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NON-SYMPLECTIC AUTOMORPHISMS OF IRREDUCIBLE HOLOMORPHIC SYMPLECTIC MANIFOLDS

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Abstract

We study automorphisms of irreducible holomorphic symplectic (IHS) manifolds of $K3^{[n]}$-type, i.e. manifolds which are deformation equivalent to the Hilbert scheme of n points on a $K3$ surface, for some $n \geq 2$. In recent years many classical theorems, concerning the classification of non-symplectic automorphisms of $K3$ surfaces, have been extended to IHS fourfolds of $K3^{[2]}$-type. Our aim is to investigate whether it is possible to further generalize these results to the case of manifolds of $K3^{[n]}$-type for any $n \geq 3$.

In the first part of the thesis we describe the automorphism group of the Hilbert scheme of n points on a generic projective $K3$ surface, i.e. a $K3$ surface whose Picard lattice is generated by a single ample line bundle. We show that, if it is not trivial, the automorphism group is generated by a non-symplectic involution, whose existence depends on some arithmetic conditions involving the number of points n and the polarization of the surface. Alongside this numerical characterization, we also determine necessary and sufficient conditions on the Picard lattice of the Hilbert scheme for the existence of the involution.

In the second part of the thesis we study non-symplectic automorphisms of prime order on IHS manifolds of $K3^{[n]}$-type. We investigate the properties of the invariant lattice and its orthogonal complement inside the second cohomology lattice of the manifold, providing a classification of their isometry classes. We then approach the problem of constructing examples (or at least proving the existence) of manifolds of $K3^{[n]}$-type with a non-symplectic automorphism inducing on cohomology each specific action in our classification. In the case of involutions, and of automorphisms of odd prime order for $n = 3, 4$, we are able to realize all possible cases. In order to do so, we present a new non-symplectic automorphism of order three on a ten-dimensional family of Lehn–Lehn–Sorger–van Straten eightfolds of $K3^{[4]}$-type. Finally, for $2 \leq n \leq 5$ we describe deformation families of large dimensions of manifolds of $K3^{[n]}$-type equipped with a non-symplectic involution and the moduli spaces which parametrize them.

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Contents

Introduction

The study of automorphisms of $K3$ surfaces has been a very active research field for decades. By means of the global Torelli theorem proved by Pjateckiĭ-Šapiro and Šafarevič [87], it is possible to reconstruct automorphisms of a $K3$ surface Σ from Hodge isometries of $H^2(\Sigma, \mathbb{Z})$ which preserve the Kähler cone. This link, together with seminal works by Nikulin [79], [78] and Mukai [77], provided the instruments to investigate finite groups of automorphisms on $K3$'s, by adopting a lattice-theoretical approach. In recent years, the interest in automorphisms has extended from $K3$ surfaces to manifolds which generalize them in higher dimension, namely irreducible holomorphic symplectic (IHS) varieties. These manifolds still have a lattice strucutre on their second cohomology group with integer coefficients (provided by the Beauville–Bogomolov–Fujiki quadratic form); however, in contrast with $K3$ surfaces, the isomorphism class of higher-dimensional IHS manifolds cannot be recovered from the integral Hodge decomposition of this lattice (the first counter-example was exhibited by Debarre [36]). Nevertheless, results by Huybrechts, Markman and Verbitsky (see [51], [69], [98]) provide a (weaker) analogous of the global Torelli theorem for irreducible holomorphic symplectic manifolds: this allows us to investigate automorphisms of IHS manifolds with methods similar to the ones which proved to be effective in the case of $K3$ surfaces, studying their action on the second cohomology lattice.

A great number of results are now known for automorphisms of prime order on IHS fourfolds which are deformations of Hilbert schemes of two points on a $K3$ surface (so-called manifolds of $K3^{[2]}$-type). The symplectic case (i.e. automorphisms which preserve the symplectic form of the manifold) is covered by Camere [27] for involutions and by Mongardi [71] for automorphisms of odd prime orders; moreover, a classification of finite groups of symplectic automorphisms on fourfolds of $K3^{[2]}$-type is provided by Höhn and Mason [47]. The fixed locus of these automorphisms is also well-understood: it consists of isolated points, $K3$ surfaces and abelian surfaces. In turn, a systematic study of non-symplectic automorphisms was started by Beauville [12], with the case of involutions, and has later seen many relevant contributions, such as the work of Boissière, Nieper-Wisskirchen and Sarti [21], which was instrumental to obtain a complete classification of the action of these automorphisms on cohomology (see [17], [16], [94]). Explicit constructions of automorphisms of prime order realizing all possible actions in this classification have been exhibited throughout the years, with the exception of the (unique) automorphism of order 23 whose existence is proved in [16]. In particular, it is worth mentioning the work by Boissière [15] on natural automorphisms, the geometric constructions on Fano varieties of lines on cubic fourfolds by Boissière, Camere and Sarti [17] and the study of induced automorphisms on moduli spaces of stable objects on $K3$ surfaces by Mongardi and Wandel [76], and by Camere, G. Kapustka, M. Kapustka and Mongardi [30] in the twisted case, as well as the involution of double EPW sextics constructed by O'Grady [85], [84].

Far less is known about automorphisms of manifolds of $K3^{[n]}$-type when $n \geq 3$. Several classification results on symplectic automorphisms are contained in [71],

[74] and, in a recent paper, Kamenova, Mongardi and Oblomkov [59] studied the fixed locus of symplectic involutions. Moreover, in his PhD thesis Rapagnetta [88] exhibited an interesting (birational) symplectic automorphism of order two of a manifold of $K3^{[3]}$-type, which plays a role in describing O'Grady's 6-dimensional IHS example (see [75]).

In the thesis we focus on non-symplectic automorphisms of manifolds of $K3^{[n]}$-type. The core of the manuscript can be divided into three parts (corresponding to Chapters 3, 4 and 5), dealing with distinct – yet closely related – problems.

In **Chapter 1** and **Chapter 2** we provide the reader with an overview of the fundamental results in the theory of lattices and of IHS manifolds respectively, which are instrumental for the study of automorphisms.

The aim of **Chapter 3** is to describe the group $\mathrm{Aut}(\Sigma^{[n]})$ of biholomorphic automorphisms of the Hilbert scheme of n points on a generic projective $K3$ surface Σ, i.e. a $K3$ surface such that $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, for an ample line bundle H. It is known that, if $H^2 = 2t$ for an integer $t \geq 1$, then the $K3$ surface Σ has no non-trivial automorphisms if $t \geq 2$; instead, if $t = 1$, Σ is isomorphic to a double cover of $\mathbb{P}^2$ ramified along a smooth sextic curve, therefore it is endowed with a covering involution (which is the only automorphism of Σ, besides the identity). In the case $n = 2$, the group $\mathrm{Aut}(\Sigma^{[2]})$ has been computed in [20] by Boissière, An. Cattaneo, Nieper-Wisskirchen and Sarti, who show that it is either trivial or generated by a non-symplectic involution. We prove that these are still the only two possibilities when $n \geq 3$. In particular, if $t = 1$ the group $\mathrm{Aut}(\Sigma^{[n]})$ is generated by the natural involution induced by the covering automorphism of Σ. For $t \geq 2$, there are no natural involutions on $\Sigma^{[n]}$ and we provide necessary and sufficient conditions to distinguish between the two possible cases $\mathrm{Aut}(\Sigma^{[n]}) = \{\mathrm{id}\}$ and $\mathrm{Aut}(\Sigma^{[n]}) \cong \mathbb{Z}/2\mathbb{Z}$. We first give a divisorial characterization, in Proposition 3.4.1 and Proposition 3.4.3, obtained by using the global Torelli theorem for IHS manifolds of $K3^{[n]}$-type: there is a (non-natural, non-symplectic) involution on $\Sigma^{[n]}$ if and only if there exists a primitive ample class $\nu \in \mathrm{NS}(\Sigma^{[n]})$ with $\nu^2 = 2$, or with $\nu^2 = 2(n-1)$ and divisibility $n-1$ in $H^2(\Sigma^{[n]}, \mathbb{Z})$ (i.e. the ideal $\left\{(\nu, l) \mid l \in H^2(\Sigma^{[n]}, \mathbb{Z})\right\} \subset \mathbb{Z}$ is generated by $n-1$). In addition to this, in Theorem 3.5.4 we also determine purely numerical necessary and sufficient conditions for the existence of an involution on $\Sigma^{[n]}$. This can be achieved by using the descriptions of Bayer and Macrì [7] for the movable and nef cones of $\Sigma^{[n]}$, and by studying how an automorphism of the Hilbert scheme acts on these cones. As an application of the numerical characterization, for any $n \geq 2$ we show how to construct an infinite sequence of values $t_{n,k} \geq 2n-2$ such that, if Σ is a $K3$ surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, $H^2 = 2t_{n,k}$, then $\Sigma^{[n]}$ admits an involution, whose action on $H^2(\Sigma^{[n]}, \mathbb{Z})$ is the opposite of the reflection with respect to a class $\nu \in \mathrm{NS}(\Sigma^{[n]})$ of square two (see Proposition 3.5.7). The results of this chapter have been the object of the paper [32].

In the last two chapters of the thesis we shift from Hilbert schemes of $K3$ surfaces to the more general setting of manifolds of $K3^{[n]}$-type, with the aim of classifying non-symplectic automorphisms of prime order. If X is deformation equivalent to the Hilbert scheme of n points on a $K3$ surface, an automorphism $\sigma \in \mathrm{Aut}(X)$ is uniquely determined by its pull-back $\sigma^* \in O(H^2(X, \mathbb{Z}))$. In turn, it is possible to describe σ^* by means of the invariant lattice $T = H^2(X, \mathbb{Z})^{\sigma^*} = \left\{v \in H^2(X, \mathbb{Z}) : \sigma^*(v) = v\right\}$ and its orthogonal complement $S = T^\perp \subset H^2(X, \mathbb{Z})$. In the case of non-symplectic automorphisms of prime order on fourfolds of $K3^{[2]}$-type, Boissière, Camere and Sarti determine in [17] all possible isometry classes for the lattices T and S. In particular, they notice that the classification is fundamentally richer for $p = 2$, rather than for p odd. For $n \geq 3$, many additional cases appear whenever p divides $2(n-1)$: this suggests that involutions deserve to be discussed separately, since 2 divides $2(n-1)$ for all $n \geq 2$. Thus, in the second part of the thesis we focus on non-symplectic automorphisms of manifolds of $K3^{[n]}$-type, distinguishing between involutions and automorphisms of prime order $p \geq 3$.

In **Chapter 4** we study non-symplectic automorphisms of odd prime order. We first investigate the properties of the pairs of lattices (T, S). The isometry classes of the two lattices depend on three numerical invariants of the automorphism: its order p and two integers m, a such that $\mathrm{rk}(S) = (p-1)m$ and $\frac{H^2(X,\mathbb{Z})}{T \oplus S} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$. We say that a triple (p, m, a), with p prime, is *admissible* for a certain $n \geq 2$ if there exists two primitive orthogonal sublattices T, S of the abstract lattice $H^2(X, \mathbb{Z})$, for any manifold X of $K3^{[n]}$-type, with the above properties and satisfying some additional necessary conditions to be the invariant and the co-invariant lattices of an automorphism of order p on a manifold of $K3^{[n]}$-type. Using classical results in lattice theory, mainly by Nikulin [79], we can determine a list of all admissible triples (p, m, a) for each value of n. Our first main result, which is purely lattice-theoretic, concerns the classification of pairs (T, S) corresponding to a given admissible triple (Theorem 4.1.12). Moreover, for $n = 3$ and $n = 4$ (which are the cases of most immediate interest) we provide in Section 4.1.4 and Appendix A the complete list of admissible triples (p, m, a) and of the corresponding (unique) pairs of lattices (T, S).

We then construct examples of non-symplectic automorphisms of odd prime order. For manifolds of $K3^{[2]}$-type, in [17] the authors prove that natural automorphisms of Hilbert schemes of points realize all but a few admissible pairs (T, S); the residual cases (except for the aforementioned automorphism of order 23) are constructed as automorphisms of Fano varieties of lines on cubic fourfolds. For $n \geq 3$, it is necessary to expand our pool of tools. *Induced* automorphisms on moduli spaces of (possibly twisted) sheaves on $K3$ surfaces, studied in [76] and [30], directly generalize natural automorphisms and allow us to realize many new pairs (T, S). In Section 4.3 we show, specifically, how to apply these constructions when $n = 3, 4$: we find that all admissible pairs of lattices (T, S) with $\mathrm{rk}(T) \geq 2$ can be realized by natural or (possibly twisted) induced automorphisms. Admissible pairs (T, S) where T has rank one require special attention. There are only four distinct triples (p, m, a) which determine pairs of lattices (T, S) with $\mathrm{rk}(T) = 1$: two for $p = 3$ and two for $p = 23$. However, for a fixed n at most two of them are admissible (no more than one for each value of $p \in \{3, 23\}$). We study these four cases in detail in Proposition 4.1.15, providing the corresponding isometry classes of the lattices T, S: even though they never correspond to natural or induced non-symplectic automorphisms, the global Torelli theorem for IHS manifolds guarantees that there exist automorphisms which realize each of them (see Proposition 4.2.4). In specific cases, it is possible to provide a geometric construction of the automorphism. In Section 4.4 we focus on one of these pairs of lattices (T, S) with $T \cong \langle 2 \rangle$, for $n = 4$, and we show that it is realized by a non-symplectic automorphism of order three on a ten-dimensional family of Lehn–Lehn–Sorger–van Straten eightfolds, obtained from an automorphism of the underlying family of cyclic cubic fourfolds. This is the first known geometric construction of a non-induced, non-symplectic automorphism of odd order on a manifold of $K3^{[4]}$-type. Moreover, thanks to it we are able to complete the list of examples of automorphisms of odd prime order $p \neq 23$ which realize all admissible pairs (T, S) for $n = 3, 4$. This part of the thesis has been the product of a collaboration with Chiara Camere ([29]).

Finally, **Chapter 5** is devoted to non-symplectic involutions. In [58], Joumaah studied moduli spaces of manifolds X of $K3^{[n]}$-type with non-symplectic involutions $i : X \to X$, providing also a classification for the invariant lattice $T = H^2(X, \mathbb{Z})^{i^*}$. This classification, however, is not entirely correct: we begin by rectifying the mistakes, providing a description of the possible signatures and discriminant forms for T and S in Section 5.1. As a lattice, $H^2(X,\mathbb{Z})$ is isometric to $L_n := U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$, therefore we can fix an isometry $\eta : H^2(X,\mathbb{Z}) \to L$ and consider T and S as sublattices of L_n. In Proposition 5.2.1 we show that, for any choice of a pair of primitive orthogonal sublattices $T, S \subset L_n$ satisfying our classification, it is possible to extend $\mathrm{id}_T \oplus (-\mathrm{id}_S)$ to a well-defined isometry of L_n, which in turn can be lifted, by the global Torelli theorem, to a non-symplectic involution i of a suitable manifold X of $K3^{[n]}$-type, such that $H^2(X,\mathbb{Z})^{i^*} \cong T$, $\left(H^2(X,\mathbb{Z})^{i^*}\right)^{\perp} \cong S$ (Proposition 5.2.3). We then use Joumaah's results on moduli spaces to study deformation families of large dimensions of manifolds of $K3^{[n]}$-type equipped with a non-symplectic involution. The period domain, for manifolds X of this type, is contained in $\mathbb{P}(S \otimes \mathbb{C}) \subset \mathbb{P}(L_n \otimes \mathbb{C})$, where S is the co-invariant lattice of i (whose isometry class, as for T, is a deformation invariant). Therefore, the moduli spaces of maximal dimension correspond to minimal invariant lattices (i.e. lattices which do not primitively contain any other lattice T in our classification). For this reason, in Section 5.3.1 we determine explicitly all possible isometry classes of pairs (T,S) with $\mathrm{rk}(T) = 1, 2$, in order to identify, at least for $n \leq 5$, the maximal deformation families of dimension $d \geq 19$ (Theorem 5.3.10). The results contained in this last chapter are part of a joint work, still in progress, with Chiara Camere and Andrea Cattaneo.

CHAPTER 1

Lattice theory

In this chapter we present a (selective) overview of the theory of lattices and of finite quadratic forms, recalling the fundamental definitions and results which we will use throughout the thesis. Our main reference for these topics will be the seminal paper [79] by Nikulin, but we will also draw from several other classical sources, such as [35], [100], [60] and [26].

1.1. Definitions and examples

DEFINITION 1.1.1. A *lattice* L is a free abelian group endowed with a symmetric, non-degenerate bilinear form $(\cdot,\cdot) : L \times L \to \mathbb{Z}$.

The *rank* of the lattice L is $\mathrm{rk}(L) = r \in \mathbb{N}$ if $L \cong \mathbb{Z}^{\oplus r}$ as groups. The bilinear form $(\cdot,\cdot)$ on L can be represented, after choosing a $\mathbb{Z}$-basis $\{e_1, \ldots, e_r\}$ for the lattice, by its *Gram matrix* $G_L = (g_{i,j}) \in \mathrm{Mat}_r(\mathbb{Z})$, where $g_{i,j} := (e_i, e_j)$; in particular, G_L is symmetric.

The *signature* of the lattice L is the signature of the bilinear form on $\mathbb{R}^r$ represented by the matrix G_L, i.e. the signature of the $\mathbb{R}$-linear extension of $(\cdot,\cdot)$ to $L \otimes_{\mathbb{Z}} \mathbb{R}$. Assuming $\mathrm{sign}(L) = (l_{(+)}, l_{(-)})$, we have $l_{(+)} + l_{(-)} = \mathrm{rk}(L)$, since the bilinear form is non-degenerate. The lattice L is said to be *positive definite* (respectively, *negative definite*) if $l_{(-)} = 0$ (respectively, $l_{(+)} = 0$); it is said to be *indefinite* if $l_{(+)}, l_{(-)} \neq 0$. Finally, L is *hyperbolic* if $l_{(+)} = 1$.

The *divisibility* $\mathrm{div}(l)$ of an element $l \in L$ is the positive generator of the ideal

$$\{(l,m) \mid m \in L\} \subset \mathbb{Z}.$$

The lattice L is *even* if the associated quadratic form is even on all elements of L, i.e. $(l,l) \in 2\mathbb{Z}$ for all $l \in L$. If t is a non-zero integer, $L(t)$ denotes the lattice of rank $r = \mathrm{rk}(L)$ and whose bilinear form is the one of L multiplied by t. If L, L' are two lattices, their orthogonal direct sum is denoted by $L \oplus L'$: it is the lattice of rank $\mathrm{rk}(L) + \mathrm{rk}(L')$ on the free abelian group $L \oplus L'$ such that, if $l_1, l_2 \in L$ and $l'_1, l'_2 \in L'$, then $(l_1 + l'_1, l_2 + l'_2) := (l_1, l_2) + (l'_1, l'_2)$.

If L is a lattice, a *sublattice* T of L is a subgroup $T \subset L$ with the property that the restriction of the bilinear form of L to T is still non-degenerate. For $T \subset L$ a sublattice, we define

$$T^\perp := \{l \in L \mid (l,t) = 0 \ \ \forall t \in T\}.$$

It is easy to check that $T^\perp$ is a sublattice of L, called the *orthogonal sublattice* of T. In particular, the sublattice $T \oplus T^\perp \subset L$ has maximal rank, i.e. $\mathrm{rk}(T) + \mathrm{rk}(T^\perp) = \mathrm{rk}(L)$.

A sublattice $T \subset L$ is called *primitive* if the quotient L/T is a free abelian group. The orthogonal complement $T^\perp$ of any sublattice $T \subset L$ is primitive; moreover, $\left(T^\perp\right)^\perp \supset T$ is the primitive sublattice of L generated by T (also called the *saturation* of T in L).

If L, L' are two lattices, a *morphism of lattices* $\varphi : L \to L'$ is a morphism of abelian groups such that $(l_1, l_2) = (\varphi(l_1), \varphi(l_2))$ for all $l_1, l_2 \in L$. Since the bilinear form of any lattice is assumed to be non-degenerate, all morphisms of lattices are injective. An *isometry* is a bijective morphism of lattices; we denote by $O(L)$ the group of isometries of a lattice L to itself. If two lattices L, L' are isometric, we write $L \cong L'$. We will often use the term *embedding* to refer to a morphism of lattices which is not necessarily surjective. An embedding $i : L \hookrightarrow L'$ is primitive if the image $i(L) \subset L'$ is a primitive sublattice.

The *dual lattice* of L is $L^\vee := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$, which admits the following equivalent description:

$$L^\vee = \{u \in L \otimes \mathbb{Q} : (u, v) \in \mathbb{Z} \ \ \forall v \in L\}\,. \tag{1}$$

Clearly, L is a subgroup of $L^\vee$. Notice that, with respect to a basis $\{e_i\}_i$ of L and the dual basis $\{e_i^* := (e_i, \cdot)\}_i$ of $L^\vee$, the matrix which represents the inclusion $L \hookrightarrow L^\vee$ is simply the Gram matrix of L with respect to $\{e_i\}_i$. Since $L \subset L^\vee$ is a subgroup of maximal rank, the quotient $A_L := L^\vee/L$ is a finite group, called the *discriminant group* of L. We denote by $\mathrm{discr}(L)$ the order of the discriminant group A_L, i.e. the index of $L \subset L^\vee$, which coincides with $|\det(G_L)|$ for any Gram matrix G_L representing the bilinear form of L with respect to a basis of the lattice. Moreover, the *length* $l(A_L)$ is defined as the minimal number of generators of A_L. If $A_L = \{0\}$, the lattice L is said to be *unimodular*. If instead $A_L \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k}$ for a prime number p and a non-negative integer k, then the lattice L is said to be *p-elementary*; in this case, $l(A_L) = k$.

Notice that the dual lattice $L^\vee$ is not actually a lattice (it is not endowed with an integer-valued bilinear form), however – using the representation (1) – we can extend the bilinear form of L by $\mathbb{Q}$-linearity to $(\cdot,\cdot) : L^\vee \times L^\vee \to \mathbb{Q}$. In particular, if we consider elements $x_1, x_2 \in L^\vee$ and $l_1, l_2 \in L$, we have:

$$(x_1 + l_1, x_2 + l_2) = (x_1, x_2) + (x_1, l_2) + (l_1, x_2) + (l_1, l_2) \equiv (x_1, x_2) \pmod{\mathbb{Z}}. \tag{2}$$

We recall the following definition.

Definition 1.1.2. A *finite bilinear form* is a symmetric bilinear form $b : A \times A \to \mathbb{Q}/\mathbb{Z}$, where A is a finite abelian group. A *finite quadratic form* is a map $q : A \to \mathbb{Q}/2\mathbb{Z}$ such that:

(*i*) $q(ka) = k^2 q(a)$ for all $k \in \mathbb{Z}$ and $a \in A$;

(*ii*) $q(a + a') - q(a) - q(a') = 2b(a, a')$ in $\mathbb{Q}/2\mathbb{Z}$, where $b : A \times A \to \mathbb{Q}/\mathbb{Z}$ is a finite bilinear form (called the bilinear form associated to q).

A finite quadratic form $q : A \to \mathbb{Q}/2\mathbb{Z}$ is said to be non-degenerate if the associated finite bilinear form b is non-degenerate, and by using b we define the orthogonal complement $H^\perp \subset A$ for any subgroup $H \subset A$. The isometry group $O(A)$ is the group of isomorphisms of A which preserve the finite quadratic form q.

Looking at the expression (2), if L is a lattice the bilinear form (with rational values) on $L^\vee$ descends to a well-defined finite bilinear form $(\cdot,\cdot) : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$. In the case where the lattice L is even, we can associate to this bilinear form on A_L a finite quadratic form q_L, defined as

$$q_L : A_L \to \mathbb{Q}/2\mathbb{Z}, \qquad q_L(x + L) := (x, x) \pmod{2\mathbb{Z}}.$$

If A_L is a finite direct sum of cyclic subgroups A_i, we write $q_L = \bigoplus_i A_i(\alpha_i)$ if the discriminant form q_L takes value $\alpha_i \in \mathbb{Q}/2\mathbb{Z}$ on a generator of the summand A_i. Notice that, for any two lattices L, L', there exists a canonical isomorphism $A_{L\oplus L'} \cong A_L \oplus A_{L'}$, which is an isometry with respect to the finite quadratic forms $q_{L\oplus L'}$ and $q_L \oplus q_{L'}$.

We recall the following result concerning finite quadratic forms.

Proposition 1.1.3. *Let q be a finite quadratic form on an abelian group A and $H \subset A$ a subgroup.*

(i) *$q = \oplus_p q_p$, where, for a prime integer p, q_p is the restriction of q to the Sylow p-subgroup $A_p \subset A$.*

(ii) *If q is non-degenerate, then $|A| = |H||H^\perp|$. Moreover, if the restriction $q|_H$ is non-degenerate, then $A = H \oplus H^\perp$ and $q = q|_H \oplus q|_{H^\perp}$.*

PROOF. See [79, Proposition 1.2.1] and [79, Proposition 1.2.2]. □

We now provide a list of examples of lattices which we will use throughout the thesis, with the associated discriminant groups.

EXAMPLE 1.1.4.

- If k is a non-zero integer, we denote by $\langle k \rangle$ the rank one lattice $L = \mathbb{Z}e$, with quadratic form $(e, e) = k$. It is positive definite if $k > 0$, negative definite if $k < 0$. The equivalence class of $\frac{e}{k} \in L \otimes \mathbb{Q}$ modulo L is a generator of $A_L \cong \frac{\mathbb{Z}}{|k|\mathbb{Z}}$, with $q_L\left(\frac{e}{k} + L\right) = \frac{1}{k}$.
- The lattice U is the unimodular, hyperbolic lattice of rank two defined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- To each of the simply laced Dynkin diagrams A_h, D_i, E_k, with $h \geq 1$, $i \geq 4$ and $k \in \{6, 7, 8\}$, we can associate a negative definite lattice of the same name. The generators of the lattice correspond bijectively to the vertices in the diagram: they are all set to have square -2. The product of two generators is 1 if the corresponding vertices in the diagram are connected by an edge, 0 otherwise. For example, we have:

$$A_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}; \qquad E_8 = \begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & 1 & & & \\ & & 1 & -2 & 0 & & & \\ & & 1 & 0 & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & 1 \\ & & & & & & 1 & -2 \end{pmatrix}.$$

- For p prime, $p \equiv 1 \pmod 4$, let

$$H_p := \begin{pmatrix} (p-1)/2 & 1 \\ 1 & -2 \end{pmatrix}.$$

It is a hyperbolic p-elementary lattice with $A_{H_p} \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$.
- For p prime, $p \equiv 3 \pmod 4$, let

$$K_p := \begin{pmatrix} -(p+1)/2 & 1 \\ 1 & -2 \end{pmatrix}.$$

It is a negative definite p-elementary lattice with $A_{K_p} \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$. In particular, $K_3 = A_2$.

REMARK 1.1.5. For any prime number p, we can define p-adic lattices and finite quadratic forms by replacing $\mathbb{Z}$ with the ring of p-adic integers $\mathbb{Z}_p$ (and $\mathbb{Q}$ with the field of p-adic numbers $\mathbb{Q}_p$) in the definitions that we have already given. Notice, in particular, that finite p-adic quadratic forms can be identified with finite quadratic forms $A \to \mathbb{Q}/2\mathbb{Z}$, where the group A is a finite abelian p-group (see [79, Section 1.7]).

1.2. Classification of discriminant forms

Let L, L' be two even lattices. We say that they have *isomorphic discriminant forms* (and we write $q_L \cong q_{L'}$) if there exists a group isomorphism $\rho : A_L \to A_{L'}$

such that $q_L(x) = q_{L'}(\rho(x)) \in \mathbb{Q}/2\mathbb{Z}$ for all $x \in A_L$. By [79, Theorem 1.3.1], L, L' have isomorphic discriminant forms if and only if there exist unimodular lattices V, V' such that $L \oplus V \cong L' \oplus V'$. Moreover, by [79, Theorem 1.1.1(a)] the signature $(v_{(+)}, v_{(-)})$ of an unimodular lattice V satisfies $v_{(+)} - v_{(-)} \equiv 0 \pmod 8$. Hence, the following definition is well-posed.

DEFINITION 1.2.1. The *signature modulo* 8 of a finite quadratic form q is

$$\operatorname{sign}(q) := l_{(+)} - l_{(-)} \pmod 8$$

where $(l_{(+)}, l_{(-)})$ is the signature of an even lattice L such that $q_L = q$.

We adopt the notation of [26]. Let p be an odd prime; we define two finite quadratic forms on $\frac{\mathbb{Z}}{p^\alpha \mathbb{Z}}$ $(\alpha \geq 1)$: they are denoted by $w^\epsilon_{p,\alpha}$, with $\epsilon \in \{-1, +1\}$. The quadratic form $w^{+1}_{p,\alpha}$ has generator value $q(1) = \frac{a}{p^\alpha} \pmod{2\mathbb{Z}}$, where a is the smallest positive even number which is a quadratic residue modulo p. Instead, for $w^{-1}_{p,\alpha}$ we have $q(1) = \frac{a}{p^\alpha}$, with a the smallest positive even number that is not a quadratic residue modulo p.

For $p = 2$, we also define finite quadratic forms $w^\epsilon_{2,\alpha}$ on $\frac{\mathbb{Z}}{2^\alpha\mathbb{Z}}$ $(\alpha \geq 1)$, where now we consider $\epsilon \in \{\pm 1, \pm 5\}$. In particular, the quadratic form $w^\epsilon_{2,\alpha}$ has $q(1) = \frac{\epsilon}{2^\alpha}$. We introduce two additional finite quadratic forms u_α, v_α, which are both defined on the group $\frac{\mathbb{Z}}{2^\alpha\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2^\alpha\mathbb{Z}}$ $(\alpha \geq 1)$. Their associated finite bilinear forms are given by the matrices

$$u_\alpha = \begin{pmatrix} 0 & \frac{1}{2^\alpha} \\ \frac{1}{2^\alpha} & 0 \end{pmatrix}, \qquad v_\alpha = \begin{pmatrix} \frac{1}{2^{\alpha-1}} & \frac{1}{2^\alpha} \\ \frac{1}{2^\alpha} & \frac{1}{2^{\alpha-1}} \end{pmatrix}.$$

The collection of finite quadratic forms $\left\{w^\epsilon_{p,\alpha}, u_\alpha, v_\alpha\right\}$, for p prime and $\alpha \geq 1$, is fundamental in the following sense.

THEOREM 1.2.2.

(*i*) *The semigroup of non-degenerate, p-adic finite quadratic forms is generated by $\left\{w^{\pm 1}_{p,\alpha}\right\}_{\alpha \geq 1}$ if p is odd, and by $\left\{w^{\pm 1}_{2,\alpha}, w^{\pm 5}_{2,\alpha}, u_\alpha, v_\alpha\right\}_{\alpha \geq 1}$ if $p = 2$.*

(*ii*) *Any non-degenerate finite quadratic form q is isomorphic to an orthogonal direct sum of the forms $w^\epsilon_{p,\alpha}, u_\alpha, v_\alpha$.*

PROOF. See [79, Theorem 1.8.1]. ☐

We point out that the representation of a finite quadratic form q as direct sum of the forms $w^\epsilon_{p,\alpha}, u_\alpha, v_\alpha$ may not be unique: there are several isomorphism relations between these forms (see [79, Proposition 1.8.2]). For instance, if p is odd we have $w^{+1}_{p,\alpha} \oplus w^{+1}_{p,\alpha} \cong w^{-1}_{p,\alpha} \oplus w^{-1}_{p,\alpha}$, while $w^\epsilon_{2,\alpha} \oplus w^{\epsilon'}_{2,\alpha} \cong w^{5\epsilon}_{2,\alpha} \oplus w^{5\epsilon'}_{2,\alpha}$ for all $\epsilon, \epsilon' \in \{\pm 1\}$.

As a consequence of Theorem 1.2.2, if p is an odd prime any non-degenerate quadratic form on $\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k}$, $k \geq 1$, is isomorphic to a direct sum of forms of type $w^{+1}_{p,1}$ and $w^{-1}_{p,1}$, with $w^{+1}_{p,1} \oplus w^{+1}_{p,1} \cong w^{-1}_{p,1} \oplus w^{-1}_{p,1}$. This means that, if S is a p-elementary lattice with discriminant group of length k, the form q_S on A_S can only be of two types, up to isometries:

$$q_S = \begin{cases} \left(w^{+1}_{p,1}\right)^{\oplus k} \\ \left(w^{+1}_{p,1}\right)^{\oplus k-1} \oplus w^{-1}_{p,1}. \end{cases}$$

REMARK 1.2.3. The signatures (mod 8) of the discriminant forms $w^\epsilon_{p,\alpha}$ are listed in [79, Proposition 1.11.2]. For p odd we have $\operatorname{sign}(w^{+1}_{p,1}) \equiv 1 - p \pmod 8$ and $\operatorname{sign}(w^{-1}_{p,1}) \equiv 5 - p \pmod 8$. Therefore, if S is p-elementary and $\operatorname{sign}(S) =$

$(s_{(+)}, s_{(-)})$, the quadratic form on the discriminant group $A_S = \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k}$ is

$$q_S = \begin{cases} \left(w_{p,1}^{+1}\right)^{\oplus k} & \text{if } s_{(+)} - s_{(-)} \equiv k(1-p) \pmod 8; \\ \left(w_{p,1}^{+1}\right)^{\oplus k-1} \oplus w_{p,1}^{-1} & \text{if } s_{(+)} - s_{(-)} \equiv k(1-p)+4 \pmod 8. \end{cases} \tag{3}$$

This means that the quadratic form of a p-elementary lattice ($p \neq 2$) is uniquely determined by its signature and its length (see [90, §1] for additional details).

EXAMPLE 1.2.4. The discriminant group of the lattice A_2 is $\mathbb{Z}/3\mathbb{Z}$ and its signature is $(0,2)$, therefore its discriminant form is $w_{3,1}^{+1}$. The lattice E_6 is also negative definite and 3-elementary, with $l(E_6) = 1$, but its signature modulo 8 is now $+2$, therefore its discriminant form is $w_{3,1}^{-1}$. The lattice $U(3)$ realizes instead the form $w_{3,1}^{+1} \oplus w_{3,1}^{-1}$, being 3-elementary of length 2 and signature $(1,1)$.

If S is a 2-elementary lattice, with $A_S \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus k}$, then q_S can be represented as a direct sum of quadratic forms $w_{2,1}^{\pm 1}, u_1, v_1$. The fundamental isomorphism relations between these four finite quadratic forms are the following:

$$\begin{aligned} u_1 \oplus u_1 &\cong v_1 \oplus v_1; \\ u_1 \oplus w_{2,1}^{\epsilon} &\cong w_{2,1}^{\epsilon} \oplus w_{2,1}^{\epsilon} \oplus w_{2,1}^{-\epsilon}; \\ v_1 \oplus w_{2,1}^{\epsilon} &\cong w_{2,1}^{-\epsilon} \oplus w_{2,1}^{-\epsilon} \oplus w_{2,1}^{-\epsilon}. \end{aligned}$$

Notice that $w_{2,1}^{1}$ (respectively, $w_{2,1}^{-1}$) is the discriminant form of the even lattice $\langle 2 \rangle$ (respectively, $\langle -2 \rangle$); u_1 is the discriminant form of $U(2)$, while v_1 is realized by the lattice D_4. This allows us to easily compute the signatures modulo 8 of the four quadratic forms, which are:

q	sign(q) (mod 8)
$w_{2,1}^{1}$	1
$w_{2,1}^{-1}$	-1
u_1	0
v_1	4

We see that, in contrast to the case p odd, the quadratic form of a 2-elementary lattice is not determined by its signature and length: for instance, the quadratic forms u_1 and $w_{2,1}^{+1} \oplus w_{2,1}^{-1}$ both have signature zero and length two, but they are not isomorphic.

1.3. Existence and uniqueness

A fundamental invariant, in the theory of lattices, is given by the *genus*.

DEFINITION 1.3.1. Two lattices L, L' belong to the same genus if $\mathrm{sign}(L) = \mathrm{sign}(L')$ and their p-adic completions $L \otimes \mathbb{Z}_p$, $L' \otimes \mathbb{Z}_p$ are isomorphic (as $\mathbb{Z}_p$-lattices) for all prime integers p.

Two lattices L, L' belong to the same genus if and only if $L \oplus U \cong L' \oplus U$, or equivalently if and only if they have the same signature and discriminant quadratic form:

THEOREM 1.3.2. *The genus of an even lattice L is determined by the triple $(l_{(+)}, l_{(-)}, q_L)$, where $(l_{(+)}, l_{(-)})$ is the signature of the lattice and q_L is its discriminant quadratic form.*

PROOF. See [79, Corollary 1.9.4]. □

Each genus contains only finitely-many isomorphism classes of lattices. It is an interesting problem to determine whether there exists an even lattice with given signature and quadratic form, and, if so, whether it is unique, up to isometries. The main results which we will need, regarding uniqueness of an indefinite lattice in its genus, are the following.

THEOREM 1.3.3. *Let L be an even lattice with discriminant quadratic form q_L and signature $(l_{(+)}, l_{(-)})$, with $l_{(+)} \geq 1$ and $l_{(-)} \geq 1$. Up to isometries, L is the unique lattice with invariants $(l_{(+)}, l_{(-)}, q_L)$ in all of the following cases:*

(i) $l_{(+)} + l_{(-)} \geq l(A_L) + 2$;
(ii) $l_{(+)} + l_{(-)} \geq 3$ *and* $\mathrm{discr}(L) \leq 127$;
(iii) $l_{(+)} + l_{(-)} \geq 3$ *and L is p-elementary, with p odd;*
(iv) *L is 2-elementary.*

PROOF. The statement combines [79, Corollary 1.13.3], [35, Chapter 15, Corollary 22], [17, Theorem 2.2] and [38, Theorem 1.5.2]. □

THEOREM 1.3.4. *If the genus of an indefinite lattice L, with $\mathrm{rk}(L) = n$ and $\mathrm{discr}(L) = d$, contains more than one isometry class, then $4^{\left[\frac{n}{2}\right]}d$ is divisible by $k^{\binom{n}{2}}$ for a non-square natural number $k \equiv 0, 1 \pmod 4$.*

PROOF. See [35, Chapter 15, Theorem 21]. □

In [79, Corollary 1.13.4]) it is shown that, if L is an even lattice of invariants $(l_{(+)}, l_{(-)}, q_L)$, then the genus defined by the triple $(l_{(+)}+1, l_{(-)}+1, q_L)$ only contains (up to isometries) the lattice $L \oplus U$, while instead $L \oplus E_8$ is the unique lattice with invariants $(l_{(+)}, l_{(-)}+8, q_L)$. These properties are used in the proof of the following splitting theorem, which we will frequently apply.

THEOREM 1.3.5. *Let L be an even lattice with $\mathrm{sign}(L) = (l_{(+)}, l_{(-)})$.*

(i) *If $l_{(+)} \geq 1$, $l_{(-)} \geq 1$ and $l_{(+)} + l_{(-)} \geq 3 + l(A_L)$, then $L = L' \oplus U$ for some even lattice L' such that $\mathrm{sign}(L') = (l_{(+)} - 1, l_{(-)} - 1)$ and $q_{L'} = q_L$.*
(ii) *If $l_{(+)} \geq 1$, $l_{(-)} \geq 8$ and $l_{(+)} + l_{(-)} \geq 9 + l(A_L)$, then $L = L' \oplus E_8$ for some even lattice L' such that $\mathrm{sign}(L') = (l_{(+)}, l_{(-)} - 8)$ and $q_{L'} = q_L$.*

PROOF. See [79, Corollary 1.13.5]. □

We now turn our attention to criteria for the existence of a lattice in a genus. We first need to recall the following result on the existence (and uniqueness) of p-adic lattices of maximal length.

THEOREM 1.3.6. *Let p be a prime and q_p a quadratic form on a finite, abelian p-group A_p. Then there exists a p-adic lattice $K(q_p)$ of rank $l(A_p)$ and discriminant form isomorphic to q_p. The p-adic lattice $K(q_p)$ is unique (up to isometries) unless $p = 2$ and there exists a finite quadratic form q'_2 such that $q_2 \cong w^{\pm 1}_{2,1} \oplus q'_2$.*

PROOF. See [79, Theorem 1.9.1]. □

The following theorem, due to Nikulin, provides necessary and sufficient conditions for the existence of an even lattice with given signature and discriminant form.

THEOREM 1.3.7. *Let q be a quadratic form on a finite abelian group A. There exists an even lattice with invariants $(l_{(+)}, l_{(-)}, q)$ if and only if the following conditions are satisfied:*

(i) $l_{(+)} \geq 0$, $l_{(-)} \geq 0$ *and* $l_{(+)} + l_{(-)} \geq l(A)$;
(ii) $l_{(+)} - l_{(-)} \equiv \mathrm{sign}(q) \pmod 8$;

(*iii*) $(-1)^{l_{(-)}} |A| \equiv \operatorname{discr}(K(q_p)) \pmod{(\mathbb{Z}_p^*)^2}$ *for all odd prime integers* p *such that* $l_{(+)} + l_{(-)} = l(A_p)$;

(*iv*) $|A| \equiv \pm \operatorname{discr}(K(q_2)) \pmod{(\mathbb{Z}_2^*)^2}$ *if* $l_{(+)} + l_{(-)} = l(A_2)$ *and* q_2 *is not of the form* $w_{2,1}^{\pm 1} \oplus q_2'$ *for any finite quadratic form* q_2'.

Proof. See [79, Theorem 1.10.1]. □

We point out that it is not necessary to check the two local conditions (*iii*) and (*iv*) if we are interested in proving the existence of a lattice L with $\operatorname{rk}(L) > l(A_L)$: in this case, Theorem 1.3.7 can be formulated as follows.

Theorem 1.3.8. *Let* q *be a quadratic form on a finite abelian group* A. *There exists an even lattice with invariants* $(l_{(+)}, l_{(-)}, q)$ *assuming that the following conditions are satisfied:*

(*i*) $l_{(+)} \geq 0$, $l_{(-)} \geq 0$ *and* $l_{(+)} + l_{(-)} > l(A)$;

(*ii*) $l_{(+)} - l_{(-)} \equiv \operatorname{sign}(q) \pmod 8$.

Proof. See [79, Corollary 1.10.2]. □

In the case of p-elementary lattices, with p odd, we already saw in Section 1.2 that the discriminant form of the lattice is uniquely determined (up to isometries) by its signature and length. This means that the genus of a p-elementary lattice L, with $p \neq 2$, depends only on the invariants $(l_{(+)}, l_{(-)}, l(A_L))$.

Theorem 1.3.9. *Let* p *be an odd prime. There exists an even hyperbolic lattice* L *with* $\operatorname{rk}(L) = r \geq 1$ *and* $l(A_L) = a \geq 0$ *if and only if the following conditions are satisfied:*

$$\begin{cases} a \leq r; \\ r \equiv 0 \pmod 2; \\ \text{if } a \equiv 0 \pmod 2, \text{ then } r \equiv 2 \pmod 4; \\ \text{if } a \equiv 1 \pmod 2, \text{ then } p \equiv (-1)^{\frac{r}{2}-1} \pmod 4; \\ \text{if } r \not\equiv 2 \pmod 8, \text{ then } r > a > 0. \end{cases}$$

Proof. See [90, §1]. □

For 2-elementary lattices, as we remarked in Section 1.2, signature and length are not enough to determine the discriminant form. We need to introduce an additional invariant.

Definition 1.3.10. Let q be a quadratic form on a finite abelian group A. We define:

$$\delta(q) = \begin{cases} 0 & \text{if } q(x) \in \mathbb{Z}/2\mathbb{Z} \text{ for all } x \in A; \\ 1 & \text{otherwise.} \end{cases}$$

If L is an even lattice, we set $\delta(L) := \delta(q_L)$.

With respect to the four fundamental finite quadratic forms $w_{2,1}^{\pm 1}, u_1, v_1$ defined in Section 1.2, which constitute the building blocks for any 2-elementary discriminant form, we have:

q	$\delta(q)$
$w_{2,1}^{1}$	1
$w_{2,1}^{-1}$	1
u_1	0
v_1	0

Theorem 1.3.11. *An even indefinite* 2*-elementary lattice* L *with signature* $(l_{(+)}, l_{(-)})$ *is uniquely determined by the invariants* $(l_{(+)}, l_{(-)}, l(A_L), \delta(L))$, *up to isometries. There exists an even* 2*-elementary lattice* L *with* $\operatorname{sign}(L) = (l_{(+)}, l_{(-)})$,

$l(A_L) = a \geq 0$ and $\delta(L) = \delta \in \{0,1\}$ if and only if $l_{(+)} \geq 0$, $l_{(-)} \geq 0$ and the following conditions are satisfied:

$$\begin{cases} a \leq l_{(+)} + l_{(-)}; \\ l_{(+)} + l_{(-)} \equiv a \pmod 2; \\ \text{if } \delta = 0, \text{ then } l_{(+)} - l_{(-)} \equiv 0 \pmod 4; \\ \text{if } a = 0, \text{ then } \delta = 0 \text{ and } l_{(+)} - l_{(-)} \equiv 0 \pmod 8; \\ \text{if } a = 1, \text{ then } l_{(+)} - l_{(-)} \equiv 1 \pmod 8; \\ \text{if } a = 2 \text{ and } l_{(+)} - l_{(-)} \equiv 4 \pmod 8, \text{ then } \delta = 0; \\ \text{if } \delta = 0 \text{ and } l_{(+)} + l_{(-)} = a, \text{ then } l_{(+)} - l_{(-)} \equiv 0 \pmod 8. \end{cases}$$

Proof. See [38, Theorem 1.5.2]. □

1.4. Orthogonal sublattices and primitive embeddings

If $T \subset L$ is a sublattice of maximal rank, we have the following sequence of inclusions:

$$T \hookrightarrow L \hookrightarrow L^\vee \hookrightarrow T^\vee$$

such that the composition is just the canonical inclusion of T in its dual $T^\vee$. Fix a basis $\{t_i\}_i$ of T and a basis $\{e_i\}_i$ of L, and denote by G_T (respectively, G_L) the Gram matrix of the lattice T (respectively, L) with respect to the chosen basis. If W is the matrix which represents the inclusion $T \hookrightarrow L$, then the transposed matrix W^t represents $L^\vee \hookrightarrow T^\vee$, therefore we conclude $G_T = W^t G_L W$. The determinant of W coincides with the index $[L : T]$, while the determinants of G_T and G_L are the discriminants of T and L respectively, therefore

$$[L : T]^2 = \frac{\operatorname{discr}(T)}{\operatorname{discr}(L)} = \frac{|A_T|}{|A_L|}.$$

In a more general setting, if $T \subset L$ is a primitive sublattice of any rank, then $T \oplus T^\perp \subset L$ has maximal rank, which implies

$$[L : (T \oplus T^\perp)]^2 = \frac{\operatorname{discr}(T \oplus T^\perp)}{\operatorname{discr}(L)} = \frac{|A_T|\,|A_{T^\perp}|}{|A_L|}. \tag{4}$$

Notice that, by the sequence of inclusions

$$T \oplus T^\vee \hookrightarrow L \hookrightarrow L^\vee \hookrightarrow \left(T \oplus T^\perp\right)^\vee \cong T^\vee \oplus \left(T^\perp\right)^\vee$$

the quotient $L/(T \oplus T^\perp)$ is isomorphic to a subgroup $M \subset A_T \oplus A_{T^\perp}$ which is isotropic (i.e. $(q_T \oplus q_{T^\perp})|_M = 0$), thus $M \subset M^\perp$ and $M^\perp/M \cong A_L$. In particular, the equality (4) implies that $|A_T|\,|A_{T^\perp}| = |A_L|\,|M|^2$. The projections

$$p_T : A_T \oplus A_{T^\perp} \to A_T, \qquad p_{T^\perp} : A_T \oplus A_{T^\perp} \to A_{T^\perp}$$

are such that $M \cong M_T := p_T(M)$ and $M \cong M_{T^\perp} := p_{T^\perp}(M)$ as groups. Moreover, the composition

$$\gamma := p_{T^\perp} \circ (p_T)^{-1}|_{M_T} : M_T \to M_{T^\perp} \tag{5}$$

is an anti-isometry, i.e. an isomorphism of groups such that $q_T(x) = -q_{T^\perp}(\gamma(x))$ for all $x \in M_T$.

