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Local expansion in Serre-Tate coordinates and p-adic iteration of Gauss-Manin connections

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Abstract

In this dissertation, we introduce the sheaf of nearly overconvergent quaternion modular forms of general weight over a normal integral model of a Shimura curve via the machinery of vector bundles with marked sections. We then study the Gauss-Manin connection and Hecke operators on the aforementioned sheaf using Serre-Tate coordinates, and finally show that the Gauss-Manin connection can be iterated p-adically.

Abstract

In questa tesi, introduciamo il fascio delle forme modulari quaternioniche quasi sovraconvergenti di peso arbitrario su un modello integrale normale di curve di Shimura attraverso lo strumento dei fibreti vettoriali con sezioni contrassegnate. Studiamo in seguito la connessione di Gauss-Manin e gli operatori di Hecke sul fascio menzionato precedentemente usando le coordinate di Serre-Tate, ed infine mostriamo che la connessione di Gauss-Manin può essere iterata p-adicamente.

Contents

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Introduction

Background and motivation

Analogous to the role played by nearly holomorphic modular forms in the study of complex L-functions, nearly overconvergent modular forms play a vital role in the study of p-adic L-functions.

In the complex setting, the notion of nearly holomorphic modular form (of weight $k \in \mathbb{N}$, order $d \in \mathbb{N}$) was first introduced by G. Shimura to prove the algebraicity of special values of L-functions [Shi76]. One key tool in the proof is the Maass-Shimura operator on the space of nearly holomorphic modular forms which raises the weight by 2 and order by 1.

It was realized later by E. Urban [Urb14] that one can give a sheaf-theoretic definition of nearly holomorphic forms using the (logarithmic) algebraic de Rham cohomology of the universal (generalized) elliptic curve over the compactified modular curve. More precisely, let $X := X_1(N)$ be the compactified modular curve of level $\Gamma_1(N)$ for some integer $N \geq 4$, and let $\mathcal{E}/X$ be the universal (generalized) elliptic curve. Also, let $\underline{\omega}$ be the Hodge line bundle over X, $\mathbb{H} := \mathbb{H}^1_{\mathrm{dR}}(\mathcal{E}/X)$ be the degree-1 relative (logarithmic) de Rham cohomology sheaf, and let $\underline{\omega}_{\mathrm{ra}}$, $\mathbb{H}^{\mathrm{ra}}$ be the real analytic sheaves over $X_{\mathbb{C}} := X \otimes_{\mathbb{Z}[\frac{1}{N}]} \mathbb{C}$ associated to $\underline{\omega}$, $\mathbb{H}$ respectively. Then the Hodge decomposition of $\mathbb{H}^{\mathrm{ra}}$ and the Kodaira-Spencer isomorphism induce a canonical isomorphism between nearly holomorphic forms of level N, weight k, order d and sections of $\mathrm{Sym}^d(\mathbb{H}^{\mathrm{ra}}) \otimes \underline{\omega}_{\mathrm{ra}}^{k-d}$ over $X_{\mathbb{C}}$. Moreover, one can recover the Maass-Shimura operator from the Gauss-Manin connection on $\mathrm{Sym}^d(\mathbb{H}^{\mathrm{ra}}) \otimes \underline{\omega}_{\mathrm{ra}}^{k-d}$ via this isomorphism.

In the p-adic setting ($p \nmid N$), the notion of nearly overconvergent modular forms (of finite order) was first introduced by R. Coleman [Col97]. Like nearly holomorphic modular forms, nearly overconvergent modular forms are equipped with a differential operator, the Atkin-Serre operator. Recently, Bertolini-Darmon-Prasanna [BDP+13], Darmon-Rotger [DR14], Harron-Xiao [HX14] and E. Urban [Urb14] independently developed a more systematic theory of (p-adic families of) nearly overconvergent modular forms (of finite order). The family nature of nearly overconvergent modular forms and the Atkin-Serre operator make the theory of nearly overconvergent modular forms a powerful tool in the study of p-adic L-functions.

To be more specific, let $\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ be the Iwasawa algebra, $\mathcal{W} := \mathrm{Spa}(\Lambda, \Lambda)$ (this is the parameterizing space of continuous characters of $\mathbb{Z}_p^\times$. for more details, see §1.7) and $\kappa : \mathbb{Z}_p^\times \to \Lambda^\times$ be the universal character, i.e. the character sending $t \in \mathbb{Z}_p^\times$ to the group-like element $[t]$ in $\Lambda^\times$. Let $X^{\mathrm{ad}}_{\mathbb{Q}_p}$, $\mathbb{H}^{\mathrm{ad}}$, $\underline{\omega}_{\mathrm{ad}}$ be the adicification of $X_{\mathbb{Q}_p}$, $\mathbb{H}$, $\underline{\omega}$ respectively. Then for affinoid subdomains $\mathcal{U} := \mathrm{Spa}(R, R^+)$ of $\mathcal{W}$ such that p is invertible in R, and for the restriction $\kappa_{\mathcal{U}}$ of κ to $\mathcal{U}$, over certain strict neighborhoods of the ordinary locus of $X^{\mathrm{ad}}_{\mathbb{Q}_p} \times_{\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)} \mathcal{U}$, Andreatta-Iovita-Stevens [AIS14] and V. Pilloni [Pil13] independently constructed the overconvergent sheaf $\mathfrak{w}^{\kappa_{\mathcal{U}}}$ of weight $\kappa_{\mathcal{U}}$ such that the specialization of $\mathfrak{w}^{\kappa_{\mathcal{U}}}$ along the classical weight

$$k : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times; a \mapsto a^k$$

recovers the restriction of $\underline{\omega}_{\mathrm{ad}}^{\otimes k}$ to certain strict neighborhoods of the ordinary locus of $X^{\mathrm{ad}}_{\mathbb{Q}_p}$. For any $d \in \mathbb{N}$, sections of $\mathrm{Sym}^d(\mathbb{H}^{\mathrm{ad}}) \otimes \mathfrak{w}^{\kappa_{\mathcal{U}}-d}$ are called the p-adic families of nearly overconvergent modular forms of weight $\kappa_{\mathcal{U}}$, order d. Using the unit root splitting over the ordinary locus, one can recover the Atkin-Serre operator from the naturally defined Gauss-Manin connection on $\mathrm{Sym}^d(\mathbb{H}^{\mathrm{ad}}) \otimes \mathfrak{w}^{\kappa_{\mathcal{U}}-d}$.

Motivated by the desire to construct triple product p-adic L-functions (associated with Coleman families), Andreatta-Iovita [AI17] introduced the (integral) nearly overconvergent sheaf $\mathbb{W}_\kappa$ of unbounded order. More precisely, let $\Lambda^0 \cong \mathbb{Z}_p[[T]]$ be the connected component of the Iwasawa algebra Λ and let $\mathcal{W}_I := \mathrm{Spa}(\Lambda_I^0[\frac{1}{\alpha}], \Lambda_I^0) \subset \mathcal{W}$ with

$$(\Lambda_I^{(0)}, \alpha) := \begin{cases} (\Lambda^{(0)}\langle \frac{T}{p}\rangle, p) \text{ if } I = [0,1], \\ (\Lambda^{(0)}\langle \frac{p}{T^{p^a}}, \frac{T^{p^b}}{p}\rangle, T), \text{ if } I = [p^a, p^b], a, b \in \mathbb{N}. \end{cases}$$

In [AIP], Andreatta-Iovita-Pilloni introduced a normal formal model $\mathfrak{X}_{r,I}$ of the strict neighborhood $\mathcal{X}_{r,I}$ of the ordinary locus of $X^{\mathrm{ad}}_{\mathbb{Q}_p} \times \mathcal{W}_I$ cut out by the existence of the canonical subgroup of certain level of $\mathcal{E}$. Denote the restriction of κ to $\mathcal{W}_I$ by κ_I. Then the unbounded overconvergent modular sheaf $\mathbb{W}_{\kappa_I}$ of weight κ_I is obtained by applying the machinery of vector bundles with marked sections to a certain subsheaf of $\mathbb{H}^{\mathrm{ad}}$ and a marked section obtained from a generator of the Cartier dual of the aforementioned canonical subgroup via the Hodge-Tate period map. The sheaf $\mathbb{W}_{\kappa_I}$ is equipped with an increasing filtration $\mathrm{Fil}_\bullet$ such that $\mathrm{Fil}_d \mathbb{W}_{\kappa_I}$ recovers the restriction of the sheaf $\mathrm{Sym}^d(\mathbb{H}^{\mathrm{ad}}) \otimes \mathfrak{w}^{\kappa_I - d}$ to the adic generic fiber $\mathcal{X}_{r,I}$. Moreover, the sheaf $\mathbb{W}_{\kappa_I}$ is equipped with a (meromorphic) connection ∇ induced from the Gauss-Manin connection on $\mathbb{H}$ and the sections of $\mathbb{W}_{\kappa_I}$ are equipped with a completely continuous operator, the Atkin operator U_p. By an elaborate calculation using the q-expansion, i.e. expansion at cusps, the authors of [AI17] are able to show that for p-adic weights θ satisfying mild conditions and sections $f \in H^0(\mathfrak{X}_{r,I}, \mathbb{W}_{\kappa_I})^{U_p=0} (:= \ker(U_p))$, there is a canonical section $\nabla^\theta(f)$, defined by p-adically iterating the Gauss-Manin connection ∇, of $\mathbb{W}_{\kappa_I + 2\theta}$ over a smaller strict neighborhood. This result is the main technical tool in the construction of triple product p-adic L-functions associated with Coleman families [AI17], and in the construction of Katz type p-adic L-functions when p is non-split in the CM field [AI].

Main results

The main topic of this dissertation is to generalize the construction of $\mathbb{W}_\kappa$ over modular curves in [AI17] to Shimura curves associated with indefinite quaternion algebras over the rational numbers $\mathbb{Q}$, and prove the iteration result using the Serre-Tate local expansion.

Let us provide more information about our results. Fix a pair $(B, \mathcal{O}_B)$ consisting of an indefinite quaternion division algebra $B/\mathbb{Q}$ and a maximal order $\mathcal{O}_B \subset B$ and fix an integer $N \geq 4$ prime to the discriminant Δ of B. Let $X := X_1^B(N)$ be the Shimura curve of level $V_1(N)$ associated with B defined over $\mathbb{Z}[\frac{1}{N\Delta}]$ and $\mathcal{A}/X$ be the universal false elliptic curve over X. Choose a rational prime p such that $(p, N\Delta) = 1$. Let $\hat{X}$ be the p-adic completion of $X_{\mathbb{Z}_p}$ and let $\mathcal{G}$ be the direct summand of $\mathcal{A}[p^\infty]$ cut out by a non-trivial idempotent in $\mathcal{O}_{B,p}$. Let $\underline{\omega}_{\mathcal{G}}$ be the sheaf of invariant differentials of $\mathcal{G}$, and $\mathbb{H}_{\mathcal{G}} := \mathbb{H}^1_{\mathrm{dR}}(\mathcal{G}/X)$. Then up to a chosen isomorphism, we have the Hodge-de Rham exact sequence

$$0 \to \underline{\omega}_{\mathcal{G}} \to \mathbb{H}_{\mathcal{G}} \to \underline{\omega}_{\mathcal{G}}^\vee \to 0,$$

as well as the Gauss-Manin connection and the Kodaira-Spencer isomorphism

$$\nabla : \ \mathbb{H}_{\mathcal{G}} \to \mathbb{H}_{\mathcal{G}} \otimes_{\mathbb{Z}_p} \Omega^1_{\hat{X}/\mathbb{Z}_p}; \quad \mathrm{KS} : \ \underline{\omega}_{\mathcal{G}}^{\otimes 2} \cong \Omega^1_{\hat{X}/\mathbb{Z}_p}.$$

For $r \in \mathbb{N}$, let $\mathfrak{X}_{r,I}$ be the admissible formal blow-up of $\hat{X} \times \mathrm{Spf}(\Lambda_I^0)$ on which the p^{r+1}-power of the Hasse invariant of $\mathcal{G}$ divides α (see Definition 2.2.9). The formal scheme $\mathfrak{X}_{r,I}$ is designed to be normal. Moreover on $\mathfrak{X}_{r,I}$, the canonical subgroups H_n of $\mathcal{G}$ of level $n \leq \begin{cases} r+a, & \text{if } I=[p^a,p^b] \\ r, & \text{if } I=[0,1] \end{cases}$ exist. The normality of $\mathfrak{X}_{r,I}$ allows us to define the level-n partial Igusa tower $\mathfrak{IG}_{n,r,I}$ over $\mathfrak{X}_{r,I}$ by considering trivializations of the Cartier dual H_n^D of H_n.

By properly modifying the sheaf $\underline{\omega}_{\mathcal{G}}$, $\mathbb{H}_{\mathcal{G}}$ using the image s of a generator of H_n^D via the Hodge-Tate period map, we obtain a full system of vector bundles with a marked section $(\mathbb{H}_{\mathcal{G}}^\sharp, \Omega_{\mathcal{G}}, s)$ over $\mathfrak{IG}_{n,r,I}$. By the machinery of vector bundles with marked sections (called VBMS machinery below, see § 1.6.1) and descent, we have the following result, which is adopted from Theorem 2.3.2.

Theorem 0.0.1. *Let $\kappa:\ \mathbb{Z}_p^\times \to \Lambda_I^\times$ be any weight. We have formal sheaves $\mathbb{W}_\kappa$ and $\mathfrak{w}^\kappa$ over $\mathfrak{X}_{r,I}$ such that*

(i) the sheaf $\mathfrak{w}^\kappa$ is a coherent $\mathcal{O}_{\mathfrak{X}_{r,I}}$-module whose sections over the generic fiber $\mathcal{X}_{r,I}$ of $\mathfrak{X}_{r,I}$ are the (p-adic families of) overconvergent quaternion modular forms;

(ii) the sheaf $\mathbb{W}_\kappa$ is equipped with an increasing filtration $\mathrm{Fil}_\bullet$ such that

(a) for each $h \in \mathbb{N}$, $\mathrm{Gr}_h\mathbb{W}_\kappa := \frac{\mathrm{Fil}_h\mathbb{W}_\kappa}{\mathrm{Fil}_{h-1}\mathbb{W}_\kappa} \cong \mathfrak{w}^\kappa \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \underline{\omega}_{\mathcal{G}}^{-2h}$. In particular, $\mathrm{Fil}_0\mathbb{W}_\kappa \cong \mathfrak{w}^\kappa$;

(b) the α-adic completion of $\lim_h \mathrm{Fil}_h\mathbb{W}_\kappa$ is $\mathbb{W}_\kappa$.

Over an explicitly defined cover of $\mathfrak{IG}_{n,r,I}$, the Gauss-Manin connection ∇ on $\mathbb{H}_{\mathcal{G}}$ induces a connection ∇ on $\mathbb{H}_{\mathcal{G}}^\sharp$ such that the section s is horizontal modulo an explicitly defined ideal. Once again by the VBMS machinery and descent, we have the following result, which is adopted from Theorem 2.3.8.

Theorem 0.0.2. *The sheaf $\mathbb{W}_\kappa$ is equipped with a meromorphic connection*

$$\nabla:\ \mathbb{W}_\kappa \mapsto \mathbb{W}_\kappa \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}[\frac{1}{\alpha}]$$

whose poles are bounded effectively by the Hasse invariant. Moreover, the connection ∇ satisfies Griffiths' transversality with respect to $\mathrm{Fil}_\bullet$ and the induced $\mathcal{O}_{\mathfrak{X}_{r,I}}$-linear map

$$\mathrm{Gr}_h(\nabla_\kappa):\ \mathrm{Gr}_h(\mathbb{W}_\kappa)[1/\alpha] \to \mathrm{Gr}_h(\mathbb{W}_\kappa)\hat{\otimes}_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}[1/\alpha]$$

is an isomorphism times with multiplication by $\mu_\kappa - h$, where $\mu_\kappa := \lim_{\gamma\to 1}\frac{\log(\kappa(\gamma))}{\log(\gamma)}$.

Over sections of $\mathbb{W}_\kappa$, we can define Hecke operators T_ℓ for $\ell \nmid p\Delta$ and the Atkin operator U_p which respect the filtration and preserve the weight. For any finite character χ of $\mathbb{Z}_p^\times$, we can define the Dirichlet twist operator θ^χ which respects the filtration and changes the weight by 2χ. For more details, see § 2.3.4.

By local calculations in the Serre-Tate coordinates, we can study pair-wise relations between these operators and the Gauss-Manin connection, as well as the iteration of the Gauss-Manin connection. More precisely, we have the following result, which is Theorem 2.5.1.

Theorem 0.0.3. *Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$. Let $q=4$ if $p=2$ and $q=p$ if $p\geq 3$. Fix an interval $I_\theta \subset [0,\infty)$ and an interval $I=[0,1]$ or $[p^a,p^b]$ for $a,b\in\mathbb{N}$, and assume that*

$$k:\ \mathbb{Z}_p^\times \to \Lambda_I, \quad \theta:\ \mathbb{Z}_p^\times \to \Lambda_{I_\theta,K}^\times, \quad \Lambda_{I_\theta,K} := \Lambda_{I_\theta}\otimes_{\mathbb{Z}_p}\mathcal{O}_K$$

are weights satisfying

(i) *there exists a constant* $c(k) \in \mathbb{N}$ *such that* $\mu_k + 2c(k) \in p\Lambda_I^0$*;*

(ii) *for any* $t \in \mathbb{Z}_p^\times$, $\theta(t) = \theta'(t)t^{c(\theta)}\chi(t)$ *for a finite character* $\chi: \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$, *a constant* $c(\theta) \in \mathbb{N}$ *(bigger than* $c(\kappa)$*) and a weight* $\theta': \mathbb{Z}_p^\times \to (\Lambda_{I_\theta,K}^0)^\times$ *such that*

$$\mu_{\theta'} \in q\Lambda_{I_\theta,K}^0, \quad \theta'(t) = \exp(\mu_\theta \log(t)) \ \forall\ t \in \mathbb{Z}_p^\times.$$

Then there exist positive integers $r' \geq r$ *and* γ *(depending on* n, r *and* p, $c(\theta)$*) such that for any* $g \in H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa)^{U_p=0}$,

(a) *the sequence* $A(g,\theta')_m := \sum_{j=1}^m \frac{(-1)^{j-1}}{j}(\nabla_{k+2c(k)}^{(p-1)} - \mathrm{Id})^j \nabla_k^{c(k)}(g)$ *and the sequence*

$$B(g,\theta')_m := \sum_{i=0}^m \frac{1}{i!}\frac{\mu_{\theta'}^i}{(p-1)^i}\left(\sum_{j_1+\ldots+j_i \leq m}\left(\prod_{a=1}^i \frac{(-1)^{j_a-1}}{j_a}\right)(\nabla_{k+2c(k)}^{(p-1)} - \mathrm{Id})^{j_1+\ldots+j_a}\right)\nabla_k^{c(k)}(g)$$

converge in $H^0(\mathfrak{X}_{r',I} \otimes_{\mathbb{Z}_p} \Lambda_{I_\theta,K}, \mathbb{W})[\frac{1}{\alpha}]$*;*

(b) *the limit*

$$\nabla_{k+2c(k)}^{\theta'}\nabla_k^{c(k)}(g) := \exp\left(\frac{\mu_{\theta'}}{p-1}\log(\nabla_{k+2c(k)}^{p-1})\right)\nabla_k^{c(k)}(g) := \lim_{m\to\infty} B(g,\theta')_m$$

belongs to $H^0(\mathfrak{X}_{r',I} \otimes_{\mathbb{Z}_p} \Lambda_{I_\theta,K}, \mathbb{W}_{k+2\theta'})[\frac{1}{\alpha}]$*;*

(c) *the element*

$$\nabla_k^\theta(g) := \theta^\chi \nabla_{k+2c(k)+2\theta'}^{c(\theta)-c(k)} \nabla_{k+2c(k)}^{\theta'} \nabla_k^{c(k)}(g)$$

is well-defined in $H^0(\mathcal{X}_{r',I} \otimes_{\mathbb{Z}_p} \Lambda_{I_\theta,K}, \mathbb{W}_{k+2\theta})$.

Organization of this dissertation

We want to briefly describe the organization of this dissertation.

Chapter 1 contains only background materials. In § 1.1, we recall the notions of formal schemes and adic spaces, which will be our working languages for p-adic geometry. In § 1.2, we briefly introduce p-divisible groups and abelian schemes, which are the main geometric objects dealt with in this dissertation. In § 1.3, we review the de Rham cohomology theory of p-divisible groups and abelian schemes and recall the definition of the Gauss-Manin connection. The following three sections contain the main technical tools used in this dissertation. § 1.4 recalls the definition of the Serre-Tate coordinates and Katz's computation of the Gauss-Manin connection in these coordinates. § 1.5 contains basic properties of canonical subgroups of p-divisible groups and the Hodge-Tate period map. § 1.6 briefly reviews the machinery of vector bundles with marked sections with an emphasis on those bundles coming from p-divisible groups and sections defined via the Hodge-Tate period map. In § 1.7, besides a brief introduction of the adic weight space, we recall the p-adic measure theory of $\mathbb{Z}_p$, which will be used in the local calculations in the Serre-Tate coordinates.

In Chapter 2, we deal with nearly overconvergent quaternion modular forms. In § 2.1, we recall the moduli description of Shimura curves and fix some conventions. In § 2.2, we introduce the Hasse invariant of a false elliptic curve using a proper chosen direct summand of the associated p-divisible group, and then define the integral model $\mathfrak{X}_{r,I}$ and the partial Igusa tower $\mathfrak{IG}_{n,r,I}$. Some useful properties of $\mathfrak{IG}_{n,r,I}$ are also proved in this section. In § 2.3, we first construct the nearly overconvergent quaternion modular sheaf $\mathbb{W}_\kappa$ using the VBMS machinery, then study the filtration of and the Gauss-Manin connection on $\mathbb{W}_\kappa$. After that, we show the functoriality of $\mathbb{W}_\kappa$, and define Hecke operators on $\mathbb{W}_\kappa$. Finally, we compare the sheaf $\mathfrak{w}^\kappa := \mathrm{Fil}_0\mathbb{W}_\kappa$ with other constructions of overconvergent quaternion modular forms. § 2.4 deals with local behaviors of the Gauss-Manin connection and Hecke operators in Serre-Tate coordinates. In § 2.5, using these local computation results, we prove that the Gauss-Manin connection iterates p-adically. In § 2.6, we briefly mention several potential applications of our results.

The Appendix A reproves the results concerning overconvergent elliptic modular forms in [AI17] using the Serre-Tate local expansion, which is included for completeness. The organization of Appendix A is almost parallel to Chapter 2. More specifically, in § A.1, we recall the definitions of the compactified modular curve, the integral model $\mathfrak{X}_{r,I}$ and the partial Igusa tower $\mathfrak{IG}_{n,r,I}$. In § A.2, we first review the filtration of and the Gauss-Manin connection on the nearly overconvergent modular sheaf $\mathbb{W}_\kappa$, and discuss the functoriality of $\mathbb{W}_\kappa$. Then we define Hecke operators over $\mathbb{W}_\kappa$. Finally, we compare $\mathfrak{w}^\kappa := \mathrm{Fil}_0\mathbb{W}_\kappa$ with other constructions of overconvergent modular forms. In § A.3, we analyze the behavior of the Gauss-Manin connection and Hecke operators in Serre-Tate local coordinates and then in § A.4, we apply these local results to show that the Gauss-Manin connection iterates p-adically.

Notational conventions

We will try to always use the standard notation.

- All rings and ring homomorphisms will be unitary. Unless explicitly stated, all rings are assumed to be commutative and all modules are assumed to be left. Moreover, ring homomorphisms of topological rings are required to be continuous.
- For any ring R, the multiplicative group of units in R will be denoted by $R^\times$. For any positive integer n, the subgroup $\{x \in R^\times : x^n = 1\}$ will be denoted by $\mu_n(R)$.
- We will use the notion of schemes as in [Har13, § 2.1-2.5]. In particular, for any scheme S, we will use the superscript $-^\vee$ to denote the dual module of any $\mathcal{O}_S$-module.
- For a finite flat commutative group scheme (resp. a p-divisible group, resp. an abelian scheme) G over a base scheme, we will use G^D to denote the Cartier dual. For more detailed definitions, see § 1.2.
- We will use the notion of formal schemes as in [Bos14, Chapters 7-8] and the notion of adic spaces as in [Hub94, § 1-2]. For more information, see § 1.1.
- We will use $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ to denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively.
- We will always fix a rational prime number p, and use $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{F}_p$ to denote the ring of p-adic integers, the field of p-adic numbers and the finite field consisting of p elements respectively.
- For any perfect field k of characteristic p, we denote the associated ring of Witt vectors by $W(k)$.

1 Preliminaries

In this chapter, we will recall background materials and fix notations which will be used freely in later parts of this dissertation.

1.1 Formal schemes and adic spaces

In this section, we recall some basic properties of formal schemes and adic spaces.

1.1.1 Formal schemes

Our main reference for formal schemes is the book [Bos14].

Definition 1.1.1. *Let A be a topological ring.*

(i) We say A is I-adic for an ideal $I \subset A$ if the topology on A is the I-adic topology and A is I-adic complete and separated, i.e. $A = \varprojlim_n A/I^n$.

(ii) We say A is an adic (resp. f-adic) ring if A is I-adic for some ideal (resp. finitely generated ideal) $I \subset A$. Such an ideal I is called an ideal of definition.

Definition 1.1.2. *Given any f-adic ring A with an ideal of definition I, the formal spectrum $\mathrm{Spf}(A)$ is the set of open prime ideals of A equipped with the Zariski topology. For any $a \in A$, we denote the open subset of $\mathrm{Spf}(A)$ on which a does not vanish by $D(a)$ and we define the structure sheaf $\mathcal{O}$ on $\mathrm{Spf}(A)$ by the rule*

$$D(a) \mapsto A\langle a^{-1}\rangle := \varprojlim_n A/I^n[a^{-1}].$$

The pair $(\mathrm{Spf}(A), \mathcal{O})$ is a locally topologically ringed space, which is referred to as the affine formal scheme associated with A and denoted by $\mathrm{Spf}(A)$ again.

A formal scheme is a locally topologically ringed space $(X, \mathcal{O}_X)$ such that any $x \in X$ admits an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic to $\mathrm{Spf}(A)$ for some f-adic ring A.

The category of formal schemes admits fiber products.

Definition 1.1.3. *Let $\mathrm{Spf}(A)$, $\mathrm{Spf}(B)$ be formal scheme over $\mathrm{Spf}(R)$. We define the fiber product $\mathrm{Spf}(A) \times_{\mathrm{Spf}(R)} \mathrm{Spf}(B)$ to be $\mathrm{Spf}(A\hat{\otimes}_R B)$, where $A\hat{\otimes}_R B$ is the completed tensor product given by $\varprojlim_{n,m} A/I^n \otimes_R B/J^m$ for any ideals of definition $I \subset A$, $J \subset B$.*

By gluing, we can define the fiber product $X \times_Z Y$ of formal schemes X, Y over Z.

We can obtain formal schemes from schemes by formal completion along closed subschemes.

Definition 1.1.4. *Let X be a locally noetherian scheme and $Y \subset X$ be a closed subscheme defined by an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. The formal completion of X along Y is the formal scheme $(Y, \mathcal{O}_Y)$ where $\mathcal{O}_Y$ is the restriction of $\varprojlim_n \mathcal{O}_X/\mathcal{I}^n$ to Y.*

In this dissertation, we will only consider admissible formal schemes.

Definition 1.1.5. *Let R be an I-adic noetherian ring which is I-torsion-free.*

(i) An I-torsion-free topological R-algebra A is called admissible if A is isomorphic to an R-algebra of the form $R\langle X_1, ..., X_n\rangle / a$ equipped with the I-adic topology.

(ii) A formal R-scheme X is called admissible if there is an open affine covering $\{U_i\}$ of X such that $U_i \cong \mathrm{Spf}(A_i)$ with the R-algebra A_i being admissible for each i.

We conclude this subsection by recalling the notion of coherent sheaves and admissible formal blow-ups.

Definition 1.1.6. *Fix an I-adic noetherian ring R which is I-torsion-free.*

(i) For any admissible R-algebra A and any finitely generated A-module M, the $\mathcal{O}$-module $\tilde{M}$ on $\mathrm{Spf}(A)$ is defined by setting $\tilde{M}(D(a)) := M \otimes_A A\langle a^{-1}\rangle$.

(ii) For an admissible formal R-scheme X,

(a) an $\mathcal{O}_X$-module $\mathcal{F}$ is called coherent if there exists an open affine cover $\{U_i = \mathrm{Spf}(A_i)\}$ of X such that $\mathcal{F}_{|U_i} \cong \tilde{M}_i$ for some finitely generated A_i-module M_i;

(b) an ideal $\mathcal{I} \subset \mathcal{O}_X$ is called open if locally on X, $\mathcal{I}$ contains $I^n\mathcal{O}_X$ for some $n \geq 1$;

(c) for any coherent open ideal $\mathcal{I} \subset \mathcal{O}_X$, the formal R-scheme

$$X_{\mathcal{I}} = \varinjlim_n \mathrm{Proj}(\oplus_{d=0}^{\infty} \mathcal{I}^d \otimes_{\mathcal{O}_X} (\mathcal{O}_X / I^n \mathcal{O}_X))$$

together with the natural morphism $X_{\mathcal{I}} \to X$ is called the formal blow-up of $\mathcal{I}$ on X. Such a formal blow-up is called an admissible formal blow-up.

1.1.2 Adic spaces

Our main reference for adic spaces is the article [Hub94].

Definition 1.1.7. *Let A be a topological ring, and let Γ be a totally ordered abelian group (whose order is extended to the monoid $\Gamma \cup \{0\}$ by setting $0 < \gamma$ for any $\gamma \in \Gamma$). A Γ-valued continuous valuation is map $|\cdot| : A \to \Gamma \cup \{0\}$ such that*

(i) $|1| = 1, \quad |0| = 0$;

(ii) for any $a, b \in A$, $|ab| = |a||b|$, $|a + b| \leq \max\{|a|, |b|\}$;

(iii) for any $\gamma \in \Gamma$, the set $\{a \in A : |a| < \gamma\}$ is open.

Two continuous valuations $|\cdot|$, $|\cdot|'$ (not necessary having the same target) are called equivalent if

for any $a, b \in A$, $|a| \leq |b|$ if and only if $|a|' \leq |b|'$.

Definition 1.1.8. *(i) Let A be a topological ring.*

(a) a subset $S \subset A$ is called bounded if for all open neighborhoods U of 0, there exists an open neighborhood V of 0 such that $VS \subset U$.

(b) an element $x \in A$ is called power-bounded if the set $\{x^n : n \geq 0\}$ is bounded.

(c) *the set of power-bounded elements in* A *is denoted by* A°. *Its subset of topologically nilpotent elements is denoted by* $A^{\circ\circ}$.

(d) *a sub-ring* $A^+ \subset A$ *is called a ring of integral elements if* $A^+ \subset A^\circ$ *is open and integrally closed.*

We will call the ring A *Huber if there is an open subring* $A_0 \subset A$ *which is f-adic. Such a ring* A_0 *is called a ring of definition. If a Huber ring* A *contains a topological nilpotent unit, then we call it Tate.*

(ii) *A pair* (A, A^+) *consisting of a Huber ring* A *and a ring of integral elements* $A^+ \subset A$ *is called an affinoid ring. A morphism between affinoid rings*

$$f: \ (A, A^+) \to (B, B^+)$$

is a continuous ring homomorphism $f : A \to B$ *such that* $f(A^+) \subset B^+$.

Definition 1.1.9. *Let* (A, A^+) *be an affinoid ring. We define the adic spectrum* $\mathrm{Spa}(A, A^+)$ *to be the set of equivalence classes of continuous valuations* $|\cdot|$ *such that* $|A^+| \le 1$, *equipped with the topology generated by subsets of the form*

$$\{x : \ |f(x)| \le |g(x)| \neq 0\} \quad \text{for some } f, g \in A.$$

For elements $s_1, s_2, ..., s_n \in A$ *and finite subsets* $T_1, ..., T_n \subset A$ *such that* $T_iA \subset A$ *is open for* $i = 1, ..., n$, *we define the associated rational subset to be*

$$U(\{T_i/s_i\}) = U(\frac{T_1}{s_1}, ..., \frac{T_n}{s_n}) = \{x \in X : \ |t_i(x)| \le |s_i(x)| \neq 0; \ \forall \ t_i \in T_i\}.$$

Proposition 1.1.10. *Let* $X = \mathrm{Spa}(A, A^+)$ *for an affinoid ring* (A, A^+) *and* $U \subset X$ *be a rational subset. There exists an affinoid ring* $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ *together with a structure morphism*

$$(A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$$

such that the corresponding map $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to X$ *factors through* U, *and is universal for all such maps. Moreover, if*

(i) *A admits a noetherian ring of definition; or*

(ii) *A is Tate and strongly noetherian, i.e, the ring* $A\langle X_1, ..., X_n\rangle$ *is noetherain for any* $n \ge 0$,

then the pre-sheaves $\mathcal{O}_X$, $\mathcal{O}_X^+$ *of topological rings on* X *defined by sending any open* $W \subset X$ *to*

$$\mathcal{O}_X(W) := \varprojlim_{U \subset W \text{ rational}} \mathcal{O}_X(U), \quad \mathcal{O}_X^+(W) := \varprojlim_{U \subset W \text{ rational}} \mathcal{O}_X^+(U)$$

are sheaves.

Proof. Here we only sketch the construction of the localization $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ of (A, A^+). For more details, see [Hub94, § 1].

Choose a ring of definition $A_0 \subset A$ and a finitely generated ideal of definition $I \subset A_0$, and assume $U = U(\frac{T_1}{s_1}, ..., \frac{T_n}{s_n})$ for some $s_1, ..., s_n$ and $T_1, ..., T_n \subset A$ such that T_iA is open for $i = 1, .., n$. Then the $IA_0[\{T_i/s_i\}]$-adic topology on $A_0[\{T_i/s_i\}] := A_0[t_i/s_i | i = 1, .., n, t_i \in T_i]$ defines a ring topology on $A[\{1/s_i\}] := A[1/s_i | i = 1, ..., n]$ such that $A_0[\{T_i/s_i\}]$ is an open subring. Take $A[\{1/s_i\}]^+$ to be the integral closure of the image of $A^+[\{T_i/s_i\}]$ in $A[\{1/s_i\}]$. The affinoid ring $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is the completion $(A\langle\{T_i/s_i\}\rangle, A\langle\{T_i/s_i\}\rangle^+)$ with respect to the topology defined above. ∎

Following [Hub94], we have the following definition of adic spaces.

Definition 1.1.11. *(i) An affinoid ring (A, A^+) is called sheafy if the structure pre-sheaf $\mathcal{O}_X$ on $X = \mathrm{Spa}(A, A^+)$ is a sheaf. For any sheafy affinoid ring (A, A^+), the associated affinoid adic space is $\mathrm{Spa}(A, A^+)$ together with the structure sheaf $\mathcal{O}_X$ and the induced valuations on $\mathcal{O}_{X,x}$ for $x \in X$.*

(ii) Let (V) denote the category whose objects are triples $(X, \mathcal{O}_X, |\cdot(x)|_{x\in X})$ where X is a topological space, $\mathcal{O}_X$ is a sheaf of complete topological rings, and for each $x \in X$, $|\cdot(x)|$ is an equivalence class of valuations on $\mathcal{O}_{X,x}$ and whose morphisms between two objects $(X, \mathcal{O}_X, |\cdot(x)|_{x\in X})$ and $(Y, \mathcal{O}_Y, |\cdot(y)|_{y\in Y})$ are maps of locally topologically ringed topological spaces $f : X \to Y$ such that for any $x \in X$, the induced valuation

$$\mathcal{O}_{Y,f(x)} \xrightarrow{f} \mathcal{O}_{X,x} \xrightarrow{|\cdot(x)|} \Gamma_x \cup \{0\}$$

is equivalent to

$$|\cdot(f(x))| : \mathcal{O}_{Y,f(x)} \to \Gamma_{f(x)} \cup \{0\}.$$

The full subcategory of adic spaces in (V) consists of objects $(X, \mathcal{O}_X, |.(x)|_{x\in X})$ which admit an open covering by spaces U_i such that the triple $(U_i, \mathcal{O}_X|_{U_i}, (|.(x)|)_{x\in U_i})$ is isomorphic to an affinoid adic space.

(iii) Let X be an adic space. A point x in X is called analytic if there exists an open neighborhood U of x such that $\mathcal{O}_X(U)$ is Tate. The open subspace of X consisting of analytic points are denoted by X^a. If $X^a = X$, we call X analytic.

Definition 1.1.12. *Let $X = \mathrm{Spa}(A, A^+)$ for an affinoid ring (A, A^+) where A is strongly noetherian or A admits a noetherian ring of definition, and M be a finite generated A-module. We can define a pre-sheaf $\tilde{M}$ by sending an open subset $W \subset X$ to*

$$\tilde{M}(W) := \varprojlim_{U \subset W \text{ rational}} M \otimes_A \mathcal{O}_X(U).$$

The pre-sheaf $\tilde{M}$ is an acyclic sheaf. More precisely,

Proposition 1.1.13. [Hub94, Theorem 2.5] *With the notation in Definition 1.1.12, $\tilde{M}$ is a sheaf of topologically complete groups and $H^i(U, \tilde{M}) = 0$ for any rational subset $U \subset X$ and $i \geq 1$.*

The following result, which is a combination of [Hub94, Proposition 4.1 & 4.2], illustrates the relation between formal schemes and adic spaces.

Proposition 1.1.14. *The assignment $\mathrm{Spf}(R) \mapsto \mathrm{Spa}(R, R)$ defines a fully faithful functor $-^{\mathrm{ad}}$ from the category of locally noetherian formal schemes to the category of adic spaces. Moreover, over any locally noetherian formal scheme X, there is a functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{ad}}$ from $\mathcal{O}_X$-modules to any $\mathcal{O}_{X^{\mathrm{ad}}}$-modules which is fully faithful when restricted to coherent $\mathcal{O}_X$-modules.*

The category of analytic adic spaces admits fiber products.

Definition 1.1.15. *Let (A, A^+) be a Tate affinoid ring, and let*

$$f : (A, A^+) \to (B, B^+), \quad g : (A, A^+) \to (C, C^+)$$

be Tate affinoid rings over (A, A^+). The Tate assumption allows one to choose a ring of definition A_0 (resp. B_0, resp. C_0) of A (resp. B, resp. C) and an ideal of definition $I \subset A_0$ such that

$$f(A_0) \subset B_0, \quad f(A_0) \subset C_0$$

and $f(I)B_0$ (resp. $f(I)C_0$) is an ideal of definition of B_0 (resp. C_0). We can equip the fiber product $D := B \otimes_A C$ a Huber ring structure by declaring that $D_0 := B_0 \otimes_{A_0} C_0$ with the ID_0-adic topology is an open subring. Let D^+ be the integral closure of $f \otimes g(B^+ \otimes_{A^+} C^+)$. The completion of (D, D^+) is the fiber product of (B, B^+) and (C, C^+) over (A, A^+).

By gluing, we can define fiber product in the category of analytic adic spaces.

We conclude this section by a discussion about adic generic fibers of formal schemes. Let A_0 be an admissible $\mathbb{Z}_p$-algebra, $A = A_0[\frac{1}{p}]$ and A^+ be the normalization of A_0 in A. As A_0 is noetherian, $\mathrm{Spa}(A_0, A_0)$ and $\mathrm{Spa}(A, A^+)$ are affinoid adic spaces according to Proposition 1.1.10. For any locally noetherain p-adic formal scheme $\mathfrak{Y}$ over $\mathrm{Spf}(A_0)$, the adic generic fiber $\mathcal{Y}$ of $\mathfrak{Y}$ is the fiber product $\mathfrak{Y}^{\mathrm{ad}} \times_{\mathrm{Spa}(A_0,A_0)} \mathrm{Spa}(A, A^+)$. Inspired by [Kas99, Lemma 9.7], we have

Lemma 1.1.16. *If $\mathfrak{Y}$ is moreover noetherian and $\mathbb{Z}_p$-flat, then for any locally free $\mathcal{O}_{\mathfrak{Y}}$-module $\mathcal{F}$ of finite rank, we have*

$$H^0(\mathfrak{Y}, \mathcal{F})[\frac{1}{p}] = H^0(\mathcal{Y}, \mathcal{F}^{\mathrm{ad}}).$$

Proof. Let $\mathrm{Spf}(R) \subset \mathfrak{Y}$ be any affine open. For some ring of integral elements $R^+ \subset R[\frac{1}{p}]$, we have

$$\mathrm{Spa}(R, R) \times_{\mathrm{Spa}(A_0,A_0)} \mathrm{Spa}(A, A^+) = \mathrm{Spa}(R[\frac{1}{p}], R^+).$$

Assume $\mathcal{F}$ is trivialized over $\mathrm{Spf}(R)$. Since $\mathfrak{Y}$ is $\mathbb{Z}_p$-flat, we have

$$\mathcal{F}(\mathrm{Spf}(R)) \hookrightarrow \mathcal{F}(\mathrm{Spf}(R))[\frac{1}{p}] = H^0(\mathrm{Spa}(R[\frac{1}{p}], R^+), \mathcal{F}^{\mathrm{ad}}).$$

Since $\mathfrak{Y}$ is noetherian, we can cover $\mathfrak{Y}$ by finitely many affine opens $\{\mathrm{Spf}(R_i)\}_{i \in I}$ such that $\mathcal{F}$ is trivialized over each $\mathrm{Spf}(R_i)$. Thus for any $f \in H^0(\mathcal{Y}, \mathcal{F}^{\mathrm{ad}})$, there exists a constant h such that

$$p^h f \in H^0(\mathrm{Spf}(R_i), \mathcal{F}) = H^0(\mathrm{Spa}(R_i, R_i), \mathcal{F}^{\mathrm{ad}}), \ \forall\, i \in I.$$

Since $\mathcal{F}$ is a coherent sheaf, we know $p^h f \in H^0(\mathfrak{Y}, \mathcal{F}) = H^0(\mathfrak{Y}^{\mathrm{ad}}, \mathcal{F}^{\mathrm{ad}})$, and the lemma follows. □

By abuse of language, we will usually omit $-^{\mathrm{ad}}$ in the notation.

1.2 Abelian schemes and p-divisible groups

In this section, we recall basic properties of abelian schemes and p-divisible groups. Throughout this section, S will be a fixed base scheme.

1.2.1 Finite flat group schemes

Our main reference for finite flat group scheme is the exposition article [Tat97].

Let Sets, Gr be the categories of sets and groups respectively, and let Sch_S be the category of S-schemes.

Definition 1.2.1. *A group scheme G/S is an S-scheme together with a lifting of the functor* $\mathrm{Hom}(-, G): \mathrm{Sch}_S \to \mathrm{Sets}$ *to* $\mathrm{Hom}(-, G): \mathrm{Sch}_S \to \mathrm{Gr}$. *A group scheme G is called*

(i) commutative if $G(T) := \mathrm{Hom}(T, G)$ is commutative for all $T \in \mathrm{Sch}_S$;

(ii) finite flat if $\mathcal{O}_G$ is locally free of finite rank as $\mathcal{O}_S$-module. This rank is a locally constant function on S with values in $\mathbb{N}$, which is called the rank of G and denoted by $[G : S]$;

- *étale if the underlying scheme is étale over S.*
- *connected if the underlying scheme is connected (assuming S is connected).*

A homomorphism of group schemes is a morphism of schemes which induces a natural transformation of the corresponding group functors.

Example 1.2.2. We list several important group schemes.

(i) The additive group over S is the group functor sending an S-scheme T to the additive group $\mathcal{O}_T(T)$. It is represented by the base change of $\mathrm{Spec}(\mathbb{Z}[X])$ to S, and denoted by $G_{a,S}$.

(ii) The multiplicative group is the group functor sending an S-scheme T to the multiplicative group $\mathcal{O}_T(T)^*$. It is represented by the base change of $\mathrm{Spec}(\mathbb{Z}[X, X^{-1}])$ to S and denoted by $G_{m,S}$.

(iii) For an abstract finite group G, the constant group scheme associated to G over S is the group functor sending an S-scheme T to the group of locally constant functions from the underlying topological space $|T|$ to G. It is represented by $\sqcup_{g\in G} S$ and denoted by $\underline{G}_S$.

(iv) For any group homomorphism $\phi: G \to G'$ of S-groups, the group functor sending an S-scheme T to $\ker(\phi: G(T) \to G'(T))$ is representable. This group scheme is called the kernel of ϕ and denoted by $\ker(\phi)$. In particular, the kernel of the n-th power map $[n]: G \to G$ is denoted by $G[n]$. If $G = G_{m,S}$, we denote $G_{m,S}[n]$ by $\mu_{n,S}$.

When S is clear from content, we usually omit it in notations.

It would be convenient to discuss finite flat group schemes using the f.p.p.f site.

Definition 1.2.3. *Let X be a S-scheme. An f.p.p.f covering of X is a family of scheme morphisms $\{f_i : X_i \to X\}$ such that each $f_i: X_i \to X$ is flat, locally of finite presentation and $\cup_{i\in I} f_i(X_i) = X$.*

The category Sch_S together with f.p.p.f coverings is called the f.p.p.f site on S. We will denote sheaves (resp. sheaves of groups) over the f.p.p.f site on S by f.p.p.f sheaves (resp. S-groups).

Proposition 1.2.4. *Let S be a locally noetherian scheme.*

(i) Let G/S be a group scheme and $H \subset G$ be a a finite flat closed subgroup. The geometric quotient $\pi: G \to G/H$ exists. If H is moreover normal, then G/H is a group scheme which fits into the exact sequence of f.p.p.f S-groups

$$0 \to H \to G \to G/H \to 0.$$

Moreover, G is finite flat if and only if G/H is finite flat and $[G : S] = [H : S][G : G/H]$.

(ii) *If G is a finite commutative group scheme of rank n, then n kills G, i.e. the n-th power map $[n]: G \to G$ is the 0 map.*

(iii) *If S is the spectrum of a Henselian local ring, then we have the connected-étale exact sequence of f.p.p.f S-groups*

$$0 \to G^0 \to G \to G^{\text{ét}} \to 0$$

where G° is the connected component of G and $G^{\text{ét}}$ is the maximal étale quotient group of G.

Proof. (i) For details, see [Tat97, § 3.4]. (ii) This is a theorem of Deligne, see [Tat97, § 3.8]. (iii) See [Tat97, § 3.7.1]. □

Theorem 1.2.5. *Let G/S be a finite flat commutative group scheme. Then the inner-Hom functor*

$$T \mapsto \mathrm{Hom}_T(G_T, G_{m,T})$$

on Sch_S is representable by a finite flat group scheme over S, which is called the Cartier dual of G and denoted by G^D. Moreover, the Cartier dual induces a duality on the category of finite flat commutative group schemes over S.

Proof. For details, see [Tat97, § 3.8]. □

1.2.2 Abelian schemes

Our main reference for abelian schemes is the lecture notes [Mil08]. The unpublished book [vdGM07] is also very useful.

We start with several definitions. Recall that S is a fixed base scheme.

Definition 1.2.6. (i) *A surjective group homomorphism $f: G \to G'$ with quasi-finite kernel between smooth S-group schemes is called an isogeny.*

(ii) *An abelian scheme of relative dimension g over S is a S-group scheme $A \to S$ which is proper, flat and finitely-presented, and has smooth and connected geometric fiber of dimension g. In particular, an abelian scheme of relative dimension 1 is called an elliptic curve, an abelian scheme of relative dimension 2 is called an abelian surface, and abelian schemes over a field are called abelian varieties.*

(iii) *A semi-abelian scheme over S is a group scheme $\pi: G \to S$ that is separated, smooth, commutative and such that for each point $s \in S$, the fiber G_s is an extension of an abelian variety by a torus*

$$0 \to T_s \to G_s \to A_s \to 0.$$

We collect several general properties of (semi-)abelian schemes here.

Proposition 1.2.7. (i) *Let A/S be an abelian scheme. Then A is commutative and any morphism $A \to G$ to a separated S-group G that maps identity to identity is a S-group homomorphism. Moreover, let $\bar{s} \in S$ be a geometric point, and $A_{\bar{s}}$ be the fiber of A at $\bar{s}$. Then the natural map $\mathrm{End}_S(A) \hookrightarrow \mathrm{End}_{\bar{s}}(A_{\bar{s}})$ is injective.*

(ii) *Let $f: A \to A'$ be any isogeny between abelian S-schemes of dimension g. Then f is finite and locally free. In particular,* $\ker(f)$ *is a finite flat group scheme over S. The rank of* $\ker(f)$ *is called the degree of f and denoted by* $\deg(f)$.

(iii) *For any abelian scheme A/S of relative dimension g, the N-power map $[N]: A \to A$ is an isogeny of degree N^{2g}. If N is invertible over S, then $A[N]$ is étale.*

(iv) *Let $f: A \to A'$ be any degree-N isogeny between abelian S-schemes. Then there exists a unique isogeny $g: A' \to A$ such that $f \circ g = [N]$ and $g \circ f = [N]$.*

(v) *There is a contravariant involution $-^D$ on the category of abelian schemes over S with isogeny. Moreover, any invertible sheaf $\mathcal{L}$ over any abelian scheme A/S defines a homomorphism $\lambda(\mathcal{L}): A \to A^D$ and for any isogeny $f: A \to B$ of abelian S-schemes with duality $f^D: B^D \to A^D$, the Cartier dual of* $\ker(f)$ *is* $\ker(f^D)$.

(vi) *Assume S is locally noetherian. Let A/S be a semi-abelian scheme and $H \subset A$ be a closed subgroup, flat and quasi-finite over S. Then A/H is a semi-abelian scheme if H is étale or A is étale locally quasi-projective.*

Proof. For details of $(i) - (iv)$, we refer to [Mil08, § 15]. For (v), see [FC13, Chapter 1.1]. For (vi), see [Lan13, Lemma 3.4]. □

Definition 1.2.8. *Let A/S be an abelian scheme. A homomorphism $\lambda: A \to A^D$ of S-groups is called a polarization of A if for each geometric point $\bar{s} \in S$, $\lambda_{\bar{s}} = \lambda(\mathcal{L}_{\bar{s}})$ for some ample invertible sheaf $\mathcal{L}_{\bar{s}}$ on $A_{\bar{s}}$. A polarization $\lambda: A \to A^D$ is called principal if λ is an isomorphism.*

Proposition 1.2.9. *Let $\pi: A \to S$ be an abelian scheme. Any polarization $\lambda: A \to A^D$ is an isogeny and $(\lambda^D)^D = \lambda$ upon the identification $(A^D)^D \cong A$.*

Let $\mathcal{L}$ be a relative ample invertible sheaf over A. Then $\lambda(\mathcal{L})$ is a polarization, which is principal if and only if $\pi_*(\mathcal{L})$ is invertible.

Proof. See the discussion in [FC13, Page 4]. □

For a polarized abelian variety, there is a natural involution on the endomorphism ring.

Definition 1.2.10. *Let k be any field and B be a (non-commutative) k-algebra. An involution on B is a k-linear map $-: B \to B$ such that*

$$\bar{1} = 1; \quad \overline{ab} = \bar{b}\bar{a}; \quad \bar{\bar{a}} = a, \ \forall\ a, b \in B.$$

If $a\bar{a} \in k$ for all $a \in B$, the involution is called standard.

If $k = \mathbb{Q}$ and B is finite dimensional, an involution $- : B \to B$ is called positive if for any $a \neq 0 \in B$, we have $Tr(\bar{a}a) > 0$ where Tr is the trace defined by left multiplication.

Definition 1.2.11. *Let A/k be an abelian variety with a polarization $\lambda: A \to A^D$. The Rosati involution associated to λ is an involution*

$$\dagger := \dagger_\lambda : \ \mathrm{End}^\circ(A) := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{End}^\circ(A), \quad \alpha \mapsto \lambda^{-1} \circ \alpha^D \circ \lambda.$$

If λ is principal, then the Rosati involution preserves the sub-ring $\mathrm{End}(A)$.

The Rosati involution is positive up to multiplication by ± 1.

Proposition 1.2.12. *Let A/k be a polarized abelian variety over an algebraically field $k = \bar{k}$. Then $\mathrm{End}^\circ(A)$ is a finite dimensional semi-simple algebra over $\mathbb{Q}$ of rank at most $4\dim_k(A)^2$, and up to multiplication by ± 1, the Rosati convolution $\dagger$ is positive. Moreover, if we have a decomposition $\mathrm{End}^\circ(A) = \prod_{i=1}^r M_{n_i}(D_i)$ for some positive integers $n_1, .., n_r$ and division algebras D_i, then there exists an isogeny*

$$A \to A_1^{n_1} \times ... \times A_r^{n_r}$$

for pair-wisely non-isogenous simple abelian varieties A_i, i.e. A_i admits no non-trivial abelian subvarieties.

Proof. For details, see [Mil08, Theorem 13.2 & Proposition 9.1]. □

1.2.3 p-divisble groups

Our treatment closely follows [Wan09].

There are two equivalent definitions of p-divisible groups.

Definition 1.2.13. *A p-divisible group G over S is an f.p.p.f group G such that:*

(i) the group G is p-divisible i.e. the p-power map $[p]: G \to G$ is an epimorphism;

(ii) the group G is p-torsion i.e. $G = \varinjlim_n G[p^n]$;

(iii) for each $n \geq 1$, $G[p^n]$ is a finite flat group scheme over S.

Equivalently, a p-divisible group G/S is an inductive system $\{G_n\}_{n\geq 1}$ of finite flat group schemes over S such that $G_n = G_{n+1}[p^n]$ and for each point $s \in S$, the rank of the fiber of $G(n)$ at s is $p^{nh(s)}$ where $h(s)$ is locally constant function on S, called the height of G.

An f.p.p.f S-group morphism $f : G \to G'$ between two p-divisible groups is called an isogeny if it is an f.p.p.f epimorphism with locally finite free kernel, i.e. the kernel is representable by a finite flat group scheme.

We list several basic properties of p-divisible groups.

Proposition 1.2.14. *(i) The category of p-divisible groups is pseudo-abelian, stable under base change and extensions. More precisely, we have the following:*

(a) Let G/S be a p-divisible group and $e \in \mathrm{End}_S(G)$ be a non-trivial idempotent. Then $\ker(e)$ and $\ker(1-e)$ are both p-divisible groups over S and $G = \ker(e) \oplus \ker(1-e)$.

(b) Let $f : S' \to S$ be a morphism of schemes and G/S be a p-divsible group. Then $f^*(G)$ is a p-divisible group over S'.

(c) If we have an exact sequence of f.p.p.f S-groups

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

and G_1,G_3 are p-divisible groups, then so is G_2 and $h(G_2) = h(G_1) + h(G_3)$.

(ii) Let G/S be a p-divisible group and H/S be a finite flat group scheme together with a monomorphism of f.p.p.f S-groups $\iota : H \to G$. Then the f.p.p.f S-group G/H is also a p-divisible group.

(iii) *If S is connected or quasi-compact, then a homomorphism $f: G \to G'$ between p-divisible groups is an isogeny if and only if there exists a homomorphism $g: G' \to G$ such that $f \circ g = p^N$ and $g \circ f = p^N$ for some positive integer N.*

(iv) *Let $G = \{G(n)\}_{n\geq 1}$ be a p-divisible group over S. The inductive system $\{G(n)^D\}_{n\geq 1}$ with transition maps $p^D: G(n)^D \to G(n+1)^D$ is also a p-divisible group over S, called the Cartier dual (or the Serre dual) of G and denoted by G^D. Moreover, the Cartier dual induces a duality on the category of p-divisible groups over S.*

(v) *Assume $S = \mathrm{Spec}(R)$ for some henselian local ring, and let $G = \{G(n)\}_{n\geq 1}$ be a p-divisible group over R. Then $G^\circ = \{G(n)^\circ\}_{n\geq 1}$ and $G^{\text{ét}} = \{G(n)^{\text{ét}}\}_{n\geq 1}$ are p-divisible groups over R and we have the connected-étale exact sequence of f.p.p.f groups*

$$0 \to G^\circ \to G \to G^{\text{ét}} \to 0.$$

(vi) *If A/S is an abelian scheme of relative dimension g, then $A[p^\infty] := \{A[p^n]\}$ is a p-divisible group of height $2g$. Moreover, $A[p^\infty]^D = A^D[p^\infty]$.*

Proof. See [Wan09, § 1.1] and the references therein. □

As we will work over formal schemes, we recall the notion of p-divisible groups over formal schemes following de Jong.

Definition 1.2.15. *Let $\mathfrak{X}$ be a locally Noetherian formal scheme. Let $\mathfrak{I} \subset \mathcal{O}_{\mathfrak{X}}$ be the maximal ideal of definition and $X_n = \mathrm{Spec}(\mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^n)$. A p-divisible group G over $\mathfrak{X}$ is a system $\{G_n\}_{n\geq 1}$ of p-divisible groups G_n/X_n together with an isomorphism $G_{n+1} \otimes_{X_{n+1}} X_n \cong G_n$ for each $n \geq 1$.*

1.2.4 Frobenius and Verschiebung

Assume S is an $\mathbb{F}_p$-scheme, and let $f: X \to S$ be a morphism of $\mathbb{F}_p$-schemes. We have a commutative diagram

$$\begin{tikzcd}
X \arrow[dr, dotted, "F"] \arrow[drr, bend left, "F_{abs}"] \arrow[ddr, bend right] & & \\
& X^{(p)} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& S \arrow[r, "F_{abs}"] & S
\end{tikzcd}$$

where the square is Cartesian (and defines $X^{(p)}$) and F_{abs} is the absolute Frobenius map. The map

$$F: X \to X^{(p)}$$

in the above diagram is called the Frobenius map (relative to S).

Proposition 1.2.16. *If G is a commutative flat S-group scheme, then we can define a morphism of S-schemes $V: G^{(p)} \to G$, called the Verschiebung map, so that both F and V are S-group homomorphisms and $V \circ F = [p]: G \to G$. Moreover, we have the following*

(i) *if G is an abelian scheme over S of relative dimension g, then F and V are isogenies of degree p^g and $V \circ F = F \circ V = [p]$.*

(ii) if G is finite locally free over S, then the Verschiebung morphism is the Cartier dual of the Frobenius morphism, i.e. $V_{G/S}^D = F_{G^D/S}$ and $(F_{G/S})^D = V_{G^D/S}$.

Proof. For details, see [vdGM07, § 5.2]. ☐

Let $G = \{G(n)\}$ be a p-divisible over S. Then $G^{(p)} = \{G(n)^{(p)}\}$ is also a p-divisible over S. Note that the Frobenius and Verschiebung morphisms are compatible for all $n \geq 1$, so we have isogenies of p-divisible groups

$$F: \; G \to G^{(p)}; \quad V: \; G^{(p)} \to G$$

such that $F \circ V = V \circ F = [p]$.

1.3 The Hodge-de Rham filtration and the Gauss-Manin connection

In this section, we recall basic properties of algebraic de Rham cohomology and Gauss-Manin connections of abelian schemes (and p-divisible groups). Throughout this section, S will be a fixed base scheme.

1.3.1 Algebraic de Rham cohomology

Our main reference for algebraic de Rham cohomology is [BBM06].

