A NON-DC FUNCTION WHICH IS DC ALONG ALL CONVEX CURVES

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Abstract. A problem asked by the authors in 1989 concerns the natural question, whether one can deduce that a continuous function f on an open convex set $D \subset \mathbb{R}^n$ is DC (i.e., is a difference of two convex functions) from the behavior of f "along some special curves φ". I.M. Prudnikov published in 2014 a theorem (working with convex curves φ in the plane), which would give a positive answer in $\mathbb{R}^2$ to our problem. However, in the present note we construct an example showing that this theorem is not correct, and thus our problem remains open in each $\mathbb{R}^n$, $n > 1$.

Introduction

A function f on an open convex set $D \subset \mathbb{R}^n$ is called a DC (or d.c.) function if it is a difference of two convex functions. Fore more information about DC functions and their applications see e.g. [3], [6], [1].

DC functions of one variable have a very simple internal characterization: they are precisely indefinite integrals of functions with locally bounded variation. However, for $n \geq 2$, no simple and useful internal characterization of DC functions is known.

Already A.D. Aleksandrov (who first studied and used DC functions of more variables) in 1949 asked whether a function on $\mathbb{R}^2$ which is (in a natural sense) DC on each line must be DC; now it is well-known that this is not the case (see e.g. [7, p. 35]). So the following rough question arises:

Is it possible to characterize DC functions on $D \subset \mathbb{R}^n$ "in the language of curves" only?

Two precise versions of this question were formulated in [7, Problems 6 and 7, p. 45]; Problem 7 is reproduced in Remark 2.2 below. A paper [4] by I.M. Prudnikov contains a claim (Theorem 1) which would give a positive answer to [7, Problem 7] for real functions in $\mathbb{R}^2$ (see Remark 2.2). However, we construct an example showing that [4, Theorem 1] is not correct, and thus our [7, Problem 7] remains open in each $\mathbb{R}^n$, $n > 1$, even for real functions.

Let us describe the claim of [4, Theorem 1]:

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For a two-dimensional compact convex set $K \subset \mathbb{R}^2$ let $r_K : [0, T_K] \to \partial K$ be a simple closed curve parametrized by the arc length. Let $D \subset \mathbb{R}^2$ be an open bounded convex set and $f \colon D \to \mathbb{R}$ a Lipschitz function.

[4, Theorem 1] asserts, that f is DC if and only if the following condition holds:

(C) There exists a constant $c = c(D, f) > 0$ such that for each two-dimensional convex compact $K \subset D$ the variation $V(\Phi', [0, T_K])$ is less than c, where $\Phi(t) := f(r_K(t)), t \in [0, T_K]$. (The variation is calculated by using only partitions consisting of points of differentiability of Φ.)

(Note that the assumption of convexity of K is omitted in the English translation of [4].)

The main aim of the present note is to show (by a counterexample) that condition (C) does not imply that f is DC. It is difficult to specify which step in the proof of [4, Theorem 1] is incorrect, since many arguments in this proof are sketched only.

1. Preliminaries

We consider $\mathbb{R}^d$ equipped with the standard Euclidean norm $\|\cdot\|$. Given $x \in \mathbb{R}^d$ and $r > 0$, by $B(x, r)$ we denote the closed ball of center x and radius r.

Let $I \subset \mathbb{R}$ be a non-degenerate interval, and $h \colon I \to \mathbb{R}^d$. If I is compact the variation $V(h, I)$ of h on I is defined in the standard way via finite partitions of I. For an arbitrary I, we put

$$V(h, I) = \sup\{V(h, [a, b]) : \ a, b \in I, \ a < b\}.$$

It is elementary to see that we always have

(1)
$$\begin{aligned} V(h, [a, b]) &\le V(h, [a, b)) + \limsup_{s \to b^-} \|h(s) - h(b)\| \\ &\le V(h, (a, b)) + \limsup_{s \to a^+} \|h(s) - h(a)\| + \limsup_{s \to b^-} \|h(s) - h(b)\|. \end{aligned}$$

Let us recall the following definition from [2]. Given a continuous mapping $F \colon [a, b] \to \mathbb{R}^d$ and a finite partition $D = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$, we define

$$P(F, D) := \sum_{i=1}^{n-1} \left\| \frac{F(t_{i+1}) - F(t_i)}{\|F(t_{i+1}) - F(t_i)\|} - \frac{F(t_i) - F(t_{i-1})}{\|F(t_i) - F(t_{i-1})\|} \right\|$$

if the quantity of the right-hand side makes sense, otherwise $P(F, D) := 0$. The quantity

$$P_a^b F := \sup_D P(F, D)$$

is called the *turn* of F on $[a, b]$.