Lemma 1.4.1. *Let L be an unimodular lattice and $T \subset L$ a primitive sublattice. Then, as groups:*

$$A_T \cong A_{T^\perp} \cong \frac{L}{T \oplus T^\perp}.$$

Proof. Since $|A_L| = 1$, we have $|M|^2 = |A_T|\,|A_{T^\perp}|$ by (4), hence $M_T = A_T$ and $M_{T^\perp} = A_{T^\perp}$. The projections $p_T|_M : M \to A_T$ and $p_{T^\perp}|_M : M \to A_{T^\perp}$ are therefore isomorphisms of groups. □

A lattice isometry $\varphi : L \to L'$ induces in a natural way a group isomorphism $\overline{\varphi} : A_L \to A_{L'}$ such that $q_L(x + L) = q_{L'}(\overline{\varphi}(x + L))$ for all $x \in L^\vee$. This isometry of finite quadratic forms is defined as $\overline{\varphi}(x + L) := \left(\varphi^{-1}\right)^\vee(x) + L'$, where $\left(\varphi^{-1}\right)^\vee : L^\vee \to (L')^\vee$ is the transposed morphism of φ^{-1}. In particular, for any lattice L we obtain a canonical homomorphism $O(L) \to O(A_L)$, $\varphi \mapsto \overline{\varphi}$.

PROPOSITION 1.4.2. *Let L be an indefinite lattice, with $\mathrm{rk}(L) \geq l(A_L) + 2$. Then the homomorphism $O(L) \to O(A_L)$, $\varphi \mapsto \overline{\varphi}$ is surjective.*

PROOF. See [38, Proposition 1.4.7]. □

When studying primitive embeddings $L \hookrightarrow L'$, we can usually consider as equivalent two embeddings whose images correspond to each other via an isometry of L'.

DEFINITION 1.4.3. Two primitive embeddings $i : L \hookrightarrow L'$, $j : L \hookrightarrow L''$ define *isomorphic primitive sublattices* if there exists an isometry $\varphi : L' \to L''$ such that $\varphi(i(L)) = j(L)$.

The following fundamental result, proved by Nikulin in [79, Proposition 1.15.1], provides a practical description of primitive embeddings.

THEOREM 1.4.4. *Let S be an even lattice of signature $(s_{(+)}, s_{(-)})$ and discriminant form q_S. For an even lattice L of invariants $(l_{(+)}, l_{(-)}, q_L)$ unique in its genus (up to isometries), primitive embeddings $i : S \hookrightarrow L$ are determined by quintuples $\Theta_i := (H_S, H_L, \gamma, T, \gamma_T)$ such that:*

- *H_S is a subgroup of A_S, H_L is a subgroup of A_L and $\gamma : H_S \to H_L$ is an isomorphism $q_S|_{H_S} \cong q_L|_{H_L}$;*
- *T is a lattice of signature $(l_{(+)} - s_{(+)}, l_{(-)} - s_{(-)})$ and discriminant form $q_T = ((-q_S) \oplus q_L)|_{\Gamma^\perp/\Gamma}$, where $\Gamma \subset A_S \oplus A_L$ is the graph of γ and $\Gamma^\perp$ is its orthogonal complement in $A_S \oplus A_L$ with respect to the finite bilinear form associated to $(-q_S) \oplus q_L$;*
- *$\gamma_T \in O(A_T)$.*

The lattice T is isomorphic to the orthogonal complement of $i(S)$ in L. Moreover, two quintuples Θ and Θ' define isomorphic primitive sublattices if and only if $\overline{\mu}(H_S) = H'_S$ for some $\mu \in O(S)$ and there exist $\phi \in O(A_L)$, $\nu : T \to T'$ isometries such that $\gamma' \circ \overline{\mu} = \phi \circ \gamma$ and $\overline{\nu} \circ \gamma_T = \gamma'_{T'} \circ \overline{\nu}$.

Notice in particular that Theorem 1.4.4 allows us to determine the discriminant forms of all possible orthogonal complements $T \subset L$ of the image of a primitive embedding $S \hookrightarrow L$. We will use this result throughout the thesis. In the case where L is unimodular, Theorem 1.4.4 admits a simpler formulation.

COROLLARY 1.4.5. *Let S be an even lattice of signature $(s_{(+)}, s_{(-)})$ and discriminant form q_S, and L an even unimodular lattice of signature $(l_{(+)}, l_{(-)})$ unique in its genus (up to isometries). There exists a primitive embedding $i : S \hookrightarrow L$ if and only if there exists an even lattice T of signature $(l_{(+)} - s_{(+)}, l_{(-)} - s_{(-)})$ and discriminant form $q_T = -q_S$. For a given lattice T with these properties, each primitive embedding $i : S \hookrightarrow L$ with $i(S)^\perp \cong T$ is determined by an isomorphism $\gamma : A_S \to A_T$ such that $q_T \circ \gamma = -q_S$.*

PROOF. See [79, Proposition 1.6.1]. □

EXAMPLE 1.4.6. In order to exemplify how Theorem 1.4.4 is applied, we classify primitive embeddings of $S = U(2)$ inside the lattice $L := U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1) \rangle$, which is unique in its genus (up to isometries) by Theorem 1.3.3. As we saw in Section 1.2, $A_S \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 2}$ with discriminant form $q_S = u_1$, while $A_L \cong \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}$

and q_L takes value $-\frac{1}{2(n-1)}$ on the generator e of the discriminant group. As a consequence, $H_L \subset A_L$ can only be 0 or the subgroup of order two generated by $(n-1)e$.

- If $H_L = 0$, then $H_S = 0$ and $\Gamma = \{(0,0)\} \subset A_S \oplus A_L$. In this case, the orthogonal complement S inside L is a lattice of signature $(2,19)$ and discriminant form $-q_S \oplus q_L$. By Theorem 1.3.3, there is only one isometry class of lattices T with these properties, and a representative is $T = U \oplus U(2) \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$. Notice that a primitive embedding $S \hookrightarrow L$ with such an orthogonal complement can be realized as follows. Let $\{e_1, e_2\}$, $\{f_1, f_2\}$ be the bases of two distinct summands U of L: we then map the first generator of $S = U(2)$ to $e_1 + f_1$, and the second generator to $e_2 + f_2$.
- If $H_L = \langle (n-1)e\rangle$, then $H_S = \langle s\rangle$, where $s \in A_S$ is an element of order two such that $q_S(s) \equiv q_L((n-1)e) = -\frac{n-1}{2} \pmod 2$: in particular, such an element exists if and only if n is odd. Let ϵ_1, ϵ_2 be the generators of A_S; then, if $n \equiv 1 \pmod 4$ we can either choose $s = \epsilon_1$ or $s = \epsilon_2$, while we need to take $s = \epsilon_1 + \epsilon_2$ if $n \equiv 3 \pmod 4$. Here $\Gamma = \langle (s, (n-1)e)\rangle$ and we compute that the quadratic form on $\Gamma^\perp/\Gamma$ is isomorphic to q_L. As a consequence, the orthogonal complement of S in L is isometric to $T = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$ (see Theorem 1.3.3). We can realize this embedding in the following way. Let $\{e_1, e_2\}$ be a basis for a summand U of L and g the generator of the summand $\langle -2(n-1)\rangle$. Let $t = \frac{n-1}{2} \in \mathbb{N}$; we then map the first generator of $S = U(2)$ to $2e_1 + te_2 + g$, and the second generator to e_2.

As a consequence of Theorem 1.4.4 we also get a characterization for the existence of an isometry $\Phi \in O(L)$ which extends an isometry $\phi \in O(T)$, where $T \subset L$ is a primitive sublattice. Recall from (5) the definition of the anti-isometry $\gamma = p_{T^\perp} \circ (p_T)^{-1}|_{M_T} : M_T \to M_{T^\perp}$.

Proposition 1.4.7. *Let L be an even lattice, $T \subset L$ a primitive sublattice and $\phi \in O(T)$. There exists an isometry $\Phi \in O(L)$ such that $\Phi|_T = \phi$ if and only if there exists $\psi \in O(T^\perp)$ such that $\overline{\psi} \circ \gamma = \gamma \circ \overline{\phi}$.*

Proof. See [79, Corollary 1.5.2]. □

In particular, for any isometry $\psi \in O(T^\perp)$ as in the statement there exists $\Phi \in O(L)$ which extends $\phi \oplus \psi \in O(T \oplus T^\perp)$.

CHAPTER 2

Irreducible holomorphic symplectic manifolds

2.1. General facts and examples

Definition 2.1.1. An *irreducible holomorphic symplectic* (IHS) manifold is a compact Kähler manifold X which is simply connected and such that $H^0(X, \Omega_X^2) = \mathbb{C}\omega_X$, where ω_X is an everywhere non-degenerate holomorphic two-form.

The holomorphic form ω_X is referred to as the *symplectic form* of X. We recall, in the following remark, the main properties of irreducible holomorphic symplectic manifolds (for more details, see for instance [48]).

Remark 2.1.2.

(*i*) Since ω_X induces a symplectic form on the tangent space T_xX, for all points $x \in X$, the complex dimension of X is even.

(*ii*) If $\dim X = 2n$, then $\chi(X, \mathcal{O}_X) = n+1$, because for $k \in \{0, \ldots, 2n\}$ we have
$$H^0(X, \Omega_X^k) = \begin{cases} \mathbb{C}\omega_X^{k/2} & \text{if } k \equiv 0 \pmod 2; \\ 0 & \text{if } k \equiv 1 \pmod 2. \end{cases}$$

(*iii*) Since ω_X^n generates $H^0(X, \Omega_X^{2n}) = H^0(X, K_X)$ and it is nowhere vanishing, it provides a trivialization of the canonical bundle, therefore $K_X \cong \mathcal{O}_X$.

(*iv*) The two-form ω_X defines an alternating homomorphism $TX \to \Omega_X^1$, which is bijective since ω_X is everywhere non-degenerate. As a consequence, $TX \cong \Omega_X^1$ and thus $H^1(TX) \cong H^{1,1}(X)$.

(*v*) Since X is simply connected, we have $H_1(X, \mathbb{Z}) = 0$ and therefore (by the universal coefficient theorem) the second cohomology group with integer coefficients $H^2(X, \mathbb{Z})$ has no torsion.

From the triviality of the canonical bundle, IHS manifolds have vanishing first Chern class: they actually constitute one of the three building blocks of compact Kähler manifolds Z such that $c_1(Z)_{\mathbb{R}} = 0$, as explained by the following theorem.

Theorem 2.1.3. *(Beauville–Bogomolov decomposition) Let Z be a compact Kähler manifold with $c_1(Z)_{\mathbb{R}} = 0$. Then there exists a finite étale covering Z' of Z with*
$$Z' \cong T \times \bigsqcup_i V_i \times \bigsqcup_j X_j$$
where T is a complex torus, V_i are Calabi-Yau manifolds and X_j are IHS manifolds.

Proof. See [10, Théorème 2]. □

Let X be an irreducible holomorphic symplectic manifold: since $K_X \cong \mathcal{O}_X$, the deformations of X are unobstructed by the Bogomolov–Tian–Todorov theorem (see [95]). This implies that there exists a universal deformation $\mathcal{X} \to \mathrm{Def}(X)$ of the manifold X, with $\mathrm{Def}(X)$ smooth of dimension $h^1(X, TX) = h^{1,1}(X) = b_2(X) - 2$.

Theorem 2.1.4. *Let $\mathcal{X} \to I$ be a smooth and proper morphism of complex manifolds and assume that the fiber $\mathcal{X}_0$ over a point $0 \in I$ is irreducible holomorphic symplectic. Then, if a fiber $\mathcal{X}_t$ is Kähler, it is an IHS manifold.*

PROOF. See [10, Proposition 9]. □

In particular, small deformations of irreducible holomorphic symplectic manifolds are again IHS, by combining Theorem 2.1.4 with [63, Theorem 15].

We now present some examples of IHS manifolds. First, we recall the definition of a $K3$ surface.

DEFINITION 2.1.5. A $K3$ surface is a compact complex smooth surface Σ such that $K_\Sigma \cong \mathcal{O}_\Sigma$ and $H^1(\Sigma, \mathcal{O}_\Sigma) = 0$.

We point out that any $K3$ surface is Kähler, even if this property is not explicitly requested in the definition (see [93]). All irreducible holomorphic symplectic manifolds of dimension two are $K3$ surfaces, and we can therefore state that IHS manifolds provide a generalization of $K3$ surfaces in higher dimensions. Notice that abelian surfaces (i.e. algebraic complex two-dimensional tori) are endowed with a non-degenerate holomorphic 2-form and therefore they are holomorphic symplectic surfaces, however they are not IHS because they are not simply connected.

EXAMPLE 2.1.6. *Hilbert schemes of points on a $K3$ surface.*
Let Σ be a $K3$ surface and $n \geq 1$ an integer. We will denote by $\Sigma^{[n]}$ the Hilbert scheme of n points on Σ, that is the Douady space parametrizing zero-dimensional subschemes $(Z, \mathcal{O}_Z)$ of the surface Σ of length n (i.e. $\dim_\mathbb{C} \mathcal{O}_Z = n$). Notice that $\Sigma^{[n]}$ is, in general, just a complex space, but it is a scheme (even projective) if the $K3$ surface Σ is projective (see [10, §6]). The Hilbert scheme $\Sigma^{[n]}$ also arises as a minimal resolution of singularities of the n-th symmetric product $\Sigma^{(n)}$, via the *Hilbert–Chow morphism*

$$\begin{aligned} \rho : \Sigma^{[n]} &\to \Sigma^{(n)} \\ [(Z, \mathcal{O}_Z)] &\mapsto \sum_{p \in \Sigma} l(\mathcal{O}_{Z,p}) p \end{aligned}$$

where $l(\mathcal{O}_{Z,p})$ is the length of $\mathcal{O}_{Z,p}$, which is zero outside the (finite) set of points p in the support of Z. It was proved by Fogarty that ρ is a resolution of the singularities of $\Sigma^{(n)}$ and that $\Sigma^{[n]}$ is smooth; it is also Kähler because Σ is Kähler (see [97]). In [10], Beauville showed that $\Sigma^{[n]}$ is an irreducible symplectic manifold of dimension $2n$, whose symplectic form is derived from the one of the underlying $K3$ surface. This result had already been obtained by Fujiki in the case $n = 2$, where the geometric description of the Hilbert scheme is particularly simple. Any irreducible holomorphic symplectic manifold which is deformation equivalent to $\Sigma^{[n]}$, for some $K3$ surface Σ, is called a manifold of *$K3^{[n]}$-type*.

EXAMPLE 2.1.7. *Generalized Kummer manifolds.*
Let A be a complex two-dimensional torus and $n \geq 1$ an integer. The Hilbert scheme $A^{[n+1]}$ is holomorphic symplectic, but it is not IHS since it is not simply connected. We consider the summation morphism

$$\begin{aligned} s : A^{[n+1]} &\to A \\ [(Z, \mathcal{O}_Z)] &\mapsto \sum_{p \in A} l(\mathcal{O}_{Z,p}) p \end{aligned}$$

and we define $K_n(A) := s^{-1}(0)$, where $0 \in A$ is the zero-point of the torus. The fiber $K_n(A)$ is now an IHS manifold of dimension $2n$, as proved by Beauville in [10], and we refer to these varieties as *generalized Kummer manifolds*. In particular, $K_1(A)$ is the Kummer $K3$ surface of the torus A, which is isomorphic to the blow-up of the quotient $A/\{\pm \mathrm{id}\}$. Irreducible holomorphic symplectic manifolds which are deformations of a generalized Kummer manifold are called IHS manifolds of *Kummer-type*.

Hilbert schemes of points on a $K3$ surface and generalized Kummer manifolds provide two distinct ways to construct irreducible holomorphic symplectic manifolds in all even complex dimensions. Up to deformation, these are actually the only known examples of IHS manifolds, except in dimension six and in dimension ten, where we have two constructions (due to O'Grady) of irreducible holomorphic symplectic manifolds which are neither of $K3^{[n]}$-type, nor of Kummer-type. For more details on the two families of O'Grady manifolds, which are obtained as desingularizations of some moduli spaces of sheaves on $K3$ surfaces and on abelian surfaces, we refer the reader to [82] and [83].

To conclude this section, we recall a geometrical construction of a manifold of $K3^{[2]}$-type, which we will need in Chapter 4.

EXAMPLE 2.1.8. *The Fano variety of lines on a cubic fourfold.*
Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold. The *Fano variety of lines* $F(Y)$ is the variety parametrizing lines contained in Y: it is again a smooth, projective manifold of dimension four (see [6, Theorem 8]). Beauville and Donagi proved in [13] that, if Y is a Pfaffian cubic fourfold, then $F(Y)$ is isomorphic to the Hilbert scheme $\Sigma^{[2]}$, for $\Sigma \subset \mathbb{P}^8$ a $K3$ surface of degree 14. As a consequence, for any cubic hypersurface $Y \subset \mathbb{P}^5$ the Fano variety of lines $F(Y)$ is an irreducible holomorphic symplectic manifold of $K3^{[2]}$-type.

2.2. Cohomology of IHS manifolds

One of the main properties of IHS manifolds is that their second cohomology group with integer coefficients (which, as we have already seen, is torsion-free) can be equipped with a non-degenerate symmetric bilinear form, which generalizes the intersection product on $H^2(\Sigma, \mathbb{Z})$ for a $K3$ surface Σ.

Let X be an irreducible holomorphic symplectic manifold of dimension $2n$ and let ω be a symplectic form on X satisfying $\int_X (\omega \wedge \overline{\omega})^n = 1$. We define the following quadratic form on the elements $\alpha \in H^2(X, \mathbb{C})$:

$$\widetilde{q}(\alpha) := \frac{n}{2} \int_X (\omega \wedge \overline{\omega})^{n-1} \wedge \alpha^2 + (1-n) \left(\int_X \omega^{n-1} \wedge \overline{\omega}^n \wedge \alpha \right) \left(\int_X \omega^n \wedge \overline{\omega}^{n-1} \wedge \alpha \right).$$

With respect to the Hodge decomposition

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

for any $\alpha \in H^2(X, \mathbb{C})$ we can write $\alpha = a\omega + \xi + b\overline{\omega}$, with $\xi \in H^{1,1}(X)$. Then, we have

$$\widetilde{q}(\alpha) = ab + \frac{n}{2} \int_X (\omega \wedge \overline{\omega})^{n-1} \wedge \xi^2.$$

In particular, $H^{1,1}(X)$ is orthogonal to $H^{2,0}(X) \oplus H^{0,2}(X)$, with respect to the bilinear form associated to $\widetilde{q}$.

THEOREM 2.2.1. *Let X be an IHS manifold of dimension $2n$. There exists a positive constant $c_X \in \mathbb{R}$ such that, for all $\alpha \in H^2(X, \mathbb{C})$:*

$$\int_X \alpha^{2n} = c_X \widetilde{q}(\alpha)^n.$$

Furthermore, the quadratic form $\widetilde{q}$ can be renormalized to a quadratic form q which is non-degenerate, primitive and integral on $H^2(X, \mathbb{Z})$.

PROOF. See [41, Theorem 4.7]. □

The (unique) renormalized quadratic form q mentioned in the theorem is such that, for any $\alpha \in H^2(X, \mathbb{Z})$:

$$\int_X \alpha^{2n} = cq(\alpha)^n$$

where $c \in \mathbb{Q}^+$ is called the *Fujiki constant* of the manifold X.

EXAMPLE 2.2.2. For manifolds of $K3^{[n]}$-type, the Fujiki constant is $c = \frac{(2n)!}{n!2^n}$ (computed by Beauville in [10]).

The integral quadratic form q on $H^2(X,\mathbb{Z})$ is called the *Beauville–Bogomolov–Fujiki quadratic form* of the irreducible holomorphic symplectic manifold X. The cohomology group $H^2(X,\mathbb{Z})$ is therefore endowed with a lattice structure, whose signature is $(3, b_2(X) - 3)$ by [10, Théorème 5]. Moreover, from the definition of $\widetilde{q}$ we deduce

$$\widetilde{q}(\omega) = 0, \qquad \widetilde{q}(\omega + \overline{\omega}) > 0.$$

REMARK 2.2.3. When using the Beauville–Bogomolov–Fujiki quadratic form q on $H^2(X,\mathbb{Z})$, for an IHS manifold X, we will often write x^2 in place of $q(x)$, to denote the value of q on an element $x \in H^2(X,\mathbb{Z})$. We will instead use the brackets $(\cdot,\cdot)$ to refer to the bilinear form associated to q.

We emphasize that the Fujiki constant and the Beauville–Bogomolov–Fujiki quadratic form are birational invariants and deformation invariants. As a consequence, for any IHS manifold X' which is deformation equivalent to X we have a lattice isometry $H^2(X',\mathbb{Z}) \cong H^2(X,\mathbb{Z})$. Also notice that the Néron–Severi group $\mathrm{NS}(X) := H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ is a sublattice of $H^2(X,\mathbb{Z})$, which can be identified with:

$$\mathrm{NS}(X) = H^2(X,\mathbb{Z}) \cap \omega^\perp$$

because $H^{1,1}(X)$ is orthogonal to $H^{2,0}(X) \oplus H^{0,2}(X)$ inside $H^2(X,\mathbb{C})$.

EXAMPLE 2.2.4. Let Σ be a $K3$ surface. Then, the pairing associated to the Beauville–Bogomolov–Fujiki quadratic form is just the intersection form on $H^2(X,\mathbb{Z})$. As a lattice, $H^2(\Sigma,\mathbb{Z})$ is unimodular and we have an isometry:

$$\big(H^2(\Sigma,\mathbb{Z}), q\big) \cong L_{K3} := U^{\oplus 3} \oplus E_8^{\oplus 2}.$$

For any $n \geq 2$, there exists a natural inclusion (see [10, Proposition 6])

$$i : H^2(\Sigma,\mathbb{Z}) \hookrightarrow H^2(\Sigma^{[n]},\mathbb{Z})$$

such that

$$H^2(\Sigma^{[n]},\mathbb{Z}) = i\left(H^2(\Sigma,\mathbb{Z})\right) \oplus \mathbb{Z}\delta$$

where 2δ is the class of the exceptional divisor E of the Hilbert–Chow morphism $\rho : \Sigma^{[n]} \to \Sigma^{(n)}$ (in particular, E is the locus in $\Sigma^{[n]}$ which parametrizes non-reduced zero-dimensional subschemes of length n). By restricting to algebraic classes, we also have

$$\mathrm{NS}(\Sigma^{[n]}) = i\,(\mathrm{NS}(\Sigma)) \oplus \mathbb{Z}\delta.$$

The class δ is such that $q(\delta) = -2(n-1)$, thus, for any manifold X of $K3^{[n]}$-type, we have

$$\big(H^2(X,\mathbb{Z}), q\big) \cong L_n := U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle.$$

In particular, $b_2(X) = \mathrm{rk}(L_n) = 23$ and $\mathrm{sign}(L_n) = (3, 20)$.

Using the Beauville–Bogomolov–Fujiki quadratic form, one can obtain the Euler characteristic of any divisor $D \in H^2(X,\mathbb{Z})$ (see [49, Example 23.19]):

$$\chi(X, D) = \binom{q(D)/2 + n + 1}{n}.$$

EXAMPLE 2.2.5. For the sake of completeness, we recall the isometry classes and Fujiki constants of the Beauville–Bogomolov–Fujiki lattices for the other known deformation types of IHS manifolds (see [89]).

- Let X be an IHS manifold deformation equivalent to a $2n$-dimensional generalized Kummer variety $K_n(A)$. Then
$$\big(H^2(X,\mathbb{Z}),q\big) \cong U^{\oplus 3} \oplus \langle -2(n+1)\rangle$$
and the Fujiki constant is $c = \frac{(2n)!}{n!2^n}(n+1)$. In particular, $b_2(X) = 7$.
- Let X be an IHS manifold deformation equivalent to a O'Grady sixfold. Then
$$\big(H^2(X,\mathbb{Z}),q\big) \cong U^{\oplus 3} \oplus \langle -2\rangle^{\oplus 2}$$
and the Fujiki constant is $c = 60$. In particular, $b_2(X) = 8$.
- Let X be an IHS manifold deformation equivalent to a O'Grady tenfold. Then
$$\big(H^2(X,\mathbb{Z}),q\big) \cong U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_2$$
and the Fujiki constant is $c = 945$. In particular, $b_2(X) = 24$.

Notice that it is possible to distinguish between the different deformation types just by looking at the rank of $H^2(X,\mathbb{Z})$ (i.e. the second Betti number).

As in the case of $K3$ surfaces, there exists a projectivity criterion for all IHS manifolds, which employs the Beauville–Bogomolov–Fujiki quadratic form.

Theorem 2.2.6. *Let X be an irreducible holomorphic symplectic manifold. Then X is projective if and only if there exists $l \in H^{1,1}(X) \cap H^2(X,\mathbb{Z})$ such that $q(l) > 0$.*

Proof. See [49, Proposition 26.13]. □

Equivalently, the projectivity criterion states that an IHS manifold X is projective if and only if the Néron–Severi sublattice $\mathrm{NS}(X) \subset H^2(X,\mathbb{Z})$ is hyperbolic.

Having introduced the Beauville–Bogomolov–Fujiki lattice, we can present an additional construction of irreducible holomorphic symplectic manifolds of $K3^{[n]}$-type.

Example 2.2.7. *Moduli spaces of sheaves on $K3$ surfaces.*
Let (Σ,α) be a twisted $K3$ surface, where $\alpha \in \mathrm{Br}(\Sigma) := H^2\left(\Sigma,\mathcal{O}_\Sigma^*\right)_{\mathrm{tor}}$ is a Brauer class. By [96, §2], if α has order k then it can be identified with a surjective homomorphism $\alpha : \mathrm{Tr}(\Sigma) \to \mathbb{Z}/k\mathbb{Z}$, where $\mathrm{Tr}(\Sigma) := \mathrm{NS}(\Sigma)^\perp \subset H^2(\Sigma,\mathbb{Z})$ is the transcendental lattice of the surface. A B-field lift of α is a class $B \in H^2(\Sigma,\mathbb{Q})$ (which can be determined via the exponential sequence) such that $kB \in H^2(\Sigma,\mathbb{Z})$ and $\alpha(v) = (kB,v)$ for all $v \in \mathrm{Tr}(\Sigma)$ (see [53, §3]). Notice that B is defined only up to an element in $H^2(\Sigma,\mathbb{Z}) + \frac{1}{k}\mathrm{NS}(\Sigma)$.

The full cohomology $H^*(\Sigma,\mathbb{Z}) = H^0(\Sigma,\mathbb{Z}) \oplus H^2(\Sigma,\mathbb{Z}) \oplus H^4(\Sigma,\mathbb{Z})$ admits a lattice structure, with pairing $(r,H,s)\cdot(r',H',s') = H\cdot H' - rs' - r's$. As a lattice, $H^*(\Sigma,\mathbb{Z})$ is isometric to the *Mukai lattice* $\Lambda_{24} = U^{\oplus 4} \oplus E_8^{\oplus 2}$. A Mukai vector $v = (r,H,s)$ is *positive* if $H \in \mathrm{Pic}(\Sigma)$ and either $r > 0$, or $r = 0$ and $H \neq 0$ effective, or $r = H = 0$ and $s > 0$. Starting from a positive vector $v = (r,H,s) \in H^*(\Sigma,\mathbb{Z})$ and a B-field lift B of α we can define the twisted Mukai vector $v_B := (r, H + rB, s + B\cdot H + r\frac{B^2}{2})$. Then, if v_B is primitive, for a suitable choice of a polarization D of Σ the coarse moduli space $M_{v_B}(\Sigma,\alpha)$ of α-twisted Gieseker D-stable sheaves with Mukai vector v_B is a projective irreducible holomorphic symplectic manifold of $K3^{[n]}$-type, with $n = \frac{v_B^2}{2} + 1$. Moreover, we have a canonical isomorphism of lattices $\theta : v_B^\perp \to H^2(M_{v_B}(\Sigma,\alpha),\mathbb{Z})$ (see [8], [101]). For the sake of readability, we do not specify the ample divisor D in the notation for $M_{v_B}(\Sigma,\alpha)$, even though the construction depends on it: we will always assume that a choice of a polarization (generic with respect to the Mukai vector v_B, in the sense of [101, Definition 3.5]) has been made.

In the case $\alpha = 0$, the setting gets simplified. Let $v \in H^*(\Sigma, \mathbb{Z})$ be a primitive, positive Mukai vector; then $M_v(\Sigma, 0)$ is isomorphic to the moduli space $M_\tau(v)$ of τ-stable objects of Mukai vector v, for $\tau \in \mathrm{Stab}(\Sigma)$ a v-generic Bridgeland stability condition on the derived category $D^b(\Sigma)$ (see [25] for details). In particular, the Hilbert scheme $\Sigma^{[n]}$ can be seen as a moduli space of stable rank one sheaves on Σ, i.e. $\Sigma^{[n]} \cong M_\tau(v_n)$ for $v_n := (1, 0, 1-n) \in \Lambda_{24}$ and τ v_n-generic inside a specific euclidean open subset of $\mathrm{Stab}(\Sigma)$.

2.3. Moduli spaces and Torelli theorems

Let X be an irreducible holomorphic symplectic manifold whose second cohomology lattice $H^2(X, \mathbb{Z})$ is isometric to a lattice L.

Definition 2.3.1. A *marking* of X is a choice of an isometry $\eta : H^2(X, \mathbb{Z}) \to L$. The pair (X, η) is called a *marked* irreducible holomorphic symplectic manifold. We say that two marked IHS manifolds (X, η) and (X', η') are *isomorphic* if there exists a biregular (i.e. biholomorphic) isomorphism $f : X \to X'$ such that $\eta' = \eta \circ f^*$.

We can quotient the set of marked IHS manifolds (X, η) of a given deformation type, with $H^2(X, \mathbb{Z}) \cong L$, by the isomorphism relation, thus obtaining

$$\mathcal{M}_L := \left\{(X, \eta) \mid \eta : H^2(X, \mathbb{Z}) \to L \text{ marking}\right\} / \cong .$$

The set $\mathcal{M}_L$ can actually be endowed with a structure of compact complex space: in order to show this, we need to introduce a period map.

Definition 2.3.2. Let X be an irreducible holomorphic symplectic manifold and $\eta : H^2(X, \mathbb{Z}) \to L$ a marking. The *period domain* Ω_L is the complex space

$$\Omega_L := \{\kappa \in \mathbb{P}(L \otimes \mathbb{C}) \mid (\kappa, \kappa) = 0, (\kappa, \overline{\kappa}) > 0\} .$$

As we have already remarked, the symplectic form $\omega \in H^{2,0}(X)$ satisfies, by definition of the Beauville–Bogomolov–Fujiki quadratic form, the two properties $(\omega, \omega) = 0$ and $(\omega, \overline{\omega}) > 0$. This implies that the choice of a marking η of X determines a point $\mathcal{P}(X, \eta) := \eta(H^{2,0}(X)) = \eta(\mathbb{C}\omega)$ in the period domain Ω_L.

Let $p : \mathcal{X} \to I$ be a flat deformation of the IHS manifold $X = p^{-1}(0)$. By Ehresmann's theorem (see [62, Theorem 2.4]), if $\eta : H^2(X, \mathbb{Z}) \to L$ is a marking of X, then there exists an open neighbourhood $J \subset I$ of the point 0 and a family of markings $F_t : \mathcal{X}_t \to L$ over J such that $F_0 = \eta$. Then, we define the map $\mathcal{P} : J \to \Omega_L$ as

$$\mathcal{P}(t) = F_t(H^{2,0}(\mathcal{X}_t)).$$

When considering the universal deformation $\mathcal{X} \to \mathrm{Def}(X)$, the map $\mathcal{P} : \mathrm{Def}(X) \to \Omega_L$ is called the (local) *period map.*

Theorem 2.3.3 (Local Torelli theorem). *Let (X, η) be a marked irreducible holomorphic symplectic manifold. The period map*

$$\mathcal{P} : \mathrm{Def}(X) \to \Omega_L$$

is a local isomorphism.

Proof. See [10, Théorème 5]. □

By means of this local isomorphism, the universal deformations can be used as local charts for $\mathcal{M}_L$, which therefore is a compact non-Hausdorff complex space of dimension $h^{1,1}(X) = b_2(X) - 2$. More specifically, by [51, Proposition 4.3], for any marked pair (X, η) there exists a holomorphic embedding $\mathrm{Def}(X) \hookrightarrow \mathcal{M}_L$, identifying $\mathrm{Def}(X)$ with an open neighbourhood of the point $(X, \eta) \in \mathcal{M}_L$. The maps $\mathcal{P} : \mathrm{Def}(X) \to \Omega_L$ can then be glued together, yielding a period map $\mathcal{P} : \mathcal{M}_L \to \Omega_L$ which is a local isomorphism by the local Torelli theorem.

THEOREM 2.3.4. *Let $\mathcal{M}_L^0$ be a connected component of the moduli space $\mathcal{M}_L$. Then the restriction of the period map $\mathcal{P}_0 : \mathcal{M}_L^0 \to \Omega_L$ is surjective.*

PROOF. See [48, Theorem 8.1]. $\square$

In dimension two, the classical Torelli theorem states that two $K3$ surfaces Σ, Σ' are isomorphic if and only if there exists an effective Hodge isometry $H^2(\Sigma, \mathbb{Z}) \to H^2(\Sigma', \mathbb{Z})$. In the case of IHS manifolds, the same statement does not hold (see [36] for a counterexample); however, a weaker version of the global Torelli has been proved, as a result of work by Huybrechts, Markman and Verbitsky.

THEOREM 2.3.5 (Global Torelli theorem). *Let $\mathcal{M}_L^0$ be a connected component of the moduli space $\mathcal{M}_L$. For each $\omega \in \Omega_L$, the fiber $P_0^{-1}(\omega)$ consists of pairwise inseparable points. If (X, η) and (X', η') are inseparable points of $\mathcal{M}_L^0$, then X, X' are bimeromorphic.*

PROOF. See [69, Theorem 2.2]. $\square$

The global Torelli theorem also admits a lattice-theoretic formulation: in order to present it, we first need to introduce the concept of monodromy operator.

DEFINITION 2.3.6. Let X, Y be holomorphic symplectic manifolds. A lattice isometry $f : H^2(X, \mathbb{Z}) \to H^2(Y, Z)$ is a *parallel transport operator* if there exists a smooth and proper family $\pi : \mathcal{X} \to B$ and a continuous path $\gamma : [0, 1] \to B$ such that $X \cong \mathcal{X}_{\gamma(0)}$, $Y \cong \mathcal{X}_{\gamma(1)}$ and f is induced by parallel transport in the local system $R^2\pi_*\mathbb{Z}$ along γ.

A parallel transport operator $f : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is called a *monodromy operator* of X.

We denote by $\mathrm{Mon}^2(X) \subset O(H^2(X, \mathbb{Z}))$ the subgroup of monodromy operators, which is of finite index (see [69, Lemma 7.5]). In particular, two marked pairs (X, η), (X', η') belong to the same connected component of $\mathcal{M}_L$ if and only if $\eta' \circ \eta^{-1}$ is a parallel transport operator. As a consequence, the number of connected components of $\mathcal{M}_L$ is $\pi_0(\mathcal{M}_L) = \left[O(H^2(X, \mathbb{Z})) : \mathrm{Mon}^2(X)\right]$.

If X is an IHS manifold and $\eta : H^2(X, \mathbb{Z}) \to L$ is a marking, we can define

$$\mathrm{Mon}^2(L) := \left\{\eta \circ \psi \circ \eta^{-1} \mid \psi \in \mathrm{Mon}^2(X)\right\} \subset O(L).$$

The group $\mathrm{Mon}^2(L) \subset O(L)$, whose elements are still called monodromy operators, is the same for any choice of a marked pair (X, η) in a connected component $\mathcal{M}_L^0 \subset \mathcal{M}_L$, but could a priori depend on $\mathcal{M}_L^0$. However, if the subgroup $\mathrm{Mon}^2(X) \subset O(H^2(X, \mathbb{Z}))$ is normal, then $\mathrm{Mon}^2(L)$ is independent on the choice of the connected component (see [69, Remark 7.6]).

EXAMPLE 2.3.7. Let Σ be a $K3$ surface. We denote by $O^+(H^2(\Sigma, \mathbb{Z}))$ the subgroup of $O(H^2(\Sigma, \mathbb{Z}))$ of orientation preserving isometries: it is a normal subgroup of index two. Then, $\mathrm{Mon}^2(\Sigma) = O^+(H^2(\Sigma, \mathbb{Z}))$ (see [23, Theorem A]), therefore the moduli space $\mathcal{M}_{L_{K3}}$ of marked $K3$ surfaces has two connected components, which correspond to each other via the map $(\Sigma, \eta) \mapsto (\Sigma, -\eta)$.

Let X be an IHS manifold of $K3^{[n]}$-type. In [68], Markman proved that $\mathrm{Mon}^2(X) \subset O(H^2(X, \mathbb{Z}))$ is a normal subgroup and provided several equivalent characterizations of monodromy operators on X, which we now recall.

PROPOSITION 2.3.8. *Let X be an irreducible holomorphic symplectic manifold of $K3^{[n]}$-type. An isometry $\psi \in O(H^2(X, \mathbb{Z}))$ is a monodromoy operator if and only if ψ is orientation preserving and it induces the action $\overline{\psi} = \pm \mathrm{id} \in O(A_{H^2(X,\mathbb{Z})})$.*

In particular, the index of $\mathrm{Mon}^2(X)$ as a subgroup of $O^+(H^2(X,\mathbb{Z}))$ is 2^{r-1}, where $r = \rho(n-1)$ is the number of distinct prime divisors of $n-1$. As a consequence, if $n = 2$ or $n-1$ is a prime power, then $\mathrm{Mon}^2(X) = O^+(H^2(X,\mathbb{Z}))$, exactly as for $K3$ surfaces.

Let $u \in H^2(X,\mathbb{Z})$ be an element with $(u,u) \neq 0$; we denote by R_u the reflection of $H^2(X,\mathbb{Q})$ with respect to u, i.e.

$$R_u(w) := w - 2\frac{(u,w)}{(u,u)}u.$$

REMARK 2.3.9. For an even lattice L we define the *real spinor norm*

$$\mathrm{sn}^L_{\mathbb{R}} : O(L \otimes \mathbb{R}) \to \mathbb{R}^* / (\mathbb{R}^*)^2 \cong \{\pm 1\}, \quad \mathrm{sn}^L_{\mathbb{R}}(\gamma) = \left(-\frac{u_1^2}{2}\right) \dots \left(-\frac{u_r^2}{2}\right)$$

where $\gamma = R_{u_1} \circ \ldots \circ R_{u_r}$ as a product of reflections with respect to vectors $u_i \in L \otimes \mathbb{R}$ (in particular, by Cartan–Dieudonné theorem, $r \leq \mathrm{rk}(L)$). Then, an isometry $\psi \in O(L)$ is orientation preserving if and only if its real spinor norm is $+1$.

Notice that, if $(u,u) = \pm 2$, the reflection R_u restricts to an integral isometry $R_u \in O(H^2(X,\mathbb{Z}))$. We define

$$\rho_u = \begin{cases} R_u & \text{if } (u,u) < 0; \\ -R_u & \text{if } (u,u) > 0. \end{cases}$$

PROPOSITION 2.3.10. *Let X be a manifold of $K3^{[n]}$-type. Then the group $\mathrm{Mon}^2(X)$ is generated by the isometries ρ_u, for all $u \in H^2(X,\mathbb{Z})$ with $(u,u) = \pm 2$.*

We can now state the Hodge-theoretic form of the global Torelli theorem.

THEOREM 2.3.11. *Let X, Y be irreducible holomorphic symplectic manifolds. If there exists a parallel transport operator $\psi : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ which is also a Hodge isometry, then X and Y are bimeromorphic. If, moreover, ψ maps a Kähler class to a Kähler class, then there exists a biregular isomorphism $f : Y \to X$ such that $f^* = \psi$.*

PROOF. See [69, Theorem 1.3]. □

In the following, we will denote by $\mathrm{Mon}^2_{\mathrm{Hdg}}(X) \subset \mathrm{Mon}^2(X)$ the subgroup of monodromy operators which preserve the Hodge decomposition.

2.4. Kähler cone and cones of divisors

For an IHS manifold X, the Beauville–Bogomolov–Fujiki quadratic form allows us to define several cones of interest, contained in $H^{1,1}(X,\mathbb{R})$, or in its intersection with $H^2(X,\mathbb{Z})$. In this section we investigate their structure.

DEFINITION 2.4.1. Let X be an irreducible holomorphic symplectic manifold. The *positive cone* $\mathcal{C}_X$ is the connected component of $\left\{x \in H^{1,1}(X,\mathbb{R}) \mid (x,x) > 0\right\}$ which contains the cone of Kähler classes $\mathcal{K}_X$.

Recall that, in the case of $K3$ surfaces, the Kähler cone coincides with the set of real $(1,1)$-classes which have positive intersection with all rational curves on the surface. Boucksom, answering a question by Huybrechts, generalized this result for any IHS manifold X (see [24, Théorème 1.2]), where we have:

$$\mathcal{K}_X = \left\{\alpha \in \mathcal{C}_X \mid \int_C \alpha > 0 \text{ for all rational curves } C \subset X\right\}.$$

DEFINITION 2.4.2. Let X be an irreducible holomorphic symplectic manifold. A *prime* divisor is an irreducible reduced effective divisor. A prime divisor $E \subset X$ is *exceptional* if $(E, E) < 0$. The *fundamental exceptional chamber* of X is the cone:

$$\mathcal{FE}_X = \{x \in \mathcal{C}_X \mid (x, E) > 0 \text{ for all prime exceptional divisors } E \subset X\}.$$

By [69, Proposition 5.6], $\mathcal{FE}_X$ is also the cone of classes $x \in \mathcal{C}_X$ such that $(x, D) > 0$ for any non-zero uniruled divisor $D \subset X$.

Of course, for a $K3$ surface Σ prime exceptional divisors are just rational curves $C \subset \Sigma$, and $\mathcal{FE}_\Sigma = \mathcal{K}_\Sigma$. For IHS manifolds X of higher dimensions, the Kähler cone is, in general, strictly contained in $\mathcal{FE}_X$: it is actually a chamber, with respect to a suitable decomposition of $\mathcal{FE}_X$, which we now present.

DEFINITION 2.4.3. Let X be an irreducible holomorphic symplectic manifold. The *birational Kähler cone* $\mathcal{BK}_X$ of X is the union of the cones $f^*\mathcal{K}_{X'}$, for all birational models $f : X \dashrightarrow X'$.

As a consequence of the following proposition, we have $\mathcal{BK}_X \subset \mathcal{FE}_X$.

PROPOSITION 2.4.4. *Let X, Y be irreducible holomorphic symplectic manifolds and $g : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ a parallel transport operator, which is also a Hodge isometry. There exists a birational map $f : Y \dashrightarrow X$ such that $g = f^*$ if and only if $g(\mathcal{FE}_X) \subset \mathcal{FE}_Y$.*

PROOF. See [69, Corollary 5.7]. □

By [69, Proposition 5.6], we also have the inclusion $\mathcal{FE}_X \subset \overline{\mathcal{BK}}_X$; as a consequence, $\overline{\mathcal{FE}}_X = \overline{\mathcal{BK}}_X$.

The positive cone $\mathcal{C}_X$ is invariant under the action of $\mathrm{Mon}^2_{\mathrm{Hdg}}(X)$, and we can consider the following *chambers* in it.

DEFINITION 2.4.5. Let X be an irreducible holomorphic symplectic manifold.

(i) An *exceptional chamber* of $\mathcal{C}_X$ is a subset of the form $g(\mathcal{FE}_X)$, for an isometry $g \in \mathrm{Mon}^2_{\mathrm{Hdg}}(X)$.

(ii) A *Kähler-type chamber* of $\mathcal{C}_X$ is a subset of the form $g(f^*(\mathcal{K}_Y))$, for an isometry $g \in \mathrm{Mon}^2_{\mathrm{Hdg}}(X)$ and a birational map $f : X \dashrightarrow Y$.

We now assume that X is projective. The group of Hodge monodromies $\mathrm{Mon}^2_{\mathrm{Hdg}}(X)$ acts transitively on the set of exceptional chambers, as proven in [69, Theorem 6.18]. Moreover, each exceptional chamber (and, in particular, $\mathcal{FE}_X$) is the interior of a fundamental domain for the action of the normal subgroup

$$W_{\mathrm{Exc}} = \langle R_E \mid E \subset X \text{ prime exceptional divisor} \rangle \subset \mathrm{Mon}^2_{\mathrm{Hdg}}(X)$$

where, as in the previous section, R_E denotes the reflection of $H^2(X, \mathbb{Z})$ with respect to the class E (if E is a prime exceptional divisor, $R_E \in \mathrm{Mon}^2_{\mathrm{Hdg}}(X)$ by [69, Proposition 6.2]). Let $\mathrm{Mon}^2_{\mathrm{Bir}} \subset \mathrm{Mon}^2_{\mathrm{Hdg}}(X)$ be the subset of monodromy operators f^* for all birational maps $f : X \dashrightarrow X$. Then, by Proposition 2.4.4, $\mathrm{Mon}^2_{\mathrm{Bir}} \subset \mathrm{Mon}^2_{\mathrm{Hdg}}(X)$ is the stabilizer of the fundamental exceptional chamber. Moreover, the following equality holds (see [69, Theorem 6.18]):

$$\mathrm{Mon}^2_{\mathrm{Hdg}}(X) = W_{\mathrm{Exc}} \rtimes \mathrm{Mon}^2_{\mathrm{Bir}}(X).$$

We now focus on Kähler-type chambers, which are the $\mathrm{Mon}^2_{\mathrm{Hdg}}(X)$-translates of the Kähler cone $\mathcal{K}_X$. As a consequence of Theorem 2.3.11, distinct Kähler-type chambers are disjoint, while the closures of two adjacent chambers intersect along a wall (since $\overline{\mathcal{BK}}_X = \overline{\mathcal{FE}}_X$).