Definition 1.3.1. *Let $f: \; X \to S$ be any morphism of schemes and let*

$$\Omega^\bullet_{X/S} := (0 \to \mathcal{O}_X \to \Omega^1_{X/S} \to \Omega^2_{X/S} \to ...)$$

be the de Rham complex. For each $i \in \mathbb{N}$, we define the i-th relative de Rham cohomology group $\mathbb{H}^i_{\mathrm{dR}}(X/S)$ to be $R^i f_*(\Omega^\bullet_{X/S})$. The filtration $F^\bullet \Omega^\bullet_{X/S}$ of $\Omega^\bullet_{X/S}$ given by

$$F^j \Omega^\bullet_{X/S} := (0 \to ... \to 0 \to \Omega^j_{X/S} \to \Omega^{j+1}_{X/S} \to ...), \quad \forall j \in \mathbb{N}$$

induces a descending, separated and exhaustive filtration

$$F^j \mathbb{H}^i_{\mathrm{dR}}(X/S) = \mathrm{Im}(R^i f_*(F^j \Omega^\bullet_{X/S}) \to R^i f_*(\Omega^\bullet_{X/S})), \quad \forall j \in \mathbb{N}$$

on $\mathbb{H}^i_{\mathrm{dR}}(X/S)$ for each $i \in \mathbb{N}$. This filtration is called the Hodge filtration. By taking the Cartan-Ellenberg resolution, we get the Hodge to de Rham spectral sequence

$$E_1^{p,q} = R^q f_* \Omega^p_{X/S} \Longrightarrow \mathbb{H}^{p+q}_{\mathrm{dR}}(X/S).$$

The algebraic de Rham cohomology of abelian schemes is well-understood. To simplify notations, for an abelian scheme $\pi: \; A \to S$, we will denote the vector bundle $\pi_* \Omega^1_{A/S}$ by $\underline{\omega}_{A/S}$.

Proposition 1.3.2. *Let $\pi: \; A \to S$ be an abelian scheme with zero section $e: \; S \to A$. The Hodge to de Rham spectral sequence of A degenerates at the first page and we have a canonical exact sequence of locally free $\mathcal{O}_S$-modules of finite rank*

$$0 \to \underline{\omega}_{A/S} \to \mathbb{H}^1_{\mathrm{dR}}(A/S) \to R^1 \pi_* \mathcal{O}_A \to 0.$$

whose formation commutes with base change on S and which is functorial with respect to A. Moreover, *(i) we have an identification* $\underline{\omega}_{A/S} = e^*\Omega^1_{A/S}$*;*

(ii) for each $i \geq 0$, $\mathbb{H}^i_{\mathrm{dR}}(A/S) = \wedge^i \mathbb{H}^1_{\mathrm{dR}}(A/S)$.

Let $\pi' : A^D \to S$ *be the dual abelian scheme. Then there is a canonical perfect pairing*

$$\mathbb{H}^1_{\mathrm{dR}}(A/S) \times \mathbb{H}^1_{\mathrm{dR}}(A^D/S) \to \mathcal{O}_S$$

which induces a perfect pairing

$$\underline{\omega}_{A/S} \times R^1\pi'_*\mathcal{O}_{A^D} \to \mathcal{O}_S.$$

Proof. The computation of algebraic de Rham cohomology is [BBM06, Proposition 2.5.2]. Details about the perfect pairing can be found in [BBM06, Chapter 5]. □

The algebraic de Rham cohomology groups are equipped with the Gauss-Manin connection.

Definition 1.3.3. *Let* X *be any* S*-scheme with relative differentials* $\Omega^1_{X/S}$*, and* $\mathcal{M}$ *be an* $\mathcal{O}_X$*-module. A connection on* $\mathcal{M}$ *over* S *is a map*

$$\nabla : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

such that for any open sub-scheme $U \subset X$*, we have*

$$\nabla(hs) = h\nabla(s) + d(h) \otimes s, \ \forall\ s \in \mathcal{M}(U),\ h \in \mathcal{O}_X(U).$$

A section $s \in \mathcal{M}(U)$ *is called horizontal with respect to* ∇ *if* $\nabla(s) = 0$.

For any $i \geq 1$*, a connection* ∇ *on* $\mathcal{M}$ *induces a map*

$$\nabla :\ \mathcal{M}\otimes_{\mathcal{O}_X}\Omega^i_{X/Y} \to \mathcal{M}\otimes_{\mathcal{O}_X}\Omega^{i+1}_{X/Y}, \quad s\otimes h \mapsto s\otimes dh + (-1)^i\nabla s\otimes h,\ \forall\ h \in \Omega^i_{X/Y}(U),\ s \in \mathcal{M}(U).$$

If the induced sequence

$$\mathcal{M} \xrightarrow{\nabla} \mathcal{M}\otimes\Omega^1_{X/Y} \xrightarrow{\nabla} \mathcal{M}\otimes\Omega^2_{X/Y}\cdots$$

is a complex, then the connection ∇ *is called flat or integrable.*

Alternatively, we have the Grothendieck's description of connections. Equip the $\mathcal{O}_X$*-module* $\mathcal{R} := \mathcal{O}_X \oplus \Omega^1_{X/S}$ *with the ring structure given by:*

$$(f,\alpha)(g,\beta) = (fg, f\beta + g\alpha).$$

We have a projection $\mathcal{R} \to \mathcal{O}_X, (f,\alpha) \mapsto f$ *together with two sections*

$$\mathcal{O}_X \to \mathcal{R},\ f \mapsto (f,0);\quad f \mapsto (f, df),$$

which corresponds to a diagram

$$X \xrightarrow{\ \Delta\ } X^1 := \mathrm{Spec}_X(\mathcal{R}) \overset{j_1}{\underset{j_2}{\rightrightarrows}} X\ .$$

Then a connection on the $\mathcal{O}_X$*-module* $\mathcal{M}$ *is an isomorphism of* $\mathcal{O}_{X^1}$*-modules*

$$\epsilon :\ j_2^*(\mathcal{M}) \cong j_1^*(\mathcal{M})$$

such that $\Delta^*(\epsilon) = \mathrm{Id}$*. Moreover,* ϵ *is flat if it satisfies suitable co-cycle conditions. The relation between* ϵ *and the usual definition* ∇ *is given by*

$$\epsilon(1\otimes m) = m\otimes 1 + \nabla(m), \quad \forall m \in \mathcal{M}.$$

For connections with logarithmic poles, a similar description is also available.

Definition 1.3.4. *Let $f: X \to Y$ be a smooth morphism of S-schemes. We have a canonical exact sequence*

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0. \tag{1.3.1}$$

The Kodaira-Spencer map is the connection map

$$\mathrm{KS}: \ f_*\Omega^1_{X/Y} \to R^1 f_*(f^*\Omega^1_{Y/S}) = R^1 f_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}.$$

The sequence 1.3.1 induces a filtration $F_K^\bullet \Omega^\bullet_{X/S}$ of $\Omega^\bullet_{X/S}$ given by

$$F^i_K \Omega^\bullet_{X/S} := \mathrm{Im}(f^*\Omega^i_{Y/S} \otimes_{\mathcal{O}_X} \Omega^{\bullet - i}_{X/S} \to \Omega^\bullet_{X/S}), \quad \forall i \in \mathbb{N}.$$

As $\mathrm{Gr}^p \Omega^\bullet_{X/S} = f^*\Omega^p_{Y/S} \otimes_{\mathcal{O}_X} \Omega^{\bullet - p}_{X/Y}$, this filtration induces a spectral sequence

$$E_1^{p,q} := \Omega^p_{Y/S} \otimes_{\mathcal{O}_Y} \mathbb{H}^q_{\mathrm{dR}}(X/Y) \Longrightarrow R^{p+q} f_* \Omega^\bullet_{X/S}.$$

The differential

$$d_1^{0,q}: \ \mathbb{H}^q_{\mathrm{dR}}(X/Y) \to \Omega^1_{Y/S} \otimes_{\mathcal{O}_Y} \mathbb{H}^q_{\mathrm{dR}}(X/Y)$$

is called the Gauss-Manin connection and denoted by $\nabla_{X/Y}$.

According to the discussion in [KO68], we have

Proposition 1.3.5. *Let $f: X \to Y$ be a smooth morphism of S-schemes. The Gauss-Manin connection*

$$\nabla_{X/Y}: \ \mathbb{H}^q_{\mathrm{dR}}(X/Y) \to \mathbb{H}^q_{\mathrm{dR}}(X/Y) \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}$$

is flat. Moreover, the Kodaira-Spencer map KS *coincides with the composition*

$$f_*\Omega^1_{X/Y} \to \mathbb{H}^1_{\mathrm{dR}}(X/Y) \xrightarrow{\nabla_{X/Y}} \mathbb{H}^q_{\mathrm{dR}}(X/Y) \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S} \to R^1 f_*\mathcal{O}_X \otimes \Omega^1_{Y/S}.$$

The Gauss-Manin connection and Kodaira-Spencer map have the following properties.

Proposition 1.3.6. *Let S be any base scheme.*

(i) Let $X \to Y$ be a smooth morphism of S-schemes with Y smooth over S. Let (Y', i_0) be a closed smooth S-subscheme of Y, $X' = X \times_Y Y'$ and let $\iota_0: i_0^*\Omega^1_{Y/S} \to \Omega^1_{Y'/S}$ be the canonical pull-back map. The Gauss-Manin connections $\nabla_{X/Y}$ and $\nabla_{X'/Y'}$ satisfy

$$\nabla_{X'/Y'} \circ i_0^* = (1 \otimes \iota_0) \circ i_0^* \nabla_{X/Y}.$$

(ii) If S is connected and $\pi: A' \to A$ is an isogeny of abelian schemes over S, then

$$\nabla_{A'/S} \circ \pi^* = (\pi^* \otimes \mathrm{Id}) \circ \nabla_{A/S}.$$

In particular, the action of the (non-commutative) endomorphism ring $\mathrm{End}_S(A)$ on $\mathbb{H}^1_{\mathrm{dR}}(A/S)$ is $\mathcal{O}_S$-linear and commutes with Gauss-Manin connections.

Proof. For details, we refer to [Mor11, § 2.1]. □

1.3.2 Universal vector extensions of p-divisible groups

Our discussion on universal extensions is close to [Wan09, § 1.2-1.3]. The book [MM06] and lecture notes [CO09] are also very helpful.

Recall that S is a fixed base scheme.

Definition 1.3.7. *To any quasi-coherent $\mathcal{O}_S$-module M, we may associate an f.p.p.f sheaf $\tilde{M}$ by the assignment*

$$\mathrm{Sch}_S \to \mathrm{Gr}; \quad T \mapsto \Gamma(T, M_T)$$

where M_T is the pull-back of M to T. If M is locally free, then $\tilde{M}$ is representable by a group scheme which is locally a product of $\mathbb{G}_a$'s. Such group schemes are called vector groups.

For any commutative f.p.p.f group G/S, a universal vector extension is an exact sequence of f.p.p.f sheaves

$$(\epsilon) \quad 0 \to V(G) \to E(G) \to G \to 0$$

such that $V(G)$ is a vector group and (ϵ) is universal for all extensions of G by S-vector groups, i.e. for any vector group V, (ϵ) induces an isomorphism

$$\mathrm{Hom}_S(V(G), V) \to \mathrm{Ext}^1_S(G, V).$$

Thanks to the discussion in [MM06, § 1.8-1.9], universal vector extensions of abelian schemes and p-divisible groups exist. More precisely,

Theorem 1.3.8. *(i) If A/S is an abelian scheme, then the universal vector extension $E(A)$ of A exists and is functorial with respect to A.*

(ii) If p is locally nilpotent on S and G/S is a p-divisible group, then the universal extension $E(G)$ of G exists and is functorial with respect to G. Moreover, if $G = A[p^\infty]$ for some abelian scheme A, then the pull-back of $E(A)$ to G coincides with $E(G)$.

One can define Lie algebra and invariant differentials of the universal vector extensions of abelian schemes and p-divisible groups.

Definition 1.3.9. *(i) For any f.p.p.f sheaf G/S and any $k \geq 1$, we set $\inf^k(G)$ to be the subsheaf of G defined as following: an element $t \in G(T)$ belongs to $\inf^k(G)(T)$ if there exists a f.p.p.f covering $\{T_i \to T\}$ and for each i, there is a closed subscheme $T_i' \to T_i$ defined by a sheaf of ideal whose $k+1$-power is zero such that $t_{|_{T_i'}} = 0$. The injective limit $\varinjlim_k \inf^k(G)$ is the infinitesimal completion of G, denoted by $\hat{G}$.*

(ii) An S-scheme G is called formally smooth if for any affine scheme X/S and a closed subscheme X_0 defined by an ideal sheaf $\mathcal{I}$ such that $\mathcal{I}^2 = 0$, any morphism of S-schemes $X_0 \to G$ lifts to a morphism $X \to G$ of S-schemes.

(iii) An f.p.p.f S-group G is an formal Lie group if G is formally smooth, $\inf^k(G)$ is representable for each $k \geq 1$ and $\hat{G} = G$.

Proposition 1.3.10. *(i) For any abelian scheme A/S, both $\hat{A}$ and $\widehat{E(A)}$ are formal Lie groups. In fact, $\hat{A}$ is just the completion of A along the zero section.*

(ii) If p is locally nilpotent on S and G/S is a p-divisible group, then both $\hat{G}$ and $\widehat{E(G)}$ are formal Lie groups.

Proof. For details, see the discussion in [Wan09, § 1.2]. □

It is known that locally a formal Lie group over S has the form $\mathrm{Spf}(\mathcal{O}_S[[X_1, ..., X_N]])$ for some $N \geq 0$.

Definition 1.3.11. *For any formal Lie group G over S with unit section s, we define the sheaf of invariant differentials $\underline{\omega}_G$ to be $s^*\Omega^1_{G/S}$ and define the Lie algebra $\mathrm{Lie}(G)$ to be $\underline{\omega}_G^\vee$.*

If p is locally nilpotent on S and G/S is a p-divisible group, we set

$$\mathrm{Lie}(G) := \mathrm{Lie}(\hat{G}), \quad \mathrm{Lie}(E(G)) := \mathrm{Lie}(\widehat{E(G)}), \quad \underline{\omega}_G := \underline{\omega}_{\hat{G}}.$$

The $\mathcal{O}_S$-rank of $\underline{\omega}_G$ is called the dimension of G.

Theorem 1.3.12. *Let S be any base scheme.*

(i) For any abelian scheme A/S, the universal vector extension is given by

$$0 \to \underline{\omega}_{A^D} \to E(A) \to A \to 0.$$

Moreover, by taking invariant differentials of the above exact sequence, one recovers the Hodge-de Rham sequence

$$0 \to \underline{\omega}_A \to \mathbb{H}^1_{\mathrm{dR}}(A/S) \to R^1\pi_*\mathcal{O}_A \to 0.$$

(ii) If p is locally nilpotent on S and G/S is a p-divisible group, then the universal vector extension is given by

$$0 \to \underline{\omega}_{G^D} \to E(G) \to G \to 0.$$

Let $\mathbb{H}^1_{\mathrm{dR}}(G/S)$ be the invariant differential of $E(G)$. Then $\mathbb{H}^1_{\mathrm{dR}}(G/S)$ admits a flat connection, called the Gauss-Manin connection, and fits in the exact sequence

$$0 \to \underline{\omega}_G \to \mathbb{H}^1_{\mathrm{dR}}(G/S) \to \mathrm{Lie}(G^D) \to 0.$$

If $G = A[p^\infty]$ for some abelian scheme A/S, then we recover the Hodge-de Rham sequence of A and the Gauss-Manin connection on $\mathbb{H}^1_{\mathrm{dR}}(A/S)$.

Proof. For details, we refer to [BBM06, Chapter 5] or [Mes72, Chapter IV]. □

We remark that by passing to the limit, similar results remain valid for p-divisible groups over an adic formal scheme on which p is topologically nilpotent.

1.4 The Gauss-Manin connection in Serre-Tate coordinates

In this section, we recall general computation results of the Gauss-Manin connection in Serre-Tate coordinates following the exposition article [Kat81].

1.4.1 Serre-Tate theorem

Let R be a ring on which p is nilpotent and Ab_R be the category of abelian schemes over R with group homomorphisms. Let $I \subset R$ be a nilpotent ideal, $R_0 = R/I$ and $\mathrm{Def}(R, R_0)$ be the category of triples (A_0, G, ϵ) consisting of an abelian scheme A_0/R_0, a p-divisible group G/R and an isomorphism of p-divisible groups

$$\epsilon : \; G \times_R R_0 \cong A_0[p^\infty]$$

with obvious morphisms. The theorem of Serre-Tate states

Theorem 1.4.1. [Kat81, Theorem 1.2.1] *The functor*

$$\mathrm{Ab}_R \to \mathrm{Def}(R, R_0), \quad A \mapsto (A \times_R R_0, A[p^\infty], \mathrm{Id})$$

induces an equivalence of categories.

Let k be a perfect field of characteristic p and denote the category of local artinian $W(k)$-algebras with residue field (isomorphic to) k together with the obvious morphisms as $\mathcal{C}_k$.

Definition 1.4.2. *For any abelian variety $\bar{A}/k$, a deformation of $\bar{A}$ to an object $R \in \mathcal{C}_k$ is a pair (A, ϵ) consisting of an abelian scheme A/R and an isomorphism*

$$\epsilon : \; A \times_R k \to \bar{A}.$$

According to Grothendieck, the deformation functor is pro-representable.

Theorem 1.4.3. [Oor71, Theorem 2.2.1] *Let $\bar{A}/k$ be an abelian variety of dimension g over a perfect field k of characteristic p. Then the functor $\widehat{\mathcal{M}}_{\bar{A}/k}$ which to an object $R \in \mathcal{C}_k$ associates the isomorphism classes of deformations of $\bar{A}$ to R is pro-represented by a complete, local noetherian ring $\mathcal{R}$ which is non-canonically isomorphic to $W(k)[[t_{ij}, 1 \le i, j \le g]]$.*

If k is algebraically closed and $\bar{A}/k$ is ordinary, thanks to the work of Serre-Tate, we have a canonical way to choose the coordinates t_{ij}.

Definition 1.4.4. *Let S be a scheme on which p is locally nilpotent or an adic formal scheme on which p is topologically nilpotent. A p-divisible group G/S is called ordinary if G sits in the middle of a short exact sequence*

$$0 \to G_1 \to G \to G_2 \to 0$$

where G_1/S, G_2/S are p-divisible groups such that G_1^D and G_2 are étale. An abelian scheme A/S is called ordinary if $A[p^\infty]$ is ordinary.

We remark that if $S = \mathrm{Spec}(R)$ for a Henselian local ring R, then G is ordinary if and only if the Cartier dual of G° is étale.

Theorem 1.4.5. *Let $\bar{A}$ be an ordinary abelian variety over an algebraically closed field k of characteristic p, and let $T_p\bar{A} := \varprojlim_n \bar{A}[p^n](k)$ be the "physical" Tate module.*

(i) There is an isomorphism of functors

$$\widehat{\mathcal{M}}_{\bar{A}/k}(-) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p\bar{A} \otimes_{\mathbb{Z}_p} T_p\bar{A}^D, \hat{G}_m(-)).$$

In particular, $\widehat{\mathcal{M}}_{\bar{A}/k}$ has the structure of a formal torus.

(ii) *For any $R \in \mathcal{C}_k$ and any deformation A of $\bar{A}/k$ to R, let*

$$q(A/R; -, -): \ T_p\bar{A} \times T_p\bar{A}^D \to \hat{G}_m(R)$$

be the corresponding bilinear form. For any $\alpha \in T_p\bar{A}$ and $\alpha' \in T_p\bar{A}^D$, we have

$$q(A/R; \alpha, \alpha') = q(A^D/R; \alpha', \alpha).$$

(iii) *Let $\bar{f}: \ \bar{A} \to \bar{B}$ be a homomorphism to another ordinary abelian variety $\bar{B}/k$, and let A/R, B/R be deformations of $\bar{A}$, $\bar{B}$ to $R \in \mathcal{C}_k$ respectively. Then $\bar{f}$ admits a lifting (necessarily unique) $f: \ A \to B$ if and only if*

$$q(A/R; \alpha, \bar{f}^D(\beta)) = q(B/R; \bar{f}(\alpha), \beta), \ \forall\ \alpha \in T_p\bar{A}, \ \ \beta \in T_p\bar{B}^D.$$

Proof. A detailed argument can be found in [Kat81, Theorem 2.1]. We sketch the construction of $q(A/R; -, -)$ here for the convenience of readers. Let $\bar{P} = \{\bar{P}_n\} \in T_p\bar{A}$, $\bar{Q} = \{\bar{Q}_n\} \in T_p\bar{A}^D$, and let A/R be a deformation of $\bar{A}/k$. For any lift P_n of $\bar{P}_n$ in $A(R)$, as

$$\hat{A}(R) := \ker(A(R) \to A(k) = \bar{A}(k))$$

is killed by p^n for $n \gg 0$, the element p^nP_n is independent of the choice of P_n. Moreover, for each $n \geq 1$, the Weil pairing $e: \ A[p^n] \times A^D[p^n] \to \mu_{p^n}$ together with the exact sequences

$$0 \to \hat{A}[p^n](R) \to A[p^n](R) \to \bar{A}[p^n](k) \to 0;$$
$$0 \to \hat{A^D}[p^n](R) \to A^D[p^n](R) \to \bar{A^D}[p^n](k) \to 0$$

induces a pairing $e: \ \hat{A}[p^n] \times A^D[p^n](k) \to \mu_{p^n}$. Therefore the element $e(p^nP_n, \bar{Q}_n) \in \mu_{p^n}(R)$ is well-defined and independent of n. The bilinear form $q(A/R; -, -)$ is obtained by setting $q(A/R, \bar{P}, \bar{Q}) := e(p^nP_n, \bar{Q}_n)$. $\square$

Remark 1.4.6. *(i) By passing to the limit, the universal deformation $\mathcal{A}/\mathcal{R}$ of $\bar{A}/k$ corresponds to a bilinear form $q(\mathcal{A}/\mathcal{R}; -, -): \ T_p\bar{A} \otimes_{\mathbb{Z}_p} T_p\bar{A}^D \to \hat{G}_m(\mathcal{R})$.*

(ii) *When k is not algebraically closed, the group structure on $\widehat{\mathcal{M}} \otimes \bar{k}$ descends to a group structure on $\widehat{\mathcal{M}}$. For details, see* [Noo92, § 1.1].

1.4.2 The Gauss-Manin connection in Serre-Tate coordinates

Let k be an algebraically closed field of characteristic p, $\bar{A}/k$ be an ordinary abelian variety of dimension g and $\pi: \ \mathcal{A} \to \mathcal{R}$ be the universal deformation of $\bar{A}/k$. Denoting $\mathbb{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{R})$ by $\mathbb{H}_{\mathcal{A}}$, we have the (formal) Hodge-de Rham sequence

$$0 \to \underline{\omega}_{\mathcal{A}/\mathcal{R}} \to \mathbb{H}_{\mathcal{A}} \to \mathrm{Lie}(\mathcal{A}^D/\mathcal{R}) \to 0$$

as well as the (formal) Gauss-Manin connection $\nabla: \ \mathbb{H}_{\mathcal{A}} \to \mathbb{H}_{\mathcal{A}} \otimes \Omega^1_{\mathcal{R}/W(k)}$. For any (formal) scheme S over k (resp. $W(k)$), let $S^{(p)}$ be the pullback of S via the absolute Frobenius morphism. Let $\widehat{\mathcal{M}}_{\bar{A}/k}$ be the deformation space of $\bar{A}/k$. Then according to the discussion in [Kat81, § 4.1], we have a canonical isomorphism

$$\widehat{\mathcal{M}}^{(p)}_{A/k} \cong \widehat{\mathcal{M}}_{A^{(p)}/k}$$

which identifies $\mathcal{A}^{(p)}$ with the universal deformation of $\bar{A}^{(p)}$.

Let $H \subset \mathcal{A}$ be the canonical subgroup of level 1 (see Proposition 1.5.5 for more details). Then $\mathcal{A}' := \mathcal{A}/H$ is a lifting of $\bar{A}^{(p)}$, and we have a unique classifying map $\widehat{\mathcal{M}}_{\bar{A}/k} \xrightarrow{\psi} \widehat{\mathcal{M}}_{\bar{A}^{(p)}/k}$ such that $\psi^*(\mathcal{A}^{(p)}) = \mathcal{A}'$. The composition map

$$\phi: \ \widehat{\mathcal{M}}_{\bar{A}/k} \xrightarrow{\psi} \widehat{\mathcal{M}}_{\bar{A}^{(p)}/k} \cong (\widehat{\mathcal{M}}_{\bar{A}/k})^{(p)} \to \widehat{\mathcal{M}}_{\bar{A}/k}$$

satisfies that $\phi^*(\mathcal{A}) = \mathcal{A}'$. Moreover, the quotient isogeny $\mathcal{A} \to \mathcal{A}'$ over $\widehat{\mathcal{M}}_{\bar{A}/k}$ induces a ϕ-linear map $F: \ \mathbb{H}_{\mathcal{A}} \to \mathbb{H}_{\mathcal{A}}$ which makes the following diagram commute

$$\begin{tikzcd}
0 \arrow[r] & \underline{\omega}_{\mathcal{A}/\mathcal{R}} \arrow[r] \arrow[d, "F"] & \mathbb{H}_{\mathcal{A}} \arrow[r] \arrow[d, "F"] & \mathrm{Lie}(\mathcal{A}^D/\mathcal{R}) \arrow[r] \arrow[d, "F"] & 0, \\
0 \arrow[r] & \underline{\omega}_{\mathcal{A}/\mathcal{R}} \arrow[r] & \mathbb{H}_{\mathcal{A}} \arrow[r] & \mathrm{Lie}(\mathcal{A}^D/\mathcal{R}) \arrow[r] & 0.
\end{tikzcd}$$

View the vector bundles $\underline{\omega}_{\mathcal{A}/\mathcal{R}}$, $\mathbb{H}_{\mathcal{A}}$ and $\mathrm{Lie}(\mathcal{A}^D/\mathcal{R})$ as $\mathcal{R}$-modules.

Proposition 1.4.7. *(i) We have a canonical isomorphism*

$$\mathrm{HT}: \ T_p\bar{A}^D \otimes_{\mathbb{Z}_p} \mathcal{R} \cong \underline{\omega}_{\mathcal{A}/\mathcal{R}}$$

which induces an isomorphism

$$T_p\bar{A}^D \cong \{x \in \mathbb{H}_{\mathcal{A}}: \ F(x) = px\}.$$

(ii) Dually, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p\bar{A}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{R} \cong \mathrm{Lie}(\mathcal{A}^D/\mathcal{R}).$$

which induces an isomorphism

$$\eta: \ \mathrm{Hom}_{\mathbb{Z}_p}(T_p\bar{A}, \mathbb{Z}_p) \cong \{x \in \mathbb{H}_{\mathcal{A}}: \ F(x) = x\}$$

via the quotient map $\mathbb{H}_{\mathcal{A}} \to \mathrm{Lie}(\mathcal{A}^D/\mathcal{R})$.

Proof. Our formulation here follows [Bro13, Lemma IV.2]. For details, see [Kat81, Lemma 4.2.1]. For more properties of the Hodge-Tate period morphism HT, see § 1.5.2. ☐

Let $\{P_1, ..., P_g\}$ be a $\mathbb{Z}_p$-basis of $T_p\bar{A}$, $\{P_1^t, ..., P_g^t\}$ be the dual basis, and $\{Q_1, ..., Q_g\}$ be a $\mathbb{Z}_p$-basis of $T_p\bar{A}^D$. By setting

$$t_{ij} = q(\mathcal{A}/\mathcal{R}; P_i, Q_j) - 1 \in \mathcal{R},$$

we have $\mathcal{R} = W(k)[[t_{ij}, 1 \leq i, j \leq g]]$.

Theorem 1.4.8. [Kat81, Theorem 4.3.1] *For $1 \leq i \leq g$, we have $\nabla(\eta(P_i^t)) = 0$ and for any $Q \in T_p\bar{A}^D$,*

$$\nabla(\mathrm{HT}(Q)) = \sum_{i=1}^{g} \eta(P_i^t) \otimes d\log q(\mathcal{A}/\mathcal{R}; P_i, Q).$$

1.5 Canonical subgroups and the Hodge-Tate period morphism

In this section, we review basic properties of canonical subgroups and the Hodge-Tate period morphism for p-divisible groups, which will be the main technical tools in this dissertation. Our exposition closely follows [AIP, Appendix A].

1.5.1 Canonical subgroups

Definition 1.5.1. *Let S be a scheme of characteristic p and G/S be a p-divisible group of height h and dimension d. The Veischiebung map $V: G^{(p)} \to G$ induces a morphism*

$$\underline{\omega}_{G/S} \xrightarrow{\mathrm{HW}(G)} \underline{\omega}_{G^{(p)}/S},$$

called the Hasse-Witt matrix of G. The determinant $\mathrm{Ha}(G) \in (\wedge^d \omega_{G/S})^{\otimes(p-1)}$ of $\mathrm{HW}(G)$ is called the Hasse invariant of G.

The Hasse invariant of an ordinary abelian scheme is invertible. More precisely,

Proposition 1.5.2. *Let S be a scheme of characteristic p and let A/S be an abelian scheme of relative dimension n. For $G := A[p^\infty]$, the following are equivalent:*

(i) the Hasse invariant $\mathrm{Ha}(G)$ is invertible;

(ii) the abelian scheme A is ordinary;

(iii) for any geometric point $\bar{s}$ of the étale site $S_{\text{ét}}$, $|A[p](\bar{s})| = p^n$;

(iv) for any geometric point $\bar{s} \in S_{\text{ét}}$,

$$G_{\bar{s}} \cong \mu_{p^\infty}^n \times (\mathbb{Q}_p/\mathbb{Z}_p)^n.$$

Proof. The key ingredient is the fact that an isogeny between abelian schemes is étale if and only if its kernel is an étale group scheme. For more details, we refer to [Sch15, Proposition III.2.5]. □

We need to lift the Hasse invariant to the p-adic setting.

Definition 1.5.3. *Let R be any p-adically complete ring. Let G/R be a p-divisible group of dimension d and $\bar{G} := G \times_R R/p$. We define the Hodge ideal $\mathrm{Hdg}(G)$ to be the inverse image of the ideal $\mathrm{Ha}(\bar{G})(\wedge^d \underline{\omega}_{\bar{G}})^{\otimes(1-p)} \subset R/p$ in R. When G is clear from context, we simply write $\mathrm{Hdg}(G)$ as Hdg.*

The following will used frequently.

Lemma 1.5.4. [AIP, Lemma A.1] *Zariski locally, the ideal Hdg is generated by two elements. If $p \in \mathrm{Hdg}^2$, then Hdg is invertible.*

In the rest of this section, we will fix an integral domain A and a non-zero element $\alpha \in A$ such that

(i) the ring A is α-adic;

(ii) there is a continuous morphism $\mathbb{Z}_p \to A$.

Proposition 1.5.5. [AIP, Corollary A.2] *Let G be a p-divisible group of height h and dimension $d \leq h$ over an α-adic complete A-algebra R which is A-torsion-free. Suppose $p \in \mathrm{Hdg}^{p^{m+1}}$ for some positive integer m. For each $1 \leq n \leq m$, G admits a canonical subgroup H_n of level n which is locally free of rank p^{nd} and*

$$H_n \equiv \mathrm{Ker}(F^n) \bmod p\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}$$

where F^n is the n-th iteration of the relative Frobenius map.

(i) Let $G' := G/H_n$. Then $\mathrm{Hdg}(G') = \mathrm{Hdg}(G)^{p^n}$ and for each $n' \leq m-n$, G' admits a canonical subgroup $H'_{n'}$ of level n' which fits into the exact sequence

$$0 \to H_n \to H_{n'+n} \to H'_{n'} \to 0.$$

In particular, H_n is a subgroup of H_{n+m}.

(ii) The conormal sheaf $\underline{\omega}_{G[p^n]/H_n}$ is annihilated by $\mathrm{Hdg}(G)^{\frac{p^n-1}{p-1}}$, and we have

$$\det \underline{\omega}_{G[p^n]/H_n} = \det \underline{\omega}_{G[p^n]}/\mathrm{Hdg}^{\frac{p^n-1}{p-1}}.$$

(iii) If $\alpha \in \mathrm{Hdg}(G)$, then $G[p^n]/H_n$ is étale over $R[\frac{1}{\alpha}]$ and locally isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^{n-d}$.

(iv) We have $\mathrm{Hdg}(G) = \mathrm{Hdg}(G^D)$, and the Weil pairing $G[p^n] \times G^D[p^n] \to \mu_{p^n}$ induces an identification $H_n(G) = H_n(G^D)^{\perp}$.

1.5.2 The Hodge-Tate period morphism

Definition 1.5.6. *Let S be any base scheme and $\frac{dt}{t}$ be the standard invariant differential on $G_{m,S}$. Then for any finite flat commutative group scheme H/S, we have a canonical map of sheaves*

$$\mathrm{HT}: \ H^D = \mathcal{H}om(H, G_m) \to \underline{\omega}_{H/S}; \quad f \mapsto f^*(\frac{dt}{t}).$$

If p is locally nilpotent on S and G/S is a p-divisible group, we define T_pG to be the sheaf on Sch_S given by $T \mapsto \varprojlim_n G[p^n](T)$. Then by taking the limit with respect to $G[p^n]$, we have a canonical map of sheaves $\mathrm{HT}: T_pG \to \underline{\omega}_{G^D}$. The analogous definition works when S is an adic formal scheme on which p is topologically nilpotent.

The following proposition plays a crucial role in this dissertation.

Proposition 1.5.7. *Let $\alpha = p$ and S be a p-adic normal formal scheme over A without A-torsion, and let G/S be a p-divisible group of height h and dimension d. Assume that there exists a positive integer n such that $p \in \mathrm{Hdg}^{p^{n+1}}$. Then the level-$n$ canonical subgroup $H_n \subset G$ exists, and we have*

$$\ker(\underline{\omega}_G \to \underline{\omega}_{H_n}) \subset p^n\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\underline{\omega}_G.$$

In particular, we have a well-defined map, which will also be denoted by HT by abuse of notation,

$$H_n^D \xrightarrow{\mathrm{HT}} \underline{\omega}_{H_n} \to \frac{\underline{\omega}_G}{p^n\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\underline{\omega}_G}.$$

Moreover assume that S is connected and $H_n^D(S) \cong (\mathbb{Z}/p^n\mathbb{Z})^d$.

(i) *Let* $\Omega_G \subset \underline{\omega}_G$ *be the* $\mathcal{O}_S$*-submodule generated by some (hence any) lift of the images of a* $\mathbb{Z}/p^n\mathbb{Z}$*-basis of* $H_n^D(S)$ *along* HT*. Then* Ω_G *is locally free of rank* d*.*

(ii) *The* $\mathcal{O}_S$*-module* $[\det(\Omega_G) : \det(\underline{\omega}_G)]$ *is an invertible ideal whose* $(p-1)$*-th power is* Hdg*. We will denote it by* $\mathrm{Hdg}^{\frac{1}{p-1}}$*.*

(iii) *Setting* $\mathcal{I}(G) := p^n \mathrm{Hdg}^{-\frac{p^n}{p-1}}$*, then* $\mathrm{Hdg}^{\frac{1}{p-1}}\underline{\omega}_G \subset \Omega_G \subset \underline{\omega}_G$*, and the map* HT *induces an isomorphism*

$$H_n^D(S) \otimes_{\mathbb{Z}} \mathcal{O}_S/\mathcal{I}(G) \cong \Omega_G \otimes_{\mathcal{O}_S} \mathcal{O}_S/\mathcal{I}(G).$$

Proof. Our statement follows the discussion in [AI17, § 6.1]. For the main part and item (i), note that S is normal, so we can reduce to the case $S = \mathrm{Spf}(R)$ for some complete discrete valuation ring R of characteristic $(0,p)$, which in turn follows from the discussion in [AIP15, § 3.2]. Item (ii) follows from [AIP, Proposition A.3] and item (iii) can be deduced directly from items (i) and (ii). □

Remark 1.5.8. *Let* $\mathbb{C}_p$ *be a complete algebraically closed non-Archimedean field over* $\mathbb{Q}_p$*(for example, the* p*-adic completion of any algebraic closure of* $\mathbb{Q}_p$*), and let* $\mathcal{O}_{\mathbb{C}_p}$ *be its ring of integers. Then for any* p*-divisible group* $G/\mathcal{O}_{\mathbb{C}_p}$*, the Hodge-Tate period morphism* HT *induces the Hodge-Tate complex*

$$0 \to \mathrm{Lie}(G)(1) \xrightarrow{\mathrm{HT}^\vee_{G^D}(1)} T_pG \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p} \xrightarrow{\mathrm{HT}_G} \underline{\omega}_{G^D} \to 0$$

whose cohomology groups are killed by $p^{\frac{1}{p-1}}$*. For details, we refer to* [SW13, Proposition 4.3.6] *and the references therein.*

1.6 Formal vector bundles with marked sections

In this section, we briefly introduce the machinery of vector bundles with marked sections (abbreviated as VBMS), following the article [AI17]. Throughout this section, S will be a fixed connected formal scheme with an invertible ideal of definition $\mathcal{I} \subset \mathcal{O}_S$, and $\bar{S}$ will be the reduction of S modulo $\mathcal{I}$. We will only consider formal schemes $f : T \to S$ with the ideal of definition $f^*(\mathcal{I}) \subset \mathcal{O}_T$.

We first recall the notion of formal vector bundles.

Definition 1.6.1. *A formal vector bundle of rank* n *over* S *is a formal scheme* $f : X \to S$ *together with extra data, called charts, consisting of an affine open covering* $\{U_i\}_{i\in I}$ *of* S *and for each* $i \in I$*, an isomorphism* $\psi_i : X_{|U_i} := f^{-1}(U_i) \cong \mathbb{A}^n_{U_i}$*, such that for any* $i, j \in I$ *and any affine open formal scheme* $U \subset U_i \cap U_j$*, the induced map* $\psi_j \circ \psi_i^{-1}$ *on* $\mathbb{A}^n_U$ *is a linear automorphism.*

A morphism $g : X \to Y$ *between two formal vector bundles over* S *of ranks* n*,* n'*, with charts* $(\{U_i\}, \psi_i)_{i\in I}$*,* $(\{U'_j\}, \psi'_j)_{j\in J}$ *respectively is a morphism of* S*-formal schemes such that for every* $i \in I$*,* $j \in J$ *and every affine open subscheme* $U \subset U_i \cap U'_j$*, the composition*

$$\mathbb{A}^n_U \xrightarrow{\psi_i^{-1}} X_{|U} \xrightarrow{g_{|U}} Y_{|U} \xrightarrow{\psi'_j} \mathbb{A}^{n'}_U$$

is a linear map.

According to [AI17, Lemma 2.2], for any locally free $\mathcal{O}_S$-module $\mathcal{E}$ of finite rank over S, there is a unique formal vector bundle $\mathcal{V}(\mathcal{E})$ over S such that $\mathcal{V}(\mathcal{E})(T) = H^0(T, t^*(\mathcal{E})^\vee)$ for any formal scheme $t: T \to S$. Moreover, the functor $\mathcal{E} \mapsto \mathcal{V}(\mathcal{E})$ induces a rank-preserving anti-equivalence between the category of locally free $\mathcal{O}_S$-modules of finite rank and the category of formal vector bundles over S of finite rank.

Definition 1.6.2. *Let $\mathcal{E}$ be locally free $\mathcal{O}_S$-module of finite rank over S and $\bar{\mathcal{E}}$ be the associated $\mathcal{O}_{\bar{S}}$-module. A set of marked sections of $\mathcal{E}$ is a set of global sections $\{s_1, .., s_d\} \subset H^0(\bar{S}, \bar{\mathcal{E}})$ such that the induced map*

$$\oplus_{i=1}^d \mathcal{O}_{\bar{S}} \to \bar{\mathcal{E}}; \quad (a_1, ..., a_d) \mapsto \sum a_i s_i$$

identifies $\mathcal{O}_{\bar{S}}^d$ with a locally direct summand of $\bar{\mathcal{E}}$.

Definition 1.6.3. *A system of vector bundles with marked sections over S (with respect to $\mathcal{I}$) is a triple $(\mathcal{E}, \mathcal{F}, \{s_1, ..., s_d\})$ consisting of a locally free $\mathcal{O}_S$-module $\mathcal{E}$ of finite rank, a locally free $\mathcal{O}_S$-submodule $\mathcal{F} \subset \mathcal{E}$ of rank d such that $\mathcal{E}/\mathcal{F}$ is also locally free, and a set of marked sections $\{s_1, ..., s_d\}$ of $\mathcal{F}$. For simplicity, we denote $(\mathcal{E}, \mathcal{E}, \{s_1, \cdots, s_d\})$ by $(\mathcal{E}, \{s_1, ...s_d\})$.*

A morphism between two systems of vector bundles with marked sections

$$f: \ (\mathcal{E}, \mathcal{F}, \{s_1, ..., s_d\}) \to (\mathcal{E}', \mathcal{F}', \{s_1', ..., s_d'\})$$

is a morphism of $\mathcal{O}_S$-modules $f: \mathcal{E} \to \mathcal{E}'$ with modulo $\mathcal{I}$ reduction $\bar{f}$ such that

$$f(\mathcal{F}) \subset \mathcal{F}'; \quad \bar{f}(s_i) = s_i', \ \forall i = 1, ..., d.$$

Lemma 1.6.4. *Let $(\mathcal{E}, \{s_1, ..., s_d\})$ be a system of vector bundle with marked sections over S.*

(i) The subfunctor $\mathcal{V}_0(\mathcal{E}, s_1, ..., s_d)$ of $\mathcal{V}(\mathcal{E})$ that associates to any formal scheme $t: T \to S$ the subset of sections $\rho \in H^0(T, t^*(\mathcal{E})^\vee)$ whose modulo $\mathcal{I}$ reduction $\bar{\rho}$ satisfies $\bar{\rho}(t^*(s_i)) = 1$ for every $i = 1, ..., d$ is represented by an open formal subscheme of an admissible formal blow up of $\mathcal{V}(\mathcal{E})$.

(ii) If there is a $\mathcal{O}_S$-submodule $\mathcal{F} \subset \mathcal{E}$ of rank d such that $(\mathcal{E}, \mathcal{F}, \{s_1, ..., s_d\})$ is a system of vector bundles with marked sections, then we have a Cartesian diagram

$$\begin{tikzcd} \mathcal{V}_0(\mathcal{E}, s_1, ..., s_d) \arrow[r] \arrow[d] & \mathcal{V}(\mathcal{E}) \arrow[d] \\ \mathcal{V}_0(\mathcal{F}, s_1, ..., s_d) \arrow[r] & \mathcal{V}(\mathcal{F}) \end{tikzcd}$$

whose vertical arrows are principal homogeneous spaces under the group of affine transformations $\mathcal{A}_S^{h-d}$. Let

$$f_0: \ \mathcal{V}_0(\mathcal{E}, s_1, ..., s_d) \to S, \quad f_0': \ \mathcal{V}_0(\mathcal{F}, s_1, ..., s_d) \to S$$

be the structure morphisms. Then $f_{0,*}\mathcal{O}_{\mathcal{V}_0(\mathcal{E}, s_1, ..., s_d)}$ is endowed with an increasing filtration $\mathrm{Fil}_\bullet$ such that

(a) the sheaf $f_{0,*}\mathcal{O}_{\mathcal{V}_0(\mathcal{E}, s_1, ..., s_d)}$ is the $\mathcal{I}$-adic completion of $\varinjlim_d \mathrm{Fil}_d f_{0,*}\mathcal{O}_{\mathcal{V}_0(\mathcal{E}, s_1, ..., s_d)}$;

(b) for each $h \in \mathbb{N}$, we have $\mathrm{Gr}_h f_{0,*}\mathcal{O}_{\mathcal{V}_0(\mathcal{E}, s_1, ..., s_d)} \cong f_{0,*}'\mathcal{O}_{\mathcal{V}_0(\mathcal{F}, s_1, ..., s_d)} \otimes_{\mathcal{O}_S} \mathrm{Sym}^h(\mathcal{E}/\mathcal{F})$.

Proof. We give a sketch here. For details of item (i), see [AI17, Lemma 2.4], and for details of item (ii), see [AI17, Lemma 2.5 & Corollary 2.6].

Let $U = \mathrm{Spf}(R) \subset S$ be an open affine such that $\mathcal{I}(U) = (\tau)$, $\mathcal{F}(U)$ is a free R-module with a basis $\{e_i\}_{i=1,..,d}$ satisfying $e_i \mod \tau = s_i$, and $\mathcal{E}(U)$ is a free R-module with a basis $\{e_i\}_{i=1,...,d'}$. Then by sending e_i to X_i, we have

$$\mathcal{V}(\mathcal{E})_{|U} = \mathrm{Spf}(R\langle X_1, ..., X_{d'}\rangle), \quad \mathcal{V}(\mathcal{F})_{|U} = \mathrm{Spf}(R\langle X_1, ..., X_d\rangle).$$

Let $Z_i = \frac{X_i - 1}{\tau}$ for each $i = 1, ..., d$. Unwinding the definition, we have

$$\mathcal{V}_0(\mathcal{E}, s_1, ..., s_d)_{|U} = \mathrm{Spf}(R\langle Z_1, ..., Z_d, X_{d+1}, ..., X_{d'}\rangle), \quad \mathcal{V}_0(\mathcal{F}, s_1, ..., s_d)_{|U} = \mathrm{Spf}(R\langle Z_1, ..., Z_d\rangle),$$

and $\mathrm{Fil}_h f_{0,*}\mathcal{O}_{\mathcal{V}_0(\mathcal{E},s_1,...,s_d)}(U)$ consists of polynomials of degree $\leq h$ in the variables $X_{d+1}, ..., X_{d'}$ with $R\langle Z_1, ..., Z_d\rangle$-coefficients. This local description allows us to prove the desired results. $\square$

In cases that we want to emphasize the ideal $\mathcal{I}$, we put it at the end of the notation. For example, we write $\mathcal{V}_0(\mathcal{E}, s_1, .., s_d)$ with respect to I as $\mathcal{V}_0(\mathcal{E}, s_1, ..., s_d; \mathcal{I})$.

We have the following functoriality result.

Lemma 1.6.5. *Let*

$$g : \ (\mathcal{E}', \mathcal{F}', \{s'_1, ..., s'_d\}) \to (\mathcal{E}, \mathcal{F}, \{s_1, ..., s_d\})$$

be a morphism of systems of vector bundles with marked sections over S. Then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{V}_0(\mathcal{E}, s_1, ..., s_d) \arrow[r] \arrow[d, "g"] & \mathcal{V}_0(\mathcal{F}, s_1, ..., s_d) \arrow[d, "g"] \\
\mathcal{V}_0(\mathcal{E}', s'_1, ..., s'_d) \arrow[r] & \mathcal{V}_0(\mathcal{F}', s_1, ..., s_d).
\end{tikzcd}$$

Consequently, the induced morphism

$$g : \ f'_{0,*}\mathcal{O}_{\mathcal{V}_0(\mathcal{E}',s'_1,...,s'_d)} \to f_{0,*}\mathcal{O}_{\mathcal{V}_0(\mathcal{E},s_1,..,s_d)}$$

is compatible with filtration, where

$$f_0 : \ \mathcal{V}_0(\mathcal{E}, s_1, ..., s_d) \to S, \quad f'_0 : \ \mathcal{V}_0(\mathcal{E}', s'_1, ..., s'_d) \to S$$

are the structure morphisms.

Fix a $\mathbb{Z}_p$-algebra A together with a non-zero divisor $\tau \in A$ such that A is τ-adic and assume S is a τ-adic formal scheme locally of finite type over $\mathrm{Spf}(A)$. Note that $\mathcal{I}$ and (τ) defines the same topology. Let $(\mathcal{E}, \mathcal{F}, \{s_1, \cdots, s_d\})$ be a system of vector bundles with marked sections and $\nabla : \ \mathcal{E} \to \mathcal{E}\hat{\otimes}_{\mathcal{O}_S}\Omega^1_{S/A}$ be a flat connection such that every s_i is horizontal for the reduction of ∇ modulo $\mathcal{I}$.

Lemma 1.6.6. *We have a flat connection*

$$\nabla_0 : \ f_{0,*}\mathcal{O}_{\mathcal{V}_0(\mathcal{E},s_1,...,s_d)} \to f_{0,*}\mathcal{O}_{\mathcal{V}_0(\mathcal{E},s_1,...,s_d)}\hat{\otimes}_{\mathcal{O}_S}\Omega^1_{S/A},$$

where $\hat{\otimes}$ is the τ-adic completed tensor product. Moreover, we have the following:

(i) The connection ∇_0 satisfies Griffiths transversality with respect to $\mathrm{Fil}_\bullet$, *i.e.*

$$\nabla_0(\mathrm{Fil}_h f_{0,*}(\mathcal{O}_{\mathcal{V}_0(\mathcal{E},s_1,...,s_d)})) \subset \mathrm{Fil}_{h+1} f_{0,*}(\mathcal{O}_{\mathcal{V}_0(\mathcal{E},s_1,...,s_d)})\hat{\otimes}_{\mathcal{O}_S}\Omega^1_{S/A}, \quad \forall \ h \in \mathbb{N}.$$

(ii) The induced map

$$\mathrm{Gr}_h(\nabla_0): \ \mathrm{Gr}_h f_{0,*}\mathcal{O}_{\mathcal{V}_0(\mathcal{E},s_1,...,s_d)} \to \mathrm{Gr}_{h+1} f_{0,*}\mathcal{O}_{\mathcal{V}_0(\mathcal{E},s_1,...,s_d)}\hat{\otimes}_{\mathcal{O}_S}\Omega^1_{S/A}$$

is $\mathcal{O}_S$-linear for each $h \in \mathbb{N}$ and $\mathrm{Gr}_\bullet(\nabla_0)$ is $\mathrm{Sym}^\bullet(\mathcal{E}/\mathcal{F})$-linear.

Proof. The connection is constructed using Grothendieck's description (see Definition 1.3.3). More precisely, let $S^1 = \mathrm{Spf}(\mathcal{O}_S \oplus \Omega^1_{S/A})$, j_1, j_2 be the two projections, and $\epsilon : j_1^*(\mathcal{E}) \to j_2^*(\mathcal{E})$ be the isomorphism of $\mathcal{O}_{S^1}$-modules corresponding to ∇. Let $\bar{\epsilon}$ be the reduction of ϵ modulo $\mathcal{I}$. The horizontal assumption on $\{s_i\}_{i=1,..,d}$ implies that

$$\bar{\epsilon}(j_1^*(s_i)) \equiv j_2^*(s_i), \ \forall\ 1 \le i \le d.$$

Then we have a morphism of systems of vector bundles with marked section over S^1

$$\epsilon : \ (j_1^*(\mathcal{E}), j_1^*(\mathcal{F}), \{j_1^*(s_1), ..., j_1^*(s_d)\}) \to (j_2^*(\mathcal{E}), j_2^*(\mathcal{F}), \{j_2^*(s_1), ..., j_2^*(s_d)\}).$$

By Lemma 1.6.5, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{V}_0(\mathcal{E}, s_1, ..., s_d) \times_{S,j_2} S^1 \arrow[r, "\epsilon"] \arrow[d] & \mathcal{V}_0(\mathcal{E}, s_1, ..., s_d) \times_{S,j_1} S^1 \arrow[d] \\
\mathcal{V}_0(\mathcal{F}, s_1, ..., s_d) \times_{S,j_2} S^1 \arrow[r, "\epsilon"] & \mathcal{V}_0(\mathcal{F}, s_1, ..., s_d) \times_{S,j_1} S^1
\end{tikzcd}$$

Once again by Grothendieck's description, we get the desired connection ∇_0. For more details and the remaining part of the proof, see [AI17, Lemma 2.9]. $\square$

The connection ∇_0 has the following functoriality.

Proposition 1.6.7. *Let*

$$g : \ (\mathcal{E}', \mathcal{F}', \{s'_1, ..., s'_d\}) \to (\mathcal{E}, \mathcal{F}, \{s_1, ..., s_d\})$$

be a morphism of systems of vector bundles with marked sections over S, and assume that we are given connections

$$\nabla' : \ \mathcal{E}' \to \mathcal{E}'\hat{\otimes}_{\mathcal{O}_S}\Omega^1_{S/A}, \quad \nabla : \ \mathcal{E} \to \mathcal{E}\hat{\otimes}_{\mathcal{O}_S}\Omega^1_{S/A}$$

such that $\nabla \circ g = (g \otimes \mathrm{Id}) \circ \nabla'$. If all s'_i are horizontal for ∇' modulo $\mathcal{I}$, then $\nabla_0 \circ g = (g \otimes \mathrm{Id}) \circ \nabla'_0$.

Proof. For details, we refer to [AI17, Corollary 2.7 & Lemma 2.9]. $\square$

1.6.1 Formal vector bundles with marked sections associated to p-divisible groups

In this dissertation, we will only consider formal vector bundles with marked sections coming from p-divisible groups. Let S be a connected normal p-adic formal scheme and let G/S is a p-divisible group of height h, dimension d. Assume that there is an integer $n \in \mathbb{N}$ such that $p \in \mathrm{Hdg}^{p^{n+1}}$. Let $H_n \subset G$ be the level-n canonical subgroup. Assume that $H_n(S) \cong (\mathbb{Z}/p^n\mathbb{Z})^d$, and let $\mathcal{I} := \mathcal{I}(G) := p^n \mathrm{Hdg}^{-\frac{p^n}{p-1}}$. Note that $\mathcal{I}(G)$ is invertible thanks to Lemma 1.5.4.

Definition 1.6.8. *We define $\mathbb{H}^\sharp_G$ to be the push-out of*

$$\begin{tikzcd} \mathrm{Hdg}(G)^{\frac{1}{p-1}}\underline{\omega}_G \arrow[r] \arrow[d] & \mathrm{Hdg}(G)^{\frac{1}{p-1}}\mathbb{H}^1_{\mathrm{dR}}(G), \\ \Omega_G & \end{tikzcd}$$

and set $P_1, ..., P_d$ to be a fixed basis of $H_n^D(S)$ and s_i to be the image of $\mathrm{HT}(P_i)$ *in $H^0(S, \Omega_G/\mathcal{I}(G)\Omega_G)$.*

Proposition 1.6.9. [AI17, Proposition 6.2-6.3] *We have an exact sequence of locally free $\mathcal{O}_S$-modules*

$$0 \to \Omega_G \to \mathbb{H}^\sharp_G \to \mathrm{Hdg}(G)^{\frac{1}{p-1}}\underline{\omega}^\vee_{G^D} \to 0$$

Moreover, $(\mathbb{H}^\sharp_G, \Omega_G, \{s_1, ..., s_d\})$ is a system of vector bundles with marked sections over S with respect to the ideal $\mathcal{I}(G)$.

If moreover $G^D[p^n](S) \cong (\mathbb{Z}/p^n\mathbb{Z})^h$, then the Gauss-Manin connection ∇ on $\mathbb{H}^1_{\mathrm{dR}}(G)$ induces a connection

$$\nabla^\sharp_G : \mathbb{H}^\sharp_G \to \mathbb{H}^\sharp_G \hat{\otimes}_{\mathcal{O}_S} \frac{1}{\mathrm{Hdg}(G)^{\frac{1}{p-1}}}\Omega^1_{S/A}$$

such that for each $1 \le i \le d$, $\nabla^\sharp_G(s_i) \equiv 0$ modulo $\mathcal{I}(G)$.

We remark that if the ideal $\mathrm{Hdg}(G)^{\frac{1}{p-1}}$ is a p-th power in $\mathcal{O}_S/p\mathcal{O}_S$, we have a connection

$$\nabla^\sharp_G : \mathbb{H}^\sharp_G \to \mathbb{H}^\sharp_G \hat{\otimes}_{\mathcal{O}_S} \Omega^1_{S/A}.$$

Let G'/S be another p-divisible group of height h and dimension d such that

$$\mathrm{Hdg}(G') \subset \mathrm{Hdg}(G), \text{ and } p \in \mathrm{Hdg}(G')^{p^{n+1}}.$$

Assume there is an isogeny $f : G' \to G$ which induces an isomorphism between the level-n canonical subgroups $H'_n \subset G'$ and $H_n \subset G$.

Proposition 1.6.10. [AI17, Proposition 6.4-6.5] *Setting $\mathcal{I} := \mathcal{I}(G')$, the isogeny $f : G' \to G$ induces a natural morphism of exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & \Omega_G \arrow[r] \arrow[d, "f^*"] & \mathbb{H}^\sharp_G \arrow[r] \arrow[d, "f^\sharp"] & \mathrm{Hdg}(G)^{\frac{1}{p-1}}\underline{\omega}^\vee_{G^D} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega_{G'} \arrow[r] & \mathbb{H}^\sharp_{G'} \arrow[r] & \mathrm{Hdg}(G')^{\frac{1}{p-1}}\underline{\omega}^\vee_{G',D} \arrow[r] & 0 \end{tikzcd}$$

where $f^* : \Omega_G \to \Omega_{G'}$ is an isomorphism, $f^*(s_i) = s'_i \bmod \mathcal{I}$ for each $i = 1, ...d$ and the right vertical arrow is induced by $f^D : G^D \to G'^{,D}$. If

$$G^D[p^n](S) \cong (\mathbb{Z}/p^n\mathbb{Z})^h, \quad G'^{,D}[p^n](S) \cong (\mathbb{Z}/p^n\mathbb{Z})^h,$$

then $f^\sharp : \mathbb{H}^\sharp_G \to \mathbb{H}^\sharp_{G'}$ is compatible with the connections $\nabla^\sharp_G$ and $\nabla^\sharp_{G'}$.

1.7 Measures on $\mathbb{Z}_p$ and the weight space

In this section, we briefly recall the p-adic measure theory of $\mathbb{Z}_p$ and the weight space associated with $\mathbb{Z}_p^\times$.

1.7.1 Measures on $\mathbb{Z}_p$

Our main reference for the measure theory on $\mathbb{Z}_p$ is the lecture note [Col04].

Definition 1.7.1. *Let $|\cdot|$ be the p-adic norm on $\mathbb{Q}_p$ normalized by $|p| = p^{-1}$. A $\mathbb{Q}_p$-Banach algebra is a $\mathbb{Q}_p$-algebra B equipped with a function $\|\cdot\| : B \to \mathbb{R}_{\geq 0}$ satisfying that*

(i) $\|1\| = 1$, and $\|a\| = 0$ if and only if $a = 0$;

(ii) for any $a, b \in B$,

$$\|ab\| \leq \|a\|\|b\|, \quad \|a+b\| \leq \max\{\|a\|, \|b\|\};$$

(iii) for any $x \in \mathbb{Q}_p$, $a \in B$, $\|xa\| = |x|\|a\|$,

and that B is complete with respect to metric induced by $\|\cdot\|$. A $\mathbb{Q}_p$-Banach algebra $(B, \|\cdot\|)$ is called uniform if the unit ball $B^\circ := \{a \in B : \|a\| \leq 1\}$ contains all power bounded elements in B.

Definition 1.7.2. *Let $(B, \|\cdot\|)$ be any uniform $\mathbb{Q}_p$-Banach algebra, and $\mathcal{C}^\circ(\mathbb{Z}_p, B)$ be the space of continuous B-valued functions on $\mathbb{Z}_p$ equipped with the norm $\|f\| := \sup_{x \in \mathbb{Z}_p} |f(x)|$. A B-valued measure on $\mathbb{Z}_p$ is a continuous B-linear function from $\mathcal{C}^\circ(\mathbb{Z}_p, B)$ to B. We denote the space of B-valued measure on $\mathbb{Z}_p$ by $\mathcal{D}_0(\mathbb{Z}_p, B)$.*

For any B-valued measure μ on $\mathbb{Z}_p$, its Amice transformation is the power series

$$A_\mu(T) = \int_{\mathbb{Z}_p} (1+T)^x d\mu := \sum_{n \geq 0} \int_{\mathbb{Z}_p} \binom{x}{n} d\mu T^n \in B[[T]].$$

The structures of $\mathcal{C}^\circ(\mathbb{Z}_p, B)$ and $\mathcal{D}^\circ(\mathbb{Z}_p, B)$ are well-understood by the following theorem of Mahler.