Analogously, the *convexity* of F over $[a, b]$ is the quantity

$$K_a^b F := \sup_D K(F, D)$$

where

$$K(F, D) := \sum_{i=1}^{n-1} \left\| \frac{F(t_{i+1}) - F(t_i)}{t_{i+1} - t_i} - \frac{F(t_i) - F(t_{i-1})}{t_i - t_{i-1}} \right\| .$$

We shall sometimes use the alternative notation $K(F, [a, b]) := K_a^b F$.

Lemma 1.1. *Let $K \subset \mathbb{R}^2$ be a two-dimensional compact convex set, and $r\colon [0, \ell] \to \partial K$ a parametrization by the arc-length of ∂K as a simple closed curve. Then r admits the right derivative $r'_+(t)$ at each $t \in [0, \ell)$, and the left derivative $r'_-(t)$ at each $t \in (0, \ell]$. Moreover,*

$$K_0^\ell r = P_0^\ell r \le 2\pi.$$

Proof. We can (an do) assume that r is counterclockwise oriented. Given a partition $D = \{0 = t_0 < t_1 < \cdots < t_n = \ell\}$, convexity of K implies that the vectors $v_i := \frac{r(t_i) - r(t_{i-1})}{\|r(t_i) - r(t_{i-1})\|}$, $i = 1, \ldots, n$, are "ordered in the counterclockwise way" on the unit circumference in the plane, and $P(r, D)$ is the length of the corresponding simple (not necessarily closed) polygonal curve inscribed in the unit circle. Thus clearly $P_0^\ell r \le 2\pi$ (= the length of the unit circumference). By [2, Theorem 4.10], r has finite "turn of tangents" $T_0^\ell r$ (for definition see [2, pp. 25–26]) which satisfies $T_0^\ell r = P_0^\ell r$. By [2, Lemma 4.4], r admits at each point both one-sided derivatives $r'_\pm(t)$ and they are equal to the corresponding "one-sided tangents" $\tau_\pm(r, t)$ (for definition see [2, p. 25]). Finally, by [2, Proposition 5.7], r has finite convexity and $K_0^\ell r = T_0^\ell r = P_0^\ell r \le 2\pi$ holds. □

2. The counterexample

There exists a Lipschitz real-valued function f on $D := (-2, 2)^2$ such that f is not DC and satisfies condition (C).

Our construction will proceed in three steps. In the first step we will construct a mapping ("curve") $\varphi\colon \mathbb{R} \to D$ which is DC, in the sense that all its components are DC functions. The second step will produce a Lipschitz function $f\colon D \to \mathbb{R}$ such that $f \circ \varphi$ is not DC (which implies that f is not DC). In the third, most difficult step, we will prove that condition (C) is satisfied.

First step: construction of φ.

Choose a sequence $a_n > 0$, $n \in \mathbb{N}$, with $\sum_{n=1}^{\infty} a_n = 1/2$. Then we can clearly choose $h_n > 0$, $n \in \mathbb{N}$, such that $\sum_{n=1}^{\infty} h_n = 1$, and $\sum_{n=1}^{\infty} h_n/a_n < \infty$.

Further set $b_1 := 0$, $b_n := 2\sum_{i=1}^{n-1} a_i$ for $n \geq 2$ and $c_n := b_n + a_n$, $n \in \mathbb{N}$. Thus $0 = b_1 < c_1 < b_2 < c_2 < \cdots < 1$, $\lim b_n = \lim c_n = 1$ and $c_n - b_n = b_{n+1} - c_n = a_n$, $n \in \mathbb{N}$.

Let g be the (unique) function on $\mathbb{R}$ such that

(i) $g(b_1) = 0$, $g(c_1) = 0$, $g(b_n) = g(c_n) = \sum_{i=1}^{n-1} h_i$ for $n \geq 2$, $g(1) = \sum_{i=1}^{\infty} h_i = 1$,

(ii) g is affine on each $[b_n, c_n]$ and $[c_n, b_{n+1}]$, $n \in \mathbb{N}$, and

(iii) g is constant on $(-\infty, 0]$ and $[1, \infty)$.

