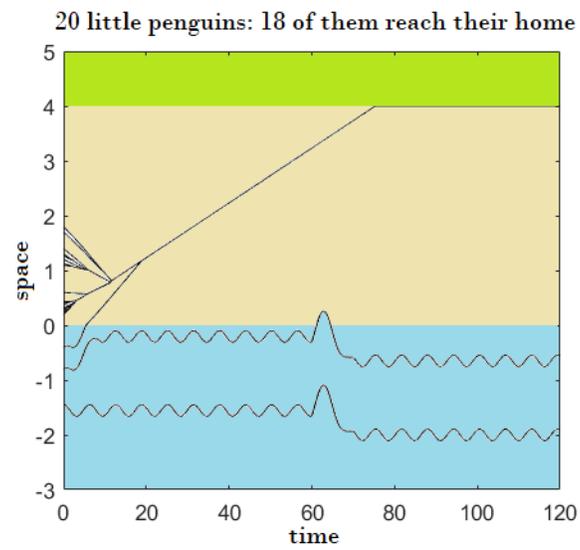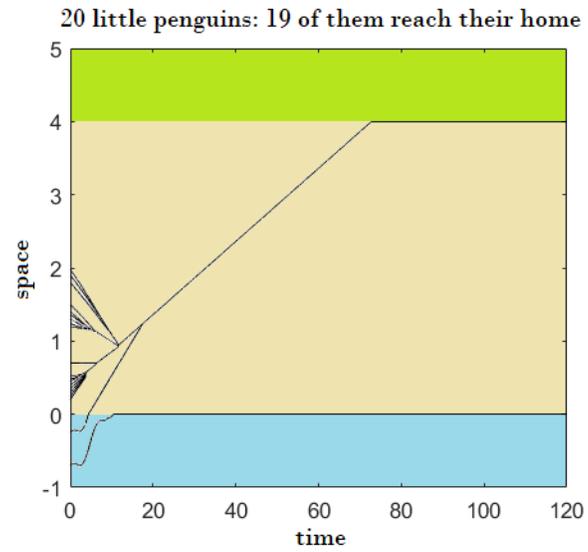
FIGURE 5. Effect of the waves on the movement of the penguins in the sea.

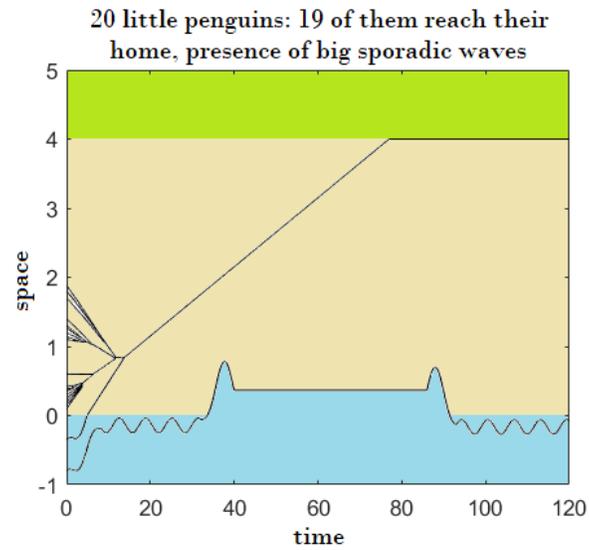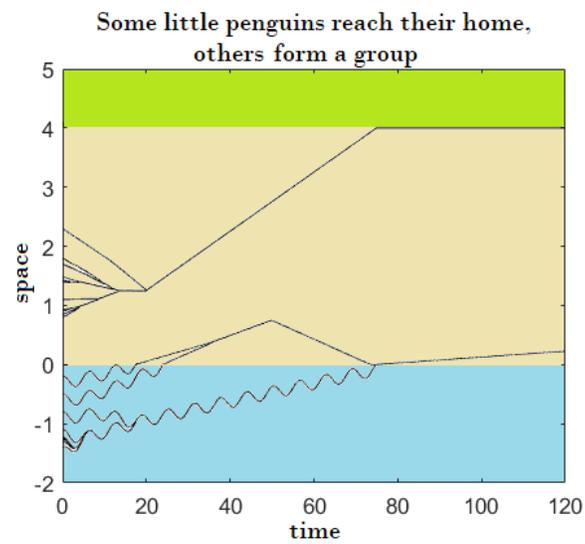FIGURE 6. Effect of the waves on the movement of the penguins in the sea.

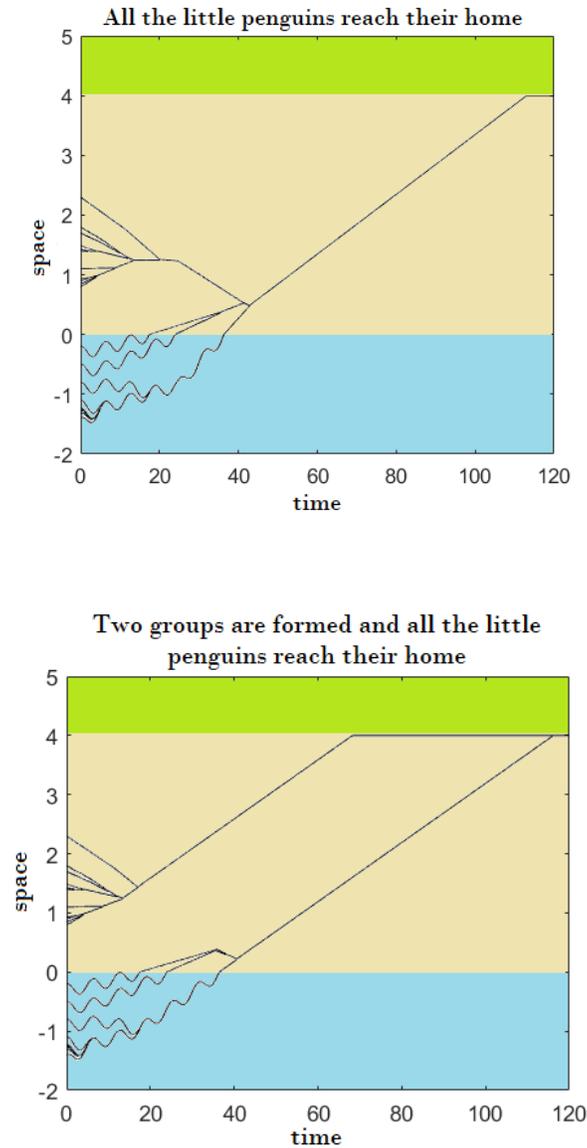FIGURE 7. The penguins form smaller groups and move towards their home.

FIGURE 8. The penguins form groups of different sizes and reach their home.

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