Theorem 1.7.3. *Let B be any uniform $\mathbb{Q}_p$-Banach algebra.*

(i) A function $f : \mathbb{Z}_p \to B$ is continuous if and only if $f(x) = \sum_{n \geq 0} b_n \binom{x}{n}$, with $b_n \in B$ and $b_n \to 0$.

(ii) The Amice transformation is an isometry from $\mathcal{D}_0(\mathbb{Z}_p, B)$ to the set

$$\{\sum_{n \geq 0} b_n T^n : b_n \in B, \sup_{n \geq 0} |b_n| < \infty\}$$

equipped with the norm given by $\|\sum_{n \geq 0} b_n T^n\| := \sup_{n \geq 0} |b_n|$.

Proof. For details, we refer to [Col04, Theorems 1.3.2 & 1.4.5]. □

For any $f \in \mathcal{C}^\circ(\mathbb{Z}_p, \mathbb{Q}_p)$ and $\mu \in \mathcal{D}^\circ(\mathbb{Z}_p, B)$, we can define a new measure $f \cdot \mu$ by setting $(f \cdot \mu)(g) := \mu(fg)$ for any $g \in \mathcal{C}^\circ(\mathbb{Z}_p, B)$. The Amice transformation of $f \cdot \mu$ is explicitly computed for typical functions in [Col04, Page 13].

Proposition 1.7.4. *Let $\mu \in \mathcal{D}^\circ(\mathbb{Z}_p, B)$ with $A_\mu(T) = \sum_{n \geq 0} b_n T^n$.*

(i) If $f(x) = x$, then $A_{f \cdot \mu}(T) = \partial A_\mu(T)$ where $\partial := (1+T)\frac{d}{dT}$;

(ii) If f is the characteristic function on $i+p^n\mathbb{Z}_p$, then

$$A_{f\cdot\mu}(T) = p^{-n}\sum_{z^p=1} z^{-i}A_\mu(z(1+T)-1).$$

Definition 1.7.5. *We define the ϕ operator on $\mathcal{D}^\circ(\mathbb{Z}_p, B)$ by*

$$\mu \mapsto \phi(\mu): \ f \mapsto \int_{\mathbb{Z}_p} f(px)d\mu$$

and the ψ-operator by

$$\mu \mapsto \psi(\mu): \ f \mapsto \int_{\mathbb{Z}_p} f(\frac{x}{p})d\mu.$$

Here when $\frac{x}{p}\notin\mathbb{Z}_p$, we understand $f(\frac{x}{p})$ to be 0.

The following proposition follows from the discussion in [Col04, Page 14].

Proposition 1.7.6. *For any $\mu\in\mathcal{D}_0(\mathbb{Z}_p, B)$, we have*

$$A_{\phi(\mu)}(T) = A_\mu((1+T)^p-1); \quad A_{\psi(\mu)}(T) = \frac{1}{p}\sum_{z^p=1} A_\mu(z(1+T)-1).$$

Moreover, $\psi\circ\phi = \mathrm{Id}$, and $\psi(\mu)=0$ if and only if μ is supported on $\mathbb{Z}_p^\times$. Actually, if f is the characteristic function of $\mathbb{Z}_p^\times$, then $\phi\circ\psi(\mu) = f\mu$.

The space $\mathcal{D}^\circ(\mathbb{Z}_p,\mathbb{Q}_p)$ is related to the Iwasawa algebra $\mathbb{Z}_p[[\mathbb{Z}_p]]$.

Definition 1.7.7. *For any profinite group G, the completed group algebra of G with $\mathbb{Z}_p$-coefficients is $\mathbb{Z}_p[[G]] := \varprojlim_{\text{open } H\lhd G}\mathbb{Z}_p[G/H]$ equipped with the projective limit topology.*

Proposition 1.7.8. *Any element $\mu\in\mathbb{Z}_p[[\mathbb{Z}_p]]$ is the projective limit of some*

$$\mu_n = \sum_{g\in\mathbb{Z}_p/p^n\mathbb{Z}_p} a_g[g] \in \mathcal{O}[\mathbb{Z}_p/p^n\mathbb{Z}_p],$$

and the map $\mathbb{Z}_p[[\mathbb{Z}_p]]\to\mathcal{D}_0(\mathbb{Z}_p,\mathbb{Q}_p)$ induced by the integration pairing

$$(\mu, f)\mapsto \int_{\mathbb{Z}_p} f d\mu := \lim_{n\to\infty}\sum_{g\in\mathbb{Z}/p^n\mathbb{Z}} a_g f_n(g)$$

is a well-defined injection whose image is just the unit ball in $\mathcal{D}^\circ(\mathbb{Z}_p,\mathbb{Q}_p)$. Here $\{f_n\}$ is any sequence of functions such that each f_n factors through $\mathbb{Z}_p/p^n\mathbb{Z}_p$ and $\lim_{n\to\infty} f_n = f$.

Proof. For details, we refer to [ST04, § 12]. □

For completeness, we record Amice's locally analytic analogue of Mahler's theorem, which is used implicitly in the discussion of the weight space. For any uniform Banach $\mathbb{Q}_p$-algebra B and any positive integer h, a function $f:\ \mathbb{Z}_p\to B$ is called h-analytic if for any $a\in\mathbb{Z}_p$, there is $f_a(T)\in B\langle T\rangle$ such that $f_a(x) = f(p^hx+a)$ for any $x\in\mathbb{Z}_p$. Then Amice's theorem, which is [Col04, Theorem 1.7.8], says

Theorem 1.7.9. *A function $f:\ \mathbb{Z}_p\to B$ is h-analytic if and only if there exists $b_n\in B$, $b_n\to 0$ as $n\to\infty$ such that $f(x) = \sum_{n\geq 0} b_n[\frac{n}{p^h}]!\binom{x}{n}$. Moreover, for each $n\geq 0$,*

$$[\frac{n}{p^h}]!\binom{x}{n}(\mathbb{Z}_p+p^hB^\circ)\subset B^\circ,$$

where $[\cdot]$ is the greatest integer function.

1.7.2 The weight space

Our exposition on the weight space follows from [AIP, § 2]. The discussion in [CHJ17, § 2.1] is also be helpful.

Let $q = 4$ if $p = 2$ and $q = p$ if $p \geq 3$. Decomposing $\mathbb{Z}_p^\times$ as $(\mathbb{Z}/q\mathbb{Z})^\times \times (1 + q\mathbb{Z}_p)$, then we have an isomorphism

$$\mathbb{Z}_p[[\mathbb{Z}_p^\times]] \cong \mathbb{Z}_p[(\mathbb{Z}/q\mathbb{Z})^\times][[T]]; \quad \exp(q) \mapsto 1 + T.$$

We will identify $\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ (resp. $\Lambda^0 := \mathbb{Z}_p[[1 + q\mathbb{Z}_p]]$) with $\mathbb{Z}_p[(\mathbb{Z}/q\mathbb{Z})^\times][[T]]$ (resp. $\mathbb{Z}_p[[T]]$) equipped with the (p, T)-adic topology.

Lemma 1.7.10. [AIP, Lemma 2.1] *The space* $\mathrm{Spa}(\Lambda, \Lambda)$ *is a noetherian adic space. Moreover, we have the following:*

(i) There is a $1 - 1$ *correspondence between the non-analytic points in* $\mathrm{Spa}(\Lambda, \Lambda)$ *and points in* $\mathrm{Spec}(\mathbb{F}_p[(\mathbb{Z}/q\mathbb{Z})^\times])$*; and*

(ii) The open subspace $\mathrm{Spa}(\Lambda, \Lambda)^a$ *of analytic points is covered by the open subsets*

$$\{x \in \mathrm{Spa}(\Lambda, \Lambda) : \ |x(p)| \neq 0\}, \quad \{x \in \mathrm{Spa}(\Lambda, \Lambda) : \ |x(T)| \neq 0\}.$$

We call $\mathrm{Spa}(\Lambda, \Lambda)^a$ the adic weight space, and the open subspace $\mathcal{W}^{\mathrm{rig}} := \{x \in \mathrm{Spa}(\Lambda, \Lambda) : |x(p)| \neq 0\}$ the rigid analytic weight space.

Definition 1.7.11. *Denote* $\mathrm{Spa}(\Lambda^{(0)}, \Lambda^{(0)})$ *by* $\mathcal{W}^{(0)}$*, and for any closed interval* $I = [p^a, p^b]$ *with* $a \in \mathbb{N} \cup \{-\infty\}$*,* $b \in \mathbb{N}$*, denote the open affinoid subspace*

$$\{x \in \mathcal{W}^{(0)} : \ |p|_x \leq |T|_x^{p^a} \neq 0; \quad |T|_x^{p^b} \leq |p|_x \neq 0\}$$

by $\mathcal{W}_I^{(0)}$*, where for* $a = -\infty$*, the first condition is understood as empty.*

We have

$$\Lambda_I^{(0)} := \mathcal{O}^+_{\mathcal{W}_I^{(0)}}(\mathcal{W}_I^{(0)}) = \begin{cases} \Lambda^{(0)}\langle \frac{T^{p^b}}{p} \rangle, & \text{if } a = -\infty, \\ \Lambda^{(0)}\langle \frac{p}{T^{p^a}}, \frac{T^{p^b}}{p} \rangle, & \text{if } a \neq -\infty. \end{cases}$$

By equipping $\Lambda_I^{(0)}[\frac{1}{p}]$ with the norm $\|a\| := \inf_{n, p^n a \in \Lambda_I^{(0)}} p^n$, we obtain a uniform $\mathbb{Q}_p$-Banach algebra whose unit ball is $\Lambda_I^{(0)}$.

The functor sending complete affinoid $(\mathbb{Z}_p, \mathbb{Z}_p)$-algebra (A, A^+) to the group of continuous homomorphisms $\mathrm{Hom}(\mathbb{Z}_p^\times, A^\times)$ is represented by the affinoid ring (Λ, Λ). Moreover, the universal character $\kappa : \ \mathbb{Z}_p^\times \to \Lambda^\times$ is given by sending $a \in \mathbb{Z}_p^\times$ to the group-like element $[a]$. For each closed interval $I \subset [0, \infty)$, let $\kappa_I : \ \mathbb{Z}_p^\times \to \Lambda_I^\times$ be the induced universal character. We define the weight of κ_I to be

$$\mu_{\kappa_I} := \lim_{a \to 1} \frac{\log \kappa_I(a)}{\log(a)}.$$

Then for any affinoid ring (A, A^+) over $(\mathbb{Z}_p, \mathbb{Z}_p)$, and for any interval $I \subset [0, p^n q^{-1}]$, the character κ_I extends uniquely to a function

$$\mathbb{Z}_p^\times(1 + qp^{n-1}A^+) \to (A^+)^\times; \quad t \mapsto \kappa_I(t) := \log(\mu_{\kappa_I} \exp(t)), \ \forall \ t \in 1 + qp^{n-1}A^+.$$

Let G_a^+ (resp. G_m^+) be the sheaf such that for any affinoid ring (A, A^+),

$$G_a^+(A, A^+) := A^+ \quad (\text{resp. } G_m^+(A, A^+) := (A^+)^\times).$$

Proposition 1.7.12. [AIP, Proposition 2.1] *For $n \geq 1$ and $I = [0, p^n q^{-1}]$, the universal character κ_I defines a unique pairing*

$$\mathcal{W}_I \times \mathbb{Z}_p^\times(1 + qp^{n-1}G_a^+) \to G_m^+$$

which restricts to a pairing

$$\mathcal{W}_I \times (1 + qp^{n-1}G_a^+) \to 1 + qG_a^+.$$

Remark 1.7.13. *Following* [KL16, § 2.6], *a quasi-Stein space in the category of locally noetherian adic space is an object X which can be written as the union of an ascending sequence*

$$X_0 \subset X_1 \subset .... \subset X_n \subset ...$$

of affinoid subspaces such that for each $i \geq 0$, the natural map $\mathcal{O}(X_{i+1}) \to \mathcal{O}(X_i)$ has dense image. Then according to [Eme07, Proposition 6.4.5] *(see also* [ST03, Theorem 4.11]*), $\mathcal{W}^{\mathrm{rig}}$ is a quasi-Stein space. In particular, $H^0(\mathcal{W}^{\mathrm{rig}}, \mathcal{O}_{\mathcal{W}})$ ia a Fréchet $\mathbb{Q}_p$-algebra which is faithfully flat over $\Lambda[1/p]$.*

2 Nearly overconvergent quaternion modular forms

In this chapter we will first discuss the Hasse invariant of a false elliptic curves and establish the integral models of strict neighborhoods of the ordinary locus of Shimura curves.

Then we will apply the VBMS machinery to (a certain direct summand of) the p-divisible group associated to the universal false elliptic curve to construct the nearly overconvergent quaternion modular sheaf $\mathbb{W}_\kappa$ for a certain weight κ.

After that, we will study the filtration of $\mathbb{W}_\kappa$, analyze the (meromorphic) Gauss-Manin connection on $\mathbb{W}_\kappa$, and define the Hecke operators, in particular the U_p-operator, on $\mathbb{W}_\kappa$.

Finally, we will study the local behavior of the Gauss-Manin connection and Hecke operator in Serre-Tate coordinates, and show that the Gauss-Manin connection iterates p-adically.

2.1 Shimura curve over $\mathbb{Q}$

In this section, we briefly recall the moduli description of Shimura curves over $\mathbb{Q}$.

2.1.1 Quaternion algebras

Definition 2.1.1. *Let F be any field. A quaternion algebra B/F is a four-dimensional central simple algebra over F.*

It is well-known, see [Voi17, § 3.5 & 6.2], that

Proposition 2.1.2. *Let F be any field. A quaternion algebra B/F is either $M_2(F)$ or a division algebra over F. Moreover, if F has characteristic other than 2, then there exists a F-basis $1, i, j, k$ of B such that*

$$a := i^2 \in F^\times, \quad b := j^2 \in F^\times, \quad k = ij = -ji;$$

while if the characteristic is 2, there exists a F-basis $1, i, j, k$ of B such that

$$a := i^2 + i \in F; \quad b := j^2 \in F^\times; \quad k = ij = j(i+1).$$

We will denote the quaternion algebra B/F by $(a, b \mid F)$ and $[a, b \mid F)$ respectively.

Definition 2.1.3. *Let B/F be a quaternion algebra. If $B = (a, b \mid F)$ with the basis $\{1, i, j, k\}$, we define the main involution on B to be*

$$- : \ B \to B; \quad \alpha = x_1 + x_2 i + x_3 j + x_4 k \mapsto \bar{\alpha} := x_1 - x_2 i - x_3 j - x_4 k;$$

if $B = [a, b \mid F)$ with a basis $\{1, i, j, k\}$, we define the main involution on B to be

$$- : \ B \to B; \quad \alpha = x_1 + x_2 i + x_3 j + x_4 k \mapsto \bar{\alpha} := \alpha + x_2.$$

The reduced trace Tr and reduced norm N on B are defined as

$$\mathrm{Tr}(\alpha) := \bar{\alpha} + \alpha; \text{ resp. } N(\alpha) := \bar{\alpha}\alpha.$$

Clearly, the main involution is standard, i.e. $x\bar{x} \in F$ for any $x \in B$.

Definition 2.1.4. *Let $\mathcal{O} \subset F$ be a subring and B/F be a quaternion algebra. An $\mathcal{O}$-order in B is a subring $L \subset B$ such that L is finitely generated as an $\mathcal{O}$-module and $L \otimes_{\mathcal{O}} F = B$.*

(i) An $\mathcal{O}$-order is called maximal if it is not strictly contained in any larger order.

(ii) An $\mathcal{O}$-order is called Eichler if it is the intersection of two maximal orders.

Definition 2.1.5. *Let B be a quaternion algebra over a field F. For any $x_1, .., x_4 \in B$, we define the discriminant of $x_1, .., x_4$ to be*

$$d(x_1, .., x_4) := \det(\mathrm{Tr}(x_i x_j)_{1 \le i,j \le 4}).$$

Let $\mathcal{O} \subset F$ be a subring and $L \subset B$ be an $\mathcal{O}$-order. The discriminant $\mathrm{disc}(L)$ of L is the ideal of $\mathcal{O}$ generated by

$$\{d(x_1, .., x_4) : \ x_1, ..., x_4 \in L\};$$

and the reduced discriminant $\mathrm{disc}'(L)$ of L is the ideal of $\mathcal{O}$ generated by

$$\{m(x_1, x_2, x_3) : \ x_1, .., x_3 \in L\}, \quad m(x_1, x_2, x_3) := \mathrm{Tr}((x_1 x_2 - x_2 x_1)\bar{x}_3).$$

Lemma 2.1.6. [Voi17, Lemma 15.4.8] *Given two projective $\mathcal{O}$-orders $L \subset L' \subset B$, we have*

$$\mathrm{disc}'(L) = [L' : L]\mathrm{disc}'(L'); \quad \mathrm{disc}'(L)^2 = \mathrm{disc}(L).$$

Quaternion algebras over $\mathbb{Q}$

The following result is well-known. For details, see for example [Ser79].

Proposition 2.1.7. *We have the following classification of quaternion algebras:*

(i) The 2-by-2 matrix ring $M_2(\mathbb{C})$ is the unique quaternion algebra over complex numbers $\mathbb{C}$.

(ii) The matrix ring $M_2(\mathbb{R})$ and the Hamilton quaternion $\mathbb{H} = (-1, -1 \mid \mathbb{R})$ are the only quaternion algebras over $\mathbb{R}$. Moreover, $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$.

(iii) Let ℓ be a rational prime and $L/\mathbb{Q}_\ell$ be a finite extension with uniformizer π. Then there exist only two quaternion algebras over L: $M_2(L)$ and $(\pi, u \mid L)$ where u is a non-square in $\mathcal{O}_L^\times$. Moreover, for any degree-2 field extension L'/L, we have that $(\pi, u \mid L) \otimes_L L' \cong M_2(L')$.

Definition 2.1.8. *A quaternion algebra B over a local field is called ramified if B is a division algebra, and unramified if B is the 2-by-2 matrix ring.*

Let F be a number field. A quaternion algebra B/F is called ramified resp. unramified at a place v if $B_v := B \otimes_F F_v$ is ramified resp. unramified. When $F = \mathbb{Q}$, B is called definite resp. indefinite if $B \otimes_{\mathbb{Q}} \mathbb{R}$ is ramified resp. unramified.

Theorem 2.1.9. *For any number field F, there is a $1-1$ correspondence between quaternion algebras over F and subsets of non-complex places of F with even cardinality given by sending a quaternion algebra B/F to the set of ramified places of B.*

Moreover if K/F is a quadratic field extension, then the following are equivalent:

(i) the field K splits B, i.e. $B \otimes_F K \cong M_2(K)$;

(ii) *no ramified finite place of B in F splits in K and no ramified archimedean place of B in F is unramified in K;*

(iii) *there exists an embedding $K \hookrightarrow B$ of F-algebras.*

Proof. For details of the $1-1$ correspondence, we refer to [Voi17, Theorem 14.1]. The equivalence between item (i) and item (ii) is an application of the $1-1$ correspondence and item (iii) of Proposition 2.1.7. The equivalence between item (iii) and item (i) is a special case of the Skolem-Noether theorem. □

Definition 2.1.10. *For any quaternion algebra $B/\mathbb{Q}$, the discriminant Δ of B is the product of all ramified non-Archimedean places of B.*

Note that B is uniquely determined by Δ, and in particular, B is definite resp. indefinite if and only if Δ has an odd resp. even number of distinct prime factors.

For any global field or non-Archimedean local field F with ring of integers $\mathcal{O}_F$ and any quaternion algebra B/F, we refer to an $\mathcal{O}_F$-order simply as an order. We record the following results.

Proposition 2.1.11. [Voi17, Proposition 23.43] *For any finite field extension $F/\mathbb{Q}_p$ with uniformizer π and for any quaternion algebra B/F, we have that:*

- *if B is ramified, there exists a unique maximal order, consisting of all $\mathcal{O}_F$-integral elements.*
- *if $B = M_2(F)$, then all maximal orders of B are conjugate to $M_2(\mathcal{O}_F)$. Moreover, any Eichler order of B with reduced discriminant π^n is conjugate to the standard Eichler order of level π^n*

$$\begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \pi^n\mathcal{O}_F & \mathcal{O}_F \end{pmatrix}.$$

Theorem 2.1.12. [Voi17, Theorem 9.5.1] *Let $B/\mathbb{Q}$ be a quaternion algebra with discriminant Δ and $L \subset B$ be a fixed order. Then the assignment $M \mapsto \{M_v\}_v$ defines a $1-1$ correspondence between the set of orders in B and the set of collections of orders $N_v \subset B_v$ indexed by places v of $\mathbb{Q}$ such that $N_v = L_v$ for all but finitely many primes.*

By [Voi17, Paragraph 23.4.19, Lemma 10.4.2 & Theorem 23.2.9], we have

Corollary 2.1.13. *Let $B/\mathbb{Q}$ be a quaternion algebra with discriminant Δ. An order $L \subset B$ is maximal if and only if $\mathrm{disc}'(L) = \Delta$. More generally, for any Eichler order $L \subset B$,*

(i) *the reduced discriminant $\mathrm{disc}'(L) = N\Delta$ for some integer $N \geq 1$, called the level of L, with $(N, \Delta) = 1$;*

(ii) *for $(v, \Delta) = 1$, L_v is conjugate to the standard Eichler order of level $v^{\mathrm{ord}_v(N)}$;*

(iii) *for $v \mid \Delta$, L_v is the maximal order.*

By [Voi17, Example 28.4.15], we know that if $B/\mathbb{Q}$ is indefinite, then all Eichler orders of the same level are conjugate.

2.1.2 False elliptic curves and false isogenies

In this subsection, $B/\mathbb{Q}$ will be an indefinite quaternion algebra of discriminant Δ together with a fixed maximal order $\mathcal{O}_B$ and a fixed isomorphism $B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$, and $B^{(p)}/\mathbb{Q}$ will be the unique definite quaternion algebra with discriminant $\begin{cases} p\Delta \text{ if } p \nmid \Delta \\ p^{-1}\Delta \text{ if } p \mid \Delta. \end{cases}$

Definition 2.1.14. *A false elliptic curve over a base scheme S is a pair (A, i) consisting of an abelian surface A/S and an injective ring homomorphism $i : \mathcal{O}_B \to \mathrm{End}_S(A)$. When i is clear from content, we omit it in the notation.*

Definition 2.1.15. *Let (A, i), (B, j) be false elliptic curves over a base scheme S. A false homomorphism $f : (A, i) \to (B, j)$ is a S-group homomorphism such that for any $x \in \mathcal{O}_B$, $f \circ i(x) = j(x) \circ f$. A false homomorphism f is called a false isogeny if f is an isogeny of abelian schemes.*

The set of all false homomorphism from (A, i) to (B, j) is denoted by $\mathrm{Hom}_{\mathcal{O}_B}((A, i), (B, j))$. If $(A, i) = (B, j)$, we denote $\mathrm{Hom}_{\mathcal{O}_B}((A, i), (B, j))$ by $\mathrm{End}_{\mathcal{O}_B}(A, i)$ and equip it with the ring structure given by composition. Moreover, we set $\mathrm{End}^{\circ}_{\mathcal{O}_B}(A, i) := \mathrm{End}_{\mathcal{O}_B}(A, i) \otimes_{\mathbb{Z}} \mathbb{Q}$.

False elliptic curves

False elliptic curves over fields are well-understood.

Proposition 2.1.16. [Phi15, Proposition 2.1.4] *Given any false elliptic curve (A, i) over $\bar{\mathbb{F}}_p$, then $\mathrm{End}^{\circ}_{\mathcal{O}_B}(A)$ is*

(i) either an imaginary quadratic field $K/\mathbb{Q}$ which splits B; or

(ii) the definite quaternion algebra $B^{(p)}$.

Moreover, A is isogenous to E^2 for some elliptic curve $E/\bar{\mathbb{F}}_p$ where E is ordinary in case (i) and supersingular in case (ii).

Let $\mathcal{H} \subset \mathbb{C}$ denote the upper half plane. View $\mathcal{O}_B$ as a subring of $M_2(\mathbb{R})$ via the fixed isomorphism $B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$ and for any $\tau \in \mathcal{H}$, set $\Lambda_\tau := \mathcal{O}_B \begin{pmatrix} \tau \\ 1 \end{pmatrix}$. According to [Phi15, Proposition 2.1.5 & 2.1.6], we have

Proposition 2.1.17. *Any false elliptic curve (A, i) over $\mathbb{C}$ is of the form $\mathbb{C}^2/\Lambda_\tau$ for some $\tau \in \mathcal{H}$. Moreover, for any false elliptic curve (A, i) over $\mathbb{C}$,*

(i) either (A, i) is simple with $\mathrm{End}^{\circ}(A) \cong B$ (this forces B to be a division algebra) and $\mathrm{End}^{\circ}_{\mathcal{O}_B}(A) = \mathbb{Q}$; or

(ii) the false elliptic curve (A, i) is isogenous to E^2 for some elliptic curve $E/\mathbb{C}$. In this case,

(a) if B is a division algebra, then E/C admits complex multiplication by an imaginary field K which splits B, and $\mathrm{End}^{\circ}_{\mathcal{O}_B}(A) \cong K$;

(b) if E/C does not admit complex multiplication, then $B = M_2(\mathbb{Q})$ and $\mathrm{End}^{\circ}_{\mathcal{O}_B}(A) = \mathbb{Q}$.

Proposition 2.1.18. [Phi15, Lemma 2.1.9] *Let (A,i), (B,j) be false elliptic curves over any algebraically closed field. If A and B are isogenous as abelian varieties, then (A,i), (B,j) are isogenous as false elliptic curves.*

Let $-$ be the main involution of B. By Theorem 2.1.9, there exists $t \in B$ such that $t^2 = -\Delta$ and the map

$$*: \ B \mapsto B; \quad a \mapsto t^{-1}\bar{a}t$$

is an involution. Note that the order $\mathcal{O}_B$ is stable under $*$ because it is stable under the main convolution and the conjugation induced by t. By the discussion in [Buz97, Page 3], we have the following result for families of false elliptic curves.

Theorem 2.1.19. *Let S be any scheme with $\Delta \in \mathcal{O}_S^\times$, and let (A,i) be any false elliptic curve over S. Then there exists a unique principal polarization $\lambda_A : \ A \to A^D$ such that on each geometric point $\bar{s} \in S$, the Rosati involution*

$$\dagger: \ \mathrm{End}^\circ(A_{\bar{s}}) \to \mathrm{End}^\circ(A_{\bar{s}})$$

induces the involution $*$ on B.

We remark that here the action $\mathcal{O}_B^{\mathrm{op}}$ on A^D is given by $a \mapsto i(a)^D$, $\forall\, a \in \mathcal{O}_B$.

False dual isogenies and false degrees

In this subsubsection, we will fix a connected base scheme S such that $\Delta \in \mathcal{O}_S^\times$. Combining [Phi15, Lemma 2.1.13, Proposition 2.1.14 & Corollary 2.1.17], we have

Lemma 2.1.20. *Let $f: \ (A,i) \to (B,j)$ be a false isogeny of false elliptic curves over S. Then the morphism*

$$f' = \lambda_A^{-1} \circ f^D \circ \lambda_B : \ (B,j) \to (A,i)$$

is also a false isogeny, referred to as the false dual isogeny of f, such that $(f')' = f$. Moreover, we have the following:

(i) If $g: \ (A,i) \to (B,j)$, $h: (B,j) \to (C,k)$ are false isogenies of false elliptic curves over S, then

$$(f+g)' = f' + g'; \quad (h \circ f)' = f' \circ h'.$$

(ii) The composition $f' \circ f: \ (A,i) \to (A,i)$ is the multiplication by an integer, called the false degree of f and denoted by $\deg'(f)$, satisfying that

$$\deg(f) = (\deg'(f))^2, \quad \deg'(f) = \deg'(f').$$

By [Phi15, Lemma 2.1.18 & Proposition 2.1.19], we have

Proposition 2.1.21. *Let (A,i), (B,j) be false elliptic curves over S. Any non-zero false homomorphism $f: \ (A,i) \to (B,j)$ is an isogeny and*

$$\deg': \ \mathrm{Hom}_{\mathcal{O}_B}(A,B) \to \mathbb{Z}, \quad f \mapsto \deg'(f)$$

is a positive definite quadratic form.

Let $f : (A, i) \to (B, j)$ be a false isogeny of false elliptic curves over S. Note that the Cartier dual of $\ker(f)$ is (canonically isomorphic to) $\ker(f^D)$.

Definition 2.1.22. *We define the Weil pairing* $\langle -, - \rangle :\ \ker(f) \times \ker(f') \to G_m$ *to be the pairing* $\ker(f) \times \ker(f^D) \to G_m$ *pre-composed with the isomorphism* $\ker(f') \xrightarrow[\cong]{\lambda_B} \ker(f^D)$.

Note that we have

$$\forall\ P \in \ker(f)(S),\ Q \in \ker(f')(S),\ x \in \mathcal{O}_B, \quad \langle i(x)P, Q\rangle = \langle P, j(x^*)Q\rangle.$$

2.1.3 Shimura curves of level $V_1(N)$

In this subsection, $B/\mathbb{Q}$ will be an indefinite quaternion division algebra of discriminant Δ together with a maximal order $\mathcal{O}_B \subset B$ and fixed isomorphisms

$$B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}), \quad \mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_v \cong M_2(\mathbb{Z}_v) \quad \forall \quad v \nmid \Delta.$$

These data induce an isomorphism $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \cong M_2(\mathbb{Z}/N\mathbb{Z})$ for any $N \in \mathbb{N}$ such that $(N, \Delta) = 1$.

Definition 2.1.23. *Let (A, i) be a false elliptic curve over a scheme S on which Δ is invertible.*

(i) A full level-N structure on (A, i) is an isomorphism of S-group schemes

$$\psi :\ (\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z})_S \xrightarrow{\sim} A[N]$$

which preserves the left action of $\mathcal{O}_B$.

(ii) An arithmetic level-N structure (also called a level $V_1(N)$-structure) on (A, i) is an inclusion

$$\psi_N :\ \mu_N \times \mu_N \hookrightarrow A[N]$$

of group schemes with left action of $\mathcal{O}_B$ over S, where we equip $\mu_N \times \mu_N$ with the $\mathcal{O}_B$-action induced by the fixed isomorphism $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \cong M_2(\mathbb{Z}/N\mathbb{Z})$ and the canonical map $\mathbb{Z}/N\mathbb{Z} \to \mathrm{End}_S(\mu_N)$.

We will use (A, i, ψ) (resp. $(A, i.\psi_N)$) to denote a false elliptic curve (A, i) with a full (resp. an arithmetic) level N structure.

The moduli problem of false elliptic curves with level $V_1(N)$-structure is representable. More precisely, as stated in [Buz97, § 2],

Theorem 2.1.24. *Assume $N \geq 4$. The functor associates to an $\mathbb{Z}[\frac{1}{N\Delta}]$-scheme S the isomorphism classes of false elliptic curves with level $V_1(N)$-structure over S is representable by a scheme $X_1^B(N)$ which is smooth, proper and of relative dimension 1 over $\mathbb{Z}[\frac{1}{N\Delta}]$ with geometrically irreducible fibers. In particular, there is a universal false elliptic curve with level $V_1(N)$-structure over $X_1^B(N)$, which will be denoted by $(\mathcal{A}, i, \psi_N)$.*

Remark 2.1.25. *Analogous to the discussion in [Kat77, § 1.4], for a false elliptic curve (A, i) over S, we can define the naive level-N structure to be an inclusion*

$$\psi_N :\ (\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z})_S \hookrightarrow A[N].$$

When S contains all N-th roots of unity, by fixing an isomorphism $\mathbb{Z}/N\mathbb{Z} \cong \mu_N$, we can identify a naive level-N structure with an arithmetic one.

On the other hand, suppose that (A, i, ψ_N) is a false elliptic with naive level-N structure over S. Then $A' := A/\mathrm{Im}(\psi_N)$ together with the induced embedding $i' : \mathcal{O}_B \to \mathrm{End}_S(A')$ is a false elliptic over S and the quotient map $\pi : (A, i) \to (A', i')$ is a false isogeny. Let $\pi' : (A', i') \to (A, i)$ be the false dual isogeny. Then $\ker(\pi')$ is the Cartier dual of $\mathrm{Im}(\psi_N)$ and we have an isomorphism

$$\psi'_N : \mu_N \times \mu_N \cong \ker(\pi') \subset A'[N].$$

The assignment $(A, i, \psi_N) \mapsto (A', i', \psi'_N)$ induces a $1-1$ correspondence between false elliptic curves together with a naive level-N structure over S and false elliptic curves together with an arithmetic level-N structure over S.

Since we will consider p-adic theory for $(p, N\Delta) = 1$, it is (almost) equivalent to choose either level-N structure. We choose the arithmetic one, because it has slightly better rationality properties.

Definition 2.1.26. *The $\mathbb{Z}[\frac{1}{N\Delta}]$-scheme $X_1^B(N)$ is called the Shimura curve of level $V_1(N)$ associated to the indefinite quaternion division algebra B.*

We include the following complex description of $X_1^B(N)$ for completeness. Let $\mathbb{A}$ (resp. $\mathbb{A}_f$) be the ring of adèles (resp. finite adèles) of $\mathbb{Q}$ and $\mathcal{O}_{B,f} = \mathcal{O}_B \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. Then we have a surjective ring(resp. group) homomorphism

$$\mathcal{O}_{B,f} \to M_2(\mathbb{Z}/N\mathbb{Z}), \quad \text{resp.} \quad \mathcal{O}_{B,f}^\times \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

Let $V_1(N)$ be the pre-image of

$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) : \ c \equiv 0, \ d \equiv 1 \mod N\}$$

and $V_\infty \subset \mathrm{GL}_2(\mathbb{R})$ be the stabilizer of $\sqrt{-1}$ under the Möbius action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{C} - \mathbb{R}$. By the discussion in [Buz97, Page 5],

Proposition 2.1.27. *There is a natural bijection between $X_1^B(N)(\mathbb{C})$ and*

$$B^\times \backslash (B \otimes_{\mathbb{Q}} \mathbb{A})^\times / V_1(N) V_\infty.$$

Let $\mathcal{O}_{B,N}$ be the standard Eichler order of level N in $\mathcal{O}_B$, i.e. the Eichler order whose image along the isomorphism $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \cong M_2(\mathbb{Z}/N\mathbb{Z})$ consists of upper triangular matrixes, and let $\Gamma_{1,N}$ be the group of units in $\mathcal{O}_{B,N}$ whose reduced norm is 1 and whose image along the fixed isomorphism has the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. Then we have the following bijection

$$X_1^B(N)(\mathbb{C}) = \Gamma_{1,N} \backslash \mathcal{H}.$$

2.1.4 Summary of notations

We fix the following notations for the remaining part of this chapter. Recall that p is a fixed rational prime.

- $B/\mathbb{Q}$ will be an indefinite quaternion division algebra of prime-to-p discriminant Δ and main involution $-$, together with a fixed isomorphism $B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$.

- $\mathcal{O}_B \subset B$ will be a maximal order with fixed isomorphisms $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_v \cong M_2(\mathbb{Z}_v)$ for all finite place $v \nmid \Delta$.

- N will be a positive integer such that $N \geq 4$, $(N, \Delta) = 1$ and $(p, N\Delta) = 1$.

- $\dagger$ will be the involution on B defined by $a^\dagger = t^{-1}\bar{a}t$ for a fixed element $t \in B$ such that $t^2 = -\Delta$.

- $e \in \mathcal{O}_{B,p}$ will be a non-trivial idempotent, and g, g' will be elements in $\mathcal{O}_{B,p}^\times$ such that

$$1 - e = g^{-1}eg, \quad e^\dagger = (g')^{-1}eg'.$$

  This is possible because all non-trivial idempotents in $\mathcal{O}_{B,p}$ are conjugate by Skolem-Noether theorem.

- All false elliptic curves will be defined with respect to $\mathcal{O}_B$.

- For any false elliptic curve (A, i) defined over a base scheme on which Δ is invertible, $\lambda_A : A \to A^D$ will be the principal polarization such that on each geometric point, the corresponding Rosati involution is compatible with $\dagger$.

- For any false isogeny $f : (A, i) \to (B, j)$ with false dual isogeny $f' : (B, j) \to (A, i)$, the Weil pairing $\langle -, - \rangle : \ker(f) \times \ker(f') \to G_m$ will satisfy

$$\langle xP, Q \rangle = \langle P, x^\dagger Q \rangle, \quad \forall\ P \in \ker f,\ Q \in \ker f',\ x \in \mathcal{O}_B.$$

- For any false elliptic curve with level $V_1(N)$-structure (A, i, ψ_N) and any finite flat group subscheme H of A of rank prime to N which is stable under the action of $\mathcal{O}_B$, the quotient of (A, i, ψ_N) by H will be the triple $(A/H, i, \psi'_N)$ where $i : \mathcal{O}_B \to \mathrm{End}(A/H)$ is the embedding induced by the quotient isogeny $\pi : A \to A/H$, and $\psi'_N : \mu_N \times \mu_N \to A/H[N]$ is the unique level $V_1(N)$-structure on $(A/H, i)$ which makes the diagram below commute:

$$\begin{tikzcd} & \mu_N \times \mu_N \arrow[dl, "\psi_N"'] \arrow[dr, "\psi'_N"] & \\ A[N] \arrow[rr, "\pi"] & & A/H[N]. \end{tikzcd}$$

- $X_1^B(N)$ will be the Shimura curve of level $V_1(N)$ and $(\mathcal{A}, i, \psi_N)$ will be the universal false elliptic curve together with the universal level $V_1(N)$-structure over $X_1^B(N)$. Via base change, we will view $X_1^B(N)$, $\mathcal{A}$ as $\mathbb{Z}_p$-schemes and we will denote the p-adic completion of $X_1^B(N)$, $\mathcal{A}$ by $\hat{X}_1^B(N)$, $\mathcal{A}$ respectively. When no confusion arises, we will simply write $X_1^B(N)$ (resp. $\hat{X}_1^B(N)$) as X (resp. $\hat{X}$).

2.2 The Hasse invariant and the partial Igusa tower

2.2.1 p-divisible groups associated to false elliptic curves

Let S be any $\mathbb{Z}_p$-scheme and (A, i) be any false elliptic over S. Given any $n \geq 1$ and any finite flat group subscheme $H \subset A[p^n]$ which is stable under the action of $\mathcal{O}_B$, we have an $\mathcal{O}_{B,p}$-action on H since $\mathbb{Z}_p$ acts on H via $\mathbb{Z}_p/p^n\mathbb{Z}_p$. Set

$$H^2 := \ker(e : \; H \to H); \quad H^1 := \ker(1 - e : \; H \to H).$$

The inclusion $H^1 \times H^2 \to H$ is an isomorphism and the chosen element g induces an isomorphism $g : \; H_1 \cong H_2$.

Lemma 2.2.1. [Kas99, Lemma 10.1] *Given any $n \geq 1$, let $H \subset A[p^n]$ be any finite flat group subscheme. Then H is stable under the $\mathcal{O}_B$-action if and only if there exist group subschemes $H_i \subset H$ for $i = 1, 2$ such that $H = H_1 \times H_2$, $(1 - e)H_1 = 0$ and $H_2 = g(H_1)$.*

In particular, for each $n \geq 1$, $A[p^n]$ is stable under the action $\mathcal{O}_B$, so we have

$$A[p^n]^1 \times A[p^n]^2 \cong A[p^n], \quad g : \; A[p^n]^1 \cong A[p^n]^2.$$

Clearly, these decompositions and isomorphisms are compatible, so we have

$$A[p^\infty]^1 \times A[p^\infty]^2 \cong A[p^\infty]; \quad g : \; A[p^\infty]^1 \cong A[p^\infty]^2.$$

Denote $A[p^\infty]^1$ by G_A and $A[p^\infty]^2$ by G'_A. Since $A[p^\infty]$ has height 4 and dimension 2, both G_A and G'_A have height 2 and dimension 1. Let

$$A[p^n]^{1,\dagger} := \ker(1 - e^\dagger : \; A[p^n] \to A[p^n]); \quad A[p^\infty]^{1,\dagger} := \ker(1 - e^\dagger : \; A[p^\infty] \to A[p^\infty]).$$

The Weil pairings $\langle -, - \rangle : \; A[p^n] \times A[p^n] \to G_m[p^n]$ for all $n \geq 1$ induce isomorphisms

$$A[p^n]^{1,D} \cong A[p^n]^{1,\dagger}, \quad G_A^D \cong A[p^\infty]^{1,\dagger}.$$

By the functoriality of de Rham cohomology, the $\mathcal{O}_B$-action on A induces an $\mathcal{O}_{B,p}^{\mathrm{op}}$-action on the Hodge-de Rham sequence

$$0 \to \underline{\omega}_{A/S} \to \mathbb{H}^1_{\mathrm{dR}}(A/S) \to \mathrm{Lie}(A^D/S) \to 0.$$

Thanks to Theorem 1.3.12, when p is locally nilpotent on S, we have

$$0 \to \underline{\omega}_{G_A/S} \to \mathbb{H}^1_{\mathrm{dR}}(G_A/S) \to \mathrm{Lie}(G_A^D) \to 0.$$

Lemma 2.2.2. *When p is locally nilpotent on S, the decomposition $A[p^\infty] = G_A \times G'_A$ induces an identification between*

$$0 \to e\underline{\omega}_{A/S} \to e\mathbb{H}^1_{\mathrm{dR}}(A/S) \to e\mathrm{Lie}(A^D/S) \to 0; \text{ and}$$
$$0 \to \underline{\omega}_{G_A/S} \to \mathbb{H}^1_{\mathrm{dR}}(G_A/S) \to \mathrm{Lie}(A[p^\infty]^{1,\dagger}/S) \to 0.$$

Moreover, let ∇_A and ∇_{G_A} be the Gauss-Manin connection on $\mathbb{H}^1_{\mathrm{dR}}(A/S)$ and $\mathbb{H}^1_{\mathrm{dR}}(G_A/S)$ respectively. Then we have $\nabla_{G_A} = \nabla_A \circ e$.

Proof. By Theorem 1.3.12, the decomposition $A[p^\infty] = G_A \times G'_A$ induces identifications

$$\underline{\omega}_{G_A/S} = e\underline{\omega}_{A/S}, \quad \mathbb{H}^1_{\mathrm{dR}}(G_A/S) = e\mathbb{H}^1_{\mathrm{dR}}(A/S), \quad \mathrm{Lie}(G_A^D/S) = e\mathrm{Lie}(A^D/S), \quad \nabla_{G_A} = \nabla_A \circ e.$$

Note that the chosen principal polarization λ_A induces an isomorphism $\mathrm{Lie}(A/S) \cong \mathrm{Lie}(A^D/S)$ compatible with the involution $\dagger$ on $\mathcal{O}_B$, so we have a non-degenerate pairing

$$\mathrm{Lie}(A^D/S) \times \underline{\omega}_{A/S} \to \mathcal{O}_S$$

such that the adjoint of $x \in \mathcal{O}_{B,p}$ is $x^\dagger$. Combining with the isomorphism $G_A^D \cong A[p^\infty]^{1,\dagger}$ induced by λ_A, the Weil pairing induces an isomorphism of $\mathcal{O}_S$-modules

$$e\mathrm{Lie}(A^D/S) \cong e^\dagger\mathrm{Lie}(A/S) \cong \mathrm{Lie}(A[p^\infty]^{1,\dagger}/S),$$

and the proof is complete. □

To simplify the notation, we will write $\mathbb{H}^1_{\mathrm{dR}}(G_A/S)$ as $\mathbb{H}_{G_A}$.

We now consider the universal false elliptic curve $(\mathcal{A}, i)$ over X.

Proposition 2.2.3. [Kas99, Corollary 3.2] *Let $\pi : (\mathcal{A}, i) \to X$ be the universal false elliptic curve. The Kodaira-Spencer morphism*

$$\mathrm{KS}: \ \pi_*\Omega^1_{\mathcal{A}/X} \to \Omega^1_{X/\mathbb{Z}_p} \otimes_{\mathcal{O}_X} R^1\pi_*\mathcal{O}_{\mathcal{A}}$$

induces an isomorphism

$$e\pi_*\Omega_{\mathcal{A}/X} \otimes e^\dagger\pi_*\Omega_{\mathcal{A}/X} \to \Omega^1_{X/\mathbb{Z}_p}.$$

By composing with the isomorphism $g' : \ e\pi_*\Omega^1_{\mathcal{A}/X} \cong e^\dagger\pi_*\Omega^1_{\mathcal{A}/X}$ induced by $g' \in \mathcal{O}_{B,p}^\times$, we have an isomorphism, also referred to as the Kodaira-Spencer isomorphism,

$$\mathrm{KS}: \ \underline{\omega}^{\otimes 2}_{G_{\mathcal{A}}/X} \xrightarrow[\cong]{\mathrm{Id}\otimes g'} e\pi_*\Omega^1_{\mathcal{A}/X} \otimes e^\dagger\pi_*\Omega^1_{\mathcal{A}/X} \xrightarrow[\cong]{\mathrm{KS}} \Omega^1_{X/\mathbb{Z}_p}. \tag{2.2.1}$$

If we change the choice of g', then KS varies by some element in $\mathbb{Z}_p^\times$.

By passing to the limit, we have the Hodge-de Rham exact sequence of $\mathcal{O}_{\hat{X}}$-modules

$$0 \to \underline{\omega}_{\mathcal{G}} \to \mathbb{H}_{\mathcal{G}} \to \mathrm{Lie}(G^\dagger) \to 0,$$

as well as the Kodaira-Spencer isomorphism and the Gauss-Manin connection

$$\mathrm{KS}: \ \underline{\omega}^{\otimes 2}_{\mathcal{G}} \to \Omega^1_{\hat{X}/\mathbb{Z}_p}; \quad \nabla: \ \mathbb{H}_{\mathcal{G}} \to \mathbb{H}_{\mathcal{G}} \otimes_{\mathcal{O}_{\hat{X}}} \Omega^1_{\hat{X}/\mathbb{Z}_p}$$

where $\mathcal{G} := G_{\mathcal{A}}$ and $\mathcal{G}^\dagger := \mathcal{A}[p^\infty]^{1,\dagger}$.

2.2.2 The Hasse invariant

Definition 2.2.4. *Let S be any $\mathbb{F}_p$-scheme and (A, i) be any false elliptic curve over S. We define* $\mathrm{H}(A)$ *to be the Hasse invariant* $\mathrm{Ha}(G_A)$ *of G_A.*

For any false elliptic curve (A, i) over a $\mathbb{F}_p$-scheme, we can canonically equip $A^{(p)}$ with a structure of false elliptic curve such that the Frobenius map $F : \ A \to A^{(p)}$ is a false isogeny whose false dual isogeny is the Verschiebung map $V : A^{(p)} \to A$. The following result is well-known.

Proposition 2.2.5. *The false elliptic curve (A, i) is ordinary if and only if $\mathrm{H}(A) \in \underline{\omega}_{G_A}^{\otimes(p-1)}$ is invertible.*

Proof. By Proposition 1.5.2, we know that (A, i) is ordinary if and only if $V : A^{(p)} \to A$ is finite étale, which is equivalent to $\ker(V)$ being an étale group. As $\ker(V) = \ker(V)^1 \times \ker(V)^2$ for

$$\ker(V)^1 = \ker(V : G_A^{(p)} \to G_A), \quad \ker(V)^2 = \ker(V : (G'_A)^{(p)} \to G'_A),$$

and $\ker(V)^1 \cong \ker(V)^2$, we know that $\ker(V)$ is étale if and only if $\ker(V)^1$ is étale, which in turn is equivalent to $\mathrm{Ha}(G_A)$ being invertible. For an alternative proof, see [Kas99, Proposition 5.1]. □

Let k be an algebraically closed field of characteristic p and (A, i) be a false elliptic curve over k. Then we have actually shown that $A[p]$ is connected if and only if $\mathrm{H}(A) = 0$. In this case, the false elliptic curve (A, i) is called supersingular. Consider the universal false elliptic curve $\mathcal{A}$ over $X_1^B(N)$. Let $\bar{\mathcal{A}}$ be the mod-p reduction of $\mathcal{A}$ and $\mathrm{H} := H(\bar{\mathcal{A}})$.

Lemma 2.2.6. [Kas99, Lemma 5.2] *The Hasse invariant* H *has simple zeroes on $X_1^B(N) \otimes_{\mathbb{Z}_p} \bar{\mathbb{F}}_p$.*

In fact, such a result holds in a broader framework, see [Kas04, Proposition 5.3].

The Hasse invariant H admits a global lift under mild conditions.

Proposition 2.2.7. [Kas99, Proposition 6.2] *If $k \geq 3$, then $H^1(X_1^B(N), \underline{\omega}_{\mathcal{G}}^k) = 0$. As a consequence, if $p \geq 5$, there exists an element $E_{p-1} \in H^0(X_1^B(N), \underline{\omega}_{\mathcal{G}}^{p-1})$ such that*

$$E_{p-1} \equiv \mathrm{H} \mod p.$$

In general, we need to consider the Hodge ideal.

Definition 2.2.8. *Let $(\mathcal{A}, i, \psi_N)$ be the universal false elliptic curve over $\hat{X} := \hat{X}_B^1(N)$. We define the Hodge ideal* $\mathrm{Hdg} := \mathrm{Hdg}(\mathcal{A}) \subset \mathcal{O}_{\hat{X}}$ *of $\mathcal{A}$ to be the inverse image of $\underline{\omega}_{\bar{\mathcal{G}}}^{\otimes(1-p)}\mathrm{H} \subset \mathcal{O}_{X_{\mathbb{F}_p}}$.*

According to Lemma 1.5.4, Hdg is locally generated by two elements, and if $p \in \mathrm{Hdg}^2$, then Hdg is invertible.

2.2.3 The ordinary locus and strict neighborhoods

Recall that if $I = [0, 1]$ or $[p^a, p^b]$ with $a, b \in \mathbb{N}$, we have

$$(\Lambda_I^0, \alpha) = \begin{cases} (\mathbb{Z}_p[[T]]\langle \frac{T}{p} \rangle, p) \text{ if } I = [0,1], \\ (\mathbb{Z}_p[[T]]\langle \frac{p}{T^{p^a}}, \frac{T^{p^b}}{p} \rangle, T) \text{ if } I = [p^a, p^b]. \end{cases}$$

Definition 2.2.9. *Let $\mathfrak{X}_I := \hat{X} \otimes_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spf}(\Lambda_I^0)$, and base change $\mathcal{A}$, $\mathcal{G} := G_{\mathcal{A}}$, $\underline{\omega}_{\mathcal{G}}$ to $\mathfrak{X}_I$. We define the ordinary locus $\mathfrak{X}_I^{\mathrm{ord}}$ to be the functor which associates to every α-adically complete Λ_I^0-algebra R the set of morphisms $f : \mathrm{Spf}(R) \to \mathfrak{X}_I$ such that $f^*(\mathrm{Hdg}) = R$.*

Moreover, for any integer $r \in \mathbb{N}$, we define the strict neighborhood $\mathfrak{X}_{r,I}$ to be the functor which associates to every α-adically complete Λ_I^0-algebra R the set of equivalence classes of pairs (f, η), where $f : \mathrm{Spf}(R) \to \mathfrak{X}_I$ is a morphism of formal schemes and $\eta \in H^0(\mathrm{Spf}(R), f^*(\underline{\omega}^{(1-p)p^{r+1}}))$ such that

$$\eta \mathrm{H}^{p^{r+1}} = \alpha \bmod p^2.$$

Here the equivalence is given by $(f, \eta) \sim (f', \eta')$ if $f = f'$ and $\eta = (1 + \frac{p^2}{\alpha} u)\eta'$ for some $u \in R$.

The equivalence relation is designed to guarantee that $\mathfrak{X}_{r,I}$ is representable. More precisely, in analogy with the modular curve case (see [AIP, Proposition 3.1] or [Sch15, Lemma III.2.13]),

Lemma 2.2.10. *We have*

(i) The functor $\mathfrak{X}_I^{\mathrm{ord}}$ is representable by the open formal subscheme of $\mathfrak{X}_I$ defined by the condition that H *is invertible.*

(ii) The functor $\mathfrak{X}_{r,I}$ is representable by an open formal subscheme of an admissible blow-up of $\mathfrak{X}_I$. Moreover, there is a natural map $\mathfrak{X}_I^{\mathrm{ord}} \to \mathfrak{X}_{r,I}$ identifying $\mathfrak{X}_I^{\mathrm{ord}}$ as an open subscheme of $\mathfrak{X}_{r,I}$.

Proof. Let $\mathrm{Spf}(A) \subset \hat{X}$ be affine open on which $\underline{\omega}_{\mathcal{G}}$ can be trivialized over $\mathrm{Spf}(A)$ and let $\mathrm{Spf}(A \otimes_{\mathbb{Z}_p} \Lambda_I^0) \subset \mathfrak{X}_I$ be the preimage of $\mathrm{Spf}(A)$. By fixing a trivialization, we will identify H as a scalar. Let $\tilde{\mathrm{H}} \in A$ be an lift of H. Then we have

$$\begin{aligned}
\mathfrak{X}_I^{\mathrm{ord}} \otimes_{\mathfrak{X}_I} \mathrm{Spf}(A \otimes_{\mathbb{Z}_p} \Lambda_I^0) &= \mathrm{Spf}(A \otimes_{\mathbb{Z}_p} \Lambda_I^0 \langle \frac{1}{\tilde{\mathrm{H}}} \rangle);\\
\mathfrak{X}_{r,I} \otimes_{\mathfrak{X}_I} \mathrm{Spf}(A \otimes_{\mathbb{Z}_p} \Lambda_I^0) &= \mathrm{Spf}\Big(A \otimes_{\mathbb{Z}_p} \Lambda_I^0 \langle X \rangle / (\tilde{\mathrm{H}}^{p^{r+1}} X - \alpha) \Big).
\end{aligned}$$

We need to check that these expressions are independent of (the choice of) the lift $\tilde{\mathrm{H}}$. The case of $\mathfrak{X}_I^{\mathrm{ord}}$ is clear. For the case of $\mathfrak{X}_{r,I}$, choose $\eta \in A \otimes_{\mathbb{Z}_p} \Lambda_I^0$ such that $\eta(\tilde{\mathrm{H}})^{p^{r+1}} = \alpha + p^2 u$ for some $u \in A \otimes_{\mathbb{Z}_p} \Lambda_I^0$, and let $\tilde{\mathrm{H}}'$ be another lift of H. It is straightforward to check that

$$(\tilde{\mathrm{H}})^{p^{r+1}} \equiv (\tilde{\mathrm{H}}')^{p^{r+1}} \mod p^2,$$

and since $\alpha \mid p$ in Λ_I^0, we have that $1 + \frac{p^2}{\alpha} u$ is a unit in $A \otimes_{\mathbb{Z}_p} \Lambda_I^0$. Combining these facts, we have that $\eta' := \eta(1 + \frac{p^2}{\alpha} u)^{-1}$ satisfies $\eta'(\tilde{\mathrm{H}}')^{p^{r+1}} \equiv \alpha \mod p^2$. Thus the local description is independent of the choice of the lift $\tilde{\mathrm{H}}$.

The final part follows from the observations that $\mathrm{Spf}(A \otimes_{\mathbb{Z}_p} \Lambda_I^0 \langle \frac{1}{\tilde{\mathrm{H}}} \rangle)$ is open in $\mathrm{Spf}(A \otimes_{\mathbb{Z}_p} \Lambda_I^0)$ as well as in $\mathrm{Spf}(A \otimes_{\mathbb{Z}_p} \Lambda_I^0 \langle X \rangle / (\tilde{\mathrm{H}}^{p^{r+1}} X - \alpha))$, and $\tilde{\mathrm{H}}$ is invertible if and only if H is invertible. □

Denote the adic generic fibers of $\mathfrak{X}_{r,I}$, $\mathfrak{X}_I^{\mathrm{ord}}$, $\mathfrak{X}_I$ by $\mathcal{X}_{r,I}$, $\mathcal{X}_I^{\mathrm{ord}}$, $\mathcal{X}_I$ respectively.

Remark 2.2.11. *The adic space $\mathcal{X}_{r,I}$ is affinoid. In fact, by Proposition 2.2.7, upon replacing* H *by a power, we may choose a global lifting $\widetilde{\mathrm{H}}$ of* H *such that $\mathcal{X}_{r,I}$ is the open subspace of $\mathcal{X}_I$ defined by $|\widetilde{\mathrm{H}}|^{p^{r+1}} \geq \max\{|T|, |p|\}$. By* [Kas99, Corollary 3.2]*, $\widetilde{\mathrm{H}}$ is a global section of an ample line bundle, so $\mathcal{X}_{r,I}$ is affinoid.*

A similar result concerning modular curves holds; see Remark A.1.6.

Following essentially the argument of [AIP, Lemma 3.4], we have the following crucial result. For the definitions of relevant commutative algebra terminologies, we refer to [Aut14].

Lemma 2.2.12. *The formal scheme $\mathfrak{X}_{r,I}$ is excellent and normal.*

Proof. By applying [Val76, Theorem 9] to $\mathbb{Z}_p$, we get that $\mathfrak{X}_{r,I}$ is excellent. To show the normality, let $\mathrm{Spf}(A) \subset \hat{X}$ be an affine open on which $\underline{\omega}_{\mathcal{G}}$ can be trivialized, and let $\tilde{\mathrm{H}}$ be a lift of H (viewed as a scalar). We have that the inverse image of $\mathrm{Spf}(A)$ in $\mathfrak{X}_{r,I}$ is $\mathrm{Spf}(R)$ for

$$R = \begin{cases} A[[T]]\langle u, v \rangle / (up - T, \tilde{\mathrm{H}}^{p^{r+1}} v - p) \text{ if } I = [0,1], \\ A[[T]]\langle u, v, w \rangle / (T^{p^a} v - p, uv - T^{p^{b-a}}, \tilde{\mathrm{H}}^{p^{r+1}} w - T) \text{ if } I = [p^a, p^b]. \end{cases}$$

We prove the normality using Serre's criterion, i.e. a ring is normal if and only if it is Cohen-Macauly and regular at codimension 1. Since A is regular, the algebra R is Cohen-Macauly. Note that for any dimension 1 local ring $(S,\mathfrak{m})$, if we have a non-zero divisor $\pi\in\mathfrak{w}$ such that S/π is reduced, then S is regular with parameter π. Let $\wp\in\mathrm{Spec}(R)$ be a codimension-1 point.

- If $\alpha\notin\wp$, then $\wp$ is a point of $\mathrm{Spec}(R[\frac{1}{\alpha}])$. As $\mathcal{X}_{r,I}$ is smooth, we have that $R[\frac{1}{\alpha}]$ is formally smooth, which implies the regularity of $\wp$.
- If $\alpha\in\wp$, then $\wp$ is a generic point of

$$R/\alpha R=\begin{cases}A/p[u,v]/(\tilde{\mathrm{H}}^{p^{r+1}}v) \text{ if } I=[0,1],\\ A/p[u,v,w]/(uv,\tilde{\mathrm{H}}^{p^{r+1}}w) \text{ if } I=[p^a,p^b].\end{cases}$$

  – If $\tilde{\mathrm{H}}\notin\wp$, then $R_\wp/\alpha R_\wp$ is reduced, so $R_\wp$ is regular of dimension 1 with parameter α.