Set

$$\varphi(t) := \begin{cases} (t, g(t)) & \text{for } t \in [0,1], \\ (0,0) & \text{for } t < 0, \\ (1,1) & \text{for } t > 1. \end{cases}$$

Obviously, g is nondecreasing and $\varphi(\mathbb{R}) \subset D$. Since $\sup\{h_n/a_n :\ n \in \mathbb{N}\} < \infty$, it is easy to see that both g and φ are Lipschitz. Further, the right derivative $\varphi'_+(t)$ equals: $(1,0)$ for $t \in [b_n, c_n)$; $(1, h_n/a_n)$ for $t \in [c_n, b_{n+1})$; and $(0,0)$ for $t \notin [0,1)$. Clearly $\lim_{s \to 1-} \varphi'_+(s) = (1,0)$ since $h_n/a_n \to 0$. Now, (1) and the definition of φ easily yield

$$V(\varphi'_+, \mathbb{R}) \leq V(\varphi'_+, [0,1)) + \|(1,0)\| = 2\sum_{n=1}^{\infty} h_n/a_n \ + 1 < \infty.$$

By [7, Theorem 2.3] we obtain that φ is DC.

Second step: construction of f.

Let $t_0^{(n)} = b_n < t_1^{(n)} < t_2^{(n)} < \cdots < t_n^{(n)} < c_n = t_{n+1}^{(n)}$ be an equidistant partition of the interval $[b_n, c_n]$ ($n \in \mathbb{N}$); so $t_{j+1}^{(n)} - t_j^{(n)} = (c_n - b_n)/(n+1) = a_n/(n+1)$.

Choose $r_n > 0$, $n \in \mathbb{N}$, so small that

$$r_n < \frac{a_n}{4(n+1)}, \ r_n < \frac{h_n}{2} \ \text{ and } \ r_n < \frac{h_{n-1}}{2} \ (n \geq 2), \tag{2}$$

$$r_n + \frac{4r_n(n+1)}{a_n}(1 - b_n) < \frac{h_n}{2}, \ \text{ and} \tag{3}$$

$$-r_n - \frac{4r_n(n+1)}{a_n} c_n > -\frac{h_{n-1}}{2}, \ \ n \geq 2. \tag{4}$$

Now, for $n \in \mathbb{N}$ and $1 \le j \le n$, set $z_j^{(n)} := \varphi(t_j^{(n)}) = (t_j^{(n)}, g(t_j^{(n)}))$. The first inequality of (2) implies that the balls

$$B_j^{(n)} := B(z_j^{(n)}, r_n) \qquad (n \in \mathbb{N},\ 1 \le j \le n)$$

are pairwise disjoint subsets of D, and so we can define $f\colon D \to \mathbb{R}$ by

$$f(x) := \begin{cases} \frac{1}{n^2}(r_n - \|x - z_j^{(n)}\|) & \text{if } x \in B_j^{(n)}, \\ 0 & \text{if } x \in D \setminus \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{n} B_j^{(n)}. \end{cases}$$

It is easy to see that f is Lipschitz with constant 1. Put $F(t) := f(\varphi(t))$, $t \in \mathbb{R}$, and notice that $F(t_j^{(n)} \pm r_n) = 0$, $F(t_j^{(n)}) = r_n/n^2$, and F is affine on $[t_j^{(n)} - r_n, t_j^{(n)}]$ and $[t_j^{(n)}, t_j^{(n)} + r_n]$ $(n \in \mathbb{N})$. Now (supposing that F'_+ exists on $\mathbb{R}$), we have

$$\begin{aligned} V(F'_+, [0,1]) &\ge \sum_{n=1}^{\infty} V(F'_+, [b_n, c_n]) \ge \sum_{n=1}^{\infty} \sum_{j=1}^{n} |F'_+(t_j^{(n)} - r_n) - F'_+(t_j^{(n)})| \\ &\ge \sum_{n=1}^{\infty} n \cdot \frac{2(r_n/n^2)}{r_n} = \infty. \end{aligned}$$

So, [7, Theorem 2.3] implies that F is not DC. Consequently f is not DC either (see e.g. [7, Theorem 4.2 and Theorem 1.20]).