DEFINITION 2.4.6. An element $s \in H^{1,1}(X, \mathbb{Q})$ with $(s,s) < 0$ is a *monodromy birationally minimal* (MBM) class if there exists a birational map $f : X \dashrightarrow Y$ and a Hodge monodromy operator $g \in \mathrm{Mon}^2_{\mathrm{Hdg}}(X)$ such that $g(s)^\perp$ supports a face of $f^*\mathcal{K}_Y$.

Monodromy birationally minimal classes were introduced and studied by Amerik and Verbitsky ([2], [3]), who showed that they behave well under deformation: if $s \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ is an integral MBM class and (X', s') is a deformation of (X, s) such that s' is still of type $(1,1)$, then s' is MBM (see [3, Theorem 2.16]). Moreover, MBM classes determine the walls which separate the Kähler-type chambers in the positive cone of an IHS manifold.

THEOREM 2.4.7. *Let X be an irreducible holomorphic symplectic manifold. The connected components of*

$$\mathcal{C}_X \setminus \bigcup_{s\ MBM} s^\perp$$

are the Kähler-type chambers of X.

PROOF. See [2, Theorem 6.2]. □

All the cones we have introduced so far live inside $H^{1,1}(X, \mathbb{R})$; we now want to study their intersections with the integral cohomology $H^2(X, \mathbb{Z})$. Recall that the Néron–Severi lattice is defined as $\mathrm{NS}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ and, since $H^1(X, \mathcal{O}_X) = 0$, the first Chern class $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ provides an isomorphism $\mathrm{Pic}(X) \cong \mathrm{NS}(X)$.

DEFINITION 2.4.8. A line bundle $L \in \mathrm{Pic}(X)$ is called *movable* if the codimension of the base locus of the linear system $|L|$ is at least two. The *movable cone* $\mathrm{Mov}(X) \subset \mathrm{NS}(X)_\mathbb{R} := \mathrm{NS}(X) \otimes \mathbb{R}$ is the cone generated by the classes of movable line bundles.

PROPOSITION 2.4.9. *Let X be a projective irreducible holomorphic symplectic manifold. The interior of the movable cone coincides with $\mathcal{FE}_X \cap \mathrm{NS}(X)_\mathbb{R}$. The group W_{Exc} acts faithfully on $\mathcal{C}_X \cap \mathrm{NS}(X)_\mathbb{R}$ and there is a bijective correspondence between the set of exceptional chambers of X and the set of chambers of $\mathcal{C}_X \cap \mathrm{NS}(X)_\mathbb{R}$ with respect to the action of W_{Exc}. In particular, $\overline{\mathrm{Mov}(X)} \subset \mathcal{C}_X$ is a fundamental domain for the action of W_{Exc} on $\mathcal{C}_X \cap \mathrm{NS}(X)_\mathbb{R}$.*

PROOF. See [69, Lemma 6.22]. □

If X is projective, the *ample cone* $\mathcal{A}_X$ (i.e. the cone in $\mathrm{NS}(X)_\mathbb{R}$ generated by ample classes) is contained inside $\mathrm{Mov}(X)$; more specifically, $\mathcal{A}_X = \mathcal{K}_X \cap \mathrm{NS}(X)_\mathbb{R}$. The *nef cone* $\mathrm{Nef}(X) \subset \mathrm{NS}(X)_\mathbb{R}$ is the closure of the ample cone. Proposition 2.4.4, together with the global Torelli theorem (Theorem 2.3.11), provides the following statement.

PROPOSITION 2.4.10. *Let X, Y be irreducible holomorphic symplectic manifolds. A birational map $f : X \dashrightarrow Y$ induces a Hodge isometry $f^* : H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ such that $f^*(\mathrm{Mov}(Y)) = \mathrm{Mov}(X)$. Moreover, if X, Y are projective and $f^*(\mathrm{Nef}(Y))$ intersects $\mathcal{A}_X$, then f is a biregular isomorphism and $f^*(\mathrm{Nef}(Y)) = \mathrm{Nef}(X)$.*

The decomposition of the positive cone $\mathcal{C}_X \subset H^{1,1}(X, \mathbb{R})$ into exceptional and Kähler-type chambers can be adapted to the integral cohomology. In order to do so, we need to define wall-divisors.

DEFINITION 2.4.11. A *wall-divisor* is a primitive class $D \in \mathrm{NS}(X)$ with $D^2 < 0$ and such that $g(D^\perp) \cap \mathcal{BK}_X = \emptyset$ for all $g \in \mathrm{Mon}^2_{\mathrm{Hdg}}(X)$.

Notice that, for any MBM class $s \in H^{1,1}(X, \mathbb{Q})$, a suitable rational multiple of s is a wall-divisor. Thus, the movable cone $\mathrm{Mov}(X)$ is one of the connected components of

$$(\mathcal{C}_X \cap \mathrm{NS}(X)_{\mathbb{R}}) \setminus \bigcup_{E \in \mathcal{P}_{\mathrm{Ex}}} E^{\perp}$$

where $\mathcal{P}_{\mathrm{Ex}}$ is the set of prime exceptional divisors, while the ample cone $\mathcal{A}_X$ is one of the connected components of

$$(\mathcal{C}_X \cap \mathrm{NS}(X)_{\mathbb{R}}) \setminus \bigcup_{\delta \in \Delta(X)} \delta^{\perp}$$

where $\Delta(X)$ is the set of wall-divisors.

As we already stated, wall-divisors are preserved under smooth deformations if their Hodge type does not change; in particular, we have the following result.

THEOREM 2.4.12. *Let (X, η), (Y, μ) be marked irreducible holomorphic symplectic manifolds in the same connected component $\mathcal{M}_L^0$ of the moduli space $\mathcal{M}_L$. If $D \in \mathrm{NS}(X)$ is a wall-divisor of X and $(\mu^{-1} \circ \eta)(D) \in \mathrm{NS}(Y)$, then $(\mu^{-1} \circ \eta)(D)$ is a wall-divisor of Y.*

PROOF. See [73, Theorem 1.3]. □

In [7], Bayer and Macrì gave a numerical characterization of wall-divisors for moduli spaces of sheaves on a projective $K3$ surface. By applying Theorem 2.4.12, their numerical description can be extended to all manifolds of $K3^{[n]}$-type: this was done explicitly by Mongardi for $n = 2, 3, 4$ (see [73]). We now explain in more detail the results of Bayer and Macrì, focusing on the case of Hilbert schemes of points on a $K3$ surface, since we will need them in the next chapter.

For a $K3$ surface Σ, the Mukai lattice $H^*(\Sigma, \mathbb{Z}) \cong U^{\oplus 4} \oplus E_8^{\oplus 2}$ carries a weight-two Hodge structure, whose $(1,1)$-part is $H^*_{\mathrm{alg}}(\Sigma, \mathbb{Z}) := H^0(\Sigma, \mathbb{Z}) \oplus \mathrm{NS}(\Sigma) \oplus H^4(\Sigma, \mathbb{Z})$. For any $n \geq 2$, the vector $v_n = (1, 0, 1-n)$ belongs to $H^*_{\mathrm{alg}}(\Sigma, \mathbb{Z})$. The isomorphism $\theta^{-1} : H^2(\Sigma^{[n]}, \mathbb{Z}) \to v_n^{\perp} \subset H^*(\Sigma, \mathbb{Z})$, which was introduced in Example 2.2.7, can be realized by mapping $H^2(\Sigma^{[n]}, \mathbb{Z}) \cong H^2(\Sigma, \mathbb{Z}) \oplus \mathbb{Z}\delta$ to $v_n^{\perp} = H^2(\Sigma, \mathbb{Z}) \oplus \mathbb{Z}(-1, 0, 1-n)$ inside the Mukai lattice. In particular, it satisfies $\theta(v_n^{\perp} \cap H^*_{\mathrm{alg}}(\Sigma, \mathbb{Z})) = \mathrm{NS}(\Sigma^{[n]})$.

THEOREM 2.4.13. *Let Σ be a $K3$ surface and $X = \Sigma^{[n]}$, for $n \geq 2$. The movable cone $\mathrm{Mov}(X)$ is one of the open chambers of the decomposition of $\overline{\mathcal{C}}_X \cap \mathrm{NS}(X)_{\mathbb{R}}$ whose walls are the linear subspaces $\theta(v_n^{\perp} \cap a^{\perp})$ for $a \in H^*_{alg}(\Sigma, \mathbb{Z})$ such that*

(i) $a^2 = -2$, $(v_n, a) = 0$, *or*
(ii) $a^2 = 0$, $(v_n, a) = 1$, *or*
(iii) $a^2 = 0$, $(v_n, a) = 2$.

PROOF. See [7, Theorem 12.3]. □

THEOREM 2.4.14. *Let Σ be a $K3$ surface and $X = \Sigma^{[n]}$, for $n \geq 2$. The ample cone $\mathcal{A}_X$ is one of the open chambers of the decomposition of $\overline{\mathcal{C}}_X \cap \mathrm{NS}(X)_{\mathbb{R}}$ whose walls are the linear subspaces $\theta(v_n^{\perp} \cap a^{\perp})$ for $a \in H^*_{alg}(\Sigma, \mathbb{Z})$ with $a^2 \geq -2$ and $0 \leq (v_n, a) \leq n-1$.*

PROOF. See [7, Theorem 12.1]. □

In particular, by studying the walls of Theorem 2.4.14 contained in $\mathrm{Mov}(\Sigma^{[n]})$ one can, in principle, determine all wall-divisors of $\Sigma^{[n]}$.

REMARK 2.4.15. The additional walls that need to be considered in the decomposition of Theorem 2.4.14, with respect to the ones already appearing in Theorem 2.4.13, are the walls which separate the Kähler chambers in $\mathcal{FE}_{\Sigma^{[n]}} \cap \mathrm{NS}(\Sigma^{[n]})_{\mathbb{R}}$. They are of the form $\theta(v_n^{\perp} \cap a^{\perp})$, where we can restrict to consider $a \in H^*_{\mathrm{alg}}(\Sigma, \mathbb{Z})$ such that:

(*i*) $a^2 = -2$, $1 \leq (v_n, a) \leq n-1$, or
(*ii*) $a^2 = 0$, $3 \leq (v_n, a) \leq n-1$, or
(*iii*) $2 \leq a^2 < \frac{n-1}{2}$ and $2a^2 + 1 \leq (v_n, a) \leq n-1$.

These bounds for a^2 and (v_n, a) are a consequence of the fact that the rank two sublattice $\langle a, v_n \rangle \subset H^*(\Sigma^{[n]}, \mathbb{Z})$ needs to be hyperbolic, in order to define a wall $\theta(v_n^{\perp} \cap a^{\perp})$ which intersects the positive cone (see [7, Theorem 5.7]).

2.5. Automorphisms

Let X be an irreducible holomorphic symplectic manifold. We denote by $\mathrm{Aut}(X)$ the group of biholomorphic automorphisms of X and by $\mathrm{Bir}(X)$ the group of birational automorphisms. Clearly, $\mathrm{Aut}(X) \subset \mathrm{Bir}(X)$, and in the case of $K3$ surfaces the two groups coincide ($K3$ surfaces have trivial canonical divisor, hence they are minimal surfaces by [11, Proposition II.3]).

THEOREM 2.5.1. *Let X be an irreducible holomorphic symplectic manifold which, together with a marking $\eta : H^2(X, \mathbb{Z}) \to L$, defines a general point in a connected component of $\mathcal{M}_L$. Then,* $\mathrm{Aut}(X) = \mathrm{Bir}(X)$.

PROOF. See [48, Proposition 9.2]. □

For all compact complex manifolds we have

$$\dim(\mathrm{Aut}(X)) = h^0(TX)$$

and therefore, if X is IHS, $\dim(\mathrm{Aut}(X)) = h^{1,0}(X) = 0$, meaning that $\mathrm{Aut}(X)$ is a discrete group. We also know, by [22, Theorem 2], that the group $\mathrm{Bir}(X)$ is finitely generated, if X is projective. We define the following homomorphism:

$$\lambda : \mathrm{Bir}(X) \to O(H^2(X, \mathbb{Z}))$$
$$f \mapsto f^*$$

where $f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is the pull-back of f, which preserves the Beauville–Bogomolov–Fujiki quadratic form. The morphism λ satisfies the following properties (see [48, Proposition 9.1]):

(*i*) $\lambda(\mathrm{Bir}(X)) = \mathrm{Mon}^2_{\mathrm{Bir}}(X) \subset \mathrm{Mon}^2_{\mathrm{Hdg}}(X)$;
(*ii*) $\lambda(\mathrm{Aut}(X)) = \left\{ g \in \mathrm{Mon}^2_{\mathrm{Hdg}}(X) \mid g(\mathcal{K}_X) \cap \mathcal{K}_X \neq \emptyset \right\}$;
(*iii*) $\lambda^{-1}(\lambda(\mathrm{Aut}(X))) = \mathrm{Aut}(X)$;
(*iv*) $\ker(\lambda) \subset \mathrm{Aut}(X)$ is finite.

Notice that properties (*ii*) and (*iii*) summarize the Hodge-theoretic global Torelli theorem (Theorem 2.3.11).

By results of Hassett and Tschinkel, the kernel of the homomorphism λ is invariant under smooth deformations of the manifold X (see [45, Theorem 2.1]), and it has been computed for all known deformation types of IHS manifolds.

THEOREM 2.5.2. *Let X be a manifold of $K3^{[n]}$-type. Then,* $\ker(\lambda) = \{\mathrm{id}_X\}$.

PROOF. See [9, Proposition 10] and [72, Lemma 1.2]. □

The theorem implies that, for manifolds of $K3^{[n]}$-type, any automorphism is completely determined by its action on the second cohomology lattice.

In addition to the action on $H^2(X, \mathbb{Z})$, another important invariant of an automorphism $f \in \mathrm{Aut}(X)$ is its action on $H^0(X, \Omega^2_X)$, i.e. on the symplectic form

ω of the IHS manifold X. Since f^* is a Hodge isometry, we have $f^*(\omega) = \xi\omega$, for $\xi \in \mathbb{C}^*$; moreover, if f is of finite order m, then $\xi^m = 1$.

DEFINITION 2.5.3. An automorphism $f \in \mathrm{Aut}(X)$ is *symplectic* if $f^*(\omega) = \omega$; otherwise, f is called *non-symplectic*.

By [9, Proposition 6], if there exists a non-symplectic $f \in \mathrm{Aut}(X)$, then the IHS manifold X is projective. Moreover, if $f^*(\omega) = \xi\omega$, with ξ a root of unity of order m, then $\varphi(m) \leq b_2(X) - \mathrm{rk}(\mathrm{NS}(X))$ (where φ denotes Euler's totient function).

DEFINITION 2.5.4. Let $f \in \mathrm{Aut}(X)$ be an automorphism of finite order of an IHS manifold X. The *invariant* lattice of f is

$$T_f = H^2(X,\mathbb{Z})^{f^*} = \left\{u \in H^2(X,\mathbb{Z}) \mid f^*(u) = u\right\}$$

and the *co-invariant* lattice of f is

$$S_f = \left(H^2(X,\mathbb{Z})^{f^*}\right)^{\perp} \subset H^2(X,\mathbb{Z}).$$

Both T_f and S_f are primitive sublattices of $H^2(X,\mathbb{Z})$, since they can be expressed as kernels of lattice isometries: in particular, if $m \in \mathbb{N}$ is the order of f, we have

$$T_f = \ker(f^* - \mathrm{id}), \qquad S_f = \ker(\mathrm{id} + f^* + \ldots + (f^*)^{m-1}). \tag{6}$$

We recall that the transcendental lattice of an irreducible holomorphic symplectic manifold X is the primitive sublattice $\mathrm{Tr}(X) = \mathrm{NS}(X)^{\perp} \subset H^2(X,\mathbb{Z})$. The following proposition explains the relative positions of the lattices T_f, S_f with respect to $\mathrm{NS}(X)$, $\mathrm{Tr}(X)$.

PROPOSITION 2.5.5. *Let X be an irreducible holomorphic symplectic manifold and $f \in \mathrm{Aut}(X)$.*

(*i*) *If f is symplectic, then $\mathrm{Tr}(X) \subset T_f$, $S_f \subset \mathrm{NS}(X)$ and S_f is negative definite.*
(*ii*) *If f is non-symplectic, then $T_f \subset \mathrm{NS}(X)$, $\mathrm{Tr}(X) \subset S_f$ and T_f is hyperbolic.*

PROOF. We discuss the two cases separately.

(*i*) If $x \in S_f$, by the description (6) we have $\sum_{i=0}^{m-1}(f^*)^i(x) = 0$, assuming that m is the order of the automorphism f. If $H^{2,0}(X)$ is generated by the symplectic form ω, we have:
$$0 = \left(\sum_{i=0}^{m-1}(f^*)^i(x), \omega\right) = \sum_{i=0}^{m-1}\big((f^*)^i(x), \omega\big) = m(x,\omega)$$
where we used the fact that f^* is an isometry of $H^2(X,\mathbb{Z})$ and $f^*\omega = \omega$. Therefore, $(x,\omega) = 0$, hence $S_f \subset \mathrm{NS}(X) = H^2(X,\mathbb{Z}) \cap \omega^{\perp}$; by passing to the orthogonal complements, we also deduce $\mathrm{Tr}(X) \subset T_f$. Notice that $T_f \otimes \mathbb{C} \subset H^2(X,\mathbb{C})$ contains the three-dimensional space $\mathbb{C}\omega \oplus \mathbb{C}\overline{\omega} \oplus \mathbb{C}\beta$, where $\beta := \sum_{i=0}^{m-1}(f^*)^i(\alpha) \neq 0$ for a Kähler class $\alpha \in \mathcal{K}_X$: this implies that the signature of T_f is $(3, \mathrm{rk}(T_f) - 3)$, because $H^2(X,\mathbb{Z})$ has signature $(3, b_2(X) - 3)$. Hence, S_f is negative definite.
(*ii*) If f is non-symplectic, for any $x \in T_f$ we have $(x,\omega) = (x, \xi\omega)$, where $\xi \in \mathbb{C}^*$ is the root of unity such that $f^*\omega = \xi\omega$. Therefore, $(x,\omega) = 0$, which gives the inclusions $T_f \subset \mathrm{NS}(X)$, $\mathrm{Tr}(X) \subset S_f$. As we already remarked, the existence of a non-symplectic automorphism implies that X is projective, hence $\mathrm{NS}(X)$ is hyperbolic by Theorem 2.2.6. Since the positive class β, defined in the symplectic case, is still contained in $T_f \otimes \mathbb{C}$, we conclude that $T_f \subset \mathrm{NS}(X)$ is also hyperbolic. □

In our work, we will be mainly interested in non-symplectic automorphisms. For results on symplectic automorphisms of IHS manifolds, with special regard to manifolds of $K3^{[n]}$-type, we refer the reader to [27] and [74].

EXAMPLE 2.5.6. *Natural automorphisms.*
Let Σ be a (smooth) $K3$ surface. An automorphism $f \in \mathrm{Aut}(\Sigma)$ induces an automorphism $f^{[n]}$ on the Hilbert scheme $\Sigma^{[n]}$, by setting $f^{[n]}(Z) = f(Z)$ for any zero-dimensional subscheme $Z \subset \Sigma$ of length n. Such an automorphism $f^{[n]}$ is said to be *natural*. In Example 2.2.4, we recalled the existence of an embedding $i : H^2(\Sigma,\mathbb{Z}) \hookrightarrow H^2(\Sigma^{[n]},\mathbb{Z})$ such that $H^2(\Sigma^{[n]},\mathbb{Z}) = i\left(H^2(\Sigma,\mathbb{Z})\right) \oplus \mathbb{Z}\delta$, where 2δ is the class of the exceptional divisor E of the Hilbert–Chow morphism $\rho : \Sigma^{[n]} \to \Sigma^{(n)}$. As observed in [22, §3], the natural automorphism $f^{[n]}$ acts as f^* on the summand $i(H^2(\Sigma^{[n]},\mathbb{Z}))$, while $\left(f^{[n]}\right)^*(\delta) = \delta$, because $f^{[n]}$ leaves the exceptional divisor E globally invariant (if $Z \subset \Sigma$ is a non-reduced subscheme, $f(Z)$ is still non-reduced). This property actually characterizes natural automorphisms.

THEOREM 2.5.7. *Let Σ be a $K3$ surface and $n \geq 2$. An automorphism $g \in \mathrm{Aut}(\Sigma^{[n]})$ is natural if and only if it leaves globally invariant the exceptional divisor $E \subset \Sigma^{[n]}$.*

PROOF. See [22, Theorem 1]. $\square$

Since $(f^{[n]})^* = (f^*, \mathrm{id})$ on $H^2(\Sigma,\mathbb{Z}) = i\left(H^2(\Sigma,\mathbb{Z})\right) \oplus \mathbb{Z}\delta$, if $T_f, S_f \subset H^2(\Sigma,\mathbb{Z})$ are the invariant and co-invariant lattices of $f \in \mathrm{Aut}(\Sigma)$, the following equalities hold:

$$T_{f^{[n]}} = i(T_f) \oplus \mathbb{Z}\delta \cong T_f \oplus \langle -2(n-1)\rangle; \qquad S_{f^{[n]}} = i(S_f) \cong S_f. \tag{7}$$

By [10, Proposition 6], the embedding $i : H^2(\Sigma,\mathbb{Z}) \hookrightarrow H^2(\Sigma^{[n]},\mathbb{Z})$ is the restriction of an injective morphism $\iota : H^2(\Sigma,\mathbb{C}) \hookrightarrow H^2(\Sigma^{[n]},\mathbb{C})$ which is compatible with the Hodge decomposition and such that $H^2(\Sigma^{[n]},\mathbb{C}) = \iota(H^2(\Sigma,\mathbb{C})) \oplus \mathbb{C}[E]$. As a consequence, if $f \in \mathrm{Aut}(\Sigma)$ is a symplectic automorphism, then $f^{[n]}$ also acts symplectically on $\Sigma^{[n]}$. In fact, if $\omega \in H^{2,0}(\Sigma)$ then $\iota(\omega) \in H^{2,0}(\Sigma^{[n]})$ and $\left(f^{[n]}\right)^*(\iota(\omega)) = \iota\left(f^*(\omega)\right)$. Analogously, if f is non-symplectic then $f^{[n]}$ is non-symplectic too.

EXAMPLE 2.5.8. *Beauville's non-natural involutions.*
By Riemann–Roch, a projective $K3$ surface Σ of degree $2n \geq 4$ admits an embedding in the projective space $\mathbb{P}^{n+1}$. A general point $Z \in \Sigma^{[n]}$ corresponds to n distinct points on Σ; the linear subspace $\langle Z\rangle \subset \mathbb{P}^{n+1}$ intersects the surface Σ along n more points, which determine another element $Z' \in \Sigma^{[n]}$. Therefore, we obtain the following birational involution of the Hilbert scheme $\Sigma^{[n]}$:

$$\begin{aligned} j_n : \Sigma^{[n]} &\dashrightarrow \Sigma^{[n]} \\ Z &\mapsto Z'. \end{aligned}$$

In the case $n = 2$, we are considering a quartic surface $\Sigma \subset \mathbb{P}^3$: a generic pair of points $p, q \in \Sigma$ defines a line $\overline{pq}$ which cuts out two additional points on Σ, thus providing the birational involution $j_2 \in \mathrm{Bir}(\Sigma^{[2]})$. As proved by Beauville in [9, §6], j_2 is actually biregular if Σ does not contain any lines; instead, for $n \geq 3$ the involution j_n never belongs to $\mathrm{Aut}(\Sigma^{[n]})$.

Notice that the automorphism $j_2 \in \mathrm{Aut}(\Sigma^{[2]})$ does not leave the exceptional divisor $E \subset \Sigma^{[2]}$ invariant, since we can find distinct points p, q on the quartic surface Σ such that $(\overline{pq} \cap \Sigma) \setminus \{p, q\}$ is supported on one point, hence j_2 is not natural.

CHAPTER 3

Automorphisms of Hilbert schemes of points on a generic projective K3 surface

In this chapter, which collects the results of the paper [32], we study the group $\mathrm{Aut}(\Sigma^{[n]})$ for a generic projective $K3$ surface Σ and any $n \geq 2$. Our aim is to generalize the following result, due to Boissière, An. Cattaneo, Nieper-Wisskirchen and Sarti (see [20, Theorem 1.1]), which completely resolves the case $n = 2$.

THEOREM 3.0.1. *Let Σ be an algebraic $K3$ surface such that $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, with $H^2 = 2t$, $t \geq 1$. Then $\Sigma^{[2]}$ admits a non-trivial automorphism if and only if one of the following equivalent conditions is satisfied:*

- (i) *t is not a square, the equation $X^2 - 4tY^2 = 5$ has no integer solutions and the equation $X^2 - tY^2 = -1$ has integer solutions;*
- (ii) *there exists an ample class $D \in \mathrm{NS}(\Sigma^{[2]})$ such that $D^2 = 2$.*

If so, the class D is unique and $\mathrm{Aut}(\Sigma^{[2]})$ is generated by a non-symplectic involution, whose action on $H^2(\Sigma^{[2]}, \mathbb{Z})$ is the reflection in the span of D.

3.1. Preliminaries

3.1.1. Pell's equations. A quick overview of the basic theory of Pell's equations can be found in [20, §2.1]. In this section we fix the notation and recall the properties that we will need for our purposes.

DEFINITION 3.1.1. A (generalized) Pell's equation is a diophantine equation in two unknowns X, Y of the form

$$X^2 - rY^2 = m$$

for $r \in \mathbb{N}$ and $m \in \mathbb{Z}$.
A solution $(X, Y) \in \mathbb{Z}^2$ of the equation is called *positive* if $X > 0, Y > 0$. If integer solutions exist, the *minimal* solution is the positive solution with minimal X.

REMARK 3.1.2. If $m > 0$ (respectively, $m < 0$), the minimal solution of Pell's equation $X^2 - rY^2 = m$ is also the solution which minimizes (respectively, maximizes) the slope $\frac{Y}{X} = \sqrt{\frac{1}{r} - \frac{m}{rX^2}}$.

Clearly, for any $r \in \mathbb{N}$ there exist solutions for Pell's equation $X^2 - rY^2 = 1$; in particular, if r is a square the only solutions are $(\pm 1, 0)$. In the case where r is the product of two non trivial integers and $m = 1$, we have the following result.

LEMMA 3.1.3. *Let $s, q \in \mathbb{N}$, with $q \neq 1$. If the equation $sX^2 - qY^2 = -1$ admits integer solutions, let (a, b) be the positive one with minimal X. Then the minimal solution of Pell's equation $X^2 - sqY^2 = 1$ is $(2sa^2 + 1, 2ab)$.*

PROOF. A more general statement can be found in [37, Lemma A.2]. For the case $s = 1$ see also [20, Lemma 2.1]. $\square$

The solutions of $X^2 - rY^2 = m$ can be divided into equivalence classes. Two solutions (x, y) and (x', y') are equivalent if there exists a solution (z, w) of Pell's equation $X^2 - rY^2 = 1$ such that

$$\begin{cases} x' = zx + rwy \\ y' = wx + zy \end{cases} \tag{8}$$

(for more details, see [33, Chapter XXXIII, §18]). We define the *fundamental solution* in an equivalence class to be the solution with smallest non-negative Y, if it is unique. If instead there are two solutions with this property in the same equivalence class, they are of the form (X, Y), $(-X, Y)$, with $X > 0$: in this case we consider (X, Y) to be the fundamental solution. By applying (8) recursively, all solutions in an equivalence class can be reconstructed from the fundamental one, after computing the minimal solution of $X^2 - rY^2 = 1$.

REMARK 3.1.4. If $m > 0$, let (X, Y) be a fundamental solution of $X^2 - rY^2 = m$ and (z, w) the minimal solution of $X^2 - rY^2 = 1$. Then, either (X, Y) or $(-X, Y)$ belongs to the closed interval on the hyperbola $X^2 - rY^2 = m$ delimited by the points $(\sqrt{m}, 0)$ and $(\sqrt{m(z+1)/2}, \sqrt{m(z-1)/(2r)})$. Notice, moreover, that all solutions (x, y) contained in this interval are such that $0 \leq \frac{y}{x} < \frac{w}{z}$.

3.1.2. Movable and nef cones of $\Sigma^{[n]}$. We consider an algebraic $K3$ surface Σ with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, where we assume that H is an ample class with $H^2 = 2t$, $t \geq 1$. Let $\Sigma^{[n]}$ be the Hilbert scheme of n points on Σ. The ample class H on Σ induces a line bundle $\widetilde{H}$ on $\Sigma^{[n]}$, whose first Chern class we denote by h. Then we can take $\{h, -\delta\}$ as a basis for the Néron-Severi lattice $\mathrm{NS}(\Sigma^{[n]}) \subset H^2(\Sigma^{[n]}, \mathbb{Z})$, where 2δ is the class of the exceptional divisor of the Hilbert–Chow morphism $\Sigma^{[n]} \to \Sigma^{(n)}$ (see Example 2.2.4).

Recall from Example 2.2.7 and Section 2.4 that $\Sigma^{[n]}$ is isomorphic to the moduli space $M_\tau(v_n)$, where $v_n = (1, 0, 1-n) \in H^*(\Sigma, \mathbb{Z})$ and τ is a v_n-generic Bridgeland stability condition. In particular, we have an isomorphism

$$\theta^{-1} : H^2(\Sigma^{[n]}, \mathbb{Z}) \longrightarrow v_n^\perp \subset H^*(\Sigma, \mathbb{Z})$$

such that $\theta(v_n^\perp \cap H^*_{\mathrm{alg}}(\Sigma, \mathbb{Z})) = \mathrm{NS}(\Sigma^{[n]})$. If $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, the basis $\{h, -\delta\}$ of $\mathrm{NS}(\Sigma^{[n]})$ is realized as

$$h = \theta(0, -H, 0), \quad -\delta = \theta(1, 0, n-1)$$

where (with an abuse of notation) we still denote by H the first Chern class of the generator of $\mathrm{Pic}(\Sigma)$.

We saw in Section 2.4 that the walls between Kähler-type chambers in the positive cone $\mathcal{C}_{\Sigma^{[n]}} \cap \mathrm{NS}(\Sigma^{[n]})_\mathbb{R}$ are linear subspaces of the form $\theta(v_n^\perp \cap a^\perp)$, where $a \in H^*_{\mathrm{alg}}(\Sigma, \mathbb{Z})$ are elements with prescribed values for a^2 and (v_n, a) (see Theorem 2.4.13 and Theorem 2.4.14). The following lemma provides a description of algebraic classes $a \in H^*_{\mathrm{alg}}(\Sigma, \mathbb{Z})$ with given square and pairing with $v_n = (1, 0, 1-n)$.

LEMMA 3.1.5. *Let Σ be an algebraic $K3$ surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, $H^2 = 2t$, $t \geq 1$. An element $a \in H^*_{alg}(\Sigma, \mathbb{Z})$ has $a^2 = 2\rho$ and $(v_n, a) = \alpha$, with $\rho, \alpha \in \mathbb{Z}$, if and only if it is of the form*

$$a = \left(\frac{X+\alpha}{2(n-1)}, -YH, \frac{X-\alpha}{2} \right) \qquad or \qquad a = -\left(\frac{X-\alpha}{2(n-1)}, -YH, \frac{X+\alpha}{2} \right)$$

where (X, Y) is a solution of Pell's equation $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ such that $X \geq 0$ and $2(n-1) \mid (X+\alpha)$ or $2(n-1) \mid (X-\alpha)$ respectively.

PROOF. Let $a = (r, cH, s) \in H^*_{\mathrm{alg}}(\Sigma, \mathbb{Z}) = H^0(\Sigma, \mathbb{Z}) \oplus \mathrm{Pic}(\Sigma) \oplus H^4(\Sigma, \mathbb{Z})$, with $r, c, s \in \mathbb{Z}$. Since $a^2 = 2(tc^2 - rs)$ and $(v_n, a) = r(n-1) - s$, we want to characterize the triples $(r, c, s) \in \mathbb{Z}^3$ such that

$$\begin{cases} tc^2 - rs = \rho \\ r(n-1) - s = \alpha. \end{cases} \tag{9}$$

Let $(r, c, s) \in \mathbb{Z}^3$ be a solution of (9). If $2r(n-1) \geq \alpha$, then

$$X = 2r(n-1) - \alpha \geq 0, \qquad Y = -c$$

satisfy $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ and $2(n-1)$ divides $X + \alpha$. If instead $2r(n-1) < \alpha$, then

$$X = -2r(n-1) + \alpha > 0, \qquad Y = c$$

satisfy $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ and $2(n-1)$ divides $X - \alpha$.

Conversely, let (X, Y) be a solution of Pell's equation $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$, with $X \geq 0$ and such that $2(n-1)$ divides either $X + \alpha$ or $X - \alpha$. In the first case, we define

$$r = \frac{X + \alpha}{2(n-1)}, \qquad c = -Y, \qquad s = r(n-1) - \alpha = \frac{X - \alpha}{2}$$

while in the second case we set

$$r = -\frac{X - \alpha}{2(n-1)}, \qquad c = Y, \qquad s = r(n-1) - \alpha = -\frac{X + \alpha}{2}.$$

Then, one can easily check that $(r, c, s) \in \mathbb{Z}^3$ is a solution of (9). □

In the case of Hilbert schemes $\Sigma^{[n]}$ with $\mathrm{rk}(\mathrm{NS}(\Sigma^{[n]})) = 2$, both the ample cone and the movable cone inside $\mathrm{NS}(\Sigma^{[n]})_{\mathbb{R}}$ are delimited by two extremal rays.

THEOREM 3.1.6. *Let Σ be an algebraic $K3$ surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, $H^2 = 2t$ and $n \geq 2$.*

(*i*) *If $t(n-1) = c^2$, for $c \in \mathbb{N}$, then $\overline{\mathrm{Mov}(\Sigma^{[n]})} = \langle h, (n-1)h - c\delta \rangle$.*
(*ii*) *If $t(n-1)$ is not a square and the equation $(n-1)X^2 - tY^2 = 1$ has integer solutions, let (z, w) be the positive solution with minimal z. Then $\overline{\mathrm{Mov}(\Sigma^{[n]})} = \langle h, (n-1)zh - tw\delta \rangle$.*
(*iii*) *If $t(n-1)$ is not a square and $(n-1)X^2 - tY^2 = 1$ has no integer solutions, then $\overline{\mathrm{Mov}(\Sigma^{[n]})} = \langle h, zh - tw\delta \rangle$ where (z, w) is the minimal solution of Pell's equation $X^2 - t(n-1)Y^2 = 1$ such that $z \equiv \pm 1 \pmod{n-1}$.*

PROOF. See [7, Proposition 13.1]. □

In all three cases of the theorem, the extremal ray of $\overline{\mathrm{Mov}(\Sigma^{[n]})}$ generated by h is also one of the two walls delimiting $\mathrm{Nef}(\Sigma^{[n]})$. If $\mathrm{Nef}(\Sigma^{[n]}) \neq \overline{\mathrm{Mov}(\Sigma^{[n]})}$, the second wall of the nef cone coincides with the wall with minimal slope, inside $\mathrm{Mov}(\Sigma^{[n]})$, among those listed in Remark 2.4.15 (in [7], these are referred to as *flopping walls*).

3.2. Ample classes and isometries of $\mathrm{NS}(\Sigma^{[n]})$

In this section we investigate the group of isometries $O(\mathrm{NS}(\Sigma^{[n]}))$. We adopt a similar approach to the one used by the authors of [20] for the case $n = 2$, in order to extend their results.

Definition 3.2.1. Let Σ be a smooth complex surface and k a non-negative integer. A line bundle $L \in \mathrm{Pic}(\Sigma)$ is *k-very ample* if the restriction $H^0(\Sigma, L) \to H^0(\Sigma, L \otimes \mathcal{O}_Z)$ is surjective for any zero-dimensional subscheme $(Z, \mathcal{O}_Z)$ of length $h^0(\mathcal{O}_Z) \leq k+1$.

We recall the following geometric interpretation of k-very ampleness. For any subscheme $(Z, \mathcal{O}_Z)$ we have the inclusion $H^0(\Sigma, L \otimes I_Z) \hookrightarrow H^0(\Sigma, L)$, which allows us to define a rational map $\gamma : \Sigma^{[k]} \dashrightarrow \mathrm{Grass}(k, H^0(\Sigma, L))$. Then, γ is an embedding if and only if L is k-very ample ([31, Main Theorem]).

Proposition 3.2.2. *Let Σ be a $K3$ surface, L a big and nef line bundle on Σ and $k \geq 0$ an integer. Then L is k-very ample if and only if $L^2 \geq 4k$ and there exist no effective divisors D such that:*

(a) $2D^2 \overset{(i)}{\leq} L \cdot D \leq D^2 + k + 1 \overset{(ii)}{\leq} 2k + 2$*;*
(b) (i) *is an equality if and only if* $L \sim 2D$ *and* $L^2 \leq 4k + 4$*;*
(c) (ii) *is an equality if and only if* $L \sim 2D$ *and* $L^2 = 4k + 4$*.*

Proof. See [61, Theorem 1.1]. □

Proposition 3.2.3. *Let Σ be an algebraic $K3$ surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, $H^2 = 2t$, $t \geq 1$. The class $ah - \delta \in \mathrm{NS}(\Sigma^{[n]})$ is ample if $a \geq n+2$. In particular, inside $\mathrm{NS}(\Sigma^{[n]})_{\mathbb{R}}$, the ample cone $\mathcal{A}_{\Sigma^{[n]}}$ is contained in $\{xh - y\delta \mid x > 0, y > 0\}$.*

Proof. If $L = aH \in \mathrm{Pic}(\Sigma)$ is n-very ample, then the element $ah - \delta \in \mathrm{NS}(\Sigma^{[n]})$ is ample, because it is the first Chern class of the line bundle $\gamma^*(\mathcal{O}_{\mathbb{G}}(1))$, where we consider the embedding $\gamma : \Sigma^{[n]} \to \mathbb{G} := \mathrm{Grass}(n, H^0(\Sigma, L))$ obtained from L (see [14, Construction 2]). We now use Knutsen's characterization of n-very ample line bundles (Proposition 3.2.2). We have $L^2 \geq 4n$ if and only if $ta^2 \geq 2n$. If $D = dH$ is effective ($d > 0$), we can reformulate condition (a) of Proposition 3.2.2 as

$$4td^2 \overset{(i)}{\leq} 2tad \leq 2td^2 + n + 1 \overset{(ii)}{\leq} 2n + 2.$$

It is easy to see that, if $a \geq n+2$, there are no values d satisfying all these inequalities, while instead the condition $L^2 \geq 4n$ holds.

Thus, for all $a \geq n+2$ the line bundle $L = aH$ is n-very ample and $ah - \delta \in \mathcal{A}_{\Sigma^{[n]}}$. Moreover, the two classes $h, -\delta \in \mathrm{NS}(\Sigma^{[n]})$ are not ample, therefore we conclude $\mathcal{A}_{\Sigma^{[n]}} \subset \{xh - y\delta \mid x > 0, y > 0\}$, because $\mathcal{A}_{\Sigma^{[n]}}$ is a convex cone. □

We are now interested in describing the isometries of the Néron-Severi lattice of $\Sigma^{[n]}$. With respect to the basis $\{h, -\delta\}$, the bilinear form on $\mathrm{NS}(\Sigma^{[n]})$ satisfies $(h,h) = 2t, (-\delta, -\delta) = -2(n-1), (h, -\delta) = 0$. Thus, in coordinates, an isometry $\phi \in O(\mathrm{NS}(\Sigma^{[n]}))$ is represented by a matrix of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A, B, C, D integers such that:

(i) $\det M = \pm 1$, i.e. $AD - BC = \pm 1$;
(ii) $(Ah - C\delta, Ah - C\delta) = 2t$, i.e. $(n-1)C^2 = t(A^2 - 1)$;
(iii) $(Bh - D\delta, Bh - D\delta) = -2(n-1)$, i.e. $(n-1)(D^2 - 1) = tB^2$;
(iv) $(Ah - C\delta, Bh - D\delta) = 0$, i.e. $(n-1)CD = tAB$.

From this list of conditions, we find two alternative forms for M:

$$\begin{pmatrix} A & B \\ C & A \end{pmatrix} \text{ or } \begin{pmatrix} A & B \\ -C & -A \end{pmatrix}, \quad \text{with } (n-1)C = tB \text{ and } (n-1)A^2 - tB^2 = n-1.$$

Matrices of the first form have determinant $+1$, while those of the second form have determinant -1. Moreover, for $M = \begin{pmatrix} A & B \\ -C & -A \end{pmatrix}$ we have $M^2 = I_2$ (from the relations between A, B, C), meaning that the isometry described by M is an involution.

Notice that we can write $O(\mathrm{NS}(\Sigma^{[n]})) \cong N \rtimes \langle s \rangle$, where N is the normal subgroup of isometries of the first form:

$$N := \left\{ \begin{pmatrix} A & B \\ C & A \end{pmatrix} \mid A, B, C \in \mathbb{Z}, (n-1)C = tB, (n-1)A^2 - tB^2 = n-1 \right\}$$

while s is the order two matrix $s := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O(\mathrm{NS}(\Sigma^{[n]}))$.

An automorphism $f \in \mathrm{Aut}(\Sigma^{[n]})$ induces, by pull-back, an isometry f^* of the lattice $H^2(\Sigma^{[n]}, \mathbb{Z})$, such that $f^*|_{\mathrm{NS}(\Sigma^{[n]})} \in O(\mathrm{NS}(\Sigma^{[n]}))$. As proved for the case $n = 2$ in [20, Proposition 4.3], there is a link between such isometries and the ample cone of $\Sigma^{[n]}$.

Proposition 3.2.4. *Let Σ be an algebraic $K3$ surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, $H^2 = 2t$, $t \geq 1$. The isometry of $\mathrm{NS}(\Sigma^{[n]})$ induced by $f \in \mathrm{Aut}(\Sigma^{[n]})$ is either the identity or the involution given, with respect to the basis $\{h, -\delta\}$, by the matrix*

$$\begin{pmatrix} A & B \\ -C & -A \end{pmatrix}, \quad \text{with } (n-1)C = tB,\ (n-1)A^2 - tB^2 = n-1,\ A > 0,\ B < 0$$

and with A, B defining the ample cone of $\Sigma^{[n]}$:

$$\mathcal{A}_{\Sigma^{[n]}} = \{ xh - y\delta \mid y > 0, (n-1)Ay < -tBx \}.$$

Proof. The isometry ϕ induced by f on the Néron-Severi group of $\Sigma^{[n]}$ can only be of the two forms described previously in this section. We look at them separately.

- $\phi = \begin{pmatrix} A & B \\ C & A \end{pmatrix}$ with $(n-1)C = tB$ and $(n-1)A^2 - tB^2 = n-1$.
  Assume $\phi \neq \pm \mathrm{id}$, i.e. $B \neq 0$. Notice that ϕ, which is the restriction to $\mathrm{NS}(\Sigma^{[n]})$ of $f^* \in \mathrm{Mon}^2_{\mathrm{Hdg}}(\Sigma^{[n]})$, maps ample classes to ample classes (see Proposition 2.4.10). Then, as a consequence of Proposition 3.2.3, $\phi(a, 1) = (aA + B, aC + A) \in \mathcal{A}_{\Sigma^{[n]}}$ for any $a \geq n+2$. Moreover, we have $\mathcal{A}_{\Sigma^{[n]}} \subset \{xh - y\delta \mid x > 0, y > 0\}$, again by Proposition 3.2.3, so we deduce $A > 0$, $C > 0$ (therefore also $B > 0$). Instead, since h is not an ample class, $\phi(1, 0) = (A, C) \notin \mathcal{A}_{\Sigma^{[n]}}$, which means that (in the plane $\mathrm{NS}(\Sigma^{[n]})_{\mathbb{R}}$):
  $$\mathcal{A}_{\Sigma^{[n]}} \subset \left\{ xh - y\delta \mid y > 0, Ay < \frac{tB}{n-1} x \right\}.$$
  Now, the class $\phi(n+2, 1) = ((n+2)A + B, (n+2)C + A)$ needs to be in the ample cone, but it does not satisfy the inequality $Ay < \frac{tB}{n-1}x$ (because of the property $(n-1)A^2 - tB^2 = n-1 > 0$), so we get a contradiction: ϕ cannot be of this form, unless $\phi = \mathrm{id}$ (we have to exclude $\phi = -\mathrm{id}$, because it does not preserve the ample cone).
- $\phi = \begin{pmatrix} A & B \\ -C & -A \end{pmatrix}$ with $(n-1)C = tB$ and $(n-1)A^2 - tB^2 = n-1$.
  We proceed as in the previous case: the classes $\phi(a, 1) = (aA + B, -aC - A)$ are ample for any $a \geq n+2$, so $A > 0$ and $C < 0$ (thus also $B < 0$; notice that we cannot have $C = 0$, otherwise $\phi(a, 1)$ would not be in the ample cone, for any positive a). In particular, all the rays through the classes

$\phi(a,1)$, for $a \geq n+2$, are contained in the ample cone $\mathcal{A}_{\Sigma^{[n]}}$. Passing to the limit $a \to +\infty$, the ray through $(A,-C)$ must be in the closure of the cone, so

$$\mathcal{A}_{\Sigma^{[n]}} \supset \mathcal{F} := \left\{ xh - y\delta \mid y > 0, Ay < -\frac{tB}{n-1}x \right\}.$$

To conclude, we observe that $\phi(1,0) = (A,-C) \notin \mathcal{A}_{\Sigma^{[n]}}$, because h is not ample, so we also have $\mathcal{A}_{\Sigma^{[n]}} \subset \mathcal{F}$. □

3.3. The automorphism group of $\Sigma^{[n]}$

The aim of this section is to classify the group structure of $\mathrm{Aut}(\Sigma^{[n]})$ and to determine some preliminary numerical conditions for the existence of a non-trivial automorphism.