  – If $\tilde{\mathrm{H}}\in\wp$, then

$$R/\tilde{\mathrm{H}}R=\begin{cases}A/(p,\tilde{\mathrm{H}})[u,v] \text{ if } I=[0,1];\\ A/(p,\tilde{\mathrm{H}})[u,v,w]/(uv) \text{ if } I=[p^a,p^b].\end{cases}$$

  Note that $\tilde{\mathrm{H}}$ has simple zeroes in A/p by Proposition 2.2.6, so $R_\wp/\tilde{\mathrm{H}}R_\wp$ is reduced, so $R_\wp$ is regular with parameter $\tilde{\mathrm{H}}$.

Now we have the regularity of all codimension-1 points of $\mathrm{Spec}(R)$, hence the normality of R by Serre's criterion. □

2.2.4 The partial Igusa towers

We call a triple (n,r,I) pre-adapted if

(i) $I=[0,1]$ and $1\le n\le r$; or

(ii) $I=[p^a,p^b]$ and $1\le n\le r+a$.

Recall that over $\mathfrak{X}_{r,I}$, we have $\alpha\in\mathrm{Hdg}^{p^{r+1}}$. Note that if $I=[0,1]$, then $\alpha=p$, while if $I=[p^a,p^b]$, then $\alpha=T$ and we have $T^a\mid p$ in Λ_I^0. Therefore for any pre-adapted triple (n,r,I), $p\in\mathrm{Hdg}^{p^{n+1}}$ over $\mathfrak{X}_{r,I}$. By Lemma 1.5.4, Hdg is invertible over $\mathfrak{X}_{r,I}$ and thanks to Proposition 1.5.5, the p-divisible group $\mathcal{G}$ admits the level-n canonical subgroup H_n^1. As $\mathcal{G}^\dagger\cong\mathcal{G}^D$, we have

$$\mathrm{Hdg}(\mathcal{G}^\dagger)=\mathrm{Hdg}(\mathcal{G}^D)=\mathrm{Hdg}(\mathcal{G}),$$

so $\mathcal{G}^\dagger$ admits the level-n canonical subgroup $H_n^{1,\dagger}$ and $(H_n^1)^D\cong\mathcal{G}^\dagger[p^n]/H_n^{1,\dagger}$.

To define the partial Igusa towers over $\mathfrak{X}_{r,I}$, we need the following lemma.

Lemma 2.2.13. *Let S be a connected locally noetherian (formal)-scheme and G be a finite étale group scheme of rank n over S. The functor sending a (formal) S-scheme T to*

(i) the set of T-group subschemes of G_T of rank m with $m\mid n$; or

(ii) the set of T-group subschemes of G_T of rank m which intersects G'_T trivially where $G'\subset G$ is a fixed rank-n' group subscheme over S; or

(iii) the set of trivializations of G_T over T

is representable by a finite étale (formal) S-scheme. Here the subscript T means base change to T.

Proof. We give an argument here for lack of reference. We deal with the first case; the other two are similar. Take a finite Galois cover $\tilde{S}$ of S with Galois group Γ such that $G_{\tilde{S}}$ is constant. Assume that $G_{\tilde{S}} = \underline{H}_{\tilde{S}}$ for some finite group H. Then H is equipped with an Γ-action. Let $\{H_i\}_{i\in I}$ be the set of all subgroups of rank m in H. Then Γ has an induced action on $\{H_i\}_{i\in I}$. For each $\gamma \in \Gamma$, we have that $\gamma * H_i = H_{\gamma(i)}$ for some $\gamma(i) \in I$.

It is straightforward to show that $\sqcup_{i\in I}\tilde{S}$ represents the functor which sends a (formal) $\tilde{S}$-scheme $\tilde{T}$ to the set of $\tilde{T}$-subgroups of rank m of $G_{\tilde{T}}$. Moreover, over $\sqcup_{i\in I}\tilde{S}$, we have

- an action of Γ which sends a point s lying in the i-th component to the point $\gamma * s$ lying in the $\gamma(i)$-th component;
- a universal object $\sqcup_{i\in I}H_i$ with H_i lying over the i-th component $\tilde{S}$. The action of Γ on $\sqcup_{i\in I}H_i$ is given by $\gamma * (a_i)_i = (\gamma * a_i)_{\gamma(i)}$.

Clearly, the structure map $\sqcup_{i\in I}\tilde{S} \to \tilde{S} \to S$ is Γ-invariant and finite étale, so according to [Hel18, Corollary 2.16], the quotient X of $\sqcup_{i\in I}\tilde{S}$ by Γ is a (formal) scheme which admits a finite étale morphism to S and fits into the Cartesian diagram below

$$\begin{tikzcd} \sqcup_{i\in I}\tilde{S} \arrow[r] \arrow[d] & \tilde{S} \arrow[d] \\ X \arrow[r] & S. \end{tikzcd}$$

Similarly, the quotient of $\sqcup_{i\in I}H_i$ by Γ is a rank-m subgroup of $H_X \subset G_X$, and the base change of H_X to $\sqcup_{i\in I}\tilde{S}$ is $\sqcup_{i\in I}H_i$.

We claim that X represents the functor which sends a (formal) S-scheme T to the set of rank-m T-group subschemes of G_T with $m \mid n$ and H_X is the universal object. In fact, given any (formal) S-scheme T with $\tilde{T} = T \times_S \tilde{S}$, by [Hel18, Proposition 2.17], we know that $\mathrm{Aut}(\tilde{T}/T) = \Gamma$ and $\tilde{T}/\Gamma = T$. Hence choosing an element $f \in X(T)$ is equivalent to choosing an Γ-invariant S-morphism $\tilde{f} : \tilde{T} \to X$, which is moreover equivalent to choosing an Γ-equivariant S-morphism

$$\tilde{f} : \tilde{T} \to \sqcup_{i\in I}\tilde{S} = X \times_S \tilde{S}.$$

By the universal property of $\sqcup_{i\in I}\tilde{S}$, choosing $\tilde{f}$ is equivalent to choosing a finite étale rank-m group subscheme $\tilde{U}$ of $G_{\tilde{T}}$ over $\tilde{T}$, which is equipped with an Γ-action compatible with the Γ-action on $\tilde{T}$. Now taking the quotient of $\tilde{U}$ by Γ, we get a rank-m group subscheme U of G_T over T, which characterizes $\tilde{U}$ as $U \times_T \tilde{T} = \tilde{U}$. Via the correspondence described above, we can easily show that $f^*(H_X) = U$, which completes the proof. □

Note that $\mathcal{X}_{r,I}$, $(H_n^1)^D$ and $\mathcal{G}[p^n]^D$ are étale and locally isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ and $(\mathbb{Z}/p^n\mathbb{Z})^2$ respectively. Inspired by the argument of [AIP, Lemma 3.2],

Lemma 2.2.14. *We have*

(i) There is a finite étale Galois cover $\mathcal{IG}'_{n,r,I}$ over $\mathcal{X}_{r,I}$ parameterizing compatible trivializations $(\mathbb{Z}/p^n\mathbb{Z})^2 \cong \mathcal{G}^D[p^n]$ and $(H_n^1)^D \cong \mathbb{Z}/p^n\mathbb{Z}$ over $\mathfrak{X}_{r,I}$. Moreover, the normalization of $\mathfrak{X}_{r,I}$ in $\mathcal{IG}'_{n,r,I}$ is well-defined and finite over $\mathfrak{X}_{r,I}$.

(ii) *There is a finite étale Galois cover $\mathcal{IG}_{n,r,I}$ over $\mathcal{X}_{r,I}$ parameterizing trivializations $(H_n^1)^D \cong \mathbb{Z}/p^n\mathbb{Z}$ over $\mathfrak{X}_{r,I}$. Moreover, the normalization of $\mathfrak{X}_{r,I}$ in $\mathcal{IG}_{n,r,I}$ is well-defined and finite over $\mathfrak{X}_{r,I}$.*

Proof. We deal with item (i) here; the proof of item (ii) is similar.

Let $\mathrm{Spf}(R) \subset \mathfrak{X}_{r,I}$ be an affine open. Note that R is α-adic, so the adic generic fiber of $\mathrm{Spf}(R)$ is $\mathrm{Spa}(R[\frac{1}{\alpha}], R)$. Since $(H_n^1)^D$ and H_n^D are étale over $R[\frac{1}{\alpha}]$, by Lemma 2.2.13 there is a finite étale Galois algebra $R'_n[\frac{1}{\alpha}]$ over $R[\frac{1}{\alpha}]$ parameterizing all compatible trivializations $(\mathbb{Z}/p^n\mathbb{Z})^2 \cong \mathcal{G}^D[p^n]$ and $\mathbb{Z}/p^n\mathbb{Z} \cong (H_n^1)^D$ over $R[\frac{1}{\alpha}]$, where R'_n is the normalization of R in $R'_n[\frac{1}{\alpha}]$. Note that R is excellent, so that R'_n is excellent and finite over R. Therefore R'_n is α-adically complete and separated, and if we equip $R'_n[\frac{1}{\alpha}]$ with topology such that R'_n is a ring of definition and (α) is an ideal of definition, we get a Tate affinoid ring $(R'_n[\frac{1}{\alpha}], R'_n)$. It is straightforward to check that $\mathrm{Spa}(R'_n[\frac{1}{\alpha}], R'_n)$ is a finite étale Galois cover of $\mathrm{Spa}(R[\frac{1}{\alpha}], R)$ parameterizing all compatible trivializations $(\mathbb{Z}/p^n\mathbb{Z})^2 \cong \mathcal{G}^D[p^n]$ and $(H_n^1)^D \cong \mathbb{Z}/p^n\mathbb{Z}$ over $\mathrm{Spa}(R[\frac{1}{\alpha}], R)$. To construct $\mathcal{IG}'_{n,r,I}$, we only need to show that we can glue $\mathrm{Spa}(R'_n[\frac{1}{\alpha}], R'_n)$. By Lemma 2.2.13, we know that $R'_n[\frac{1}{\alpha}]$ can be glued. To show that R'_n can be glued, it suffices to show that for any $0 \neq f \in R$, $R'_n\langle\frac{1}{f}\rangle$ is the normalization of $R\langle\frac{1}{f}\rangle$ in $R'_n[\frac{1}{\alpha}] \otimes_R R\langle\frac{1}{f}\rangle$. As $R\langle\frac{1}{f}\rangle$ is flat over R, $R'_n\langle\frac{1}{f}\rangle = R'_n \otimes_R R\langle\frac{1}{f}\rangle$ is a subring of $R'_n[\frac{1}{\alpha}] \otimes_R R\langle\frac{1}{f}\rangle$. Thus $R'_n\langle\frac{1}{f}\rangle$ is normal and finite over $R\langle\frac{1}{f}\rangle$ thanks to the normality of R and the fact that for any normal, excellent ring C and any ideal $I \subset C$, the I-adically completion of $C[\frac{1}{g}]$ is normal for any $g \in C$ (for details, see [AIP, Proposition 3.4]). This argument actually shows that $\mathrm{Spf}(R'_n)$ can be glued into a normal formal scheme over $\mathfrak{X}_{r,I}$, which is the normalization of $\mathfrak{X}_{r,I}$ in $\mathcal{IG}'_{n,r,I}$. □

Definition 2.2.15. *Let (n, r, I) be any pre-adapted triple. We define the formal partial Igusa tower of level n $\mathfrak{IG}_{n,r,I}$ over $\mathfrak{X}_{r,I}$ to be the normalization of $\mathfrak{X}_{r,I}$ in $\mathcal{IG}_{n,r,I}$. We will also denote the normalization of $\mathfrak{X}_{r,I}$ in $\mathcal{IG}'_{n,r,I}$ by $\mathfrak{IG}'_{n,r,I}$.*

The Galois group of $\mathcal{IG}_{n,r,I}$ over $\mathcal{X}_{r,I}$ is $(\mathbb{Z}/p^n\mathbb{Z})^\times$. More precisely, for any $\mathcal{IG}_{n,r,I}$-point $\alpha :\ \mathbb{Z}/p^n\mathbb{Z} \cong (H_n^1)^D$ and any $g \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, $g * \alpha$ is the composition

$$\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\times g} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\alpha} H_n^D.$$

By the normality of $\mathfrak{IG}_{n,r,I}$ and $\mathfrak{IG}'_{n,r,I}$, we have that the $(\mathbb{Z}/p^n\mathbb{Z})^\times$-action on $\mathcal{IG}_{n,r,I}$ extends to $\mathfrak{IG}_{n,r,I}$, and

$$H_n^D(\mathfrak{IG}_{n,r,I}) \cong \mathbb{Z}/p^n\mathbb{Z}, \quad \mathcal{G}[p^n]^D(\mathfrak{IG}'_{n,r,I}) \cong (\mathbb{Z}/p^n\mathbb{Z})^2, \quad (H_n^1)^D(\mathfrak{IG}'_{n,r,I}) \cong \mathbb{Z}/p^n\mathbb{Z}.$$

We can describe $\mathfrak{IG}_{n,r,I}$ using the universal false elliptic curve $\mathcal{A}$ instead of $\mathcal{G}$. Setting

$$H_n^2 := g(H_n^1), \quad H_n := H_n^1 \times H_n^2,$$

then H_n is a rank p^{2n} finite flat $\mathcal{O}_B$-submodule of $\mathcal{A}$ according to Lemma 2.2.1. By Lemma 2.2.1 and Proposition 1.5.5, we have the following:

(i) over $\mathcal{X}_{r,I}$, a trivialization $\mathbb{Z}/p^n\mathbb{Z} \cong (H_n^1)^D$ is equivalent to an isomorphism $(\mathbb{Z}/p^n\mathbb{Z})^2 \cong H_n^D$ of $\mathcal{O}_B$-modules;

(ii) modulo $p\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}$,

$$H_n^1 \equiv \ker(F^n : \mathcal{G}_{\mathcal{A}} \to \mathcal{G}_{\mathcal{A}}^{(p^n)}), \quad H_n \equiv \ker(F^n :\ \mathcal{A} \to \mathcal{A}^{(p^n)}).$$

In particular, H_n depends only on $\mathcal{A}$.

We will call H_n the level-n canonical subgroup of $\mathcal{A}$. For a more systematic treatment of canonical subgroups of π-divisible groups, we refer to [Bra13, § 2.2].

Adapting the proof of [AIP, Proposition 3.3 & 3.6] (recorded as Proposition A.1.9 in this dissertation), we can show

Proposition 2.2.16. *Let H_1 be the level-1 canonical subgroup of the universal false elliptic curve $(\mathcal{A}, i)$ and $f : (\mathcal{A}, i) \to (\mathcal{A}/H_1, i)$ be the quotient false isogeny.*

(i) For each $r \geq 1$, f induces a morphism $\phi : \mathfrak{X}_{r+1,I} \to \mathfrak{X}_{r,I}$ which is finite flat of degree p and whose reduction modulo $p\mathrm{Hdg}^{-1}$ is the Frobenius map relative to Λ_I^0. The morphism is characterized by $\phi^*(\mathcal{A}_{|\mathfrak{X}_{r,I}}) = \mathcal{A}_{|\mathfrak{X}_{r+1,I}}/H_1$.

(ii) Given any pre-adapted triple (n, r, I), f induces a finite $(\mathbb{Z}/p^n\mathbb{Z})^\times$-equivariant morphism $\Phi : \mathfrak{IG}_{n,r+1,I} \to \mathfrak{IG}_{n,r,I}$ which makes the diagram below commute

$$\begin{tikzcd} \mathfrak{IG}_{n,r+1,I} \arrow[r, "\Phi"] \arrow[d] & \mathfrak{IG}_{n,r,I} \arrow[d] \\ \mathfrak{X}_{r+1,I} \arrow[r, "\phi"] & \mathfrak{X}_{r,I}. \end{tikzcd}$$

Proof. For item (i), given any affine open $\mathrm{Spf}(R) \subset \mathfrak{X}_{r+1,I}$, the quotient $(\mathcal{A}' := \mathcal{A}/H_1, i, \psi'_N)$ of $(\mathcal{A}, i, \psi_N)$ by H_1 determines a morphism $g : \mathrm{Spf}(R) \to \mathfrak{X}_I$. Note that $G_{\mathcal{A}/H_1} = G_{\mathcal{A}}/H_1^1$ where H_1^1 is the canonical subgroup of $G_{\mathcal{A}}$. So we have

$$\mathrm{H}(\mathcal{A}') \equiv \mathrm{H}(\mathcal{A})^p \mod p\mathrm{Hdg}(\mathcal{A})^{-1}; \quad \underline{\omega}_{G_{\mathcal{A}'}} \equiv \underline{\omega}_{G_{\mathcal{A}}}^{\otimes p} \mod p\mathrm{Hdg}(\mathcal{A})^{-1}. \tag{2.2.2}$$

Note that by definition, there exists $\eta \in \underline{\omega}_{G_{\mathcal{A}}}^{(1-p)p^{r+2}}(R)$ such that

$$\mathrm{H}(\mathcal{A})^{p^{r+2}}\eta \equiv \alpha \mod p^2.$$

By taking p^{r+1}-th powers of equations 2.2.2, we have

$$\mathrm{H}(\mathcal{A}')^{p^{r+1}} \equiv \mathrm{H}(\mathcal{A})^{p^{r+2}} \mod ((p\mathrm{Hdg}(\mathcal{A})^{-1})^{p^{r+1}}, p^2 p\mathrm{Hdg}(\mathcal{A})^{-1}),$$

and we can choose $\eta' \in \underline{\omega}_{G_{\mathcal{A}'}}^{\otimes(1-p)p^{r+1}}$ such that

$$\eta' \equiv \eta \mod ((p\mathrm{Hdg}(\mathcal{A})^{-1})^{p^{r+1}}, p^2 p\mathrm{Hdg}(\mathcal{A})^{-1}).$$

Combining these facts, we have

$$\mathrm{H}(\mathcal{A}')^{p^{r+1}}\eta' \equiv \alpha \mod p^2.$$

By definition, the pair (g, η') determines a morphism $\phi : \mathrm{Spf}(R) \to \mathfrak{X}_{r,I}$ characterized by $\phi^*((\mathcal{A}, i, \psi_N)) = (\mathcal{A}', i, \psi'_N)$.

Let $\mathrm{Spf}(A) \subset \hat{X}$ be an affine open on which H admits a lift $\tilde{\mathrm{H}}$. Then we have that the inverse image of $\mathrm{Spf}(A)$ in $\mathfrak{X}_{r+1,I}$ resp. $\mathfrak{X}_{r,I}$ is

$$A \otimes_{\mathbb{Z}_p} \Lambda_I^0\langle X\rangle/(\tilde{\mathrm{H}}^{p^{r+2}}X - \alpha) \text{ (resp. } A \otimes_{\mathbb{Z}_p} \Lambda_I^0\langle Y\rangle/(\tilde{\mathrm{H}}^{p^{r+1}}Y - \alpha)),$$

and the morphism

$$\phi : A \otimes_{\mathbb{Z}_p} \Lambda_I^0\langle X\rangle/(\tilde{\mathrm{H}}^{p^{r+1}}X - \alpha) \to A \otimes_{\mathbb{Z}_p} \Lambda_I^0\langle Y\rangle/(\tilde{\mathrm{H}}^{p^{r+2}}Y - \alpha)$$

is Λ_I^0-linear. Moreover, modulo $p\tilde{\mathrm{H}}^{-1}$, we have that ϕ is the Frobenius map (relative to Λ_I^0) on $A \otimes_{\mathbb{Z}_p} \Lambda_I^0/p\tilde{\mathrm{H}}^{-1}$ and $\phi(X) = Y$. In particular, ϕ is finite flat of degree p modulo $\frac{p}{\mathrm{Hdg}(\mathcal{A})}$. Note that $A \otimes_{\mathbb{Z}_p} \Lambda_I^0$ is $p\mathrm{Hdg}^{-1}$-adic, so we have that ϕ is also finite flat of degree p.

By gluing, we get the desired finite flat degree-p morphism $\phi: \mathfrak{X}_{r+1,I} \to \mathfrak{X}_{r,I}$.

For item (ii), we first construct a morphism $\Phi : \mathcal{IG}_{n,r+1,I} \to \mathcal{IG}_{n,r,I}$ on generic fibers. Let $\mathrm{Spa}(C, C^+)$ be any affionid and $x : \mathrm{Spa}(C, C^+) \to \mathcal{IG}_{n,r+1,I}$ be any point. The point x determines a false elliptic curve (A, i) and a trivialization $\gamma_n : \mathbb{Z}/p^n\mathbb{Z} \cong (H_n^1(A))^D$ of the level-n canonical subgroup $H_n^1(A) \subset G_A$. As $p \in \mathrm{Hdg}(A')^{p^{n+1}} = \mathrm{Hdg}(A)^{p^{n+2}}$, the level-$n$ canonical subgroup of (A', i) exists and is isomorphic to $H_{n+1}(A)/H_1(A)$. Moreover, since f has false degree p, the false dual isogeny $f' : (A', i) \to (A, i)$ of f induces an isomorphism $H_n^1(A') \underset{\cong}{\xrightarrow{f'}} H_n^1(A)$. The composition

$$\gamma' : \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\gamma} H_n^1(A)^D \xrightarrow{(f')^D} H_n^1(A')^D,$$

determines a $\mathrm{Spa}(C, C^+)$-point $((A', i), \gamma')$ of $\mathcal{IG}_{n,r,I}$. The morphism Φ defined by sending $((A, i), \gamma)$ to $((A', i), \gamma')$ is clearly $(\mathbb{Z}/p^n\mathbb{Z})^\times$-equivariant and compatible with $\phi: \mathcal{X}_{r+1,I} \to \mathcal{X}_{r,I}$. By normality, we can extend Φ to a morphism $\Phi : \mathfrak{IG}_{n,r+1,I} \to \mathfrak{IG}_{n,r,I}$, which is the desired morphism. □

Let $h_n : \mathfrak{IG}_{n,r,I} \to \mathfrak{IG}_{n-1,r,I}$ be the structure morphism. By normality, we have

$$h_{n,*}(\mathcal{O}_{\mathfrak{IG}_{n,r,I}})^{1+p^{n-1}\mathbb{Z}/p^n\mathbb{Z}} = \mathcal{O}_{\mathfrak{IG}_{n-1,r,I}}, \ \forall\, n \geq 2, \ h_{1,*}(\mathcal{O}_{\mathfrak{IG}_{1,r,I}})^{(\mathbb{Z}/p\mathbb{Z})^\times} = \mathcal{O}_{\mathfrak{X}_{r,I}}.$$

Moreover, inspired by the argument in the proof of [AIP, Proposition 3.5]

Proposition 2.2.17. *We have*

$$\mathrm{Tr}_{1/0}(h_{1,*}(\mathcal{O}_{\mathfrak{IG}_{1,r,I}})) = \mathcal{O}_{\mathfrak{X}_{r,I}}, \quad \mathrm{Hdg}^{p^{n-1}}\mathcal{O}_{\mathfrak{IG}_{n-1,r,I}} \subset \mathrm{Tr}_{n/n-1}(h_{n,*}(\mathcal{O}_{\mathfrak{IG}_{n,r,I}})) \quad \forall\, n \geq 2.$$

Proof. The case $n = 1$ is straightforward, as the degree $p - 1$ is prime to p.

For $n \geq 2$, let $\mathrm{Spf}(R) \subset \mathfrak{X}_{r,I}$ be an affine open on which $\mathrm{Hdg}(\mathcal{A})$ is generated by $\tilde{\mathrm{H}}$ (possible by Lemma 1.5.4), let $\mathrm{Spf}(B_i)$ be its inverse image in $(H_i^1(\mathcal{A}))^D$ for $i = n-1, n$ and let $\mathrm{Spf}(C)$ be its inverse image in $(H_n^1/H_{n-1}^1)^D$. Let $D(B_n/B_{n-1})$ be the different ideal of $B_{n-1} \to B_n$ and $D(C/R)$ be the different ideal of $R \to C$. Note that over the f.p.p.f topology, the morphism $(H_n^1)^D \to (H_{n-1}^1)^D$ is a homogeneous space of $(H_n^1/H_{n-1}^1)^D$, so we have an identification of $B_n \otimes_{B_{n-1}} B_n = C \otimes_R B_n$-modules

$$D(B_n/B_{n-1}) \otimes_{B_{n-1}} B_n = D(C/R) \otimes_R B_n.$$

Let $\mathcal{A}' = \mathcal{A}/H_{n-1}$. As $G_{\mathcal{A}'}$ is of dimension 1 and $H_1^1(\mathcal{A}') \cong H_n^1/H_{n-1}^1$, by Proposition 1.5.5, we have that

$$\underline{\omega}_{(H_1^1(\mathcal{A}'))^D} = \underline{\omega}_{G_{\mathcal{A}'}^D[p]}/\mathrm{Hdg}(G_{\mathcal{A}'}), \quad \mathrm{Hdg}(\mathcal{A})^{p^{n-1}} = \mathrm{Hdg}(\mathcal{A}'),$$

and the annihilator of $\underline{\omega}_{(H_1^1(\mathcal{A}'))^D}$ is $\mathrm{Hdg}(\mathcal{A})^{p^{n-1}}R$, which in turn imply that the annihilator of $\Omega^1_{(H_1^1(\mathcal{A}'))^D/\mathfrak{X}_{r,I}}$ is $\mathrm{Hdg}(\mathcal{A})^{p^{n-1}}C$. According to the relation between the different ideal and differentials (see [Far10, § 1 & 2]), we have $D(C/R) = \mathrm{Hdg}(\mathcal{A})^{p^{n-1}}C$. By faithfully flat descent, we have $D(B_n/B_{n-1}) = \mathrm{Hdg}(\mathcal{A})^{p^{n-1}}B_n$. Upon shrinking $\mathrm{Spf}(R)$, we may assume B_n is a free B_{n-1}-module. Under this assumption, we have an isomorphism

$$D(B_n/B_{n-1})^{-1} \cong \mathrm{Hom}_{B_{n-1}}(B_n, B_{n-1}); \quad x \mapsto \mathrm{Tr}_{B_n/B_{n-1}}(\cdot x).$$

The existence of surjective morphisms $B_n \to B_{n-1}$ as B_{n-1}-modules implies that

$$\mathrm{Tr}_{B_n/B_{n-1}}(D(B_n/B_{n-1})^{-1}) = B_{n-1}.$$

Since $D(B_n/B_{n-1}) = \mathrm{Hdg}(\mathcal{A})^{p^{n-1}} B_n$, we have $\mathrm{Tr}_{B_n/B_{n-1}}(B_n) = \mathrm{Hdg}(\mathcal{A})^{p^{n-1}} B_{n-1}$.

Note that for each $n \geq 1$, $(H_n^1)^D$ is locally isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$, so over $\mathcal{X}_{r,I}$, we have an open immersion $\mathcal{IG}_{n,r,I} \to (H_n^1)^D$ for each $n \geq 1$, and a Cartesian diagram

$$\begin{tikzcd} \mathcal{IG}_{n,r,I} \arrow[r] \arrow[d] & (H_n^1)^D \arrow[d] \\ \mathcal{IG}_{n-1,r,I} \arrow[r] & (H_{n-1}^1)^D \end{tikzcd}$$

for each $n \geq 2$. By taking normalizations, for each $n \geq 2$, we have a commutative diagram

$$\begin{tikzcd} \mathfrak{IG}_{n,r,I} \arrow[r] \arrow[d] & (H_n^1)^D \arrow[d] \\ \mathfrak{IG}_{n-1,r,I} \arrow[r] & (H_{n-1}^1)^D \end{tikzcd}$$

over $\mathfrak{X}_{r,I}$ which induces injective morphisms $\mathcal{O}_{(H_i^1)^D} \hookrightarrow \mathcal{O}_{\mathfrak{IG}_{i,r,I}}$ for $i = n, n-1$. Bear this in mind, the desired result follows directly from the fact that $\mathrm{Tr}_{B_n/B_{n-1}}(B_n) = \mathrm{Hdg}(\mathcal{A})^{p^{n-1}} B_{n-1}$. □

Lemma 2.2.18. *Over $\mathfrak{IG}_{1,r,I}$, there exists an invertible sheaf $\underline{\delta}$ whose $(p-1)$-power is* Hdg. *Moreover, the natural morphism $\mathfrak{IG}_{1,r,I} \to \mathfrak{X}_{r,I}$ is flat.*

Proof. Assume $\mathrm{Spf}(R) \subset \mathfrak{X}_{r,I}$ is an affine open on which Hdg is generated by $\tilde{\mathrm{H}}$. By an explicit calculation using Oort-Tate theory (see for example [Bra13, Proposition 3.4]), $(H_1^1)^D$ has the form $\mathrm{Spec}(R[x]/(x^p - \tilde{\mathrm{H}}x))$. Let $\mathrm{Spf}(C) \subset \mathfrak{IG}_{1,r,I}$ be the preimage of $\mathrm{Spf}(R)$. As $(H_1^1)^D(C) \cong \mathbb{Z}/p\mathbb{Z}$, there exists $z \in C$ such that $z^{p-1} = \widetilde{\mathrm{H}}$.

Let $C' := R[x]/(x^{p-1} - \widetilde{\mathrm{H}})$. Then C' is Cohen-Macauly because R is normal. For any height-1 prime $\wp \subset C'$,

- if $\widetilde{\mathrm{H}} \notin \wp$, the étaleness of H_1^D over $B[\frac{1}{\widetilde{\mathrm{H}}}]$ implies $C'_\wp$ is regular;
- if $\widetilde{\mathrm{H}} \in \wp$, then $C'_\wp/xC'_\wp$ is reduced, so $C'_\wp$ is regular with parameter x.

By Serre's criterion of normality, we have that C' is normal. Since $(H_1^1)^D \cong \mathbb{Z}/p\mathbb{Z}$ over $\mathcal{X}_{r,I}$, the morphism

$$C' \to C, \quad x \mapsto z$$

is an isomorphism after inverting α. Note that both C and C' are normal, so we have

$$C' \cong C, \quad x \mapsto z,$$

which implies the flatness of C over R and completes the proof. □

We remark that an analogous result holds in the modular curve case; see Lemma A.1.10.

We have the following results describing the difference between the ordinary locus and its strict neighborhoods. The modular curve analogue, which is a combination of [AI17, Lemma 3.3 & 3.4], is recorded as Proposition A.1.11.

Proposition 2.2.19. *Fix a pre-adapted triple (n, r, I).*

(i) Let $\iota : \mathfrak{X}^{\mathrm{ord}}_{r,I} \to \mathfrak{X}_{r,I}$ be the natural map, and $\mathfrak{IG}^{\mathrm{ord}}_{n,r,I} := \mathfrak{IG}_{n,r,I} \times_{\mathfrak{X}_{r,I}} \mathfrak{X}^{\mathrm{ord}}_{r,I}$ be the ordinary locus of $\mathfrak{IG}_{n,r,I}$. Then for each $h \in \mathbb{N}$, the kernels of

$$\mathcal{O}_{\mathfrak{X}_{r,I}}/\alpha^h\mathcal{O}_{\mathfrak{X}_{r,I}} \to \iota_*(\mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}_{r,I}}/\alpha^h\mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}_{r,I}}); \quad \mathcal{O}_{\mathfrak{IG}_{n,r,I}}/\alpha^h\mathcal{O}_{\mathfrak{IG}_{n,r,I}} \to \iota_*(\mathcal{O}_{\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}}/\alpha^h\mathcal{O}_{\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}})$$

are annihilated by $\mathrm{Hdg}^{hp^{r+1}}$ and $\mathrm{Hdg}^{hp^{r+1}+\frac{p^n-p}{p-1}}$ respectively.

(ii) Let $\eta : \mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I} \to \mathfrak{X}_I \to \hat{X}$ be the structure map. The cokernel of the induced map $\eta^*(\Omega^1_{\hat{X}/\mathbb{Z}_p}) \to \Omega^1_{\mathfrak{IG}_{n,r,I}/\Lambda^0_I}$ is annihilated by a power (depending on n) of $\underline{\delta}$.

Proof. Here we only consider the case $I = [p^a, p^b]$; the case $I = [0, 1]$ is similar. As in Lemma 2.2.10, let $\mathrm{Spf}(A) \subset \hat{X}$ be an affine open on which $\underline{\omega}_{\mathcal{G}}$ can be trivialized and H admits a lift $\widetilde{\mathrm{H}}$. The inverse image of $\mathrm{Spf}(A)$ in $\mathfrak{X}_I$ (resp. $\mathfrak{X}_{r,I}$) is $\mathrm{Spf}(A \otimes_{\mathbb{Z}_p} \Lambda^0_I)$(resp. $\mathrm{Spf}(R)$) where

$$A \otimes_{\mathbb{Z}_p} \Lambda^0_I = A[[T]]\langle u, v\rangle/(T^{p^a}v - p, uv - T^{p^{b-a}}) \text{ (resp. } R = A \otimes_{\mathbb{Z}_p} \Lambda^0_I\langle \frac{T}{\widetilde{\mathrm{H}}^{p^{r+1}}}\rangle).$$

For item (i), since the inverse image of $\mathrm{Spf}(A)$ in $\mathfrak{X}^{\mathrm{ord}}_I$ is $\mathrm{Spf}(R^{\mathrm{ord}})$ for $R^{\mathrm{ord}} = A \otimes_{\mathbb{Z}_p} \Lambda^0_I\langle \frac{1}{\widetilde{\mathrm{Ha}}}\rangle$, by setting $w = \frac{T}{\widetilde{\mathrm{H}}^{p^{r+1}}} \in R$, the kernel of the natural map

$$R/T^hR \to R^{\mathrm{ord}}/T^hR^{\mathrm{ord}}$$

is generated by $(w^h, Tw^{h-1}, ..., T^{h-1}w) = w^h$. The result for $\mathfrak{X}_{r,I}$ follows directly.

Note that by Lemma 2.2.18, $\mathfrak{IG}_{1,r,I}$ is flat over $\mathfrak{X}_{r,I}$. Combining with the result for $\mathfrak{X}_{r,I}$, we have the result for $\mathfrak{IG}_{1,r,I}$. For $n \geq 2$, according to the argument in Proposition 2.2.17, $\mathfrak{IG}_{n,r,I}$ is the normalization of $\widetilde{\mathfrak{IG}}_{n,r,I} := \mathfrak{IG}_{1,r,I} \times_{(H^1_1)^D} (H^1_n)^D$ in $\mathcal{IG}_{n,r,I}$. Moreover, the different ideal $\mathcal{D}(H^D_n/H^D_1)$ equals to $\mathrm{Hdg}^{\frac{p^n-p}{p-1}}\mathcal{O}_{H_n}$ and by descent, the different ideal $\mathcal{D}(\widetilde{\mathfrak{IG}}_{n,r,I}/\mathfrak{IG}_{1,r,I})$ equals to $\mathrm{Hdg}^{\frac{p^n-p}{p-1}}\mathcal{O}_{\widetilde{\mathfrak{IG}}_{n,r,I}}$. Thus we have $\underline{\delta}^{p^n-p}\mathcal{O}_{\mathfrak{IG}_{n,r,I}} \subset \mathcal{O}_{\widetilde{\mathfrak{IG}}_{n,r,I}}$. Note that $\widetilde{\mathfrak{IG}}_{n,r,I}$ is flat over $\mathfrak{IG}_{1,r,I}$ because $(H^1_n)^D$ is flat over $(H^1_1)^D$, so the kernel of

$$\mathcal{O}_{\widetilde{\mathfrak{IG}}_{n,r,I}}/T^h\mathcal{O}_{\widetilde{\mathfrak{IG}}_{n,r,I}} \to \iota_*(\mathcal{O}_{\widetilde{\mathfrak{IG}}^{\mathrm{ord}}_{n,r,I}}/T^h\mathcal{O}_{\widetilde{\mathfrak{IG}}^{\mathrm{ord}}_{n,r,I}})$$

is killed by $\mathrm{Hdg}^{hp^{r+1}}$, and the desired result for $\mathfrak{IG}_{n,r,I}$ follows.

Note that for given any morphism $f : X \to Y$ of formal schemes over any base scheme S, we have a right exact sequence

$$f^*(\Omega^1_{Y/S}) \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0.$$

As the pull-back of $\Omega^1_{\hat{X}/\mathbb{Z}_p}$ along the natural map $\mathfrak{X}_I \to \hat{X}$ is just $\Omega^1_{\mathfrak{X}_I/\Lambda^0_I}$, it suffices to show the relative differentials $\Omega^1_{\mathfrak{X}_{r,I}/\mathfrak{X}_I}$ and $\Omega^1_{\mathfrak{IG}_{n,r,I}/\mathfrak{X}_{r,I}}$ are annihilated by powers of Hdg. By an explicit calculation using the second exact sequence of Kähler differentials, we obtain the desired result for $\Omega^1_{\mathfrak{X}_{r,I}/\mathfrak{X}_I}$. Since the pre-image of $\mathrm{Spf}(R)$ in $\mathfrak{IG}_{1,r,I}$ is $\mathrm{Spf}(R[x]/(x^{p-1} - \widetilde{\mathrm{H}}))$, we know $\Omega^1_{\mathfrak{IG}_{1,r,I}/\mathfrak{X}_{r,I}}$ is annihilated by $\underline{\delta}^{p-2}$. By the relation between different ideal and differential forms, we know that $\Omega^1_{\mathfrak{IG}_{n,r,I}/\mathfrak{IG}_{1,r,I}}$ is annihilated by $\mathrm{Hdg}^{2\frac{p^n-p}{p-1}}$. Combining all these, we get the desired result. □

We remark that according to our argument, there exists a positive integer c_n and a positive integer c_s such that the relative differentials $\Omega^1_{\mathfrak{IG}_{n,r,I}/\mathfrak{X}_{r,I}}$ and $\Omega^1_{\mathfrak{X}_{r,I}/\mathfrak{X}_I}$ are annihilated by Hdg^{c_n} and Hdg^{c_s} respectively.

2.3 The nearly overconvergent quaternion modular sheaf

Fix a pre-adapted triple (n, r, I), and let $\mathfrak{IG}_{n,r,I} \xrightarrow{g_n} \mathfrak{X}_{r,I}$ be the formal partial Igusa tower of level n. Let $(\mathcal{A}, i)$ be the universal false elliptic curve, $\mathcal{G} := G_{\mathcal{A}}$ and let $H_n^1 \subset \mathcal{G}[p^n]$, $H_n \subset \mathcal{A}[p^n]$ be the level-n canonical subgroups. Let Hdg be the Hodge ideal associated to $\mathcal{A}$. Recall that $H_n^D(\mathfrak{IG}_{n,r,I}) \cong \mathbb{Z}/p^n\mathbb{Z}$ and over $\mathfrak{IG}_{n,r,I}$, there is an ideal $\underline{\delta}$ whose $(p-1)$-th power is Hdg. Let $s := s_n := \mathrm{HT}(P_n)$ where $P_n \in H_n^D(\mathfrak{IG}_{n,r,I})$ is the universal section. By applying the VBMS machinery in § 1.6.1 to $(\mathbb{H}_{\mathcal{G}}^\sharp := \underline{\delta}\mathbb{H}_{\mathcal{G}}, \Omega_{\mathcal{G}} := \underline{\delta\omega_{\mathcal{G}}}, s)$ with respect to the ideal $\underline{\beta}_n := p^n \mathrm{Hdg}^{-\frac{p^n}{p-1}}$, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{V}_0(\mathbb{H}_{\mathcal{G}}^\sharp, s) \arrow[d] \arrow[dr, "\pi"] & & \\
\mathcal{V}_0(\Omega_{\mathcal{G}}, s) \arrow[r, "\pi'"] & \mathfrak{IG}_{n,r,I} \arrow[r, "g_n"] & \mathfrak{X}_{r,I}.
\end{tikzcd}$$

Let $\mathcal{T} \subset \mathcal{T}^{\mathrm{ext}}$ be formal groups over $\mathfrak{X}_{r,I}$ such that

$$\mathcal{T}(S) = 1 + \rho^*(\underline{\beta}_n)\mathcal{O}_S \subset \mathcal{T}(S)^{\mathrm{ext}} = \mathbb{Z}_p^\times(1 + \rho^*(\underline{\beta}_n)\mathcal{O}_S) \subset G_m(S)$$

for any $\mathfrak{X}_{r,I}$-formal scheme $\rho : S \to \mathfrak{X}_{r,I}$. We define actions of $\mathcal{T}$ (resp. $\mathcal{T}^{\mathrm{ext}}$) on $\mathcal{V}_0(\mathbb{H}_{\mathcal{G}}^\sharp, s)$ and $\mathcal{V}_0(\Omega_{\mathcal{G}}, s)$ over $\mathfrak{IG}_{n,r,I}$ (resp. $\mathfrak{X}_{r,I}$) as follows:

(i) for any $\mathfrak{IG}_{n,r,I}$-formal scheme $\rho : S \to \mathfrak{IG}_{n,r,I}$, a point $v \in \mathcal{V}_0(\mathbb{H}_{\mathcal{G}}^\sharp, s)(S)$ is an $\mathcal{O}_S$-linear map $v : \rho^*(\mathbb{H}_{\mathcal{G}}^\sharp) \to \mathcal{O}_S$ whose reduction $\bar{v}$ modulo $\rho^*(\underline{\beta}_n)$ sends $\rho^*(s)$ to 1. For any $t \in \mathcal{T}(S)$, we define $t * v := t^{-1}v$. The action of $\mathcal{T}$ on $\mathcal{V}_0(\Omega_{\mathcal{G}}, s)$ is similar.

(ii) for any $\mathfrak{X}_{r,I}$-formal scheme $u : S \to \mathfrak{X}_{r,I}$, a point in $\mathcal{V}_0(\mathbb{H}_{\mathcal{G}}^\sharp, s)(S)$ is a pair (ρ, v) consisting of a lift $\rho : S \to \mathfrak{IG}_{n,r,I}$ of u and an $\mathcal{O}_S$-linear map $v : \rho^*(\mathbb{H}_{\mathcal{G}}^\sharp) \to \mathcal{O}_S$ such that $\bar{v}(\rho^*(s)) = 1$. For any $\lambda \in \mathbb{Z}_p^\times$ with image $\bar{\lambda}$ in $(\mathbb{Z}/p^n\mathbb{Z})^\times$, we set $\lambda * (\rho, v) := (\bar{\lambda} \circ \rho, \lambda^{-1}v)$. This makes sense because the automorphism $\bar{\lambda} : \mathfrak{IG}_{n,r,I} \to \mathfrak{IG}_{n,r,I}$ induces isomorphisms

$$\bar{\lambda}^* : \Omega_{\mathcal{G}} \xrightarrow{\cong} \Omega_{\mathcal{G}}; \quad \mathbb{H}_{\mathcal{G}}^\sharp \xrightarrow{\cong} \mathbb{H}_{\mathcal{G}}^\sharp$$

such that modulo $\underline{\beta}_n$, $\bar{\lambda}^*(s) = \bar{\lambda}s$. The action of $\mathcal{T}^{\mathrm{ext}}$ on $\mathcal{V}_0(\Omega_{\mathcal{E}}, s)$ is similar.

We let $\mathcal{T}^{\mathrm{ext}}$ act on functions via pull-back, i.e. $(t * f)(\rho, v) = f(t * (\rho, v))$.

Definition 2.3.1. *Let $\kappa_I : \mathbb{Z}_p^\times \to \Lambda_I^\times$ be the universal character, $\kappa_{I,f}$ be the restriction of κ_I to $(\mathbb{Z}/q\mathbb{Z})^\times$, and $\kappa_I^0 := \kappa_I(\kappa_{I,f})^{-1}$.*

- *We define $\mathfrak{w}^{\kappa_{I,f}}$ to be the coherent $\mathcal{O}_{\mathfrak{X}_{r,I}}$-module $(g_{i,*}(\mathcal{O}_{\mathfrak{IG}_{i,r,I}}) \otimes_{\Lambda_I^0} \Lambda_I)[(\kappa_{I,f})^{-1}]$ where $i = 2$ if $p = 2$ and $i = 1$ if $p \geq 3$.*
- *We define $\mathfrak{w}^{\kappa_I,0} := f'_{0,*}(\mathcal{O}_{\mathcal{V}_0(\Omega_E,s)})[(\kappa_I^0)^{-1}]$, where $f'_0 := g_n \circ \pi' : \mathcal{V}_0(\Omega_{\mathcal{E}}, s) \to \mathfrak{X}_{r,I}$ is the structure morphism.*
- *We define $\mathbb{W}_{\kappa_I}^0 := f_{0,*}(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}_{\mathcal{E}}^\sharp,s)})[(\kappa_I^0)^{-1}]$, where $f_0 := g_n \circ \pi : \mathcal{V}_0(\mathbb{H}_{\mathcal{E}}^\sharp, s) \to \mathfrak{X}_{r,I}$ is the structure morphism.*
- *We define $\mathfrak{w}^{\kappa_I} := \mathfrak{w}^{\kappa_I,0} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \mathfrak{w}^{\kappa_{I,f}}$, $\quad \mathbb{W}_{\kappa_I} := \mathbb{W}_{\kappa_I}^0 \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \mathfrak{w}^{\kappa_{I,f}}$.*

Here $-[\chi]$ means taking χ-isotypic components with respect to the $(\mathbb{Z}/p^i\mathbb{Z})^\times$-action for $\mathfrak{w}^{\kappa_{I,f}}$ and the $\mathcal{T}^{\mathrm{ext}}$-action for other sheaves. When the interval I is clear, we omit it in notations.

2.3.1 Local descriptions of $\mathbb{W}_\kappa$

We call a triple (n,r,I) adapted if

- $I=[0,1]$, $4\le n\le r$ if $p=2$ or $2\le n\le r$ if $p\ge 3$; or
- $I=[p^a,p^b]$, $b+4\le n\le r+a$ if $p=2$ or $b+2\le n\le r+a$ if $p\ge 3$.

Then following essentially the idea of [AI17, Theorem 3.11], we have

Theorem 2.3.2. *Let (n,r,I) be an adapted triple. The filtration $\mathrm{Fil}_\bullet\pi_*(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}},s)})$ (see Lemma 1.6.4 (ii)) induces a filtration $\mathrm{Fil}_\bullet := g_{n,*}(\mathrm{Fil}_\bullet)$ on $f_{0,*}(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}},s)})$ which is preserved by the action of $\mathcal{T}^{\mathrm{ext}}$. For every $h\ge 0$, we define*

$$\mathrm{Fil}_h\mathbb{W}^0_\kappa := \mathrm{Fil}_h f_{0,*}(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}},s)})[(\kappa^0)^{-1}];\quad \mathrm{Fil}_h\mathbb{W}_\kappa := \mathrm{Fil}_h\mathbb{W}^0_\kappa\otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}}\mathfrak{w}^{\kappa_f}.$$

Then we obtain an increasing filtration $\mathrm{Fil}_\bullet$ of $\mathbb{W}^{(0)}_\kappa$ such that

(i) for each $h\in\mathbb{N}$, $\mathrm{Fil}_h\mathbb{W}^0_\kappa$ is a locally free $\mathcal{O}_{\mathfrak{X}_{r,I}}$-module of rank $h+1$ for the Zariski topology on $\mathfrak{X}_{r,I}$;

(ii) for each $h\in\mathbb{N}$, $\mathrm{Gr}_h\mathbb{W}^{(0)}_\kappa\cong\mathfrak{w}^{\kappa,(0)}\otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}}\underline{\omega}_{\mathcal{G}}^{-2h}$. In particular, $\mathrm{Fil}_0\mathbb{W}^{(0)}_\kappa\cong\mathfrak{w}^{\kappa,(0)}$;

(iii) $\mathbb{W}^{(0)}_\kappa$ is the α-adic completion of $\varinjlim_{h\to\infty}\mathrm{Fil}_h\mathbb{W}^{(0)}_\kappa$.

The rest of this subsection is devoted to the proof of this theorem.

Let R be an α-torsion-free α-adic Λ^0_I-algebra, and let $\rho:\ \mathrm{Spf}(R)\to\mathfrak{X}_{r,I}$ be a morphism of Λ^0_I-formal schemes such that $\rho^*(\underline{\omega}_{\mathcal{G}})$ (resp. $\rho^*(\mathbb{H}_{\mathcal{G}})$) is a free R-module of rank 1 (resp. 2) with a generator ω(resp. a basis $\{\omega,\eta\}$). Let

$$\rho_n:\ \mathrm{Spf}(R_n) := \mathrm{Spf}(R)\times_{\mathfrak{X}_{r,I}}\mathfrak{IG}_{n,r,I}\to\mathfrak{IG}_{n,r,I}$$

be the base change of ρ to $\mathrm{Spf}(R_n)$. Shrinking $\mathrm{Spf}(R)$ if necessary, we will assume that $\rho_n^*(\underline{\beta}_n)$(resp. $\rho_n^*(\underline{\delta})$) is generated by β_n(resp. δ) such that $\rho_n^*(\Omega_{\mathcal{G}})$ and $\rho_n^*(\mathbb{H}^\sharp_{\mathcal{G}})$ have bases $\{f:=\delta\omega\}$ and $\{f,e:=\delta\eta\}$ respectively with $f \mod (\beta_n)=\rho_n^*(s)$.

Lemma 2.3.3. *Let X,Y,Z be variables such that $1+\beta_nZ=X$. By assigning f,e to X,Y respectively, we have*

$$\rho^*(\mathcal{V}(\mathbb{H}^\sharp_{\mathcal{G}}))=\mathrm{Spf}(R_n\langle X,Y\rangle),\quad \rho^*(\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}},s))=\mathrm{Spf}(R_n\langle Z,Y\rangle).$$

Moreover, we have that

(i) for any $t\in\mathcal{T}(R_n)$ and $f=\sum_{i,j\ge 0}a_{ij}Z^iY^j\in R_n\langle Z,Y\rangle$, one has

$$t^{-1}*f=\sum_{s\ge 0,j\ge 0}f_{s,j}(t)Z^sY^j,\quad f_{s,j}(t)=\sum_{i\ge s}a_{ij}\binom{i}{s}t^{s+j}(\frac{t-1}{\beta_n})^{i-s};\tag{2.3.1}$$

(ii) for any $\lambda\in\mathbb{Z}_p^\times$ with image $\bar\lambda\in(\mathbb{Z}/p^n\mathbb{Z})^\times$ and for any $r\in R_n$, one has

$$\lambda*(1+\beta_nZ)=\lambda^{-1}u(\bar\lambda)(1+\beta_nZ),\quad \lambda*Y=\lambda^{-1}u(\bar\lambda)Y;\quad \lambda*r=(\bar\lambda)^*(r)\tag{2.3.2}$$

with $u:\ (\mathbb{Z}/p^n\mathbb{Z})^\times\to R_n^\times;\quad \bar\lambda\mapsto\frac{\bar\lambda^*(\delta)}{\delta}$.

Proof. Let $f^\vee$, $e^\vee$ be the dual R_n-basis of $\rho_n^*(f)$, $\rho_n^*(e)$ respectively. Then by the constructions in Lemma 1.6.4, for any morphism $u:\ \mathrm{Spf}(R_n') \to \mathrm{Spf}(R_n)$, one has

$$\mathcal{V}(\mathbb{H}_{\mathcal{G}}^\sharp)(R_n') = \{au^*(f^\vee) + bu^*(e^\vee):\ a, b \in R_n'\} = \mathrm{Spf}(R_n\langle X, Y\rangle)(R_n'),$$
$$\mathcal{V}_0(\mathbb{H}_{\mathcal{G}}^\sharp, s)(R_n') = \{au^*(f^\vee) + bu^*(e^\vee):\ a \in 1 + \beta_n R_n', b \in R_n'\} = \mathrm{Spf}(R_n\langle Z, Y\rangle)(R_n')$$

by viewing the element $au^*(f^\vee) + bu^*(e^\vee)$ as the R_n-morphism

$$R_n\langle X, Y\rangle \to R_n',\ X \mapsto a, Y \mapsto b; \quad R_n\langle Z, Y\rangle \to R_n',\ Z \mapsto \frac{a-1}{\beta_n}, Y \mapsto b.$$

If $t \in \mathcal{T}(R_n)$, then

$$t^{-1} * X = tX, \quad t^{-1} * Y = tY, \quad t^{-1} * Z = tZ + \frac{t-1}{\beta_n},$$

and equation 2.3.1 follows.

For equation 2.3.2, note that $\bar{\lambda}^*(\omega) = \omega$ because ω is defined over $\mathfrak{X}_{r,I}$. As $f = \delta\omega$, for any $\lambda \in \mathbb{Z}_p^\times$, $\bar{\lambda}^*(f) = \mu(\bar{\lambda})f$. Moreover, as $X(\rho, v) = v(\rho^*(f))$ for any points (ρ, v), we have

$$\lambda * X(\rho, v) = X(\lambda * (\rho, v)) = X(\bar{\lambda} \circ \rho, \lambda^{-1}v) = \lambda^{-1}v(\rho^*(\bar{\lambda})^*(f)) = \lambda^{-1}u(\bar{\lambda})v(\rho^*(f))$$

This implies that $\lambda * X = \lambda^{-1}u(\bar{\lambda})X$. The cases of Y is similar and the case of r follows directly from definition. □

Lemma 2.3.4. *Let $\pi' :\ \mathcal{V}_0(\Omega_{\mathcal{G}}, s) \to \mathfrak{IG}_{n,r,I}$ and $\pi :\ \mathcal{V}_0(\mathbb{H}_{\mathcal{G}}^\sharp, s) \to \mathfrak{IG}_{n,r,I}$ be the structure morphisms and $V = \frac{Y}{1+\beta_n Z}$. Then*

$$\rho_n^*(\pi'_*(\mathcal{O}_{\mathcal{V}_0(\Omega_{\mathcal{G}},s)})[(\kappa^0)^{-1}] = R_n(1+\beta_n Z)^\kappa := R_n\kappa(1+\beta_n Z),$$
$$\rho_n^*(\pi_*(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}_{\mathcal{G}}^\sharp,s)}))[(\kappa^0)^{-1}] = \{\sum_{m=0}^{\infty} a_m(1+\beta_n Z)^\kappa V^m:\ a_m \in R_n,\ a_m \to 0\}.$$

Proof. Note that $1 + \beta_n Z$ is invertible because $p^{n-1} \mid \beta_n$, so V is well-defined in $R\langle Z, Y\rangle$ and $R\langle Z, V\rangle = R\langle Z, Y\rangle$. Also, we have that $(1+\beta_n Z)^\kappa := \kappa(1+\beta_n Z) = \kappa^0(1+\beta_n Z)$. We claim that

$$R_n\langle Z\rangle^{\mathcal{T}} = R_n; \quad R_n\langle Z, Y\rangle^{\mathcal{T}} = R_n\langle V\rangle.$$

For any $t = (1+\beta_n u) \in \mathcal{T}(R_n)$ and $g(Z) = \sum_{n\geq 0} a_n Z^n \in R_n\langle Z\rangle^{\mathcal{T}}$, by equation 2.3.2, we have

$$g(Z) = t^{-1} * g(Z) = \sum_{n\geq 0}^{\infty} a_n(u+tZ)^n.$$

Evaluating at $Z = 0$, we have $a_0 = g(0) = \sum_{n\geq 0}^{\infty} a_n u^n$ for any $u \in R_n$, which forces $a_n = 0$ for all $n \geq 1$. Similarly, note that V is fixed by the action of $\mathcal{T}$, so if $g(Z, V) = \sum_{i,j}^{\infty} a_{i,j} Z^i V^j \in R_n\langle Z, V\rangle$ is fixed by the $\mathcal{T}$-action, then

$$g(Z, V) = t^{-1} * g(Z, V) = \sum_{i,j}^{\infty} a_{i,j}(t^{-1} * Z)^i V^j = \sum_{i,j}^{\infty} a_{i,j}(tZ + u)^i V^j.$$

Evaluating at $Z = 0$, we get that $a_{i,j} = 0$ if $i > 0$.

Note that $(1+\beta_n Z)^\kappa$ is invertible and $t * (1+\beta_n Z)^\kappa = (\kappa^0)^{-1}(t)(1+\beta_n Z)^\kappa$, so we have

$$R_n\langle Z\rangle[(\kappa^0)^{-1}] = R_n(1+\beta_n Z)^\kappa, \quad R_n\langle Z, Y\rangle[(\kappa^0)^{-1}] = R_n\langle V\rangle(1+\beta_n Z)^\kappa.$$

□

Let $\mathfrak{w}'^{,\kappa,0} := \pi'_*(\mathcal{O}_{\mathcal{V}_0(\Omega_{\mathcal{G}},s)})[(\kappa^0)^{-1}]$ and $\tilde{\mathbb{W}}^0_\kappa := \pi_*(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}},s)})[(\kappa^0)^{-1}]$ with respect to the $\mathcal{T}$-action. By Lemma 2.3.4, $\tilde{\mathbb{W}}^0_\kappa$ admits an increasing filtration $\mathrm{Fil}_\bullet$.

Lemma 2.3.5. *For each $h \in \mathbb{N}$, we have that $\mathrm{Fil}_h\tilde{\mathbb{W}}^0_\kappa$ is a locally free $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}$-module of rank $h+1$ and*

$$\mathrm{Gr}_h\tilde{\mathbb{W}}^0_\kappa \cong \mathfrak{w}'^{,\kappa,0} \otimes (\underline{\omega}_{\mathcal{G}})^{-2h}.$$

Proof. Over $\mathrm{Spf}(R_n)$, we have

$$\mathrm{Fil}_h\tilde{\mathbb{W}}^0_\kappa = \{\sum_{i=0}^h a_i(1+\beta_n Z)^\kappa V^i,\ a_i \in R_n\},$$

where $V = \frac{Y}{1+\beta_n Z}$, and Y, $1+\beta_n Z$ are defined by $e = \delta\eta \rightsquigarrow Y$, $f = \delta\omega \rightsquigarrow 1+\beta_n Z$ respectively. By Lemma 2.3.4, $(1+\beta_n Z)^\kappa$ is a local generator of $\mathfrak{w}'^{,\kappa,0}$, and the image of $(1+\beta_n Z)^\kappa V^h$ is a local generator of $\mathrm{Gr}_h\tilde{\mathbb{W}}^0_\kappa$. Recall that we have the Hodge-de Rham exact sequence

$$0 \to \underline{\omega}_{\mathcal{G}} \to \mathbb{H}_{\mathcal{G}} \to \underline{\omega}^{-1}_{\mathcal{G}^D} \to 0.$$

Note that ω is a basis of $\underline{\omega}_{\mathcal{G}}$ and the image of η is a basis of $\underline{\omega}^{-1}_{\mathcal{G}^D}$, so we may identify the image of $(1+\beta_n Z)^\kappa V^h$ in the graded piece Gr_h as a local generator of $\mathfrak{w}'^{,\kappa,0} \otimes (\underline{\omega}_{\mathcal{G}})^{-h} \otimes (\underline{\omega}_{\mathcal{G}^D})^{-h}$. Note that the chosen principal polarization and the chosen element g' induce an isomorphism $\underline{\omega}_{\mathcal{G}} \cong \underline{\omega}_{\mathcal{G}^D}$, we have $\mathrm{Gr}_h\tilde{\mathbb{W}}^0_\kappa \cong \mathfrak{w}'^{,\kappa,0} \otimes (\underline{\omega}_{\mathcal{G}})^{-2h}$. $\square$

Recall that $q = p$ if $p \geq 3$ and $q = 4$ if $p = 2$.