Third step: proof of (C).

In the sequel we will essentially use the following proposition which is a special case of [8, Theorem 4.1].

Proposition 2.1. *Let $D \subset \mathbb{R}^n$ be an open convex set and let f be a non-constant DC function on D which admits a Lipschitz control function γ. Let $\varphi : [\alpha, \beta] \to D$ be Lipschitz. Then*

$$K_\alpha^\beta(f \circ \varphi) \le (\mathrm{Lip} f + \mathrm{Lip}\gamma) K_\alpha^\beta \varphi + 2\,\mathrm{Lip}\gamma\,\mathrm{Lip}\varphi\,.$$

Here the notion of a *control function* is used, which is essential in the theory of DC mappings (see [7]). If f is a real DC function then γ is a control function for f iff both $\pm f + \gamma$ are convex. So,

(5) if $f = g - h$, with g, h convex, then $g + h$ is a control function for f.

Remark 2.2. Proposition 2.1 naturally motivates the following problem, which is a special case of [7, Problem 7].

Problem. Let $D \subset \mathbb{R}^n$ be an open convex set and let $f : D \to \mathbb{R}$ be a function. Suppose that there are $a \ge 0$, $b \ge 0$ such that

$$K_\alpha^\beta(f \circ \varphi) \le a K_\alpha^\beta \varphi + b \mathrm{Lip}\varphi$$

whenever $\varphi : [\alpha, \beta] \to D$ is Lipschitz. Is then f DC on D?

Let us note that if $n = 2$ and a, b from the above problem exist, then condition (C) from the introduction holds with $C(D, f) = 2\pi a + b$. Indeed, using Proposition 2.1 with $\varphi(t) = r_K(t)$, $t \in [\alpha, \beta] = [0, T_K]$, we obtain for $\Phi = f \circ r_K$

$$K_\alpha^\beta \Phi \le a K_\alpha^\beta r_K + b \mathrm{Lip}\, r_K \le 2\pi a + b$$

by Lemma 1.1 and 1-lipschitzness of r_K. So $K_\alpha^\beta \Phi < \infty$, and therefore $K_\alpha^\beta \Phi = V(\Phi'_+, [\alpha, \beta))$ (see, e.g., [8, Theorem 3.1]). Using (1) and the equality $\Phi'_-(\beta) = \lim_{t \to \beta-} \Phi'_+(t)$ (see, e.g., [8, Proposition 3.4]), we obtain

$$V(\Phi', [0, T_K]) \le V(\Phi'_+, [\alpha, \beta)) \le 2\pi a + b.$$

So, if [4, Theorem 1] were correct, the above problem would have a positive answer for $n = 2$.

We continue with the following claim.

Claim 1. *Let $[u, v] \subset (-2, 2)$, and let $h \colon [u, v] \to (-2, 2)$ be a continuous function which is convex or concave. Then there exists at most one $n \in \mathbb{N}$ such that the graph of h intersects both $B_j^{(n)}$ and $B_k^{(n)}$ for some $1 \le j < k \le n$.*

Proof of Claim 1. Assume that the assertion of the claim is false. First, let h be convex. Then we can choose $1 \le m < n$, $1 \le j < k \le n$, $1 \le l \le m$ and $x_1, x_2, x \in [u, v]$ such that

$$(x_1, h(x_1)) \in B_j^{(n)},\ (x_2, h(x_2)) \in B_k^{(n)} \text{ and } (x, h(x)) \in B_l^{(m)}. \tag{6}$$

Clearly $b_n < x_1 < x_2 < c_n$ and $b_m < x < c_m$. Using convexity of h, we obtain

$$\begin{aligned} \frac{(g(c_n) - r_n) - h(x)}{x_1 - x} &\le \frac{h(x_1) - h(x)}{x_1 - x} \le \frac{h(x_2) - h(x_1)}{x_2 - x_1} \\ &\le \frac{(g(b_n) + r_n) - (g(b_n) - r_n)}{\frac{a_n}{2(n+1)}} = \frac{4(n+1) r_n}{a_n}, \end{aligned}$$

and hence

$$g(c_n) - r_n - h(x) \le \frac{4(n+1) r_n}{a_n} (x_1 - x) \le \frac{4(n+1) r_n}{a_n} c_n\,. \tag{7}$$