We recall that, if Σ is a projective $K3$ surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, $H^2 = 2t$, the automorphism group $\mathrm{Aut}(\Sigma)$ is trivial if $t \geq 2$. Instead, if $t = 1$ there exists a double covering $\Sigma \to \mathbb{P}^2$, which is ramified over a smooth curve of degree six, and $\mathrm{Aut}(\Sigma) = \{\mathrm{id}, \iota\}$, where ι is the (non-symplectic) covering involution (see [**91**, §5] and [**52**, Corollary 15.2.12]). Proposition 3.2.4 allows us to provide a result on the order of the automorphism group $\mathrm{Aut}(\Sigma^{[n]})$.

Proposition 3.3.1. *Let Σ be an algebraic $K3$ surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, $H^2 = 2t$. If $t \geq 2$, the automorphism group $\mathrm{Aut}(\Sigma^{[n]})$ is either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$, generated by a non-natural, non-symplectic involution.*

Proof. The map $\lambda : \mathrm{Aut}(\Sigma^{[n]}) \to O(H^2(\Sigma^{[n]},\mathbb{Z}))$, sending an automorphism of $\Sigma^{[n]}$ to its action on the second cohomology lattice, is injective by Theorem 2.5.2. Instead, if we only consider $\Psi : \mathrm{Aut}(\Sigma^{[n]}) \to O(\mathrm{NS}(\Sigma^{[n]}))$, $f \mapsto f^*|_{\mathrm{NS}(\Sigma^{[n]})}$, its kernel is the set of natural automorphisms (generalization of [**20**, Lemma 2.4], using Theorem 2.5.7). Under the hypothesis $t \geq 2$, the identity is the only automorphism of Σ, therefore $\mathrm{Aut}(\Sigma^{[n]})$ is in one-to-one correspondence with the image of Ψ, which is either trivial or generated by an isometry of order two, by Proposition 3.2.4. If $\mathrm{Aut}(\Sigma^{[n]})$ contains an involution f, then it is non-natural and non-symplectic: in fact, if it were symplectic the co-invariant lattice $S_f = \left(H^2(\Sigma^{[n]},\mathbb{Z})^{f^*}\right)^\perp \subset H^2(\Sigma^{[n]},\mathbb{Z})$ would be contained in $\mathrm{NS}(\Sigma^{[n]})$ and it would be of rank eight (see Proposition 2.5.5 and [**74**, Corollary 5.1]), while here $\mathrm{rk}(\mathrm{NS}(\Sigma^{[n]})) = 2$. □

Remark 3.3.2. If $t = 1$, the map Ψ is no longer injective: $\ker(\Psi) = \left\{\mathrm{id}, \iota^{[n]}\right\}$, where $\iota^{[n]}$ is the natural (non-symplectic) automorphism of order two induced by the covering involution $\iota \in \mathrm{Aut}(\Sigma)$. Nevertheless, any automorphism $f \in \mathrm{Aut}(\Sigma^{[n]}) \setminus \ker(\Psi)$ is a non-natural, non-symplectic involution and the quotient $\mathrm{Aut}(\Sigma^{[n]})/\ker(\Psi)$ is either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$, following the proof of Proposition 3.3.1.

Let now $f \in \mathrm{Aut}(\Sigma^{[n]})$ be a non-natural automorphism (that is, if $t \geq 2$, any non-trivial automorphism). As we showed in Proposition 3.2.4, f induces an isometry of $\mathrm{NS}(\Sigma^{[n]})$ of the form

$$\phi = \begin{pmatrix} A & B \\ -C & -A \end{pmatrix}, \quad (n-1)C = tB,\ (n-1)A^2 - tB^2 = n-1 \tag{10}$$

with coefficients $A > 0, B < 0, C < 0$ uniquely determined by the ample cone $\mathcal{A}_{\Sigma^{[n]}}$.

One can check that this isometry is the reflection of the Néron-Severi lattice in the line spanned by the class of coordinates $(-B, A-1)$. As it was done in [**20**] for the case $n = 2$, we denote by (b,a) the primitive generator of this line: in particular,

$(b,a) := \frac{1}{d}(-B, A-1)$, with $d = \gcd(-B, A-1)$. By computing explicitly the matrix of the reflection whose fixed line is $\langle (b,a)\rangle$, we find the relations

$$A = \frac{tb^2 + (n-1)a^2}{tb^2 - (n-1)a^2}, \qquad B = -\frac{2ab(n-1)}{tb^2 - (n-1)a^2}. \tag{11}$$

Since f is a non-symplectic involution, the invariant lattice $T_f = H^2(\Sigma^{[n]}, \mathbb{Z})^{f^*}$ is contained in $\mathrm{NS}(\Sigma^{[n]})$ by Proposition 2.5.5. Thus, T_f is the lattice of rank one generated by (b,a). Moreover, the transcendental lattice $\mathrm{Tr}(\Sigma^{[n]})$ is contained in $S_f = T_f^{\perp}$, hence $f^*|_{\mathrm{Tr}(\Sigma^{[n]})} = -\,\mathrm{id}$, because the non-natural involution f^* acts as $-\,\mathrm{id}$ on $S_f = \ker(\mathrm{id} + f^*)$ (see Section 2.5).

Lemma 3.3.3. *Let ϕ be the isometry of $\mathrm{NS}(\Sigma^{[n]})$ of the form (10) induced by a non-natural automorphism $f \in \mathrm{Aut}(\Sigma^{[n]})$ and let $bh - a\delta$ be the primitive generator of the invariant lattice $T_f \subset \mathrm{NS}(\Sigma^{[n]})$. Then:*

(i) *$tb^2 - (n-1)a^2$ divides b;*
(ii) *there exists an even negative integer β such that $B = (n-1)\beta$.*

Proof. The isometry $f^* \in O(H^2(\Sigma^{[n]}, \mathbb{Z}))$ has restrictions $f^*|_{\mathrm{NS}(\Sigma^{[n]})} = \phi$ and $f^*|_{\mathrm{Tr}(\Sigma^{[n]})} = -\,\mathrm{id}_{\mathrm{Tr}(\Sigma^{[n]})}$. As we saw in Example 2.2.4, the following isometry of lattices holds:

$$H^2(\Sigma^{[n]}, \mathbb{Z}) \cong L_n = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle.$$

We embed $\mathrm{NS}(\Sigma^{[n]}) = \mathbb{Z}h \oplus \mathbb{Z}(-\delta)$ inside L_n by mapping h to $e_1 + te_2$ (where $\{e_1, e_2\}$ is a basis for the first summand U) and $-\delta$ to the generator g of the component $\langle -2(n-1)\rangle$. We take $\{e_1 + te_2, e_2, g\}$ as a basis for the lattice $U \oplus \langle -2(n-1)\rangle$, which contains $\mathrm{NS}(\Sigma^{[n]})$ as a sublattice of rank two. Notice that $w := e_1 - te_2$ belongs to the transcendental lattice, therefore $f^*(w) = -w$. By writing $w = (e_1 + te_2) - 2te_2$ we compute:

$$2tf^*(e_2) = (A+1)(e_1 + te_2) - 2te_2 - Cg$$

which can also be written, using relations (11), as

$$f^*(e_2) = \frac{b^2}{tb^2 - a^2(n-1)}(e_1 + te_2) - e_2 + \frac{ab}{tb^2 - a^2(n-1)}g.$$

Since the coefficients of this expression need to be integers, $tb^2 - (n-1)a^2$ divides $\gcd(b^2, ab) = b$. Also notice that, due to the condition $(n-1)A^2 - tB^2 = n-1$, we have: $d(tb^2 - a^2(n-1)) = 2a(n-1)$. Therefore, since $\frac{ab}{tb^2 - a^2(n-1)}$ is integer, by multiplying both numerator and denominator by d we deduce that $2(n-1)$ divides $B = -db < 0$. $\square$

Using Lemma 3.3.3, we can determine the group $\mathrm{Aut}(\Sigma^{[n]})$ in some cases where we have a simple description of the nef cone of $\Sigma^{[n]}$.

Proposition 3.3.4. *Let Σ be an algebraic $K3$ surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, $H^2 = 2t$, $t \geq 1$. If $n \geq \frac{t+3}{2}$, all automorphisms of $\Sigma^{[n]}$ are natural.*

Proof. Let $f \in \mathrm{Aut}(\Sigma^{[n]})$ be a non-natural automorphism. Its action on $\mathrm{NS}(\Sigma^{[n]})$ is non-trivial, therefore it is an isometry of the form (10), by Proposition 3.2.4. In particular, $\mathrm{Nef}(\Sigma^{[n]})$ is generated over $\mathbb{R}^+$ by h and $(n-1)Ah + tB\delta$. However, if $n \geq \frac{t+3}{2}$ we also have the description of the nef cone given in [8, Proposition 10.3]: it is generated by h and $h - \frac{2t}{t+n}\delta$. Thus, we need the two classes $(n-1)Ah + tB\delta$ and $h - \frac{2t}{t+n}\delta$ to be proportional, i.e.

$$\frac{(n-1)A}{t+n} = -\frac{tB}{2t}.$$

By using relations (11), this becomes $tb^2 + (n-1)a^2 = (t+n)ab$, therefore

$$tb^2 - (n-1)a^2 = (t+n)ab - 2(n-1)a^2 = a((t+n)b - 2(n-1)a).$$

As we proved in Lemma 3.3.3, $tb^2 - (n-1)a^2$ divides b. However, from the last expression for $tb^2 - (n-1)a^2$, this implies $a \mid b$, so necessarily $a = 1$, since $\gcd(a, b) = 1$ by definition (also, a and b are both positive).

Therefore, b is an integer solution of the equation:

$$tb^2 - (t+n)b + n - 1 = 0.$$

Assuming $n \geq 2$, a simple computation shows that this equation admits an integer solution only if n is odd and $n+1 = 2t$: in this case, $b = 2$. However, if that is the case we have $tb^2 - (n-1)a^2 = 2t + 2$, which does not divide $b = 2$ for any $t \geq 1$, thus we get a contradiction. $\square$

Notice that the condition $n \geq \frac{t+3}{2}$ is satisfied by all values $n \geq 2$ when $t = 1$, therefore the case $t = 1$ is now completely resolved. If instead $t \geq 2$, we already know from Proposition 3.3.1 that there are no non-trivial natural automorphisms on $\Sigma^{[n]}$.

Corollary 3.3.5. *Let Σ be an algebraic $K3$ surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, $H^2 = 2t$ and let $n \geq 2$.*

- *If $t = 1$, then $\mathrm{Aut}(\Sigma^{[n]}) = \left\{\mathrm{id}, \iota^{[n]}\right\}$, where $\iota^{[n]}$ is the natural involution induced by the covering involution $\iota \in \mathrm{Aut}(\Sigma)$.*
- *If $2 \leq t \leq 2n - 3$, then $\mathrm{Aut}(\Sigma^{[n]}) = \{\mathrm{id}\}$.*

From now on, we will assume $t \geq 2$. In Proposition 3.3.1 and Lemma 3.3.3 we proved that the isometry $\phi \in O(\mathrm{NS}(\Sigma^{[n]}))$ induced by a non-trivial automorphism of $\Sigma^{[n]}$ is of the form (10) with $B = (n-1)\beta$ and $\beta < 0$ even. We therefore rewrite the matrix of ϕ and the conditions on its coefficients in the following way:

$$\phi = \begin{pmatrix} A & (n-1)\beta \\ -t\beta & -A \end{pmatrix}, \quad \text{with } A^2 - t(n-1)\beta^2 = 1,\ A > 0,\ \beta < 0 \text{ even} \tag{12}$$

and with $\mathcal{A}_{\Sigma^{[n]}} = \{xh - y\delta \mid y > 0, Ay < -t\beta x\}$. We will return to numerical conditions for the existence of an automorphism on $\Sigma^{[n]}$ in Section 3.5.

3.4. Invariant polarizations

In this section we study the properties of the generator of the invariant lattice of a non-natural involution on $\Sigma^{[n]}$ and, vice versa, we show that the existence of an ample divisor in $\mathrm{NS}(\Sigma^{[n]})$ with such properties guarantees that $\mathrm{Aut}(\Sigma^{[n]})$ is non-trivial.

Recall that the divisibility of an element l in a lattice L is the positive generator of the ideal $(l, L) \subset \mathbb{Z}$. For the remaining of the chapter, the notation $\mathrm{div}(x)$ will always refer to the divisibility in the lattice $H^2(\Sigma^{[n]}, \mathbb{Z}) \cong L_n$, even if the element x is stated to belong to a proper sublattice of $H^2(\Sigma^{[n]}, \mathbb{Z})$, such as $\mathrm{NS}(\Sigma^{[n]})$.

Proposition 3.4.1. *Let Σ be an algebraic $K3$ surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, $H^2 = 2t$, $t \geq 2$ and $f \in \mathrm{Aut}(\Sigma^{[n]})$ an involution. Let ν be the primitive generator of the rank one invariant lattice $T_f \subset H^2(\Sigma^{[n]}, \mathbb{Z})$. Then one of the following holds:*

- *f^* acts as $-\mathrm{id}$ on the discriminant group of $H^2(\Sigma^{[n]}, \mathbb{Z})$ and $(\nu, \nu) = 2$;*
- *-1 is a quadratic residue modulo $n-1$, f^* acts as id on the discriminant group of $H^2(\Sigma^{[n]}, \mathbb{Z})$, $(\nu, \nu) = 2(n-1)$ and $\mathrm{div}(\nu) = n-1$.*

Proof. The generator ν of T_f coincides with the ample class $bh - a\delta \in \mathrm{NS}(\Sigma^{[n]})$ defined in Section 3.3. We recall that $f^*|_{\mathrm{NS}(\Sigma^{[n]})}$ is the reflection fixing the line $\langle \nu \rangle$, while f^* acts as $-\mathrm{id}$ on $\mathrm{Tr}(\Sigma^{[n]})$, therefore f^* can also be regarded as the opposite of the reflection of $H^2(\Sigma^{[n]}, \mathbb{Z})$ defined by ν, i.e.

$$f^* = -R_\nu : H^2(\Sigma^{[n]}, \mathbb{Z}) \to H^2(\Sigma^{[n]}, \mathbb{Z}), \quad m \mapsto 2\frac{(m,\nu)}{(\nu,\nu)}\nu - m. \tag{13}$$

Moreover, f^* is a monodromy operator, therefore by Proposition 2.3.8 we have

$$f^* \in \widetilde{O}^+(L_n) := \left\{\psi \in O^+(L_n) \mid \overline{\psi} = \pm \mathrm{id}\right\}$$

where $O^+(L_n) \subset O(L_n)$ is the subgroup of orientation-preserving isometries and $\overline{\psi}$ denotes the isometry of the discriminant group A_{L_n} induced by ψ (see Section 1.4).

The elements $l \in L_n$ such that $\overline{R_l} = \mathrm{id}$ or $\overline{(-R_l)} = \mathrm{id}$ were studied in [43, Proposition 3.1], [43, Proposition 3.2] respectively. By applying these results, and observing that ν has square $(\nu,\nu) = 2(tb^2 - (n-1)a^2) > 0$, we can conclude the following:

(i) if $\overline{f^*} = -\mathrm{id}$, then $(\nu,\nu) = 2$;

(ii) if $\overline{f^*} = \mathrm{id}$, then $(\nu,\nu) = 2(n-1)$ and $\mathrm{div}(\nu) = n-1$ or $\mathrm{div}(\nu) = 2(n-1)$.

We deduce that the square of the generator of the invariant lattice T_f can only be 2 or $2(n-1)$. The existence of a primitive element $l \in L_n$ with given square and divisibility depends on whether some arithmetic conditions are satisfied: this is proved in [44, Proposition 3.6], where an explicit description of the lattice $l^\perp$ is also provided. Using this result we find out that case (ii) is admissible only if -1 is a quadratic residue modulo $n-1$ and the divisibility of ν is $n-1$. This concludes the proof. □

REMARK 3.4.2. By Proposition 3.4.1, the square of $\nu = bh - a\delta$ can be determined by looking at the action of f^* on A_{L_n}. A generator for $A_{L_n} \cong A_{\langle -2(n-1)\rangle}$ is given by the equivalence class, in the quotient $L_n^\vee/L_n$, of $-\frac{\delta}{2(n-1)} \in L_n \otimes \mathbb{Q}$, because $-\delta$ is the generator of the summand $\langle -2(n-1)\rangle$ in L_n, via the embedding $\mathrm{NS}(\Sigma^{[n]}) \hookrightarrow L_n$ presented in the proof of Lemma 3.3.3. Thus, since $f^*|_{\mathrm{NS}(\Sigma^{[n]})} = \phi$ has the form (12), we obtain:

$$\overline{f^*}\left(-\frac{\delta}{2(n-1)}\right) = \frac{1}{2(n-1)}\left((n-1)\beta h - A(-\delta)\right) \equiv -\frac{A}{2(n-1)}(-\delta) \pmod{L_n}$$

where we used the fact that β is even, by Lemma 3.3.3. On the other hand, we know that f^* acts as $\pm\mathrm{id}$ on A_{L_n}, being a monodromy operator, hence either $A+1$ or $A-1$ is divisible by $2(n-1)$. In particular, if $2(n-1)$ divides $A-1$, then $\overline{f^*} = -\mathrm{id}$ and ν has square 2; instead, if $2(n-1)$ divides $A+1$ then $\overline{f^*} = +\mathrm{id}$ and ν has square $2(n-1)$. Notice that $2(n-1)$ can divide both $A-1$ and $A+1$ only for $n=2$: in this case, $-\mathrm{id} = +\mathrm{id}$ on $A_{L_2} \cong \mathbb{Z}/2\mathbb{Z}$ and ν has square $2 = 2(n-1)$, as already stated in Theorem 3.0.1.

By Proposition 3.4.1, if $t \geq 2$ and $\mathrm{Aut}(\Sigma^{[n]}) \neq \{\mathrm{id}\}$ there exists a primitive ample class $\nu \in \mathrm{NS}(\Sigma^{[n]})$ with $(\nu,\nu) = 2$ or $(\nu,\nu) = 2(n-1)$, $\mathrm{div}(\nu) = n-1$. We now show that the converse holds for any manifold of $K3^{[n]}$-type.

PROPOSITION 3.4.3. *Let X be an irreducible holomorphic symplectic manifold deformation equivalent to the Hilbert scheme of n points on a $K3$ surface, $n \geq 2$. Then X admits a non-symplectic involution if there exists a primitive ample class $\nu \in \mathrm{NS}(X)$ with either*

- $(\nu,\nu) = 2$, *or*
- $(\nu,\nu) = 2(n-1)$ *and* $\mathrm{div}(\nu) = n-1$.

PROOF. Let $\gamma := -R_\nu \in O(H^2(X,\mathbb{Q}))$ be the opposite of the reflection defined by ν, as in (13). If ν is an element as in the statement, γ defines an integral isometry

$\gamma \in O(H^2(X,\mathbb{Z}))$. Moreover, γ induces $\pm\,\mathrm{id}$ on the discriminant group of $H^2(X,\mathbb{Z})$ by [43, Proposition 3.1, Proposition 3.2] and it belongs to $O^+(H^2(X,\mathbb{Z}))$ because $\gamma = -R_\nu$ with $(\nu,\nu) > 0$ (see [69, §9]). This implies that γ is a monodromy operator by Proposition 2.3.8. Let ω_X be the everywhere non-degenerate closed two-form on X which generates $H^{2,0}(X)$. After extending γ to $H^2(X,\mathbb{C})$ by $\mathbb{C}$-linearity, $\gamma(\omega_X) = -\omega_X$, since ω_X belongs to $\mathrm{NS}(X)^\perp$, thus γ is an isomorphism of integral Hodge structures. Moreover, γ preserves the Kähler cone of X, because it fixes the ample class ν: by Theorem 2.3.11 we can conclude that there exists an automorphism $f \in \mathrm{Aut}(X)$ whose action on $H^2(X,\mathbb{Z})$ is γ. In particular, f is a non-symplectic involution, since the map $a : \mathrm{Aut}(X) \to O(H^2(X,\mathbb{Z}))$ is injective and $\gamma(\omega_X) = -\omega_X$. □

REMARK 3.4.4. If Σ is a $K3$ surface with Picard number one and $X = \Sigma^{[n]}$, the non-symplectic involution $f \in \mathrm{Aut}(X)$ constructed in the proof of Proposition 3.4.3 is non-natural, since its action on $\mathrm{NS}(\Sigma^{[n]})$ is non-trivial. As a consequence, using also Proposition 3.4.1, the existence of a primitive ample class with square 2, or with square $2(n-1)$ and divisibility $n-1$, is equivalent to the existence of a non-symplectic, non-natural involution.

3.5. Numerical conditions

In this last section of the chapter we apply the divisorial results of Section 3.4 to provide a completely numerical characterization for the existence of a non-trivial automorphism on $\Sigma^{[n]}$.

PROPOSITION 3.5.1. *Let Σ be an algebraic $K3$ surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, $H^2 = 2t$, $t \geq 2$. If $\mathrm{Aut}(\Sigma^{[n]}) \neq \{\mathrm{id}\}$, then $t(n-1)$ is not a square and, if $n \neq 2$, the equation $(n-1)X^2 - tY^2 = 1$ has no integer solutions. The minimal solution (z,w) of Pell's equation $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$ satisfies $w \equiv 0 \pmod 2$ and $z \equiv \pm 1 \pmod{2(n-1)}$. Moreover,*

$$\mathrm{Nef}(\Sigma^{[n]}) = \overline{\mathrm{Mov}(\Sigma^{[n]})} = \langle h, zh - tw\delta \rangle.$$

The automorphism group of $\Sigma^{[n]}$ is generated by a non-natural, non-symplectic involution whose action on $\mathrm{NS}(\Sigma^{[n]}) = \mathbb{Z}h \oplus \mathbb{Z}(-\delta)$ is given by the matrix

$$\begin{pmatrix} z & -(n-1)w \\ tw & -z \end{pmatrix}.$$

PROOF. If the automorphism group is non-trivial, by Proposition 3.3.1 it is generated by a non-natural, non-symplectic involution whose action on $\mathrm{NS}(\Sigma^{[n]})$ is of the form (12). In particular, the pair (A,β) is a solution of Pell's equation $X^2 - t(n-1)Y^2 = 1$ with $\beta \neq 0$: as a consequence, $t(n-1)$ cannot be a square (as we remarked in Section 3.1.1). Moreover, since the action of the involution on $\mathrm{NS}(\Sigma^{[n]})$ is non-trivial, it needs to exchange the two extremal rays of $\mathrm{Mov}(\Sigma^{[n]})$, which therefore need to be of the same type with respect to the classification of walls of Theorem 2.4.13. The extremal ray generated by h corresponds to a wall $\theta(v_n^\perp \cap a^\perp)$ with $a \in H^*_{\mathrm{alg}}(\Sigma,\mathbb{Z})$ isotropic such that $(v_n, a) = 1$: in [7, Theorem 5.7] it is referred to as a wall of Hilbert–Chow type, since the corresponding divisorial contraction is the Hilbert–Chow morphism $\Sigma^{[n]} \to \Sigma^{(n)}$. Thus, the second wall of $\mathrm{Mov}(\Sigma^{[n]})$ needs to be of this type too. This happens if and only if $\mathrm{Mov}(\Sigma^{[n]})$ is as in case (iii) of Theorem 3.1.6, where in particular (by Lemma 3.1.5) we need to ask that the minimal solution (z,w) of $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$ is such that w is even and $z \equiv \pm 1 \pmod{2(n-1)}$. If so, $\mathrm{Mov}(\Sigma^{[n]})$ is the interior of the cone spanned by h and $zh - tw\delta$ (Theorem 3.1.6).

We know that $\mathcal{A}_{\Sigma^{[n]}} = \{xh - y\delta \mid y > 0, Ay < -t\beta x\}$, with $\beta < 0$ even and $A > 0$, therefore the two extremal rays of the nef cone are generated by h and

$Ah - t(-\beta)\delta$. Moreover, either $A+1$ or $A-1$ is divisible by $2(n-1)$ by Remark 3.4.2, meaning $A \equiv \pm 1 \pmod{(n-1)}$. Since $\mathrm{Nef}(\Sigma^{[n]}) \subset \overline{\mathrm{Mov}(\Sigma^{[n]})}$, the minimality of the slope $\frac{w}{z}$ implies $A = z, \beta = -w$, thus $\mathrm{Nef}(\Sigma^{[n]}) = \overline{\mathrm{Mov}(\Sigma^{[n]})}$. □

REMARK 3.5.2. If $n = 2$, the equality $\mathrm{Nef}(\Sigma^{[n]}) = \overline{\mathrm{Mov}(\Sigma^{[n]})}$ when the automorphism group is non-trivial can be easily deduced from Theorem 3.0.1 and [7, Lemma 13.3]. In general, for any n, the equality also follows from the fact that the non-trivial action on $\mathrm{NS}(\Sigma^{[n]})$ of a biregular involution needs to exchange the two extremal rays of both $\mathrm{Mov}(\Sigma^{[n]})$ and $\mathrm{Nef}(\Sigma^{[n]})$ (see Proposition 2.4.10). Since the two cones share one of the extremal rays (the wall spanned by h), they share the other one too.

In order to convert the divisorial results of Section 3.4 into purely numerical conditions, we will need the following lemma.

LEMMA 3.5.3. *Let Σ be an algebraic $K3$ surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, $H^2 = 2t$, $t \geq 1$. A primitive element $\nu \in \mathrm{NS}(\Sigma^{[n]})$ has $(\nu,\nu) = 2(n-1)$ and $\mathrm{div}(\nu) = n-1$ if and only if it is of the form $\nu = (n-1)Yh - X\delta$, for a solution (X,Y) of Pell's equation $X^2 - t(n-1)Y^2 = -1$.*

PROOF. Assume that there exists $\nu = yh - x\delta \in \mathrm{NS}(\Sigma^{[n]})$ as in the statement: since $(\nu,\nu) = -2((n-1)x^2 - ty^2) = 2(n-1)$, we deduce that (x,y) is an integer solution of $(n-1)X^2 - tY^2 = -(n-1)$. The canonical embedding of $\mathrm{NS}(\Sigma^{[n]})$ inside $H^2(\Sigma^{[n]},\mathbb{Z}) \cong U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$ (see the proof of Lemma 3.3.3) maps ν to $y(e_1 + te_2) + xg$, where $\{e_1, e_2\}$ is a basis for one of the summands U and g generates $\langle -2(n-1)\rangle$. Notice that, with respect to the bilinear form on $H^2(\Sigma^{[n]},\mathbb{Z})$, we have $(\nu, e_2) = y$, therefore y is a multiple of $\mathrm{div}(\nu) = n-1$. We conclude that $(x, \frac{y}{n-1})$ is an integer solution of $X^2 - t(n-1)Y^2 = -1$.

Conversely, assume that $X^2 - t(n-1)Y^2 = -1$ admits integer solutions and let (X,Y) be one of them. We set $\nu := (n-1)Yh - X\delta \in \mathrm{NS}(\Sigma^{[n]})$: it is a primitive class of square $2(n-1)$. Via the usual embedding $\mathrm{NS}(\Sigma^{[n]}) \hookrightarrow H^2(\Sigma^{[n]},\mathbb{Z})$, which maps ν to $(n-1)Y(e_1 + te_2) + Xg$, we can easily compute the ideal

$$\left\{(\nu, m) \mid m \in H^2(\Sigma^{[n]},\mathbb{Z})\right\} = \{(n-1)(Yp - 2Xq) \mid p, q \in \mathbb{Z}\} \subset \mathbb{Z}.$$

From this description it is clear that $(n-1) \mid \mathrm{div}(\nu)$. Moreover, since $\mathrm{div}(\nu) \mid (\nu,\nu)$, we conclude that $\mathrm{div}(\nu) = n-1$ if Y is odd, while $\mathrm{div}(\nu) = 2(n-1)$ if Y is even. However, there are no solutions (X,Y) of $X^2 - t(n-1)Y^2 = -1$ with Y even, since -1 is not a quadratic residue modulo 4, thus $\mathrm{div}(\nu) = n-1$. □

We can now state and prove our main result.

THEOREM 3.5.4. *Let Σ be an algebraic $K3$ surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, $H^2 = 2t$, $t \geq 2$ and $n \geq 2$. Let (z,w) be the minimal solution of Pell's equation $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$. The Hilbert scheme $\Sigma^{[n]}$ admits a (non-symplectic, non-natural) involution if and only if*

(*i*) *$t(n-1)$ is not a square;*
(*ii*) *if $n \neq 2$, the equation $(n-1)X^2 - tY^2 = 1$ has no integer solutions;*
(*iii*) *for all integers ρ, α as follows:*
   (a) *$\rho = -1$ and $1 \leq \alpha \leq n-1$, or*
   (b) *$\rho = 0$ and $3 \leq \alpha \leq n-1$, or*
   (c) *$1 \leq \rho < \frac{n-1}{4}$ and $\max\left\{4\rho + 1, \left\lceil 2\sqrt{\rho(n-1)}\right\rceil\right\} \leq \alpha \leq n-1$*

   if Pell's equation

$$X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$$

is solvable, the minimal solution (X,Y) *with* $X \equiv \pm\alpha \pmod{2(n-1)}$ *satisfies* $\frac{Y}{X} \geq \frac{w}{2z}$*;*

(*iv*) *there exist integer solutions either for the equation* $(n-1)X^2 - tY^2 = -1$ *or for the equation* $X^2 - t(n-1)Y^2 = -1$.

PROOF. By Proposition 3.5.1, if there exists a non-trivial automorphism on $\Sigma^{[n]}$ then $t(n-1)$ is not a square and, if $n \geq 3$, the equation $(n-1)X^2 - tY^2 = 1$ has no integer solutions. Moreover, $\mathrm{Nef}(\Sigma^{[n]}) = \overline{\mathrm{Mov}(\Sigma^{[n]})} = \langle h, zh - tw\delta\rangle$. The equality $\mathrm{Nef}(\Sigma^{[n]}) = \overline{\mathrm{Mov}(\Sigma^{[n]})}$ implies that there are no flopping walls inside $\mathrm{Mov}(\Sigma^{[n]})$. In Section 3.1.2 we recalled the description of the elements $a \in H^*_{\mathrm{alg}}(\Sigma, \mathbb{Z})$ such that $\theta(v_n^\perp \cap a^\perp)$ is a flopping wall. By Lemma 3.1.5, the existence of a similar algebraic Mukai vector a corresponds to the existence of an integer solution (X,Y) with $X \equiv \pm\alpha \pmod{2(n-1)}$ to one of Pell's equations $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$, where the possible values of $a^2 = 2\rho$ and $(v_n, a) = \alpha$ are listed in Remark 2.4.15. In particular, a solution (X,Y) for one of these equations yields the wall generated by $Xh - 2tY\delta$: this wall lies in the movable cone if and only if $X, Y > 0$ and $\frac{Y}{X} < \frac{w}{2z}$. By Remark 3.1.2, we can restrict to consider the equations where the RHS term $\alpha^2 - 4\rho(n-1)$ is positive. In fact, since w is even by Proposition 3.5.1, the pair $(z, \frac{w}{2})$ is the minimal solution of $X^2 - 4t(n-1)Y^2 = 1$, therefore

$$\frac{w}{2z} = \sqrt{\frac{1}{4t(n-1)} - \frac{1}{4t(n-1)z^2}}$$

is strictly inferior to $\frac{Y}{X} = \sqrt{\frac{1}{4t(n-1)} - \frac{m}{4t(n-1)X^2}}$ for all solutions (X,Y) of Pell's equation $X^2 - 4t(n-1)Y^2 = m$, if $m \leq 0$. We also notice that, for each equation with $\alpha^2 - 4\rho(n-1) > 0$, it is sufficient to check whether the wall defined by the minimal solution (with $X \equiv \pm\alpha$ modulo $2(n-1)$) lies inside the movable cone, since the other walls corresponding to positive solutions of the same equation will all have greater slopes (Remark 3.1.2).

Finally, if $\Sigma^{[n]}$ admits an involution then, by Proposition 3.4.1, there exists a primitive element $\nu \in \mathrm{NS}(\Sigma^{[n]})$ with either $(\nu,\nu) = 2$ or $(\nu,\nu) = 2(n-1)$ and $\mathrm{div}(\nu) = n-1$. If we write $\nu = bh - a\delta$, we have $(\nu,\nu) = -2((n-1)a^2 - tb^2)$.

- If $(\nu,\nu) = 2$, then (a,b) is an integer solution of $(n-1)X^2 - tY^2 = -1$.
- If $(\nu,\nu) = 2(n-1)$ and $\mathrm{div}(\nu) = n-1$, then by Lemma 3.5.3 $(a, \frac{b}{n-1})$ is an integer solution of $X^2 - t(n-1)Y^2 = -1$.

We now want to show that the numerical conditions in the statement are sufficient to prove the existence of a non-trivial automorphism on $\Sigma^{[n]}$. By Theorem 3.1.6, from hypotheses (*i*) and (*ii*) we deduce that the closure of the movable cone of $\Sigma^{[n]}$ is $\overline{\mathrm{Mov}(\Sigma^{[n]})} = \langle h, zh - tw\delta\rangle$, with (z,w) as in the statement. Moreover, as we explained in the first part of the proof, hypothesis (*iii*) guarantees that all classes in the movable cone are ample, i.e.

$$(\star) \qquad \mathcal{A}_{\Sigma^{[n]}} = \mathrm{Mov}(\Sigma^{[n]}) = \{xh - y\delta \mid y > 0, zy < twx\}.$$

- If $(n-1)X^2 - tY^2 = -1$ admits integer solutions, let (a,b) be the positive solution with minimal X. By Lemma 3.1.3 the minimal solution of Pell's equation $X^2 - t(n-1)Y^2 = 1$ is $(Z,W) = \big(2(n-1)a^2 + 1, 2ab\big)$. Notice, in particular, that $Z \equiv 1 \pmod{n-1}$, therefore $(z,w) = (Z,W)$. Moreover, $w = 2ab$ is even and $z \equiv 1 \pmod{2(n-1)}$: this implies, as explained in the proof of Proposition 3.5.1, that both the extremal rays of the movable cone correspond to divisorial contractions of Hilbert–Chow type. We now set $\nu := bh - a\delta \in \mathrm{NS}(\Sigma^{[n]})$, which is a primitive class of square 2. In particular ν is ample, using ($\star$), because $a > 0$ and

$$za - twb = a(2(n-1)a^2 + 1 - 2tb^2) = -a < 0.$$

- If instead there are integer solutions for $X^2 - t(n-1)Y^2 = -1$, let (a, b) be again the minimal one. By Lemma 3.1.3 the minimal solution of Pell's equation $X^2 - t(n-1)Y^2 = 1$ is $(Z, W) = (2a^2 + 1, 2ab)$. Here $Z \equiv -1 \pmod{n-1}$, since $a^2 = t(n-1)b^2 - 1$, therefore $(z, w) = (Z, W)$ with w even and $z \equiv -1 \pmod{2(n-1)}$. Let $\nu := (n-1)bh - a\delta \in \mathrm{NS}(\Sigma^{[n]})$: by Lemma 3.5.3 it is a primitive class of square $2(n-1)$ and divisibility $n-1$. Moreover ν is ample, using the description ($\star$), because $a > 0$ and

$$za - tw(n-1)b = a(2a^2 + 1 - 2t(n-1)b^2) = -a < 0.$$

Therefore, if one of the two equations in hypothesis (iv) admits integer solutions, we can construct a primitive ample class ν with either $(\nu, \nu) = 2$ or $(\nu, \nu) = 2(n-1)$ and $\mathrm{div}(\nu) = n-1$. We conclude, by Proposition 3.4.3 and Remark 3.4.4, that the Hilbert scheme $\Sigma^{[n]}$ admits a non-symplectic, non-natural involution, which acts on $H^2(\Sigma^{[n]}, \mathbb{Z})$ as $-R_\nu$. $\square$

REMARK 3.5.5. In the proof of Theorem 3.5.4 we showed that, if condition (iv) holds, then the solution (z, w) of $X^2 - t(n-1)Y^2 = 1$ appearing in the statement is the minimal solution of the equation and $z \equiv \pm 1 \pmod{2(n-1)}$. Using formula (8) we deduce that all solutions of Pell's equation $X^2 - 4t(n-1)Y^2 = 1$ have $X \equiv \pm 1 \pmod{2(n-1)}$, since $(z, \frac{w}{2})$ is the minimal solution. As a consequence, applying again (8), the congruence class modulo $2(n-1)$ of X is constant (up to sign) for all solutions (X, Y) of $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ in the same equivalence class (see Section 3.1.1). Therefore, if there exist solutions (X, Y) of $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ with $X \equiv \pm\alpha \pmod{2(n-1)}$, there is also a fundamental solution with this property. Moreover, by Remark 3.1.4, the minimal solution (X, Y) of this type has slope $\frac{Y}{X}$ strictly inferior to $\frac{w}{2z}$. In order to verify if hypothesis (iii) of Theorem 3.5.4 holds, it is therefore sufficient to check that $4t(n-1)Y^2 + \alpha^2 - 4\rho(n-1)$ is not the square of an integer $X \equiv \pm\alpha \pmod{2(n-1)}$ for all integers Y such that $1 \le Y \le \sqrt{\frac{(z-1)(\alpha^2 - 4\rho(n-1))}{8t(n-1)}}$ and ρ, α as in the statement of the theorem.

REMARK 3.5.6. For $n = 2$, Theorem 3.5.4 coincides with part (i) of Theorem 3.0.1. In fact, the only equation that needs to be considered in condition (iii) of Theorem 3.5.4 is $X^2 - 4tY^2 = 5$, corresponding to $\rho = -1$, $\alpha = 1$. If there exist integer solutions (X, Y) for this equation, they all have X odd, i.e. $X \equiv \pm\alpha \pmod{2(n-1)}$, and the minimal solution satisfies $\frac{Y}{X} < \frac{w}{2z}$ (Remark 3.1.4). Condition (iii) of Theorem 3.5.4 is therefore equivalent to asking that Pell's equation $X^2 - 4tY^2 = 5$ has no solutions, as requested in Theorem 3.0.1.

As an application of Theorem 3.5.4, we prove that for any $n \ge 2$ there exist infinitely many values of t such that $\Sigma^{[n]}$ admits a non-symplectic involution f with $T_f \cong \langle 2 \rangle$. In order to do so, we consider a specific sequence of integers $\{t_{n,k}\}_{k \in \mathbb{N}}$ and we show that all these $t_{n,k}$'s are admissible if k is sufficiently large.

PROPOSITION 3.5.7. *Let Σ be an algebraic K3 surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, $H^2 = 2t$ and assume $t = (n-1)k^2 + 1$ for k, n positive integers, $n \ge 2$. If $k \ge \frac{n+3}{2}$, there exists a non-symplectic involution $f \in \mathrm{Aut}(\Sigma^{[n]})$ with $T_f \cong \langle 2 \rangle$.*

PROOF. We need to verify that the four conditions of Theorem 3.5.4 are satisfied. If $t = (n-1)k^2 + 1$, for $k \in \mathbb{N}$, it is easy to check that $t(n-1)$ is not a square and that $(X, Y) = (k, 1)$ is a solution of Pell's equation $(n-1)X^2 - tY^2 = -1$.

Moreover, if $n \geq 3$ the equation $(n-1)X^2 - tY^2 = 1$ has no integer solutions. In fact, if the equation was solvable we would be able to find solutions of Pell's equation $X^2 - t(n-1)Y^2 = n-1$ with $X \equiv 0 \pmod{n-1}$. Since $t(n-1) > (n-1)^2$, the primitive solutions of this new equation are among the convergents of the continued fraction expansion of $\sqrt{t(n-1)}$ (see for instance [33, Chapter XXXIII, §16]), which is

$$\sqrt{t(n-1)} = \left[k(n-1); \overline{2k, 2k(n-1)}\right].$$

The evaluation of $X^2 - t(n-1)Y^2$ on the convergents (X, Y) of $\sqrt{t(n-1)}$ is periodic with the same period of the continued fraction, i.e. two, and it has alternately values $-n+1$ and 1. As a consequence, $X^2 - t(n-1)Y^2 = n-1$ has solutions only if $n-1 = h^2$, with $h \in \mathbb{N}$, and in this case the only fundamental solution is $(h, 0)$. Knowing the convergents of $\sqrt{t(n-1)}$ we can determine the minimal solution of $X^2 - t(n-1)Y^2 = 1$, which is $(z, w) = (2k^2(n-1)+1, 2k)$. Since $z \equiv 1 \pmod{n-1}$, we conclude (using formula (8)) that all solutions (X, Y) of $X^2 - t(n-1)Y^2 = n-1$ have $X \equiv h \pmod{n-1}$: in particular, this congruence class is never zero, because $h^2 = n-1$ with $n \geq 3$.

It remains to check whether condition (iii) of Theorem 3.5.4 holds. Assuming $k \geq \frac{n+3}{2}$, we have $4t(n-1) > (\alpha^2 - 4\rho(n-1))^2$ for all α, ρ as in the statement of the theorem (notice that $\alpha^2 - 4\rho(n-1)$ can be at most $(n-1)^2 + 4(n-1)$). As before, from this condition we deduce that the solutions of $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ are encoded in the convergents of the continued fraction of

$$\sqrt{4t(n-1)} = \left[2k(n-1); \overline{k, 4k(n-1)}\right].$$

The quadratic form $X^2 - 4t(n-1)Y^2$ takes values $-4(n-1)$ and 1, respectively, on the first two convergents. As a consequence, if Pell's equation $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ is solvable, then $\alpha^2 - 4\rho(n-1) = h^2$ for some $h \in \mathbb{N}$ and the minimal solution is $(X, Y) = (hz, h\frac{w}{2})$, with $(z, \frac{w}{2})$ the minimal solution of Pell's equation $X^2 - 4t(n-1)Y^2 = 1$. Since $\frac{Y}{X} = \frac{w}{2z}$, if the solution (X, Y) defines a wall in the positive cone of $\Sigma^{[n]}$, this wall does not intersect the interior of the movable cone. Hence, condition (iii) is satisfied. □

To conclude, we provide a list of the first values of t such that there exists a non-natural automorphism $f \in \mathrm{Aut}(\Sigma^{[n]})$, for $2 \leq n \leq 12$ (more details on the case $n = 2$ can be found in [20]). By Corollary 3.3.5 we have $t \geq 2n-2$. In particular, if -1 is a quadratic residue modulo $n-1$ we know, by Proposition 3.4.1, that the generator of the invariant lattice T_f can either have square 2 or $2(n-1)$: for these values of n we determine the smallest t in each of the two cases.

n	first t s.t. $T_f \cong \langle 2 \rangle$	first t s.t. $T_f \cong \langle 2(n-1) \rangle$
2	2	
3	19	13
4	19	/
5	37	/
6	46	34
7	55	/
8	64	/
9	73	/
10	82	/
11	91	73
12	100	/

For $n = 3$, in [46, Example 14] Hassett and Tschinkel proved the existence of a non-natural automorphism $f \in \mathrm{Aut}(\Sigma^{[3]})$ with $T_f \cong \langle 2\rangle$ when Σ is a $K3$ surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}H$, $H^2 = 114$ (i.e. $t = 57$). However, as shown in the table above, this is not the smallest t for which a similar automorphism exists on $\Sigma^{[3]}$.

We notice an interesting pattern in the second column of the table: except for $n = 2$ and $n = 4$, the first value of t such that there exists an automorphism $f \in \mathrm{Aut}(\Sigma^{[n]})$ with invariant lattice $T_f \cong \langle 2\rangle$ is always of the form $t = 9n - 8$. In particular, this is one of the values of t considered in Proposition 3.5.7, corresponding to $k = 3$. We conjecture that this value of t is admissible for all $n \geq 3$. If so, the lower bound on k provided in Proposition 3.5.7 for the existence of the non-natural involution might be significantly improved. By Theorem 3.5.4, this is equivalent to proving the following number theoretical conjecture.

Conjecture 3.5.8. For all integers $n \geq 3$:

- the equation $(n-1)X^2 - (9n-8)Y^2 = 1$ has no integer solutions;
- Pell's equation $X^2 - 4(9n-8)(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ has no positive solutions with $X \equiv \pm\alpha \pmod{2(n-1)}$ and $\frac{Y}{X} < \frac{3}{18n-17}$, for all integers ρ, α as follows:
  (1) $\rho = -1$ and $1 \leq \alpha \leq n-1$, or
  (2) $\rho = 0$ and $3 \leq \alpha \leq n-1$, or
  (3) $1 \leq \rho < \frac{n-1}{4}$ and $\max\left\{4\rho + 1, \left\lceil 2\sqrt{\rho(n-1)}\right\rceil\right\} \leq \alpha \leq n-1$.

CHAPTER 4

Non-symplectic automorphisms of odd prime order on manifolds of $K3^{[n]}$-type

The aim of this chapter is to classify non-symplectic automorphisms of odd prime order on irreducible holomorphic symplectic manifolds which are deformations of Hilbert schemes of any number n of points on $K3$ surfaces, extending results already known for $n = 2$ (see [**17**]). In order to do so, we study the properties of the invariant lattice of the automorphism (and its orthogonal complement) inside the second cohomology lattice of the manifold. We also explain how to construct automorphisms with specific actions on cohomology and, for $n = 4$, we present a geometric construction of non-symplectic automorphisms on the Lehn–Lehn–Sorger–van Straten eightfold, which come from automorphisms of the underlying cubic fourfold. These results, which appear in the paper [**29**], are the product of a collaboration with Chiara Camere.

4.1. Monodromies induced by automorphisms of odd prime order

In this section we explain how to classify, for any n, the isometry classes of the invariant and co-invariant lattices by use of numerical parameters related to their signatures and lengths. This classification is explicitly discussed for $n = 3, 4$ in Section 4.1.4. Moreover, in Section 4.1.3 we study in greater depth the cases where the invariant lattice has rank one.

4.1.1. Discriminant groups of invariant and co-invariant sublattices. Let X be a manifold of $K3^{[n]}$-type with an action of a finite group $G = \langle\sigma\rangle$, where σ is a non-symplectic automorphism of prime order $p \geq 3$. In particular, p can be at most 23, since this is the rank of the lattice $H^2(X, \mathbb{Z})$ (see Example 2.2.4). Following the notation of Section 2.5, we will denote by $T = T_\sigma = H^2(X, \mathbb{Z})^{\sigma^*}$ the invariant sublattice of $H^2(X, \mathbb{Z})$ and by $S = S_\sigma = T^\perp \subset H^2(X, \mathbb{Z})$ the co-invariant lattice.

Remark 4.1.1. After choosing a marking

$$\eta : H^2(X, \mathbb{Z}) \to L_n = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$$

the invariant and co-invariant lattices of an automorphism of X can also be regarded as primitive sublattices $T, S \subset L_n$. We point out that a different marking η' will produce a pair of sublattices (T', S') of L_n which is isomorphic to (T, S) in the sense of Definition 1.4.3. For this reason, we are interested in classifying the pairs (T, S) only up to isomorphisms of primitive sublattices in L_n.