Lemma 2.3.6. *Let (n, r, I) be an adapted triple.*

(i) The natural morphism $\mathcal{O}_{\mathfrak{IG}_{n,r,I}} \to \pi'_*(\mathcal{O}_{\mathcal{V}_0(\Omega_{\mathcal{G}},s)})$ induces an isomorphism

$$\mathcal{O}_{\mathfrak{IG}_{n,r,I}}/q \cong \mathfrak{w}'^{,\kappa,0}/q.$$

(ii) Let $\mathcal{O}^{\circ\circ}_{\mathfrak{X}_{r,I}} \subset \mathcal{O}_{\mathfrak{X}_{r,I}}$ be the subsheaf of topological nilpotent elements. Then we have

$$\kappa^0_I(\mathbb{Z}_p^\times) - 1 \subset \mathcal{O}^{\circ\circ}_{\mathfrak{X}_{r,I}}, \quad \kappa^0_I(1+p^{n-1}\mathbb{Z}_p) - 1 \subset \mathrm{Hdg}^{\frac{p^n-p}{p-1}}\mathcal{O}^{\circ\circ}_{\mathfrak{X}_{r,I}}.$$

Proof. For (i), note that locally, say over $\mathrm{Spf}(R_n)$, $\mathfrak{w}'^{,\kappa,0}$ is $R_n\kappa(1+\beta_n Z)$. Now by Proposition 1.7.12, we have $\kappa(1+\beta_n Z) \equiv 1 \bmod q$ as $p^{n-1} \mid \beta_n$. We are done.

For (ii), note that for any integer $i \geq 1$

$$\kappa_I(\exp q^{p^i}) = (1+T)^{p^i} = 1 + \sum_{k=1}^{i}\binom{i}{k}T^i \equiv 1 \bmod (T^{p^i}, T^{p^{i-1}}p, ..., p^iT),$$

so we have

$$\kappa_I(1+qp^i\mathbb{Z}_p) - 1 \subset (T^{p^i}, T^{p^{i-1}}p, ..., p^iT)\mathcal{O}_{\mathfrak{X}_{r,I}} \subset (T^{p^i}, p)\mathcal{O}_{\mathfrak{X}_{r,I}}.$$

As κ^0_I is the connected part of κ_I, we have that $\kappa^0_I(\mathbb{Z}_p^\times) = \kappa_I(1+q\mathbb{Z}_p)$, and $\kappa^0_I(\mathbb{Z}_p^\times) - 1 \subset \mathcal{O}^{\circ\circ}_{\mathfrak{X}_{r,I}}$. By the adapted assumption on (n, r, I), we have

$$T \in (p), \quad p\mathrm{Hdg}^{-p^n} \subset \mathfrak{X}^{\circ\circ}_{r,I}, \quad \text{if } I = [0,1];$$
$$p\mathrm{Hdg}^{-p^n} \subset \mathfrak{X}^{\circ\circ}_{r,I}, \quad T^{p^{n-3}}\mathrm{Hdg}^{-p^n} \subset \mathfrak{X}^{\circ\circ}_{r,I}, \quad \text{if } I = [p^a, p^b].$$

Note that $\frac{p^n-p}{p-1} \leq p^n$, so we have

$$\kappa_I^0(1+p^{n-1}\mathbb{Z}_p) - 1 \subset \mathrm{Hdg}^{\frac{p^n-p}{p-1}} \mathcal{O}_{\mathfrak{X}_{r,I}}^{\circ\circ}.$$

$\square$

Now assume $\rho : \mathrm{Spf}(R) \to \mathfrak{X}_{r,I}$ is an open immersion and Hdg is generated by $\tilde{\mathrm{H}}$ over $\mathrm{Spf}(R)$. By Proposition 2.2.17, we can inductively construct $\{c_i\}_{i=0,\ldots,n}$ such that

$$c_0 = 1,\ c_1 = \frac{1}{p-1}, \ldots, c_i \in \tilde{\mathrm{H}}^{-\frac{p^i-p}{p-1}} R_i; \quad \mathrm{Tr}_{i/i-1}(c_i) = c_{i-1},\ \forall i = 1, .., n.$$

Generalizing the descent argument in [AIP, Lemma 5.4], we have

Proposition 2.3.7. *Let (n, r, I) be an adapted triple and let b_n be a section of $\mathfrak{w}'^{,\kappa,0}(R_n)$ which corresponds to 1 via the isomorphism $\mathfrak{w}'^{,\kappa,0}/q \cong \mathcal{O}_{\mathfrak{IG}_{n,r,I}}/q\mathcal{O}_{\mathfrak{IG}_{n,r,I}}$ (in Lemma 2.3.6). Then $\mathfrak{w}^{\kappa,0}(R)$ is a free R-module of rank 1. Moreover, for any lift $\tilde{\sigma}$ of $\sigma \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, the element*

$$b := \sum_{\sigma \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \kappa_I^0(\tilde{\sigma})\tilde{\sigma} * (c_n b_n)$$

is a well-defined generator of $\mathfrak{w}^{\kappa,0}(R)$. Consequently, we have

$$\mathbb{W}_\kappa^0(R) = \{\sum_{h \geq 0} a_h b V^h :\ a_h \in R \ \lim_{h\to\infty} a_h \to 0\}$$

Proof. Note that for any $\lambda \in 1 + p^n\mathbb{Z}_p$, $\lambda * b_n = (\kappa_I^0)^{-1}(\lambda)b_n$, so b is independent of the lift $\tilde{\sigma}$. Write b_n as $1 + qh_n$ with $h_n \in \mathfrak{w}'^{,\kappa,0}(R_n)$. We have

$$b = \sum_{\sigma \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \kappa_I^0(\tilde{\sigma})\tilde{\sigma} * c_n + \sum_{\sigma \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \kappa_I^0(\tilde{\sigma})\tilde{\sigma} * (qc_n h_n).$$

As $\widetilde{\mathrm{H}}^{\frac{p^n-p}{p-1}} \mid \widetilde{\mathrm{H}}^{p^{n+1}} \mid p$, we have

$$\sum_{\sigma \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \kappa_I^0(\tilde{\sigma})\tilde{\sigma} * (qc_n h_n) \in R_n^{\circ\circ} R_n\langle Z\rangle.$$

As (n, r, I) is adapted, by Lemma 2.3.6, we have

$$\sum_{\sigma \in 1+p^{i-1}\mathbb{Z}/p^i\mathbb{Z}} \kappa_I^0(\tilde{\sigma})\tilde{\sigma} * c_i \in c_{i-1} + R_n^{\circ\circ} R_n\langle Z\rangle$$

for each $1 \leq i \leq n$. Taking the summation of all $1 \leq i \leq n$, we have

$$\sum_{\sigma \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \kappa_I^0(\tilde{\sigma})\tilde{\sigma} * c_n \in 1 + R_n^{\circ\circ} R_n\langle Z\rangle.$$

So $b \in 1 + R_n^{\circ\circ} R_n\langle Z\rangle$. Moreover, it is easy to check

$$\gamma * b = (\kappa_I^0)^{-1}(\lambda)b,\ \forall\ \gamma \in \mathbb{Z}_p^\times,$$

so $b \in \mathfrak{w}'^{,\kappa,0}(R_n) = R_n\kappa(1 + \beta_n Z)$. As $b \equiv 1 \mod R_n^{\circ\circ}$, we have that b is also a generator of the R_n-module $\mathfrak{w}'^{,\kappa,0}(R_n)$. For any element $b' \in \mathfrak{w}^{\kappa,0}(R) \subset \mathfrak{w}'^{,\kappa,0}(R_n)$, if we write b' as $u'b$ for

$u' \in R_n$, then u' is invariant under the $(\mathbb{Z}/p^n\mathbb{Z})^\times$-action, hence belonging to R. This shows the R-module $\mathfrak{w}^{\kappa,0}(R)$ is free of rank 1 with generator b.

Note that V is fixed by the action of $\mathcal{T}^{\text{ext}}$, so the image of bV^h along $\mathrm{Gr}_h\tilde{\mathbb{W}}_\kappa^0 \cong \mathfrak{w}'^{,\kappa,0} \otimes \underline{\omega}_{\mathcal{G}}^{-2h}$ is a basis of $\mathfrak{w}^{\kappa,0} \otimes \underline{\omega}_{\mathcal{G}}^{-2h}(R)$. By induction, we have

$$\{\sum_{i=0}^{h} a_i b V^i :\ a_i \in R\} = \mathrm{Fil}_h \mathbb{W}_\kappa^0(R).$$

As $\tilde{\mathbb{W}}_\kappa^0$ is the α-adic completion of $\lim_h \mathrm{Fil}_h\tilde{\mathbb{W}}_\kappa^0$, we have that $\mathbb{W}_\kappa^0(R)$ is the α-adic completion of $\lim_h \mathrm{Fil}_h\mathbb{W}_\kappa^0(R)$, i.e.

$$\mathbb{W}_\kappa^0(R) = \{\sum_{h\geq 0} a_h b V^h :\ a_h \in R,\ \lim_{h\to\infty} a_h \to 0\}.$$

$\square$

2.3.2 The Gauss-Manin connection on $\mathbb{W}_\kappa$

Fix a pre-adapted triple (n,r,I). By the construction of $\mathfrak{IG}'_{n,r,I}$, we have

$$\mathcal{G}[p^n]^D(\mathfrak{IG}'_{n,r,I}) \cong (\mathbb{Z}/p^n\mathbb{Z})^2, \quad (H_n^1)^D(\mathfrak{IG}'_{n,r,I}) \cong \mathbb{Z}/p^n\mathbb{Z}.$$

So by Proposition 1.6.9, over $\mathfrak{IG}'_{n,r,I}$, the Gauss-Manin connection on $\mathbb{H}_\mathcal{G}$ induces a connection

$$\nabla^\sharp :\ \mathbb{H}_\mathcal{G}^\sharp \to \mathbb{H}_\mathcal{G}^\sharp \hat{\otimes}_{\mathcal{O}_{\mathfrak{IG}'_{n,r,I}}} \frac{1}{\underline{\delta}} \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda_I^0}$$

such that $s = \mathrm{HT}(P_n)$ is horizontal with respect to $\nabla^\sharp$ mod $\underline{\beta}_n$. Now the VBMS machinery implies that over $\mathfrak{IG}'_{n,r,I}$, we have an integrable connection

$$\nabla^\sharp :\ \pi_*(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}_\mathcal{G}^\sharp, s)}) \to \pi_*(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}_\mathcal{G}^\sharp, s)}) \otimes \frac{1}{\underline{\delta}} \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda_I^0},$$

which satisfies Griffiths' transversality.

Theorem 2.3.8. *Let $i = 2$ if $p = 2$, $i = 1$ if $p \geq 3$ and assume (n,r,I) is adapted.*

(i) The connection $\nabla^\sharp$ on $\pi_*(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}_\mathcal{G}^\sharp, s)})$ over $\mathfrak{IG}'_{n,r,I}$ descends to a connection

$$\nabla_\kappa^0 :\ \mathbb{W}_\kappa^0 \to \mathbb{W}_\kappa^0 \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}[1/\alpha]$$

which satisfies Griffiths' tranversality and the induced $\mathcal{O}_{\mathfrak{X}_{r,I}}$-linear map

$$\mathrm{Gr}_h(\nabla_\kappa^0) : \mathrm{Gr}_h(\mathbb{W}_\kappa^0)[1/\alpha] \to \mathrm{Gr}_{h+1}(\mathbb{W}_\kappa^0) \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}[1/\alpha]$$

is an isomorphism times with multiplication by $\mu_\kappa - h$.

(ii) Tensoring ∇_κ^0 with the connection ∇^f induced by the derivation $d :\ \mathcal{O}_{\mathfrak{IG}_{i,r,I}} \to \Omega^1_{\mathfrak{IG}_{i,r,I}/\lambda_I^0}$, we have a connection

$$\nabla_\kappa := \nabla_\kappa^0 \otimes \nabla^f :\ \mathbb{W}_\kappa[1/\alpha] \to \mathbb{W}_\kappa \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}[1/\alpha]$$

which satisfies Griffiths' transversality and the induced $\mathcal{O}_{\mathfrak{X}_{r,I}}$-linear map

$$\mathrm{Gr}_h(\nabla_\kappa) :\ \mathrm{Gr}_h(\mathbb{W}_\kappa)[1/\alpha] \to \mathrm{Gr}_{h+1}(\mathbb{W}_\kappa) \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}[1/\alpha]$$

is an isomorphism times with multiplication by $\mu_\kappa - h$.

(iii) *Let $k \in \mathbb{N}$ be a specialization of κ and denote the specialization of an object along k by a subscript k. Then we have a canonical identification*

$$(\mathrm{Sym}^k(\mathbb{H}_{\mathcal{G}}) \otimes_{\Lambda_I^0} \Lambda_I[1/\alpha])_k \cong (\mathrm{Fil}_k(\mathbb{W}_\kappa)[1/\alpha])_k$$

which is compatible with connection and filtration. Here on the left, we consider the Gauss-Manin connection and the Hodge filtration.

The proof of this theorem depends on the following local calculations.

Let $\mathrm{Spf}(R)$ be an α-torsion-free formal scheme over Λ_I^0 and $\rho : \mathrm{Spf}(R) \to \mathfrak{X}_{r,I}$ be a morphism of Λ_I^0-formal schemes such that $\rho^*(\underline{\omega}_{\mathcal{G}}) \subset \rho^*(\mathbb{H}_{\mathcal{G}})$ are free R-modules of rank-1 (resp. 2) with bases ω and $\{\omega, \eta\}$ respectively. Consider the pull-backs

$$\mathrm{Spf}(R_n') := \mathfrak{IG}'_{n,r,I} \times_{\mathfrak{X}_{r,I}} \mathrm{Spf}(R) \xrightarrow{\rho_n'} \mathfrak{IG}'_{n,r,I}, \quad \mathrm{Spf}(R_n) := \mathfrak{IG}_{n,r,I} \times_{\mathfrak{X}_{r,I}} \mathrm{Spf}(R) \xrightarrow{\rho_n} \mathfrak{IG}_{n,r,I},$$

and assume that $\rho_n^*(\underline{\beta}_n)$ (resp. $\rho_n^*(\underline{\delta})$) are generated by β_n (resp. δ) such that $\{f = \delta\omega\}$ and $\{f, e = \delta\eta\}$ are bases of $\rho_n^*(\Omega_{\mathcal{G}}) \subset \rho_n^*(\mathbb{H}_{\mathcal{G}}^\sharp)$ respectively with $f \mod \beta_n = \rho_n^*(s)$.

Equip $\tilde{R}_n' = R_n' \oplus \frac{1}{\delta}\Omega^1_{R_n'/\Lambda_I^0}$ with the ring structure given by $(a, f)(b, g) = (ab, ag + bf)$ and the R_n'-algebra structures given by

$$j_1, \text{ (resp.) } j_2 : R_n' \to \tilde{R}_n'; \quad a \mapsto (a, 0), \text{ (resp.) } a \mapsto (a, da),$$

and let $\Delta : \tilde{R}_n' \to R_n'$ be the natural projection. According to Definition 1.3.3, the connection $\nabla^\sharp$ over $\mathrm{Spf}(R_n')$ is a $\tilde{R}_n'$-linear isomorphism

$$\epsilon^\sharp : \ j_2^*(\rho_n'^{,*}(\mathbb{H}_{\mathcal{G}}^\sharp)) \cong j_1^*(\rho_n'^{,*}(\mathbb{H}_{\mathcal{G}}^\sharp))$$

such that $\Delta^*(\epsilon^\sharp) = \mathrm{Id}$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\tilde{R}_n')$ be the matrix of $\epsilon^\sharp$ with respect to the basis $\{f \otimes 1, e \otimes 1\}$ and $\{1 \otimes f, 1 \otimes e\}$ (viewed as column vectors) and write $a = 1 + a_0$, $d = 1 + d_0$. Recall that $X = 1 + \beta_n Z$.

Lemma 2.3.9. *We have that $a_0, b, c, d_0 \in \frac{1}{\mathrm{Hdg}}\Omega^1_{R_n/\Lambda_I^0}$, and for each $h \in \mathbb{N}$,*

$$\nabla^\sharp(X^\kappa V^h) = (\mu_\kappa - h)X^\kappa V^h \otimes a_0 + hX^\kappa V^h \otimes d_0 + (\mu_\kappa - h)X^\kappa V^{h+1} \otimes b + hX^\kappa V^{h-1} \otimes c. \quad (2.3.3)$$

Proof. Since $\Delta^*(\epsilon^\sharp) = \mathrm{Id}$, we have that $\Delta(a) = \Delta(d) = 1$ and $\Delta(b) = \Delta(c) = 0$, which means $a_0, b, c, d_0 \in \Omega^1_{R_n'/\Lambda_I^0}$. In particular, $a_0^2 = b^2 = c^2 = d_0^2 = 0$.

To simplify notations, we omit $\rho_n'^{,*}$ in the following. By definition, over R_n (in fact, R works), there exists $x, y, z, w \in \Omega^1_{R_n/\Lambda_I^0}$ such that

$$\nabla(\omega) = \omega \otimes x + \eta \otimes y, \quad \nabla(\eta) = \omega \otimes z + \eta \otimes w.$$

Since $f = \delta\omega$ and $g = \delta\eta$ with $\delta \in R_n$, over R_n' we have

$$\nabla^\sharp(f) = \nabla(\delta\omega) = f \otimes \frac{d\delta}{\delta} + f \otimes x + e \otimes y,$$
$$\nabla^\sharp(e) = \nabla(\delta\eta) = e \otimes \frac{d\delta}{\delta} + f \otimes z + e \otimes w.$$

Note that δ^{p-1} generates the ideal Hdg, so we have

$$\frac{d\delta}{\delta} = (p-1)^{-1}\frac{d\delta^{p-1}}{\delta^{p-1}} \in \frac{1}{\mathrm{Hdg}}\Omega^1_{R_n/\Lambda_I^0}, \quad a_0, b, c, d_0 \in \frac{1}{\mathrm{Hdg}}\Omega^1_{R_n/\Lambda_I^0}.$$

Since X and Y are obtained from f and e respectively, we have

$$\epsilon^\sharp(X) = aX + bY; \quad \epsilon^\sharp(Y) = cX + dY.$$

Recall that $\mu_\kappa = \lim_{a\to 1} \frac{\log \kappa(a)}{\log a}$. As $a_0^2 = b^2 = a_0 b = 0$, the element

$$(a + bV)^\kappa = (1 + (a_0 + bV))^\kappa := \exp(\mu_\kappa \log(1 + (a_0 + bV)))$$

is well-defined, and equals to $1 + \mu_\kappa(a_0 + bV)$. Moreover, note that $(a + bV)(1 - (a_0 + bV)) = 1$, so for any $h \geq 0$, we have that $(a + bV)^{-h} = 1 - h(a_0 + bV)$ and

$$(c + dV)^h = (V + (c + d_0)V)^h = V^h + hV^{h-1}(c + d_0V) = (1 + hd_0)V^h + chV^{h-1}.$$

By construction, we then have

$$\begin{aligned}\epsilon^\sharp(X^\kappa V^h) &= (aX + bY)^\kappa(\frac{cX + dY}{aX + bY})^h = X^\kappa(a + bV)^{\kappa-h}(c + dV)^h \\ &= X^\kappa(V^h + (hd_0 + (\mu_\kappa - h)a_0)V^h + b(\mu_\kappa - h)V^{h+1} + chV^{h-1}).\end{aligned}$$

Note that $\nabla^\sharp(X^\kappa V^h) = \epsilon^\sharp(X^\kappa V^h) - X^\kappa V^h$, so we have

$$\nabla^\sharp(X^\kappa V^h) = (\mu_\kappa - h)X^\kappa V^h \otimes a_0 + hX^\kappa V^h \otimes d_0 + (\mu_\kappa - h)X^\kappa V^{h+1} \otimes b + hX^\kappa V^{h-1} \otimes c.$$

□

As a consequence of Lemma 2.3.4 and Lemma 2.3.9, $\nabla^\sharp$ induces a flat connection over $\mathfrak{IG}_{n,r,I}$

$$\nabla_\kappa^0 : \quad \tilde{\mathbb{W}}_\kappa^0 \to \tilde{\mathbb{W}}_\kappa^0 \hat{\otimes}_{\mathcal{O}_{\mathfrak{IG}_{n,r,I}}} \frac{1}{\mathrm{Hdg}}\Omega^1_{\mathfrak{IG}_{n,r,I}/\Lambda_I^0}.$$

As $\nabla^\sharp$ satisfies the Griffiths' tranversality , so is ∇_κ^0.

Proposition 2.3.10. *The connection ∇_κ^0 induces a connection*

$$\nabla_\kappa^0 : \ \mathbb{W}_\kappa^0 \to \mathbb{W}_\kappa^0 \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}[1/\alpha]$$

such that the induced $\mathcal{O}_{\mathfrak{X}_{r,I}}$-linear map

$$\mathrm{Gr}_h(\nabla_\kappa^0) : \mathrm{Gr}_h(\mathbb{W}_\kappa^0)[1/\alpha] \to \mathrm{Gr}_{h+1}(\mathbb{W}_\kappa^0) \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}[1/\alpha]$$

is an isomorphism times with multiplication by $\mu_\kappa - h$.

Proof. Note that the base change of $\mathbb{W}_\kappa^0$ from $\mathfrak{X}_{r,I}$ to $\mathfrak{IG}_{n,r,I}$ is isomorphic to $\tilde{\mathbb{W}}_\kappa^0$ by the argument of Proposition 2.3.7, so we have

$$\nabla_\kappa^0 : \ \mathbb{W}_\kappa^0 \to \mathbb{W}_\kappa^0 \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \frac{1}{\mathrm{Hdg}}\Omega^1_{\mathfrak{IG}_{n,r,I}/\Lambda_I^0}.$$

By Proposition 2.2.19, there is a constant c_n such that $\mathrm{Hdg}^{c_n}\Omega^1_{\mathfrak{IG}_{n,r,I}/\Lambda_I^0} \subset g_n^*(\Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0})$. Note that after inverting (any lift of) H, $g_n : \mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I}$ is finite étale with Galois group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ and $\Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}$ has no Hdg-torsion, so by faithfully flat descent, we get the desired connection

$$\nabla^0_\kappa : \mathbb{W}^0_\kappa \to \mathbb{W}^0_\kappa \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \frac{1}{\mathrm{Hdg}^{1+c_n}}\Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0} \subset \mathbb{W}^0_\kappa \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}[\frac{1}{\alpha}].$$

More concretely, let $\mathrm{Spf}(R) \subset \mathfrak{X}_{r,I}$ be an affine open. By Proposition 2.3.7, we have

$$\mathbb{W}^0_\kappa(R) = \{\sum_{h\geq 0} a_h U_h,\ a_h \in R, a_h \to 0\}$$

where $U_h = s(1+\beta_n Z)^\kappa V^h$ for some $s \in R$ such that $\frac{\lambda^*(s)}{s}(\frac{\lambda^*(\delta)}{\delta})^\kappa = 1$ for any $\lambda \in (\mathbb{Z}/p^n\mathbb{Z})^\times$. Then we have

$$\nabla^0_\kappa(U_h) = U_h \otimes (d\log s + \mu_\kappa d\log\delta + \mu_\kappa x + h(w-x)) + (\mu_\kappa - h)U_{h+1} \otimes y + hU_{h-1} \otimes z.$$

Since $\frac{\lambda^*(s)}{s}(\frac{\lambda^*(\delta)}{\delta})^\kappa = 1$ for any $\lambda \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, we have $d\log s + \mu_\kappa d\log\delta \in \Omega^1_{R/\Lambda_I^0}[1/\alpha]$. So ∇^0_κ induces a R-linear map

$$\mathrm{Gr}_h(\nabla^0_\kappa) : \mathrm{Gr}_h(\mathbb{W}^0_\kappa)(R) \to \mathrm{Gr}_{h+1}(\mathbb{W}^0_\kappa)(R) \otimes_R \Omega^1_{R/\Lambda_I^0}[1/\alpha], \quad U_h \mapsto (\mu_\kappa - h)U_{h+1} \otimes y.$$

The Kodaira-Spencer isomorphism

$$\mathrm{KS} : \underline{\omega}_{\mathcal{G}} \to \underline{\omega}^\vee_{\mathcal{G}^D} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{Z}_p}, \quad \omega \mapsto \eta \otimes y$$

implies that y is a generator of the free $\mathcal{O}_X$-module $\Omega^1_{X/\mathbb{Z}_p}$, hence a generator of the $R[\frac{1}{\alpha}]$-module $\Omega^1_{R/\Lambda_I^0}[1/\alpha]$ (via the structure map $\eta : \mathfrak{X}_{r,I} \to \hat{X}$), which completes the proof. □

Lemma 2.3.11. *The derivation $d : \mathcal{O}_{\mathfrak{IG}_{i,r,I}} \to \Omega^1_{\mathfrak{IG}_{i,r,I}/\lambda_I^0}$ induces a connection*

$$\nabla^f : \mathfrak{w}^{\kappa_f} \to \mathfrak{w}^{\kappa_f} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}[1/\alpha].$$

Proof. Let $\mathrm{Spf}(R) \subset \mathfrak{X}_{r,I}$ be an affine open and $\mathrm{Spf}(R_i)$ be the preimage of $\mathrm{Spf}(R)$ in $\mathfrak{IG}_{i,r,I}$. The derivation on R_i defines a map

$$d : R_i \otimes_{\Lambda_I^0} \Lambda_I \to \Omega^1_{R_i/\Lambda_I^0} \otimes_{\Lambda_I^0} \Lambda_I \subset \Lambda_I \otimes_{\Lambda_I^0} R_i \otimes_R \frac{1}{\widetilde{\mathrm{H}}^{c_i}}\Omega^1_{R/\Lambda_I^0},$$

where the inclusion is due to Proposition 2.2.19 and the constant c_i is independent of R.

Note that after inverting $\widetilde{\mathrm{H}}$, H_i^D is étale and locally isomorphic to $\mathbb{Z}/p^i\mathbb{Z}$, so $R_i[\frac{1}{\widetilde{\mathrm{H}}}]$ is finite étale over $R[\frac{1}{\widetilde{\mathrm{H}}}]$ with Galois group $(\mathbb{Z}/p^i\mathbb{Z})^\times$. By taking $[\kappa_f^{-1}]$-isotypic component with respect to the $(\mathbb{Z}/p^i\mathbb{Z})^\times$-action, we get

$$\nabla^f : \mathfrak{w}^{\kappa_f} \to \mathfrak{w}^{\kappa_f} \otimes_{\mathfrak{X}_{r,I}} \frac{1}{\mathrm{Hdg}^{c_i}}\Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}$$

□

Proof of Theorem 2.3.8. By Proposition 2.3.10 and Lemma 2.3.11, we are reduced to showing item (iii). By abuse of notation, we assume that $\kappa \in \mathbb{N}$ is a classical weight. Then over $\mathfrak{IG}_{n,r,I}$, we have an isomorphism

$$\mathrm{Sym}^\kappa(\mathbb{H}^\sharp_{\mathcal{G}}) \cong \mathrm{Fil}_\kappa \tilde{\mathbb{W}}^0_\kappa; \quad f = \delta\omega \mapsto X,\ e = \delta\eta \mapsto Y$$

which is compatible with filtration and connection. Note that $\mathbb{H}_{\mathcal{G}}^{\sharp} = \underline{\delta}\mathbb{H}_{\mathcal{E}}$, so we have isomorphisms of $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}$-modules

$$\mathrm{Sym}^{\kappa}(\mathbb{H}_{\mathcal{G}}^{\sharp}) \cong \underline{\delta}^{\kappa}\mathrm{Sym}^{\kappa}(\mathbb{H}_{\mathcal{G}}^{\sharp}),\ \text{and } \mathrm{Fil}_{\kappa}(\tilde{\mathbb{W}}_{\kappa}^{0}) \cong \underline{\delta}^{\kappa}\mathrm{Sym}^{\kappa}(\mathbb{H}_{\mathcal{G}}).$$

Note that after inverting Hdg, $\mathfrak{IG}_{n,r,I}$ is étale over $\mathfrak{X}_{r,I}$, so

$$\mathfrak{w}^{\kappa_f} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \mathcal{O}_{\mathfrak{IG}_{n,r,I}}[\frac{1}{\mathrm{Hdg}}] \cong \mathcal{O}_{\mathfrak{IG}_{n,r,I}} \otimes_{\Lambda_I^0} \Lambda_I[\frac{1}{\mathrm{Hdg}}],$$

Composing these isomorphisms, we get an isomorphism

$$\mathrm{Fil}_{\kappa}(\mathbb{W}_{\kappa}) \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \mathcal{O}_{\mathfrak{IG}_{n,r,I}}[\frac{1}{\mathrm{Hdg}}] \cong \mathrm{Sym}^{\kappa}(\mathbb{H}_{\mathcal{G}}) \otimes_{\Lambda_I^0} \Lambda_I \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \mathcal{O}_{\mathfrak{IG}_{n,r,I}}[\frac{1}{\mathrm{Hdg}}],$$

which by descent induces the desired isomorphism

$$\mathrm{Sym}^{\kappa}(\mathbb{H}_{\mathcal{G}}) \otimes_{\Lambda_I^0} \Lambda_I[\frac{1}{\mathrm{Hdg}}] \cong \mathrm{Fil}_{\kappa}(\mathbb{W}_{\kappa})[\frac{1}{\mathrm{Hdg}}].$$

□

Iteration of the Gauss-Manin connection

We remark that the arguments in the proofs of Theorem 2.3.8, Proposition 2.3.10, Lemma 2.3.11 actually imply that there exists a constant c_f such that

$$\mathrm{Sym}^2(\underline{\omega}_{\mathcal{G}}) \otimes_{\Lambda_I^0} \Lambda_I \subset \frac{1}{\mathrm{Hdg}^{c_f+2}}\mathrm{Fil}_0(\mathbb{W}_2),$$

and there exist constants c_n, c_i such that

$$\nabla_{\kappa}^{0}:\ \mathbb{W}_{\kappa}^{0} \to \mathbb{W}_{\kappa}^{0} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \frac{1}{\mathrm{Hdg}^{1+c_n}}\Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0};\quad \nabla^{f}:\ \mathfrak{w}^{\kappa_f} \mapsto \mathfrak{w}^{\kappa_f} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \frac{1}{\mathrm{Hdg}^{c_i}}\Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}.$$

Note that by the argument of Proposition 2.2.19, there is a constant c_s such that Hdg^{c_s} annihilates $\Omega^1_{\mathfrak{X}_{r,I}/\mathfrak{X}_I}$, so the Kodaira-Spencer isomorphism induces a morphism

$$\mathrm{KS}^{-1}:\ \Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0} \to \frac{1}{\mathrm{Hdg}^{c_s}}\mathrm{Sym}^2(\underline{\omega}_{\mathcal{G}}).$$

Also note that the multiplicative structures on $\pi_*(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}_{\mathcal{G}}^{\sharp},s)})$ and $\mathcal{O}_{\mathfrak{IG}_{i,r,I}}$ induce a map

$$\mathbb{W}_{\kappa} \otimes \mathrm{Fil}_0(\mathbb{W}_2) \to \mathbb{W}_{\kappa+2}.$$

Combining all these facts, the Gauss-Manin connection

$$\nabla_{\kappa}:\ \mathbb{W}_{\kappa} \to \mathbb{W}_{\kappa} \otimes_{\mathfrak{X}_{r,I}} \frac{1}{\mathrm{Hdg}^{1+c_n+c_i}}\Omega_{\mathfrak{X}_{r,I}/\Lambda_I^0}$$

induces a map, also called the Gauss-Manin connection,

$$\nabla_{\kappa}:\ \mathbb{W}_{\kappa} \to \frac{1}{\mathrm{Hdg}^{c(n)}}\mathbb{W}_{\kappa+2},$$

where $c(n) := 3 + c_n + c_i + c_f + c_s$.

To consider ∇_{κ} for different κ at the same time, we introduce the following definition.

Definition 2.3.12. *Let $f_0 : \mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}}, s) \to \mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I}$ be the structure map. We define*

$$\mathbb{W}^0 := f_{0,*}\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}},s)}; \quad \mathbb{W} := \mathbb{W}^0 \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \mathfrak{w}^{\kappa_f};$$
$$\mathbb{W}^{0,\prime}_{\kappa} := \sum_{m\in\mathbb{Z}} \mathbb{W}^0_{\kappa+2m} \subset \mathbb{W}^0; \quad \mathbb{W}'_{\kappa} := \mathbb{W}^{0,\prime}_{\kappa} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \mathfrak{w}^{\kappa_f},$$

Note that for $m \neq m'$, $\mathbb{W}_{\kappa+2m} \cap \mathbb{W}_{\kappa+2m'} = (0)$, so we have a well-defined connection

$$\nabla_\kappa : \mathbb{W}'_\kappa \to (\frac{1}{\mathrm{Hdg}^{c(n)}})\mathbb{W}'_\kappa; \quad \nabla_\kappa |_{\mathbb{W}_{\kappa+2m}} = \nabla_{\kappa+2m}.$$

For each $M \geq 0$, we denote the iteration

$$\mathbb{W}_\kappa \xrightarrow{\nabla_\kappa} \frac{1}{\mathrm{Hdg}^{c(n)}}\mathbb{W}_{\kappa+2} \xrightarrow{\nabla_{\kappa+2}} \frac{1}{\mathrm{Hdg}^{2c(n)}}\mathbb{W}_{\kappa+4}\ldots \xrightarrow{\nabla_{\kappa+2M-2}} \frac{1}{\mathrm{Hdg}^{Mc(n)}}\mathbb{W}_{\kappa+2M}$$

by ∇^M_κ. When the weight κ is clear, we simply write ∇_κ as ∇.

2.3.3 Functoriality of $\mathbb{W}_\kappa$

Functoriality on the interval I

Let (n, r, I) and (n, r, J) be adapted triples with $J \subset I$. Let $(\mathcal{A}_?, i)$ resp. $\mathcal{G}_?$ be the universal false elliptic curve, resp. the associated p-divisible group over $\mathfrak{X}_{r,?}$ and let $(\Omega_{\mathcal{G}_?}, \mathbb{H}^\sharp_{\mathcal{G}_?}, s_?)$ be the associated system of vector bundles with marked sections over $\mathfrak{IG}_{n,r,?}$ for $? = I, J$. Then the natural morphism $\Lambda^0_I \to \Lambda^0_J$ induces the following commutative diagram

$$\begin{tikzcd}
\mathfrak{IG}_{n,r,J} \arrow[r] \arrow[d] & \mathfrak{IG}_{n,r,I} \arrow[d] \\
\mathfrak{X}_{r,J} \arrow[r, "\theta_{J,I}"] & \mathfrak{X}_{r,I},
\end{tikzcd}$$

such that

$$\theta^*_{J,I}(\mathcal{G}_I) = \mathcal{G}_J; \quad \theta^*_{J,I}(\Omega_{\mathcal{G}_I}) = \Omega_{\mathcal{G}_J}; \quad \theta^*_{J,I}(\mathbb{H}^\sharp_{\mathcal{G}_I}) = \mathbb{H}^\sharp_{\mathcal{G}_J}; \quad \theta^*_{J,I}(s_I) = s_J.$$

So we have the following commutative diagram compatible with the action of $\mathcal{T}^{\mathrm{ext}}$

$$\begin{tikzcd}
\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}_J}, s_J) \arrow[r] \arrow[d] & \mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}_I}, s_I) \arrow[d] \\
\mathcal{V}_0(\Omega_{\mathcal{G}_J}, s_J) \arrow[r] \arrow[d] & \mathcal{V}_0(\Omega_{\mathcal{G}_I}, s_I) \arrow[d] \\
\mathfrak{IG}_{n,r,J} \arrow[r] \arrow[d] & \mathfrak{IG}_{n,r,I} \arrow[d] \\
\mathfrak{X}_{r,J} \arrow[r, "\theta_{J,I}"] & \mathfrak{X}_{r,I}.
\end{tikzcd} \tag{2.3.4}$$

Lemma 2.3.13. *The commutative diagram 2.3.4 induces a canonical isomorphism*

$$\theta^0 : \theta^*_{J,I}(\mathbb{W}^0_{\kappa,I}) \to \mathbb{W}^0_{\kappa,J}$$

which is compatible with filtration and connection. Tensoring θ^0 with the canonical morphism

$$\theta^f : \theta^*_{J,I}(\mathfrak{w}_I^{\kappa_f}) \to \mathfrak{w}_J^{\kappa_f},$$

we get a canonical morphism $\theta : \theta^*_{J,I}(\mathbb{W}_{\kappa,I}) \to \mathbb{W}_{\kappa,J}$ which is an isomorphism after inverting α.

Proof. The existence of $\theta^0 : \theta^*_{J,I}(\mathbb{W}^0_{\kappa_I}) \to \mathbb{W}^0_{\kappa_J}$ and the compatibility with filtration and connection follow directly from the diagram 2.3.4.

Note that $\mathrm{Fil}_0\mathbb{W}^0_{\kappa_I} = \mathfrak{w}^{\kappa_I,0}$ is invertible, and $\theta^*_{J,I}(\mathfrak{w}^{\kappa_I,0})^{-1} = \theta^*_{J,I}(\mathfrak{w}^{\kappa_I^{-1},0})$. So combining the morphisms

$$\theta^0 : \theta^*_{J,I}(\mathfrak{w}^{\kappa_I,0}) \to \mathfrak{w}^{\kappa_J,0}; \quad \theta^0 : \theta^*_{J,I}(\mathfrak{w}^{\kappa_I^{-1},0}) \to \mathfrak{w}^{\kappa_J^{-1},0},$$

we get a sequence of morphisms

$$\mathcal{O}_{\mathfrak{X}_{r,J}} \to \mathfrak{w}^{\kappa_J,0} \otimes_{\mathcal{O}_{r,J}} \theta^*_{J,I}(\mathfrak{w}^{\kappa_I^{-1},0}) \to \mathcal{O}_{\mathfrak{X}_{r,J}}$$

whose composition is the identity map. As all three sheaves are locally free of rank 1 and $\mathfrak{X}_{r,J}$ is noetherian, we have that $\theta^0 : \theta^*_{J,I}(\mathfrak{w}^{\kappa_I,0}) \to \mathfrak{w}^{\kappa_J,0}$ is an isomorphism. Since

$$\mathrm{Gr}_h\mathbb{W}^0_{\kappa_I} \cong \mathfrak{w}^{\kappa_I,0} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \underline{\omega}_{\mathcal{G}_I}^{-2h}$$

for each $h \in \mathbb{N}$, by induction we have that θ induces an isomorphism

$$\theta^0 : \theta^*_{J,I}(\mathrm{Fil}_h\mathbb{W}^0_{\kappa_I}) \to \mathrm{Fil}_h\mathbb{W}^0_{\kappa_J}.$$

Since $\mathbb{W}^0_{\kappa,I}$ is the α-adic completion of $\lim_{h\to\infty}\mathrm{Fil}_h\mathbb{W}^0_{\kappa_I}$, we see that $\theta^0 : i^*_{J,I}(\mathbb{W}^0_{\kappa_I}) \to \mathbb{W}^0_{\kappa_J}$ is an isomorphism.

Note that the lower square of the diagram 2.3.4 induces a canonical morphism

$$\theta^f : \theta^*_{J,I}\mathfrak{w}^{\kappa_I,f} \to \mathfrak{w}^{\kappa_J,f}$$

which is compatible with connection. As $\mathfrak{w}^{\kappa_I,f}[\frac{1}{\alpha}]$ is free of rank 1 as an $\mathcal{O}_{\mathcal{X}_{r,I}} \otimes_{\Lambda_I^0} \Lambda_I$-module, a similar argument as was used for θ^0 shows that θ^f is an isomorphism after inverting α.

Combining the results for θ^0 and θ^f, we have that θ is compatible with filtration and connection, and becomes an isomorphism after inverting α. □

Functoriality on n

Let (n, r, I), (n', r, I) be adapted triples such that $n \le n'$. Let $(\mathcal{A}, i)$ resp. $\mathcal{G}$ be the universal false elliptic curve, resp. the associated p-divisible group over $\mathfrak{X}_{r,I}$. Let $(\Omega_{\mathcal{G},?}, \mathbb{H}^\sharp_{\mathcal{G},?}, s_?)$ be the associated system of vector bundles with marked sections on $\mathfrak{IG}_{?,r,I}$ for $? = n, n'$. Via the natural map $\theta_{n',n} : \mathfrak{IG}_{n',r,I} \to \mathfrak{IG}_{n,r,I}$, we have

$$\theta^*_{n',n}(\Omega_{\mathcal{G},n}) = \Omega_{\mathcal{G},n'}, \ \theta^*_{n',n}(\mathbb{H}^\sharp_{\mathcal{G},n}) = \mathbb{H}^\sharp_{\mathcal{G},n'}, \quad \theta^*_{n',n}(s_n) \equiv s_{n'} \mod \underline{\beta}_n.$$

Combining with the fact that $\underline{\beta}_{n'} \subset \underline{\beta}_n$, we get a commutative diagram

$$\begin{tikzcd}
\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G},n'}, s_{n'}) \arrow[r] \arrow[d] & \mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G},n}, s_n) \arrow[d] \\
\mathcal{V}_0(\Omega_{\mathcal{G},n'}, s_{n'}) \arrow[r] \arrow[d] & \mathcal{V}_0(\Omega_{\mathcal{G},n}, s_n) \arrow[d] \\
\mathfrak{IG}_{n',r,I} \arrow[r, "\theta_{n',n}"] \arrow[d] & \mathfrak{IG}_{n,r,I} \arrow[dl] \\
\mathfrak{X}_{r,I}, &
\end{tikzcd} \tag{2.3.5}$$

which is clearly compatible with the action of $\mathbb{Z}_p^\times(1+\underline{\beta}_{n'}G_a)$.

Lemma 2.3.14. *The diagram 2.3.5 induces canonical isomorphisms*

$$\theta : \ \mathbb{W}^0_{\kappa,n} \to \mathbb{W}^0_{\kappa,n'}, \quad \theta : \ \mathbb{W}_{\kappa,n} \to \mathbb{W}_{\kappa,n'}$$

which are compatible with filtration and connection.

Proof. The diagram 2.3.5 induces a morphism

$$\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G},n}, s_n)} \to \mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G},n'}, s_{n'})}$$

which commutes with the $\mathbb{Z}_p^\times(1+\underline{\beta}_{n'}G_a)$-action. By taking the $(\kappa_I^0)^{-1}$-isotypic component, we get the desired morphism

$$\theta^0 : \ \mathbb{W}^0_{\kappa,n} \to \mathbb{W}^0_{\kappa,n'},$$

which is clearly compatible with filtration and connection. Tensoring

$$\theta^0 : \ \mathfrak{w}^{\kappa,n,0} = \mathrm{Fil}_0\mathbb{W}^0_{\kappa,n} \to \mathfrak{w}^{\kappa,n',0} = \mathrm{Fil}_0\mathbb{W}^0_{\kappa,n'}$$

with $(\mathfrak{w}^{\kappa,n,0})^{-1} = \mathfrak{w}^{\kappa^{-1},n,0}$, we get a sequence of morphisms

$$\mathcal{O}_{\mathfrak{X}_{r,I}} \to \mathfrak{w}^{\kappa,n',0} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \mathfrak{w}^{\kappa^{-1},n,0} \to \mathcal{O}_{\mathfrak{X}_{r,I}}$$

whose composition is the identity map. This forces

$$\theta^0 : \ \mathfrak{w}^{\kappa,n,0} \to \mathfrak{w}^{\kappa,n',0}$$

to be an isomorphism. By induction, the isomorphisms

$$\mathrm{Gr}_h\mathbb{W}^0_{\kappa,n} \cong \mathfrak{w}^{\kappa,n,0} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \underline{\omega}_{\mathcal{G}}^{-2h}, \ \forall \ h \in \mathbb{N}$$

imply that θ^0 is an isomorphism restricted on each filtered submodule Fil_h. Note that $\mathbb{W}^0_{\kappa,n}$ is the α-adic completion of $\lim_h \mathrm{Fil}_h\mathbb{W}^0_{\kappa,n}$, so we see that

$$\theta : \ \mathbb{W}^0_{\kappa,n} \to \mathbb{W}^0_{\kappa,n'}$$

is an isomorphism. The case of $\mathbb{W}_\kappa$ follows directly. □

Functoriality on r

Let (n, r, I) and (n, r', I) be adapted triples such that $r' \geq r$. Let $(\mathcal{A}_?, i)$ be the universal false elliptic curve over $\mathfrak{X}_{?,I}$ and $(\mathbb{H}^\sharp_{\mathcal{G},?}, \Omega_{\mathcal{G},?}, s_?)$ be the associated system of vector bundles with marked sections on $\mathfrak{IG}_{n,?,I}$ for $? = r', r$. The natural map $\theta : \mathfrak{X}_{r',I} \to \mathfrak{X}_{r,I}$ induces the following commutative diagram

$$\begin{tikzcd}
\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G},r'}, s_{r'}) \arrow[r] \arrow[d] & \mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G},r'}, s_r) \arrow[d] \\
\mathcal{V}_0(\Omega_{\mathcal{G},r'}, s_{r'}) \arrow[r] \arrow[d] & \mathcal{V}_0(\Omega_{\mathcal{G},r}, s_r) \arrow[d] \\
\mathfrak{IG}_{n,r',I} \arrow[r] \arrow[d] & \mathfrak{IG}_{n,r,I} \arrow[d] \\
\mathfrak{X}_{r',I} \arrow[r, "\theta"] & \mathfrak{X}_{r,I},
\end{tikzcd} \tag{2.3.6}$$

which is compatible with the $\mathcal{T}^{\text{ext}}$-action. Analogous to the proof of Lemma 2.3.14, we can show

Lemma 2.3.15. *The diagram 2.3.6 induces an isomorphism*

$$\theta^0 : \theta^*(\mathbb{W}^0_{\kappa,r}) \to \mathbb{W}^0_{\kappa,r'}$$

which is compatible with connection and filtration, and a canonical morphism

$$\theta^f : \theta^*(\mathfrak{w}^{\kappa_f,r}) \to \mathfrak{w}^{\kappa_f,r'}$$

which becomes an isomorphism after inverting α. We moreover have a canonical morphism

$$\theta := \theta^0 \otimes \theta^f : \theta^*(\mathbb{W}_{\kappa,r}) \to \mathbb{W}_{\kappa,r'}$$

compatible with connection and filtration which becomes an isomorphism after inverting α.

2.3.4 Hecke operators

The Frobenius and the U_p-operator

Let (n, r, I) be a pre-adapted triple. Let $H_1 \subset \mathcal{A}$ be the level-1 canonical subgroup of the universal false elliptic curve $(\mathcal{A}, i)$. According to Proposition 2.2.16, the quotient false isogeny $f : (\mathcal{A}, i) \to (\mathcal{A}' := \mathcal{A}/H_1, i)$ induces a $(\mathbb{Z}/p^n\mathbb{Z})^\times$-equivaraint commutative diagram

$$\begin{tikzcd}
\mathfrak{IG}_{n,r+1,I} \arrow[r, "\Phi"] \arrow[d] & \mathfrak{IG}_{n,r,I} \arrow[d] \\
\mathfrak{X}_{r+1,I} \arrow[r, "\phi"] & \mathfrak{X}_{r,I}
\end{tikzcd} \tag{2.3.7}$$

where Φ is characterized by $\Phi^*((\mathcal{A}, i)_{/\mathfrak{IG}_{n,r,I}}) = (\mathcal{A}', i)_{/\mathfrak{IG}_{n,r+1,I}}$.

Lemma 2.3.16. *When $n = 2$ if $p = 2$ and $n = 1$ if $p \geq 3$, the diagram 2.3.7 induces a morphism of $\mathcal{O}_{\mathfrak{X}_{r+1,I}}$-modules*

$$\mathcal{V}^f : \phi^*(\mathfrak{w}^{\kappa_f}) \to \mathfrak{w}^{\kappa_f}$$

which is compatible with connection and becomes an isomorphism after inverting α.

Proof. Let $n = 2$ if $p = 2$, $n = 1$ if $p \geq 3$. Let $\pi_n : \mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I}$ be the structure morphism and let

$$\phi^*(\mathfrak{w}^{\kappa_f}) \to \phi^*\pi_{n,*}(\mathcal{O}_{\mathfrak{IG}_{n,r,I}}) \otimes_{\Lambda_I^0} \Lambda_I \to \pi_{n,*}\Phi^*(\mathcal{O}_{\mathfrak{IG}_{n,r,I}}) \otimes_{\Lambda_I^0} \Lambda_I$$

be the map induced by the diagram 2.3.7. By taking $[\kappa_f^{-1}]$-isotypic component with respect to the $(\mathbb{Z}/p^n\mathbb{Z})^\times$-action, we get the desired morphism $\mathcal{V}^f$ together with the compatibility with connection.

After inverting α (or even Hdg), $\mathfrak{w}^{\kappa_f}[\frac{1}{\alpha}]$ is a free of rank-1 $\mathcal{O}_{\mathfrak{X}_{r,I}} \otimes_{\Lambda_I^0} \Lambda_I$-module with inverse $\mathfrak{w}^{\kappa_f^{-1}}[\frac{1}{\alpha}]$, so tensoring $\mathcal{V}^f$ with $\phi^*(\mathfrak{w}^{\kappa_f^{-1}})[\frac{1}{\alpha}]$, we get a sequence of morphisms

$$\mathcal{O}_{\mathcal{X}_{r+1,I}} \to \mathfrak{w}^{\kappa_f} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \phi^*(\mathfrak{w}^{\kappa_f^{-1}})[\frac{1}{\alpha}] \to \mathcal{O}_{\mathcal{X}_{r+1,I}}$$

whose composition is the identity map, which implies that $\mathcal{V}^f$ is an isomorphism after inverting α. □

Let $H_{n+1}^1 \subset \mathcal{G}$ be the level-$(n+1)$ canonical subgroup. As shown in the proof of Proposition 2.2.16, $H_n'^{,1} := H_{n+1}^1/H_1^1 \subset \mathcal{G}' := G_{\mathcal{A}'}$ is the level-n canonical subgroup, and over $\mathfrak{IG}_{n+1,r+1,I}$, the false dual $f' : (\mathcal{A}', i) \to (\mathcal{A}, i)$ of the quotient isogeny $f : (\mathcal{A}, i) \to (\mathcal{A}', i)$ induces an isomorphism $H_n' \cong H_n$. Over $\mathcal{IG}_{n+1,r+1,I}$, we have the universal trivialization $\gamma_{n+1} : \mathbb{Z}/p^{n+1}\mathbb{Z} \to (H_{n+1}^1)^D$. Therefore there is a unique trivialization $\gamma_n' : \mathbb{Z}/p^n\mathbb{Z} \cong (H_n')^D$ over $\mathcal{IG}_{n,r,I}$ defined by the commutative diagram

$$\begin{tikzcd} \mathbb{Z}/p^n\mathbb{Z} \arrow[r, "\mathrm{Id}"] \arrow[d, "\gamma_n"] & \mathbb{Z}/p^n\mathbb{Z} \arrow[r, "1\mapsto p"] \arrow[d, "\gamma_n'"] & \mathbb{Z}/p^{n+1}\mathbb{Z} \arrow[d, "\gamma_{n+1}"] \\ H_n^{1,D} \arrow[r, "(f')^D"] & (H_n^{1,\prime})^D \arrow[r, "f^D"] & H_{n+1}^{1,D}. \end{tikzcd} \tag{2.3.8}$$

Let $p_{n+1,n} : \mathfrak{IG}_{n+1,r+1,I} \to \mathfrak{IG}_{n,r+1,I}$ be the forgetful map. The composition

$$t_1 : \mathfrak{IG}_{n+1,r+1,I} \xrightarrow{p_{n+1,n}} \mathfrak{IG}_{n,r+1,I} \xrightarrow{\Phi} \mathfrak{IG}_{n,r,I}$$

is the normalization of the morphism on adic generic fibers sending $(\mathcal{A}, i, \gamma_{n+1})$ to $(\mathcal{A}', i, \gamma_n')$.

Lemma 2.3.17. *Let $\mathcal{G}' := G_{\mathcal{A}'}$. Over $\mathfrak{IG}_{n+1,r+1,I}$, the quotient false isogeny $f : (\mathcal{A}, i) \to (\mathcal{A}', i)$ induces an isomorphism*

$$\frac{f^*}{p} : \Omega_{\mathcal{G}'} \to \Omega_{\mathcal{G}}$$

and moreover a commutative diagram

$$\begin{tikzcd} p_{n+1,n}^*(\mathcal{V}_0(\Omega_{\mathcal{G}}, s_n)) \arrow[rr] \arrow[dr] & & t_1^*(\mathcal{V}_0(\Omega_{\mathcal{G}}, s_n)) \arrow[dl] \\ & \mathfrak{IG}_{n+1,r+1,I} \arrow[d] & \\ & \mathfrak{X}_{r+1,I} & \end{tikzcd} \tag{2.3.9}$$

which is compatible with the action of $\mathbb{Z}_p^\times(1 + \underline{\beta}_{n+1} G_a)$. If (n, r, I) is adapted, then the induced morphism of $\mathcal{O}_{\mathfrak{X}_{r,I}}$-modules $\mathcal{V}^0 : \phi^*(\mathfrak{w}^{\kappa,0}) \to \mathfrak{w}^{\kappa,0}$ is an isomorphism.

Proof. Note that $\mathcal{G}' = \mathcal{G}/H_1^1$, so we have

$$f^* : \ \Omega_{\mathcal{G}'} \to \Omega_{\mathcal{G}}, \ s'_n \mapsto ps_{n+1} \mod \underline{\beta}_{n+1}\Omega_{\mathcal{G}}.$$

As both $\Omega_{\mathcal{G}'}$ and $\Omega_{\mathcal{G}}$ are locally free of rank-1 $\mathcal{O}_{\mathfrak{IG}_{n+1,r+1,I}}$-modules, we have that f^* is injective with image $p\Omega_{\mathcal{G}}$, so $\frac{f^*}{p}$ is a well-defined isomorphism.

Moreover, as $pt_1^*(\underline{\beta}_n) = \underline{\beta}_{n+1}$, we know that $\frac{f^*}{p}(s'_n) \equiv s_n \mod t_1^*(\underline{\beta}_n)\Omega_{\mathcal{G}}$. Therefore $\frac{f^*}{p}$ induces a map of $\mathfrak{IG}_{n+1,r+1,I}$-formal schemes

$$\mathcal{V}_0(\Omega_{\mathcal{G}}, s_n; t_1^*(\underline{\beta}_n)) \to t_1^*(\mathcal{V}_0(\Omega_{\mathcal{G}}, s_n)) = \mathcal{V}_0(\Omega_{\mathcal{G}'}, s_{n'}; t_1^*(\underline{\beta}_n)).$$

Here the notation $\mathcal{V}_0(\omega_{\mathcal{G}}, s_n; t_1^*(\underline{\beta}_n))$ emphasizes that we view s_n as a marked section modulo $t_1^*(\underline{\beta}_n)$. On the other hand, as $\underline{\beta}_n \subset t_1^*(\underline{\beta}_n)$, we have a morphism

$$p_{n+1,n}^*(\mathcal{V}_0(\Omega_{\mathcal{G}}, s_n; \underline{\beta}_n)) \to \mathcal{V}_0(\Omega_{\mathcal{G}}, s_n; t_1^*(\underline{\beta}_n)).$$

Combining these two morphisms, we get the desired diagram 2.3.9 which is equivariant with respect to the action of $\mathbb{Z}_p^\times(1 + \underline{\beta}_{n+1}G_a)$.

If (n, r, I) is adapted, then by taking the $[(\kappa^0)^{-1}]$-isotypic component with respect to the $\mathbb{Z}_p^\times(1 + \underline{\beta}_{n+1}G_a)$-action and applying the argument in the proof of Lemma 2.3.14, we get the desired isomorphism of $\mathcal{O}_{\mathfrak{X}_{r+1,I}}$-modules

$$\mathcal{V}^0 : \ \phi^*(\mathfrak{w}^{\kappa,0}) \to \mathcal{O}_{t_1^*(\mathcal{V}_0(\Omega_{\mathcal{G}}, s_n))}[(\kappa^0)^{-1}] \to \mathcal{O}_{p_{n+1,n}^*(\mathcal{V}_0(\Omega_{\mathcal{G}}, s_n))}[(\kappa^0)^{-1}] \cong \mathfrak{w}^{\kappa,0}.$$

$\square$

Definition 2.3.18. *We define $\mathcal{V} : \ \phi^*(\mathfrak{w}^\kappa) \to \mathfrak{w}^\kappa$ to be $\mathcal{V}^0 \otimes \mathcal{V}^f$ and define the V_p-operator to be the morphism on global sections*

$$H^0(\mathfrak{X}_{r,I}, \mathfrak{w}^\kappa) \xrightarrow{\phi^*} H^0(\mathfrak{X}_{r+1,I}, \phi^*(\mathfrak{w}^\kappa)) \xrightarrow{\mathcal{V}} H^0(\mathfrak{X}_{r+1,I}, \mathfrak{w}^\kappa).$$

We can also define the U_p-operator over $\mathbb{W}_\kappa$, whose restriction to $\mathfrak{w}^\kappa$ is a left inverse of V_p.

Lemma 2.3.19. *For an adapted triple (n, r, I), the false dual isogeny $f' : \ (\mathcal{A}', i) \to (\mathcal{A}, i)$ induces a morphism $\mathcal{U}^0 : \ \mathbb{W}_\kappa^0 \to \phi^*(\mathbb{W}_\kappa^0)$ of $\mathfrak{X}_{r+1,I}$-modules which is compatible with filtration and connection, and whose restriction*

$$\mathcal{U}^0 : \ \mathfrak{w}^{\kappa,0} = \mathrm{Fil}_0\mathbb{W}_\kappa^0 \to \phi^*(\mathrm{Fil}_0\mathbb{W}_\kappa^0) = \phi^*(\mathfrak{w}^{\kappa,0})$$

is inverse to $\mathcal{V}^0$.

Proof. Let $(\mathbb{H}_{\mathcal{G}'}^\sharp, \Omega_{\mathcal{G}'}, s'_n := \mathrm{HT}(\gamma'_n(1)))$ be the system of vector bundles with marked sections associated to $\mathcal{G}'$ over $\mathfrak{IG}_{n+1,r+1,I}$. Then we have

$$t_1^*(\mathcal{A}/\mathfrak{IG}_{n,r,I}) = \mathcal{A}', \quad t_1^*((\Omega_{\mathcal{G},n}, s_n)) = (\Omega_{\mathcal{G}'}, s'_n), \quad t_1^*(\mathbb{H}_{\mathcal{G},n}^\sharp, s_n) = (\mathbb{H}_{\mathcal{G}'}^\sharp, s'_n).$$

Note that $pt_1^*(\underline{\beta}_n) = \underline{\beta}_{n+1}$, so $f' : \ \mathcal{G}' \to \mathcal{G}$ induces morphisms of formal schemes

$$\mathcal{V}_0(\mathbb{H}_{\mathcal{G}'}^\sharp, s'_n; t_1^*(\underline{\beta}_n)) = t_1^*(\mathcal{V}_0(\mathbb{H}_{\mathcal{G}}^\sharp, s_n)) \to \mathcal{V}_0(\mathbb{H}_{\mathcal{G}}^\sharp, s_n; t_1^*(\underline{\beta}_n)); \quad t_1^*(\mathcal{V}_0(\Omega_{\mathcal{G}}, s_n)) \to \mathcal{V}_0(\Omega_{\mathcal{G}}, s_n; t_1^*(\underline{\beta}_n))$$

which commute with the action of $\mathbb{Z}_p^\times(1 + \underline{\beta}_{n+1}G_a)$.