Notice that, in case $m < n - 1$, we have $g(c_{n-1}) = g(c_m) + \sum_{i=m}^{n-2} h_i > g(c_m) + \frac{h_m}{2}$. Consequently, we have

$$\begin{aligned} g(c_{n-1}) + \frac{h_{n-1}}{2} &\ge g(c_m) + \frac{h_m}{2} \ge h(x) - r_m + \frac{h_m}{2} \overset{\text{by (2)}}{>} h(x) \\ &\overset{\text{by (7)}}{\ge} g(c_n) - r_n - \frac{4(n+1) r_n}{a_n} c_n \overset{\text{by (4)}}{>} g(c_n) - \frac{h_{n-1}}{2} = g(c_{n-1}) + \frac{h_{n-1}}{2}, \end{aligned}$$

which is a contradiction.

Now, let h be concave. Then we choose $1 \le n < m$, $1 \le j < k \le n$, $1 \le l \le m$ and $x_1, x_2, x \in [u,v]$ such that (6) holds.

Similarly as above, concavity of h and our construction imply that

$$\frac{4(n+1)r_n}{a_n} = \frac{(g(b_n)+r_n)-(g(b_n)-r_n)}{\frac{a_n}{2(n+1)}} \ge \frac{h(x_2)-h(x_1)}{x_2-x_1}$$
$$\ge \frac{h(x)-h(x_2)}{x-x_2} \ge \frac{h(x)-g(b_n)-r_n}{x-x_2},$$

and hence

$$h(x)-g(b_n)-r_n \le \frac{4(n+1)r_n}{a_n}(x-x_2) \le \frac{4(n+1)r_n}{a_n}(1-b_n)\,. \tag{8}$$

Notice that $g(b_m) = g(b_n) + \sum_{i=n}^{m-1} h_i \ge g(b_n) + \frac{h_n}{2} + \frac{h_{m-1}}{2}$. Thus we have

$$g(b_n)+\frac{h_n}{2} \le g(b_m)-\frac{h_{m-1}}{2} \le h(x)+r_m-\frac{h_{m-1}}{2} \overset{\text{by (2)}}{<} h(x)$$
$$\overset{\text{by (8)}}{\le} g(b_n)+r_n+\frac{4(n+1)r_n}{a_n}(1-b_n) \overset{\text{by (3)}}{<} g(b_n)+\frac{h_n}{2},$$

which is a contradiction. □

Claim 2. *f is differentiable at $\varphi(1)=(1,1)$ with $f'(\varphi(1))=0$.*

Proof. Recall that $f \equiv 0$ outside the balls $B_j^{(n)}$ ($1 \le j \le n < \infty$), and $f(\varphi(1)) = 0$. Let $P_1\colon \mathbb{R}^2 \to \mathbb{R}$ denote the canonical projection on the first coordinate. For $x \in B_j^{(n)}$, we have $\|x-\varphi(1)\| \ge 1-P_1(x) \ge 1-(t_j^{(n)}+r_n) > 1-c_n > b_{n+1}-c_n = a_n$, and hence (by (2))

$$0 \le \frac{f(x)}{\|x-\varphi(1)\|} \le \frac{f(z_j^{(n)})}{a_n} = \frac{r_n}{n^2 a_n} < \frac{1}{4(n+1)n^2}\,.$$

Since $\varphi(1) \notin B_j^{(n)}$ ($1 \le j \le n < \infty$), it is easy to see that $\lim_{x\to\varphi(1)} \frac{f(x)}{\|x-\varphi(1)\|} = 0$. We are done. □

Convention. In what follows, given a mapping $F\colon [0,\ell] \to \mathbb{R}^d$ such that $F(\ell) = F(0)$, if necessary we consider F extended to $\mathbb{R}$ as a (uniquely determined) ℓ-periodic function. So, for instance, if the right derivative $F'_+(t)$ exists at each $t \in [0,\ell)$ then the variation $V(F'_+,[0,\ell])$ is calculated with $F'_+(\ell) := F'_+(0)$.

It is easy to see that this "circular variation" $V(F'_+,[0,\ell])$ does not depend on shifts of the parameter. More precisely, if F is extended periodically, $d \in \mathbb{R}$ and $G(t) := F(t+d)$, then $V(G'_+,[0,\ell]) = V(F'_+,[0,\ell])$.