We collect in the next proposition several results proved by Boissière–Nieper-Wißkirchen–Sarti [**21**] and Tari [**94**] (recall also Proposition 2.5.5).

Proposition 4.1.2. *Let X be a manifold of $K3^{[n]}$-type and $G = \langle\sigma\rangle$ a group of prime order p acting non-symplectically on X. Then:*

- *there exists a positive integer $m := m_G(X)$ such that $\mathrm{rk}(S) = (p-1)m$;*
- *S has signature $(2, (p-1)m - 2)$ and T has signature $(1, 22 - (p-1)m)$;*

- $\frac{H^2(X,\mathbb{Z})}{T\oplus S}$ *is a* p*-torsion group, i.e.* $\frac{H^2(X,\mathbb{Z})}{T\oplus S} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$ *for some non-negative integer* $a := a_G(X)$*;*
- $a \leq m$.

As we explained in Section 1.4, if we consider T and S as primitive sublattices of L_n, there exists an isotropic subgroup $M \subset A_T \oplus A_S$ such that $M^\perp/M \cong A_{L_n}$. Denoting by p_T and p_S the two projections from $A_T \oplus A_S$ to A_T and A_S respectively, their restrictions to M are injective: the isomorphic images are $M_T := p_T(M) \subset A_T$ and $M_S := p_S(M) \subset A_S$. Since the discriminant groups are finite, we conclude $p^a \mid \mathrm{discr}(T)$, $p^a \mid \mathrm{discr}(S)$. Moreover, $q_T|_{M_T} \cong -q_S|_{M_S}$, because of the anti-isometry $\gamma = p_S \circ (p_T)^{-1}|_{M_T} : M_T \to M_S$.

LEMMA 4.1.3. *Let* σ *be a non-symplectic automorphism of prime order* $p \geq 3$ *of a manifold of* $K3^{[n]}$*-type and let* $\psi = \sigma^* \in \mathrm{Mon}^2(L_n)$*. Then:*

(i) the action of ψ *on* $M^\perp \subset A_T \oplus A_S$ *is trivial;*
(ii) the co-invariant lattice $S = \left((L_n)^\psi\right)^\perp$ *is* p*-elementary.*

PROOF.

(i) The isometry ψ acts trivially on $A_{L_n} \cong M^\perp/M$, from Proposition 2.3.8 and because $\psi^p = \mathrm{id}$ with p odd (therefore ψ cannot induce $-\mathrm{id}$ on A_{L_n}). This implies that, for any element $(x,y) \in M^\perp \subset A_T \oplus A_S$, we have $\overline{\psi}(x,y) - (x,y) \in M$, where $\overline{\psi} \in O(A_{L_n})$ is the isometry induced by ψ. Moreover, ψ acts trivially on the discriminant group A_T (because $\psi|_T = \mathrm{id}_T$), thus $\overline{\psi}(x,y) - (x,y) = (0, \overline{\psi}(y) - y)$ (the natural inclusions of A_T and A_S in $A_{T\oplus S} \cong A_T \oplus A_S$ are ψ-equivariant). Since M is the graph in $A_T \oplus A_S$ of the anti-isometry $\gamma : M_T \to M_S$, we deduce that $\overline{\psi}(y) = y$ for any $y \in p_S(M^\perp)$. This means that the action of ψ is trivial on $M^\perp$, not only on the quotient $M^\perp/M$.

(ii) For any $n \geq 2$, the lattice L_n admits a natural primitive embedding inside the Mukai lattice $\Lambda_{24} = U^{\oplus 4} \oplus E_8^{\oplus 2}$ (see [69, Corollary 9.5]). As we remarked in the previous point of the proof, the action of ψ on the discriminant group A_{L_n} is trivial: this allows us to extend ψ to an isometry $\rho \in O(\Lambda_{24})$ such that $\rho|_{L_n{}^\perp} = \mathrm{id}$, by Proposition 1.4.7. The lattice Λ_{24} is unimodular, therefore both the invariant lattice $T_\rho = (\Lambda_{24})^\rho \subset \Lambda_{24}$ and the co-invariant $S_\rho = (T_\rho)^\perp \subset \Lambda_{24}$ are p-elementary (see for instance [94, Lemme 2.10]). Since $L_n{}^\perp \subset T_\rho$, passing to the orthogonal complements we have $S_\rho \subset L_n$, and therefore $S = S_\rho$ is p-elementary. □

For fixed values of $n \geq 2$ and $p \geq 3$ prime, we write $2(n-1) = p^\alpha\beta$ with α, β integers, $\alpha \geq 0$ and $(p,\beta) = 1$. Then $A_{L_n} \cong \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}} \cong \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$ is an orthogonal splitting (see Proposition 1.1.3): in particular, we now show that there exists a subgroup of A_T isomorphic to the summand $\frac{\mathbb{Z}}{\beta\mathbb{Z}}$.

LEMMA 4.1.4. *Let* $(A_T)_p$ *and* $(A_S)_p$ *be the Sylow* p*-subgroups of* A_T *and* A_S *respectively. Then*

$$A_T = (A_T)_p \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}, \qquad A_S = (A_S)_p.$$

Moreover, $|A_T| = p^a\beta t$ *and* $|A_S| = p^a s$ *for some positive integers* t, s *such that* $ts = p^\alpha$*.*

PROOF. Since $A_{L_n} \cong M^\perp/M$ and $|M| = p^a$, we deduce $\left|M^\perp\right| = p^{a+\alpha}\beta$. Moreover, $(p,\beta) = 1$, thus there exists a unique subgroup $N \subset M^\perp$ of order β, such that the restriction to N of the projection $M^\perp \to M^\perp/M$ is injective. Using the fact that there is also a unique subgroup of order β inside A_{L_n}, we conclude that

N is isomorphic to the component $\frac{\mathbb{Z}}{\beta\mathbb{Z}}$ of A_{L_n}. By Lemma 4.1.3, the action of the automorphism σ on $N \subset M^\perp$ is trivial and any element of $p_S(N)$ is of p-torsion: we are lead to conclude $p_S(N) = 0$, because $(p, \beta) = 1$. Thus, N is contained in A_T.

Since $M_T \subset (A_T)_p$, $M_S \subset (A_S)_p$ we can write $|A_T| = p^a\beta t$ and $|A_S| = p^a s$, with t, s positive integers. From

$$[L_n : (T \oplus S)]^2 = \frac{|A_T|\,|A_S|}{|A_{L_n}|}$$

(see Section 1.4) we get $ts = p^\alpha$. The two integers t, s are therefore powers of p with non-negative exponents. $\square$

We are now ready to describe the structures of the two discriminant groups A_T and A_S.

Proposition 4.1.5. *Let X be a manifold of $K3^{[n]}$-type and $G = \langle\sigma\rangle$ a group of odd prime order p acting non-symplectically on X. Then one of the following cases holds:*

(*i*) $A_S = M_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$, $A_T \cong M_T \oplus A_{L_n} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$;

(*ii*) $\alpha = 1$, $a = 0$, $A_S \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$, $A_T \cong \frac{\mathbb{Z}}{\beta\mathbb{Z}}$;

(*iii*) $\alpha \geq 1$, $a \geq 1$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$, $A_T \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$.

Proof. If $a = 0$, the group M is trivial and $A_{L_n} \cong A_T \oplus A_S$. By Lemma 4.1.4 we deduce that there are only two possibilities: $A_S = 0$, $A_T \cong A_{L_n}$ or $A_S \cong \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$, $A_T \cong \frac{\mathbb{Z}}{\beta\mathbb{Z}}$. The second case, though, is admissible only for $\alpha = 1$, because we know that S is p-elementary by Lemma 4.1.3.

From now on we will assume $a \geq 1$. Let us first consider the case $\alpha = 0$: this implies $\beta = 2(n-1)$, $t = s = 1$. Then, by using Lemma 4.1.4 we conclude $A_S = M_S$ and $A_T = M_T \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}} \cong M_T \oplus A_{L_n}$.

If $\alpha = 1$ we have $2(n-1) = p\beta$ and $ts = p$. There are two possibilities:

- $t = p, s = 1$. In this case, $A_S = M_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$, therefore $q_S|_{M_S} = q_S$ is non-degenerate and the same holds for $q_T|_{M_T}$, since $q_T|_{M_T} \cong -q_S|_{M_S}$. Then, by Proposition 1.1.3 we can write $A_T = M_T \oplus M_T^\perp$, which implies $(A_T)_p \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$ and therefore $A_T \cong M_T \oplus A_{L_n}$.
- $t = 1, s = p$. Now $A_T = M_T \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$, since $(A_T)_p = M_T$. Hence $q_T|_{M_T}$ is non-degenerate, which again implies that also $q_S|_{M_S}$ is non-degenerate, i.e. $A_S = M_S \oplus M_S^\perp$. We are lead to conclude $A_S = M_S \oplus \frac{\mathbb{Z}}{p\mathbb{Z}} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$.

Therefore, if $\alpha = 1$ (and $a \geq 1$) both cases (i), (iii) appearing in the statement are admissible.

Now assume $\alpha \geq 2$. Set $H := (A_T)_p \oplus A_S \subset A_T \oplus A_S$ and let $H[p] \subset H$ be the p-torsion subgroup. Since $M^\perp/M \cong A_{L_n} \cong \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$, there exists an element $x \in H$ of order at least p^α: the quotient $\langle x\rangle/(\langle x\rangle \cap H[p])$ has then order at least $p^{\alpha-1}$, which shows that $[H : H[p]] \geq p^{\alpha-1}$. On the other hand, $[H : H[p]] \leq p^\alpha$: in fact, $|H[p]| \geq p^{2a}$, because $M_T \oplus M_S \subset H[p]$, and $|H| = p^a t \cdot p^a s = p^{2a+\alpha}$ (by Lemma 4.1.4). We conclude that the index $[H : H[p]]$ is either p^α or $p^{\alpha-1}$.

If $[H : H[p]] = p^\alpha$, then $H[p] = M_T \oplus M_S$. By construction $H = H_p$, therefore:

$$H \cong \bigoplus_{i=1}^{2a+\alpha} \left(\frac{\mathbb{Z}}{p^i\mathbb{Z}}\right)^{\oplus m_i}, \qquad H[p] \cong \bigoplus_{i=1}^{2a+\alpha} \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus m_i} \tag{14}$$

for suitable integers $m_i \geq 0$ such that $\sum_i im_i = 2a+\alpha$ and $\sum_i m_i = 2a$. Thus, the coefficients m_i must satisfy $\alpha = \sum_i (i-1)m_i$. Furthermore, since we know that H contains an element of order at least p^α, there exists $j \geq \alpha$ such that $m_j \geq 1$. This leaves us with two possibilities for the choice of the coefficients m_i.

- $H \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus 2a-1} \oplus \frac{\mathbb{Z}}{p^{\alpha+1}\mathbb{Z}}$. Then, we have either $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^{\alpha+1}\mathbb{Z}}$, $(A_T)_p = M_T$ or $A_S = M_S, (A_T)_p \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^{\alpha+1}\mathbb{Z}}$. Both cases are not admissible: by Proposition 1.1.3 (as we remarked discussing $\alpha = 1$) we would need to be able to write, respectively, $A_S = M_S \oplus M_S^\perp$ and $A_T = M_T \oplus M_T^\perp$, but now this is not possible.
- $H \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus 2a-2} \oplus \frac{\mathbb{Z}}{p^2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$. Disregarding the cases where $A_S = M_S$ or $(A_T)_p = M_T$ (which can be excluded as in the previous point) we are left with two alternatives:
  - $(A_T)_p \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^2\mathbb{Z}}$;
  - $(A_T)_p \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^2\mathbb{Z}}$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$.

  In both cases, though, the lattice S is not p-elementary, contradicting Lemma 4.1.3.

We conclude that $[H : H[p]] = p^{\alpha-1}$. We can again write H and $H[p]$ as in (14), where now $\sum_i im_i = 2a+\alpha$, $\sum_i m_i = 2a+1$ and as before there exists $j \geq \alpha$ such that $m_j \geq 1$. We then deduce $H \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus 2a} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$, which gives rise to four possible conclusions:

- $(A_T)_p \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$, meaning $A_T \cong M_T \oplus A_{L_n}$ and $A_S = M_S$;
- $(A_T)_p \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$;
- $(A_T)_p \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$;
- $(A_T)_p \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$.

The last two cases are excluded because S is p-elementary by Lemma 4.1.3. □

REMARK 4.1.6. We can make some additional remarks on the structures of the discriminant groups A_T, A_S after recalling the following result.

THEOREM 4.1.7. *Let M be a lattice and $\psi \in O(M)$ of prime order $p \neq 2$. Then $p^m \operatorname{discr}(S_\psi)$ is a square in $\mathbb{Z}$, where $m = \frac{\operatorname{rk}(S_\psi)}{p-1}$.*

PROOF. See [94, Théorème 2.23]. □

Let X be a manifold of $K3^{[n]}$-type and $\psi \in \operatorname{Mon}^2(L_n)$ the isometry induced on L_n by an automorphism $\sigma \in \operatorname{Aut}(X)$ of prime order $p \geq 3$. From Proposition 4.1.5 we know that $\operatorname{discr}(S) = |A_S|$ is either p^a or p^{a+1}. In particular:

- if $p \nmid 2(n-1)$ (i.e. $\alpha = 0$), the groups A_T, A_S are as in Proposition 4.1.5, case (i), therefore a and m must be of same parity by Theorem 4.1.7.
- If $p \mid 2(n-1)$ (i.e. $\alpha \geq 1$), a and m are not required to have same parity: the structures of A_T and A_S are the ones given in Proposition 4.1.5 case (i) if a and m have same parity, the ones of cases (ii) or (iii) if a and m have different parity.

4.1.2. Admissible triples. We are now interested in studying primitive embeddings of lattices $T, S \hookrightarrow L_n$ satisfying Proposition 4.1.2 and Proposition 4.1.5, assuming $p \geq 3$. For the purposes of this work, we restrict to $\alpha \leq 1$: notice that, since $2(n-1) = p^\alpha\beta$, the first instance with $\alpha \geq 2$ occurs for $n = 10$, i.e. on manifolds of complex dimension 20.

Our main result is Theorem 4.1.12, in which we show that, under suitable hypotheses, given the values (p, m, a) defined in Proposition 4.1.2 the isometry classes of T and S are uniquely determined. To do so we first need to provide a characterization of primitive embeddings $S \hookrightarrow L_n$ for lattices S as above (Lemma 4.1.8 and Proposition 4.1.9). Finally, in Proposition 4.1.14 we describe all possible structures, up to isometries, for the discriminant quadratic forms q_S and q_T.

We recall that, by Proposition 4.1.2, the lattice S has signature $(2, (p-1)m-2)$ and it is p-elementary by Lemma 4.1.3, with discriminant group $A_S = \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k}$, where k is the length of A_S. Then (see Remark 1.2.3) there are only two non-isometric possible forms q_S, the ones in (3).

Since $L_n = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$, the quadratic form q_{L_n} on $A_{L_n} = \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}$ is such that $q_{L_n}(1) = -\frac{1}{2(n-1)} \in \mathbb{Q}/2\mathbb{Z}$. If we write $2(n-1) = p^\alpha\beta$, with $(p, \beta) = 1$, then a trivial computation shows:

$$q_{L_n} = \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}\left(-\frac{1}{2(n-1)}\right) \cong \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}\left(-\frac{\beta}{p^\alpha}\right) \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}\left(-\frac{p^\alpha}{\beta}\right).$$

In Section 1.2 we defined the two non-isomorphic finite quadratic forms $\omega_{p,\alpha}^{\pm 1}$ on $\frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$. Denoting by $q_{\alpha,\beta}$ the quadratic form $\frac{\mathbb{Z}}{\beta\mathbb{Z}}\left(-\frac{p^\alpha}{\beta}\right)$, we conclude:

$$q_{L_n} = \begin{cases} w_{p,\alpha}^{+1} \oplus q_{\alpha,\beta} & \text{if } \left(\frac{-\beta}{p}\right) = +1 \\ w_{p,\alpha}^{-1} \oplus q_{\alpha,\beta} & \text{if } \left(\frac{-\beta}{p}\right) = -1 \end{cases} \tag{15}$$

where the parentheses denote the Legendre symbol.

LEMMA 4.1.8. *Let S be an even lattice with discriminant group $A_S = \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k}$ of genus $(2, (p-1)m-2, q_S)$. Let $L_n = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$ and let $e \in A_{L_n}$ be the generator of the component $\frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$ of $A_{L_n} \cong \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}\left(-\frac{\beta}{p^\alpha}\right) \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}\left(-\frac{p^\alpha}{\beta}\right)$. Then:*

- (i) *If $\alpha = 0$, primitive embeddings of S in L_n compatible with Proposition 4.1.5 are determined by pairs (T, γ_T), with T a lattice of signature $(1, 22-(p-1)m)$, $q_T = (-q_S) \oplus q_{L_n}$ and $\gamma_T \in O(A_T)$. Two pairs (T, γ_T) and $(T', \gamma'_{T'})$ determine isomorphic sublattices in L_n if and only if there exists an isometry $\nu : T \to T'$ such that $\overline{\nu} \circ \gamma_T = \gamma'_{T'} \circ \overline{\nu}$.*
- (ii) *If $\alpha = 1$, primitive embeddings $S \hookrightarrow L_n$ compatible with Proposition 4.1.5 are determined by triples (x, T, γ_T), with T of signature $(1, 22-(p-1)m)$, $\gamma_T \in O(A_T)$ and either:*
  - (a) *$x = 0$, $q_T = (-q_S) \oplus q_{L_n}$, or*
  - (b) *$x \in A_S[p]$ with $q_S(x) = -\frac{\beta}{p}$ (mod $2\mathbb{Z}$) and $\Gamma^\perp/\Gamma \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k-1} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$, where $\Gamma \subset A_S \oplus A_{L_n}$ is the subgroup generated by (x, e) and $\Gamma^\perp$ is its orthogonal complement with respect to the form $(-q_S) \oplus q_{L_n}$; moreover, $q_T = ((-q_S) \oplus q_{L_n})|_{\Gamma^\perp/\Gamma}$.*

  Two triples (x, T, γ_T) and $(x', T', \gamma'_{T'})$ determine isomorphic sublattices in L_n if and only if there exists $\mu \in O(S)$ and an isometry $\nu : T \to T'$, such that $\overline{\mu}(x) = x'$ and $\overline{\nu} \circ \gamma_T = \gamma'_{T'} \circ \overline{\nu}$.

PROOF. Each primitive embedding $i : S \hookrightarrow L_n$ is determined by a quintuple $\Theta_i = (H_S, H_{L_n}, \gamma, T, \gamma_T)$ as in Theorem 1.4.4, since L_n is unique in its genus (up to isometries) by Theorem 1.3.3. Recalling that T is the orthogonal complement of $i(S)$ in L_n, we ask $\mathrm{sign}(T) = (1, 22-(p-1)m)$. We will discuss separately the cases $\alpha = 0$ and $\alpha = 1$.

- (i) $\alpha = 0$. Since p and β are coprime, the only possibility is: $H_S = \{0\}$, $H_{L_n} = \{0\}$ and $\gamma = \mathrm{id}$. The embedding $S \hookrightarrow L_n$ is therefore determined by the pair (T, γ_T). In particular, we have $\Gamma = \{(0,0)\}$, $\Gamma^\perp = A_S \oplus A_{L_n}$, thus $A_T = A_S \oplus A_{L_n}$ and $q_T = (-q_S) \oplus q_{L_n}$. This is coherent with case (i) of Proposition 4.1.5.
- (ii) $\alpha = 1$. We have again the case $H_S = \{0\}, H_{L_n} = \{0\}$, $\gamma = \mathrm{id}$ (which means that S and T are as in case (i) of Proposition 4.1.5, hence $l(A_T) = k+1$). This case corresponds to the triples where $x = 0$ and it is described as for $\alpha = 0$. Alternatively, provided that there exists an element $x \in A_S$ of order p such that $q_S(x) = q_{L_n}(e)$, with e as in the statement, we can also take $H_S = \langle x\rangle, H_{L_n} = \langle e\rangle$, $\gamma : x \mapsto e$. Notice that such an element x does not exist only if $k = 0$ or if $q_S = w^{\xi}_{p,1}, q_{L_n} = w^{-\xi}_{p,1} \oplus q_{1,\beta}$, with $\xi \in \{\pm 1\}$: in all other cases, using the isomorphism $w^{+1}_{p,1} \oplus w^{+1}_{p,1} \cong w^{-1}_{p,1} \oplus w^{-1}_{p,1}$, we can write the form q_S as in (3), where at least one of the direct summands is of the same type of the $w^{\epsilon}_{p,1}$ appearing in q_{L_n} (the component corresponding to the subgroup H_{L_n}). In this setting, the graph Γ of γ is the subgroup of $A_S \oplus A_{L_n}$ generated by (x, e). In particular, since $\Gamma \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$, the quotient $\Gamma^\perp/\Gamma$ cannot be isomorphic to $A_S \oplus A_{L_n}$, which implies that we are not in case (i) of Proposition 4.1.5. Nevertheless, if $\alpha = 1$ and $k \geq 1$ the structures of the discriminant groups can also be as in cases (ii) or (iii), where $l(A_T) = \max\{1, k-1\}$: the embedding is admissible if $\Gamma^\perp/\Gamma \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k-1} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$ and, if so, the quadratic form on A_T is $q_T = ((-q_S) \oplus q_{L_n})|_{\Gamma^\perp/\Gamma}$.

Finally, for both values of α the stated results about isomorphic sublattices follow directly from Theorem 1.4.4. $\square$

Lemma 4.1.8 allows us to list all possible primitive embeddings $i : S \hookrightarrow L_n$ satisfying Proposition 4.1.5 for a given lattice S. We now prove that, adding some extra hypotheses, the number of distinct isometry classes for $i(S)^\perp$ is actually very limited.

PROPOSITION 4.1.9. *Let S and L_n be as in Lemma 4.1.8, with $\alpha \leq 1$ and $k \leq 21 - \alpha - (p-1)m$.*

- (i) *If $\alpha = 0$ or $k = 0$, or if $\alpha = 1$ and $q_S = w^{\xi}_{p,1}, q_{L_n} = w^{-\xi}_{p,1} \oplus q_{1,\beta}$ for $\xi \in \{\pm 1\}$, all primitive embeddings of S in L_n compatible with Proposition 4.1.5 define isomorphic sublattices. In particular, the orthogonal complement T is uniquely determined by the genus of S.*
- (ii) *Otherwise, provided that the natural homomorphism $O(S) \to O(A_S)$ is surjective, there are at most two distinct isomorphism classes for the orthogonal complement T of the image of a compatible embedding $S \hookrightarrow L_n$, one with $l(A_T) = k+1$ and one with $l(A_T) = \max\{1, k-1\}$.*

PROOF. If $\alpha = 0$ or $k = 0$, or if $\alpha = 1$ and $q_S = w^{\xi}_{p,1}, q_{L_n} = w^{-\xi}_{p,1} \oplus q_{1,\beta}$ for $\xi \in \{\pm 1\}$, by Lemma 4.1.8 a (compatible) primitive embedding of S in L_n is characterized by a pair (T, γ_T), with T a lattice of signature $(1, 22-(p-1)m)$, $q_T = (-q_S) \oplus q_{L_n}$ and $\gamma_T \in O(A_T)$. In this case, then, $l(A_T) = \max\{1, k+\alpha\}$, because $(p, \beta) = 1$. If such an indefinite lattice T exists and if $l(A_T) \leq \mathrm{rk}(T) - 2$

(i.e. if $k \leq 21 - \alpha - (p-1)m$), then T is uniquely determined, up to isometries, by Theorem 1.3.3. This assumption also guarantees that the natural morphism $O(T) \to O(A_T)$ is surjective (Proposition 1.4.2), therefore different choices of γ_T give isomorphic primitive sublattices S in L_n.

Assume now that we are not in one of the cases of point (i) (in particular, let $\alpha = 1$ and $k \geq 1$); moreover, suppose that $O(S) \to O(A_S)$ is surjective and $k \leq 20 - (p-1)m$. A compatible embedding $i : S \hookrightarrow L_n$ is determined by a triple (x, T, γ_T) as in Lemma 4.1.8. We make a distinction:

- Triples $(0, T, \gamma_T)$ correspond to embeddings where $q_T = (-q_S) \oplus q_{L_n}$, so $l(A_T) = k+1$. Then, as before, from the assumption $k \leq 20-(p-1)m$ it follows that all these embeddings define isomorphic sublattices in L_n and that T is uniquely determined.
- If $x \neq 0$, the triple (x, T, γ_T) was obtained, in the proof of Lemma 4.1.8, from a quintuple $\Theta_i = (H_S, H_{L_n}, \gamma, T, \gamma_T)$, with $H_S = \langle x \rangle \subset A_S$, $H_{L_n} = \langle e \rangle \subset A_{L_n}$. If we now consider a different quintuple $\Theta_{i'}$, with $H'_S = \langle x' \rangle$ and $x' \neq 0$, the embeddings i, i' will define isomorphic sublattices of L_n. This follows from Lemma 4.1.8 and Theorem 1.4.4, because, under our assumptions, two different subgroups $H_S, H'_S \subset A_S$ as above are conjugated by an automorphism of S. In fact, the restrictions $q_S|_{H_S}$ and $q_S|_{H'_S}$ are non-degenerate and isomorphic, since they both are $\frac{\mathbb{Z}}{p\mathbb{Z}}\left(-\frac{\beta}{p}\right)$, hence, by Proposition 1.1.3 and the classification of p-elementary forms, also the forms on $H_S^\perp$ and $(H'_S)^\perp$ will coincide. This implies that there exists an automorphism of A_S which exchanges H_S and H'_S: by the surjectivity of $O(S) \to O(A_S)$, this automorphism is induced by an automorphism of S. We conclude that the isometry class of S as a primitive sublattice of L_n does not depend on the choice of $x \neq 0$, nor of T, γ_T, since $k \leq 20-(p-1)m$ and here $l(A_T) = \max\{1, k-1\}$, so $l(A_T) \leq \mathrm{rk}(T)-2$. $\square$

Adopting the terminology used in [17], we provide the following definition.

Definition 4.1.10. Let (p, m, a) be a triple of integers, with $3 \leq p \leq 23$ prime, $m \geq 1$, $(p-1)m \leq 22$ and $0 \leq a \leq \min\{m, 23-(p-1)m\}$. The triple is said to be *admissible*, for a given integer $n \geq 2$, if there exist two orthogonal sublattices $T, S \subset L_n = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1) \rangle$ such that: $\mathrm{sign}(T) = (1, 22-(p-1)m)$, $\mathrm{sign}(S) = (2, (p-1)m-2)$, $\frac{L_n}{T \oplus S} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$ and the discriminant groups A_T and A_S are as in Proposition 4.1.5.

Remark 4.1.11. The condition $\frac{L_n}{T \oplus S} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$ implies that all admissible triples of the form $(p, m, 0)$ will define orthogonal sublattices $T, S \subset L_n$ such that $L_n = T \oplus S$.

We can now rephrase Proposition 4.1.9 in the following way, taking into account the uniqueness of S too.

Theorem 4.1.12. *Let $2(n-1) = p^\alpha \beta$, with $(p, \beta) = 1$ and $\alpha \leq 1$. If (p, m, a) is an admissible triple, there exists a unique even p-elementary lattice S as in Definition 4.1.10, up to isometries. Its primitive embedding in L_n and its orthogonal complement $T \subset L_n$ are uniquely determined (up to isometries of L_n) by (p, m, a), if $l(A_T) \leq 21-(p-1)m$.*

Proof. By Proposition 4.1.5 and Remark 4.1.6 the discriminant group A_S is $\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$ if m and a have same parity, otherwise $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$. Moreover, since the triple (p, m, a) fixes the signature of S, it also fixes the quadratic form on A_S, as we explained in Remark 1.2.3. Thus, if $\mathrm{rk}(S) \geq 3$ the lattice S is unique, up to isometries of S, by Theorem 1.3.3. The same is shown to hold also for the remaining cases, i.e. the triples $(3,1,0)$ and $(3,1,1)$, where S is positive definite of rank two: by [35, Table 15.1], we have $S \cong A_2(-1)$.

For any co-invariant lattice S corresponding to an admissible triple (p,m,a), we now show that the homomorphism $O(S) \to O(A_S)$ is surjective. By Definition 4.1.10 we have $\mathrm{sign}(S) = (2,(p-1)m-2)$, hence the two triples $(3,1,0)$ and $(3,1,1)$ are the only ones where S is not indefinite: for them, the surjectivity of $O(S) \to O(A_S)$ follows from [79, Remark 1.14.6], because $S \cong A_2(-1)$. For all other admissible triples, S is indefinite and by Proposition 1.4.2 we only need to show that $\mathrm{rk}(S) \geq l(A_S)+2$. As recalled in the first part of the proof, the length of A_S is either a or $a+1$, thus we want to prove that $\mathrm{rk}(S) \geq a+3$. We have

$$\mathrm{rk}(S) = (p-1)m \geq 2m \geq 2a$$

because $p \geq 3$ and $a \leq m$. Hence, if $a \geq 3$ the condition $\mathrm{rk}(S) \geq a+3$ is satisfied for any m. If instead $a \leq 2$ (i.e. $a+3 \leq 5$) the inequality holds for all $m \geq 3$. The only triples left are the ones with $m \in \{1,2\}$ and $a \in \{0,1,2\}$. For all these triples, $(p-1)m \geq a+3$ whenever $p \geq 5$, therefore we only need to check the triples with $p=3$. We already discussed the two admissible triples with $p=3$, $m=1$, where $S \cong A_2(-1)$. Instead, if $m=2$ then $\mathrm{rk}(S) = (p-1)m = 4 \geq a+3$ for $a = 0,1$. Finally, the triple $(3,2,2)$ defines a 3-elementary lattice S with $\mathrm{sign}(S) = (2,2)$ and $l(A_S) = 2$, since m and a have same parity, hence $\mathrm{rk}(S) \geq l(A_S)+2$.

The statement now follows from Remark 4.1.6 and Proposition 4.1.9, under the assumption $l(A_T) \leq 21-(p-1)m$. □

REMARK 4.1.13. If both triples (p,m,a), $(p,m,a+1)$ are admissible, with m and a of different parity, then they determine the same lattice S, up to isometries. As a matter of fact, in both cases the signature of S is $(2,(p-1)m-2)$ and, by Remark 4.1.6, its discriminant group is $\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$. Notice, however, that the invariant lattices T corresponding to the two triples are non-isometric.

To conclude this subsection, we use our results to list all possible quadratic forms q_S, q_T, up to isometries, on the discriminant groups A_S, A_T: by Lemma 4.1.8, we will need to discuss separately the cases $\alpha = 0$ and $\alpha = 1$ and to distinguish on whether $-\beta$ is a quadratic residue modulo p. This classification of quadratic forms is needed for listing admissible pairs of lattices (T,S) for specific values of n and p.

PROPOSITION 4.1.14. *Let $2(n-1) = p^\alpha\beta$, with $(p,\beta)=1$ and $\alpha \leq 1$. Let (p,m,a) be an admissible triple and T, S lattices corresponding to it. Then one of the following holds:*

- *(i)* $q_T = (-q_S) \oplus q_{L_n}$, *with* $q_S = \left(w_{p,1}^{+1}\right)^{\oplus a}$ *or* $q_S = \left(w_{p,1}^{+1}\right)^{\oplus a-1} \oplus w_{p,1}^{-1}$*;*
- *(ii)* $\alpha = 1$, $-\beta$ *is a quadratic residue modulo* p *and*
  - (a) $q_S = \left(w_{p,1}^{+1}\right)^{\oplus a+1}, q_T = \left(-w_{p,1}^{+1}\right)^{\oplus a} \oplus q_{1,\beta}$, *or*
  - (b) $a \geq 1, q_S = \left(w_{p,1}^{+1}\right)^{\oplus a} \oplus w_{p,1}^{-1}, q_T = \left(-w_{p,1}^{+1}\right)^{\oplus a-1} \oplus (-w_{p,1}^{-1}) \oplus q_{1,\beta}$.
- *(iii)* $\alpha = 1$, $-\beta$ *is not a quadratic residue modulo* p *and*
  - (a) $a \geq 1, q_S = \left(w_{p,1}^{+1}\right)^{\oplus a+1}, q_T = \left(-w_{p,1}^{+1}\right)^{\oplus a-1} \oplus (-w_{p,1}^{-1}) \oplus q_{1,\beta}$, *or*
  - (b) $q_S = \left(w_{p,1}^{+1}\right)^{\oplus a} \oplus w_{p,1}^{-1}, q_T = \left(-w_{p,1}^{+1}\right)^{\oplus a} \oplus q_{1,\beta}$.

PROOF. As explained in the proof of Lemma 4.1.8, case (i) corresponds to embeddings $S \hookrightarrow L_n$ determined by quintuples $(H_S, H_{L_n}, \gamma, T, \gamma_T)$ with $H_S = 0$, $H_{L_n} = 0$. Moreover, the quadratic form q_S is as in (3), with $k = l(A_S) = a$ by Proposition 4.1.5. This is the only possibility when $\alpha = 0$. If instead $\alpha = 1$, there may also be compatible embeddings $S \hookrightarrow L_n$ corresponding to quintuples with

$H_S \neq 0$ (see again Lemma 4.1.8): in this case, by the surjectivity of $O(S) \to O(A_S)$ (see Theorem 4.1.12), as we showed in the proof of Proposition 4.1.9 the subgroup H_S can be regarded, up to changing the generators of A_S, as one of the direct summands in the representation (3) of the quadratic form q_S. On the discriminant group of the orthogonal complement $T \subset L_n$, the quadratic form is then $q_T = ((-q_S) \oplus q_{L_n})|_{\Gamma^\perp/\Gamma}$, with q_{L_n} as in (15) and q_S as in (3), where now $k = l(A_S) = a+1$ by Proposition 4.1.5.

Let's assume that $-\beta$ is a quadratic residue modulo p, so that $q_{L_n} = w_{p,1}^{+1} \oplus q_{1,\beta}$, and suppose $q_S = \left(w_{p,1}^{+1}\right)^{\oplus a+1}$. Adopting the same notation used in the previous proofs, let $x \in A_S$ be the generator of the subgroup corresponding to one of the summands $w_{p,1}^{+1}$ in q_S and e be the generator of $\mathbb{Z}/p\mathbb{Z} \subset A_{L_n}$. Then $H_S = \langle x \rangle$, $H_{L_n} = \langle e \rangle$, $\gamma : x \mapsto e$ and the graph of γ is $\Gamma = \langle (x, e) \rangle \subset A_S \oplus A_{L_n}$. A direct computation shows that, with respect to the quadratic form $(-q_S) \oplus q_{L_n}$ on $A_S \oplus A_{L_n}$, the orthogonal of Γ is

$$\Gamma^\perp = \left(H_S^\perp \oplus H_{L_n}^\perp\right) + \Gamma.$$

This implies that the quadratic form $q_T \cong ((-q_S) \oplus q_{L_n})|_{\Gamma^\perp/\Gamma}$ is isometric to the restriction of $(-q_S) \oplus q_{L_n}$ to $H_S^\perp \oplus H_{L_n}^\perp$, therefore $q_T = \left(-w_{p,1}^{+1}\right)^{\oplus a} \oplus q_{1,\beta}$.

If instead $q_S = \left(w_{p,1}^{+1}\right)^{\oplus a} \oplus w_{p,1}^{-1}$, we need to ask $a \geq 1$, otherwise it is not possible to find subgroups $H_S \subset A_S$ and $H_{L_n} \subset A_{L_n}$ such that $q_S|_{H_S} \cong q_{L_n}|_{H_{L_n}}$. As in the previous case, we can assume $H_S = \langle x \rangle$, $H_{L_n} = \langle e \rangle$, $\gamma : x \mapsto e$ with $x \in A_S$ generator of one of the components $w_{p,1}^{+1}$ in q_S and $e \in A_{L_n}$ generator of the summand $w_{p,1}^{+1}$ of q_{L_n}. Since Γ, $\Gamma^\perp$ are the same as above, the form q_T still arises as the restriction of $(-q_S) \oplus q_{L_n}$ to $H_S^\perp \oplus H_{L_n}^\perp$, and therefore $q_T = \left(-w_{p,1}^{+1}\right)^{\oplus a-1} \oplus (-w_{p,1}^{-1}) \oplus q_{1,\beta}$.

The two cases where $q_{L_n} = w_{p,1}^{-1} \oplus q_{1,\beta}$ (i.e. $-\beta$ is not a quadratic residue modulo p) can be discussed in an analogous way. □

4.1.3. A special case: $\mathrm{rk}(T) = 1$**.** In this subsection we focus on the cases where the invariant lattice T has rank one, which correspond to maximal dimensional families of manifolds of $K3^{[n]}$-type equipped with a non-symplectic automorphism. Since T has rank one, $\mathrm{rk}(S) = (p-1)m = 22$: for p odd, this can only happen if $p = 3, m = 11$ or $p = 23, m = 1$. As before, we write $2(n-1) = p^\alpha\beta$, with $(p, \beta) = 1$.

If $\alpha = 0$, then a must be odd, because it needs to be of the same parity as m (Remark 4.1.6); in particular, $a \geq 1$. Moreover $A_T \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$, by Proposition 4.1.5: since $\mathrm{rk}(T) = 1$, then necessarily $\alpha = 0$, $a = 1$. We conclude $T \cong \langle 2p(n-1) \rangle$ ($\alpha = 0$ means that p and $\beta = 2(n-1)$ are coprime).

If instead $\alpha \geq 1$, there are a priori two possibilities:

- a odd. Then $A_T \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a} \oplus \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$ with $\alpha \geq 1$ and $a \geq 1$. As a consequence $l(A_T) \geq 2$, so T cannot have rank one.
- a even. By the classification provided in Proposition 4.1.5, T cannot be of rank one if $a > 0$. Hence $\alpha = 1$, $a = 0$, $T \cong \langle \beta \rangle = \langle \frac{2(n-1)}{p} \rangle$.

Moreover, we need to impose conditions on the orthogonal lattice S, using again Proposition 4.1.5. Since $\mathrm{rk}(T) = 1$, we can also use [44, Proposition 3.6] to determine the existence and the structure of such primitive sublattices $T, S \subset L_n$. We do it separately for the two possible cases we found.

- $\alpha = 0, a = 1, T = \langle h \rangle$, with $h \in L_n$ primitive of square $h^2 = 2p(n-1)$.
  By [44, Proposition 3.6], the orthogonal lattice S has discriminant $\frac{4p(n-1)^2}{f^2}$,

where f is the divisibility of h in L_n. By Proposition 4.1.5 we know that $A_S \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$, therefore $\mathrm{discr}(S) = p$ and we need $f = 2(n-1)$. By applying again [44, Proposition 3.6], we can conclude that such a T exists if and only if $-p$ is a quadratic residue modulo $4(n-1)$.

- $\alpha = 1, a = 0, T = \langle h \rangle$, with $h \in L_n$ primitive of square $h^2 = \frac{2(n-1)}{p}$. We have $A_S \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$, by Proposition 4.1.5, and $p = \mathrm{discr}(S) = \frac{4(n-1)^2}{pf^2}$, so $f = \frac{2(n-1)}{p}$. Here $p^2 \nmid 4(n-1)$, so p is invertible modulo $\frac{4(n-1)}{p}$, hence by [44, Proposition 3.6] such a T exists if and only if $-p$ is a quadratic residue modulo $\frac{4(n-1)}{p}$.

We summarize these results as follows.

Proposition 4.1.15. *Let $p \geq 3$ be a prime and $2(n-1) = p^\alpha \beta$ with $(p, \beta) = 1$. A triple (p, m, a), with $(p-1)m = 22$, is admissible if and only if $\alpha \in \{0, 1\}$, $a = 1 - \alpha$ and $-p$ is a quadratic residue modulo $\frac{4(n-1)}{p^\alpha}$.*

If this happens, then one of the following holds:

(i) $\alpha = 0, p = 3, m = 11, a = 1, T \cong \langle 6(n-1) \rangle, S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$;
(ii) $\alpha = 1, p = 3, m = 11, a = 0, T \cong \langle \beta \rangle = \langle \frac{2(n-1)}{3} \rangle, S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$;
(iii) $\alpha = 0, p = 23, m = 1, a = 1, T \cong \langle 46(n-1) \rangle, S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus K_{23}$;
(iv) $\alpha = 1, p = 23, m = 1, a = 0, T \cong \langle \beta \rangle = \langle \frac{2(n-1)}{23} \rangle, S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus K_{23}$.

Proof. The explicit description of the lattice S in the four cases is obtained by combining [44, Proposition 3.6] (where S is represented as $U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus B$ for a negative definite even lattice B of rank 2 depending on p, n, f) with the results on lattice isomorphisms of Theorem 1.3.5 and Theorem 1.3.9, which guarantee the uniqueness, up to isometries, of p-elementary lattices of signature $(2, 20)$ and length one, when $p = 3$ or $p = 23$. The lattice K_{23} was defined in Example 1.1.4. □

From Proposition 4.1.15 it follows, for instance, that the triple $(3, 11, 1)$ is admissible for $n = 2$, as already observed in [17], because -3 is a quadratic residue modulo 4: in this case we have $T \cong \langle 6 \rangle$. Similarly, $(3, 11, 0)$ is admissible when $n = 4$ (here $\alpha = 1$ and, again, -3 is a quadratic residue modulo 4), with $T \cong \langle 2 \rangle$. Instead, $(3, 11, 1)$ is not admissible when $n = 3$, because -3 is not a quadratic residue modulo 8.

The triple $(23, 1, 1)$ was already found to be admissible for $n = 2$ in [16] (where the authors also gave the isomorphism classes of T, S). By our proposition, this triple is admissible for $n = 3, 4$ too, since $-23 \equiv 1$ both modulo 8 and modulo 12. Finally, the smallest value of n such that $(23, 1, 0)$ is admissible is $n = 24$, since $2(n-1) = 46 = 23 \cdot 2$ and -23 is a quadratic residue modulo 4.

4.1.4. Admissible triples for $n = 3, 4$. In this section we provide a complete classification of admissible triples (p, m, a) for $n = 3, 4$. In both cases, for any odd prime number p we have $\alpha \leq 1$, therefore Theorem 4.1.12 allows us to exhibit the lattices T, S (up to isometries) for each triple. This classification of the two lattices is achieved by direct computation for all possible triples (p, m, a), checking for each of them if lattices T, S as in Definition 4.1.10 exist or not. To do so, we apply Theorem 1.3.7, which provides necessary and sufficient conditions for the existence of an even lattice with given signature and discriminant form.

Manifolds of $K3^{[3]}$-type.

- For $p = 23$ there is only one admissible triple, namely $(23, 1, 1)$, as we already observed in Section 4.1.3: the isometry classes of S and T are given in Proposition 4.1.15, case (iii).

- For all primes $5 \leq p \leq 19$, the admissible triples and the lattices S are the ones listed for $n = 2$ in the tables of [17, Appendix A], while the lattices T can be obtained from the corresponding ones in the tables by switching $\langle -2 \rangle$ with $\langle -4 \rangle$ in their description, since $L_3 = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -4 \rangle$. Notice that, with respect to [17, Table 5], by Remark 4.1.6 we can now say that the triple $(13, 1, 0)$ is not admissible, neither for $n = 2$ nor for $n = 3$: in fact, for these values of n we have $\alpha = 0$ for all possible primes p, hence m and a need to have the same parity.

  Example: $(p, m, a) = (5, 5, 3)$. This triple is not admissible for $n = 2$ and it is checked to be still not admissible for $n = 3$. In fact, for these values of p, m, a the lattice S would be isomorphic to $U(5) \oplus E_8^{\oplus 2} \oplus H_5$ (by Proposition 4.1.5 and Theorem 1.3.3), whose discriminant group is $A_S \cong A_{U(5)} \oplus A_{H_5} \cong \frac{\mathbb{Z}}{5\mathbb{Z}} \left(\frac{2}{5}\right)^{\oplus 3}$ (recall the definition of the lattice H_5 from Example 1.1.4). As we pointed out at the beginning of Section 4.1.2, $A_{L_3} \cong \frac{\mathbb{Z}}{4\mathbb{Z}} \left(-\frac{1}{4}\right)$, therefore if S admitted an embedding in L_3 the quadratic form on T would be $q_T = \frac{\mathbb{Z}}{5\mathbb{Z}} \left(\frac{8}{5}\right)^{\oplus 3} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \left(-\frac{1}{4}\right)$ by Lemma 4.1.8 and $\operatorname{sign}(T) = (1, 2)$. By Theorem 1.3.7, a lattice T with these invariants exists only if its 5-adic completion $T_5 := T \otimes \mathbb{Z}_5$ is such that $|A_T| \equiv \operatorname{discr}(K) \bmod (\mathbb{Z}_5^*)^2$, where K is the unique 5-adic lattice of rank $l(A_{T_5})$ and discriminant form q_{T_5} (see Theorem 1.3.6). In our case, since $A_{T_5} \cong (A_T)_5 \cong \frac{\mathbb{Z}}{5\mathbb{Z}} \left(\frac{8}{5}\right)^{\oplus 3}$, using Theorem 1.2.2 we compute $K = \langle 5 \cdot \frac{1}{8} \rangle^{\oplus 3}$, where $\frac{1}{8} \in \mathbb{Z}_5^*$. As a consequence, $|A_T| = 4 \cdot 5^3$ and $\operatorname{discr}(K) = \left(\frac{5}{8}\right)^3$: these two values do not satisfy the relation $|A_T| \equiv \operatorname{discr}(K) \bmod (\mathbb{Z}_5^*)^2$, because $2^{11} \notin (\mathbb{Z}_5^*)^2$. We conclude that a lattice T with such signature and quadratic form does not exist.
- For $p = 3$, Table 1 in Appendix A lists all admissible triples, with the corresponding isomorphism classes for T, S: as for larger primes, we can find many similarities with the analogous table for $n = 2$ ([17, Table 1]). However, there are also some significant differences.
  - As we observed in Section 4.1.3, there are no admissible triples with $m = 11$.
  - The triple $(3, 9, 5)$ is now admissible: here $S = U(3)^{\oplus 2} \oplus E_6 \oplus E_8$, while $\operatorname{sign}(T) = (1, 4)$ and $q_T = -q_S \oplus q_{L_3} \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \left(\frac{4}{3}\right)^{\oplus 5} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \left(-\frac{1}{4}\right)$. The existence of a lattice T with these invariants is proved by applying Theorem 1.3.7 and there is a unique isometry class in the genus of T by Theorem 1.3.4. In particular, we can take $T = U(3) \oplus \Omega$, where Ω is the even lattice of rank three whose bilinear form is defined by the matrix
  $$\Omega = \begin{pmatrix} -6 & 0 & -3 \\ 0 & -6 & 9 \\ -3 & 9 & -18 \end{pmatrix}.$$
  We have $\operatorname{sign}(\Omega) = (0, 3)$ and $q_\Omega = \frac{\mathbb{Z}}{3\mathbb{Z}} \left(\frac{4}{3}\right)^{\oplus 2} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}} \left(\frac{2}{3}\right) \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \left(-\frac{1}{4}\right)$, therefore $q_{U(3) \oplus \Omega} \cong -q_S \oplus q_{L_3}$ (using [79, Proposition 1.8.2]).
  - An additional new admissible triple is $(p, m, a) = (3, 8, 6)$: here we compute $S = U(3)^{\oplus 2} \oplus E_6^{\oplus 2}$, therefore $\operatorname{sign}(T) = (t_{(+)}, t_{(-)}) = (1, 6)$ and $q_T = \frac{\mathbb{Z}}{3\mathbb{Z}} \left(\frac{4}{3}\right)^{\oplus 6} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \left(-\frac{1}{4}\right)$. In this case, the strict inequality $t_{(+)} + t_{(-)} > l(A_T)$ holds: since moreover $t_{(+)} - t_{(-)} \equiv \operatorname{sign}(q_T) \pmod 8$, such a lattice T exists by Theorem 1.3.8 and again it is unique (up to isometries) by Theorem 1.3.4. A representative of this genus is $T = U(3) \oplus A_2 \oplus \Omega$.