As $p^{n-1} \mid t_1^*(\underline{\beta}_n) = p^n \mathrm{Hdg}^{-\frac{p^{n+1}}{p-1}}$, similar to the proof of Lemma 2.3.14, we have

$$\mathcal{O}_{\mathcal{V}_0(\Omega_{\mathcal{G}}, s_n; t_1^*(\underline{\beta}_n))}[(\kappa^0)^{-1}] = \mathfrak{w}^{\kappa,0}, \quad \mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}}, s_n; t_1^*(\underline{\beta}_n))}[(\kappa^0)^{-1}] = \mathbb{W}^0_\kappa.$$

On the other hand, by the definition of t_1, we have a morphism

$$\phi^*(\mathbb{W}^0_\kappa) \to \phi^*(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}}, s_n)})[(\kappa^0)^{-1}] \to \mathcal{O}_{t_1^*(\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}}, s_n))}[(\kappa^0)^{-1}].$$

which is compatible with filtration. Applying the argument in the proof of Lemma 2.3.14 once again, we have

$$\phi^*(\mathbb{W}^0_\kappa) \cong \mathcal{O}_{t_1^*(\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}}, s_n))}[(\kappa^0)^{-1}].$$

Pre-composing its inverse with the identification $\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}}, s_n; t_1^*(\underline{\beta}_n))}[(\kappa^0)^{-1}] = \mathbb{W}^0_\kappa$, we get

$$\mathcal{U}^0 : \ \mathbb{W}^0_\kappa = \mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}}, s_n; t_1^*(\underline{\beta}_n))}[(\kappa^0)^{-1}] \to \phi^*(\mathbb{W}^0_\kappa).$$

Moreover, as the composition

$$p^*_{n+1,n}(\mathcal{V}_0(\Omega_{\mathcal{G}}, s_n)) \to t_1^*(\mathcal{V}_0(\Omega_{\mathcal{G}}, s_n)) \to \mathcal{V}_0(\Omega_{\mathcal{G}}, s_n; t_1^*(\underline{\beta}_n))$$

is just the natural morphism, the restriction of $\mathcal{U}^0$ on $\mathfrak{w}^{\kappa,0}$ is a left inverse of $\mathcal{V}^0$. $\square$

Definition 2.3.20. *We define the U_p operator on global sections of $\mathbb{W}^0_\kappa$ as the composition*

$$H^0(\mathfrak{X}_{r+1,I}, \mathbb{W}^0_\kappa) \xrightarrow{\mathcal{U}^0} H^0(\mathfrak{X}_{r+1,I}, \phi^*(\mathbb{W}^0_\kappa)) \xrightarrow{\frac{1}{p}\mathrm{Tr}_{\mathfrak{X}_{r+1,I}/\mathfrak{X}_{r,I}}} H^0(\mathfrak{X}_{r,I}, \mathbb{W}^0_\kappa)(\to H^0(\mathfrak{X}_{r+1,I}, \mathbb{W}^0_\kappa)).$$

On the generic fiber $\mathcal{X}_{r,I}$, we define the U_p-operator as the composition

$$H^0(\mathcal{X}_{r+1,I}, \mathbb{W}_\kappa) \xrightarrow{\mathcal{U}^0 \otimes (\mathcal{V}^f)^{-1}} H^0(\mathcal{X}_{r+1,I}, \phi^*(\mathbb{W}_\kappa)) \xrightarrow{\frac{1}{p}\mathrm{Tr}_{\mathfrak{X}_{r+1,I}/\mathfrak{X}_{r,I}}} H^0(\mathcal{X}_{r,I}, \mathbb{W}_\kappa)(\to H^0(\mathcal{X}_{r+1,I}, \mathbb{W}_\kappa)).$$

Proposition 2.3.21. *For each $m \in \mathbb{N}$, we have*

$$U_p(H^0(\mathcal{X}_{r,I}, \mathrm{Fil}_m(\mathbb{W}_\kappa))) \subset H^0(\mathcal{X}_{r,I}, \mathrm{Fil}_m(\mathbb{W}_\kappa)).$$

Moreover, we have the following:

(i) the induced map on the graded piece $H^0(\mathfrak{X}_{r,I}, \mathrm{Gr}_h(\mathbb{W}^0_\kappa))$ is 0 modulo $(\frac{p}{\mathrm{Hdg}^2})^h$;

(ii) for any classical specialization $k \in \mathbb{N}$ of κ, the identification

$$(\mathrm{Sym}^k(\mathbb{H}_{\mathcal{G}}) \otimes_{\Lambda^0_I} \Lambda_I[1/\alpha])_k \cong (\mathrm{Fil}_k(\mathbb{W}_\kappa)[1/\alpha])_k$$

is compatible with U_p-operators. Here the subscript k means the specialization of the object along k.

Proof. Every statement except item (i) is clear from definition. Note that $\mathrm{Hdg}(\mathcal{A}') = \mathrm{Hdg}(\mathcal{A})^p$, so by Proposition 1.6.10, $f' : (\mathcal{A}', i) \to (\mathcal{A}, i)$ induces the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega_{\mathcal{G}} \arrow[r] \arrow[d, "\cong"] & \mathbb{H}^\sharp_{\mathcal{G}} \arrow[r] \arrow[d] & \underline{\delta}\underline{\omega}^{-1}_{\mathcal{G}^D} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega_{\mathcal{G}'} \arrow[r] & \mathbb{H}^\sharp_{\mathcal{G}'} \arrow[r] & \underline{\delta}^p \underline{\omega}^{-1}_{(\mathcal{G}')^D} \arrow[r] & 0 \end{tikzcd}$$

where the first vertical arrow is an isomorphism and the third vertical arrow is induced by $(f')^D: (\mathcal{G}')^D \to \mathcal{G}^D$. Via the chosen principal polarization, we have

$$(\mathcal{G}')^D \cong \ker(1-e^\dagger: \mathcal{A}'[p^\infty] \to \mathcal{A}'[p^\infty]); \quad \mathcal{G}^D \cong \ker(1-e^\dagger: \mathcal{A}[p^\infty] \to \mathcal{A}[p^\infty])$$

which identifies $(f')^D$ with the isogeny induced by $f: (\mathcal{A}, i) \to (\mathcal{A}', i)$. By construction, f mod $\frac{p}{\mathrm{Hdg}}$ is the Frobenius map, which is zero on tangent spaces, so we have

$$f^*(\underline{\omega}_{\mathcal{G}^D}^{-1}) \equiv 0 \bmod \frac{p}{\mathrm{Hdg}}\underline{\omega}_{(\mathcal{G}')^D}^{-1}, \quad f^*(\underline{\delta\omega}_{\mathcal{G}^D}^{-1}) \equiv 0 \mod \frac{p}{\underline{\delta}^{p-2}}\underline{\omega}_{(\mathcal{G}')^D}^{-1} = \frac{p}{\mathrm{Hdg}^2}\underline{\delta}^p\omega_{(\mathcal{G}')^D}^{-1}.$$

By unwinding the definition of $\mathcal{U}_0$, the local description of $\tilde{\mathbb{W}}_\kappa^0$ (see Lemma 2.3.4) implies that

$$\mathcal{U}^0: \mathrm{Gr}_h\mathbb{W}_\kappa^0 \to \mathrm{Gr}_h\phi^*(\mathbb{W}_\kappa^0)$$

is 0 modulo $\frac{p^h}{\mathrm{Hdg}(\mathcal{A})^{-2h}}$, which completes the proof of item (i). □

As promised, we have

Lemma 2.3.22. *The composition*

$$U_p \circ V_p: H^0(\mathcal{X}_{r,I}, \mathfrak{w}^\kappa) \to H^0(\mathcal{X}_{r,I}, \mathfrak{w}^\kappa)$$

is the identity map.

Proof. Note that $\mathcal{U} \circ \mathcal{V} = \mathrm{Id}: \mathfrak{w}^\kappa \to \mathfrak{w}^\kappa$, so the composition $U_p \circ V_p$ is just

$$H^0(\mathcal{X}_{r,I}, \mathfrak{w}^\kappa) \xrightarrow{\phi^*} H^0(\mathcal{X}_{r+1,I}, \phi^*(\mathfrak{w}^\kappa)) \xrightarrow{\frac{1}{p}\mathrm{Tr}_{\mathfrak{X}_{r+1,I}/\mathfrak{X}_{r,I}}} H^0(\mathcal{X}_{r,I}, \mathfrak{w}^\kappa)$$

Since the morphism $\phi: \mathfrak{X}_{r+1,I} \to \mathfrak{X}_{r,I}$ is flat of degree p, we are done. □

The U_p-operator admits a description using the Hecke correspondence.

Definition 2.3.23. *For any positive integer d prime to N, we define $\langle d\rangle: X_1^B(N) \to X_1^B(N)$ to be the diamond operator sending (A, i, ψ_N) to $(A, i, d\psi_N)$. Similar definition applies to $\mathfrak{X}_{r,I}$, and we extend $\langle d\rangle$ to $\mathfrak{IG}_{n,r,I}$ by acting on the trivializations trivially.*

Definition 2.3.24. *Let $\mathcal{Z}_{r,I}$ be the adic space over the generic fiber $\mathcal{X}_{r,I}$ of $\mathfrak{X}_{r,I}$ which parameterizes all rank-p subgroups $D^1 \subset \mathcal{G}[p]$ intersecting the level-1 canonical subgroup $H_1^1 \subset \mathcal{G}[p]$ trivially (see Lemma 2.2.13). Equivalently, $\mathcal{Z}_{r,I}$ is the adic space over $\mathcal{X}_{r,I}$ which parameterizes all non-trivial cyclic $\mathcal{O}_B$-submodule of $\mathcal{A}[p]$ intersecting the level-1 canonical subgroup $H_1 \subset \mathcal{A}[p]$ trivially.*

Let

$$p_1: \mathcal{Z}_{r,I} \to \mathcal{X}_{r,I}; \ ((\mathcal{A}, i), \mathcal{D}^1) \to (\mathcal{A}, i)$$

be the structure map. Note that p_1 is finite étale of degree p.

Lemma 2.3.25. *The map defined on points by $u: (\mathcal{A}, i) \mapsto (\mathcal{A}/H_1, \mathcal{G}[p]/H_1^1)$ induces a commutative diagram*

$$\begin{tikzcd} \mathcal{X}_{r+1,I} \arrow[r, "u", "\cong"'] \arrow[d, "\phi"] & \mathcal{Z}_{r,I} \arrow[d, "p_1"] \\ \mathcal{X}_{r,I} \arrow[r, "\mathrm{Id}"] & \mathcal{X}_{r,I} \end{tikzcd}$$

where the upper horizontal arrow is an isomorphism with inverse

$$p_2: \ \mathcal{Z}_{r,I} \to \mathcal{X}_{r+1,I}; \ ((\mathcal{A}, i), \mathcal{D}^1) \mapsto \langle p^{-1} \rangle (\mathcal{A}/\mathcal{D}, i).$$

Moreover, the quotient false isogeny $f^a: \ (\mathcal{A}, i) \to (\mathcal{A}/\mathcal{D}, i)$ *induces a morphism*

$$p_2^*(\mathbb{W}_\kappa) \to p_1^*(\mathbb{W}_\kappa)$$

such that the composition

$$H^0(\mathcal{X}_{r+1,I}, \mathbb{W}_\kappa) \xrightarrow{p_2^*} H^0(\mathcal{X}_{r+1,I}, p_2^*(\mathbb{W}_\kappa)) \to H^0(\mathcal{X}_{r,I}, p_1^*(\mathbb{W}_\kappa)) \xrightarrow{\frac{1}{p}\mathrm{Tr}_{p_1}} H^0(\mathcal{X}_{r,I}, \mathbb{W}_\kappa)$$

coincides with U_p*.*

Proof. By definition, the morphism $p_1 \circ u: \ \mathcal{X}_{r+1,I} \to \mathcal{X}_{r,I}$ sends $(\mathcal{A}, i)$ to $(\mathcal{A}/H_1, i)$, hence coincides with ϕ.

Let $\mathcal{D} = \mathcal{D}^1 \times g(\mathcal{D})$. The assignment $(\mathcal{A}, \mathcal{D}^1) \mapsto \mathcal{A}/\mathcal{D}$ on points induces $p_2: \ \mathcal{Z}_{r,I} \to \mathcal{X}_I$. Note that

$$p_2 \circ u: \ \mathcal{X}_{r+1,I} \to \mathcal{X}_I; \quad (\mathcal{A}, i) \mapsto (\mathcal{A}/H_1, \mathcal{G}[p]/H_1^1) \mapsto \langle p^{-1} \rangle (\mathcal{A}/\mathcal{A}[p], i) = (\mathcal{A}, i)$$

is an open immersion, so we have $\Omega_{\mathcal{Z}_{r,I}/\mathcal{X}_{r+1,I}} = 0$. As p_1 is étale and $p_1 \circ u = \phi$, we have that the relative differential $\Omega_{\mathcal{X}_{r+1,I}/\mathcal{X}_{r,I}}$ along ϕ is zero, which implies ϕ is étale. Clearly ϕ and p_1 both have degree p, so u is an isomorphism, which in turn implies that the map p_2 induces an isomorphism $\mathcal{Z}_{r,I} \to \mathcal{X}_{r+1,I}$.

Let $\mathcal{Z}_{n,r,I} := \mathcal{IG}_{n,r,I} \times_{\mathcal{X}_{r,I}, \pi_1} \mathcal{Z}_{r,I}$. Over $\mathcal{Z}_{n,r,I}$, the false quotient isogeny

$$f^a: \ (\mathcal{A}, i) \to (\mathcal{A}/\mathcal{D}, i)$$

induces an isomorphism between the level-n canonical subgroups $H_n \subset \mathcal{A}$ and $H_n' \subset \mathcal{A}/\mathcal{D}$, so the universal trivialization $\gamma: \ \mathbb{Z}/p^n\mathbb{Z} \cong H_n^{1,D}$ induces a trivialization $\gamma': \ \mathbb{Z}/p^n\mathbb{Z} \cong (H_n^{1,\prime})^D$. The pair $(\langle p^{-1} \rangle \mathcal{A}/\mathcal{D}, \gamma')$ over $\mathcal{Z}_{n,r,I}$ induces a commutative diagram compatible with the $(\mathbb{Z}/p^n\mathbb{Z})^\times$-action

$$\begin{tikzcd}
 & \mathcal{Z}_{n+1,r,I} \arrow[dl] \arrow[dr] \arrow[dd, dotted] & \\
\mathcal{IG}_{n+1,r+1,I} \arrow[rr, "t_1"] \arrow[dd] & & \mathcal{IG}_{n,r,I} \arrow[dd] \\
 & \mathcal{Z}_{r,I} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\
\mathcal{X}_{r+1,I} \arrow[rr, "\phi"] & & \mathcal{X}_{r,I}.
\end{tikzcd}$$

Note that $f^a: \ (\mathcal{A}, i) \to (\mathcal{A}/\mathcal{D}, i)$ on $\mathcal{Z}_{n+1,r,I}$ is the pull-back of $f': \ (\mathcal{A}/H_1, i) \to (\mathcal{A}, i)$ over $\mathcal{IG}_{n+1,r+1,I}$, so the final statement follows from the corresponding statement for $\mathcal{U}$. $\square$

Correspondences of degree prime to $p\Delta$

Let $(\mathcal{A}, i, \psi_N)$ be the universal false elliptic curve together with the universal $V_1(N)$-level structure over $\mathfrak{X}_{r,I}$. Let ℓ be a rational prime such that $(p\Delta, \ell) = 1$, and let $\mathfrak{C}_{r,I}$ be the formal scheme over $\mathfrak{X}_{r,I}$ parameterizing closed $\mathcal{O}_B$-submodules $\mathcal{C} \subset \mathcal{A}[\ell]$ which are locally free of rank ℓ^2 and intersect with $\mathrm{Im}(\psi_N)$ trivially (see Lemma 2.2.13).

Note that $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{F}_\ell \cong M_2(\mathbb{F}_\ell)$. Let x be any non-trivial idempotent element in $M_2(\mathbb{F}_\ell)$ and

$$\mathcal{A}[\ell]^1 := \ker(1 - x : \ \mathcal{A}[\ell] \to \mathcal{A}[\ell]); \quad \mathcal{A}[\ell]^2 := \ker(x : \ \mathcal{A}[\ell] \to \mathcal{A}[\ell]),$$

Then $\mathcal{A}[\ell] = \mathcal{A}[\ell]^1 \times \mathcal{A}[\ell]^2$. Moreover, there exists $y \in M_2(\mathbb{F}_\ell)$ such that $1 - x = yxy^{-1}$ and such an element y induces an isomorphism $\mathcal{A}[\ell]^1 \cong \mathcal{A}[\ell]^2$. From this, we can show that giving a $\mathcal{O}_B$-submodule $\mathcal{C} \subset \mathcal{A}[\ell]$ of rank ℓ^2 which intersects $\mathrm{Im}(\psi_N)$ trivially is equivalent to giving a subgroup $\mathcal{C}^1 \subset \mathcal{A}[\ell]^1$ of rank ℓ which intersects $\mathrm{Im}(\psi_N)$ trivially. As a consequence, Lemma 2.2.13 implies that the structure map

$$p_1 : \ \mathfrak{C}_{r,I} \to \mathfrak{X}_{r,I}, (\mathcal{A}, i, \psi_N, \mathcal{C}) \mapsto (\mathcal{A}, i, \psi_N)$$

is finite étale of degree $\ell + 1$ if $(\ell, Np\Delta) = 1$ or ℓ if $\ell \mid N$.

Lemma 2.3.26. *Let $\mathcal{C} \subset \mathcal{A}[\ell]$ be the universal $\mathcal{O}_B$-submodule of rank ℓ^2 which intersects $\mathrm{Im}(\psi_N)$ trivially, and let $\mathfrak{C}_{n,r,I} := \mathfrak{IG}_{n,r,I} \times_{\mathfrak{X}_{r,I}, p_1} \mathfrak{C}_{r,I}$. Then the quotient false isogeny*

$$f : \ (\mathcal{A}, i) \to (\mathcal{A}/C, i)$$

induces morphisms

$$p_2 : \ \mathfrak{C}_{r,I} \to \mathfrak{X}_{r,I}, \quad p_{2,n} : \ \mathfrak{C}_{n,r,I} \to \mathfrak{IG}_{n,r,I}$$

and a commutative diagram

$$\begin{tikzcd}
p_1^*(\mathcal{V}_0(\mathbb{H}_{\mathcal{G}}^\sharp, s_n)) \arrow[r] \arrow[d] & \mathcal{V}_0(\mathbb{H}_{\mathcal{G}}^\sharp, s_n) \arrow[d] \\
p_1^*(\mathcal{V}_0(\Omega_{\mathcal{G}}, s_n)) \arrow[r] \arrow[d] & \mathcal{V}_0(\Omega_{\mathcal{G}}, s_n) \arrow[d] \\
\mathfrak{C}_{n,r,I} \arrow[r, "p_{2,n}"] \arrow[d] & \mathfrak{IG}_{n,r,I} \arrow[d] \\
\mathfrak{C}_{r,I} \arrow[r, "p_2"] & \mathfrak{X}_{r,I}
\end{tikzcd} \tag{2.3.10}$$

which is compatible with the $\mathcal{T}^{\mathrm{ext}}$-action. The diagram 2.3.10 induces a morphism of $\mathcal{O}_{\mathfrak{C}_{r,I}}$-modules

$$f^* : \ p_2^*(\mathbb{W}_\kappa) \to p_1^*(\mathbb{W}_\kappa)$$

which is compatible with connection and filtration.

Proof. The forgetful morphism $p_1 : \ \mathfrak{C}_{r,I} \to \mathfrak{X}_{r,I}$ determines (an equivalence class of) a pair $(\mathrm{Hdg}(\mathcal{A}), \eta)$ with $\eta \in H^0(\mathfrak{C}_{r,I}, \underline{\omega}_{\mathcal{G}}^{(1-p)p^{r+1}})$ and $\mathrm{Hdg}(\mathcal{A})^{p^{r+1}}\eta \equiv \alpha \bmod p^2$. As f induces an isomorphism $\mathcal{G} := G_{\mathcal{A}} \cong G_{\mathcal{A}/\mathcal{C}} = \mathcal{G}/\mathcal{C}^1$, we have $\mathrm{Hdg}(\mathcal{A}) = \mathrm{Hdg}(\mathcal{A}/\mathcal{C})$, so there exists a pair $(\mathrm{Hdg}(\mathcal{A}/C), \eta')$ such that

$$\mathrm{Hdg}(\mathcal{A}/C)^{p^{r+1}}\eta' \equiv \alpha \mod p^2.$$

This implies that the morphism $p_2 : \mathfrak{C}_{r,I} \to \mathfrak{X}_I$ characterized by $p_2^*(\mathcal{A}) = \mathcal{A}/\mathcal{C}$ factors through $\mathfrak{X}_{r,I}$. Note that f induces an isomorphism between the level-n canonical subgroups $H_n^1 \subset \mathcal{G}$ and $H_n^{1,\prime} \subset G_{\mathcal{A}/\mathcal{C}}$, so the universal section $\gamma : \mathbb{Z}/p^n\mathbb{Z} \to H_n^{1,D}$ over $\mathfrak{IG}_{n,r,I}$ induces a section $\gamma' : \mathbb{Z}/p^n\mathbb{Z} \to (H_n^{1,\prime})^D$ over $\mathfrak{C}_{n,r,I}$ such that $f^D \circ \gamma' = \gamma$. The triple $(\mathcal{A}/\mathcal{C}, i, \gamma')$ determines a morphism $p_{2,n} : \mathfrak{C}_{n,r,I} \to \mathfrak{IG}_{n,r,I}$. Together with the morphism of vector bundles over $\mathfrak{C}_{n,r,I}$,

$$f^* : \Omega_{\mathcal{G}/\mathcal{C}^1} \to \Omega_{\mathcal{G}};\ \mathrm{HT}(\gamma'(1)) \mapsto \mathrm{HT}(\gamma(1)),$$

we have the diagram 2.3.10.

By taking the $[(\kappa^0)^{-1}]$-isotypic component with respect to the $\mathcal{T}^{\mathrm{ext}}$-action on the induced morphism

$$p_2^*(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}_{\mathcal{G}}^\sharp, s_n)}) \to \mathcal{O}_{p_1^*(\mathcal{V}_0(\mathbb{H}_{\mathcal{G}}^\sharp, s_n))},$$

and noting that p_1 is finite flat, we get a morphism $p_2^*(\mathbb{W}_\kappa^0) \to p_1^*(\mathbb{W}_\kappa^0)$ which is compatible with connection and filtration.

Let $i = 2$ if $p = 2$ and $i = 1$ if $p \geq 3$. By taking the $[(\kappa^f)^{-1}]$-isotypic component with respect to the $(\mathbb{Z}/p^i\mathbb{Z})^\times$-action on the induced morphism

$$p_2^*(\mathcal{O}_{\mathfrak{IG}_{i,r,I}}) \otimes_{\Lambda_I^0} \Lambda_I \to \mathcal{O}_{p_1^*(\mathfrak{IG}_{i,r,I})} \otimes_{\Lambda_I^0} \Lambda_I,$$

we get a morphism $p_2^*(\mathfrak{w}^{\kappa_f}) \to p_1^*(\mathfrak{w}^{\kappa_f})$ which is compatible with connection.

By tensoring these two morphisms, we get the desired morphism $f^* : p_2(\mathbb{W}_\kappa) \to p_1^*(\mathbb{W}_\kappa)$ which is compatible with connection and filtration. □

Since p_1 is étale, the trace operator Tr_{p_1} is well-defined.

Definition 2.3.27. *For any rational prime ℓ such that $(\ell, p\Delta) = 1$, we define the Hecke operator T_ℓ to be the composition*

$$H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa) \xrightarrow{p_2^*} H^0(\mathfrak{C}_{r,I}, p_2^*(\mathbb{W}_\kappa)) \xrightarrow{f^*} H^0(\mathfrak{C}_{r,I}, p_1^*(\mathbb{W}_\kappa)) \xrightarrow{\frac{1}{\ell}\mathrm{Tr}_{p_1}} H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa).$$

Twists by finite characters

Let (n, r, I) be a pre-adapted triple. Let $(\mathcal{A}, i)$ be the universal false elliptic curve, $\mathcal{G}$ be the associated p-divisible group, and $H_n^1 \subset \mathcal{G}[p^n]$, $H_n \subset \mathcal{A}[p^n]$ be the level-n canonical subgroups respectively. Let $(\mathcal{A}', i)$ be the quotient of $(\mathcal{A}, i)$ by H_n, $G_{\mathcal{A}'}$ be the p-divisible group associated with $\mathcal{A}'$, and let $H_n'^{,1} \subset G_{\mathcal{A}'}$, $H_n' \subset \mathcal{A}'$ be the level-n canonical subgroup respectively. Then the quotient false isogeny $f : (\mathcal{A}, i) \to (\mathcal{A}', i)$ has false degree p^n. Moreover, let $f' : (\mathcal{A}', i) \to (\mathcal{A}, i)$ be the false dual isogeny and $H_n'' := \ker f'$. Then we have that $\mathcal{A}'[p^n] = H_n'' \times H_n'$ and the false isogeny f' induces an isomorphism $H_n' \cong H_n$.

The Weil pairing induces isomorphisms

$$H_n'' \cong H_n^D \cong (H_n')^D, \quad (H_n^{1,\prime})^D \cong H_n''^{,\dagger} := \ker(1 - e^\dagger : H_n'' \to H_n'').$$

Together with the isomorphism

$$g' : H_n''^{,\dagger} \cong H_n''^{,1} := \ker(1 - e : H_n'' \to H_n''),$$

we have $(H_n^{1,\prime})^D \cong H_n''^{,1}$. Over $\mathfrak{IG}_{2n,r+n,I}$, the section $\gamma' : \mathbb{Z}/p^n\mathbb{Z} \to (H_n'^{,1})^D = H_n''^{,1}$ defined by the diagram 2.3.8 induces maps

$$s : \mathbb{Z}/p^n\mathbb{Z} \to H_n''^{,1}, \quad s^D : H_n'^{,1} \to \mu_{p^n}.$$

Let $K/\mathbb{Q}_p$ be any finite extension containing all p^n-th roots of unity and $\mathcal{O}_K$ be the ring of integers in K. Base change $\mathfrak{X}_{r,I}$, $\mathfrak{I}\mathfrak{G}_{n,r,I}$ from $\mathbb{Z}_p$ to $\mathcal{O}_K$ (but we do not change the notations). A chosen p^n-th root of unity ζ determines an isomorphism

$$\mathbb{Z}/p^n\mathbb{Z} \cong \mathrm{Hom}_{\mathfrak{I}\mathfrak{G}_{2n,r+n,I}}(\mathbb{Z}/p^n\mathbb{Z}, \mu_{p^n}), \quad j \mapsto (j \mapsto \zeta^j).$$

Inspired by the idea of [AI17, Proposition 3.26], we have

Lemma 2.3.28. *The map*

$$\eta: \ \mathrm{Hom}_{\mathcal{O}_B}(H''_n, H'_n) \cong \mathrm{Hom}(H''^{,1}_n, H'^{,1}_n) \to \mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, \mu_{p^n}) \cong \mathbb{Z}/p^n\mathbb{Z}, \ u \mapsto s^D \circ u \circ s$$

is a bijection.

Proof. The first isomorphism follows directly from Lemma 2.2.1. As s and s^D are isomorphisms on $\mathcal{IG}_{2n,r+n,I}$ and $\mathfrak{I}\mathfrak{G}_{2n,r+n,I}$ is α-torsion free, we know η is injective and is a bijection when restricted to the generic fiber. It suffices to show that any morphism $u: \ H''^{,1}_n \to H'^{,1}_n$ defined on the generic fiber extends to $\mathfrak{I}\mathfrak{G}_{2n,r+n,I}$. As $H''^{,1}_n$ is finite flat over the normal formal scheme $\mathfrak{I}\mathfrak{G}_{2n,r+n,I}$, it suffices to show that u extends in codimension 1. By localization, we may and do assume $(\mathcal{A}, i)$, $(\mathcal{A}', i)$, $H''^{,1}_n$ and $H'^{,1}_n$ are defined over a discrete valuation ring R of mixed characteristic $(0, p)$ and that $u: \ H''^{,1}_n \to H'^{,1}_n$ is defined over $L := \mathrm{Frac}(R)$. Let

$$G_{n,L} := \mathrm{Im}(\mathrm{Id} \times u : H''^{,1}_{n,L} \to H''^{,1}_{n,L} \times H'^{,1}_{n,L} = G_{\mathcal{A}'}[p^n]_L)$$

and G_n be the scheme theoretic closure of $G_{n,L}$ in $G_{\mathcal{A}'}[p^n]$. We only need to show that the induced projection $G_n \subset G_{\mathcal{A}'}[p^n] \to H''^{,1}_n$ is an isomorphism. Note that

$$H''^{,1}_n = \mathcal{G}[p^n]/H^1_n, \quad H'^{,1}_n = H^1_{2n}/H^1_n,$$

so for any $m \le n-1$, we have

$$\begin{aligned} &H''^{,1}_n[p^m] = \mathcal{G}[p^m]/H^1_m; \quad H''^{,1}_n[p^{m+1}]/H''^{,1}_n[p^m] \cong \mathcal{G}[p]/H^1_1; \\ &H'^{,1}_n[p^m] = H^1_{n+m}/H^1_n \cong H^1_m; \quad H'^{,1}_n[p^{m+1}]/H'^{,1}_n[p^m] = H^1_{n+m+1}/H^1_{n+m} \cong H^1_1 \end{aligned}$$

where the isomorphisms are induced by appropriate false dual isogenies. By [Tat97, Proposition 4.2.1] and an easy induction argument, we are reduced to showing that $\mathrm{Hom}_R(H''^{,1}_1, H'^{,1}_1)$ has p different elements. As $H''^{,1}_1$ has rank p and $(H'^{,1}_1)^D \cong H''^{,1}_1$, by Oort-Tate theory, we have

$$H''^{,1}_1 \cong \mathrm{Spec}(R[x]/(x^p - \tilde{\mathrm{H}}^{p^n} x)); \quad H'^{,1}_1 \cong \mathrm{Spec}(R[y]/(y^p + \frac{p}{\tilde{\mathrm{H}}^{p^n}} y))$$

where $\tilde{\mathrm{H}}$ is a lift of the Hasse invariant of $(\mathcal{A}, i)$. Note that $-p$ admits all $p-1$-th root in R as R contains all p-th roots of unity, and that $H''^{,1}_1(R) \cong \mathbb{Z}/p\mathbb{Z}$ by the definition of $\mathfrak{I}\mathfrak{G}_{2n,r+n,I}$, so $\tilde{\mathrm{H}}$ and $\frac{-p}{\tilde{\mathrm{H}}^{2p^n}}$ both admit all $p-1$-th roots over R. For each $(p-1)$-th root v of $\frac{-p}{\tilde{\mathrm{H}}^{2p^n}}$, the morphism $y \mapsto vx$ induces a group homomorphism $H''^{,1}_1 \to H'^{,1}_1$. Together with the trivial group homomorphism $y \mapsto 0$, we get p different elements of $\mathrm{Hom}_R(H''^{,1}_1, H'^{,1}_1)$. In particular, any element in $\mathrm{Hom}_L(H''^{,1}_1, H'^{,1}_1)$ admits a unique extension to R. So we have $G_1 \cong H''^{,1}_1$ and the proposition follows. □

For any $j \in \mathbb{Z}/p^n\mathbb{Z}$, let $\rho_j: H_n'' \to H_n'$ be the morphism of $\mathcal{O}_B$-modules such that $\eta(\rho_j) = j$, and let

$$H_{\rho_j} := (\rho_j \times \mathrm{Id})(H_n'') \subset H_n' \times H_n'' = \mathcal{A}'[p^n].$$

Clearly we have $H_{\rho_j} \times H_n' = \mathcal{A}'[p^n]$ as $\mathcal{O}_B$-modules. Set $\mathcal{A}_j' := (\mathcal{A}'/H_{\rho_j}, i)$. The quotient false isogeny $f_j': \mathcal{A}' \to \mathcal{A}_j'$ induces an isomorphism between H_n' and its image $H_{n,j}' := f_j'(H_n')$, which implies that $H_{n,j}' \subset \mathcal{A}_j'$ is the level-n canonical subgroup. Note that $H_{n,j}'$ is the kernel of the false dual isogeny $\mathcal{A}_j' \to \mathcal{A}'$, so $\mathrm{Hdg}(\mathcal{A}_j')^{p^n} = \mathrm{Hdg}(\mathcal{A}') = \mathrm{Hdg}(\mathcal{A})^{p^n}$, which implies that $\mathrm{Hdg}(\mathcal{A}_j') = \mathrm{Hdg}(\mathcal{A})$.

For each $j \in \mathbb{Z}/p^n\mathbb{Z}$, let $t_j: \mathfrak{IG}_{2n,r+n,I} \to \mathfrak{IG}_{n,r+n,I}$ be the normalization of the morphism $\mathcal{IG}_{2n,r+n,I} \to \mathcal{IG}_{n,r+n,I}$ defined by sending $(\mathcal{A}, i, \gamma_{2n})$ to $(\mathcal{A}_j', i, \gamma_j')$, where γ_{2n} is the universal trivialization of $(H_{2n}^1)^D$ and γ_j' is the unique trivialization of $(H_{n,j}'^{,1})^D$ such that $f_j'^{,D}(\gamma_j') = \gamma'$. Let $p_k: \mathfrak{IG}_{k,r+n,I} \to \mathfrak{X}_{r+n,I}$ be the forgetful map and let $v_j := p_n \circ t_j: \mathfrak{IG}_{2n,r+n,I} \to \mathfrak{X}_{r+n,I}$.

Lemma 2.3.29. *Let (n, r, I) be an adapted triple. Over $\mathfrak{IG}_{2n,r+n,I}$, the false isogeny*

$$f_j: (\mathcal{A}, i) \xrightarrow{f} (\mathcal{A}', i) \xrightarrow{f_j'} (\mathcal{A}_j', i)$$

induces a morphism of $\mathcal{O}_{\mathfrak{IG}_{2n,r+n,I}}$-modules

$$f_j^*: v_j^*(\mathbb{W}_\kappa^0) \to p_{2n}^*(\mathbb{W}_\kappa^0)$$

which preserves the filtration $\mathrm{Fil}_\bullet \mathbb{W}_\kappa$ and the Gauss-Manin connection ∇_κ^0. After inverting α, f_j induces a morphism

$$f_j^*: v_j^*(\mathbb{W}_\kappa)[\frac{1}{\alpha}] \to p_{2n}^*(\mathbb{W}_\kappa)[\frac{1}{\alpha}]$$

which preserves the filtration $\mathrm{Fil}_\bullet \mathbb{W}_\kappa$ and the Gauss-Manin connection ∇_κ.

Proof. Let $(\mathbb{H}_{\mathcal{G}}^\sharp, \Omega_{\mathcal{G}}, s)$ be the system of vector bundles with marked sections associated to $\mathcal{G}$ over $\mathfrak{IG}_{n,r+n,I}$, and let $\mathcal{G}_j' := G_{\mathcal{A}_j'}$. Over $\mathfrak{IG}_{2n,r+n,I}$, we have

$$t_j^*(\mathcal{A}, i, \gamma) = (\mathcal{A}_j', i, \gamma_j'), \quad t_j^*(\Omega_{\mathcal{G}}, s) = (\Omega_{\mathcal{G}_j'}, s_j'), \quad t_j^*(\mathbb{H}_{\mathcal{G}}^\sharp, s) = (\mathbb{H}_{\mathcal{G}_j'}^\sharp, s_j')$$

where $s_j' = \mathrm{HT}(\gamma_j'(1))$. Proceeding as in the proof of Lemma 2.3.17, we have

$$\frac{f^*}{p^n}: \Omega_{\mathcal{G}'} \xrightarrow{\cong} \Omega_{\mathcal{G}}.$$

Together with the isomorphism

$$f_j'^{,*}: \Omega_{\mathcal{G}_j'} \cong \Omega_{\mathcal{G}'},$$

we get an isomorphism

$$\frac{f_j^*}{p^n}: \Omega_{\mathcal{G}_j'} \cong \Omega_{\mathcal{G}}, \quad s_j' \mapsto p_{2n,n}^*(s),$$

where $p_{2n,n}: \mathfrak{IG}_{2n,r+n,I} \to \mathfrak{X}_{r+n,I}$ is the forgetful map. We further have a well-defined morphism $\frac{f_j^*}{p^n}: \underline{\omega}_{\mathcal{G}_j'} \to \underline{\omega}_{\mathcal{G}}$, because

$$\mathrm{Hdg}(\mathcal{A}) = \mathrm{Hdg}(\mathcal{A}_j') = \underline{\delta}^{p-1}, \quad \Omega_{\mathcal{G}} = \underline{\delta \omega}_{\mathcal{G}}, \quad \Omega_{\mathcal{G}_j'} = \underline{\delta \omega}_{\mathcal{G}_j'}.$$

Let f'_j be the false dual isogeny $(\mathcal{A}'_j, i) \to (\mathcal{A}, j)$. By duality, we have a well-defined morphism

$$\frac{f_j'^{,*}}{p^n} : \underline{\omega}^{-1}_{\mathcal{G}^D} \to \underline{\omega}^{-1}_{(\mathcal{G}'_j)^D}.$$

Combining these together, we have well-defined maps of $\mathfrak{IG}_{2n,r+n,I}$-modules

$$\frac{f_j^*}{p^n} : \mathbb{H}_{\mathcal{G}'_j} \to \mathbb{H}_{\mathcal{G}}; \quad \mathbb{H}^\sharp_{\mathcal{G}'_j} = \underline{\delta}\mathbb{H}_{\mathcal{G}'_j} \to \mathbb{H}^\sharp_{\mathcal{G}} = \underline{\delta}\mathbb{H}_{\mathcal{G}'_j}.$$

The induced morphism of $\mathfrak{IG}_{2n,r+n,I}$-formal schemes

$$p^*_{2n,n}(\mathcal{O}_{\mathcal{V}_0}(\mathbb{H}^\sharp_{\mathcal{G}}, s)) \to t^*_j(\mathcal{O}_{\mathcal{V}_0}(\mathbb{H}^\sharp_{\mathcal{G}}, s))$$

is clearly compatible with the filtration and the $\mathcal{T}$-action.

When (n, r, I) is adpated, we have $p^*_n(\mathbb{W}^0_\kappa) = \tilde{\mathbb{W}}^0_\kappa$, so by taking the $[(\kappa^0)^{-1}]$-isotypic component with respect to the $\mathcal{T}$-action, we get the desired morphism

$$f^*_j : v^*_j(\mathbb{W}^0_\kappa) \to p^*_{2n}(\mathbb{W}^0_\kappa)$$

which preserves the filtration and the Gauss-Manin connection.

The multiplicative structure on $\mathcal{O}_{\mathcal{IG}_{2n,r+n,I}} \otimes_{\Lambda^0_I} \Lambda_I$ induces an isomorphism

$$\mathfrak{w}^{\kappa_f}[\frac{1}{\alpha}] \otimes_{\mathcal{O}_{\mathcal{X}_{r+n,I}}} \mathcal{O}_{\mathcal{IG}_{2n,r+n,I}} \cong \mathcal{O}_{\mathcal{IG}_{2n,r+n,I}} \otimes_{\Lambda^0_I} \Lambda_I, \quad a \otimes b \mapsto ab.$$

Together with the natural identification

$$v^*_j(\mathcal{O}_{\mathfrak{X}_{r+n,I}}) = \mathcal{O}_{\mathfrak{IG}_{2n,r+n,I}} = p^*_{2n}(\mathcal{O}_{\mathfrak{X}_{r+n,I}}),$$

we obtain a morphism $v^*_j(\mathfrak{w}^{\kappa_f})[\frac{1}{\alpha}] \to p^*_{2n}(\mathfrak{w}^{\kappa_f})[\frac{1}{\alpha}]$. Tensoring with f^*_j, we get the desired morphism

$$f^*_j : v^*_j(\mathbb{W}_\kappa)[\frac{1}{\alpha}] \to p^*_{2n}(\mathbb{W}_\kappa)[\frac{1}{\alpha}]$$

which is compatible with the filtration and the Gauss-Manin connection. □

For any primitive character $\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to (\Lambda_I[\zeta])^\times$, $g(\chi) := \sum_{j \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j)\zeta^j$ be the associated Gauss sum.

Corollary 2.3.30. *By taking global sections, we get a morphism*

$$\theta^\chi : H^0(\mathcal{X}_{r+n,I}, \mathbb{W}_\kappa) \xrightarrow{g(\chi^{-1})(\sum_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j) f^*_j v^*_j)} H^0(\mathcal{X}_{r+n,I}, p_{2n,*}p^*_{2n}(\mathbb{W}_{\kappa+2\chi}))$$
$$\xrightarrow{\frac{1}{p^{2n}}\mathrm{Tr}} H^0(\mathcal{X}_{r+n,I}, \mathbb{W}_{\kappa+2\chi})$$

which preserves the filtration $\mathrm{Fil}_\bullet$.

Proof. Only the statement about weight needs an argument. For any $a \in (\mathbb{Z}/p^{2n}\mathbb{Z})^\times$, we claim that the following diagram is commutative

$$\begin{tikzcd} \mathfrak{IG}_{2n,r+n,I} \arrow[r, "a"] \arrow[d, "t_{a^2j}"'] & \mathfrak{IG}_{2n,r+n,I} \arrow[d, "t_j"] \\ \mathfrak{IG}_{n,r+n,I} \arrow[r, "a"] & \mathfrak{IG}_{n,r+n,I}. \end{tikzcd} \tag{2.3.11}$$

It suffices to check commutativity on points. Let $(\mathcal{A}, i, \gamma)$ be the universal false elliptic curve together with the universal section $\gamma : \mathbb{Z}/p^{2n}\mathbb{Z} \to H^D_{2n}$ over $\mathfrak{IG}_{2n,r+n,I}$. We have

$$a * (\mathcal{A}, i, \gamma) = (\mathcal{A}, i, a\gamma), \quad t_j : (\mathcal{A}, i, \gamma) \mapsto (\mathcal{A}', i, \gamma') \mapsto (\mathcal{A}'_j, i, \gamma_j).$$

If we replace γ by $a\gamma$, we are changing γ' to $a\gamma'$ and $s, s^\vee$ to $as, as^\vee$ respectively, hence changing H_{ρ_j} to $H_{\rho_{a^2j}}$ and γ_j to $a\gamma_{a^2j}$, so we have

$$t_j(\mathcal{A}, i, a\gamma) = (\mathcal{A}'_{a^2j}, i, a\gamma_{a^2j}) = a * (t_{a^2j}(\mathcal{A}, i, \gamma)),$$

and the commutativity of the diagram 2.3.11 follows.

For any $\tilde{a} \in \mathbb{Z}_p^\times$ whose image modulo p^n is a and for any $s \in H^0(\mathcal{X}_{r,I}, \mathbb{W}_\kappa)$, we have

$$\begin{aligned}
&\tilde{a} * ((\sum_{j \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j) f_j^* v_j^*)(s)) \\
&= \sum_{j \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j) \tilde{a} * (f_j^*(s)) \\
&= \sum_{j \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j) f_{a^2j}^*(a * (s)) \\
&= \sum_{j \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \kappa^{-1}(a) \chi(j) f_{a^2j}^*(s) \\
&= \sum_{j \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \kappa^{-1}(\tilde{a}) \chi^{-2}(a) \chi(j) f_j^*(s) \\
&= (\kappa + 2\chi)^{-1}(\tilde{a}) (\sum_{j \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j) f_j^* v_j^*)(s),
\end{aligned}$$

which proves the statement about weight. □

2.3.5 Comparison with other constructions of overconvergent quaternion modular forms

In [Bra13], the author constructs overconvergent quaternion modular forms of general weight in the spirit of test objects, which is analogous to the construction of overconvergent elliptic modular forms in [AIS14]. Another construction of overconvergent quaternion modular forms of general weight is given in [CHJ17, § 2.8], using differential forms by adapting the method in [Pil13]. The equivalence between these two constructions can be shown using the argument in [AIS14, Appendix B].

In [CHJ17, Chapter 2], a construction of overconvergent quaternion modular forms using the perfectoid language is spelled out in details. Moreover, in § 2.8 of `op.cit`, the equivalence between the pecfectoid construction and Pilloni type construction is explained in details.

As explained in [Bra13, § 4.2.2], when restricted to the ordinary locus, all overconvergent quaternion modular forms mentioned above recover Katz's p-adic quaternion modular forms.

At integral level, we may adapt the formulation in [AIP, § 5] to define overconvergent quaternion modular forms as follows: Let $f_n : \mathfrak{F}_{n,r,I} \to \mathfrak{IG}_{n,r,I}$ be the formal scheme such that

$$\mathfrak{F}_{n,r,I}(R) = \{(P, \omega) \in (H_n^{1,D}(R) - H_n^{1,D}[p^{n-1}](R)) \times \underline{\omega}_{\mathcal{G}}(R) : \ \mathrm{HT}(P) = \omega \text{ in } \underline{\omega}_{\mathcal{G}}/p^n \mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\}$$

for any formal scheme $\mathrm{Spf}(R) \to \mathfrak{X}_{r,I}$ where R is α-adically complete and α-torsion-free, and define the $\mathcal{T}^{\mathrm{ext}}(R)$-action on $\mathfrak{F}_{n,r,I}(R)$ by

$$(\lambda x) * (P, \omega) = (\lambda P, (\lambda x)\omega), \ \forall\, \lambda \in \mathbb{Z}_p^\times,\ x \in \mathcal{T}(R),\ (P,\omega) \in \mathfrak{F}_{n,r,I}(R).$$

Let $g_n : \ \mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I}$ be the structure map. The overconvergent quaternion modular sheaf for the universal character κ_I is defined by

$$\mathfrak{w}_{\kappa_I} := \mathfrak{w}_{\kappa_I,0} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \mathfrak{w}^{\kappa_{I,f}}, \quad \mathfrak{w}_{\kappa_I,0} := (g_n \circ f_n)_* \mathcal{O}_{\mathfrak{F}_{n,r,I}}[(\kappa^0)^{-1}].$$

We can moreover define the U_p, V_p-operators on $\mathfrak{w}_{\kappa_I}$ using the formulation in `op.cit` § 5.4 and the Hecke operators away from $p\Delta$ adapting the formulation in [Pil13, § 4.1].

Lemma 2.3.31. *Let (n, r, I) be any pre-adapted triple. There is a canonical isomorphism between $\mathcal{V}_0(\Omega_{\mathcal{G}}, s)$ and $\mathfrak{F}_{n,r,I}$ which is compatible with the $\mathcal{T}^{\mathrm{ext}}$-action. As a corollary, we have an isomorphism of $\mathcal{O}_{\mathfrak{X}_{r,I}}$-modules $\mathfrak{w}^{\kappa_I} \cong \mathfrak{w}_{\kappa_I}$ which is compatible with the Hecke operators.*

Proof. Let $P_n \in H_n^{1,D}(\mathfrak{IG}_{n,r,I})$ be the universal section and $s = \mathrm{HT}(P_n)$. Let $\mathrm{Spf}(R) \to \mathfrak{X}_{r,I}$ be an affine formal scheme with R α-adically complete and α-torsion-free. Then $\mathcal{V}_0(\Omega_{\mathcal{G}}, s)(R)$ consists of pairs (ρ, ν) where $\rho :\ \mathrm{Spf}(R) \to \mathfrak{IG}_{n,r,I}$ is an $\mathfrak{X}_{r,I}$-morphism and $\nu \in \mathrm{Hom}_R(\rho^*\Omega_{\mathcal{G}}, R)$ such that

$$\nu(\rho^*(s)) \bmod \rho^*(p^n \mathrm{Hdg}^{-\frac{p^n}{p-1}}) = 1.$$

By the definition of $\mathfrak{IG}_{n,r,I}$, we have a $1-1$ correspondence between $\rho :\ \mathrm{Spf}(R) \to \mathfrak{IG}_{n,r,I}$ and elements in $H_n^{1,D}(R) - H_n^{1,D}[p^{n-1}](R)$ given by $\rho \mapsto \rho^*(P_n)$. Noting that $\Omega_{\mathcal{G}}$ and $\underline{\omega}_{\mathcal{G}}$ are invertible $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}$-modules, and $\Omega_{\mathcal{G}} = \mathrm{Hdg}^{\frac{1}{p-1}} \underline{\omega}_{\mathcal{G}}$, we have an equivalence of functors compatible with $\mathcal{T}^{\mathrm{ext}}$-actions given by

$$\mathcal{V}_0(R) \mapsto \mathfrak{F}_{n,r,I}(R);\ (\rho, \nu) \mapsto (\rho^*(P_n), \nu^\vee),$$

where $\nu^\vee$ is the dual element of ν in $\Omega_{\mathcal{G}}$ such that $\nu(\nu^\vee) = 1$. This functor gives the desired isomorphisms

$$\mathfrak{w}^{\kappa_I,0} \cong \mathfrak{w}_{\kappa_I,0}; \quad \mathfrak{w}^{\kappa_I} \cong \mathfrak{w}_{\kappa_I}.$$

For the compatibility of Hecke operators, see the proof of Lemma A.2.24 below. □

We can also compare $\mathfrak{w}^\kappa$ with Katz's p-adic quaternion modular forms at the integral level.

Note that $\mathfrak{IG}^{\mathrm{ord}}_{n,r,I} \to \mathfrak{X}^{\mathrm{ord}}_I$ is finite étale with Galois group $(\mathbb{Z}/p^n\mathbb{Z})^\times$, so the inverse limit $\pi :\ \mathfrak{IG}^{\mathrm{ord}}_{\infty,r,I} := \varprojlim_{n\geq 0} \mathfrak{IG}^{\mathrm{ord}}_{n,r,I} \to \mathfrak{X}^{\mathrm{ord}}_I$ is well-defined and equipped with an action of $\mathbb{Z}_p^\times$. Adapting the argument of [Pil13, Proposition 6.1], we have

Proposition 2.3.32. *There is a canonical isomorphism*

$$\mathfrak{w}^{\kappa_I,0}_{|_{\mathfrak{X}^{\mathrm{ord}}_{r,I}}} \cong \pi_*(\mathcal{O}_{\mathfrak{IG}^{\mathrm{ord}}_{\infty,r,I}})[(\kappa_I^0)^{-1}].$$

Proof. Let $\mathrm{Spf}(R) \to \mathfrak{X}^{\mathrm{ord}}_I$ be an affine formal scheme with R α-adically complete and α-torsion-free. A point in $\mathfrak{IG}^{\mathrm{ord}}_{\infty,r,I}(R)$ is a compatible sequence of generators $P = \{P_n \in H_n^{1,D}(R)\}$, and the $\mathbb{Z}_p^\times$-action is given by $\lambda * P = \{\lambda P_n\}$. Set $\mathrm{HT}(P) := \varprojlim_n \mathrm{HT}(P_n)$. Then we have an $\mathbb{Z}_p^\times$-equivariant morphism of $\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}$-formal schemes

$$\mathfrak{IG}^{\mathrm{ord}}_{\infty,r,I} \to \mathfrak{F}^{\mathrm{ord}}_{n,r,I} := \mathfrak{F}_{n,r,I} \otimes_{\mathfrak{IG}_{n,r,I}} \mathfrak{IG}^{\mathrm{ord}}_{n,r,I}; \quad P \mapsto (P_n, \mathrm{HT}(P)).$$

As the action of $\mathcal{T}^{\mathrm{ext}}$ on $\mathfrak{F}_{n,r,I}$ is analytic (see Lemma 2.3.3), by taking the $[(\kappa_I^0)^{-1}]$-isotypic component we have an induced morphism

$$\mathfrak{w}^{\kappa_I,0}_{|_{\mathfrak{X}^{\mathrm{ord}}_{r,I}}} = \mathcal{O}_{\mathfrak{F}^{\mathrm{ord}}_{n,r,I}}[(\kappa_I^0)^{-1}] \to \pi_*\mathcal{O}_{\mathfrak{IG}^{\mathrm{ord}}_{\infty,r,I}}[(\kappa_I^0)^{-1}].$$

As $\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}$ is étale over $\mathfrak{X}^{\mathrm{ord}}_I$ and $(\pi_*\mathcal{O}_{\mathfrak{IG}^{\mathrm{ord}}_{\infty,r,I}})^{\mathbb{Z}_p^\times} = \mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}_I}$, we have that $\mathfrak{w}^{\kappa_I,0}_{|_{\mathfrak{X}^{\mathrm{ord}}_I}}$ is an invertible $\mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}_I}$-module with inverse $\mathfrak{w}^{\kappa_I^{-1},0}_{|_{\mathfrak{X}^{\mathrm{ord}}_{r,I}}}$. Moreover, we have a sequence of morphisms

$$\mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}_I} = \mathfrak{w}^{\kappa_I,0}_{|_{\mathfrak{X}^{\mathrm{ord}}_{r,I}}} \otimes_{\mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}_I}} \mathfrak{w}^{\kappa_I^{-1},0}_{|_{\mathfrak{X}^{\mathrm{ord}}_{r,I}}} \to \pi_*\mathcal{O}_{\mathfrak{IG}^{\mathrm{ord}}_{\infty,r,I}}[(\kappa_I^0)^{-1}] \otimes_{\mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}_I}} \pi_*\mathcal{O}_{\mathfrak{IG}^{\mathrm{ord}}_{\infty,r,I}}[\kappa_I^0] \to \mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}_I} \tag{2.3.12}$$

whose composition is the identity map. By construction, we have an isomorphism of the adic generic fibers

$$\mathcal{IG}^{\mathrm{ord}}_{\infty,r,I} \cong \varprojlim_{n\geq 1} \mathcal{F}^{\mathrm{ord}}_{n,r,I},$$

so we have

$$\mathfrak{w}^{\kappa_I,0}_{|_{\mathfrak{X}^{\mathrm{ord}}_{r,I}}}[\frac{1}{\alpha}] \cong \pi_*\mathcal{O}_{\mathfrak{IG}^{\mathrm{ord}}_{\infty,r,I}}[\frac{1}{\alpha}][(\kappa_I^0)^{-1}].$$

Combining with equation 2.3.12, we have the desired isomorphism

$$\mathfrak{w}^{\kappa_I,0}_{|_{\mathfrak{X}^{\mathrm{ord}}_{r,I}}} \cong \pi_*\mathcal{O}_{\mathfrak{IG}^{\mathrm{ord}}_{\infty,r,I}}[(\kappa_I^0)^{-1}].$$

□

2.4 Expansion in Serre-Tate coordinates

Let $k = \bar{\mathbb{F}}_p$ and $W = W(k)$ be the associated ring of Witt vectors. Let $(\bar{A}/k, i, \psi_N)$ be an ordinary false elliptic curve $\bar{A}/k$ together with a $V_1(N)$-level structure ψ_N. Let $\{P_1, P_2\}$ be a basis of $T_p\bar{A}$ such that

$$eP_1 = P_1, \quad (1-e)P_2 = P_2.$$

Let P_1^D, P_2^D be the image of P_1,P_2 in $T_p\bar{A}^D$ via $\lambda_{\bar{A}}$, and let Q_1,Q_2 be the dual basis of P_1,P_2. Let $\tilde{\mathcal{R}}$ be the deformation space of $\bar{A}/k$ viewed simply as an abelain variety, and set

$$q_{ij} = q(\tilde{\mathcal{A}}/\tilde{\mathcal{R}}; P_i, P_j^D), \quad t_{ij} = q_{ij} - 1, \quad i,j \in \{1,2\}.$$

By Theorem 1.4.8, if we let $\omega_i = \mathrm{HT}(P_i^D),\ \eta_i = \eta(Q_i),\ i = 1,2$, then we have

$$\tilde{\mathcal{R}} = W[[t_{ij}:\ i,j = 1,2]], \quad \nabla\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = 0, \quad \nabla\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix} d\log q_{11} & d\log q_{21}\\ d\log q_{12} & d\log q_{22}\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}.$$

Consider the subfunctor $\mathcal{M}^f$ of $\mathrm{Spf}(\tilde{\mathcal{R}})$ which sends an artinian local ring R with residue field k to the set of false elliptic curves over R with reduction $\bar{A}$.

Proposition 2.4.1. *The subfunctor $\mathcal{M}^f$ is a dimension-1 formal torus pro-representable by* $\mathrm{Spf}(\mathcal{R})$ *where $\mathcal{R}$ is the quotient of $\tilde{\mathcal{R}}$ by the closed ideal generated by the relations*

$$q(\tilde{\mathcal{A}}/\tilde{\mathcal{R}}; bP, Q^D) = q(\tilde{\mathcal{A}}/\tilde{\mathcal{R}}; P, b^\dagger Q^D), \quad \forall\ b \in \mathcal{O}_B,\ P \in T_p\bar{A},\ Q^D \in T_p\bar{A}^D.$$

In particular, $\mathcal{R} = W[[t]]$ where t is the image of t_{11}.

Proof. This is a direct consequence of Theorem 1.4.5(iii). For more details, see [Bro13, Proposition V.1] or [Mor11, Proposition 3.3]. □

Let $\mathcal{A}^f/\mathcal{R}$ be the universal false deformation of $\bar{A}/k$. One can recover the Hodge exact sequence and the Gauss-Manin connection of $\mathcal{A}^f/\mathcal{R}$ by restricting the Hodge sequence and the Gauss-Manin connection of $\tilde{\mathcal{A}}/\tilde{\mathcal{R}}$ to $\mathcal{R}$.

Lemma 2.4.2. [Bro13, Lemma V.2] *For any element*

$$\eta \in \mathbb{H}^1_{\mathrm{dR}}(\tilde{\mathcal{A}}/\tilde{\mathcal{R}}), \quad P \in T_p\bar{A}^D, \quad Q \in \mathrm{Hom}(T_p\bar{A}, \mathbb{Z}_p), \quad b \in \mathcal{O}_B,$$

we have

$$\nabla(\eta)_{|\mathcal{R}} = \nabla(\eta_{|\mathcal{R}}); \quad \omega_{bP}|_{\mathcal{R}} = b^\dagger \omega_P|_{\mathcal{R}}; \quad \eta_{b^*Q}|_{\mathcal{R}} = b^\dagger \eta_Q|_{\mathcal{R}}.$$

In particular, as $e^\dagger P_i^D = P_i^D$, we have

$$e\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}\omega_1\\ 0\end{pmatrix}$$

Denote the restriction of ω_1 to $\mathcal{R}$ by ω and the restriction of $\eta(e^\dagger Q_1)$ to $\mathcal{R}$ by η. Then ω and $\{\omega, \eta\}$ are bases of $\underline{\omega}_{\mathcal{G}}$ and $\mathbb{H}_{\mathcal{G}}$ respectively. Moreover, by Theorem 1.4.8, we have

$$\nabla\omega = \eta \otimes d\log(1+t); \quad \nabla(\eta) = 0.$$

Apply Lemma 2.3.9 to X,Y corresponding to $f = \delta\omega$, $e = \delta\eta$ respectively and let $V = \frac{Y}{X}$. Then we have

$$\nabla^\sharp(X^\kappa V^h) = X^k V^h \otimes \mu_\kappa d\log(\delta) + (\mu_\kappa - h)X^\kappa V^{h+1} \otimes d\log(1+t). \tag{2.4.1}$$

On the other hand, composing with the isomorphism $g' : \underline{\omega}_{\mathcal{G}} \cong e^\dagger \mathbb{H}_{\mathcal{G}}$, we have

$$c\mathrm{KS}(\omega^{\otimes 2}) = d\log(1+t)$$

for some $c \in \mathbb{Z}_p^\times$. Upon replacing g' by $c^{-1}g'$, we may and will assume $\mathrm{KS}(\omega^{\otimes 2}) = d\log(1+t)$.

Note that $(\bar{A}, i, \psi_N)$ corresponds to a point $x \in X(k)$ where $X = X_1^B(N)$, so by the discussion in [Mor11, Page 24], we have

Lemma 2.4.3. *There exists a canonical isomorphism* $\hat{\mathcal{O}}_{X,x}\hat{\otimes}_{\mathbb{Z}_p} W \cong W[[t]]$.