We are going to prove that *our (Lipschitz non-DC) function f satisfies the following, a bit stronger variant of property* (C):

(C$'$) *There exists a constant $c > 0$ such that for each two-dimensional compact convex set $K \subset D$ and for each arc-length parametrization $r\colon [0,\ell] \to \partial K$ of ∂K as a simple closed curve, and for $\Phi := f \circ r$, we have (in the sense of our Convention):*
(a) both one-sided derivatives $\Phi'_+(t)$ and $\Phi'_-(t)$ exist finite at each $t \in [0,\ell]$;
(b) $V(\Phi'_+, [0,\ell]) \le c$ and $V(\Phi'_-, [0,\ell]) \le c$.

To prove this, let $K \subset D$ and $r\colon [0,\ell] \to \partial K$ be as in (C$'$). We can clearly assume that the parametrization r is counterclockwise. The boundary ∂K is the union of four parts:

$$\partial K = G_1 \cup G_2 \cup G_3 \cup G_4,$$

where:

- G_1 and G_3 are the graphs of a continuous convex and a continuous concave function, respectively, both defined on an interval $[\alpha,\beta] \subset (-2,2)$;
- G_2, G_4 are two (possibly degenerate) vertical line segments.

We can (and do) assume that r "starts with the convex part", that is, for some $0 < \ell_1 \le \ell_2 < \ell_3 \le \ell$ we have

$$r([0,\ell_1]) = G_1,\ r([\ell_1,\ell_2]) = G_2,\ r([\ell_2,\ell_3]) = G_3,\ r([\ell_3,\ell]) = G_4.$$

(i) The "convex part" $G_1 = r([0,\ell_1])$.
First suppose that there exists $t_0 \in (0,\ell_1]$ such that $r(t_0) = \varphi(1) = (1,1)$. Given $\varepsilon \in (0,t_0)$, the set

$$L_\varepsilon := \{(n,j) :\ n,j \in \mathbb{N},\ j \le n,\ r([0,t_0-\varepsilon]) \cap B_j^{(n)} \ne \emptyset\}$$

is finite. (This follows from the construction of the balls $B_j^{(n)}$, since $(1,1) \notin r([0,t_0-\varepsilon])$.) Let $n_\varepsilon := \max\{n \in \mathbb{N} :\ (n,j) \in L_\varepsilon \text{ for some } j \le n\}$, and apply Claim 1 to choose an $m \le n_\varepsilon$ such that $(n,j) \in L_\varepsilon$ for at most one j whenever $n \le n_\varepsilon$ and $n \ne m$. Let $P_1\colon \mathbb{R}^2 \to \mathbb{R}$ be as in the proof of Claim 2. Since the projections

$$I_j^{(n)} := P_1(B_j^{(n)})\,, \quad (n,j) \in L_\varepsilon,$$

are finitely many pairwise disjoint compact subintervals of $(-2,2)$, there exist open intervals $U_j^{(n)}$, $(n,j) \in L_\varepsilon$, such that $I_j^{(n)} \subset U_j^{(n)} \subset \overline{U_j^{(n)}} \subset (-2,2)$, $(n,j) \in L_\varepsilon$, and the intervals $\overline{U_j^{(n)}}$, $(n,j) \in L_\varepsilon$, are pairwise disjoint.

Let $J_j^{(n)}$, $(n,j) \in L_\varepsilon$, be relatively open subintervals of $[0,t_0-\varepsilon]$ such that

$$K_j^{(n)} := r^{-1}(B_j^{(n)}) \cap [0,t_0-\varepsilon] \subset J_j^{(n)} \subset \overline{J_j^{(n)}} \subset r^{-1}(U_j^{(n)} \times (-2,2))$$

for each $(n,j) \in L_\varepsilon$. (This is clearly possible since $P_1 \circ r : [0,\ell_1] \to [\alpha,\beta]$ is an increasing homeomorphism.) Notice that

$$f(x) = \max\left\{\tfrac{1}{n^2}\big(r_n - \|x - z_j^{(n)}\|\big), 0\right\}, \quad x \in U_j^{(n)} \times (-2,2),\ (n,j) \in L_\varepsilon.$$