Manifolds of $K3^{[4]}$-type.

- For $p = 23$ we have that $(23, 1, 1)$ is the only admissible triple (see Section 4.1.3): the isomorphism classes of T, S were obtained in Proposition 4.1.15.
- For primes $5 \leq p \leq 19$, again the lattices T, S and all admissible triples are the ones listed in the tables of [17, Appendix A] (apart from $(13, 1, 0)$, which is not admissible), up to replacing the $\langle -2 \rangle$ summand with a $\langle -6 \rangle$ summand in T.
- The last prime we need to consider is $p = 3$. This is the first case we encounter where an odd p divides $2(n-1)$: in particular, $2(n-1) = 6 = 3^\alpha \beta$ with $\alpha = 1$ and $\beta = 2$. Since we have $\alpha = 1$, by Lemma 4.1.8 and Proposition 4.1.9 we know that we can expect to have many more admissible triples than the ones we found for $p = 3$ and $n = 2, 3$: in fact, the same lattice S might be embedded in L_4 in two non-isomorphic ways. Table 2 (Appendix A) contains the list of all admissible triples and of the corresponding isomorphism classes for the lattices T, S. In particular, the triple $(3, 11, 0)$ is admissible thanks to Proposition 4.1.15; some other triples, such as $(3, 8, 6)$ and $(3, 8, 7)$, are excluded again by use of Theorem 1.3.7.

4.2. Existence of automorphisms

The classification of admissible lattices T, S presented in Section 4.1 does not tell us which cases can be realized by actual automorphisms. In this section we provide several tools to construct non-symplectic automorphisms of odd prime order on manifolds of $K3^{[n]}$-type, which are valid for any $n \geq 2$. In particular, we are interested in two types of manifolds: Hilbert schemes of points on $K3$ surfaces and moduli spaces of (possibly twisted) sheaves on $K3$'s. Moreover, in Section 4.2.3 we show that the existence of automorphisms which realize admissible pairs (T, S) where T has rank one can always be proved using the global Torelli theorem for IHS manifolds.

4.2.1. Natural automorphisms. Let Σ be a $K3$ surface. As we saw in Section 2.5.6, an automorphism $\varphi \in \mathrm{Aut}(\Sigma)$ induces a natural automorphism $\varphi^{[n]}$ on the Hilbert scheme $\Sigma^{[n]}$, which maps a zero-dimensional subscheme $\xi \subset \Sigma$ of length n to its schematic image $\varphi(\xi)$. In particular, if $T_\varphi, S_\varphi \subset H^2(\Sigma, \mathbb{Z}) \cong L_{K3}$ are the invariant and co-invariant lattices of φ, we have

$$T_{\varphi^{[n]}} \cong T_\varphi \oplus \langle -2(n-1) \rangle, \quad S_{\varphi^{[n]}} \cong S_\varphi.$$

We conclude that all admissible triples (p, m, a) where $T \cong T_{K3} \oplus \langle -2(n-1) \rangle$ and $S \cong S_{K3}$, with T_{K3}, S_{K3} the invariant lattice and its orthogonal complement for the action of a non-symplectic automorphism on a $K3$ surface, are realized by natural automorphisms. All isomorphism classes of the pairs (T_{K3}, S_{K3}) can be found in [4] (order $p = 3$) and [5] (prime order $5 \leq p \leq 19$), therefore it is immediate to check, for any n, which admissible cases have a natural realization.

In the tables of Appendix A we mark with the symbol ♣ the triples realized by natural automorphisms. For $n = 4$ (Table 2), it may not always be immediate to recognize the lattices T_{K3} of [4, Table 2] as direct summands in the lattices T we provide, since we often choose different representatives in the same isomorphism classes. In particular, we recall the following isometries: $U \oplus E_6 \oplus A_2 \cong U(3) \oplus E_8$; $U \oplus A_2^{\oplus 3} \cong U(3) \oplus E_6; U(3) \oplus A_2^{\oplus 3} \cong U \oplus E_6^\vee(3)$ (they can all be proved using Theorem 1.3.3). The reason why we adopt different genus representatives for these lattices will become clear in Section 4.3.1 (Lemma 4.3.1).

4.2.2. Induced automorphisms. A direct generalization of the notion of natural automorphisms is given by *induced automorphisms*, which were first introduced and studied in [86], [76] and later extended to the case of twisted $K3$ surfaces in [30]. Recall the theory of moduli spaces of twisted sheaves on $K3$ surfaces, which we presented in Example 2.2.7. Let (Σ, α) be a twisted $K3$ surface and φ be an automorphism of Σ: the Brauer class α is invariant with respect to φ if and only if $\alpha \circ \varphi^*|_{\mathrm{Tr}(\Sigma)} = \alpha$.

PROPOSITION 4.2.1. *Let (Σ, α) be a twisted $K3$ surface, φ an automorphism of Σ, v a positive Mukai vector and B a B-field lift of α such that v_B is primitive. If v_B and α are φ-invariant, then φ induces (via pullback of sheaves) an automorphism $\widehat{\varphi}$ of $M_{v_B}(\Sigma, \alpha)$.*

PROOF. See [76, Proposition 1.32] and [30, §3]. □

The automorphisms $\widehat{\varphi}$ arising in this way are called *twisted induced* (or just *induced* in the non-twisted case, i.e. if $\alpha = 0$). As an application of the twisted version of the global Torelli theorem for $K3$ surfaces (see [50, Corollary 5.4]), it is possible to characterize twisted induced automorphisms by studying their action on the Mukai lattice.

PROPOSITION 4.2.2. *Let σ be an automorphism of finite order on a manifold X of $K3^{[n]}$-type acting trivially on A_{L_n}. Then σ admits a realization as a twisted induced automorphism on a suitable moduli space $M_{v_B}(\Sigma, \alpha)$ if and only if the invariant lattice of the extension of σ to the Mukai lattice contains primitively a copy of $U(d)$, for some multiple d of the order of the Brauer class α.*

PROOF. See [30, Theorem 3.4]. □

Let $v = (r, H, s)$ be a positive Mukai vector. If $B \in H^2(\Sigma, \mathbb{Q})$ is a B-field lift of α such that v_B is primitive, then the transcendental lattice of the moduli space $M_{v_B}(\Sigma, \alpha)$ is isomorphic to $\ker(\alpha) \subset \mathrm{Tr}(\Sigma)$, which (if $\alpha \neq 0$) is a sublattice of index equal to the order of α. By [101, §3], $\mathrm{Pic}(M_{v_B}(\Sigma, \alpha)) \cong v_B^\perp \cap \mathrm{Pic}(\Sigma, \alpha)$ inside $H^*(\Sigma, \mathbb{Z})$, where $\mathrm{Pic}(\Sigma, \alpha) \cong \mathrm{Pic}(\Sigma) \oplus U$ if $\alpha = 0$, otherwise $\mathrm{Pic}(\Sigma, \alpha)$ is generated by $\mathrm{Pic}(\Sigma)$ and the vectors $(0, 0, 1), (k, kB, 0)$ by [67, Lemma 3.1], assuming the order of α is k.

If $\alpha = 0$, then $M_v(\Sigma, 0)$ is isomorphic to the moduli space $M_\tau(v)$ of τ-stable objects of Mukai vector v, for some primitive positive Mukai vector v and $\tau \in \mathrm{Stab}(\Sigma)$ a v-generic Bridgeland stability condition. By our previous discussion, the transcendental lattice of $M_\tau(v)$ coincides with $\mathrm{Tr}(\Sigma)$, while its Picard lattice is isomorphic to $v^\perp \cap (\mathrm{Pic}(\Sigma) \oplus U)$. In particular, the summand U in $\mathrm{Pic}(\Sigma) \oplus U$ is just $H^0(\Sigma, \mathbb{Z}) \oplus H^4(\Sigma, \mathbb{Z})$, which is the orthogonal complement of $H^2(\Sigma, \mathbb{Z}) \cong L_{K3}$ inside $H^*(\Sigma, \mathbb{Z}) \cong \Lambda_{24}$. Since L_{K3} is unimodular, the action of $\varphi \in \mathrm{Aut}(\Sigma)$ on L_{K3} extends to an action on Λ_{24} which is trivial on $(L_{K3})^\perp$ (by Proposition 1.4.7). Let $T_{K3}, S_{K3} \subset L_{K3}$ and $\widehat{T}, \widehat{S} \subset \Lambda_{24}$ be the invariant and co-invariant lattices of these two actions: by what we stated, $\widehat{T} = T_{K3} \oplus U$ and $\widehat{S} = S_{K3}$. The induced automorphism $\widehat{\varphi}$ acts on $H^2(M_\tau(v), \mathbb{Z}) \cong L_n = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1) \rangle$: its invariant lattice is $T \cong \left(v^\perp\right)^\varphi = \widehat{T} \cap v^\perp$ (see [76, Lemma 1.34]). We rephrase the results of [76, §2,3] as follows.

PROPOSITION 4.2.3. *Let (p, m, a) be an admissible triple for a certain $n \geq 2$, with (T, S) the corresponding pair of lattices; consider the canonical primitive embeddings $S \hookrightarrow L_n \hookrightarrow \Lambda_{24}$ and define $\widehat{T} := S^\perp \subset \Lambda_{24}$. Then the triple (p, m, a) is realized by an induced automorphism if $\widehat{T} \cong U \oplus T_{K3}$, $S \cong S_{K3}$, with (T_{K3}, S_{K3}) the invariant lattice and its orthogonal complement for the action of a non-symplectic* *automorphism on a $K3$ surface, and there exists a primitive vector $v \in \widehat{T}$ of square $2(n-1)$ such that $T \cong v^\perp \cap \widehat{T}$.*

In particular, all natural automorphisms can be considered as induced, since $\langle -2(n-1)\rangle$ is the orthogonal in U of an element of square $2(n-1)$.

In Section 4.3 we will apply the theory of induced (and twisted induced) automorphisms to construct geometric realizations of several admissible triples for manifolds of type $K3^{[3]}$ and $K3^{[4]}$.

4.2.3. Existence for $\mathrm{rk}(T) = 1$. The global Torelli theorem (Theorem 2.3.11) can be applied to prove the existence of automorphisms of manifolds of $K3^{[n]}$-type realizing the pairs of lattices (T, S) classified in Proposition 4.1.15, i.e. for $\mathrm{rk}(T) = 1$.

PROPOSITION 4.2.4. *Let (p, m, a) be an admissible triple as in Proposition 4.1.15, for a certain $n \geq 2$, and let T, S be the lattices associated to it. There exists a manifold X of $K3^{[n]}$-type and a non-symplectic automorphism $f \in \mathrm{Aut}(X)$ of order p such that $T_f \cong T$ and $S_f \cong S$.*

PROOF. We discuss separately the four possible cases classified in Proposition 4.1.15, keeping the same numbering.

Case (ii): $(3, 11, 0)$. Here we have $2(n-1) = 3\beta$, with $(3, \beta) = 1$; the invariant and co-invariant lattices are $T = \langle \beta \rangle$ and $S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$, which by Proposition 4.1.15 can be seen as orthogonal sublattices of $L_n = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$. We first construct a monodromy of the lattice L_n having invariant lattice T and co-invariant lattice S. The triple has $a = 0$, therefore $L_n = T \oplus S$ (see Remark 4.1.11): an isometry $\phi \in O(L_n)$ can then be represented as $\phi = \gamma \oplus \psi$, with $\gamma \in O(T)$ and $\psi \in O(S)$. Moreover, since we want ϕ to be of order 3 with invariant lattice T, we will need $\gamma = \mathrm{id}_T$ and ψ of order 3 with no non-zero fixed points.

By [4, Theorem 3.3], there exist a $K3$ surface Σ and a non-symplectic automorphism $\varphi \in \mathrm{Aut}(\Sigma)$ of order 3 with invariant lattice $T_{K3} = U$ and co-invariant $S_{K3} = U^{\oplus 2} \oplus E_8^{\oplus 2}$. Thus, the natural automorphism $\varphi^{[n]}$ on the Hilbert scheme $\Sigma^{[n]}$ will have invariant lattice $T' = U \oplus \langle -2(n-1)\rangle$ and co-invariant lattice $S' = S_{K3} = U^{\oplus 2} \oplus E_8^{\oplus 2}$. Notice that $T' \oplus S' = L_n$, meaning that the triple $(3, 10, 0)$ is realized by a natural automorphism for all $n \geq 2$. Moreover, since $\varphi^{[n]}$ has odd order, it induces a monodromy of L_n which acts as $+\mathrm{id}$ on the discriminant $A_{L_n} \cong A_{T'} \oplus A_{S'}$ (Proposition 2.3.8). The restriction of this monodromy to S' is therefore an isometry $\mu \in O(S')$ of order 3, with no non-zero fixed vectors, such that $\overline{\mu} = \mathrm{id}_{A_{S'}}$ and $\mathrm{sn}_{\mathbb{R}}^{S'}(\mu) = 1$ (recall the definition of real spinor norm from Remark 2.3.9). On our original lattice $S = S' \oplus A_2$ we now consider the isometry $\psi = \mu \oplus \rho_0$, where ρ_0 acts on $A_2 = \left(\mathbb{Z}e_1 \oplus \mathbb{Z}e_2, \left(\begin{smallmatrix} -2 & 1 \\ 1 & -2 \end{smallmatrix}\right)\right)$ as

$$\rho_0(e_1) = e_2, \qquad \rho_0(e_2) = -e_1 - e_2.$$

It is easy to check that ρ_0 is an isometry of order 3 without non-zero fixed points, inducing the identity on the discriminant group A_{A_2} (this isometry was also used in [39, §6.6]). Notice that, since A_2 is negative definite, $\mathrm{sn}_{\mathbb{R}}^{A_2}(\rho_0) = 1$. We can then conclude that $\psi = \mu \oplus \rho_0$ is an isometry of S of order 3 with no non-zero fixed points, which induces the identity on the discriminant group. Moreover, since ψ is defined as an orthogonal sum, $\mathrm{sn}_{\mathbb{R}}^{S}(\psi) = \mathrm{sn}_{\mathbb{R}}^{S'}(\mu) \cdot \mathrm{sn}_{\mathbb{R}}^{A_2}(\rho_0) = 1$ (the reflections appearing in the factorisation of $\psi_{\mathbb{R}}$ are the extensions to $S \otimes \mathbb{R}$ of the ones which factorise $\mu_{\mathbb{R}}$ and $(\rho_0)_{\mathbb{R}}$ as transformations of the orthogonal subspaces $S' \otimes \mathbb{R}, A_2 \otimes \mathbb{R} \subset S \otimes \mathbb{R}$). By the same reasoning, $\mathrm{sn}_{\mathbb{R}}^{L_n}(\phi) = \mathrm{sn}_{\mathbb{R}}^{L_n}(\mathrm{id}_T \oplus \psi) = \mathrm{sn}_{\mathbb{R}}^{S}(\psi) = 1$. Thus, ϕ is a monodromy operator, thanks to Proposition 2.3.8, with invariant lattice T and co-invariant S. By generalizing [17, Proposition 5.3], there exists a manifold X of $K3^{[n]}$-type and a marking $\eta : H^2(X, \mathbb{Z}) \to L_n$ such that $\eta(\mathrm{NS}(X)) = T$. The monodromy ϕ is a Hodge isometry, since it preserves $H^{2,0}(X) = \mathbb{C}\omega_X$ (because $\mathrm{NS}(X) = \omega_X^\perp \cap H^2(X, \mathbb{Z})$). Moreover, since $\mathrm{rk}(T) = 1$, ϕ fixes a Kähler class (the generator of $\eta(\mathrm{NS}(X)) = T$). The Hodge-theoretic global Torelli theorem (Theorem 2.3.11) allows us to conclude that there exists an automorphism $f \in \mathrm{Aut}(X)$ such that $\eta \circ f^* \circ \eta^{-1} = \phi$.

Case (i)*:* $(3, 11, 1)$. In this case, $T = \langle 6(n-1)\rangle$ and $S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$. Now $T \oplus S$ is a proper sublattice of L_n, because $a = 1$; however, we can still consider the isometry $\phi = \mathrm{id}_T \oplus \psi \in O(T \oplus S)$ defined above. Since $\overline{\psi} = \mathrm{id}_{A_S}$, the isometry ϕ can be extended to $\Phi \in O(L_n)$ by Proposition 1.4.7. As recalled in Section 4.1.1, $A_{L_n} \cong M^\perp/M$, with $M, M^\perp$ subgroups of $A_T \oplus A_S$, meaning that $\overline{\Phi} = \mathrm{id}_{A_{L_n}}$, since $\overline{\phi} = \mathrm{id} \in O(A_{T\oplus S})$. Moreover, we also have $\mathrm{sn}_{\mathbb{R}}^{L_n}(\Phi) = \mathrm{sn}_{\mathbb{R}}^{T\oplus S}(\phi) = \mathrm{sn}_{\mathbb{R}}^{S}(\psi) = 1$ (see for instance the proof of [28, Proposition 3.5]). Thus, $\Phi \in \mathrm{Mon}^2(L_n)$, and it still has invariant lattice T and co-invariant lattice S. We can now apply Theorem 2.3.11 in the same way as before to conclude that, also in this case, there exists an automorphism of a suitable manifold of $K3^{[n]}$-type inducing Φ.

Cases (iii)*,* (iv)*:* $(23, 1, 0)$ *and* $(23, 1, 1)$. These two cases can be realized by generalizing [16], where the authors proved the existence of an automorphism of order 23 on a manifold of $K3^{[2]}$-type, having invariant lattice $T \cong \langle 46\rangle$ and co-invariant lattice $S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus K_{23}$. In Proposition 4.1.15 we showed that, if a triple (p, m, a) with $p = 23$ is admissible, then $m = 1$ and $a \in \{0, 1\}$; moreover, in this case the two orthogonal sublattices of L_n are $S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus K_{23}$ and either $T = \langle 46(n-1)\rangle$ if $a = 1$ (as we have for $n = 2$), or $T = \langle \frac{2(n-1)}{23}\rangle$ if $a = 0$. We notice in particular that S does not depend on n and in [16, Proposition 5.3] it was proved that such lattice admits an isometry ψ of order 23 inducing the identity on A_S. Thus, $\mathrm{id}_T \oplus \psi \in O(T \oplus S)$ can be extended to an isometry $\phi \in O(L_n)$ such that $\overline{\phi} = \mathrm{id} \in O(A_{L_n})$ (if $a = 0$ we have $L_n = T \oplus S$, so $\mathrm{id}_T \oplus \psi$ is already an isometry of L_n with this property; otherwise, if $a = 1$, we apply again Proposition 1.4.7). Following the same proof of [16, Theorem 6.1], there exists an automorphism realizing the triple. We point out that, while for $n = 2$ the monodromies of L_n are just the isometries preserving the positive cone, for higher values of n the isometry also needs to induce $\pm\mathrm{id}$ on A_{L_n} (see Proposition 2.3.8). This, however, is not a problem since we know that $\overline{\phi} = \mathrm{id} \in O(A_{L_n})$. □

We remarked in Section 4.1.3 that the triple $(3, 11, 0)$ is admissible for $n = 4$, therefore we can now conclude that it is realized by an automorphism: we mark this case with the symbol ★ in the corresponding table of Appendix A. We will see an explicit geometric realization of it in Section 4.4.1.

4.3. Induced automorphisms for $n = 3, 4$, $p = 3$

The new admissible triples $(3, m, a)$ that appear passing from $n = 2$ to $n = 3$ and, more significantly, to $n = 4$ (see Section 4.1.4 and Appendix A) cannot be realized by natural automorphisms. However, in this section we will show that all of them but one admit a realization using (possibly twisted) induced automorphisms, which were discussed in Section 4.2.2.

4.3.1. Induced automorphisms for $n = 4$. Let T, S be the lattices associated to an admissible triple $(3, m, a)$ for $n = 4$ such that $S = S_{K3}$, where S_{K3} is the co-invariant lattice of a non-symplectic automorphism φ of order 3 on a $K3$ surface Σ (see [4] for a complete classification of these lattices). Let $\widehat{T}$ be the orthogonal complement of S in the Mukai lattice Λ_{24}: since $S = S_{K3}$, we have $\widehat{T} \cong T_{K3} \oplus U$, with T_{K3} the invariant lattice of φ. Then the following result holds.

LEMMA 4.3.1. *If $T_{K3} \cong U(3) \oplus W$, for some even lattice W, and $T \cong U \oplus W \oplus \langle -6 \rangle$, then the triple $(3, m, a)$ is realized by an automorphism induced by φ on a suitable moduli space $M_\tau(v)$ of dimension eight.*

PROOF. If $T_{K3} \cong U(3) \oplus W$, then $\widehat{T} \cong U \oplus U(3) \oplus W$ is the invariant lattice of the extended action of φ to Λ_{24}. Let v be a primitive Mukai vector of square six in the component $U(3)$ of $\widehat{T}$: then $v^\perp \cap \widehat{T} \cong U \oplus W \oplus \langle -6 \rangle$. Proposition 4.2.3 allows us to conclude. □

REMARK 4.3.2. Lemma 4.3.1 holds not only for $n = 4$, but for any n such that $n \equiv 1 \pmod 3$, since this is the condition which guarantees the existence of a primitive element of square $2(n-1)$ in the lattice $U(3)$.

THEOREM 4.3.3. *For $n = 4$, all admissible triples $(3, m, a) \neq (3, 11, 0), (3, 10, 3), (3, 9, 4), (3, 8, 5)$ admit a geometric realization via non-twisted induced automorphisms.*

PROOF. Except for the four cases excluded in the statement, the only admissible triples in Table 2 of Appendix A which cannot be realized by a natural automorphism and do not satisfy the hypotheses of Lemma 4.3.1 are $(3, 8, 1)$, $(3, 7, 0)$, $(3, 4, 1)$ and $(3, 3, 0)$.

Consider the triple $(3, 8, 1)$. Here we have $S = U^{\oplus 2} \oplus E_6^{\oplus 2}$ and $T = \langle 2 \rangle \oplus E_6$. By [4, Theorem 3.3] there exists a $K3$ surface Σ and a non-symplectic automorphism of order three $\varphi \in \mathrm{Aut}(\Sigma)$ with $S_{K3} = S$ and $T_{K3} = U \oplus A_2^{\oplus 2}$: in order to show that the triple $(3, 8, 1)$ is realized by an automorphism induced by φ we need to prove the existence of a primitive Mukai vector $v \in \widehat{T} = U^{\oplus 2} \oplus A_2^{\oplus 2}$ of square six and orthogonal complement $v^\perp \cap \widehat{T}$ isometric to T (see Proposition 4.2.3). We describe primitive embeddings $\langle 6 \rangle \hookrightarrow \widehat{T}$ using Theorem 1.4.4, since $\widehat{T}$ is unique in its genus (up to isometries) by Theorem 1.3.3. The discriminant groups of the two lattices $\langle 6 \rangle$ and $\widehat{T}$ are:

$$A_{\langle 6 \rangle} = \langle s \rangle \cong \frac{\mathbb{Z}}{6\mathbb{Z}} \left(\frac{1}{6} \right), \qquad A_{\widehat{T}} = \langle t_1, t_2 \rangle \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \left(\frac{4}{3} \right) \oplus \frac{\mathbb{Z}}{3\mathbb{Z}} \left(\frac{4}{3} \right).$$

We consider the isometric subgroups $H := \langle 2s \rangle \subset A_{\langle 6 \rangle}$ and $H' := \langle t_1 + t_2 \rangle \subset A_{\widehat{T}}$. Let $\gamma : H \to H'$ be the isomorphism sending the generator of H to the generator of H' (both these elements have order three and quadratic form $\frac{2}{3} \bmod 2\mathbb{Z}$). The graph of γ is the subgroup $\Gamma = \langle 2s + t_1 + t_2 \rangle \subset A_{\langle 6 \rangle}(-1) \oplus A_{\widehat{T}}$ and its orthogonal complement is $\Gamma^\perp = \langle s + t_1, s + t_2 \rangle$. Passing to the quotient $\Gamma^\perp / \Gamma$, the class of the element $s + t_2$ becomes the opposite of the class of $s + t_1$, meaning that

$$\frac{\Gamma^\perp}{\Gamma} = \langle [s + t_1] \rangle \cong \frac{\mathbb{Z}}{6\mathbb{Z}} \left(\frac{7}{6} \right).$$

This quotient coincides with the discriminant group of $T = \langle 2 \rangle \oplus E_6$: by Theorem 1.4.4, this implies that there exists a primitive embedding $\langle 6 \rangle \hookrightarrow \widehat{T}$ with orthogonal complement T, thus the triple $(3, 8, 1)$ has an induced realization by Proposition 4.2.3. Moreover, this computation guarantees that the triple $(3, 4, 1)$ is also realized by an induced automorphism, since in this case both $T = \langle 2 \rangle \oplus E_6 \oplus E_8$ and $T_{K3} = U \oplus A_2^{\oplus 2} \oplus E_8$ differ from the ones of $(3, 8, 1)$ only for an additional copy of the unimodular lattice E_8.

With a similar approach it is possible to show that the admissible triples $(3, 7, 0)$ and $(3, 3, 0)$ are realized by induced automorphisms too: here $T = \langle 2 \rangle \oplus E_8$, $T_{K3} = U \oplus E_6$ and $T = \langle 2 \rangle \oplus E_8^{\oplus 2}$, $T_{K3} = U \oplus E_6 \oplus E_8$ respectively. □

All the cases which can be realized by non-natural, non-twisted induced automorphisms are marked with the symbol ♮ in Table 2 of Appendix A.

4.3.2. Twisted induced automorphisms for $n = 3, 4$. Both for $n = 3$ and $n = 4$, in Section 4.1.4 we have found admissible triples for $p = 3$ where the lattice S is different from all possible co-invariant lattices S_{K3} of non-symplectic automorphisms of order three on $K3$ surfaces, classified in [4]. Thus, we cannot realize these cases in a natural way, nor using induced automorphisms on moduli spaces of ordinary sheaves on $K3$'s (Proposition 4.2.3). However, we prove that (excluding $(3, 11, 0)$ for $n = 4$, which will be discussed in Section 4.4.1) they all admit a geometric realization using twisted induced automorphisms.

We are interested in the following triples (p, m, a): $(3, 9, 5)$ and $(3, 8, 6)$ for $n = 3$; $(3, 10, 3), (3, 9, 4), (3, 8, 5)$ for $n = 4$. For each of these cases, let T, S be the corresponding lattices in Table 1 ($n = 3$) or Table 2 ($n = 4$) of Appendix A. Notice that S is always of the form $S = U(3)^{\oplus 2} \oplus W$, where W is one of the lattices $E_8^{\oplus 2}$, $E_6 \oplus E_8$, $E_6^{\oplus 2}$.

Let Σ be a $K3$ surface with transcendental lattice $\mathrm{Tr}(\Sigma) = S_{K3} \cong U \oplus U(3) \oplus W$, where S_{K3} is the co-invariant lattice of a non-symplectic automorphism $\varphi \in \mathrm{Aut}(\Sigma)$ of order three: the existence of (Σ, φ) is guaranteed, in all cases, by [4, Theorem 3.3] and [4, Table 2]. This $K3$ surface has $\mathrm{Pic}(\Sigma) = T_{K3} \cong U(3) \oplus M$, for an even lattice M which is either $0, A_2, A_2^{\oplus 2}$.

Proposition 4.3.4. *Let S and (Σ, φ) be as above. Then there exists a φ-invariant Brauer class $\alpha \in \mathrm{Br}(\Sigma)[3]$ whose kernel in $\mathrm{Tr}(\Sigma)$ is isomorphic to S.*

Proof. As we recalled in Section 4.2.2, a Brauer class $\alpha \in \mathrm{Br}(\Sigma)$ of order three corresponds to a surjective homomorphism $\alpha : \mathrm{Tr}(\Sigma) \to \mathbb{Z}/3\mathbb{Z}$, which is φ-invariant if and only if $\alpha \circ \varphi^*|_{\mathrm{Tr}(\Sigma)} = \alpha$.

We consider $\alpha := (e_1, -) : \mathrm{Tr}(\Sigma) \to \mathbb{Z}/3\mathbb{Z}$, where $\{e_1, e_2\}$ is a basis for the summand U in $\mathrm{Tr}(\Sigma) \cong U \oplus U(3) \oplus W$. The kernel of this homomorphism is $\ker(\alpha) = K \oplus U(3) \oplus W$, with $K = \{v \in U \mid (e_1, v) \equiv 0 \,(\mathrm{mod}\ 3\mathbb{Z})\}$. In particular $K = \langle e_1, 3e_2 \rangle \cong U(3)$, thus $\ker(\alpha) \cong S$.

We now want to check that $\alpha \circ \varphi^*|_{\mathrm{Tr}(\Sigma)} = \alpha$. By [4, Examples 1.1], recalling that W is a direct sum of copies of E_6 and E_8, the action of the automorphism φ on $\mathrm{Tr}(\Sigma) \cong U \oplus U(3) \oplus W$ can be expressed as

$$\varphi^*|_{\mathrm{Tr}(\Sigma)} = \pi \oplus \rho$$

where ρ is a suitable isometry of order three of W with no non-zero fixed points and π is the isometry of $U \oplus U(3)$ which, with respect to a basis $\{e_1, e_2, f_1, f_2\}$, is given by:

$$e_1 \mapsto e_1 - f_1, \qquad e_2 \mapsto -2e_2 - f_2,$$
$$f_1 \mapsto -2f_1 + 3e_1, \qquad f_2 \mapsto f_2 + 3e_2.$$

We have $(e_1, \pi(e_i)) \equiv (e_1, e_i)\ (\mathrm{mod}\ 3\mathbb{Z})$ and $(e_1, \pi(f_i)) \equiv (e_1, f_i) \equiv 0\ (\mathrm{mod}\ 3\mathbb{Z})$, for $i = 1, 2$, therefore the Brauer class α is invariant with respect to φ. □

Theorem 4.3.5. *The admissible triples $(p, m, a) = (3, 9, 5), (3, 8, 6)$ for $n = 3$ and $(p, m, a) = (3, 10, 3), (3, 9, 4), (3, 8, 5)$ for $n = 4$ admit a geometric realization using twisted induced automorphisms.*

Proof. Fix a triple (p, m, a) as in the statement, let T, S be the invariant and co-invariant lattices associated to it and Σ, φ, α as in Proposition 4.3.4. We want to construct a moduli space $M_v(\Sigma, \alpha)$ having T as Picard lattice and S as transcendental lattice, and on which φ induces an automorphism.

We are considering α of the form $(e_1, -) : \mathrm{Tr}(\Sigma) \to \mathbb{Z}/3\mathbb{Z}$, where e_1 is a generator of U inside $\mathrm{Tr}(\Sigma) \cong U \oplus U(3) \oplus W$. As a consequence, recalling Section 4.2.2, the element $B = \frac{e_1}{3} \in \mathrm{Tr}(\Sigma) \otimes \frac{1}{3}\mathbb{Z} \subset H^2(X, \mathbb{Q})$ is a B-field lift of α, with the properties $B^2 = 0$ and $B \cdot L = 0$ for any $L \in \mathrm{Pic}(\Sigma)$.

Assume first that (p, m, a) is one of the three admissible triples for $n = 4$. We already remarked that $\mathrm{Pic}(\Sigma) = T_{K3} \cong U(3) \oplus M$: this means that we can find a primitive divisor H in the summand $U(3)$ of $\mathrm{Pic}(\Sigma)$ with $H^2 = 6$. Moreover, up to taking its opposite we can assume that H is effective (by Riemann–Roch). Let $v = (0, H, 0) \in H^*(\Sigma, \mathbb{Z})$ be the primitive positive Mukai vector defined by H, and $B = \frac{e_1}{3}$ the selected B-field lift of α: by the properties of B, the twisted Mukai vector v_B (defined in Example 2.2.7) coincides with v, and therefore it has square six and it is invariant with respect to φ. By the previous discussion, φ induces a non-symplectic automorphism of order three on the moduli space of twisted sheaves $M_v(\Sigma, \alpha)$, which is a manifold of $K3^{[4]}$-type. The transcendental lattice of $M_v(\Sigma, \alpha)$ is $\ker(\alpha) \cong S$ (Proposition 4.3.4), while its Picard group is isomorphic to the intersection $v_B^{\perp} \cap \langle \mathrm{Pic}(\Sigma), (0, 0, 1), (3, 3B, 0)\rangle$. Since $3B = e_1 \in \mathrm{Tr}(\Sigma)$, the lattice generated by $(0, 0, 1)$ and $(3, 3B, 0)$ is orthogonal to $\mathrm{Pic}(\Sigma)$; moreover, it is isomorphic to $U(3)$, by the fact that $B^2 = 0$. Thus

$$\mathrm{Pic}\left(M_v(\Sigma, \alpha)\right) \cong \left(H^{\perp} \cap \mathrm{Pic}(\Sigma)\right) \oplus U(3) \cong \langle -6\rangle \oplus M \oplus U(3)$$

which is exactly the lattice T corresponding to (p, m, a) (see Table 2 of Appendix A).

Consider now the case where (p, m, a) is one of the admissible triples $(3, 9, 5)$, $(3, 8, 6)$ for $n = 3$. In this case $\mathrm{Pic}(\Sigma) \cong U(3) \oplus A_2 \oplus M'$, therefore, if $\{e_1, e_2\}$ is a basis for $U(3)$ and $\{\delta_1, \delta_2\}$ a basis for A_2, we can take the primitive element of square four $\widetilde{H} = e_1 + e_2 + \delta_1 \in \mathrm{Pic}(\Sigma)$. Let H be the effective divisor between $\widetilde{H}$ and $-\widetilde{H}$. As before, $v = v_B = (0, H, 0)$ is a primitive positive Mukai vector invariant with respect to φ. Then φ induces an automorphism on $M_v(\Sigma, \alpha)$, which is a manifold of $K3^{[3]}$-type with transcendental lattice $\ker(\alpha) \cong S$ and

$$\mathrm{Pic}(M_v(\Sigma, \alpha)) \cong v_B^{\perp} \cap \langle \mathrm{Pic}(\Sigma), (0, 0, 1), (3, 3B, 0)\rangle \cong (H^{\perp} \cap \mathrm{Pic}(\Sigma)) \oplus U(3).$$

It can be shown that the orthogonal complement of $e_1 + e_2 + \delta_1$ in $U(3) \oplus A_2$ is isomorphic to the lattice Ω defined in Section 4.1.4, thus $\mathrm{Pic}(M_v(\Sigma, \alpha)) \cong \Omega \oplus M' \oplus U(3)$, which is the lattice T corresponding to the triple (p, m, a) in Table 1 of Appendix A.

To conclude the proof, we need to show that the automorphism induced by φ on $M_v(\Sigma, \alpha)$ leaves the whole Picard lattice invariant. Both for $n = 3$ and $n = 4$, the direct summand $U(3)$ in $\mathrm{Pic}\left(M_v(\Sigma, \alpha)\right)$ is the lattice $\langle (0, 0, 1), (3, 3B, 0)\rangle$. Notice that φ acts as the identity on $H^4(\Sigma, \mathbb{Z})$, therefore $(0, 0, 1)$ is fixed. Moreover, it maps $(3, 3B, 0)$ to $(3, 3\varphi^*(B), 0)$, but these two classes coincide in $H^2(M_v(\Sigma, \alpha), \mathbb{Z})$ since they correspond to each other via the Hodge isometry (equivariant with respect to the action of φ)

$$\exp(\varphi^*(B) - B) : \widetilde{H}(\Sigma, B, \mathbb{Z}) \to \widetilde{H}(\Sigma, \varphi^*(B), \mathbb{Z})$$

between the two Hodge structures of Σ defined by the B-field lifts $B, \varphi^*(B)$ of α (see [53, §2] and [30, Remark 2.4]; here we use $B^2 = \varphi^*(B)^2 = 0$). Since φ^* also fixes $\mathrm{Pic}(\Sigma)$, we get the result. □

In Table 1 and Table 2 of Appendix A we use the symbol $\diamondsuit$ to mark the five admissible triples which can be realized only via twisted induced automorphisms.

4.4. Automorphisms of the LLSvS eightfold

Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold which does not contain a plane and $M_3(Y) = \mathrm{Hilb}^{gtc}(Y)$ the irreducible component of $\mathrm{Hilb}^{3n+1}(Y)$ containing twisted cubic curves on Y. The manifold $M_3(Y)$ is smooth, projective of dimension ten and it is called the *Hilbert scheme of generalized twisted cubics on Y*. In [65], Lehn, Lehn, Sorger and van Straten proved that there exist an irreducible holomorphic symplectic manifold Z_Y of dimension eight, a closed Lagrangian embedding $j : Y \hookrightarrow Z_Y$ and a morphism $u : M_3(Y) \to Z_Y$ which factorizes as $\Phi \circ a$, where $a : M_3(Y) \to Z'_Y$ is a $\mathbb{P}^2$-bundle to an eight-dimensional manifold Z'_Y and $\Phi : Z'_Y \to Z_Y$ is the contraction of an extremal divisor $D \subset Z'_Y$ to the image $j(Y) \subset Z_Y$. Moreover, by work of Addington and Lehn ([1]) Z_Y is a manifold of $K3^{[4]}$-type.

We recall some details about the construction. For all curves $C \in M_3(Y)$ the linear span $\langle C \rangle \subset \mathbb{P}^5$ is a $\mathbb{P}^3$. In particular, C lies on the cubic surface $S_C = Y \cap \langle C \rangle$, which is integral since Y does not contain any plane. A point $p \in D \subset Z'_Y$ is defined by the datum $(y, \mathbb{P}(W))$, with $y \in Y$ and $\mathbb{P}(W) \subset \mathbb{P}^5$ a three-dimensional linear subspace through y contained in the tangent space T_yY (here and in the following $W \in \mathrm{Grass}(\mathbb{C}^6, 4)$). The curves of $M_3(Y)$ parametrized by this datum are non-Cohen–Macaulay: an element C in the fiber $a^{-1}(p)$ is a singular cubic curve C^0 cut out on Y by a plane through y contained in $\mathbb{P}(W) \subset T_yY$, together with an embedded point in y. The contraction $\Phi|_D : D \to j(Y)$ sends $p = (y, \mathbb{P}(W))$ to $j(y)$.

Instead, a point $p \in Z'_Y \setminus D$ corresponds to the choice of the following data:

- a three-dimensional linear subspace $\mathbb{P}(W) \subset \mathbb{P}^5$;
- a linear determinantal representation for the surface $S = \mathbb{P}(W) \cap Y$, i.e. the orbit $[A]$ of a 3×3-matrix A with coefficients in W^* such that $\det(A) = 0$ is an equation for S in $\mathbb{P}(W)$, where the orbit is taken with respect to the action of $(\mathrm{GL}_3 \times \mathrm{GL}_3)/\Delta$, $\Delta := \{(tI_3, tI_3) \mid t \in \mathbb{C} \setminus \{0\}\}$ (see [65, §3]).

Then, any curve C in the fiber $a^{-1}(p)$ lies on S and is arithmetically-Cohen–Macaulay. The generators of the homogeneous ideal $I_{C/S}$ are the three minors of a 3×2-matrix A_0, whose columns are independent linear combinations of the columns of A. The morphism Φ maps $Z'_Y \setminus D$ isomorphically to $Z_Y \setminus j(Y)$.

As showed in [40, §3] (see also [17, §6.2]), one can construct non-symplectic automorphisms of the Fano variety of lines $F(Y)$ (which is a manifold of $K3^{[2]}$-type, see Example 2.1.8) starting from automorphisms of the cubic fourfold Y. It is therefore natural to ask whether a similar approach can be used to produce automorphisms on Z_Y: the answer is positive, we will show how to do so and how to choose Y in order to construct a non-symplectic automorphism on Z_Y realizing the admissible triple $(3, 11, 0)$ for $n = 4$.

By [70], automorphisms of a cubic hypersurface $Y \subset \mathbb{P}^5$ are restrictions of linear automorphisms of $\mathbb{P}^5$. The list of all automorphisms of prime order on smooth cubic fourfolds was provided in [42, Theorem 3.8].

Lemma 4.4.1. *Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold not containing a plane and $\sigma \in \mathrm{PGL}(6)$ be an automorphism such that $\sigma(Y) = Y$. Then, σ induces an automorphism $\check{\sigma}$ of $M_3(Y)$ such that $a(\check{\sigma}(C)) = a(\check{\sigma}(C'))$ if $a(C) = a(C')$.*

Proof. We begin by looking at curves in the fibers of a over $D \subset Z'_Y$. Let $p \in D$ be a point corresponding to $(y, \mathbb{P}(W))$, and $C_1, C_2 \in a^{-1}(p)$: as explained above, each C_i consists of a plane cubic curve C_i^0, singular in y, together with an embedded point at y. In particular, $C_i^0 = \pi_i \cap Y$, with π_1, π_2 two-dimensional subspaces inside $\mathbb{P}(W)$ tangent to Y in y. Then, $\sigma(C_i^0)$ are again plane cubic curves, cut out on Y by two planes through $\sigma(y)$ inside $\sigma(\mathbb{P}(W)) \subset T_{\sigma(y)}Y$. Let $\check{\sigma}(C_i)$ be $\sigma(C_i^0)$, with the unique non-reduced structure at $\sigma(y)$: then $\check{\sigma}(C_1)$, $\check{\sigma}(C_2)$ are elements of $M_3(Y)$ in the fiber $a^{-1}(p')$, with p' defined by $(\sigma(y), \sigma(\mathbb{P}(W)))$.

Consider now a point $p \in Z'_Y \setminus D$, corresponding to $\mathbb{P}(W) \subset \mathbb{P}^5$ and the orbit of a 3×3-matrix $A = (w_{i,j})$, with $w_{i,j} \in W^*$. Denote $\mathbb{P}(W') := \sigma(\mathbb{P}(W))$ and let S be the integral cubic surface $\mathbb{P}(W) \cap Y$, which is the vanishing locus in $\mathbb{P}(W)$ of

$g := \det(A) \in S^3W^*$. Then, the surface $\sigma(S) \subset \mathbb{P}(W')$ is the vanishing locus of $g\circ\sigma^{-1}$, which is the determinant of the matrix $\sigma^*A := \left(w_{i,j} \circ \sigma^{-1}\right)$ with coefficients in $(W')^*$.

Two elements $C_1, C_2 \in a^{-1}(p)$ are aCM cubic curves on S. The generators of $I_{C_i/S}$ are given by the three minors of a 3×2-matrix A_i whose two columns are in the span of the columns of A. Then, $\sigma(C_1), \sigma(C_2)$ are aCM curves on $\sigma(S)$: by pullback, the generators of $I_{\sigma(C_i)/\sigma(S)}$ are the minors of σ^*A_i, whose columns are again linear combinations of the columns of σ^*A. Thus, $\check{\sigma}(C_i) := \sigma(C_i) \in M_3(Y)$, for $i = 1, 2$, belongs to the fiber of a over the point defined by $\mathbb{P}(W')$ and $[\sigma^*A]$. $\square$

As a consequence of Lemma 4.4.1, there exists an automorphism σ' of the manifold Z'_Y such that $\sigma' \circ a = a \circ \check{\sigma}$. Moreover, from the previous proof, σ' leaves the divisor D invariant. Since $\Phi : Z'_Y \to Z_Y$ is a contraction of D, σ' descends to an automorphism $\widetilde{\sigma} \in \mathrm{Aut}(Z_Y)$ (see for instance [66, Lemma 3.2]).

By [99, Proposition 4.8], there exists a dominant rational map of degree six

$$\psi : F(Y) \times F(Y) \dashrightarrow Z_Y$$

such that

$$\psi^*(\omega_{Z_Y}) = \mathrm{pr}_1^*(\omega_{F(Y)}) - \mathrm{pr}_2^*(\omega_{F(Y)}) \tag{16}$$

where $\omega_{F(Y)}$ and ω_{Z_Y} are the symplectic forms on $F(Y)$ and Z_Y respectively. The rational map ψ is defined as follows. Let $(l, l') \in F(Y) \times F(Y)$ be a generic element, so that the span $\langle l, l'\rangle$ is a $\mathbb{P}^3$, and let x be a point on l. The plane $\langle x, l'\rangle$ intersects the cubic fourfold Y along the union of the line l' and a conic Q passing through x. Then $C := l \cup_x Q$ is a rational cubic curve contained in Y. We set $\psi(l, l') := u(C) \in Z_Y$, which is well-defined since all reducible cubic curves C arising from different choices of the point $x \in l$ belong to the same fiber of u.

Lemma 4.4.2. *Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold not containing a plane, $\sigma \in \mathrm{PGL}(6)$ such that $\sigma(Y) = Y$ and $\widetilde{\sigma} \in \mathrm{Aut}(Z_Y)$ the automorphism induced by σ on Z_Y. Then $\psi\left(\sigma(l), \sigma(l')\right) = \widetilde{\sigma}\left(\psi(l, l')\right)$.*

Proof. As we recalled, $\psi(l, l') = u(C)$ with $C = l \cup_x Q$, $x \in l$ and $Y \cap \langle x, l'\rangle = l' \cup Q$; moreover, $\widetilde{\sigma}\left(\psi(l, l')\right) = u\left(\check{\sigma}(C)\right)$ by Lemma 4.4.1. In turn, $\psi\left(\sigma(l), \sigma(l')\right) = u(C')$, where $C' = \sigma(l) \cup_{\sigma(x)} Q'$ and $Y \cap \langle \sigma(x), \sigma(l')\rangle = \sigma(l') \cup Q'$. However, the intersection $Y \cap \langle \sigma(x), \sigma(l')\rangle$ coincides with $\sigma\left(Y \cap \langle x, l'\rangle\right)$. As a consequence, $Q' = \sigma(Q)$ and so $C' = \check{\sigma}(C)$. $\square$

Thanks to this equivariance of the map ψ and the relation (16) we deduce that, if $\widetilde{\sigma} \neq \mathrm{id}$ and σ acts non-symplectically on $F(Y)$, then $\widetilde{\sigma}$ is also non-symplectic and of the same order of σ.