Proof. Let R be any artinian ring with residue field k, and let A^f/R be a false deformation of $\bar{A}/k$ to R. By Hensel's lemma, we can uniquely lift ψ_N to a $V_1(N)$-level structure $\psi_{N,R}$. By the moduli interpretation of $X_1^B(N)$, we get a morphism $\mathcal{O}_{X,x} \to R$. By passing to the inverse limit, $\mathcal{A}^f/\mathcal{R}$ has a universal $V_1(N)$-level structure Ψ_N and we have a morphism $\hat{\mathcal{O}}_{X,x} \to \mathcal{R}$, which induces a morphism $\hat{\mathcal{O}}_{X,x}\hat{\otimes}_{\mathbb{Z}_p} W \to \mathcal{R}$. Let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$. By the universal property, we have a morphism $\mathcal{R} \to \mathcal{O}_{X,x}/\mathfrak{m}_x^n \otimes_{\mathbb{Z}_p} W$. Passing to the inverse limit, we get a morphism $\mathcal{R} \to \hat{\mathcal{O}}_{X,x}\hat{\otimes}_{\mathbb{Z}_p} W$. It is easy to check these two morphisms are inverse to each other, and the lemma follows from Proposition 2.4.1. □

Convention 2.4.4. *Throughout this section, $k = \bar{F}_p$, $W = W(k)$, and we will always base change $X := X_1(N)$, $\hat{X}$, $\mathfrak{X}_{r,I}$, $\mathfrak{IG}_{n,r,I}$ and $\Lambda_I^{(0)}$ from $\mathbb{Z}_p$ to W, and denote them using the same notation.*

Equip $W[[t]]$ with the p-adic topology and let $R_t = \Lambda_I^0[[t]]$. By Lemma 2.4.3, we have morphisms

$$\rho: \operatorname{Spf}(W[[t]]) \to \hat{X}, \quad \operatorname{Spf}(R_t) \to \mathfrak{X}_I^{\mathrm{ord}} (\to \mathfrak{X}_{r,I} \ \forall r \in \mathbb{N})$$

such that $\rho^*((\mathcal{A}, i, \psi_N)) = (\mathcal{A}^f, i, \Psi_N)$. Denote $\rho^*(\mathfrak{w}^\kappa)$, $\rho^*(\mathbb{W}_\kappa)(R_t)$, $\rho^*(\mathbb{W})(R_t)$ by $\mathfrak{w}^\kappa(t)$, $\mathbb{W}_\kappa(t)$, $\mathbb{W}(t)$ respectively. We have the following Serre-Tate expansion principle for Shimura curves. For the modular curve analogue, we refer to Lemma A.3.3.

Lemma 2.4.5. *The evaluation maps*

$$H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa) \to \mathbb{W}_\kappa(t); \quad H^0(\mathfrak{X}_I^{\mathrm{ord}}, \mathbb{W}) \to \mathbb{W}(t),$$
$$H^0(\mathfrak{X}_{r,I}^{\mathrm{ord}}, \mathbb{W}/p^N\mathbb{W}) \to \mathbb{W}(t)/p^N\mathbb{W}(t), \quad \forall\ N \geq 1,$$

are injective.

Proof. As $\mathcal{X}_{r,I}$ is irreducible, it is easy to deduce the injectivity of

$$H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa) \to \mathbb{W}_\kappa(t)$$

from Lemma 1.1.16 and the fact that $\mathfrak{w}^{\kappa_f}$ is locally free of finite rank when restricted to $\mathcal{X}_{r,I}$. Recall that $\mathbb{W}^0 = f_{0,*}\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}},s)}$ where $f_0: \mathcal{V}_0(\mathbb{H}_{\mathcal{G}^\sharp,s}) \to \mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I}$ is the structure map, and locally, $\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{G}},s)}$ is a ring of formal power series over $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}$. As $\mathfrak{IG}_{n,r,I}^{\mathrm{ord}} \to \mathfrak{X}_{r,I}^{\mathrm{ord}}$ is flat, we can deduce that

$$H^0(\mathfrak{X}_I^{\mathrm{ord}}, \mathbb{W}) \to \mathbb{W}(t), \quad H^0(\mathfrak{X}_{r,I}^{\mathrm{ord}}, \mathbb{W}/p^N\mathbb{W}) \to \mathbb{W}(t)/p^N\mathbb{W}(t), \quad \forall\ N \geq 1$$

are injective from the fact that $\mathfrak{IG}_{n,r,I}^{\mathrm{ord}}$ is geometrically irreducible, which is shown in [Hid12, Chapter 7]. □

We summarize some running notations for the rest of this section:

- $k = \bar{F}_p$, $W = \bar{F}_p$, and we base change X, $\hat{X}$, $\mathfrak{X}_{r,I}$, $\mathfrak{IG}_{n,r,I}$,$\Lambda_I^{(0)}$ from $\mathbb{Z}_p$ to W.
- $(\bar{A}/k, i, \psi_N)$ is a fixed ordinary false elliptic curve with $V_1(N)$-level structure ψ_N and universal false deformation $(\mathcal{A}^f/\mathcal{R}, i, \Psi_N)$.
- $P \in eT_p\bar{A}(k)$ is a fixed generator, $P^D := \lambda_{\bar{A}}(P) \in T_p\bar{A}^D$, and Q is the dual basis of P.
- $1 + t = q(\mathcal{A}^f/\mathcal{R}; P, P^D)$, $\mathcal{R} = W[[t]]$, $\omega = \mathrm{HT}(P^D)$, and $\eta = \eta(e^\dagger Q)$. By our convention on g', we have $\mathrm{KS}(\omega^{\otimes 2}) = d\log(1+t)$.
- $R_t = \Lambda_I^0[[t]]$, $\tilde{R}_t = R_t \otimes_{\Lambda_I^0} \Lambda_I$, and $\rho: \operatorname{Spf}(R_t) \to \mathfrak{X}_I^{\mathrm{ord}}$ is the morphism characterized by $\rho^*(\mathcal{A}, i, \psi_N) = (\mathcal{A}^f, i, \Psi_N)$.
- $\mathfrak{w}^\kappa(t) = \rho^*(\mathfrak{w}^\kappa)(R_t)$, $\mathbb{W}_\kappa(t) = \rho^*(\mathbb{W}_\kappa)(R_t)$, and $\mathbb{W}(t) = \rho^*(\mathbb{W})(R_t)$.

2.4.1 The Gauss-Manin connection in Serre-Tate coordinates

Recall that $(\mathbb{H}^\sharp_{\mathcal{G}}, \Omega_{\mathcal{G}}, s)$ is the system of vector bundles with marked sections associated to $\mathcal{G}$ over $\mathfrak{IG}_{n,r,I}$ with respect to the ideal $\underline{\beta}_n$.

Lemma 2.4.6. *Let* $\operatorname{Spf}(R_{n,t}) := \mathfrak{IG}_{n,r,I} \times_{\mathfrak{X}_{r,I},\rho} \operatorname{Spf}(R_t)$ *and base change* ρ *to* $\operatorname{Spf}(R_{n,t})$. *Equip* $\sqcup_{g \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \operatorname{Spf}(R)$ *with the* $(\mathbb{Z}/p^n\mathbb{Z})^\times$*-action given by left translation.*

(*i*) *We have an* $(\mathbb{Z}/p^n\mathbb{Z})^\times$*-equivariant isomorphism* $\mathrm{Spf}(R_{n,t}) \cong \sqcup_{g\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\mathrm{Spf}(R_t)$.

(*ii*) *There exists* $\delta \in R_{n,t}^\times$ *such that* $\delta\omega \bmod p^n = \rho^*(s)$ *and each component* δ_g *of* δ *belongs to* $\mathcal{T}^{\mathrm{ext}}(R_t)$. *In particular,* $\kappa^0(\delta)$ *is well-defined.*

Proof. Let $\mathcal{C}_n \subset \mathcal{A}^f$ be the level-n canonical subgroup. As $\mathcal{A}/\mathcal{R}$ is ordinary, $\mathcal{C}_n^D$ is étale. By the equivalence between the category of finite étale algebras over $W[[t]]$ and the category of finite étale algebras over k, we know that $\mathcal{C}_n^{1,D} \cong \mathbb{Z}/p^n\mathbb{Z}$ as finite flat group schemes over $W[[t]]$.

(*i*) Note that $\mathrm{Spf}(R_{n,t}) = \mathfrak{IG}^{\mathrm{ord}}_{n,r,I} \times_{\mathfrak{X}_I^{\mathrm{ord}},\rho} \mathrm{Spf}(R_t)$ because the morphism ρ factors through $\mathfrak{X}_I^{\mathrm{ord}}$. By construction, $\mathfrak{IG}^{\mathrm{ord}}_{n,I}$ is the finite étale cover of $\mathfrak{X}_I^{\mathrm{ord}}$ with Galois group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ which parameterizes all trivializations of $H_n^{1,D}$ over $\mathfrak{X}_I^{\mathrm{ord}}$. The fact $\mathcal{C}_n^{1,D} \cong \mathbb{Z}/p^n\mathbb{Z}$ over $\mathrm{Spf}(R_t)$ implies that $\mathrm{Spf}(R_{n,t}) \cong \sqcup_{g\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\mathrm{Spf}(R_t)$.

(*ii*) Note that $\bar{A}^D/k$ and $\bar{A}/k$ are ordinary. We can view $\mathcal{C}_n^D/k$ as the maximal étale quotient of $\bar{A}^D[p^n]$, so we have

$$T_p\bar{A}^D(k) = \varprojlim_{n\geq 1} \bar{A}^D[p^n](k) = \varprojlim_{n\geq 1} \mathcal{C}_n^D(k).$$

Let P_n^D be the image of P^D in $\mathcal{C}_n^D(k)$. Since $\mathrm{Spf}(R_t)$ is connected, P_n^D can be viewed as a generator of

$$\mathcal{C}_n^{1,D}(k) = \mathcal{C}_n^{1,D}(W[[t]]) = \mathcal{C}_n^{1,D}(R_t).$$

Let $\gamma:\ \mathbb{Z}/p^n\mathbb{Z} \to H_n^{1,D}$ be the universal trivialization. By item (*i*),

$$\mathcal{C}_n^{1,D}(R_{n,t}) = \sqcup_{g\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\mathcal{C}_n^{1,D}(R_t).$$

By considering the action of $(\mathbb{Z}/p^n\mathbb{Z})^\times$, we can show that there exists $\bar{\delta} \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ such that $\rho^*(\gamma(1)) = (\bar{\delta}gP_n^D)_{g\in(\mathbb{Z}/p^n\mathbb{Z})^\times}$.

Note that $\omega \mod p^n = \mathrm{HT}(P_n^D)$ and $s = \mathrm{HT}(\gamma(1))$, so we have

$$\rho^*(s) = (\bar{\delta}g\omega \bmod p^n)_{g\in(\mathbb{Z}/p^n\mathbb{Z})^\times}.$$

Consequently, we can choose $\delta = (\delta_g)_{g\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \in R_{n,t}^\times$ with $\delta_g \bmod p^n = \bar{\delta}g$ such that $\delta\omega \bmod p^n = \rho^*(s)$. Since $\kappa^0(\delta_g)$ is well-defined for each g, we have $\kappa^0(\delta)$ is well-defined. □

Recall that by assumption, $\mathrm{KS}(\omega^{\otimes 2}) = d\log(1+t)$. Take δ as in Lemma 2.4.6, and let $X = 1 + p^nZ$, Y correspond to $f = \delta\omega$, $e = \delta\eta$ respectively and set $V := \frac{Y}{X}$. Then by Proposition 2.3.7, we have

Proposition 2.4.7. *Let*

$$\partial:\ R_t \to R_t \text{ (resp. } \tilde{R}_t \to \tilde{R}_t),\quad \sum_{i\geq 0} a_it^i \mapsto \sum_{i\geq 0} ia_i(1+t)t^{i-1}.$$

(*i*) *Let* $U_{\kappa^0,h} = \kappa^0(\delta)^{-1}(1+p^nZ)^\kappa(\frac{Y}{1+p^nZ})^h$. *Then*

$$\mathbb{W}_\kappa^0(t) = \{\sum_{h\geq 0} a_hU_{\kappa^0,h}:\ a_h \in R, a_h \to \infty\},$$

and for any $a \in R_t$ *and* $h \geq 0$, *we have*

$$\nabla_\kappa^0(aU_{\kappa^0,h}) = 2_f(\delta^{-1})(\partial(a)U_{\kappa^0+2,h} + a(\mu_\kappa - h)U_{(\kappa+2)^0,h+1}).$$

(ii) *Let* $\mathfrak{w}^{\kappa_f}(t) := \rho^*(\mathfrak{w}^{\kappa_f})(R_t)$. *Then* $\mathfrak{w}^{\kappa_f}(t)$ *is a free* $\tilde{R}_t$*-module of rank* 1. *Moreover, we can choose a basis* a_{κ_f} *such that* $\nabla^f_\kappa(a_{\kappa_f}) = 0$.

(iii) *Let* $U_{\kappa,h} = U_{\kappa^0,h} \otimes a_{\kappa_f}$. *Then* $\mathbb{W}_\kappa(t) = \{\sum_{h\geq 0} a_h U_{\kappa,h} :\ a_h \in \tilde{R}_t,\ a_h \to 0\}$, *and for any* $a \in \tilde{R}_t$ *and* $h \geq 0$, *we have*

$$\nabla_\kappa(aU_{\kappa,h}) = \partial(a)U_{\kappa,h} + a(\mu_\kappa - h)U_{\kappa+2,h+1}. \tag{2.4.2}$$

Proof. (i) Since $\kappa^0(\delta)$ is well-defined, by equation 2.3.2 in the proof of Lemma 2.3.4, we have

$$\lambda * (U_{\kappa^0,h}) = \kappa^0(\lambda^{-1})U_{\kappa^0,h},\ \forall\ \lambda \in \mathbb{Z}_p^\times.$$

Note that $U_{\kappa^0,0} = \kappa^0(\delta)^{-1}(1+p^nZ)^{\kappa^0} \equiv 1 \mod R_t^{\circ\circ}$ and $\rho:\ \mathrm{Spf}(R_t) \to \mathfrak{X}_{r,I}$ is flat, so by Proposition 2.3.7, we have

$$\mathbb{W}^0_\kappa(t) = \{\sum_{h\geq 0} a_h U_{\kappa^0,h} :\ a_h \in R_t, a_h \to \infty\}.$$

By equation 2.4.1 and Leibniz's rule, we have

$$\begin{aligned}\nabla^0_\kappa(U_{\kappa^0,h}) = U_{\kappa^0,h} \otimes (d\log(\kappa^0)^{-1}(\delta) + \mu_\kappa d\log\delta) + (\mu_\kappa - h)U_{\kappa^0,h+1} \otimes d\log(1+t)\\ = (\mu_\kappa - h)U_{\kappa^0,h+1} \otimes d\log(1+t);\end{aligned}$$

and for any $a \in R_t$,

$$\nabla^0_\kappa(aU_{\kappa^0,h}) = (\partial(a)U_{\kappa^0,h} + (\mu_\kappa - h)U_{\kappa^0,h+1}) \otimes d\log(1+t).$$

Note that $f = \delta\omega$, so we have

$$\begin{aligned}&d\log(1+t) = \mathrm{KS}(\omega^{\otimes 2}) = \delta^{-2}\mathrm{KS}(f^{\otimes 2}),\\ &\nabla^0_\kappa(U_{\kappa^0,h}) = (\mu_\kappa - h)\delta^{-2}(1+p^nZ)^2U_{(\kappa^0),h+1} = 2_f(\delta^{-1})(\mu_\kappa - h)U_{(\kappa+2)^0,h+1}.\end{aligned}$$

Thus for any $a \in R_t$, we have

$$\nabla^0_\kappa(aU_{\kappa^0,h}) = 2_f(\delta^{-1})(\partial(a)U_{(\kappa+2)^0,h} + a(\mu_\kappa - h)U_{(\kappa+2)^0,h+1}).$$

(ii) Let $i=2$ if $p=2$ and $i=1$ if $p\geq 3$. By Lemma 2.4.6, there exist orthogonal idempotent elements $e_g \in R_{i,t}, g \in (\mathbb{Z}/p^i\mathbb{Z})^\times$ such that $R_{i,t} = \oplus R_t e_g$ and for any $g \in (\mathbb{Z}/p^i\mathbb{Z})^\times$, $g*e_h = e_{g^{-1}h}$. It is routine to check that

$$\mathfrak{w}^{\kappa_f}(t) = \{\sum_{g\in(\mathbb{Z}/p^i\mathbb{Z})^\times} ae_g \otimes \kappa_f(g^{-1}), a \in \tilde{R}_t\},$$

and that $a_{\kappa_f} := \sum_{g\in(\mathbb{Z}/p^i\mathbb{Z})^\times} e_g \otimes \kappa_f(g^{-1})$ is a basis of $\mathfrak{w}^{\kappa_f}(t)$ such that $\nabla^f_\kappa(a_{\kappa_f}) = 0$.

(iii) It suffices to show

$$\delta^{-2}\kappa^0(\delta^{-1})a_{\kappa_f} = (\kappa+2)^0(\delta^{-1})a_{(\kappa+2)_f},$$

which is straightforward. □

2.4.2 Hecke operators in Serre-Tate coordinates

Hecke operators away from $p\Delta$

Let ℓ be a rational prime such that $(\ell, p\Delta) = 1$; let $a_\ell := \ell$ if $\ell \mid N$ or $a_\ell := \ell + 1$ if $\ell \nmid N$, and let $\{D_j\}_{j=1,..,a_\ell}$ be the set of $\mathcal{O}_B$-submodules which are locally free of rank ℓ^2 and intersect $\mathrm{Im}(\psi_N)$ trivially. Then each D_j deforms uniquely to a $\mathcal{O}_B$-module $\mathcal{D}_j \subset \mathcal{A}^f$ of rank ℓ^2 such that $\mathcal{D}_j \cap \mathrm{Im}(\Psi_N) = (0)$. Let $(\bar{A}/D_j, i, \psi_j)$ be the quotient of $(\bar{A}, i, \psi_N)$ by D_j, and let $(\mathcal{A}_j^f/\mathcal{R}_j, i, \Psi_j)$ be the universal false deformation of $(\bar{A}/D_j, i, \psi_j)$. Then $(\mathcal{A}^f/\mathcal{D}_j, i)$ over $\mathcal{R}$ induces a morphism

$$G_j : \mathrm{Spf}(\mathcal{R}) \to \mathrm{Spf}(\mathcal{R}_j); \quad G_j^*(\mathcal{A}_j^f) = \mathcal{A}^f/\mathcal{D}_j.$$

Let $g_j : \mathcal{A}^f \to \mathcal{A}^f/\mathcal{D}_j$ be the quotient false isogeny. Then we have (see also [Bro13, Lemma V.4])

Lemma 2.4.8. *Let $P_j \in T_p\bar{A}/D_j(k)$ be the unique element whose image P_j^D via $\lambda_{\bar{A}/D_j}$ satisfies that $g_j^D(P_j^D) = P^D$, Q_j be the dual basis of P_j, and*

$$\omega_j := \mathrm{HT}(P_j^D), \ \eta_j := \eta(Q_j), \ 1 + t_j := q(\mathcal{A}_j^f/\mathcal{R}_j; P_j, P_j^D) \in \mathcal{R}_j.$$

Then we have

$$g_j^*(G_j^*(\omega_j)) = \omega, \quad g_j^*(G_j^*(\eta_j)) = \ell\eta, \quad G_j^*(1 + t_j) = (1 + t)^{\frac{1}{\ell}}.$$

Proof. Let $g_j' : \mathcal{A}^f/\mathcal{D}_j \to \mathcal{A}^f$ be the false dual isogeny. As $g_j' \circ g_j = g_j \circ g_j' = \ell$, both g_j and g_j' induce isomorphisms on the Tate modules, so P_j is well-defined and $g_j(P) = \ell P_j$.

By functoriality of the Hodge-Tate period map, the following diagram is commutative

$$\begin{tikzcd}
T_p(\bar{A}/D_j)(k) \arrow[r, "\lambda_{\bar{A}/D_j}"] \arrow[d, "g_j'"] & T_p(\bar{A}/D_j)^D(k) \arrow[r, "\mathrm{HT}"] \arrow[d, "g_j^D"] & \underline{\omega}_{\mathcal{A}/\mathcal{D}_j} \arrow[d, "g_j^*"] \\
T_p\bar{A}(k) \arrow[r, "\lambda_{\bar{A}}"] & T_p\bar{A}^D(k) \arrow[r, "\mathrm{HT}"] & \underline{\omega}_{\mathcal{A}}.
\end{tikzcd}$$

So we have $g_j^*(G_j^*(\omega_j)) = \omega$ and dually, $g_j^*(G_j^*(\eta_j)) = \ell\eta$. By Theorem 1.4.5 (*iii*), we have

$$\begin{aligned}
&G_j^*(1 + t_j) = q(\mathcal{A}^f/\mathcal{D}_j/\mathcal{R}; P_j, P_j^D) = q(\mathcal{A}^f/\mathcal{D}_j/\mathcal{R}; \frac{1}{\ell}g_j(P), P_j^D) \\
&= q(\mathcal{A}^f/\mathcal{R}; \frac{1}{\ell}P, g_j^D(P_j^D)) \\
&= q(\mathcal{A}^f/\mathcal{R}; \frac{1}{\ell}P, P^D) \\
&= (1 + t)^{\frac{1}{\ell}}.
\end{aligned}$$

□

Let $R_{t_j} := \Lambda_I[[t_j]]$, $\tilde{R}_{t_j} := R_{t_j} \otimes_{\Lambda_I^0} \Lambda_I$ and let $\rho_j : \mathrm{Spf}(R_{t_j}) \to \mathfrak{X}_{r,I}$ be the morphism characterized by $\rho_j^*(\mathcal{A}^f, i, \psi_N) = (\mathcal{A}_j^f, i, \Psi_j)$. Denote $\rho_j^*(\mathbb{W}_\kappa)(R_{t_j})$ by $\mathbb{W}_\kappa(t_j)$. Then based on Lemma 2.4.8, we have

Proposition 2.4.9. *For each $j = 1, ..., a_\ell$, we can choose a basis $\{U_{\kappa,h,j}\}_{h\in\mathbb{N}}$ for $\mathbb{W}_\kappa(t_j)$ such that given any $f \in H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa)$ with $\rho_j^*(f) = \sum_{h\geq 0} f_j(t_j)U_{\kappa,h,j}$ for $j = 1, ..., a_\ell$, we have*

$$\rho^*(T_\ell(f)) = \sum_{j=1}^{a_\ell}\sum_{h\geq 0} f_j((1+t)^{1/l} - 1)l^h U_{\kappa,h}.$$

Proof. Let $\mathrm{Spf}(R_{n,t_j}) := \mathrm{Spf}(R^j) \times_{\rho_j,\mathfrak{X}_{r,I}} \mathfrak{IG}_{n,r,I}$. According to Proposition 2.4.7, we can choose $\delta_j \in (R_{n,t_j})^\times$ such that $\delta_j\omega_j \equiv \rho_j^*(s) \mod p^n$, and by sending $\delta_j\omega_j$ to $X_j = 1 + p^n Z_j$ and $\delta_j\eta_j$ to Y_j, the set $\{U_{\kappa,h,j}\}_{h\geq 0}$ where

$$U_{\kappa,h,j} = (\kappa^0)^{-1}(\delta_j)(1+p^nZ_j)^\kappa(\frac{Y_j}{1+p^nZ_j})^h \otimes \sum_{g\in(\mathbb{Z}/q\mathbb{Z})^\times} e_g \otimes \kappa_f^{-1}(g)$$

is a basis of $\mathbb{W}_\kappa(t_j)$. Note that by definition and Lemma 2.4.8, we have

$$g_j^*G_j^*(\rho_j^*(s)) = \rho^*(s); \quad g_j^*G_j^*(\omega_j) = \omega, \quad g_j^*G_j^*(\eta_j) = \ell\eta,$$

so we have

$$g_j^*G_j^*(U_{\kappa,h,j}) = l^hU_{\kappa,h}, \quad \forall 1 \leq j \leq a_\ell.$$

Unwinding the definition of T_ℓ, we have $\rho^*(T_\ell(f)) = \frac{1}{\ell}\sum_{j=1}^{a_l} g_j^*G_j^*(\rho_j^*(f))$. By Lemma 2.4.8, $G_j^*(t_j) = (1+t)^{1/\ell} - 1$, so we have

$$\rho^*(T_\ell(f)) = \sum_{j=1}^{a_\ell}\sum_{h\geq 0} f_j((1+t)^{1/l} - 1)l^h U_{\kappa,h}.$$

□

Now we can discuss the relation between ∇_κ and T_ℓ.

Proposition 2.4.10. *Let $f \in H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa)$, then*

$$\nabla_\kappa \circ T_\ell(f) = \ell T_\ell \circ \nabla_\kappa(f).$$

Proof. By Lemma 2.4.5, it suffices to show the equality in Serre-Tate coordinates. Assume that for $j = 1, ..., a_\ell$, $\rho_j^*(f) = \sum_{h\geq 0} f_j(t_j)U_{\kappa,h,j}$. Then we have

$$\rho^*(\nabla_\kappa \circ T_\ell(f)) = \frac{1}{\ell}\nabla_\kappa(\sum_{j=1}^{a_\ell}\sum_{h\geq 0} f_j((1+t)^{1/\ell} - 1)\ell^h U_{\kappa,h})$$

$$= \frac{1}{\ell}\sum_{j=1}^{a_\ell}\sum_{h\geq 0}((1+t)^{1/\ell} f_j'((1+t)^{1/\ell} - 1)\ell^{h-1}U_{\kappa,h} + (\mu_\kappa - h)f_j((1+t)^{1/\ell} - 1)\ell^h U_{\kappa+2,h+1});$$

$$\rho_j^*(\nabla_\kappa(f)) = \sum_{h\geq 0}\nabla_\kappa(f_j(t_j)U_{\kappa,h,j})$$

$$= \sum_{h\geq 0}((1+t_j)f_j'(t_j)U_{\kappa+2,h,j} + f_j(t_j)(\mu_\kappa - h)U_{\kappa+2,h+1,j});$$

$$\rho^*(T_\ell \circ \nabla_\kappa(f)) = \frac{1}{\ell}\sum_{j=1}^{a_\ell}\sum_{h\geq 0}(1+t)^{1/\ell} f_j'((1+t)^{1/\ell} - 1)\ell^h U_{\kappa+2,h,j}$$

$$+ \frac{1}{\ell}\sum_{j=1}^{a_\ell}\sum_{h\geq 0} f_j((1+t)^{1/\ell} - 1)(\mu_\kappa - h)\ell^{h+1}U_{\kappa+2,h+1,i} = \ell\rho^*(\nabla_\kappa \circ T_\ell(f)).$$

□

The Frobenius and the U_p-operator

Let $\mathcal{C} \subset \mathcal{A}^f$ be the level-1 canonical subgroup. Then the quotient false isogeny

$$f_0 : \mathcal{A}^f \to \mathcal{A}^f/\mathcal{C}$$

is a lift of the Frobenius morphism

$$F : \bar{A} \to \bar{A}^{(p)}.$$

Let $(\bar{A}^{(p)}, i, \psi_N^{(p)})$ be the quotient of $(\bar{A}/k, i, \psi_N)$ by $\ker(F)$ and $(\mathcal{A}^{f,(p)}/\mathcal{R}^{(p)}, i, \Psi_N^{(p)})$ be the universal false deformation of $(\bar{A}^{(p)}, i, \psi_N^{(p)})$. Then we have a morphism $\phi : \mathrm{Spf}(\mathcal{R}) \to \mathrm{Spf}(\mathcal{R}^{(p)})$ characterized by $\phi^*(\mathcal{A}^{f,(p)}) = \mathcal{A}^f/\mathcal{C}$.

Since $\bar{A}/k$ is ordinary, we have an isomorphism $F : T_p\bar{A}(k) \cong T_p\bar{A}^{(p)}(k)$. Then again following [Bro13, Lemma V.6 & V.7], we have that

Lemma 2.4.11. *Let $P^{(p)} := F(P)$, $P^{D,(p)} := \lambda_{\bar{A}^{(p)}}(P^{(p)})$, and $Q^{(p)}$ be the dual basis of $P^{(p)}$. Moreover, let*

$$1 + t^{(p)} := q(\mathcal{A}^{f,(p)}/\mathcal{R}^{(p)}; P^{(p)}, P^{D,(p)}), \quad \omega^{(p)} := \mathrm{HT}(P^{D,(p)}), \quad \eta^{(p)} := \eta(Q^{(p)}).$$

Then we have

$$\phi^*(1 + t^{(p)}) = (1 + t)^p, \quad f_0^*\phi^*(\omega^{(p)}) = p\omega, \quad f_0^*\phi^*(\eta^{(p)}) = \eta.$$

Proof. Let $V : \bar{A}^{(p)} \to \bar{A}$ be the Verschiebung map. By Theorem 1.4.5, we have

$$\begin{aligned}
\phi^*(1 + t^{(p)}) &= \phi^*(q(\mathcal{A}^{f,(p)}/\mathcal{R}^{(p)}; P^{(p)}, P^{D,(p)})) \\
&= q(\mathcal{A}^f/\mathcal{C}/\mathcal{R}; F(P), P^{D,(p)}) = q(\mathcal{A}^f/\mathcal{R}; P, V(P^{D,(p)})) \\
&= q(\mathcal{A}^f/\mathcal{R}; P, pP^D) = (1 + t)^p; \\
f_0^*\phi^*(\omega^{(p)}) &= \mathrm{HT}(V(P^{D,(p)})) = p\mathrm{HT}(P^D) = p\omega; \\
f_0^*\phi^*(\eta^{(p)}) &= \eta(F^*(Q^{(p)})) = \eta(Q) = \eta.
\end{aligned}$$

□

Let $R_{t^{(p)}} := \Lambda_I[[t^{(p)}]]$, and let $\rho^{(p)}$ be the morphism

$$\rho^{(p)} : \mathrm{Spf}(R_{t^{(p)}}) \to \mathfrak{X}_{r,I}, \quad (\rho^{(p)})^*(\mathcal{A}^f, i, \Psi_N) = (\mathcal{A}^{f,(p)}, i, \Psi_N^{(p)}).$$

Denote $(\rho^{(p)})^*(\mathfrak{w}^\kappa)(R_{t^{(p)}})$ by $\mathfrak{w}^\kappa(t^{(p)})$.

Proposition 2.4.12. *We can choose a basis $\{U_{\kappa,0}^{(p)}\}$ of $\mathfrak{w}^\kappa(t^{(p)})$ such that for any $f \in H^0(\mathfrak{X}_{r,I}, \mathfrak{w}_\kappa)$ with $(\rho^{(p)})^*(f) = F(t^{(p)})U_{\kappa,0}^{(p)}$, we have $\rho^*(V_p(f)) = F((1 + t)^p - 1)U_{\kappa,0}$.*

Proof. By Lemma 2.4.11, we have

$$\rho^*(V_p(f)) = \frac{1}{p} f_0^*\phi^*((\rho^{(p)})^*(f)), \quad \frac{1}{p} f_0^*\phi^*(\omega^{(p)}) = \omega.$$

Then the desired result follows from a similar argument in the proof of Lemma 2.4.8 using Proposition 2.4.7. □

To deal with the operator U_p and twists by finite characters, we need to describe étale cyclic rank-p^{2n} $\mathcal{O}_B$-submodules of $(\mathcal{A}^f, i)/\mathcal{R}$. Fix a primitive p^n-th root of unity ζ; set $W_n := W[\zeta]$; and let $(1+t)^{p^{-n}}$ be a p^n-th root of $1+t$.

Proposition 2.4.13. *Over $W[[t]][\zeta, (1+t)^{p^{-n}}]$, the connected-étale exact sequence*

$$0 \to \mathcal{A}^f[p^n]^\circ \to \mathcal{A}^f[p^n] \to \mathcal{A}^f[p^n]^{ét} \to 0 \tag{2.4.3}$$

admits p^n different splittings as $\mathcal{O}_B$-modules.

Proof. Let $H_n \subset \mathcal{A}^f[p^n]$ be the level-n canonical subgroup. Since $\bar{A}/k$ is ordinary, we have that $H_n \cong \mathcal{A}^f[p^n]^\circ$ and that $\lambda_{\mathcal{A}^f}$ induces an isomorphism $\mathcal{A}^f[p^n]^{ét} \cong H_n^D$. Taking kernels with respect to the endomorphism $1-e$, we obtain

$$0 \to H_n^1 \to G_{\mathcal{A}^f}[p^n] \to H_n^{D,\dagger} \to 0 \tag{2.4.4}$$

from the sequence 2.4.3. Moreover, a splitting of the sequence 2.4.3 as $\mathcal{O}_B$-modules is equivalent to a splitting of the sequence 2.4.4. With the help of the isomorphism $g' : (H_n^1)^D \cong H_n^{D,\dagger}$, we are reduced to the argument in Proposition A.3.6 below. □

Note that we have $\bar{A}[p^n] \cong (\mu_p^n)^2 \oplus (\mathbb{Z}/p^n\mathbb{Z})^2$ as $\mathcal{O}_B$-modules. Let $(\bar{A}', i, \psi'_N)$ be the quotient of $(\bar{A}, i, \psi_N)$ by $(\mathbb{Z}/p^n\mathbb{Z})^2$, and let $(\mathcal{A}'^{,f}/\mathcal{R}', i, \Psi')$ be the universal false deformation of $(\bar{A}', i, \psi'_N)$. Let $\mathcal{C}_j \subset \mathcal{A}^f$ be a rank-p^{2n} étale cyclic $\mathcal{O}_B$-module and let $(\mathcal{A}^f/\mathcal{C}_j, i, \Psi_j)$ be the quotient of $(\mathcal{A}^f, i, \Psi_N)$ by $\mathcal{C}_j$. Then $\langle p^{-1}\rangle(\mathcal{A}^f/\mathcal{C}_j, i, \Psi_j)$ over $W[[t]][\zeta, (1+t)^{p^{-n}}]$ induces a morphism

$$F_j : \operatorname{Spf}(W[[t]][\zeta, (1+t)^{p^{-n}}]) \to \operatorname{Spf}(\mathcal{R}')$$

characterized by $F_j^*(\mathcal{A}'^{,f}, i, \Psi') = \langle p^{-1}\rangle(\mathcal{A}^f/\mathcal{C}_j, i, \Psi_j)$. Note that the map $T_p\bar{A}(k) \to T_p\bar{A}'(k)$ induced by the quotient false isogeny $f : (\bar{A}, i) \to (\bar{A}', i)$ is multiplication by p^n times an isomorphism. As stated in [Bro13, Lemma V.9 & V.10] that

Lemma 2.4.14. *Let $P' = \frac{f(P)}{p^n}$, $P'^{,D} := \lambda_{\bar{A}'}(P')$, and let Q' be the dual basis of P'. Also let*

$$1+t' := q(\mathcal{A}'^{,f}/\mathcal{R}'; P', P'^{,D}), \quad \omega' := \mathrm{HT}(P'), \ \eta' := \eta(Q').$$

Then up to reordering $\mathcal{C}_j$, we have

$$F_j^*(1+t')^{p^n} = 1+t; \quad F_j^*(1+t')|_{t=0} = \zeta^{-j}.$$

Let $f_j : (\mathcal{A}^f, i) \to (\mathcal{A}^f/\mathcal{C}_j, i)$ be the quotient isogeny. We have that

$$f_j^* F_j^*(\omega') = \omega; \quad f_j^* F_j^*(\eta') = p^n \eta.$$

Proof. All statements except the equality $F_j^*(1+t')|_{t=0} = \zeta^{-j}$ follow from similar arguments as in the proofs of Lemma 2.4.8 and Lemma 2.4.11.

Let A^f_{can} be the canonical false lifting of (A, i) to W. Then $A^f_{\mathrm{can}}[p^n] \cong (\mu_{p^n})^2 \oplus (\mathbb{Z}/p^n\mathbb{Z})^2$ as $\mathcal{O}_B$-modules. Let $H_{n,j}$ be the étale subgroup corresponding to $\mathbb{Z}/p^n\mathbb{Z} \to \mu_{p^n} : 1 \mapsto \zeta^j$ via the construction in Proposition 2.3.28. Then it suffices to show $(1+t')(A^f_{\mathrm{can}}/H_{n,j}) = \zeta^{-j}$. Over W_n, the quotient false isogeny $A^f_{\mathrm{can}} \to A^f_{\mathrm{can}}/H_{n,j}$ induces the commutative diagram below

$$\begin{tikzcd}
0 \arrow[r] & (\mu_{p^\infty})^2 \arrow[r] \arrow[d, "="] & A^f_{\mathrm{can}}[p^\infty] \cong (\mu_{p^\infty})^2 \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^2 \arrow[r] \arrow[d, "f_j"] & (\mathbb{Q}_p/\mathbb{Z}_p)^2 \arrow[r] \arrow[d, "p^n"] & 0 \\
0 \arrow[r] & (\mu_{p^\infty})^2 \arrow[r] & A^f_{\mathrm{can}}/H_{n,j}[p^\infty] \arrow[r] & (\mathbb{Q}_p/\mathbb{Z}_p)^2 \arrow[r] & 0.
\end{tikzcd}$$

Taking kernels with respect to the endomorphism $1-e$, we have

$$\begin{tikzcd}
0 \arrow[r] & \mu_{p^\infty} \arrow[r] \arrow[d, "="] & \mu_{p^\infty} \oplus \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] \arrow[d, "f_j"] & \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] \arrow[d, "p^n"] & 0 \\
0 \arrow[r] & \mu_{p^\infty} \arrow[r] & \mu_{p^\infty} \oplus \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] & \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] & 0,
\end{tikzcd}$$

where the middle vertical arrow is the quotient by the subgroup generated by $(\zeta^j, 1/p^n)$. Unwinding the definition of Serre-Tate coordinates as in the proof of Lemma A.3.7, we have $(1+t')(A^f_{\mathrm{can}}/H_{n,j}) = \zeta^{-i}$. □

Now we apply Proposition 2.4.14 to rank-p^2 étale sub-$\mathcal{O}_B$-modules of $(\mathcal{A}^f, i)$.

Proposition 2.4.15. *Let $R_{t'} := \Lambda_I^0[[t']]$ and let $\rho' :$ $\mathrm{Spf}(R_{t'}) \to \mathfrak{X}_{r,I}$ be the morphism characterized by $(\rho')^*(\mathcal{A}, i, \psi_N) = (\mathcal{A}'^{,f}, i, \Psi'_N)$. Denote $\rho'^{,*}(\mathbb{W}_\kappa)(R_{t'})$ by $\mathbb{W}_\kappa(t')$. Then we have the following:*

(i) we can choose a basis $\{U'_{\kappa,h}\}_{h\geq 0}$ of $\mathbb{W}_\kappa(t')$ such that for any $f \in H^0(\mathcal{X}_{r,I}, \mathbb{W}_\kappa)$ such that $\rho'^{,*}(f) = \sum_{h\geq 0} F_h(t')U'_{\kappa,h}$, we have

$$\rho^*(U_p(f)) = \frac{1}{p}\sum_{i=1}^{p}\sum_{h\geq 0} F_h(\zeta^{-i}(1+t)^{1/p}-1)p^h U_{\kappa,h}.$$

(ii) for any $f \in H^0(\mathcal{X}_{r,I}, \mathbb{W}_\kappa)$, we have

$$pU_p(\nabla_\kappa(f)) = \nabla_\kappa \circ U_p(f).$$

(iii) for any $f \in H^0(\mathcal{X}_{r,I}, \mathfrak{w}_\kappa)$, if $\rho^*(f) = F(t)U_{\kappa,0}$, then

$$\rho^*(V_pU_p(f)) = \frac{1}{p}\sum_{i=0}^{p-1} F(\zeta^i(1+t)-1)U_{\kappa,0}.$$

Moreover, if $F(t) \in R_t$, then $\frac{1}{p}\sum_{i=0}^{p-1} F(\zeta^i(1+t)-1) \in R$.

Proof. (i)By Proposition 2.4.7 and Lemma 2.4.14, a similar argument as in the proof of Proposition 2.4.9 allows us to choose a basis $\{U'_{\kappa,h}\}_{h\geq 0}$ of $\mathbb{W}_\kappa(t')$ such that for each $h \geq 0$,

$$f_j^* F_j^*(U'_{\kappa,h}) = p^h U_{\kappa,h}.$$

By Lemma 2.3.25, we have

$$\rho^*(U_p(f)) = \frac{1}{p}\sum_{i=1}^{p} f_i^* F_i^*(\rho'(f)) = \frac{1}{p}\sum_{i=1}^{p} F_i^*(F(t'))p^h U_{\kappa,h}.$$

Since $F_i^*(F(t')) = F(\zeta^{-i}(1+t)^{1/p}-1)$, we are done.

(ii) Based on item (i), we can get the equality by a similar argument as in the proof of Proposition 2.4.10.

(*iii*) As twisting by the diamond operator does not affect Serre-Tate expansion, we may assume $\langle p\rangle^*\rho^*(f) = F(t)U_\kappa$. Note that $(\bar{A}^{(p)})' = \bar{A}/\bar{A}[p] \cong \langle p\rangle\bar{A}$, so applying (*i*) to $\langle p\rangle^*\rho^*$, we have

$$(\rho^{(p)})^*(U_p(f)) = \frac{1}{p}\sum_{i=1}^{p} F(\zeta^i(1+t_\sigma)^{1/p}-1)U_\kappa;$$

$$\begin{aligned}\rho^*(V_pU_p(f)) &= \frac{1}{p}\sum_{i=1}^{p} F(\zeta^i(1+(1+t)^p-1)^{1/p})U_\kappa\\ &= \frac{1}{p}\sum_{i=1}^{p} F(\zeta^i(1+t)-1)U_\kappa.\end{aligned}$$

If $F(t)\in R_t$, then we can view $F(t)$ as a Λ_I^0-valued measure on $\mathbb{Z}_p$, so by Proposition 1.7.6, we have

$$\frac{1}{p}\sum_{i=0}^{p-1} F(\zeta^i(1+t)-1)\in R_t.$$

□

Twists by finite characters

Let (n,r,I) be an adapted triple and ζ_n be a fixed primitive p^n-th root of unity.

Proposition 2.4.16. *Let $\chi:\ (\mathbb{Z}/p^n\mathbb{Z})^\times \to W_n^\times$ be a primitive character. If $f\in H^0(\mathfrak{X}_{r,I},\mathbb{W}_\kappa)$ such that $\rho^*(f)=\sum_{h\geq 0}F_h(t)U_{\kappa,h}$, then we have*

$$\rho^*(\theta^\chi(f)) = \frac{g(\chi^{-1})}{p^n}\sum_{h\geq 0}\sum_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\chi(j)F_h(\zeta_n^{-j}(1+t)-1)U_{\kappa+2\chi,h}.$$

Proof. Here we use the notations in Lemma 2.3.29. Recall that

$$\mathrm{Spf}(R_{2n,t}) := \mathrm{Spf}(R_t)\times_{\rho,\mathfrak{X}_{r+n,I}}\mathfrak{IG}_{2n,r+n,I} = \sqcup_{a\in(\mathbb{Z}/p^{2n}\mathbb{Z})^\times}\mathrm{Spf}(R_t)e_a,$$

where e_a's are index symbols such that $b*e_a = e_{ab}$ for any $b\in(\mathbb{Z}/p^{2n}\mathbb{Z})^\times$. Let $\mathcal{C}\subset\mathcal{A}^f$ be the level-n canonical subgroup, and $\{\mathcal{C}_i'\}_{i=1,\dots,p^n}$ be the rank-p^{2n} étale cyclic $\mathcal{O}_B$-submodules of $\mathcal{A}^f/\mathcal{C}[p^n]$ over (a certain extension of) R_t, indexed in the way such that $\mathcal{C}_j'$ is the pull-back of $H_{\rho_j}\subset\mathcal{A}'$ over $\mathfrak{IG}_{2n,r+n,I}$ to $\mathrm{Spf}(R_t)e_1$. This convention forces the pull-back of H_{ρ_j} to $\mathrm{Spf}(R_t)e_a$ to be $\mathcal{C}'_{a^2j}$.

For $a\in(\mathbb{Z}/p^{2n}\mathbb{Z})^\times$ and $1\leq j\leq p^n$, let $u_j:\ \mathrm{Spf}(R_t)e_a\to\mathrm{Spf}(R_t)$ be the morphism characterized by $u_j^*(\mathcal{A}^f)=(\mathcal{A}^f/\mathcal{C})/\mathcal{C}'_{a^2j}$ and let $f_{a^2j}:\ \mathcal{A}^f\to(\mathcal{A}^f/\mathcal{C})/\mathcal{C}'_{a^2j}$ be the quotient false isogeny. By Lemma 2.4.11 and Lemma 2.4.14, we have

$$u_j^*(F_h(t)) = F_h(\zeta_n^{-a^2j}(1+t)-1);\quad f_{a^2j}^*u_j^*(\omega)=p^n\omega;\quad f_{a^2j}^*u_j^*(\eta)=p^n\eta.$$

By Proposition 2.4.7, viewing e_a as the corresponding idempotent in $R_{2n,t}$, we have

$$\mathrm{Tr}_{R_{2n,t}/R_t}(\chi(j)\frac{f_{a^2j}^*u_j^*}{p^n}(F_h(t)U_{\kappa,h})e_a) = p^n\chi(j)F_h(\zeta_n^{-j}(1+t)-1)U_{\kappa+2\chi,h}.$$

Unwinding the definition of θ^χ, we find that

$$\rho^*(\theta^\chi(f)) = \frac{g(\chi^{-1})}{p^n}\sum_{h\geq 0}\sum_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\chi(j)F_h(\zeta_n^{-j}(1+t)-1)U_{\kappa+2\chi,h}.$$

□

Proposition 2.4.17. *Let $\zeta_1 = \zeta_n^{p^{n-1}}$ and let $\chi : (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times \to W_n^\times$ be any primitive character. Then for any $f \in H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa)$, we have*

$$U_p(\theta^\chi(f)) = 0, \quad \theta^\chi \nabla_\kappa(f) = \nabla_{\kappa+2\chi} \circ \theta^\chi(f).$$

If $U_p(f) = 0$, then $\theta^{\chi^{-1}}\theta^\chi(f) = f$.

Proof. By Proposition 2.4.5, it suffices to show the equalities in Serre-Tate local coordinates. Let ρ' and t' as defined in Proposition 2.4.15, and assume $(\rho')^*(f) = \sum_{h\geq 0} F_h(t')U'_{\kappa,h}$. We have

$$(\rho')^*(\theta^\chi(f)) = \frac{g(\chi^{-1})}{p^n} \sum_{h\geq 0} \sum_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j)F_h(\zeta^{-j}(1+t')-1)U'_{\kappa+2\chi,h};$$

$$\rho^*(U_p(\theta^\chi(f))) = \frac{g(\chi^{-1})}{p^{n+1}} \sum_{h\geq 0} \sum_{i=0}^{p-1} \sum_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j)F_h(\zeta_n^{-j}\zeta_1^i(1+t)-1)U_{\kappa+2\chi,h}.$$

To show $\rho^*(U_p(\theta^\chi(f))) = 0$, it suffices to show that for each $h \geq 0$,

$$\sum_{i=0}^{p-1} \sum_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j)F_h(\zeta_n^{-j}\zeta_1^i(1+t)-1) = 0.$$

For $r \geq 0$, if $p \nmid r$, $\sum_{i=0}^{p-1} \zeta_1^{ir} = 0$, while if $p \mid r$, $\sum_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j)\zeta_n^{-jr} = 0$, so we have

$$\sum_{i=0}^{p-1} \sum_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j)\zeta_n^{-jr}\zeta_1^{ir}(1+t)^r = (1+t)^r \sum_{i=0}^{p-1} \zeta_1^{ir} \sum_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j)\zeta_n^{-jr} = 0.$$

By the binomial expansion, we have

$$\sum_{i=0}^{p-1} \sum_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j)(\zeta_n^{-j}\zeta_1^i(1+t)-1)^r = 0, \ \forall r \geq 0.$$

Note that $F_h(t)$ is a power series for each $h \geq 0$, so we have $U_p(\theta^\chi(f)) = 0$.

If moreover $U_p(f)$=0, then by Proposition 1.7.6 and Proposition 2.4.15 (i), for each $h \geq 0$, F_h is the Amice transformation of an Λ_I^0-valued measure μ_{F_h} supported on $\mathbb{Z}_p^\times$. For $i \in \mathbb{Z}/p^n\mathbb{Z}$, let $\mathbf{1}_i$ be the characteristic function of $i + p^n\mathbb{Z}_p$ and let $G(\chi) := \sum_{i\in\mathbb{Z}/p^n\mathbb{Z}} \chi(i)\mathbf{1}_i$. It is easy to see that $G(\chi)G(\chi^{-1})$ is the characteristic function on $\mathbb{Z}_p^\times$. So by Proposition 1.7.4 and 1.7.6, we have

$$F_h = A_{G(\chi)G(\chi^{-1})\mu_{F_h}} = \frac{g(\chi)g(\chi^{-1})}{p^{2n}} \sum_{r,s\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \chi^{-1}(r)\chi(s)F_h(\zeta_n^{-r-s}(1+X)-1).$$

This implies $\rho^*(\theta^{\chi^{-1}}\theta^\chi(f)) = \rho^*(f)$, because we have

$$(\rho')^*(\theta^{\chi^{-1}}\theta^\chi(f)) = \frac{g(\chi)g(\chi^{-1})}{p^{2n}} \sum_{h\geq 0} \sum_{r,s\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(r)\chi^{-1}(s)F_h(\zeta_n^{-r-s}(1+t')-1)U'_{\kappa,h}.$$

Assume that $\rho^*(f) = \sum_{h\geq 0} F'_h(t)U_{\kappa,h}$. Since we have $\mu_{\kappa+2\chi} = \mu_\kappa$, both $\rho^*(\nabla_{\kappa+2\chi}\theta^\chi(f))$ and $\rho^*(\theta^\chi\nabla_\kappa(f))$ are equal to

$$\frac{g(\chi^{-1})}{p^n} \sum_{h\geq 0} \sum_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j)\zeta^{-j}(1+t)F'_h(\zeta_n^{-j}(1+t)-1)U_{\kappa+2\chi+2,h}$$

$$+\frac{g(\chi^{-1})}{p^n} \sum_{h\geq 0} \sum_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j)F'_h(\zeta_n^{-j}(1+t)-1)(\mu_{\kappa+2\chi}-h)U_{\kappa+2\chi+2,h+1}$$

□

2.4.3 Continuity of the Gauss-Manin connection

Recall that $\mathbb{W}'_\kappa(t) := \rho^*(\mathbb{W}'_\kappa)(R_t)$ and $\mathbb{W}(t) := \rho^*(\mathbb{W})(R_t)$. As in Definition 2.3.12, for each $N \geq 0$, ∇^N is the iteration

$$\mathbb{W}_\kappa \xrightarrow{\nabla_\kappa} \mathbb{W}_{\kappa+2} \xrightarrow{\nabla_{\kappa+2}} \mathbb{W}_{\kappa+4} \cdots \xrightarrow{\nabla_{\kappa+2N-2}} \mathbb{W}_{\kappa+2N}.$$

Proposition 2.4.18. *Assume that $\mu_\kappa \in p\Lambda_I^0$. For any integer $N \geq 1$ and $f \in H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa)$ such that $U_p(f) = 0$, we have*

$$(\nabla^{p-1} - \mathrm{Id})^{Np}(\rho^*(f)) \in p^N \mathbb{W}(t) \cap \mathbb{W}'_\kappa(t).$$

To show this proposition, we need several lemmas. In the following, we will freely apply the notation of Proposition 2.4.7.

Lemma 2.4.19. *For any $g(t) \in \tilde{R}_t$ and $N \geq 1$, we have*

$$\nabla^N(g(t)U_{\kappa,h}) = \sum_{j=0}^{N} a_{N,\kappa,h,j}\partial^{N-j}(g(t))U_{\kappa+2N,j+h}$$

with $a_{N,\kappa,h,j} = 1$ if $j = 0$ and for $1 \leq j \leq N$,

$$a_{N,\kappa,h,j} = \binom{N}{j}\prod_{i=0}^{j-1}(\mu_\kappa - h + N - 1 - i).$$

In particular,

(i) if $\mu_\kappa \in \Lambda_I^0$, then $a_{N,\kappa,h,j} \in \Lambda_I^0$ for all $0 \leq j \leq N$;

(ii) if $\mu_\kappa \in p\Lambda_I^0$, then $a_{p,\kappa,h,j} \in p\Lambda_I^0$ for $1 \leq j \leq p$.

Proof. Provided with Proposition 2.4.7 (*iii*), the induction argument in [AI17, Lemma 3.36] carries over verbatim. □

Lemma 2.4.20. *Let $U = U_{1,0}$ and $V = \frac{Y}{1+p^n Z}$. For any $a, r, s, j, h \geq 0$, we have*

$$U_{\kappa+a,h+j} = U^a U_{\kappa,h+j}, \quad \nabla^s(U^r) = \prod_{i=0}^{s-1}(r+i)U^{r+2s}V. \tag{2.4.5}$$

Moreover, we have the following:

(i) if i, r are positive integers such that $(p-1)p^{i-1} \mid r$, then $U^r - 1 \in p^i\mathbb{W}(t)$, i.e. $\frac{U^r-1}{p^i}$ is a well-defined element in $\mathbb{W}(t)$;

(ii) for any integer $s \geq 1$ and $r \in \mathbb{N}$, we can find polynomials $P_{r,s,i}(U) \in \mathbb{Z}_p[U]$ such that

$$\nabla^s(G_r(U)) = \sum_{i \geq \max\{r-s,0\}}^{r} P_{r,s,i}(U)G_i(U)V^s,$$

where $G_r(U) = \frac{(U^{2(p-1)}-1)^{pr}}{p^r}$. If $s \geq p$ or $p \mid r$, then we can choose $P_{r,s,i}(U)$ whose coefficients are divisible by p.

Proof. The equalities 2.4.5 follow directly from Proposition 2.4.7 and Lemma 2.4.19.

For (i), by the construction of U, it suffices to show that if $(p-1)p^{i-1} \mid r$, then

$$p^i \mid a^r - 1, \forall a \in \mathcal{T}^{\mathrm{ext}}(R_t)(:= \mathbb{Z}_p^\times(1+p^n R_t)).$$

Note that $(\mathbb{Z}/p^i\mathbb{Z})^\times$ has order $(p-1)p^{i-1}$, so if $a \in \mathbb{Z}_p^\times$, then $p^i \mid a^r - 1$. Note that for any $1 \leq j \leq r$, we have $p^i \mid \binom{r}{j} p^n a$, so $p^i \mid a^r - 1$ for $a \in 1 + p^n R_t$. For any $a \in \mathbb{Z}_p^\times(1+p^n R_t)$, write $a = a_1 a_2$ with $a_1 \in \mathbb{Z}_p^\times$ and $a_2 \in 1 + p^n R_t$. As $a^r - 1 = (a_1^r - 1)a_2^r + (a_2^r - 1)$, we have $p^i \mid (a^r - 1)$.

For (ii), let $F(U) := 2(p-1)(U^{2(p-1)} - 1)^{p-1} \in \mathbb{Z}[U]$. Then by Leibniz's rule, we have

$$\begin{aligned}
\nabla(G_r(U)) &= \frac{\nabla((U^{2(p-1)}-1)^{pr})}{p^r} \\
&= \frac{2(p-1)pr(U^{2(p-1)}-1)^{p(r-1)+p-1}U^{2p}V}{p^r} \\
&= rG_{r-1}(U)F(U)U^{2p}V.
\end{aligned}$$

Inductively, for any integer $u \geq 1$, we can find elements $P'_{r,u,i}(U) \in \mathbb{Z}[U]$ such that

$$\nabla^u(G_r(T)) = \sum_{i \geq \max\{r-u,0\}}^{r-1} P'_{r,u,i}(T) G_i(T) V^s.$$

Since $U^{2p} = (U^{2(p-1)} - 1)U^2 + U^2$, we have

$$\begin{aligned}
\nabla(G_r(U)) &= \frac{2(p-1)pr(U^{2(p-1)}-1)^{pr-1}U^{2p}V}{p^r} \\
&= 2(p-1)r\frac{p(U^{2(p-1)}-1)^{pr}U^2V + p(U^{2(p-1)}-1)^{pr-1}U^2V}{p^r} \\
&= 2p(p-1)rG_r(U)U^2V + 2(p-1)r\frac{p(U^{2(p-1)}-1)^{p(r-1)}(U^{2(p-1)}-1)^{p-1}U^2V}{p^r} \\
&= 2p(p-1)G_r(U)U^2V + rG_{r-1}(U)F(U)U^2V.
\end{aligned}$$

Let $S_{u,v} = \sum_{i \geq \max\{v-u,0\}}^{v} q\mathbb{Z}[U]G_i(U)V^u$. Note that

$$V^u V^{u'} = V^{u+u'}, \quad G_i(U)G_j(U) = G_{i+j}(U),$$

so $S_{u,v}S_{u',v'} \subset pS_{u+u',v+v'}$. The above calculation implies that

$$\nabla S_{u,v} \subset S_{u+1,v}, \quad p\nabla^s(G_r(U)) \in S_{s,r}$$

and if $p \mid r$, $\nabla^s(G_r(U)) \in S_{s,r}$ for any $s \in \mathbb{N}$. Since only terms of degree prime-to-p appear in $F(U)$, there exists $F'(U) \in \mathbb{Z}_p[U]$ such that $\nabla(F'(U)) = F(U)U^2V$. Therefore we have

$$\nabla(G_1(U)) - \nabla(F'(U)) \in S_{1,1}, \quad \nabla^s(G_1(U)) - \nabla^s(F'(U)) \in S_{s,1}.$$

Now assume $s \geq p$. We proceed by induction. The case $r = 0$ is obvious. For $r \geq 1$, assume the result holds for $r-1$. By Equation 2.4.5, we have that if $s \geq p$, then $\nabla^s(F'(U)) \in S_{s,0}$, which implies

$$\sum_{i=p}^{s} \binom{s}{i} \nabla^{s-i}(G_{r-1}(U))(\nabla^i(G_1(U)) - \nabla^i(F'(U))) \in S_{s,r}.$$

If $s-i<p$ and $1\le i\le p$, we have $p \mid \binom{s}{i}$, which implies

$$\binom{s}{i}\nabla^{s-i}(G_{r-1}(U))\nabla^{i}(F'(U)) \in S_{s,r}.$$

If $s-i\ge p$, then by induction hypothesis

$$\binom{s}{i}\nabla^{s-i}(G_{r-1}(U))\nabla^{i}(F'(U)) \in S_{s,r}.$$

Combining these together, we have

$$\sum_{i=1}^{s}\binom{s}{i}\nabla^{s-i}(G_{r-1}(U))\nabla^{i}(F'(U)) \in S_{s,r}.$$

Note that for any $s\ge 1$, $\nabla^s(G_1(U))-\nabla^s(F'(U))\in S_{s,1}$, so

$$\sum_{i=1}^{s}\binom{s}{i}\nabla^{s-i}(G_{r-1}(U))(\nabla^{i}(G_1(U))-\nabla^{i}(F'(U))) \in S_{s,r}.$$

By induction hypothesis again, we have $\nabla^s(G_{r-1}(U))G_1(U)\in S_{s,r}$. By Leibniz's rule, we have

$$\begin{aligned}\nabla^s(G_r(U)) &= \sum_{i=0}^{s}\binom{s}{i}\nabla^{s-i}(G_{r-1}(U))\nabla^{i}(G_1(U)),\\ &= \sum_{i=1}^{s}\binom{s}{i}\nabla^{s-i}(G_{r-1}(U))(\nabla^{i}(G_1(U))-\nabla^{i}(F'(U)))\\ &\quad +\sum_{i=1}^{s}\binom{s}{i}\nabla^{s-i}(G_{r-1}(U))\nabla^{i}(F'(U))\\ &\quad +\nabla^s(G_{r-1}(U))G_1(U),\end{aligned}$$

which implies that $\nabla^s(G_r(U))\in S_{s,r}$. □

For simplicity, we will use the notation $g\in R_t^{U_p}$ to mean that $\sum_{\zeta^p=1} g(\zeta(1+t)-1)=0$.