It is easy to see that the formula

$$f(x) = \Big(f(x) + \tfrac{1}{n^2}\|x - z_j^{(n)}\|\Big) - \tfrac{1}{n^2}\|x - z_j^{(n)}\|, \quad x \in U_j^{(n)} \times (-2,2),$$

is a representation of f as a difference of two continuous convex functions. It follows from (5) that f is controlled on each $U_j^{(n)} \times (-2,2)$, $(n,j) \in L_\varepsilon$, by the function

$$\gamma(x) = f(x) + \tfrac{2}{n^2}\|x - z_j^{(n)}\| = \max\left\{\tfrac{r_n}{n^2} + \tfrac{1}{n^2}\|x - z_j^{(n)}\|,\ \tfrac{2}{n^2}\|x - z_j^{(n)}\|\right\},$$

which is Lipschitz with constant $\frac{2}{n^2}$. Denote $\Phi := f \circ r$.

Fix $(n,j) \in L_\varepsilon$. If $J_j^{(n)}$ is open in $\mathbb{R}$, put $\mathcal{J}_j^{(n)} := J_j^{(n)}$. Otherwise, since the length of $J_j^{(n)}$ is at most ℓ_1, there exists an open (in $\mathbb{R}$) interval $\mathcal{J}_j^{(n)}$ of length smaller than ℓ, such that $J_j^{(n)} \subset \mathcal{J}_j^{(n)}$ and $r(\overline{\mathcal{J}_j^{(n)}}) \subset U_j^{(n)} \times (-2,2)$. Then (we use Convention above for r and Φ) $K(r, \overline{\mathcal{J}_j^{(n)}}) \le K_0^\ell r \le 2\pi$ by Lemma 1.1, and applying Proposition 2.1 we obtain

(9)

$$K\big(\Phi, \overline{\mathcal{J}_j^{(n)}}\big) \le \big(\mathrm{Lip}(f) + \mathrm{Lip}(\gamma)\big) K\big(r, \overline{\mathcal{J}_j^{(n)}}\big) + 2\mathrm{Lip}(\gamma)\,\mathrm{Lip}(r) \le \frac{3}{n^2}2\pi + \frac{4}{n^2}.$$

Hence by [8, Proposition 3.4, p. 328], both derivatives $\Phi'_\pm$ exist at each point of $\mathcal{J}_j^{(n)}$, and we have

$$V(\Phi'_\pm, J_j^{(n)}) \le V(\Phi'_\pm, \mathcal{J}_j^{(n)}) = K\big(\Phi, \overline{\mathcal{J}_j^{(n)}}\big) \le \frac{6\pi + 4}{n^2}. \tag{10}$$

Since $f \equiv 0$ outside all the balls $B_j^{(n)}$, $1 \le j \le n < \infty$, we have that $\Phi \equiv 0$ outside the compact sets $K_j^{(n)}$, $(n,j) \in L_\varepsilon$. Therefore both derivatives $\Phi'_\pm$ exist at each point of $(0, t_0 - \varepsilon)$, and

$$\Phi'_\pm(x) = 0, \quad x \in (0, t_0 - \varepsilon) \setminus \bigcup\{K_j^{(n)} : (n,j) \in L_\varepsilon\}.$$

Choose relatively open subintervals $M_j^{(n)}$, $(n,j) \in L_\varepsilon$, of $[0, t_0 - \varepsilon]$ such that $K_j^{(n)} \subset M_j^{(n)} \subset \overline{M_j^{(n)}} \subset J_j^{(n)}$. Consider the extreme points of all the intervals $M_j^{(n)}$, $(n,j) \in L_\varepsilon$, and index them in an increasing order to obtain the points

$$0 \le u_1 < v_1 < u_2 < v_2 < \cdots < u_d < v_d \le t_0 - \varepsilon,$$

where $d = \operatorname{card} L_\varepsilon$. Then for each $(n,j) \in L_\varepsilon$ there is a unique $1 \le k \le d$ such that $\inf M_j^{(n)} = u_k$ and $\sup M_j^{(n)} = v_k$.