Proposition 4.4.3. *Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold not containing a plane. The transcendental lattices of $F(Y)$ and Z_Y have the same rank.*

Proof. Let $\Gamma_\psi \subset F(Y) \times F(Y) \times Z_Y$ be the closure of the graph of the map $\psi : F(Y) \times F(Y) \dashrightarrow Z_Y$ and let V be a desingularization of Γ_ψ. We consider the projections $\pi_F : V \to F(Y) \times F(Y)$, $\pi_Z : V \to Z_Y$ which arise from the inclusion $\Gamma_\psi \subset F(Y) \times F(Y) \times Z_Y$. Let $\mathrm{Tr}_{\mathbb{C}}(F(Y)) \subset H^2(F(Y), \mathbb{C})$ and $\mathrm{Tr}_{\mathbb{C}}(Z_Y) \subset H^2(Z_Y, \mathbb{C})$ be the complexifications of the transcendental lattices of $F(Y)$ and Z_Y respectively. If we define $\mathcal{T} := (\pi_F)_*\left(\pi_Z^*\left(\mathrm{Tr}_{\mathbb{C}}(Z_Y)\right)\right)$, using the relation (16) we deduce:

$$\mathcal{T} \subset \mathrm{Tr}_{\mathbb{C}}(F(Y)) \oplus \mathrm{Tr}_{\mathbb{C}}(F(Y)) \subset H^2(F(Y) \times F(Y), \mathbb{C}).$$

In particular, by the fact that $\psi^*(\omega_{Z_Y}) \in \mathcal{T}$ and the transcendental is the minimal Hodge substructure (in the second cohomology) containing holomorphic two-forms,

$(\mathrm{pr}_i)_*(\mathcal{T}) = \mathrm{Tr}_{\mathbb{C}}(F(Y))$ for $i = 1$ or $i = 2$. This implies that the ranks of $\mathrm{Tr}(Z_Y)$ and $\mathrm{Tr}(F(Y))$ coincide. □

4.4.1. The case of cyclic cubic fourfolds. Let $\sigma \in \mathrm{PGL}(6)$ be the following automorphism of order three:

$$\sigma(x_0 : \ldots : x_5) = (x_0 : \ldots : x_4 : \xi x_5) \tag{17}$$

with $\xi = e^{\frac{2\pi i}{3}}$. We consider the ten-dimensional family $\mathcal{C}$ of smooth hypersurfaces $Y \subset \mathbb{P}^5$ of equations

$$Y : x_5^3 + F_3(x_0, \ldots, x_4) = 0$$

with F_3 an homogeneous polynomial of degree three. Cubic fourfolds $Y \in \mathcal{C}$ are called *cyclic*: they arise as triple coverings of $\mathbb{P}^4$ ramified along the smooth cubic threefold of equation $F_3 = 0$. Any $Y \in \mathcal{C}$ is invariant with respect to σ, thus $\sigma|_Y \in \mathrm{Aut}(Y)$.

Remark 4.4.4. In [17, Example 6.4] it was proved that σ induces a non-symplectic automorphism of order three on the Fano variety of lines $F(Y)$, whose invariant lattice is $T' \cong \langle 6 \rangle$. This allows us to deduce that a very general Y in the family $\mathcal{C}$ does not contain any plane. In fact, if there existed a plane $\pi \subset Y$, it would define an algebraic class in $H^{2,2}(Y)$. In particular, the second Néron–Severi group $\mathrm{NS}_2(Y) = H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y)$ would contain $\langle H^2, \pi \rangle$, where H is an ample line bundle on Y, thus $\mathrm{rk}(\mathrm{NS}_2(Y)) \geq 2$ (references in [67]). By applying the Abel–Jacobi map $H^{2,2}(Y) \to H^{1,1}(F(Y))$ (see [13]), the Picard group of $F(Y)$ would also have at least rank two, while we know that $\mathrm{Pic}(F(Y)) \cong T'$, for a very general choice of Y.

We can then construct the manifold Z_Y, for Y very general in the family $\mathcal{C}$, and consider $\widetilde{\sigma} \in \mathrm{Aut}(Z_Y)$: by our remarks at the end of the previous subsection, it is a non-symplectic automorphism of order three. We obtain the following result as a corollary of Proposition 4.4.3.

Corollary 4.4.5. *Let Y be a cubic fourfold in the family $\mathcal{C}$ not containing a plane and $\widetilde{\sigma} \in \mathrm{Aut}(Z_Y)$ the automorphism induced by $\sigma \in \mathrm{Aut}(Y)$ of the form (17). Then, the invariant lattice of $\widetilde{\sigma}$ is $T \cong \langle 2 \rangle$.*

Proof. As explained in Remark 4.4.4, the very general cubic fourfold $Y \in \mathcal{C}$ is such that $F(Y)$ has transcendental lattice of rank 22. This, together with Proposition 4.4.3, allows us to conclude that the invariant lattice of $\widetilde{\sigma}$ has the same rank of the invariant lattice of the automorphism induced by σ on $F(Y)$, namely one. Therefore, by Proposition 4.1.15, $T \cong \langle 2 \rangle$. □

At the end of this section we will present a more geometric proof of Corollary 4.4.5, using Theorem 4.4.8. In order to do so, we first need to study the fixed locus of the automorphism $\widetilde{\sigma}$.

Let $H \subset \mathrm{Fix}(\sigma)$ be the hyperplane $\{x_5 = 0\} \subset \mathbb{P}^5$. The intersection $Y_H := Y \cap H$ is the smooth cubic threefold defined by $F_3(x_0, \ldots, x_4) = 0$ inside H. We denote by Z_H the image via the map $u : M_3(Y) \to Z_Y$ of the set of twisted cubics contained in Y_H: in [92, Proposition 2.9] the authors prove that Z_H is a Lagrangian subvariety of Z.

Lemma 4.4.6. *Let Y be a cubic fourfold in the family $\mathcal{C}$ not containing a plane. Then Z_H is contained in the fixed locus of $\widetilde{\sigma}$ and any fixed point in $j(Y)$ belongs to Z_H.*

Proof. Let $j(y)$ be a point in the image of the embedding $j : Y \hookrightarrow Z_Y$ such that $\widetilde{\sigma}(j(y)) = j(y)$. In the proof of Lemma 4.4.1 we showed that $\breve{\sigma} \in \mathrm{Aut}(M_3(Y))$ maps the fiber of $u : M_3(Y) \to Z_Y$ over the point $j(y)$ to the fiber over $j(\sigma(y))$.

Therefore, since $\widetilde{\sigma} \circ u = u \circ \check{\sigma}$, we need $\sigma(y) = y$, i.e. $y \in Y_H$. We conclude $\mathrm{Fix}(\widetilde{\sigma}) \cap j(Y) = j(Y_H)$. Clearly, since $H \subset \mathrm{Fix}(\sigma)$, we have $Z_H \subset \mathrm{Fix}(\widetilde{\sigma})$. Moreover, $Z_H \cap j(Y) \cong Y_H$ (see [92, §3]), thus $Z_H \cap j(Y) = j(Y_H)$. $\square$

PROPOSITION 4.4.7. *For Y in the family $\mathcal{C}$ not containing a plane, the fixed locus of the automorphism $\widetilde{\sigma}$ is Z_H.*

PROOF. By Lemma 4.4.6, we need to prove that there are no fixed points outside Z_H, i.e. points $p \in Z_Y \setminus j(Y)$ fixed by $\widetilde{\sigma}$ such that the curves in the fiber $u^{-1}(p)$ are not contained in H. Notice that a point p of this type corresponds to $(\mathbb{P}(W), [A])$, with $\sigma(\mathbb{P}(W)) = \mathbb{P}(W)$ but $\sigma|_{\mathbb{P}(W)} \neq \mathrm{id}$. A vector space $W \in \mathrm{Grass}(\mathbb{C}^6, 4)$ is σ-invariant if and only if it can be written as $W = W_1 \oplus W_\xi$, where we set $W_t := \{w \in W \mid \sigma(w) = tw\}$. The condition $\sigma|_{\mathbb{P}(W)} \neq \mathrm{id}$ implies $W_\xi \neq 0$, therefore W_ξ is the whole one-dimensional eigenspace of $\mathbb{C}^6$ with respect to the eigenvalue ξ of σ, while W_1 is a three-dimensional subspace of the eigenspace of $\mathbb{C}^6$ where σ acts as the identity. Let $y_0, y_1, y_2 \in W^*$ be the dual elements of a basis of W_1. Then, we can take y_0, y_1, y_2, x_5 as coordinates on $\mathbb{P}(W)$, so that the action of σ on it is $\sigma(y_0 : y_1 : y_2 : x_5) = (y_0 : y_1 : y_2 : \xi x_5)$.

We showed in the proof of Lemma 4.4.1 that, for a point p as above, we have $\widetilde{\sigma}(p) = (\mathbb{P}(W), [\sigma^* A])$. Therefore, p is fixed if and only if the matrices A and $\sigma^* A$ define the same $\mathbb{P}^2$ of generalized aCM twisted cubics on the surface $S = \mathbb{P}(W) \cap Y$, whose equation in $\mathbb{P}(W)$ is of the form $g := x_5^3 + f(y_0, y_1, y_2) = 0$, where f is the restriction of F_3 to $\mathbb{P}(W_1)$. Fix a curve C in the fiber $u^{-1}(p)$: its equations in $\mathbb{P}(W)$ are given by the three minors of a 3×2-matrix A_0 with linear entries in W^*. The matrix A_0, up to a change of basis, can only be of eight different types, listed in [65, §1]. Since the curve C lies on S, the polynomial g defining the surface belongs to $I_{C/S}$, i.e. it is a combination of the minors of A_0 (see [65, §3.1]). We recall that A is a linear determinantal representation of the surface S therefore, without loss of generality, it is of the form

$$A = \left(\begin{array}{c|c} A_0 & \begin{matrix} * \\ * \\ * \end{matrix} \end{array} \right)$$

where the last column is uniquely determined by g (up to a combination of the columns of A_0). By [65, §3.1], the matrices A, $\sigma^* A$ define the same $\mathbb{P}^2$ of cubics on S if and only if the columns of $\sigma^* A_0$ belong to the span of the columns of A.

Assume A_0 is of the most general form, i.e. the form $A^{(1)} = \begin{pmatrix} w_0 & w_1 & w_2 \\ w_1 & w_2 & w_3 \end{pmatrix}^t$ of [65, §1], where $w_0, \ldots, w_3$ are suitable coordinates for $\mathbb{P}(W)$: in this case C is a smooth twisted cubic curve.

Let $M := (a_{i,j} \mid b_i)_{\substack{i=0,1,2,3 \\ j=0,1,2}} \in \mathrm{GL}_4(\mathbb{C})$ be the matrix defining the change of coordinates from $\{w_i\}_{i=0}^3$ to $\{y_0, y_1, y_2, x_5\}$. Then

$$A = \left(\begin{array}{cc|c} \sum_{j=0}^2 a_{0,j} y_j + b_0 x_5 & \sum_{j=0}^2 a_{1,j} y_j + b_1 x_5 & \sum_{j=0}^2 c_{0,j} y_j + d_0 x_5 \\ \sum_{j=0}^2 a_{1,j} y_j + b_1 x_5 & \sum_{j=0}^2 a_{2,j} y_j + b_2 x_5 & \sum_{j=0}^2 c_{1,j} y_j + d_1 x_5 \\ \sum_{j=0}^2 a_{2,j} y_j + b_2 x_5 & \sum_{j=0}^2 a_{3,j} y_j + b_3 x_5 & \sum_{j=0}^2 c_{2,j} y_j + d_2 x_5 \end{array} \right)$$

where the parameters $c_{i,j}$ and d_i are determined by g. Once we apply the automorphism we get:

$$\sigma^* A = \left(\begin{array}{cc|c} \sum_{j=0}^2 a_{0,j} y_j + \xi^2 b_0 x_5 & \sum_{j=0}^2 a_{1,j} y_j + \xi^2 b_1 x_5 & \sum_{j=0}^2 c_{0,j} y_j + \xi^2 d_0 x_5 \\ \sum_{j=0}^2 a_{1,j} y_j + \xi^2 b_1 x_5 & \sum_{j=0}^2 a_{2,j} y_j + \xi^2 b_2 x_5 & \sum_{j=0}^2 c_{1,j} y_j + \xi^2 d_1 x_5 \\ \sum_{j=0}^2 a_{2,j} y_j + \xi^2 b_2 x_5 & \sum_{j=0}^2 a_{3,j} y_j + \xi^2 b_3 x_5 & \sum_{j=0}^2 c_{2,j} y_j + \xi^2 d_2 x_5 \end{array} \right)$$

where the first two columns form the matrix $\sigma^* A_0$, whose minors define the curve $\sigma(C) \subset S$.

The first column of $\sigma^* A_0$ is a $\mathbb{C}$-linear combination of the columns of A if and only if the following linear system of 12 equations admits a solution $(h, k, t) \in \mathbb{C}^3$:

$$\begin{cases} a_{0,j} = ha_{0,j} + ka_{1,j} + tc_{0,j} & \text{for } j = 0, 1, 2 \\ a_{1,j} = ha_{1,j} + ka_{2,j} + tc_{1,j} & \text{for } j = 0, 1, 2 \\ a_{2,j} = ha_{2,j} + ka_{3,j} + tc_{2,j} & \text{for } j = 0, 1, 2 \\ \xi^2 b_0 = hb_0 + kb_1 + td_0 \\ \xi^2 b_1 = hb_1 + kb_2 + td_1 \\ \xi^2 b_2 = hb_2 + kb_3 + td_2 \end{cases} \tag{18}$$

Notice that the system made of the last three equations always admits a unique solution, namely $(h, k, t) = (\xi^2, 0, 0)$. In fact, the determinant of its matrix of coefficients is different from zero, because it coincides with the coefficient of x_5^3 in the expression of the determinant of A (and $\sigma^* A$), which needs to be a (non-zero) scalar multiple of g. Now, the triple $(h, k, t) = (\xi^2, 0, 0)$ is a solution for the whole system (18) only if $a_{0,j} = a_{1,j} = a_{2,j} = 0$ $\forall j = 0, 1, 2$, which is not possible since the matrix M needs to be invertible. We conclude that the columns of $\sigma^* A_0$ can never be combinations of the columns of A if A_0 is of the form $A^{(1)}$. The remaining cases, i.e. A_0 of the forms $A^{(2)}, \ldots, A^{(8)}$ of [65, §1], can be discussed in an entirely similar way. $\square$

Let us fix a cubic fourfold $Y \in \mathcal{C}$ not containing a plane and choose a marking $\eta_0 : H^2(Z_Y, \mathbb{Z}) \to L_4 = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -6 \rangle$. We define $\rho := \eta_0 \circ (\widetilde{\sigma})^* \circ \eta_0^{-1} \in O(L_4)$. Following [19] and [18], a $(\rho, \langle 2 \rangle)$-*polarization* of an IHS manifold X of $K3^{[4]}$-type is given by a marking $\eta : H^2(X, \mathbb{Z}) \to L_4$ and an automorphism $g \in \mathrm{Aut}(X)$ of order three such that $g|_{H^{2,0}(X)} = \xi \,\mathrm{id}$ and $\eta \circ g^* = \rho \circ \eta$ (in particular, the invariant lattice of g is isometric to $\langle 2 \rangle$, by Corollary 4.4.5). We consider the following equivalence relation: two $(\rho, \langle 2 \rangle)$-polarized eightfolds (X, η, g), (X', η', g') are equivalent if there exists an isomorphism $f : X \to X'$ such that $\eta' = \eta \circ f^*$ and $g' = f \circ g \circ f^{-1}$. Let $\mathcal{M}_{\langle 2 \rangle}^{\rho, \xi}$ be the set of equivalence classes of $(\rho, \langle 2 \rangle)$-polarized manifolds of $K3^{[4]}$-type and $\mathcal{U} \subset \mathcal{M}_{\langle 2 \rangle}^{\rho, \xi}$ the subset which parametrizes manifolds $(Z_Y, \eta, \widetilde{\sigma})$, with Y cyclic cubic fourfold not containing a plane and σ as in (17).

For any smooth cubic threefold $\mathcal{J} \subset \mathbb{P}^4$, we denote by $Y(\mathcal{J})$ the cubic fourfold which arises as triple covering of $\mathbb{P}^4$ ramified along $\mathcal{J}$. Using Proposition 4.4.7 we can prove the following result.

THEOREM 4.4.8. *Let $\mathcal{J}, \mathcal{J}'$ be smooth cubic threefolds such that $Y(\mathcal{J}), Y(\mathcal{J}')$ do not contain a plane. If $(Z_{Y(\mathcal{J})}, \eta, \widetilde{\sigma})$, $(Z_{Y(\mathcal{J}')}, \eta', \widetilde{\sigma}')$ are equivalent as $(\rho, \langle 2 \rangle)$-polarized manifolds, then $\mathcal{J} \cong \mathcal{J}'$. In particular, $\mathcal{U} \subset \mathcal{M}_{\langle 2 \rangle}^{\rho, \xi}$ has dimension ten.*

PROOF. Consider $(Z_Y, \eta, \widetilde{\sigma}) \in \mathcal{U}$ and let $Z_H \subset Z_Y$ be the fixed locus of $\widetilde{\sigma}$: by [92, Theorem 3.3] and [54, §6.3], Z_H also arises as resolution of the unique singular point of the theta divisor in the intermediate Jacobian $\mathrm{J}(Y_H)$ of the cubic threefold Y_H. This implies that Z_H is a variety of maximal Albanese dimension and $\mathrm{Alb}(Z_H) \cong \mathrm{J}(Y_H)$ (see for instance [57, §1] and references therein). By the Torelli theorem for cubic threefolds ([34, Theorem 13.11]), we conclude that the eightfold Z_Y and the action of the automorphism $\widetilde{\sigma}$ uniquely determine the threefold $\mathcal{J} = Y_H$ up to isomorphisms. The moduli space $\mathcal{C}_3^{sm}$ of smooth cubic threefolds is ten-dimensional and, for $\mathcal{J} \in \mathcal{C}_3^{sm}$ very general, the cubic fourfold $Y(\mathcal{J})$ does not contain a plane (see Remark 4.4.4). Since $\mathcal{M}_{\langle 2 \rangle}^{\rho, \xi}$ is ten-dimensional too by [19, Corollary 6.5], ten is also the dimension of the subset $\mathcal{U} \subset \mathcal{M}_{\langle 2 \rangle}^{\rho, \xi}$. $\square$

Theorem 4.4.8 allows us to provide the following alternative proof of Corollary 4.4.5. The automorphism $\widetilde{\sigma}$ corresponds to an admissible triple $(3, m, a)$, where $m - 1$ coincides with the dimension of the moduli space $\mathcal{U}$ by [**17**, §4]. Since the dimension of $\mathcal{U}$ is ten, we can use Proposition 4.1.15 to deduce $m = 11, a = 0$. Hence the invariant lattice of $\widetilde{\sigma}$ is $T \cong \langle 2 \rangle$.

CHAPTER 5

Non-symplectic involutions on manifolds of $K3^{[n]}$-type

This chapter presents the classification of non-symplectic involutions of manifolds of $K3^{[n]}$-type, thus generalizing to all even dimensions the classification which is already known for $n = 1$ by foundational work of Nikulin [80] on K3 surfaces and for $n = 2$ by work of Beauville [12] and of Boissière, Camere and Sarti [17]. The core of the classification result contained in this chapter comes from Joumaah's PhD thesis [58]; on the other hand, the main result in loc. cit. is not entirely correct, so one of our goals is to prove a revised version of it, in order to obtain the correct classification of non symplectic involutions in Proposition 5.1.5.

The prime $p = 2$ is somewhat different with respect to odd primes, that were considered in the previous chapter, because for $n \geq 2$ it always divides $2(n-1)$, which is the discriminant of the Beauville–Bogomolov–Fujiki lattice $L_n = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$. This divisibility, as already observed in Section 4.1 for odd primes, allows for a richer classification.

The results of this chapter have been obtained in collaboration with Chiara Camere and Andrea Cattaneo.

5.1. Involutions of the lattice L_n

5.1.1. Invariant and co-invariant lattices. Let (X, i) be a pair consisting of an IHS manifold X of $K3^{[n]}$-type and a non-symplectic involution $i \in \mathrm{Aut}(X)$.

By [69, Corollary 9.5], we have a primitive embedding $H^2(X,\mathbb{Z}) \cong L_n \hookrightarrow \Lambda$, where $\Lambda := \Lambda_{24} = U^{\oplus 4} \oplus E_8^{\oplus 2}$ is the Mukai lattice. We now fix $n \geq 2$ and we write $L := L_n$ for the sake of simplicity. We denote by $L^\perp$ the orthogonal complement of L inside Λ. As a consequence of Corollary 1.4.5 we have

$$A_{L^\perp} \cong \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}\left(\frac{1}{2(n-1)}\right). \tag{19}$$

Since $L^\perp \subset \Lambda$ has rank one, we deduce that $L^\perp \cong \langle 2(n-1)\rangle$.

After choosing a marking $\eta : H^2(X,\mathbb{Z}) \to L$, we can consider the action $i^* \in O(L)$. By Proposition 2.3.8, i^* satisfies the following properties: it has spin norm equal to 1 (equivalently, it is orientation preserving) and it induces $\pm\,\mathrm{id}$ on the discriminant group A_L. This means that $\pm i^* \in \widetilde{O}(L)$, where for any lattice N we define the stable orthogonal group $\widetilde{O}(N) = \{g \in O(N) \mid \overline{g} = \mathrm{id} \in O(A_N)\}$ (as in the previous chapters, $\overline{g}$ denotes the isometry of A_N induced by g). Let $\psi = \pm i^*$ be such that $\psi \in \widetilde{O}(L)$. We now show that one between the invariant and the co-invariant lattice of i^* is 2-elementary. Recall from Section 2.5 (see in particular (6)) that the co-invariant lattice of the involution i^* coincides with $\ker(\mathrm{id} + i^*) \subset H^2(X,\mathbb{Z})$ and therefore it is equal to the invariant lattice of $-i^*$, which we denote by $H^2(X,\mathbb{Z})^{-i^*}$.

Proposition 5.1.1. *Let X be a manifold of $K3^{[n]}$-type and $i \in \mathrm{Aut}(X)$ be a non-symplectic involution. Then one of the following holds:*

(*i*) *i^* acts as* id *on the discriminant group of $H^2(X,\mathbb{Z})$ and $H^2(X,\mathbb{Z})^{-i^*}$ is 2-elementary;*
(*ii*) *i^* acts as* $-$id *on the discriminant group of $H^2(X,\mathbb{Z})$ and $H^2(X,\mathbb{Z})^{i^*}$ is 2-elementary.*

PROOF. Consider $\psi \in \widetilde{O}(L)$ as above: in both cases we want to show that the invariant lattice of $-\psi$ is 2-elementary. By Proposition 1.4.7, we can extend ψ to an isometry $\tau \in \widetilde{O}(\Lambda)$ such that $\tau|_{L^\perp} = \mathrm{id}_{L^\perp}$ and with the following properties:

- $L^\psi \subset \Lambda^\tau$;
- $L^{-\psi} \subset \Lambda^{-\tau}$;
- $L^\perp \subset \Lambda^\tau$.

As a consequence, $L^\psi \oplus L^\perp \subset \Lambda^\tau$ is a finite index sublattice and moreover, inside Λ:

$$\Lambda^{-\tau} = (\Lambda^\tau)^\perp \subset (L^\psi \oplus L^\perp)^\perp = (L^\psi)^\perp \cap L \subset L.$$

Hence $L^{-\psi} = \Lambda^{-\tau}$. The invariant and co-invariant lattices of an involution of an even unimodular lattice are 2-elementary by [43, Lemma 3.5]: this concludes the proof. □

With the same notation used above, we remark the following facts.

LEMMA 5.1.2.
(*i*) *The lattice L^ψ is primitively embedded in Λ^τ.*
(*ii*) *The lattice $L^\perp$ is primitively embedded in Λ^τ.*
(*iii*) *The lattices L^ψ and $L^\perp$ are the orthogonal complement of each other in Λ^τ.*

PROOF.
(*i*) As $L^\psi \subset L$ and $L \subset \Lambda$ are primitive, we deduce that $L^\psi \subset \Lambda$ is primitive. The claim follows then from the inclusion $L^\psi \subset \Lambda^\tau$.
(*ii*) This follows from the fact that $L^\perp \subset \Lambda$ is primitive and $L^\perp \subset \Lambda^\tau \subset \Lambda$.
(*iii*) Since $(L^\psi, L^\perp) = 0$, we deduce that $L^\perp \subset (L^\psi)^{\perp_{\Lambda^\tau}}$. Moreover, both $L^\perp$ and $(L^\psi)^{\perp_{\Lambda^\tau}}$ are primitive sublattices of Λ^τ: since they have the same rank, they must coincide. □

LEMMA 5.1.3. *Let X be a manifold of $K3^{[n]}$-type and $i \in \mathrm{Aut}(X)$ be a non-symplectic involution. If (X,i) is deformation equivalent to $(\Sigma^{[n]}, \varphi^{[n]})$, for a $K3$ surface Σ and $\varphi \in \mathrm{Aut}(\Sigma)$, then $i^* \in \widetilde{O}(H^2(X,\mathbb{Z}))$.*

PROOF. As shown in [17, §4], the isomorphism classes of the invariant and co-invariant lattices of a non-symplectic automorphism are deformation invariant. For the pair $(\Sigma^{[n]}, \varphi^{[n]})$, the action of the natural involution on the exceptional divisor of the Hilbert–Chow morphism is trivial by Theorem 2.5.7. Let $\delta \in H^2(\Sigma^{[n]}, \mathbb{Z})$ be the class whose double is the exceptional divisor. From $i^*(2\delta) = 2\delta$ we get that the image of $L + \frac{1}{2(n-1)}\delta \in A_L$ is $L + \frac{1}{2(n-1)}\delta$, hence the action of i^* on A_L is trivial. □

COROLLARY 5.1.4. *Let X be a manifold of $K3^{[n]}$-type and $i \in \mathrm{Aut}(X)$ be a non-symplectic involution. If (X,i) is deformation equivalent to $(\Sigma^{[n]}, \varphi^{[n]})$, for a $K3$ surface Σ and $\varphi \in \mathrm{Aut}(\Sigma)$, then the co-invariant lattice $H^2(X,\mathbb{Z})^{-i^*}$ is 2-elementary.*

5.1.2. Discriminant groups. We explain in this section the inaccuracies in the proof of [58, Proposition 5.1.1] and provide the necessary corrections. Adopting our notation, which differs from the one used by Joumaah, let X be a manifold of $K3^{[n]}$-type with a non-symplectic involution $i \in \mathrm{Aut}(X)$. Let $T = L^{i^*}$, $S = L^{-i^*}$ be, respectively, the invariant and co-invariant lattices of the involution. The aim of

[58, Proposition 5.1.1] is to classify the discriminant groups A_T, A_S. In order to do so, Joumaah considers the isotropic subgroup $M \subset A_T \oplus A_S$, which is isomorphic to $\frac{L}{T\oplus S} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^a$ for some $a \geq 0$, and its projections $M_T := p_T(M) \subset A_T$, $M_S := p_S(M) \subset A_S$. In particular, $M \cong M_T \cong M_S$ as groups, $\frac{M^\perp}{M} \cong A_L$ and we have an anti-isometry $\gamma := p_S \circ p_T^{-1} : M_T \to M_S$ (see Section 1.4; we defined γ in (5)).

The following proposition provides the complete classification for the discriminant groups A_T, A_S.

Proposition 5.1.5. *Let X be a manifold of $K3^{[n]}$-type, for $n \geq 2$, and let $\alpha \geq 1$ and β odd such that $2(n-1) = 2^\alpha \beta$. Let $G \subset \mathrm{Aut}(X)$ be a group of order 2 acting non-symplectically on X. Denote by $T, S \subset L := L_n$, respectively, the invariant and co-invariant sublattices for the action of G, with $\frac{L}{T\oplus S} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^a$ for some $a \geq 0$. Then one of the following cases holds:*

(i) $A_T \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus a} \oplus \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}$, $A_S \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus a}$ *or vice versa;*

(ii) $a \geq 1$, $A_T \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}$, $A_S \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus a+1}$ *or vice versa;*

(iii) $\alpha = 1$, $a = 0$, $A_T \cong \frac{\mathbb{Z}}{\beta\mathbb{Z}}$, $A_S \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$ *or vice versa.*

Proof. Let i be the non-symplectic involution generating the group G and, as before, let $\psi = \pm i^*$ be the isometry such that $\psi \in \widetilde{O}(L)$, i.e. $\overline{\psi} = \mathrm{id} \in O(A_L)$. Let T, S be the invariant and co-invariant lattices of i^*, as in the statement. If $\psi = i^*$, then $T = L^\psi$, $S = L^{-\psi}$; if instead $\psi = -i^*$, then $T = L^{-\psi}$, $S = L^\psi$. As we showed in Proposition 5.1.1, the lattice $L^{-\psi}$ is 2-elementary, therefore $A_{L^{-\psi}}$ coincides with its Sylow 2-subgroup (it actually coincides with its 2-torsion part). Moreover, by using the same arguments of Lemma 4.1.4, we have $A_{L^\psi} = (A_{L^\psi})_2 \oplus \frac{\mathbb{Z}}{\beta\mathbb{Z}}$. It is now easy to check that the discriminant groups of the lattices L^ψ, $L^{-\psi}$ satisfy the same classification presented in Proposition 4.1.5 for automorphisms of odd prime order. □

Remark 5.1.6. Assume that $i^* \in \widetilde{O}(L)$, so that $\psi = i^*$. As in the proof of Proposition 4.1.5, let $H := (A_T)_2 \oplus A_S \subset A_T \oplus A_S$ and denote by $H[2] \subset H$ the subgroup of elements of order 2 in H. If $\alpha > 1$, Joumaah correctly highlighted in his proof that the index $[H : H[2]]$ needs to be $2^{\alpha-1}$ and therefore $H \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{2a} \oplus \frac{\mathbb{Z}}{2^\alpha\mathbb{Z}}$. However, contrary to what he stated, this does not necessarily imply that $H = M_T \oplus M_S \oplus \frac{\mathbb{Z}}{2^\alpha\mathbb{Z}}$, from which he inferred $A_T \cong M_T \oplus A_L, A_S = M_S$ as the only possibility for the discriminant groups. To show this, we exhibit two lattices T, S which are the invariant and co-invariant lattices of a non-symplectic involution of a manifold of $K3^{[3]}$-type and whose discriminant groups are in contrast with [58, Proposition 5.1.1].

For $n = 3$ we have $2(n-1) = 4$, meaning $\alpha = 2$, $\beta = 1$. In [55] the authors described a 20-dimensional family of manifolds of $K3^{[3]}$-type, called *double EPW cubes*, with polarization of degree four and divisibility two (see [55, Proposition 5.3]), whose members are always endowed with a non-symplectic involution i. As a consequence, the invariant lattice of i is $T \cong \langle 4\rangle$ and the co-invariant lattice is $S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$. In particular, the discriminant groups are

$$A_T = \langle t\rangle \cong \frac{\mathbb{Z}}{4\mathbb{Z}}\left(\frac{1}{4}\right), \qquad A_S = \langle s_1, s_2\rangle \cong \frac{\mathbb{Z}}{2\mathbb{Z}}\left(-\frac{1}{2}\right) \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}\left(-\frac{1}{2}\right).$$

In this case $H = A_T \oplus A_S$, since $\beta = 1$. Moreover

$$16 = |A_T \oplus A_S| = [L : T \oplus S]^2\, |A_L| = 2^{2a} \cdot 4$$

therefore $a = 1$. Looking at the discriminant quadratic forms on A_T and A_S, the only possible choice for the subgroups of order two $M_T \subset A_T$ and $M_S \subset A_S$, with

$M_T \cong M_S(-1)$, is the following:

$$M_T = \langle 2t \rangle \subset A_T, \quad M_S = \langle s_1 + s_2 \rangle \subset A_S$$

which implies $M = \langle 2t + s_1 + s_2 \rangle \subset A_T \oplus A_S$. One can check, by computing $M^\perp \subset A_T \oplus A_S$, that $\frac{M^\perp}{M} \cong A_L \cong \frac{\mathbb{Z}}{4\mathbb{Z}}\left(-\frac{1}{4}\right)$.

This is therefore a case where $\alpha = 2 > 1$ and $[H : H[2]] = 2 = 2^{\alpha-1}$. However, it is not possible to write $H = A_T \oplus A_S$ as $H = M_T \oplus M_S \oplus \frac{\mathbb{Z}}{2^\alpha\mathbb{Z}}$ (even though they are isomorphic as groups) and it is not true that $A_T \cong M_T \oplus A_L$, $A_S = M_S$.

REMARK 5.1.7. In the case of manifolds of $K3^{[2]}$-type, it was proved in [17, Lemma 8.1] (extending results from [21, §6]) that the discriminant groups can only be $A_S \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus a}$, $A_T \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus a+1}$ or vice versa. This is coherent with the classification of Proposition 5.1.5 (if $n = 2$ we have $2(n-1) = 2$, hence $\alpha = \beta = 1$).

5.2. Existence of involutions

In this section we show that the lattice-theoretic conditions of Proposition 5.1.1 are actually sufficient to give rise to a geometric realization. First, we prove that every 2-elementary sublattice of $L = L_n$ is the invariant (or co-invariant) lattice of some involution of L, and finally that we can generically lift this abstract involution to an involution of a manifold of $K3^{[n]}$-type.

PROPOSITION 5.2.1. *Let S be an even 2-elementary lattice, primitively embedded into an even lattice N. Then $\mathrm{id}_{S^\perp} \oplus (-\mathrm{id}_S)$ (respectively, $(-\mathrm{id}_{S^\perp}) \oplus \mathrm{id}_S$) extends to an isometry $\rho \in \widetilde{O}(N)$ (respectively, $-\rho \in \widetilde{O}(N)$).*

PROOF. By [79, Theorem 1.1.2], we can primitively embed N into an even unimodular lattice V of high enough rank. We fix such a primitive embedding and consider the orthogonal complements $N^{\perp_V}$ and $S^{\perp_V}$ of N and S inside V. Obviously, V is an overlattice of $S \oplus S^{\perp_V}$. We want to show that $\alpha := \mathrm{id}_{S^{\perp_V}} \oplus (-\mathrm{id}_S)$ extends to V. A completely analogous proof will show that also $(-\mathrm{id}_{S^{\perp_V}}) \oplus \mathrm{id}_S$ extends, as in the statement. Let $M_V \cong V/(S \oplus S^{\perp_V})$ be the isotropy subgroup of $A_S \oplus A_{S^{\perp_V}}$ and let p_S, $p_{S^{\perp_V}}$ be the two projections to A_S and $A_{S^{\perp_V}}$:

$$\begin{tikzcd} M_V = V/(S \oplus S^{\perp_V}) \arrow[d, "p_{S^{\perp_V}}"] \arrow[drr, "p_S"] & & \subset A_S \oplus A_{S^{\perp_V}} \\ M_{S^{\perp_V}} \subset A_{S^{\perp_V}} & & M_S \subset A_S. \end{tikzcd}$$

Since V is unimodular, we have $M_{S^{\perp_V}} = A_{S^{\perp_V}}$ and $M_S = A_S$ by Lemma 1.4.1. As before, let $\gamma : A_{S^{\perp_V}} \to A_S$ be the anti-isometry given by $p_S \circ (p_{S^{\perp_V}})^{-1}$. By Proposition 1.4.7, the existence of an extension of α to V is equivalent to the commutativity of the diagram

$$\begin{tikzcd} A_{S^{\perp_V}} \arrow[r, "\gamma"] \arrow[d, "\overline{\mathrm{id}}_{S^{\perp_V}}"] & A_S \arrow[d, "-\overline{\mathrm{id}}_S"] \\ A_{S^{\perp_V}} \arrow[r, "\gamma"] & A_S. \end{tikzcd}$$

This diagram is commutative because $-\gamma = \gamma$, since S is 2-elementary, hence we get the extension $\widetilde{\alpha} \in O(V)$ of α to V.

As $S^{\perp_N} \oplus N^{\perp_V} \subset S^{\perp_V}$, we deduce that $N^{\perp_V}$ is invariant for the action of $\widetilde{\alpha}$. Let ρ be the restriction $\widetilde{\alpha}|_N$. Since $\rho \oplus \mathrm{id}_{N^{\perp_V}}$ extends to $\widetilde{\alpha} \in O(V)$, we have a commutative diagram

$$\begin{tikzcd} A_N \arrow[r, "\pi"] \arrow[d, "\overline{\rho}"'] & A_{N^{\perp_V}} \arrow[d, "\overline{\mathrm{id}}_{N^{\perp_V}}"] \\ A_N \arrow[r, "\pi"] & A_{N^{\perp_V}} \end{tikzcd}$$

where $\pi := p_{N^{\perp_V}} \circ (p_N)^{-1}$. Hence $\overline{\rho} = \mathrm{id}_{A_N}$, i.e. $\rho \in \widetilde{O}(N)$. □

REMARK 5.2.2. This is in some sense a converse of [43, Lemma 3.5]. See also [38, Proposition 1.5.1].

We now come to the second part of the section. First, we recall some results on lattice-polarized manifolds of $K3^{[n]}$-type.

Let T be a hyperbolic lattice which admits a primitive embedding $j : T \hookrightarrow L$, with $\mathrm{rk}(T) \leq 20$. We identify T with the sublattice $j(T) \subset L$ and we denote by S its orthogonal complement in L. Following [58, §4.1], we say that T is *admissible* if it is the invariant lattice of a monodromy operator $\rho \in \mathrm{Mon}^2(L)$ of order two. In particular, T and S are as in Proposition 5.1.5, therefore one of them is 2-elementary. This implies, by Proposition 5.2.1, that ρ is the unique extension of $\mathrm{id}_T \oplus (-\mathrm{id}_S)$ to L.

Let X be a manifold of $K3^{[n]}$-type and $i \in \mathrm{Aut}(X)$ be a non symplectic involution acting on it. Joumaah says that the pair (X, i) is *of type* T if it admits a (ρ, T)-polarization, i.e. a marking $\eta : H^2(X, \mathbb{Z}) \to L$ such that $\eta \circ i^* = \rho \circ \eta$ (see Section 4.4.1). If (X, i) and (X', i') are two pairs of type T, they are said to be isomorphic if there exists an isomorphism $f : X \to X'$ such that $i' = f \circ i \circ f^{-1}$. The monodromy operators $f^* \in \mathrm{Mon}^2(L)$ induced by these isomorphisms of pairs are the isometries contained in

$$\mathrm{Mon}^2(L, T) := \left\{ g \in \mathrm{Mon}^2(L) \mid g \circ \rho = \rho \circ g \right\} = \left\{ g \in \mathrm{Mon}^2(L) \mid g(T) = T \right\}.$$

In particular, for any $g \in \mathrm{Mon}^2(L, T)$ we have that $g|_T \in O(T)$ and $g|_S \in O(S)$. We can then define the following subgroups:

$$\Gamma_T := \left\{ g|_T \mid g \in \mathrm{Mon}^2(L, T) \right\} \subset O(T), \qquad \Gamma_S := \left\{ g|_S \mid g \in \mathrm{Mon}^2(L, T) \right\} \subset O(S).$$

Notice that local deformations of a pair (X, i) of type T are parametrized by $H^{1,1}(X)^{i^*}$ (more details on this are provided in [12, Theorem 2] and [17, §4]).

Inside the moduli space $\mathcal{M}_L$ defined in Section 2.3, let $\mathcal{M}_{T,\rho}$ be the subspace of (ρ, T)-polarized marked manifolds $(X, \eta) \in \mathcal{M}_L$ (notice that $\mathcal{M}_{T,\rho} = \mathcal{M}_T^{\rho, -1}$, using the notation of Section 4.4.1). Since the symplectic form ω_X generating $H^{2,0}(X)$ is orthogonal to the Néron–Severi group (which contains T), for any $(X, \eta) \in \mathcal{M}_{T,\rho}$ the period point $\eta(H^{2,0}(X))$ belongs to

$$\Omega_S := \{ \kappa \in \mathbb{P}(S \otimes \mathbb{C}) \mid (\kappa, \kappa) = 0, (\kappa, \overline{\kappa}) > 0 \} = \Omega_L \cap \mathbb{P}(S \otimes \mathbb{C}).$$

Moreover, by [58, Proposition 4.6.7], the period map (see Section 2.3) restricts to a holomorphic surjective morphism

$$\mathcal{P} : \mathcal{M}_{T,\rho} \longrightarrow \Omega_S^0 := \Omega_S \setminus \bigcup_{\delta \in \Delta(S)} (\delta^\perp \cap \Omega_S),$$

where $\Delta(S)$ is the set of wall divisors (i.e. primitive integral MBM classes) contained in S. This restriction is equivariant with respect to the action of $\mathrm{Mon}^2(L, T)$, hence we also obtain a surjection

$$\mathcal{P} : \mathcal{M}_{T,\rho} / \mathrm{Mon}^2(L, T) \longrightarrow \Omega_S^0 / \Gamma_S.$$

PROPOSITION 5.2.3. *Let $\rho \in O(L)$ be an involution whose invariant lattice T is hyperbolic with $\mathrm{rk}(T) \leq 20$. Assume also that $\pm\rho \in \widetilde{O}(L)$. Then there exists a marked manifold (X, η) of $K3^{[n]}$-type with an involution i such that $\eta \circ i^* = \rho \circ \eta$.*

PROOF. Let $S \subset L$ be the co-invariant lattice of ρ, i.e. the orthogonal complement of T. By [**17**, Proposition 5.3], the very general point $\omega \in \Omega_S$ is the image under the period map of a T-polarized marked manifolds of $K3^{[n]}$-type (X, η) with $\mathrm{NS}(X) = \eta^{-1}(T)$. We can then consider $\alpha := \eta^{-1} \circ \rho \circ \eta \in O(H^2(X, \mathbb{Z}))$, which is an involution, and we observe that:

- (i) α induces a Hodge isometry on $H^2(X, \mathbb{C})$, since the point $\eta(H^{2,0}(X))$ is invariant for the action of ρ on Ω_S;
- (ii) α is effective, because the equality $\mathrm{NS}(X) = \eta^{-1}(T) = \eta^{-1}(L^\rho)$ implies that there is an α-fixed Kähler (even ample) class on X;
- (iii) $\pm\rho \in \widetilde{O}(L)$.

Hence, α is a monodromy operator by Proposition 2.3.10 and, by Theorem 2.3.11, there exists $i \in \mathrm{Aut}(X)$ such that $i^* = \alpha$. Since the map $\mathrm{Aut}(X) \to O(H^2(X, \mathbb{Z}))$ is injective for manifolds of $K3^{[n]}$-type (see Theorem 2.5.2), this automorphism i is both unique and an involution. It is then straightforward to check that $\eta \circ i^* = \rho \circ \eta$. □

5.3. Geography in small dimensions

The aim of this section is to make some remarks on which families of large dimensions one can expect from the results of the previous section. We first classify the admissible invariant lattices of rank one and two, and then we describe the geography of the cases for manifolds of $K3^{[n]}$-type, $n \leq 5$. We conclude by recalling known examples of non-symplectic involutions and by presenting some new ones.

5.3.1. Invariant sublattices of rank one and two. Let T, S be the invariant and co-invariant lattices of a non-symplectic involution of a manifold of $K3^{[n]}$-type. As we saw in Proposition 5.1.1, either S or T is 2-elementary, depending on the action of the involution on the discriminant group of L (which is id or $-\mathrm{id}$ respectively). Assume that S is 2-elementary and consider it embedded in the Mukai lattice Λ (the case where T is 2-elementary is similar). Starting from the signature of $S^{\perp_\Lambda}$, we can use Theorem 1.3.11 to deduce the possible isometry classes for $S^{\perp_\Lambda}$. As observed in Lemma 5.1.2, we have that T is the orthogonal complement in $S^{\perp_\Lambda}$ of $L^\perp$: since we know this last explicitly (see (19)), we can use Theorem 1.4.4 to classify all primitive embeddings $L^\perp \hookrightarrow S^{\perp_\Lambda}$ and to compute, in each case, the discriminant group of the orthogonal complement, i.e. A_T.

5.3.1.1. *Invariant sublattice of rank one.* In this subsection we prove the following proposition, which describes the pairs T and S that can occur when $\mathrm{rk}(T) = 1$.

PROPOSITION 5.3.1. *Let X be a manifold of $K3^{[n]}$-type for some $n \geq 2$, and let i be a non-symplectic involution. Assume that the invariant lattice $T \subset H^2(X, \mathbb{Z})$ has rank one, then one of the following holds:*

- (i) *if i^* acts as id on $A_{H^2(X,\mathbb{Z})}$, then -1 is a quadratic residue modulo $n-1$ and*
$$T \cong \langle 2(n-1) \rangle, \qquad S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle \oplus \langle -2 \rangle;$$
- (ii) *if i^* acts as $-\mathrm{id}$ on $A_{H^2(X,\mathbb{Z})}$, then $T \cong \langle 2 \rangle$ and*
  - (a) *either $S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1) \rangle \oplus \langle -2 \rangle$;*
  - (b) *or $n \equiv 0 \pmod 4$ and*
$$S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \begin{pmatrix} -\frac{n}{2} & n-1 \\ n-1 & -2(n-1) \end{pmatrix}.$$

Proof. This result generalizes Proposition 3.4.1, which holds for non-natural involutions of Hilbert schemes of points on a generic projective $K3$ surface.

We deal first with the case where T, S are the invariant and co-invariant lattices of an involution whose action on the discriminant A_L is the identity. This means that S is 2-elementary and that $T \oplus L^\perp \subset S^{\perp_\Lambda}$. Since both T and $L^\perp$ have signature $(1,0)$, we deduce that $S^{\perp_\Lambda}$ has signature $(2,0)$. By [35, Table 15.1], there is only one possible choice for $S^{\perp_\Lambda}$, which embeds in Λ in a unique way by [79, Theorem 1.1.2]: this is enough to claim that there is only one possible choice for S, up to isometries, which explicitly is

$$S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle \oplus \langle -2 \rangle, \qquad S^{\perp_\Lambda} = \langle 2 \rangle \oplus \langle 2 \rangle.$$

We have then to look at how $L^\perp \cong \langle 2(n-1) \rangle$ embeds primitively in $S^{\perp_\Lambda}$. A pair (x,y) gives the coordinates of a primitive vector in $S^{\perp_\Lambda} = \langle 2 \rangle \oplus \langle 2 \rangle$ of square $2(n-1)$ if and only if $\gcd(x,y) = 1$ and $x^2 + y^2 = n-1$. Moreover, the isometry group of $S^{\perp_\Lambda}$ acts on these coordinates either by permutation or by exchanging sign. The orthogonal complement of $L^\perp$ in $S^{\perp_\Lambda}$, which is T, is then a lattice isometric to $\langle 2(n-1) \rangle$, generated by $(-y, x)$. Notice that there exist two coprime integers x, y such that $x^2 + y^2 = n-1$ if and only if -1 is a quadratic residue modulo $n-1$ (to see this, combine [56, Proposition 5.1.1] and [81, Theorem 3.20]).