Observe that both $\Phi'_\pm$ are identically null: on $(0,u_1]$ (if $u_1>0$); on $[v_d,t_0-\varepsilon)$ (if $v_d<t_0-\varepsilon$); on all intervals $[v_{k-1},u_k]$, $2\le k\le d$; on a neighborhood of any of the points u_k,v_k $(1\le k\le d)$ that belongs to $(0,t_0-\varepsilon)$. Now, we can use this observation together with the additivity of variation and (1) to write

$$\begin{aligned}V\big(\Phi'_\pm,(0,t_0-\varepsilon)\big) &= V\big(\Phi'_\pm,(u_1,v_d)\big)\\ &= V\big(\Phi'_\pm,(u_1,v_1]\big)+\sum_{k=2}^{d-1}V\big(\Phi'_\pm,[u_k,v_k]\big)+V\big(\Phi'_\pm,[u_d,v_d)\big)\\ &= \sum_{k=1}^{d}V\big(\Phi'_\pm,(u_k,v_k)\big)\le\sum_{(n,j)\in L_\varepsilon}V(\Phi'_\pm,J_j^{(n)})\,.\end{aligned}$$

Thus, by (10) and the properties of L_ε (see the text after its definition), we obtain

$$\begin{aligned}V\big(\Phi'_\pm,(0,t_0-\varepsilon)\big) &\le \sum_{n=1}^{n_\varepsilon}\frac{6\pi+4}{n^2}+m\frac{6\pi+4}{m^2}\\ &\le (6\pi+4)\left(1+\sum_{n=1}^{\infty}\frac{1}{n^2}\right)=:M.\end{aligned}$$

Consequently, $V\big(\Phi'_\pm,(0,t_0)\big)\le M$ since $\varepsilon\in(0,t_0)$ was arbitrary.

We claim that $V\big(\Phi'_\pm,(0,\ell_1)\big)\le M+1$. This is obvious if $t_0=\ell_1$. If $t_0\ne\ell_1$, we have $0<t_0<\ell_1$, $\Phi'(t_0)=0$ (by Lemma 1.1 and Claim 2), and $\Phi\equiv0$ on $[t_0,\ell_1)$. Since Φ is Lipschitz with constant 1, we can use (1) to get

$$V\big(\Phi'_\pm,(0,\ell_1)\big)=V\big(\Phi'_\pm,(0,t_0]\big)\le V\big(\Phi'_\pm,(0,t_0)\big)+1\le M+1,$$

and we are done.

In case that t_0 does not exist, G_1 intersects at most finitely many balls $B_j^{(n)}$, and we can get the same estimate $V(\Phi'_\pm,(0,\ell_1))\le M+1$ directly (as above with ℓ_1 in place of $t_0-\varepsilon$).

(ii) The “concave part” $G_3=r([\ell_2,\ell_3])$.
This part can be treated in the very same way to obtain $V(\Phi'_\pm,(\ell_2,\ell_3))\le M+1$.

(iii) The two vertical segments $G_2=r([\ell_1,\ell_2])$ and $G_4=r([\ell_3,\ell])$.
By our construction (see the first inequality of (2)), each of the two vertical segments intersects at most one of the balls $B_j^{(n)}$, $1\le j\le n<\infty$. As in (9), for such n we have $K(\Phi,[\ell_1,\ell_2])\le\frac{6\pi+4}{n^2}\le M+1$. As above, it follows that $V(\Phi'_\pm,(\ell_1,\ell_2))\le M+1$ provided $\ell_1<\ell_2$; and in the same way $V(\Phi'_\pm,(\ell_3,\ell))\le M+1$ provided $\ell_3<\ell$.

(iv) Conclusion of the proof of (C$'$).
For simplicity denote $\ell_0 := 0$ and $\ell_4 := \ell$. Recall that Φ is Lipschitz with constant 1. Now, combining (1) with [8, Propositions 3.4, pp. 328–329], we obtain for $i = 1, 2, 3, 4$ that the derivatives $\Phi'_\pm(\ell_i)$ exist, and

$$V(\Phi'_+, [\ell_{i-1}, \ell_i]) \leq V(\Phi'_+, (\ell_{i-1}, \ell_i)) + 2 \leq M + 3.$$

Thus

$$V(\Phi'_+, [0, \ell]) = \sum_{i=1}^{4} V(\Phi'_+, [\ell_{i-1}, \ell_i]) \leq 4(M + 3)$$

and, symmetrically, $V(\Phi'_-, [0, \ell]) \leq 4(M + 3)$. The proof is complete.

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