We now consider the case where the action of i^* on A_L is $-\mathrm{id}$. We have that T is 2-elementary of signature $(1,0)$, hence $T \cong \langle 2 \rangle$. It follows that T embeds in a unique way in the Mukai lattice, with orthogonal complement

$$T^{\perp_\Lambda} \cong U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle.$$

We now want to describe the different embeddings of $L^\perp \cong \langle 2(n-1) \rangle$ in $T^{\perp_\Lambda}$. Since $T^{\perp_\Lambda}$ is unique in its genus up to isometries (see Theorem 1.3.3), by Theorem 1.4.4 we have only two possibilities: they correspond to the two possible choices of a subgroup of $A_{T^{\perp_\Lambda}} \cong \mathbb{Z}/2\mathbb{Z}$. Choosing the trivial subgroup, we see that the orthogonal complement of $L^\perp$ in $T^{\perp_\Lambda}$, i.e. S, has discriminant group

$$A_S = \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}} \left(-\frac{1}{2(n-1)} \right) \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \left(-\frac{1}{2} \right),$$

and signature $(2,20)$. By Theorem 1.3.3, there exists only one lattice with these invariants, up to isometries, which is

$$S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1) \rangle \oplus \langle -2 \rangle.$$

The last possibility corresponds to the choice of the whole $A_{T^{\perp_\Lambda}}$, but in this case we must have $n \equiv 0 \pmod 4$. This leads us to

$$A_S = \frac{\mathbb{Z}}{(n-1)\mathbb{Z}} \left(-\frac{n}{2(n-1)} \right),$$

where S has again signature $(2,20)$. By the same argument as above, there exists only one isometry class of lattices in this genus. A representative, which can be computed by applying [44, Proposition 3.6], is

$$S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \begin{pmatrix} -\frac{n}{2} & n-1 \\ n-1 & -2(n-1) \end{pmatrix}. \qquad \square$$

Remark 5.3.2. The three cases of Proposition 5.3.1 can be distinguished also by looking at the generator $t \in H^2(X, \mathbb{Z})$ of the invariant lattice T. In fact, by [44, Proposition 3.6], we have that:

- in case (1), t has square $2(n-1)$ and divisibility $n-1$;
- in case (2a), t has square 2 and divisibility 1;
- in case (2b), t has square 2 and divisibility 2.

Recall from Proposition 3.4.3 that, by the global Torelli theorem for IHS manifolds, the existence of a primitive ample class $t \in \mathrm{NS}(X)$ with one of these three combinations of square and divisibility is sufficient to prove the existence of a non-symplectic involution on X, whose invariant lattice is $T = \langle t \rangle$.

5.3.1.2. *Invariant sublattice of rank two.* The aim of this subsection is to provide some results for $\mathrm{rk}(T) = 2$. In particular, we describe the discriminant groups of the invariant and co-invariant lattices in complete generality, but we address the problem of their realization and uniqueness only for $n \leq 5$.

Assume that $\mathrm{rk}(T) = 2$, so that the signature of T is $(1,1)$. We first consider the case where the induced action on A_L is the identity, hence S is a 2-elementary lattice of signature $(2,19)$ and $S^{\perp_\Lambda}$ is 2-elementary of signature $(2,1)$. It follows from [79, Theorem 1.1.2] that $S^{\perp_\Lambda}$ has a unique embedding in the Mukai lattice, up to isometries. By Theorem 1.3.11 we have then two possibilities:

$$S^{\perp_\Lambda} = U \oplus \langle 2 \rangle \qquad \text{or} \qquad S^{\perp_\Lambda} = U(2) \oplus \langle 2 \rangle$$

which are both unique in their genera (up to isometries) by Theorem 1.3.3. We start with $S^{\perp_\Lambda} = U \oplus \langle 2 \rangle$, and look for a primitive embedding of $L^\perp = \langle 2(n-1) \rangle$ in $S^{\perp_\Lambda}$. By Theorem 1.4.4 we need to consider pairs of isomorphic subgroups in $A_{L^\perp}$ and $A_{S^{\perp_\Lambda}} = \frac{\mathbb{Z}}{2\mathbb{Z}}\left(\frac{1}{2}\right)$. In particular, for the choice of the trivial subgroup we have

$$A_T = \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}\left(-\frac{1}{2(n-1)}\right) \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}\left(\frac{1}{2}\right).$$

A possible realization for this lattice T is given by $T = \langle -2(n-1) \rangle \oplus \langle 2 \rangle$; if $n \leq 5$, this is the only isometry class in the genus by Theorem 1.3.3.

The other possibility is to consider the subgroup of $A_{L^\perp}$ generated by the class of $n-1$: in order for it to have the same discriminant form of $A_{S^{\perp_\Lambda}}$ we need $n \equiv 2 \pmod 4$, and in this case we have

$$A_T = \frac{\mathbb{Z}}{(n-1)\mathbb{Z}}\left(\frac{n-2}{2(n-1)}\right).$$

A lattice T with this discriminant form and signature $(1,1)$ is the following:

$$T = \begin{pmatrix} -2h & k \\ k & 2 \end{pmatrix}$$

where we write $n-1 = k^2 + 4h$, with k, h non-negative integers and k maximal. This is the only isometry class in the genus of T if $n \leq 17$, by Theorem 1.3.3. For $n = 2$, this lattice is isometric to U.

If instead we consider $S^{\perp_\Lambda} = U(2) \oplus \langle 2 \rangle$, then we have more possibilities because there are more subgroups inside its discriminant group, which is

$$A_{S^{\perp_\Lambda}} = \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 3}, \quad \text{with quadratic form } q_{S^{\perp_\Lambda}} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

It is easy to see that we can discard the choice corresponding to the trivial subgroup, as it gives rise to a lattice T of length 4, hence the only relevant subgroups of $A_{S^{\perp_\Lambda}}$ are those of order two. Up to isomorphism, we have the two following possibilities.

(*i*) The subgroup is $\langle (0,0,1) \rangle \subset A_{S^{\perp_\Lambda}}$ with $q((0,0,1)) = 1/2$. This case can occur only if $n \equiv 2 \pmod 4$, and gives

$$A_T = \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}, \quad \text{with quadratic form } q_T = \begin{pmatrix} \frac{n-2}{2(n-1)} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

For $n = 2$, the lattice $U(2)$ realizes this genus; for $n = 6$, we can consider the lattice whose bilinear form is given by the matrix $\begin{pmatrix} 2 & 4 \\ 4 & -2 \end{pmatrix}$.

(ii) The subgroup is $\langle v \rangle \cong \mathbb{Z}/2\mathbb{Z} \subset A_{S^{\perp_\Lambda}}$, for an element $v \neq (0,0,1)$ such that $q(v) = (n-1)/2$. This case gives
$$A_T = \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}} \left(-\frac{1}{2(n-1)} \right) \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \left(\frac{1}{2} \right).$$
A possible realization for this lattice is given by $T = \langle -2(n-1) \rangle \oplus \langle 2 \rangle$; if $n \leq 5$, this is the only isometry class in the genus by Theorem 1.3.3.

For $n \leq 5$, we summarize these results as follows.

PROPOSITION 5.3.3. *Let X be a manifold of $K3^{[n]}$-type for $2 \leq n \leq 5$, and let i be a non-symplectic involution. Assume that the invariant lattice $T \subset H^2(X,\mathbb{Z})$ has rank two and that i^* acts as* id *on $A_{H^2(X,\mathbb{Z})}$, then one of the following holds:*

(i) *$T \cong \langle 2 \rangle \oplus \langle -2(n-1) \rangle$ and $S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle$;*
(ii) *$T \cong \langle 2 \rangle \oplus \langle -2(n-1) \rangle$ and $S \cong U \oplus U(2) \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle$;*
(iii) *$n = 2$, $T \cong U$ and $S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle$;*
(iv) *$n = 2$, $T \cong U(2)$ and $S \cong U \oplus U(2) \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle$.*

We assume now that the action on the discriminant group is $-\,$id. In this case, T is 2-elementary of signature $(1,1)$, so $T^{\perp_\Lambda}$ is also 2-elementary and its signature is $(3,19)$. This implies that S (which is a sublattice of $T^{\perp_\Lambda}$) has signature $(2,19)$. By Theorem 1.3.11, there exist only three 2-elementary lattices of signature $(1,1)$, namely U, $U(2)$ and $\langle 2 \rangle \oplus \langle -2 \rangle$. Every such lattice, by Theorem 1.4.4, embeds in the Mukai lattice in a unique way, hence the orthogonal complement is uniquely determined too. We analyse the three cases separately: in each of them, there is only one isometry class in the genus of S by Theorem 1.3.3.

Assume first that $T = U$. This implies that $T^{\perp_\Lambda} \cong U^{\oplus 3} \oplus E_8^{\oplus 2}$, which is unimodular. So $L^\perp \cong \langle 2(n-1) \rangle$ embeds in an essentially unique way in $T^{\perp_\Lambda}$ and its orthogonal complement, i.e. S, is
$$S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1) \rangle.$$

Assume now that $T = U(2)$. Then $T^{\perp_\Lambda} = U(2) \oplus U^{\oplus 2} \oplus E_8^{\oplus 2}$ has discriminant group
$$A_{T^{\perp_\Lambda}} = \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}, \quad \text{with quadratic form } q_{T^{\perp_\Lambda}} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$
As before, we look at the cyclic subgroups of $A_{T^{\perp_\Lambda}}$: a direct computation gives rise to two different cases (see also Example 1.4.6).

(i) If we choose the trivial subgroup we have $A_S = \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$, with quadratic form
$$q_S = \begin{pmatrix} -\frac{1}{2(n-1)} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}.$$
We conclude
$$S = U \oplus U(2) \oplus E_8^{\oplus 2} \oplus \langle -2(n-1) \rangle.$$

(ii) If $n \equiv 1, 3 \pmod 4$, we can choose a subgroup of order two and we have
$$A_S = \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}} \left(-\frac{1}{2(n-1)} \right),$$
which corresponds to
$$S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1) \rangle.$$

Assume finally that $T = \langle 2\rangle \oplus \langle -2\rangle$. Then $T^{\perp_\Lambda} = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle 2\rangle \oplus \langle -2\rangle$, whose discriminant group is

$$A_{T^{\perp_\Lambda}} = \frac{\mathbb{Z}}{2\mathbb{Z}}\left(\frac{1}{2}\right) \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}\left(-\frac{1}{2}\right).$$

The same kind of computations yield three cases:

(i) The discriminant group is

$$A_S = \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}\left(-\frac{1}{2(n-1)}\right) \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}\left(\frac{1}{2}\right) \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}\left(-\frac{1}{2}\right),$$

which corresponds to

$$S = U \oplus E_8^{\oplus 2} \oplus \langle 2\rangle \oplus \langle -2\rangle \oplus \langle -2(n-1)\rangle.$$

(ii) If $n \equiv 0, 2 \pmod 4$ we can have

$$A_S = \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}\left(-\frac{1}{2(n-1)}\right),$$

which is realized by

$$S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle.$$

(iii) If $n \equiv 1 \pmod 4$ we can have

$$A_S = \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}\left(\frac{n-2}{2(n-1)}\right).$$

For $n = 5$, a representative of the unique isometry class in this genus is

$$S = U \oplus E_8^2 \oplus \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

The next proposition summarizes all possible pairs of lattices T, S corresponding to involutions whose action on the discriminant group A_L is $-\mathrm{id}$, for $n \leq 5$.

Proposition 5.3.4. *Let X be a manifold of $K3^{[n]}$-type for $2 \leq n \leq 5$, and let i be a non-symplectic involution. Assume that the invariant lattice $T \subset H^2(X, \mathbb{Z})$ has rank two and that i^* acts as $-\mathrm{id}$ on $A_{H^2(X,\mathbb{Z})}$, then one of the following holds:*

(i) *$T \cong U$ and $S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$;*
(ii) *$T \cong U(2)$ and $S \cong U \oplus U(2) \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$;*
(iii) *$T \cong \langle 2\rangle \oplus \langle -2\rangle$ and $S \cong U \oplus E_8^{\oplus 2} \oplus \langle 2\rangle \oplus \langle -2\rangle \oplus \langle -2(n-1)\rangle$;*
(iv) *$n \in \{3, 5\}$, $T \cong U(2)$ and $S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$;*
(v) *$n \in \{2, 4\}$, $T \cong \langle 2\rangle \oplus \langle -2\rangle$ and $S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$;*
(vi) *$n = 5$, $T \cong \langle 2\rangle \oplus \langle -2\rangle$ and $S \cong U \oplus E_8^2 \oplus \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.*

Remark 5.3.5. For $n = 2$, the isometries id and $-\mathrm{id}$ of $A_L \cong \mathbb{Z}/2\mathbb{Z}$ coincide, hence Proposition 5.3.3 and Proposition 5.3.4 give the same classification (to check this, recall that $U(2) \oplus \langle -2\rangle \cong \langle 2\rangle \oplus \langle -2\rangle \oplus \langle -2\rangle$ by Theorem 1.3.11).

5.3.2. Deformation types for families of large dimensions. The lattice computations of Section 5.3.1.1 and Section 5.3.1.2 allow us to determine all moduli spaces $\mathcal{M}_{T,\rho}$, for T an admissible invariant sublattice of rank one or two inside L (recall the definitions from Section 5.2). By construction, the moduli spaces $\mathcal{M}_{T,\rho}$ arise as subspaces of the complex space $\mathcal{M}_L$, which parametrizes marked IHS manifolds of $K3^{[n]}$-type (see Section 2.3). The following fact was remarked in [5, Theorem 9.5] for $K3$ surfaces, and it can be easily generalized to manifolds of $K3^{[n]}$-type.

LEMMA 5.3.6. *Let $T', T'' \subset L$ be the invariant lattices of two monodromy operators $\rho', \rho'' \in \mathrm{Mon}^2(L)$, respectively, and let $S' = (T')^\perp, S'' = (T'')^\perp$ be their orthogonal complements in L. The moduli space $\mathcal{M}_{T',\rho'}$ is in the closure of $\mathcal{M}_{T'',\rho''}$ if and only if $S' \subset S''$ and $(\rho'')|_{S'} = (\rho')|_{S'}$.*

REMARK 5.3.7. In our setting we can slightly improve the result of Lemma 5.3.6. In fact, as observed in Section 5.2, the orthogonal sublattices $T, S \subset L$ determine the involution $\rho \in \mathrm{Mon}^2(L)$ as the unique extension of $\mathrm{id}_T \oplus (-\mathrm{id}_S)$ to L. So, if we assume that $S' \subset S''$, then

$$(\rho'')|_{S'} = (-\mathrm{id}_{S''})|_{S'} = -\mathrm{id}_{S'} = (\rho')|_{S'}.$$

In the case of involutions we can then say that $\mathcal{M}_{T',\rho'}$ is in the closure of $\mathcal{M}_{T'',\rho''}$ if and only if $S' \subset S''$.

In this sense, the moduli spaces $\mathcal{M}_{T,\rho}$ of maximal dimension (where maximality is with respect to this notion) correspond to minimal (with respect to inclusion) admissible sublattices $T \subset L$. This is the reason why, in the previous section, we investigated in detail admissible invariant lattices of low rank. Any of these admissible lattices T will give rise to at least one (but there could be more a priori, depending on the number of connected components of the moduli space) projective family of dimension $21 - \mathrm{rk}(T)$, whose generic member has a non-symplectic involution with invariant lattice T. We are now interested in computing the number of irreducible components for some of these moduli spaces.

We adopt the notation of [58, Chapter 4]. Let $T \subset L$ be an admissible sublattice, i.e. the (hyperbolic) invariant lattice of an involution $\rho \in \mathrm{Mon}^2(L)$, and let $\mathcal{C}_T$ be one of the two connected components of the cone $\{x \in T \otimes \mathbb{R} \mid (x, x) > 0\}$. The definition of Kähler-type chamber that was given in Section 2.4 can be adapted as follows. The *Kähler-type chambers of T* are the connected components of

$$\mathcal{C}_T \setminus \bigcup_{\delta \in \Delta(T)} \delta^\perp$$

where $\Delta(T)$ is the set of wall divisors in T. As before, let Γ_T be the image of the restriction map $\mathrm{Mon}^2(L,T) \to O(T)$: the subgroup $\Gamma_T \subset O(T)$ has finite index and it conjugates invariant wall-divisors, therefore it also acts on the set $\mathrm{KT}(T)$ of Kähler-type chambers of T (see [58, §4.7]). In [58, Theorem 4.8.11], Joumaah proved that the quotient $\mathrm{KT}(T)/\Gamma_T$ is in one-to-one correspondence with the set of distinct deformation types of marked manifolds $(X, \eta) \in \mathcal{M}_{T,\rho}$.

PROPOSITION 5.3.8. *Let $T \cong U(2)$ be a primitive sublattice of $L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$ with orthogonal complement $S \cong U \oplus U(2) \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$. Let $\rho_1 \in \mathrm{Mon}^2(L)$ be the involution which extends $\mathrm{id}_T \oplus (-\mathrm{id}_S)$. Then, for any $n \geq 2$ there is a single deformation type of marked manifolds of $K3^{[n]}$-type $(X, \eta) \in \mathcal{M}_{T,\rho_1}$.*

PROOF. As we recalled above, the number of deformation types of (ρ_1, T)-polarized marked manifolds of $K3^{[n]}$-type is equal to the number of orbits of Kähler-type chambers of T, with respect to the action of the subgroup $\Gamma_T \subset O(T)$. For $T \cong U(2)$ as in the statement, an element $\delta \in T$ of coordinates (a, b) with respect to a basis has square $4ab$ and divisibility in L equal to $\gcd(a, b)$ (see Example 1.4.6). In particular, the divisibility can only be one if δ is primitive. However, a direct computation using Theorem 2.4.14 shows that, if δ is a wall-divisor with $\mathrm{div}(\delta) = 1$, then $\delta^2 = -2$ (see [73, Remark 2.5]). We conclude that there are no wall-divisors $\delta \in T$, since $T \cong U(2)$ contains no elements of square -2. $\square$

As we showed in Example 1.4.6 and in Subsection 5.3.1.2, when n is odd there is a second way to embed the lattice $U(2)$ in L, which is not isometric to the one studied in Proposition 5.3.8.

Proposition 5.3.9. *For n odd, let $T \cong U(2)$ be a primitive sublattice of $L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$ with orthogonal complement $S \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$. Let $\rho_2 \in \mathrm{Mon}^2(L)$ be the involution which extends $\mathrm{id}_T \oplus (-\mathrm{id}_S)$. Then, if $n = 5$ there are three distinct deformation types of marked manifolds $(X,\eta) \in \mathcal{M}_{T,\rho_2}$.*

Proof. As in the proof of Proposition 5.3.8, we need to study the Kähler-type chambers of T and therefore determine whether the lattice contains any wall-divisors. Up to isometries, the embedding $U(2) \hookrightarrow L$ in the statement can be realized as explained in Example 1.4.6. Let $t = \frac{n-1}{2} \in \mathbb{N}$ and consider the map

$$j : U(2) \hookrightarrow L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle, \quad (a,b) \mapsto 2ae_1 + (at+b)e_2 + ag$$

where $\{e_1, e_2\}$ is a basis for one of the summands U of L and g is a generator of $\langle -2(n-1)\rangle$. We then have $j(U(2))^\perp \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$, as requested. In particular, if $n = 5$ (i.e. $t = 2$) one can show that the divisibility in L of $(a,b) \in T = j(U(2))$ is $\gcd(2a,b)$, hence, if the element is primitive, it can only be one or two. We compute explicitly all possible pairs $(\delta^2, \mathrm{div}(\delta))$ for wall-divisors $\delta \in L_5 = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -8\rangle$. This is an application of Theorem 2.4.14 and Theorem 2.4.12, which gives the following results:

δ^2	$\mathrm{div}(\delta)$
-2	1
-8	2
-8	4
-8	8
-16	2
-40	4
-72	8
-136	8
-200	8

Since for any $\delta \in T$ we have $\delta^2 \in 4\mathbb{Z}$, the only possible pairs $(\delta^2, \mathrm{div}(\delta))$ for wall-divisors $\delta \in T$ are $(\delta^2, \mathrm{div}(\delta)) = (-8,2), (-16,2)$. Each of the two admissible pairs $(\delta^2, \mathrm{div}(\delta))$ yields a single wall-divisor $\delta \in T$, whose orthogonal complement $\delta^\perp$ intersects the positive cone of T in its interior. We therefore have two (distinct) walls, which cut out three Kähler-type chambers in $\mathcal{C}_T$. These three chambers correspond to three distinct orbits, with respect to the action of the group Γ_T on $\mathrm{KT}(T)$. This is due to the fact that an isometry $\gamma \in \Gamma_T$ permutes the walls of the chambers, which in our case are generated by primitive vectors having all different squares. □

By Proposition 5.3.1, there are two distinct (ρ, T)-polarizations with $T \cong \langle 2\rangle$. In the following, we will denote them by $(\rho_a, \langle 2\rangle)$ and $(\rho_b, \langle 2\rangle)$, where the orthogonal complement S of the admissible sublattice $T \subset L$ is as in case (iia) and (iib), respectively, of the proposition. In particular, for all $n \geq 2$ the moduli space $\mathcal{M}_{\langle 2\rangle,\rho_a}$ is non-empty, while $\mathcal{M}_{\langle 2\rangle,\rho_b} = \emptyset$ if $n \not\equiv 0 \pmod 4$. Instead, again by Proposition 5.3.1, there is only one (ρ, T)-polarization with $T \cong \langle 2(n-1)\rangle$: we denote by $\mathcal{M}_{\langle 2(n-1)\rangle,\rho}$ the corresponding moduli space, which is non-empty if and only if -1 is a quadratic residue modulo $n-1$. Finally, for $T \cong U(2)$, we have the two polarizations $(\rho_1, U(2))$, $(\rho_2, U(2))$ which we studied in Proposition 5.3.8 and Proposition 5.3.9, respectively.

THEOREM 5.3.10. *Let (X, η) be a marked manifold of $K3^{[n]}$-type for $2 \le n \le 5$, and let $i \in \mathrm{Aut}(X)$ be a non-symplectic involution such that the pair (X, i) deforms in a family of dimension ≥ 19. Then (X, η) belongs to the closure of one of the following moduli spaces.*

$n = 2$: $\mathcal{M}_{\langle 2\rangle,\rho_a}$ *or* $\mathcal{M}_{U(2),\rho_1}$*;*
$n = 3$: $\mathcal{M}_{\langle 2\rangle,\rho_a}$*,* $\mathcal{M}_{\langle 4\rangle,\rho}$ *or* $\mathcal{M}_{U(2),\rho_1}$*;*
$n = 4$: $\mathcal{M}_{\langle 2\rangle,\rho_a}$*,* $\mathcal{M}_{\langle 2\rangle,\rho_b}$ *or* $\mathcal{M}_{U(2),\rho_1}$*;*
$n = 5$: $\mathcal{M}_{\langle 2\rangle,\rho_a}$*,* $\mathcal{M}_{U(2),\rho_1}$ *or* $\mathcal{M}_{U(2),\rho_2}$*.*

All these moduli spaces are irreducible with the exception of $\mathcal{M}_{U(2),\rho_2}$ for $n = 5$, which has three distinct irreducible components.

PROOF. Since (X, i) deforms in a family of dimension at least 19, it is a pair of type T, for some admissible lattice T with $\mathrm{rk}(T) \le 2$. At the level of period domains, the list in the statement is an easy consequence of Lemma 5.3.6 and of Propositions 5.3.1, 5.3.3 and 5.3.4. Moreover, the period map is generically injective when restricted to manifolds polarized with a lattice of rank one, and the same is true in the case of $U(2)$ by Proposition 5.3.8 and by [58, Corollary 4.9.6], with the exception of $n = 5$ and $\mathcal{M}_{U(2),\rho_2}$ as explained in Proposition 5.3.9. □

5.3.3. Examples. Even when we limit ourselves to dimensions smaller than ten, we observe that we lack the description of most of the projective families listed in Theorem 5.3.10. Indeed, while for $n = 2$ both families have been described, respectively in [85] and [86], for $n \ge 3$ the family of $(\langle 2\rangle, \rho_a)$-polarized manifolds of $K3^{[n]}$-type is still unknown. In fact, when $n \ge 3$ the only explicit examples which have been found are for $n = 3$, $T \cong \langle 4\rangle$ (see [55] and Section 5.1.2) and $n = 4$, $T \cong \langle 2\rangle$ with polarization ρ_b (involution of the Lehn–Lehn–Sorger–van Straten eightfold; see for instance [64]), in addition to the involutions of Hilbert schemes of points on generic $K3$ surfaces which we constructed in Chapter 3.

We conclude by observing that all families of dimension 19 can in fact be realized as families of moduli spaces of stable twisted sheaves on a $K3$ surface (see Example 2.2.7).

PROPOSITION 5.3.11. *For $n \ge 2$, let (X, η) be a very general element in the moduli space $\mathcal{M}_{U(2),\rho_1}$ of Proposition 5.3.8, such that $\eta(\mathrm{NS}(X)) \cong U(2)$. Then, the manifold X is isomorphic to a moduli space of twisted sheaves on a very general projective $\langle 2(n-1)\rangle$-polarized $K3$ surface.*

PROOF. Let Σ be a generic projective $K3$ surface of degree $2(n-1)$, i.e. $\mathrm{Pic}(\Sigma) = \mathbb{Z}L$ with $L = \mathcal{O}_\Sigma(H)$ for an effective, ample divisor H with $H^2 = 2(n-1)$. Let $\{e_1, e_2\}$ generate one of the summands U in $\mathrm{Tr}(\Sigma) \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$, and consider the Brauer class of order two:

$$\alpha : \mathrm{Tr}(\Sigma) \to \mathbb{Z}/2\mathbb{Z}, \qquad v \mapsto (e_1, v).$$

Clearly, $B = \frac{e_1}{2} \in H^2(\Sigma, \mathbb{Q})$ is a B-field lift of α such that $B^2 = 0$ and $B \cdot H = 0$, since $2B \in \mathrm{Tr}(\Sigma)$. Consider the primitive positive Mukai vector $v = (0, H, 0)$: then

$$v_B = (0, H, B \cdot H) = v$$

and the moduli space $M_{v_B}(\Sigma, \alpha)$ is a manifold of $K3^{[n]}$-type with

$$\mathrm{Tr}(M_{v_B}(\Sigma, \alpha)) \cong \ker(\alpha) \cong U \oplus U(2) \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle.$$

Moreover, $\mathrm{Pic}(\Sigma, \alpha) = \langle (0, H, 0), (0, 0, 1), (2, 2B, 0)\rangle \cong \langle 2\rangle \oplus U(2)$, thus

$$\mathrm{Pic}(M_{v_B}(\Sigma, \alpha)) \cong v_B^\perp \cap \mathrm{Pic}(\Sigma, \alpha) \cong \langle (0, 0, 1), (2, e_1, 0)\rangle \cong U(2).$$

Hence, the moduli space $Y = M_{v_B}(\Sigma, \alpha)$ constructed above has $\mathrm{Pic}(Y) \cong T$, $\mathrm{Tr}(Y) \cong S$ for the lattices T, S of Proposition 5.3.8. By the same proposition we know that the moduli space $\mathcal{M}_{U(2),\rho_1}$ is irreducible. For $(X,\eta) \in \mathcal{M}_{U(2),\rho_1}$ very general we also have $\mathrm{Pic}(X) \cong T$ and $\mathrm{Tr}(X) \cong S$ (via the marking η). Hence, the statement follows from the generic injectivity of the period map for $U(2)$-polarized manifolds of $K3^{[n]}$-type (see [58, Corollary 4.9.6]). $\square$

Remark 5.3.12. For $(X,\eta) \in \mathcal{M}_{U(2),\rho_1}$, let $i \in \mathrm{Aut}(X)$ be the non-symplectic involution such that $\eta \circ i^* = \rho_1 \circ \eta$. Even though, for (X,η) very general, the manifold X is isomorphic to $Y = M_{v_B}(\Sigma,\alpha)$ as in the previous proposition, if $n \geq 3$ we cannot realize the automorphism i as a twisted induced involution on Y (in the sense of Section 4.2.2), since the group of automorphisms of the $K3$ surface Σ is trivial (see [91, §5]).

Proposition 5.3.13. *For $n = 5$, let $\mathcal{M}_{U(2),\rho_2}$ be the moduli space of Proposition 5.3.9. There exists an irreducible component $\mathcal{M}^0 \subset \mathcal{M}_{U(2),\rho_2}$ such that, for the very general element $(X,\eta) \in \mathcal{M}^0$ with $\eta(\mathrm{NS}(X)) \cong U(2)$, the manifold X is isomorphic to a moduli space Y of twisted sheaves on a very general projective $\langle 2\rangle$-polarized $K3$ surface. Moreover, the non-symplectic involution $i \in \mathrm{Aut}(X)$ such that $\eta \circ i^* = \rho_2 \circ \eta$ is realized by a twisted induced automorphism on Y.*

Proof. Let Σ be the double cover of $\mathbb{P}^2$ branched along a smooth sextic curve. In particular, $\mathrm{Pic}(\Sigma) \cong \langle 2\rangle$ and $\mathrm{Tr}(\Sigma) \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2\rangle$. If we denote by g the generator of the summand $\langle -2\rangle$ inside $\mathrm{Tr}(\Sigma)$, then the (non-primitive) index two sublattice $U^{\oplus 2}E_8^{\oplus 2} \oplus \langle 2g\rangle \subset \mathrm{Tr}(\Sigma)$ is isometric to $S = U^{\oplus 2} \oplus E_8^{\oplus} \oplus \langle -8\rangle$. Let α be the following Brauer class of order two:

$$\alpha : \mathrm{Tr}(\Sigma) \to \mathbb{Z}/2\mathbb{Z}, \qquad \lambda + mg \mapsto m$$

where $\lambda \in U^{\oplus 2} \oplus E_8^{\oplus 2}$ and $m \in \mathbb{Z}$. Clearly, $\ker(\alpha) = U^{\oplus 2}E_8^{\oplus 2} \oplus \langle 2g\rangle \cong S$. Let $\{e_1, e_2\}$ generate a summand U inside $H^2(\Sigma,\mathbb{Z}) \cong U^{\oplus 3} \oplus E_8^{\oplus 2}$. We can assume that $e_1 + e_2$ is the generator of $\mathrm{Pic}(\Sigma)$ and therefore $g = e_1 - e_2$. Notice that the rational class $B = \frac{e_2}{2} \in H^2(\Sigma,\mathbb{Q})$ is a B-field lift for α, since $\alpha(x) = (e_2, x) \in \mathbb{Z}/2\mathbb{Z}$ for all $x \in \mathrm{Tr}(\Sigma)$. Consider the (non-primitive) positive Mukai vector $v = (0, 2(e_1+e_2), 0) \in H^*(\Sigma,\mathbb{Z})$. When twisting v with respect to the B-field lift B, we obtain $v_B = (0, 2(e_1+e_2), 1)$, which is now primitive of square 8. Hence, the moduli space $M_{v_B}(\Sigma,\alpha)$ is a manifold of $K3^{[5]}$-type with transcendental lattice isomorphic to S. Moreover

$$\mathrm{Pic}(\Sigma,\alpha) = \langle (0, e_1+e_2, 0), (0,0,1), (2, e_2, 0)\rangle$$

thus

$$\mathrm{Pic}(M_{v_B}(\Sigma,\alpha)) \cong v_B^{\perp} \cap \mathrm{Pic}(\Sigma,\alpha) \cong \langle (0,0,1), (2,e_2,0)\rangle \cong U(2).$$

Since Σ is a double cover of the plane, it is equipped with a non-symplectic involution ι, which acts as id on $H^0(\Sigma,\mathbb{Z}) \oplus \mathrm{Pic}(\Sigma) \oplus H^4(\Sigma,\mathbb{Z})$ and as $-\mathrm{id}$ on $\mathrm{Tr}(\Sigma)$. This implies that both the Brauer class $\alpha : \mathrm{Tr}(\Sigma) \to \mathbb{Z}/2\mathbb{Z}$ and the twisted Mukai vector $v_B = (0, 2(e_1+e_2), 1)$ are ι-invariant. Then, as we recalled in Section 4.2.2, the moduli space $Y = M_{v_B}(\Sigma,\alpha)$ comes with a (non-symplectic) induced involution $\widetilde{\iota}$. In particular, the invariant lattice of $\widetilde{\iota}$ is the whole $\mathrm{Pic}(M_{v_B}(\Sigma,\alpha))$, since ι acts trivially on $\langle (0,0,1), (2,e_2,0)\rangle$ by [30, Remark 2.4] (the two classes $(2,e_2,0)$ and $(2,\iota^*(e_2),0) = (2,e_1,0)$ coincide in $H^2(M_{v_B}(\Sigma,\alpha),\mathbb{Z})$). As in Proposition 5.3.11, the statement follows from the generic injectivity of the period map, after recalling that $\mathcal{M}_{U(2),\rho_2}$ has three irreducible components by Proposition 5.3.9. $\square$

APPENDIX A

Invariant and co-invariant lattices for $n = 3, 4$, $p = 3$

The two tables in the following pages list all admissible triples (p, m, a) (see Definition 4.1.10) and the corresponding isometry classes for the invariant and co-invariant lattices $T, S \subset L_n \cong H^2(X, \mathbb{Z})$ of non-symplectic automorphisms of order $p = 3$ on manifolds X of $K3^{[n]}$-type, for $n = 3, 4$. This classification is discussed in Section 4.1.4.

The symbol ♣ denotes the cases which can be realized by natural automorphisms (see Section 4.2.1). The cases marked by $\natural$ (respectively, $\diamondsuit$) correspond to admissible triples that admit a realization via induced automorphisms on moduli spaces of ordinary (respectively, twisted) sheaves on $K3$ surfaces (see Section 4.3). Finally, the admissible triple $(3, 11, 0)$ for $n = 4$ is realized by the automorphism constructed in Section 4.4.1 on a ten-dimensional family of Lehn–Lehn–Sorger–van Straten eightfolds (see also Section 4.2.3).

p	m	a	S	T
♣ 3	10	0	$U^{\oplus 2} \oplus E_8^{\oplus 2}$	$U \oplus \langle -4 \rangle$
♣ 3	10	2	$U \oplus U(3) \oplus E_8^{\oplus 2}$	$U(3) \oplus \langle -4 \rangle$
♣ 3	9	1	$U^{\oplus 2} \oplus E_6 \oplus E_8$	$U \oplus A_2 \oplus \langle -4 \rangle$
♣ 3	9	3	$U \oplus U(3) \oplus E_6 \oplus E_8$	$U(3) \oplus A_2 \oplus \langle -4 \rangle$
♢ 3	9	5	$U(3)^{\oplus 2} \oplus E_6 \oplus E_8$	$U(3) \oplus \Omega$
♣ 3	8	2	$U^{\oplus 2} \oplus E_6^{\oplus 2}$	$U \oplus A_2^{\oplus 2} \oplus \langle -4 \rangle$
♣ 3	8	4	$U \oplus U(3) \oplus E_6^{\oplus 2}$	$U(3) \oplus A_2^{\oplus 2} \oplus \langle -4 \rangle$
♢ 3	8	6	$U(3)^{\oplus 2} \oplus E_6^{\oplus 2}$	$U(3) \oplus A_2 \oplus \Omega$
♣ 3	7	1	$U^{\oplus 2} \oplus A_2 \oplus E_8$	$U \oplus E_6 \oplus \langle -4 \rangle$
♣ 3	7	3	$U \oplus U(3) \oplus A_2 \oplus E_8$	$U \oplus A_2^{\oplus 3} \oplus \langle -4 \rangle$
♣ 3	7	5	$U^{\oplus 2} \oplus A_2^{\oplus 5}$	$U(3) \oplus A_2^{\oplus 3} \oplus \langle -4 \rangle$
♣ 3	7	7	$U \oplus U(3) \oplus A_2^{\oplus 5}$	$U(3) \oplus E_6^{\vee}(3) \oplus \langle -4 \rangle$
♣ 3	6	0	$U^{\oplus 2} \oplus E_8$	$U \oplus E_8 \oplus \langle -4 \rangle$
♣ 3	6	2	$U \oplus U(3) \oplus E_8$	$U \oplus E_6 \oplus A_2 \oplus \langle -4 \rangle$
♣ 3	6	4	$U^{\oplus 2} \oplus A_2^{\oplus 4}$	$U \oplus A_2^{\oplus 4} \oplus \langle -4 \rangle$
♣ 3	6	6	$U \oplus U(3) \oplus A_2^{\oplus 4}$	$U(3) \oplus A_2^{\oplus 4} \oplus \langle -4 \rangle$
♣ 3	5	1	$U^{\oplus 2} \oplus E_6$	$U \oplus E_8 \oplus A_2 \oplus \langle -4 \rangle$
♣ 3	5	3	$U \oplus U(3) \oplus E_6$	$U \oplus A_2^{\oplus 2} \oplus E_6 \oplus \langle -4 \rangle$
♣ 3	5	5	$U \oplus U(3) \oplus A_2^{\oplus 3}$	$U \oplus A_2^{\oplus 5} \oplus \langle -4 \rangle$
♣ 3	4	2	$U^{\oplus 2} \oplus A_2^{\oplus 2}$	$U \oplus E_6^{\oplus 2} \oplus \langle -4 \rangle$
♣ 3	4	4	$U \oplus U(3) \oplus A_2^{\oplus 2}$	$U \oplus E_6 \oplus A_2^{\oplus 3} \oplus \langle -4 \rangle$
♣ 3	3	1	$U^{\oplus 2} \oplus A_2$	$U \oplus E_6 \oplus E_8 \oplus \langle -4 \rangle$
♣ 3	3	3	$U \oplus U(3) \oplus A_2$	$U \oplus E_6^{\oplus 2} \oplus A_2 \oplus \langle -4 \rangle$
♣ 3	2	0	$U^{\oplus 2}$	$U \oplus E_8^{\oplus 2} \oplus \langle -4 \rangle$
♣ 3	2	2	$U \oplus U(3)$	$U \oplus E_6 \oplus E_8 \oplus A_2 \oplus \langle -4 \rangle$
♣ 3	1	1	$A_2(-1)$	$U \oplus E_8^{\oplus 2} \oplus A_2 \oplus \langle -4 \rangle$

TABLE 1. $n = 3$. See Section 4.1.4 for the definition of the lattice Ω.

p	m	a	S	T
$\bigstar$ 3	11	0	$U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$	$\langle 2 \rangle$
$\clubsuit$ 3	10	0	$U^{\oplus 2} \oplus E_8^{\oplus 2}$	$U \oplus \langle -6 \rangle$
$\natural$ 3	10	1	$U \oplus U(3) \oplus E_8^{\oplus 2}$	$U \oplus \langle -6 \rangle$
$\clubsuit$ 3	10	2	$U \oplus U(3) \oplus E_8^{\oplus 2}$	$U(3) \oplus \langle -6 \rangle$
$\diamondsuit$ 3	10	3	$U(3)^{\oplus 2} \oplus E_8^{\oplus 2}$	$U(3) \oplus \langle -6 \rangle$
$\clubsuit$ 3	9	1	$U^{\oplus 2} \oplus E_6 \oplus E_8$	$U \oplus A_2 \oplus \langle -6 \rangle$
$\natural$ 3	9	2	$U \oplus U(3) \oplus E_6 \oplus E_8$	$U \oplus A_2 \oplus \langle -6 \rangle$
$\clubsuit$ 3	9	3	$U \oplus U(3) \oplus E_6 \oplus E_8$	$U(3) \oplus A_2 \oplus \langle -6 \rangle$
$\diamondsuit$ 3	9	4	$U(3)^{\oplus 2} \oplus E_6 \oplus E_8$	$U(3) \oplus A_2 \oplus \langle -6 \rangle$
$\natural$ 3	8	1	$U^{\oplus 2} \oplus E_6^{\oplus 2}$	$\langle 2 \rangle \oplus E_6$
$\clubsuit$ 3	8	2	$U^{\oplus 2} \oplus E_6^{\oplus 2}$	$U \oplus A_2^{\oplus 2} \oplus \langle -6 \rangle$
$\natural$ 3	8	3	$U \oplus U(3) \oplus E_6^{\oplus 2}$	$U \oplus A_2^{\oplus 2} \oplus \langle -6 \rangle$
$\clubsuit$ 3	8	4	$U \oplus U(3) \oplus E_6^{\oplus 2}$	$U(3) \oplus A_2^{\oplus 2} \oplus \langle -6 \rangle$
$\diamondsuit$ 3	8	5	$U(3)^{\oplus 2} \oplus E_6^{\oplus 2}$	$U(3) \oplus A_2^{\oplus 2} \oplus \langle -6 \rangle$
$\natural$ 3	7	0	$U^{\oplus 2} \oplus A_2 \oplus E_8$	$\langle 2 \rangle \oplus E_8$
$\clubsuit$ 3	7	1	$U^{\oplus 2} \oplus A_2 \oplus E_8$	$U \oplus E_6 \oplus \langle -6 \rangle$
$\natural$ 3	7	2	$U \oplus U(3) \oplus A_2 \oplus E_8$	$U \oplus E_6 \oplus \langle -6 \rangle$
$\clubsuit$ 3	7	3	$U \oplus U(3) \oplus A_2 \oplus E_8$	$U(3) \oplus E_6 \oplus \langle -6 \rangle$
$\natural$ 3	7	4	$U^{\oplus 2} \oplus A_2^{\oplus 5}$	$U \oplus A_2^{\oplus 3} \oplus \langle -6 \rangle$
$\clubsuit$ 3	7	5	$U^{\oplus 2} \oplus A_2^{\oplus 5}$	$U(3) \oplus A_2^{\oplus 3} \oplus \langle -6 \rangle$
$\natural$ 3	7	6	$U \oplus U(3) \oplus A_2^{\oplus 5}$	$U \oplus E_6^{\vee}(3) \oplus \langle -6 \rangle$
$\clubsuit$ 3	7	7	$U \oplus U(3) \oplus A_2^{\oplus 5}$	$U(3) \oplus E_6^{\vee}(3) \oplus \langle -6 \rangle$
$\clubsuit$ 3	6	0	$U^{\oplus 2} \oplus E_8$	$U \oplus E_8 \oplus \langle -6 \rangle$
$\natural$ 3	6	1	$U \oplus U(3) \oplus E_8$	$U \oplus E_8 \oplus \langle -6 \rangle$
$\clubsuit$ 3	6	2	$U \oplus U(3) \oplus E_8$	$U(3) \oplus E_8 \oplus \langle -6 \rangle$
$\natural$ 3	6	3	$U^{\oplus 2} \oplus A_2^{\oplus 4}$	$U \oplus E_6 \oplus A_2 \oplus \langle -6 \rangle$
$\clubsuit$ 3	6	4	$U^{\oplus 2} \oplus A_2^{\oplus 4}$	$U(3) \oplus E_6 \oplus A_2 \oplus \langle -6 \rangle$
$\natural$ 3	6	5	$U \oplus U(3) \oplus A_2^{\oplus 4}$	$U \oplus A_2^{\oplus 4} \oplus \langle -6 \rangle$
$\clubsuit$ 3	6	6	$U \oplus U(3) \oplus A_2^{\oplus 4}$	$U(3) \oplus A_2^{\oplus 4} \oplus \langle -6 \rangle$
$\clubsuit$ 3	5	1	$U^{\oplus 2} \oplus E_6$	$U \oplus E_8 \oplus A_2 \oplus \langle -6 \rangle$
$\natural$ 3	5	2	$U \oplus U(3) \oplus E_6$	$U \oplus E_8 \oplus A_2 \oplus \langle -6 \rangle$
$\clubsuit$ 3	5	3	$U \oplus U(3) \oplus E_6$	$U(3) \oplus E_8 \oplus A_2 \oplus \langle -6 \rangle$
$\natural$ 3	5	4	$U \oplus U(3) \oplus A_2^{\oplus 3}$	$U \oplus E_6 \oplus A_2^{\oplus 2} \oplus \langle -6 \rangle$
$\clubsuit$ 3	5	5	$U \oplus U(3) \oplus A_2^{\oplus 3}$	$U(3) \oplus E_6 \oplus A_2^{\oplus 2} \oplus \langle -6 \rangle$
$\natural$ 3	4	1	$U^{\oplus 2} \oplus A_2^{\oplus 2}$	$\langle 2 \rangle \oplus E_6 \oplus E_8$
$\clubsuit$ 3	4	2	$U^{\oplus 2} \oplus A_2^{\oplus 2}$	$U \oplus A_2^{\oplus 2} \oplus E_8 \oplus \langle -6 \rangle$
$\natural$ 3	4	3	$U \oplus U(3) \oplus A_2^{\oplus 2}$	$U \oplus E_6^{\oplus 2} \oplus \langle -6 \rangle$
$\clubsuit$ 3	4	4	$U \oplus U(3) \oplus A_2^{\oplus 2}$	$U(3) \oplus E_6^{\oplus 2} \oplus \langle -6 \rangle$
$\natural$ 3	3	0	$U^{\oplus 2} \oplus A_2$	$\langle 2 \rangle \oplus E_8^{\oplus 2}$
$\clubsuit$ 3	3	1	$U^{\oplus 2} \oplus A_2$	$U \oplus E_6 \oplus E_8 \oplus \langle -6 \rangle$
$\natural$ 3	3	2	$U \oplus U(3) \oplus A_2$	$U \oplus E_6 \oplus E_8 \oplus \langle -6 \rangle$
$\clubsuit$ 3	3	3	$U \oplus U(3) \oplus A_2$	$U(3) \oplus E_6 \oplus E_8 \oplus \langle -6 \rangle$
$\clubsuit$ 3	2	0	$U^{\oplus 2}$	$U \oplus E_8^{\oplus 2} \oplus \langle -6 \rangle$
$\natural$ 3	2	1	$U \oplus U(3)$	$U \oplus E_8^{\oplus 2} \oplus \langle -6 \rangle$
$\clubsuit$ 3	2	2	$U \oplus U(3)$	$U(3) \oplus E_8^{\oplus 2} \oplus \langle -6 \rangle$
$\clubsuit$ 3	1	1	$A_2(-1)$	$U \oplus E_8^{\oplus 2} \oplus A_2 \oplus \langle -6 \rangle$

TABLE 2. $n = 4$.

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