Lemma 2.4.21. *For any $r,n\in\mathbb{N}$ such that $(p-1)p^{n-1}\mid r$ and for any $f(t)\in {R_t}^{U_p}$, we have*

$$\partial^r(f(t))\equiv f(t) \bmod (p^n).$$

Proof. Assume that f is the Amice transformation A_{μ_f} of μ_f. By Proposition 1.7.6, we have that μ_f is supported on $\mathbb{Z}_p^\times$. By Proposition 1.7.4, we have that $\partial^r(f(t))=A_{x^r\mu_f}$. Note that for any $x\in\mathbb{Z}_p^\times$, $x^r\equiv 1 \bmod p^i$, so there exists a continuous function $a(x)$ on $\mathbb{Z}_p^\times$ such that $x^r\mu_f=\mu_f+p^ia(x)\mu_f$, which implies

$$\partial^r(f)-f=A_{x^r\mu_f}-A_{\mu_f}=A_{p^ia(x)}=p^iA_{a(x)}\equiv 0 \mod p^n.$$

□

Proof of Proposition 2.4.18. Clearly we have $(\nabla^{p-1}-\mathrm{Id})^{Np}(\rho^*(f))\in\mathbb{W}'_\kappa(t)$. It suffices to show that $(\nabla^{p-1}-\mathrm{Id})^{Np}(\rho^*(f))\in p^N\mathbb{W}(t)$. By Proposition 2.4.15 (i), $U_p(f)=0$ implies that $\rho^*(f)=\sum_{n\ge0}F_hU_{\kappa,h}$ with $F_h\in R_t^{U_p}$. To show the proposition, it suffices to use the argument in the proof of [AI17, Proposition 3.37] to show that for any $f(t) \in R_t^{U_p}$, there exists polynomials $g_{j,r,N} \in R_t^{U_p}[U]$ such that $p^{N-2j-r} g_{j,r,N} \in R_t^{U_p}[U]$ and

$$(\nabla^{p-1} - \mathrm{Id})^{Np}(f(t)U_{\kappa,h}) = p^N \sum_{j=0}^{(p-1)Np} \sum_{r=0}^{N} p^{N-2j-r} G_r(U) g_{j,r,N} U_{\kappa,h+j}.$$

The case $N = 0$ is obvious. Assume $g_{j,r,N}$ is chosen for N. Then by Leibniz's rule, we have

$$\begin{aligned}
&(\nabla^{p-1} - \mathrm{Id})^p (g_{j,r,N} G_r(U) U_{\kappa,h+j}) \\
&= \sum_{s=0}^{p} (-1)^{p-s} \binom{p}{s} \sum_{x=0}^{(p-1)s} \binom{(p-1)s}{x} \nabla^x (G_r(U)) \nabla^{(p-1)s-x} (g_{j,r,N} U_{\kappa,h+j}) \\
&= \sum_{s=1}^{p} (-1)^{p-s} \binom{p}{s} \sum_{x=1}^{(p-1)s} \binom{(p-1)s}{x} \nabla^x (G_r(U)) \nabla^{(p-1)s-x} (g_{j,r,N} U_{\kappa,h+j}) \\
&+ G_r(U) (\nabla^{p-1} - \mathrm{Id})^p (g_{j,r,N} U_{\kappa,h+j}).
\end{aligned}$$

By Lemma 2.4.19 and Leibniz's rule, for any $g(t) \in R_t^{U_p}$, we have

$$\begin{aligned}
&(\nabla^{p-1} - \mathrm{Id})^p (g(t) U_{\kappa,h}) = \sum_{s=0}^{p} (-1)^{p-s} \binom{p}{s} \nabla^{(p-1)s} (g(t) U_{\kappa,h}) \\
&= \sum_{s=0}^{p} (-1)^{p-s} \binom{p}{s} \sum_{x=0}^{(p-1)s} a_{(p-1)s,\kappa,h,x} \partial^{(p-1)s-x} (g(t)) U_{\kappa,h+x} U^{2(p-1)s} \\
&= \sum_{s=1}^{p} (-1)^{p-s} \binom{p}{s} \sum_{x=1}^{(p-1)s} a_{(p-1)s,\kappa,h,x} \partial^{(p-1)s-x} (g(t)) U_{\kappa,h+x} U^{2(p-1)s} \\
&+ \sum_{s=0}^{p} (-1)^{p-s} \binom{p}{s} (\partial^{(p-1)s} g(t) - g(t)) U_{\kappa,h} U^{2(p-1)s} + p G_1(U) g(t) U_{\kappa,h}
\end{aligned}$$

By Lemma 2.4.19, it is easy to deduce $p \mid \binom{p}{s} a_{(p-1)s,\kappa,h,x}$. By Lemma 2.4.21, we can verify $\partial^k g(t) \in R_t^{U_p}$ for any $k \in \mathbb{N}$, and $p^2 \mid \binom{p}{s} (\partial^{(p-1)s} g(t) - g(t))$ for $0 \leq s \leq p$. Combining these, we have

$$(\nabla^{p-1} - \mathrm{Id})^p (g(t) U_{\kappa,h}) = \sum_{x=1}^{(p-1)s} g_x(t) U_{\kappa,h+x} U^{2(p-1)s} + p g_0(t) U_{\kappa,h} + p G_1(U) g(t) U_{\kappa,h}$$

such that if $p^y \mid g(t)$, then $p^{y+1} \mid g_x(t)$ for $0 \leq x \leq (p-1)s$. By assumption, we can write $g_{j,r,N} = \sum_y g_y U^y$ with $g_y \in R_t^{U_p}$ such that $p^{N-2j-r} g_y \in R_t^{U_p}$. The above argument shows that $p^N p^{N-2j-r} G_r(U) (\nabla^{p-1} - \mathrm{Id})^p (g_{j,r,N})$ can be written as

$$p^{N+1} (p^{N+1-2j-(r+1)} g_{j,r+1,N} G_{r+1}(U) + \sum_{x=0}^{(p-1)p} p^{N+1-2(j+x)-r} g_{j+x,r,N+1} G_r(U))$$

where

$$\begin{aligned}
&g_{j+x,r,N+1} \in R_t^{U_p}, \quad p^{N+1-2(j+x)-r} g_{j+x,r,N+1} \in R_t^{U_p}, \ \forall\, 0 \leq x \leq (p-1)p; \\
&g_{j,r+1,N+1} \in R_t^{U_p}, \quad p^{(N+1)-2j-(r+1)} g_{j,r+1,N+1} \in R_t^{U_p}.
\end{aligned}$$

For any $1 \le s \le p$ and $1 \le x \le (p-1)s$, by Lemma 2.4.19 and Lemma 2.4.20 (ii), we have

$$\begin{aligned}
&\nabla^x(G_r(U))\nabla^{(p-1)s-x}(g_{j,r,N}U_{\kappa,h+j})\\
&= (\sum_{z\ge\max\{r-x,0\}}^{r} P_{r,x,z}(U)G_z(U)V^x)(\sum_{w=0}^{(p-1)s-x} f_w(U)U_{\kappa,h+j+w})\\
&= \sum_{z,w} P_{r,x,z}(U)f_w(U)G_z(U)U_{\kappa,h+j+w+x}
\end{aligned}$$

where $P_{r,x,z}(U) \in Z_p[U]$ and $f_w(U) \in R_t^{U_p}[U]$. Moreover, $p^{N-2j-r}f_w(U) \in R_t^{U_p}[U]$ and when $x \ge p$, the coefficients of $P_{r,x,z}$ are divisible by p. It is easy to deduce

$$(-1)^{p-s}\binom{p}{s}\binom{(p-1)s}{x}P_{r,x,z}(U) = pP'_{r,x,z}[U]$$

for some $P'_{r,x,z}(U) \in \mathbb{Z}_p[U]$. Note that when $x \ge 1$, $w \ge 0$, $z \ge r-x$, we have

$$N-2j-r = (N+1)-2(j+w+x)-z+C_{w,z}$$

for some $C_{w,z} \ge 0$. So for each $1 \le s \le p$, $1 \le x \le (p-1)s$, we have

$$\begin{aligned}
&p^N p^{N-2j-r}(-1)^{p-s}\binom{p}{s}\binom{(p-1)s}{x}\nabla^x(G_r(U))\nabla^{(p-1)s-x}(g_{j,r,N}U_{\kappa,h+j})\\
&= p^N p^{N-2j-r}\sum_{z\ge\max\{r-x,0\}}^{r}\sum_{w=0}^{(p-1)s-x}(-1)^{p-s}\binom{p}{s}\binom{(p-1)s}{x}P_{r,x,z}(U)f_w(U)G_z(U)U_{\kappa,h+j+w+x}\\
&= P^{N+1}p^{N+1-2(j+w+x)-z}\sum_{z\ge\max\{r-x,0\}}^{r}\sum_{w=0}^{(p-1)s-x}P'_{r,x,z}(U)p^{C_{w,z}}G_z(U)f(U)U_{\kappa,h+j+w+x}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&p^N(\nabla^{p-1}-\mathrm{Id})^p(p^{N-2j-r}g_{j,r,N}G_r(U)U_{\kappa,h+j})\\
&= p^{N+1}\sum_{j=0}^{(p-1)p}\sum_{y=0}^{r+1}p^{N+1-2(j+z)-y}f_{j+x,y,N+1}G_z(U)U_{\kappa,h+j+x}
\end{aligned}$$

where $f_{j+x,y,N+1} \in R_t^{U_p}[U]$ and $p^{N+1-2(j+z)-y}f_{j+x,y,N+1}$ for each x, y. Taking summation, we obtain Proposition 2.4.18. □

2.5 p-adic iteration of the Gauss-Manin connection

Recall that $q = p$ if $p \ge 3$ and $q = 4$ if $p = 2$. Bearing the Serre-Tate local calculations in mind, we have (for the corresponding statement for modular curve, see Theorem A.4.1)

Theorem 2.5.1. *Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$. Fix an interval $I_\theta \subset [0,\infty)$ and an interval $I = [0,1]$ or $[p^a, p^b]$ for $a, b \in \mathbb{N}$, and assume that*

$$k:\ \mathbb{Z}_p^\times \to \Lambda_I; \quad \theta:\ \mathbb{Z}_p^\times \to \Lambda_{I_\theta,K}^\times, \quad \Lambda_{I_\theta,K} := \Lambda_{I_\theta}\otimes_{\mathbb{Z}_p}\mathcal{O}_K$$

are weights satisfying

(i) *there exists* $c(k) \in \mathbb{N}$ *such that* $\mu_k + 2c(k) \in p\Lambda_I^0$*;*

(ii) *for any* $t \in \mathbb{Z}_p^\times$, $\theta(t) = \theta'(t)t^{c(\theta)}\chi(t)$ *for a finite character* $\chi : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$*, a non-negative integer* $c(\theta) \geq c(k)$ *and a weight* $\theta' : \mathbb{Z}_p^\times \to (\Lambda^0_{I_\theta,K})^\times$ *such that*

$$\mu_{\theta'} \in q\Lambda^0_{I_\theta,K}, \quad \theta'(t) = \exp(\mu_\theta \log(t)) \ \forall \ t \in \mathbb{Z}_p^\times.$$

Then we have positive integers $r' \geq r$ *and* γ *(depending on* n*,* r *and* p*,* $c(\theta)$*) such that for any* $g \in H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa)^{U_p=0}$*,*

(a) *the sequence* $A(g,\theta')_m := \sum_{j=1}^m \frac{(-1)^{j-1}}{j}(\nabla^{(p-1)}_{k+2c(k)} - \mathrm{Id})^j \nabla_k^{c(k)}(g)$ *and the sequence*

$$B(g,\theta')_m := \sum_{i=0}^m \frac{1}{i!}\frac{\mu^i_{\theta'}}{(p-1)^i}\Bigg(\sum_{j_1+\ldots+j_i\leq m}(\prod_{a=1}^i \frac{(-1)^{j_a-1}}{j_a})(\nabla^{(p-1)}_{k+2c(k)} - \mathrm{Id})^{j_1+\ldots+j_a}\Bigg)\nabla_k^{c(k)}(g)$$

converge in $\mathrm{Hdg}^{-\gamma}H^0(\mathfrak{X}_{r',I} \otimes_{\mathbb{Z}_p} \Lambda_{I_\theta,K}, \mathbb{W})$*;*

(b) *the limit*

$$\nabla^{\theta'}_{k+2c(k)}\nabla_k^{c(k)}(g) := \exp(\frac{\mu_{\theta'}}{p-1}\log(\nabla^{p-1}_{k+2c(k)}))\nabla_k^{c(k)}(g) := \lim_{m\to\infty} B(g,\theta')_m$$

belongs to $\mathrm{Hdg}^{-\gamma}H^0(\mathfrak{X}_{r',I} \otimes_{\mathbb{Z}_p} \Lambda_{I_\theta,K}, \mathbb{W}_{k+2\theta'})$*;*

(c) *the element*

$$\nabla_k^\theta(g) := \theta^\chi \nabla^{c(\theta)-c(k)}_{k+2c(k)+2\theta'}\nabla^{\theta'}_{k+2c(k)}\nabla_k^{c(k)}(g)$$

is well-defined in $H^0(\mathcal{X}_{r',I} \otimes_{\mathbb{Z}_p} \Lambda_{I_\theta,K}, \mathbb{W}_{k+2\theta})$*.*

Before proving the theorem, we want to point out that the element $\nabla^\theta_\kappa g$ gives what we want. For any specialization $x \in \mathbb{Z}$ of k, and $y \in \mathbb{Z}$ of θ, and any $f \in H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa)^{U_p}$, let us denote the specialization of f at x by f_x, and the specialization of $\nabla^\theta_\kappa(f)$ at the weight $x+2y$ by $\nabla^\theta_\kappa(f)_{x,y}$.

Proposition 2.5.2. *Notations as in Theorem 2.5.1. Let* $x \in \mathbb{Z}$ *be any specialization of* k*,* $y' \in (p-1)\mathbb{N}$ *be a specialization of* $\chi\theta'$*, and let* $y := y' + c(\theta)$ *be the corresponding specialization of* θ*. Then for any* $f \in H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa)^{U_p}$*, we have*

$$\nabla_k^\theta(f)_{x,y} = \nabla_x^y(f_x).$$

Proof. By Proposition 1.3.6, the Gauss-Manin connection is functorial, so finite iterations of Gauss-Manin connections are functorial. By definition, twist by finite characters are functorial. Moreover, by Proposition 2.4.17, twist by finite character commutes with finite iteration of the Gauss-Manin connection. Note that $q \mid \mu_{\theta'}$ implies $q \mid y'$, so the specializations y' kills χ. From these observation, we are reduced to the case $c(\theta) = c(k) = 0$, $\chi = \mathrm{Id}$, $\theta = \theta'$ and $y \in (p-1)\mathbb{N}$. In this case, we have

$$\nabla_k^\theta(f) = \lim_m B(f,\theta)_m = \sum_{i=0}^m \frac{1}{i!}\frac{\mu^i_\theta}{(p-1)^i}\Bigg(\sum_{j_1+\ldots+j_i\leq m}(\prod_{a=1}^i \frac{(-1)^{j_a-1}}{j_a})(\nabla_k^{(p-1)} - \mathrm{Id})^{j_1+\ldots+j_a}\Bigg)(f).$$

The specialization of each $B(f,\theta)_m$ at (x,y) is

$$B(f_x,y)_m = \sum_{i=0}^m \frac{1}{i!}\frac{y^i}{(p-1)^i}\Bigg(\sum_{j_1+\ldots+j_i\leq m}(\prod_{a=1}^i \frac{(-1)^{j_a-1}}{j_a})(\nabla_x^{(p-1)} - \mathrm{Id})^{j_1+\ldots+j_a}\Bigg)(f).$$

Using Serre-Tate coordinates, it is easy to show $\lim_m B(f_x,y) = \nabla_x^y f_x$. As specialization is continuous with respect to the p-adic topology, we are done. ∎

The rest of this section is devoted to the proof of Theorem 2.5.1.

Proposition 2.5.3. *If $\mu_\kappa \in p\Lambda_I^0$, then for any integer $N \geq 1$ and $g \in H^0(\mathfrak{X}_{r,I}^{\mathrm{ord}}, \mathbb{W}_\kappa)^{U_p}$, we have*

$$(\nabla^{p-1} - \mathrm{Id})^{Np}(g) \subset p^N H^0(\mathfrak{X}_{r,I}^{\mathrm{ord}}, \mathbb{W}) \cap H^0(\mathfrak{X}_{r,I}^{\mathrm{ord}}, \mathbb{W}'_\kappa)$$

Proof. This follows directly from Lemma 2.4.5 and Proposition 2.4.18. □

Adapting the idea of [AI17, Proposition 4.4], we have

Proposition 2.5.4. *Let (n, r, I) be an adapted triple and s be a non-negative integer. There exists a positive integer $r' \geq r$ (depending on n, r, p and s) such that: for any $f \in \mathrm{Hdg}^{-s} H^0(\mathfrak{X}_{r,I}, \mathbb{W})$ such that $f|_{\mathfrak{X}_I^{\mathrm{ord}}} \in H^0(\mathfrak{X}_{r,I}^{\mathrm{ord}}, \alpha^j \mathbb{W})$, we have $f \in H^0(\mathfrak{X}_{r',I}, \alpha^{[j/2]}\mathbb{W})$.*

Proof. Let $\mathrm{Spf}(R) \subset \mathfrak{X}_{r,I}$ be an affine open, $\mathrm{Spf}(R^{\mathrm{ord}}) := \mathrm{Spf}(R) \times_{\mathfrak{X}_{r,I}} \mathfrak{X}_I^{\mathrm{ord}}$, and

$$\mathrm{Spf}(R_n) := \mathrm{Spf}(R) \times_{\mathfrak{X}_{r,I}} \mathfrak{IG}_{n,r,I}, \quad \mathrm{Spf}(R_n^{\mathrm{ord}}) := \mathrm{Spf}(R^{\mathrm{ord}}) \times_{\mathfrak{X}_I^{\mathrm{ord}}} \mathfrak{IG}_{n,r,I}^{\mathrm{ord}}.$$

Upon shrinking $\mathrm{Spf}(R)$, by Lemma 2.3.3, we have

$$\mathbb{W}^0(R) = R_n\langle Z, Y\rangle \to \mathbb{W}^0(R^{\mathrm{ord}}) = R_n^{\mathrm{ord}}\langle Z, Y\rangle.$$

Let $i = 1$ if $p \geq 3$ and $i = 2$ if $p \geq 2$. By the definition of $\mathfrak{w}^{\kappa_f}$, we have the following commutative diagram with injective vertical arrows

$$\begin{array}{ccc} \mathfrak{w}^{\kappa_f}(R) & \longrightarrow & \mathfrak{w}^{\kappa_f}(R^{\mathrm{ord}}) \\ \downarrow & & \downarrow \\ R_i \otimes_{\Lambda_I^0} \Lambda_I & \longrightarrow & R_i^{\mathrm{ord}} \otimes_{\Lambda_I^0} \Lambda_I. \end{array}$$

Note that by Proposition 2.2.19, $\underline{\delta}^{jp^{r+1}(p-1)+p^n-p}$ kills the kernel of $R_n/\alpha^j \to R_n^{\mathrm{ord}}/\alpha^j$, so the kernel of $\mathbb{W}^0(R)/\alpha^j\mathbb{W}^0(R) \to \mathbb{W}^0(R^{\mathrm{ord}})/\alpha^j\mathbb{W}^0(R^{\mathrm{ord}})$ is killed by $\underline{\delta}^{jp^{r+1}(p-1)+p^n-p}$ and $\mathfrak{w}^{\kappa_f}(R) \to \mathfrak{w}^{\kappa_f}(R^{\mathrm{ord}})$ is injective. Combining these, we have the kernel of

$$\mathbb{W}(R)/\alpha^j\mathbb{W}(R) \to \mathbb{W}(R^{\mathrm{ord}})/\alpha^j\mathbb{W}(R^{\mathrm{ord}})$$

is killed by $\underline{\delta}^{jp^{r+1}(p-1)+p^n-p}$, which implies

$$\underline{\delta}^{jp^{r+1}(p-1)+p^n-p+(p-1)s} f \subset \alpha^j\mathbb{W}.$$

Write $s = s_0 j + s_1$ for $s_0, s_1 \in \mathbb{N}$, and take any $r' \geq r$ such that

$$(p-1)p^{r'+1} \geq 2(p^{r+1}(p-1) + p^n - p + (p-1)(s_0 + s_1)).$$

On $\mathfrak{X}_{r',I}$, we have

$$\alpha \in \underline{\delta}^{(p-1)p^{r'+1}} \subset \underline{\delta}^{2(p^{r+1}(p-1)+p^n-p+(p-1)(s_0+s_1))},$$

which implies $f \in H^0(\mathfrak{X}_{r',I}, \alpha^{[j/2]}\mathbb{W})$. □

Recall that we have a constant $c(n)$ such that $\nabla(\mathbb{W}'_\kappa) \subset \frac{1}{\mathrm{Hdg}^{c(n)}}\mathbb{W}'_\kappa$.

Corollary 2.5.5. *If $\mu_\kappa \in p\Lambda_I^0$, then there exist positive integers $r' \geq r$, γ depending on n,r,p, and there exists a constant c depending on I such that for every integer $N \geq 1$ and every $f \in H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa)^{U_p=0}$, we have*

$$\mathrm{Hdg}^{c(n)(p-1)^2}(\nabla^{p-1} - \mathrm{Id})^N(f) \in p^{[N/c]}H^0(\mathfrak{X}_{r',I}, \mathbb{W}) \cap H^0(\mathfrak{X}_{r',I}, \mathbb{W}'_\kappa),$$

and for positive integers h and j_1,...,j_h such that $N = j_1 + j_2 ... + j_h$, we have

$$\mathrm{Hdg}^\gamma \frac{q^h}{h!}(\prod_{i=1}^h \frac{(\nabla^{p-1} - \mathrm{Id})^{j_i}}{j_i})(f) \in p^{[N/cp]}H^0(\mathfrak{X}_{r',I}, \mathbb{W}) \cap H^0(\mathfrak{X}_{r',I}, \mathbb{W}'_\kappa).$$

Proof. Write $N = p[N/p] + N_0$. We have

$$(\nabla^{(p-1)} - \mathrm{Id})^N = (\nabla^{(p-1)} - \mathrm{Id})^{p[N/p]}(\nabla^{(p-1)} - \mathrm{Id})^{N_0}.$$

Thanks to Proposition 2.4.15, we have $U_p((\nabla^{(p-1)} - \mathrm{Id})^{N_0} f) = 0$, so by Proposition 2.5.3, we have

$$(\nabla^{(p-1)} - \mathrm{Id})^N(f) \in H^0(\mathfrak{X}_{r,I}^{\mathrm{ord}}, p^{[N/p]}\mathbb{W}).$$

Note that for any positive integer j, we have

$$(p-1)j + \frac{p^2}{p-1} \geq p^2 \mathrm{ord}_p(j); \quad \mathrm{ord}_p(\frac{q^j}{j!}) \geq \frac{j}{p-1},$$

so for any $h \geq 1$ and positive integers $j_1, .., j_h$ such that $\sum_{i=1}^h j_i = N$, we have

$$[N/p] - \sum_{i=1}^h v_p(j_i) + v_p(\frac{q^h}{h!}) \geq [N/p^2].$$

By Proposition 2.5.3, we have

$$\frac{q^h}{h!}(\prod_{i=1}^h \frac{(\nabla^{p-1} - \mathrm{Id})^{j_i}}{j_i})(f) \in H^0(\mathfrak{X}_I^{\mathrm{ord}}, [N/p^2]\mathbb{W}).$$

Note that $\mathrm{Hdg}^{N(p-1)c(n)}(\nabla^{(p-1)} - \mathrm{Id})^N : \mathbb{W}'_\kappa \to \mathbb{W}'_\kappa$ is well-defined and that

$$N(p-1)c(n) \leq (p-1)p[N/p]c(n) + c(n)(p-1)^2 \leq (p-1)p^2[N/p^2]c(n) + c(n)p(p-1),$$

so by proposition 2.5.4 and the fact $\alpha \mid p$, there exists positive integer $r' \geq r$ such that

$$\begin{aligned} \mathrm{Hdg}^{c(n)(p-1)^2}(\nabla^{(p-1)} - \mathrm{Id})^N(g) &\subset H^0(\mathfrak{X}_{r',I}, \alpha^{[N/2p]}\mathbb{W}) \cap H^0(\mathfrak{X}_{r',I}, \mathbb{W}'_\kappa), \\ \mathrm{Hdg}^{c(n)p(p-1)}\frac{q^h}{h!}(\prod_{i=1}^h \frac{(\nabla^{p-1} - \mathrm{Id})^{j_i}}{j_i})(f) &\subset H^0(\mathfrak{X}_{r',I}, \alpha^{[N/2p^2]}\mathbb{W}) \cap H^0(\mathfrak{X}_{r',I}, \mathbb{W}'_\kappa). \end{aligned} \tag{2.5.1}$$

Note that if $I = [0, 1]$, $\alpha = p$; and if $I = [p^a, p^b]$, $\alpha = T$ and $T^{p^a} \mid p \mid T^{p^b}$, so by choosing

$$c = \begin{cases} 2p \text{ if } I = [0, 1]; \\ 2p^{1+b-a} \text{ if } I = [p^a, p^b] \end{cases}, \quad \gamma = c(n)p(p-1),$$

we get the desired result from the inclusions 2.5.1. $\square$

Now we are ready to prove Theorem 2.5.1.

Proof of Theorem 2.5.1. By Proposition 2.4.15, we have

$$\mathrm{Hdg}^{c(n)c(k)}\nabla_k^{c(k)}(g) \in H^0(\mathfrak{X}_{r,I} \otimes_{\mathbb{Z}_p} \mathcal{O}_K, \mathbb{W}_{k+2c(k)})^{U_p}.$$

Thus by Corollary 2.5.5, there exist constants γ and r' depending on n, r, p and $c(\kappa)$ such that

$$A(g,\theta')_m, B(g,\theta')_m \in \mathrm{Hdg}^{-\gamma}H^0(\mathfrak{X}_{r',I}, \mathbb{W}),$$

and $\{B(g,\theta')_m\}_m$ is a Cauchy sequence in $\mathrm{Hdg}^{-\gamma}H^0(\mathfrak{X}_{r',I} \otimes_{\mathbb{Z}_p} \Lambda_{I_\theta,K}, \mathbb{W})$ with respect to the p-adic topology. Therefore $\nabla_{k+2c(k)}^{\theta'}\nabla_k^{c(k)}(g)$ is well-defined in $\mathrm{Hdg}^{-\gamma}H^0(\mathfrak{X}_{r',I} \otimes_{\mathbb{Z}_p} \Lambda_{I_\theta,K}, \mathbb{W})$. Moreover, it is easy to check that

$$t * \nabla_{k+2c(k)}^{\theta'}\nabla_k^{c(k)}(g) = (k+2c(k)+2\theta')(t)\nabla_{k+2c(k)}^{\theta'}\nabla_k^{c(k)}(g), \quad \forall\, t \in \mathbb{Z}_p^\times.$$

Note that by Lemma 2.3.3, the action of $\mathcal{T}^{\mathrm{ext}}$ on $\mathbb{W}$ is analytic, so we have

$$\nabla_{k+2c(k)}^{\theta'}\nabla_k^{c(k)}(g) \in \mathrm{Hdg}^{-\gamma}H^0(\mathfrak{X}_{r',I} \otimes_{\mathbb{Z}_p} \Lambda_{I_\theta,K}, \mathbb{W}_{k+2c(k)+2\theta'})$$

According to Corollary 2.3.30, upon enlarging γ, we have

$$\nabla_k^{\theta}(g) \in \mathrm{Hdg}^{-\gamma}H^0(\mathfrak{X}_{r',I} \otimes_{\mathbb{Z}_p} \Lambda_{I_\theta,K}, \mathbb{W}_{k+2\theta}).$$

□

2.6 Possible applications

We want to briefly mention several possible applications of our results.

The de Rham Eichler-Shimura isomorphisms

The classical Eichler-Shimura isomorphism relates modular forms with modular symbols. As p-adic modular forms and rigid analytic modular symbols can be put into p-adic families, one can establish a family version Eichler-Shimura isomorphism in the p-adic setting. In [AIS15], Andreatta-Iovita-Stevens developed a theory of overconvergent Eichler-Shimura isomorphisms for modular curves . Later, a generalization of such a theory to Shimura curves over $\mathbb{Q}$ was achieved by Salazar-Gao [GS17].

Let us provide more details about Salazar-Gao's result. Recall that $X := X_1^B(N)$ is the moduli space of false elliptic curves with a $V_1(N)$-level structure. Let $X(N,p)$ be the moduli space of false elliptic curves with a $V_1(N)$-level structure and an order p^2 $\mathcal{O}_B$-submodule. Choose a finite extension K of $\mathbb{Q}_p$ and denote the absolute Galois group of K by G_K. For large enough rational number ω such that $p^\omega \in K$, let $X(\omega)/K$ be the strict neighborhood of $X_{\mathbb{Q}_p}^{\mathrm{ad}}$ defined by $|\mathrm{H}| < p^\omega$. Then the existence of canonical subgroup allows us to identify $X(\omega)$ with a strict neighborhood of the connected component of $X(N,p)_{\mathbb{Q}_p}^{\mathrm{ad}}$. Let $U \subset \mathcal{W}^{\mathrm{rig}}$ be an wide open disk defined over L such that $X(\omega) \times U \subset \mathfrak{X}_{r,I}$ for some r and I. Let Λ_U be the algebra of bounded by 1 rigid analytic functions on U and $B_U := \Lambda_U \otimes_{\mathcal{O}_K} K$. Let $T_0 := \mathbb{Z}_p^\times \times \mathbb{Z}_p$ be the compact subset of $\mathbb{Z}_p^2$, endowed with a natural action of $\mathbb{Z}_p^\times$ and the Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$. Let D_U be the space of B_U-valued distribution on T_0, homogeneous of degree κ_U for the action of $\mathbb{Z}_p^\times$. Note that D_U has a natural action of the fundamental group Γ of $X(N,p)(\mathbb{C})$. Denote the push-out of $\mathfrak{w}^{\kappa_I}|_{X(\omega)\times U}$ to $X(\omega)$ by $\mathfrak{w}^{\kappa_U}$.

Theorem 2.6.1. [GS17, Theorem 0.1] *Assume $p > 3$. Let $h > 0$ and assume that U satisfies:*

- *There exists $k_0 \in U(K)$ such that $k_0 > h - 1$;*
- *Both $H^1(\Gamma, D_U)$ and $H^0(X(\omega), \mathfrak{w}^{\lambda_U+2})$ has slope $\leq h$ decomposition.*

Then there exists a finite set $Z \subset U(\mathbb{C}_p)$ such that for any wide open disk $V \subset U$ defined over K satisfying $V(\mathbb{C}_p) \cap Z = \emptyset$, there is a finite free $\Lambda_V[\frac{1}{p}]\hat{\otimes}_K\mathbb{C}_p$-module, $S_V^{\leq h}$, with the trivial semilinear action of G_K, and a Hecke and Galois equivariant exact sequence

$$\begin{aligned} 0 \to S_V^{\leq h}(\epsilon \cdot \kappa_V) \to H^1(\Gamma, D_V)^{\leq h}\hat{\otimes}_K\mathbb{C}_p \\ \to H^0(X(\omega), \mathfrak{w}^{\kappa_V+2})^{\leq h}\hat{\otimes}_K\mathbb{C}_p \to 0. \end{aligned}$$

Here ϵ is the p-adic cyclotomic character of and we view κ_V as a character of G_K via ϵ. Moreover, there exists $b \in B_V$ such that the above sequence localized at b splits canonically and uniquely as G_K-modules.

We want to mention that Chojecki-Hansen-Johansson [CHJ17] independently gave another proof of the above theorem using the pro-étale site and the perfectoid Shimura curves introduced by P. Scholze [Sch13][Sch15].

In the viewpoint of p-adic Hodge theory, the result of Salazar-Gao should be understood as a Hodge-Tate overconvergent Eichler-Shimura isomorphism. It is natural (also useful) to pursue de Rham and crystalline overconvergent Eichler-Shimura isomorphisms. Using the results of [AI17] (recorded also in the Appendix A below), Andreatta-Iovita manages to establish de Rham overconvergent Eichler-Shimura isomorphisms for modular curves. So the results of this thesis should allow us to establish de Rham Eichler-Shimura isomorphisms for Shimura curves over $\mathbb{Q}$.

Based on the recent progress on crystalline comparison theorems, in particular a crystalline nature refinement of the pro-étale site, crystalline Eichler-Shimura isomorphisms should also be accessible.

Explicit reciprocity law for diagonal classes

Let $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ be finite slope p-adic families of elliptic modular forms satisfying reasonable conditions. Then according to [Urb14] and [AI17], one can construct a triple product p-adic L-function $\mathcal{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})$ whose specializations at unbalanced classical points (k, l, m) interpolates the algebraic part of the special value of the complex triple product Garrett-Rankin L-function $L(f_k, g_l, h_m, s)$ at $s = \frac{k+l+m-2}{2}$. Under the extra assumption that $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ are ordinary, using the overconvegent Eichler-Shimura isomorphism, Bertolini-Seveso-Venerruci [BSV] associate a big representation $V(\mathbf{f}, \mathbf{g}, \mathbf{h})$ of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ to $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ (more precisely, the associated test vectors), whose specializations at unbalanced classical points (k, l, m) recover the Galois representation $V(f_k, g_l, h_m)$ associated to f_k, g_l, h_m constructed using the classical Eichler Shimura theory. They also construct a canonical class $\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h})$ in $H^1(\mathbb{Q}, V(\mathbf{f}, \mathbf{g}, \mathbf{h}))$ (actually, a related Selmer group) using the Gross-Kudla-Schoen diagonal cycles in the Kuga-Sato varieties fibred over the modular curve. Under several technical conditions including the Heegner hypothesis, Bertolini-Seveso-Venerucci showed the following explicit reciprocity law

Theorem 2.6.2. [BSV, Theorem A-B] *The image of $\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h})$ under Perrin-Riou's big logarithm coincides with the triple product p-adic L-function $\mathcal{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})$. Moreover, for any unbalanced* *classical point* (k,l,m)*, the specialization of* $\kappa(\mathbf{f},\mathbf{g},\mathbf{h})$ *at* (k,l,m) *is crystalline if and only if* $L(f_k, g_l, h_m, \frac{k+l+m-2}{2}) = 0$.

Using the results in the thesis, one can construct triple product p-adic L-functions $\mathcal{L}_p^B(\mathbf{f}', \mathbf{g}', \mathbf{h}')$ associated with finite slope families of quaternion modular forms $\mathbf{f}'$, $\mathbf{g}'$, $\mathbf{h}'$ in a manner similar to [AI17]. By the p-adic Jacquet-Langlands correspondence developed by Greenberg-Seveso [GS16], hopefully we can choose appropriate $\mathbf{f}^{\mathbf{B}}$, $\mathbf{g}^{\mathbf{B}}$, $\mathbf{h}^{\mathbf{B}}$ such that $\mathcal{L}_p^B(\mathbf{f}^{\mathbf{B}}, \mathbf{g}^{\mathbf{B}}, \mathbf{h}^{\mathbf{B}})$ and $\mathcal{L}_p(\mathbf{f},\mathbf{g},\mathbf{h})$ are the same up to explicit factors. Using the Eichler-Shimura theory for Shimura curves over $\mathbb{Q}$, hopefully we can generalize the method of Bertolini-Seveso-Venerucci to associate a $G_{\mathbb{Q}}$-representation $V(\mathbf{f}^{\mathbf{B}}, \mathbf{g}^{\mathbf{B}}, \mathbf{h}^{\mathbf{B}})$ to $\mathbf{f}^{\mathbf{B}}$, $\mathbf{g}^{\mathbf{B}}$, $\mathbf{h}^{\mathbf{B}}$ which satisfies similar interpolation properties. Then by considering the Kuga-Sato varieties fibred over the Shimura curve, we can construct the distinguished class $\kappa(\mathbf{f}^{\mathbf{B}}, \mathbf{g}^{\mathbf{B}}, \mathbf{h}^{\mathbf{B}})$ in $H^1(\mathbb{Q}, V(\mathbf{f}^{\mathbf{B}}, \mathbf{g}^{\mathbf{B}}, \mathbf{h}^{\mathbf{B}}))$. Finally using the theory of (ϕ,Γ)-modules, it might be possible to establish an explicit reciprocity law for the finite slope families $\mathbf{f}^{\mathbf{B}}$, $\mathbf{g}^{\mathbf{B}}$, $\mathbf{h}^{\mathbf{B}}$. Consequently, we may relax the Heegner hypothesis and the ordinary assumption in [BSV] and [DR14], [DR17].

It is also interesting to generalize the study of triple product p-adic L-functions to totally real number fields F. In this direction, Molina-Salazar [MS18] managed to construct triple product p-adic L-functions associated with Hida families when p splits completely in F.

Comparison with the perfectoid approach

Let K be an imaginary quadratic field. Let (f,χ,p) be a triple consisting of a cuspidal newform f (of level $\Gamma_1(N)$, weight $k \geq 2$), a Hecke operator χ of K of infinity type $(-k-j, j)$ adapted to f in a reasonable sense and a rational prime $p \geq 5$, inert or ramified in K. Then based on the p-adic iteration result of Gauss-Manin connection for modular forms in [AI17] (see also the Appendix A below), Andreatta-Iovita [AI] constructed a Katz-type p-adic L-function $L_p(f,\chi)$ using an elaborating analysis on the reduction of the supersingular CM points, and then obtained a p-adic Gross Zagier for $L_p(f,\chi)$. This result is complementary to the result of Bertolini-Darmon-Prasanna [BDP+13], where p is required to split completely.

With the help of the trivialization of the Hodge-Tate exact sequence over the perfectoid modular curve, and the p-adic de Rham comparison theorem, D. Kriz [Kri18] give a new splitting of (certain base change of) the Hodge-de Rham exact sequence and a new Maass-Shimura operator on overconvergent modular forms. Using the local coordinates, the fake Hasse invariants, provided by the Hodge-Tate priod map (and other nice properties of this map proven in [CS17]), he then showed that this Maass-Shimura operator iterates p-adically, and then obtained his approximate p-adic L-functions $L_p(f,\chi)$ from the above triple (f,χ,p). When$k=2$, Kriz also established a p-adic Gross-Zagier for $L_p(f,\chi)$.

These two approaches have their own advantages. Using the relation between the perfectoid modular curve and the inverse limit of the partial Igusa tower, hopefully we can compare them to get a deeper understanding of the whole picture. In particular, we may understand the non-automorphic nature, pointed out in [AI], of the euler factor at p of $L_p(f,\chi)$ better.

APPENDIX

A Nearly overconvergent modular forms

In this appendix, we briefly revise nearly overconvergent modular forms. We first recall the construction of the nearly overconvergent modular sheaf $\mathbb{W}_\kappa$ of general weight κ via applying the VBMS machinery to the universal elliptic curve over the (appropriately modified) modular curve.

Then we review the filtration of and the (meromorphic) Gauss-Manin connection on $\mathbb{W}_\kappa$ and discuss Hecke operators, in particular the U_p-operator, on $\mathbb{W}_\kappa$. These two steps largely follow [AI17, § 3].

Finally, we will study the local behavior of the Gauss-Manin connection and Hecke operator in Serre-Tate coordinates, and show that the Gauss-Manin connection iterates p-adically.

A.1 The modular curve and partial Igusa towers

Definition A.1.1. *Let S be any base scheme.*

- *A DR (Deligne-Rapoport) semistable genus-1 curve over S is a proper flat and finitely presented morphism $f: C \to S$ such that for each point $s \in S$, the fiber C_s is either a smooth and geometrically connected curve of genus 1 or becomes isomorphic to a standard polygon, i.e. a n-gon, over a finite extension of $k(s)$.*

- *A generalized elliptic curve is a triple $(E, +, e)$ consisting of a DR semistable genus-1 curve E/S with relative smooth locus E^{sm}, an S-morphism $+: E^{sm} \times_S E \to E$ and a section $e: S \to E^{sm}$ such that:*

  (i) the morphism $+$ induces a commutative group scheme structure on E^{sm} with identity e.

  (ii) on each n-gon geometric fiber, the translation action by each rational point in the smooth locus induces a rotation on the graph of irreducible components.

  It is known that for any DR semistable genus-1 curve E/S with all geometric fiber irreducible and any $e \in E^{sm}(S)$, there exists a unique $+: E^{sm} \times_S E \to E$ such that $(E, +, e)$ is a generalized elliptic curve.

(iii) let N be a positive integer. A $\Gamma_1(N)$-level structure on a generalized elliptic curve E/S is a group homomorphism $\psi_N : \mathbb{Z}/N\mathbb{Z} \to E^{sm}[N](S)$ such that (the closed subscheme induced by) the effective Cartier divisor $\sum_{a\in\mathbb{Z}/N\mathbb{Z}}[\psi_N(a)]$ is a group subscheme of $E^{sm}[N]$ which meets all irreducible components of all geometric fibers $E_{\bar{s}}$.

Note that for any generalized elliptic curve, the identity component $(E^{sm})^0$ of E^{sm} is a semi-abelian scheme.

Given any generalized elliptic curve E with a $\Gamma_1(N)$-level structure ψ_N and a closed subgroup $H \subset E^{sm}$ which intersects the image of ψ_N trivially, we define the quotient of (E, ψ_N) by H to be E/H (provided it is a generalized elliptic curve, for example H has trivial intersection with the identity components on non-smooth geometric fibers) equipped with the unique $\Gamma_1(N)$-level structure ψ'_N which fits in the commutative diagram

$$\begin{tikzcd} & \mathbb{Z}/N\mathbb{Z} \arrow[dl, "\psi_N"'] \arrow[dr, "\psi'_N"] & \\ E^{sm} \arrow[rr] & & E^{sm}/H. \end{tikzcd}$$

When the $\Gamma_1(N)$-level structure is clear (or irrelevant), we usually omit it in the notation.

According to [KM85, Chapter 8] (see also the introduction of [Con07]),

Theorem A.1.2. *We have*

(i) Let $N \geq 4$. The functor on $\mathbb{Z}[1/N]$-schemes which to S assigns the isomorphism classes of elliptic curves E/S with $\Gamma_1(N)$-structures is represented by an affine, normal, smooth scheme $Y_1(N)$ of pure relative dimension 1 over $\mathbb{Z}[1/N]$.

(ii) Let $N \geq 5$. The functor on $\mathbb{Z}[1/N]$-schemes which to S assigns the isomorphism classes of generalized elliptic curve E/S together with $\Gamma_1(N)$-structure is represented by a proper smooth scheme $X_1(N)$ of pure relative dimension 1 over $\mathbb{Z}[1/N]$. Moreover, $Y_1(N) \subset X_1(N)$ is an open subscheme.

(iii) For $N \geq 5$, the cuspidal locus $Z = X_1(N) - Y_1(N)$ is an étale relative Cartier divisor over $\mathbb{Z}[1/N]$.

One can describe the cuspidal locus explicitly using Tate curves.

Definition A.1.3. *The Tate elliptic curve* $\mathrm{Tate}(q) \subset P^2_{\mathbb{Z}((q))}$ *is defined by the inhomogeneous equation*

$$y^2 + xy = x^3 + B(q)x + C(q)$$

over $\mathbb{Z}((q))$ where

$$B(q) = -5\sum_{n\geq 1}\sigma_3(n)q^n,\ C(q) = \sum_{n\geq 1}\left(\frac{-5\sigma_3(n) - 7\sigma_5(n)}{12}\right)q^n,\ \sigma_k(n) := \sum_{1\leq d|n} d^k, \forall\ k \geq 1, n \geq 1.$$

Thanks to the paper [DR73](see also [Con07, § 2.4]),

Proposition A.1.4. *We have*

(i) the Tate curve $\mathrm{Tate}(q^n)/\mathbb{Z}((q))$ extends uniquely to a generalized elliptic curve $\mathrm{Tate}(q^n)$ over $\mathbb{Z}[[q]]$ with n-gon geometric fiber over $q = 0$. Moreover, over $\mathbb{Z}[[q]]$:

(a) there is a unique homomorphism $\mu_N \to \mathrm{Tate}(q)^{sm}$ lifting $\mu_N \to C_1^{sm}$.

(b) there is an explicit isomorphism of formal groups $\hat{G}_m \to \mathrm{Tate}(q)^{\wedge}_0$ which identifies $\frac{dt}{t}$ with the canonical differential form $\frac{dx}{2x+y}$ on $\mathrm{Tate}(q)$ over $\mathbb{Z}((q))$.

(ii) Over $\mathbb{Z}[1/N, \mu_N]$, the cuspidal locus is a disjoint union of sections, which is naturally identified with level $\Gamma_1(N)$-structures of $\mathrm{Tate}(q)$ over $\mathbb{Z}[[q^{1/N}]][1/N, \zeta_N]$ modulo automorphisms given by

$$q^{1/N} \mapsto \zeta_N^a q^{\pm\frac{1}{N}},$$

where ζ_N is a primitive N-th root of unity.

Let $\pi: \mathcal{E} \to X := X_1(N)$ be the universal generalized elliptic curve, $\pi: \mathcal{E}^{sm} \to X$ be the restriction to the smooth locus and $\tilde{Z} = \pi^{-1}(Z)$ be the inverse image of the cuspidal locus Z in $\mathcal{E}^{sm}$. Then (see for example in [BDP+13, Page 1044]) we have the logarithmic Hodge-de Rham sequence

$$0 \to \underline{\omega}_{\mathcal{E}} \to \mathbb{H}_{\mathcal{E}} \to \underline{\omega}_{\mathcal{E}}^{\vee} \to 0$$

and the logarithmic Gauss-Manin connection

$$\nabla: \mathbb{H}_{\mathcal{E}} \to \mathbb{H}_{\mathcal{E}} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{Z}[1/N]}(\log(Z))$$

where $\underline{\omega}_{\mathcal{E}} := \pi_* \Omega^1_{\mathcal{E}^{sm}/X}(\log(\tilde{Z}))$ and $\mathbb{H}_{\mathcal{E}} := R^1\pi_*\Omega^\bullet_{\mathcal{E}^{sm}/X}(\log(\tilde{Z}))$. By abuse of notation, we will sometimes denote the universal semi-abelian scheme $(\mathcal{E}^{sm})^0$ by $\mathcal{E}$.

A.1.1 Strict neighborhoods of the ordinary locus

In the rest of this chapter, $N \geq 5$ will be a fixed prime-to-p integer and prime to p, X (resp. Y) will be the base change of $X_1(N)$ (resp. $Y_1(N)$) to $\mathbb{Z}_p$, and $\hat{X}$ (resp. $\hat{Y}$) will be the formal completion of X (resp. Y) along its special fiber. Let $\mathcal{E} \to \hat{X}$ be the universal semi-abelian scheme with universal $\Gamma_1(N)$-level structure ψ_N; let $\mathrm{Ha} := \mathrm{Ha}(\mathcal{E}) \in H^0(\hat{X}, \underline{\omega}_{\mathcal{E}}^{p-1} \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$ be the unique extension of the Hasse invariant of the mod-p reduction of $\mathcal{E}_{|_{\hat{Y}}}$ (for more details, see [KM85, § 12.4]); and let $\mathrm{Hdg} := \mathrm{Hdg}(\mathcal{E}) \subset \mathcal{O}_{\hat{X}}$ be the Hodge ideal generated by the pre-image of $\mathrm{Ha} \cdot \underline{\omega}_{\mathcal{E}}^{\otimes(1-p)}$.

For $I = [0,1]$ or $[p^a, p^b]$ with $a, b \in \mathbb{N}$, we choose $\alpha = p$ if $0 \in I$ and $\alpha = T$ otherwise, and set $\mathfrak{X}_I := \hat{X} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spf}(\Lambda_I^0)$. Base change $\mathcal{E}$, $\underline{\omega} := \underline{\omega}_{\mathcal{E}}$, $\mathbb{H}_{\mathcal{E}}$ and Hdg to $\mathfrak{X}_I$.

Definition A.1.5. *We define the ordinary locus $\mathfrak{X}_I^{\mathrm{ord}}$ to be the formal scheme which associates to every α-adically complete Λ_I^0-algebra R the set of morphisms $f: \mathrm{Spf}(R) \to \mathfrak{X}_I$ such that $f^*(\mathrm{Hdg}) = R$. Moreover, for any integer $r \in \mathbb{N}$, we define $\mathfrak{X}_{r,I} \to \mathfrak{X}_I$ to be the formal scheme which associates to every α-adically complete Λ_I^0-algebra R the set of equivalence classes of pairs (f, η) consisting of a morphism $f: \mathrm{Spf}(R) \to \mathfrak{X}_I$ of formal schemes and an element $\eta \in H^0(\mathrm{Spf}(R), f^*(\underline{\omega}^{(1-p)p^{r+1}}))$ such that*

$$\eta \mathrm{Hdg}^{p^{r+1}} = \alpha \bmod p^2.$$

Here the equivalence is given by $(f, \eta) \sim (f', \eta')$ if $f = f'$ and $\eta = (1 + \frac{p^2}{\alpha} u)\eta'$ for some $u \in R$.

Analogous to Remark 2.2.11, we have

Remark A.1.6. *In fact, $\mathfrak{X}_I^{\mathrm{ord}}$ is an open subscheme of $\mathfrak{X}_I$, $\mathfrak{X}_{r,I}$ is an open formal subscheme of an admissible blow-up of $\mathfrak{X}_I$ and there is a natural map $\mathfrak{X}_I^{\mathrm{ord}} \to \mathfrak{X}_{r,I}$ identifying $\mathfrak{X}_I^{\mathrm{ord}}$ as an open subscheme of $\mathfrak{X}_{r,I}$.*

Moreover, the adic generic fiber $\mathcal{X}_{r,I}$ of $\mathfrak{X}_{r,I}$ is affinoid because, upon replacing Ha *by a power* Ha^{p^n}*, we may choose a global lifting* $\widetilde{\mathrm{Ha}}$ *of* Ha. *Then $\mathcal{X}_{r,I}$ is the open subspace of $\mathcal{X}_I$ defined by $|\widetilde{\mathrm{Ha}}| \geq \max\{|T|, |p|\}$. As $\widetilde{\mathrm{Ha}}$ is a global section of an ample line bundle, this implies that $\mathcal{X}_{r,I}$ is affinoid.*

Parallel to Lemma 2.2.12, we have

Lemma A.1.7. [AIP, Lemma 3.4] *For each $r \geq 1$, the formal scheme $\mathfrak{X}_{r,I}$ is normal and excellent.*

Recall that a triple (n, r, I) is called pre-adapted if

- $I = [0, 1]$, and $1 \leq n \leq r$; or
- $I = [p^a, p^b]$ for $a, b \in \mathbb{N}$, and $1 \leq n \leq r + a$.

Proposition A.1.8. [AIP, Proposition 3.2] *For any pre-adapted triple (n, r, I), the universal semi-abelian scheme $\mathcal{E}$ over $\mathfrak{X}_{r,I}$ admits a canonical subgroup $H_n \subset \mathcal{E}[p^n]$ which is locally free of rank p^n and whose reduction modulo $p\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}$ is $\ker(F^n)$.*

A.1.2 Partial Igusa towers

Fix a pre-adapted triple (n, r, I). As shown in [AIP, Lemma 3.2], since $H_n^D \cong \mathbb{Z}/p^n\mathbb{Z}$ over $\mathcal{X}_{r,I}$, there is a finite étale Galois cover $\mathcal{IG}_{n,r,I}$ over $\mathcal{X}_{r,I}$ parameterizing trivializations $(H_n^1)^D \cong \mathbb{Z}/p^n\mathbb{Z}$ over $\mathfrak{X}_{r,I}$. Moreover, the normalization of $\mathfrak{X}_{r,I}$ in $\mathcal{IG}_{n,r,I}$ is well-defined and finite over $\mathfrak{X}_{r,I}$. We denote this normalization by $\mathfrak{IG}_{n,r,I}$, and call it the formal partial Igusa tower of level n over $\mathfrak{X}_{r,I}$.

Moreover, note that H_n^D is étale over $\mathcal{X}_{r,I}$ and locally isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$, so that the $(\mathbb{Z}/p^n\mathbb{Z})^\times$-action on $\mathcal{IG}_{n,r,I}$ given by defining $g * \alpha$ to be the composition

$$\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\times g} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\alpha} H_n^D$$

for any trivialization $\alpha : \mathbb{Z}/p^n\mathbb{Z} \cong H_n^D$ and any $g \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ identifies $(\mathbb{Z}/p^n\mathbb{Z})^\times$ with the Galois group of $\mathcal{IG}_{n,r,I}$ over $\mathcal{X}_{r,I}$. By normality, the $(\mathbb{Z}/p^n\mathbb{Z})^\times$-action on $\mathcal{IG}_{n,r,I}$ extends to $\mathfrak{IG}_{n,r,I}$. Moreover, since H_n^D is étale over $\mathfrak{X}_{r,I}^{\mathrm{ord}}$, the ordinary locus $\mathfrak{IG}_{n,r,I}^{\mathrm{ord}} := \mathfrak{IG}_{n,r,I} \otimes_{\mathfrak{X}_{r,I}} \mathfrak{X}_I^{\mathrm{ord}}$ is finite étale over $\mathfrak{X}_I^{\mathrm{ord}}$ with Galois group $(\mathbb{Z}/p^n\mathbb{Z})^\times$.

Parallel to Proposition 2.2.16, we have

Proposition A.1.9. [AIP, Propositions 3.3 & 3.6] *For any pre-adapted triple (n, r, I),*

(i) the quotient isogeny $\mathcal{E} \to \mathcal{E}/H_1$ induces a finite flat degree-p morphism $\phi : \mathfrak{X}_{r+1,I} \to \mathfrak{X}_{r,I}$ satisfying that $\phi^*(\mathcal{E}) = \mathcal{E}/H_1$ and the reduction modulo $p\mathrm{Hdg}^{-1}$ of ϕ is the Froebnius map relative to Λ_I^0.

(ii) there is a finite $(\mathbb{Z}/p^n\mathbb{Z})^\times$-equivariant morphism $\Phi : \mathfrak{IG}_{n,r+1,I} \to \mathfrak{IG}_{n,r,I}$ which makes the following diagram commutative:

$$\begin{array}{ccc} \mathfrak{IG}_{n,r+1,I} & \xrightarrow{\Phi} & \mathfrak{IG}_{n,r,I} \\ \downarrow & & \downarrow \\ \mathfrak{X}_{r+1,I} & \xrightarrow{\phi} & \mathfrak{X}_{r,I} \end{array}$$

As mentioned after Lemma 2.2.18, and stated in [AI17, Remark 2], we have

Lemma A.1.10. *Over $\mathfrak{IG}_{1,r,I}$, there exists an invertible sheaf $\underline{\delta}$ whose $(p-1)$-power is Hdg. Moreover, the natural morphism $\mathfrak{IG}_{1,r,I} \to \mathfrak{X}_{r,I}$ is flat.*

Proof. The existence of $\underline{\delta}$ follows from Proposition 1.5.7 as $\mathfrak{IG}_{1,r,I}$ is normal. To show flatness, for any affine open $\mathrm{Spf}(R) \subset \mathfrak{X}_{r,I}$ small enough such that over $\mathrm{Spf}(R)$, Hdg is generalized by a local lift $\widetilde{\mathrm{Ha}}$ of Ha, by the discussion in [AIS14, Proposition 6.2], we know that H_1^D over R has the form $\mathrm{Spec}(R[x]/(x^p - \widetilde{\mathrm{Ha}}x))$. With these preparations, the argument of Lemma 2.2.18 carries over verbatim. □

Similar to Proposition 2.2.19, we have

Proposition A.1.11. [AI17, Lemma 3.3 & 3.4] *Fix a pre-adapted triple (n,r,I).*

(i) Let $\iota: \mathfrak{X}_{r,I}^{\mathrm{ord}} \to \mathfrak{X}_{r,I}; \quad \mathfrak{IG}_{n,r,I}^{\mathrm{ord}} \to \mathfrak{IG}_{n,r,I}$ be the natural maps. Then for each $h \in \mathbb{N}$, the kernels of

$$\mathcal{O}_{\mathfrak{X}_{r,I}}/\alpha^h\mathcal{O}_{\mathfrak{X}_{r,I}} \to \iota_*(\mathcal{O}_{\mathfrak{X}_{r,I}^{\mathrm{ord}}}/\alpha^h\mathcal{O}_{\mathfrak{X}_{r,I}^{\mathrm{ord}}}); \quad \mathcal{O}_{\mathfrak{IG}_{n,r,I}}/\alpha^h\mathcal{O}_{\mathfrak{IG}_{n,r,I}} \to \iota_*(\mathcal{O}_{\mathfrak{IG}_{n,r,I}^{\mathrm{ord}}}/\alpha^h\mathcal{O}_{\mathfrak{IG}_{n,r,I}^{\mathrm{ord}}})$$

are annihilated by $\mathrm{Hdg}^{hp^{r+1}}$ and $\mathrm{Hdg}^{hp^{r+1}+\frac{p^n-p}{p-1}}$ respectively.

(ii) Let $\eta: \mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I} \to \mathfrak{X}_I \to \hat{X}$ be the composition of natural maps, then the cokernel of the induced map $\eta^*(\Omega^1_{\hat{X}/\mathbb{Z}_p}) \to \Omega^1_{\mathfrak{IG}_{n,r,I}/\Lambda_I^0}$ is annihilated by a power (depending on n) of Hdg.

A.2 The nearly overconvergent modular sheaf

Fix a pre-adapted triple (n,r,I). Let $\mathfrak{IG}_{n,r,I} \xrightarrow{g_n} \mathfrak{X}_{r,I}$ be the formal partial Igusa tower of level n. Let $\mathcal{E} \to \mathfrak{X}_{r,I}$ be the universal semi-abelian scheme, $H_n \subset \mathcal{E}$ be the level-n canonical subgroup, and Hdg be the Hodge ideal of $\mathcal{E}$. Over $\mathfrak{IG}_{n,r,I}$, we have an ideal $\underline{\delta}$ whose $(p-1)$-th power is Hdg, and that $H_n^D(\mathfrak{IG}_{n,r,I}) \cong \mathbb{Z}/p^n\mathbb{Z}$. Let $s := s_n := \mathrm{HT}(P_n)$ where $P_n \in H_n^D(\mathfrak{IG}_{n,r,I})$ is the universal section. By applying the VBMS machinery in § 1.6.1 to the system of vector bundles with marked sections $(\mathbb{H}_{\mathcal{E}}^\sharp := \underline{\delta}\underline{\omega_{\mathcal{E}}}, \Omega_{\mathcal{E}} := \underline{\delta}\underline{\omega_{\mathcal{E}}}, s)$ with respect to the ideal $\underline{\beta}_n := p^n\mathrm{Hdg}^{-\frac{p^n}{p-1}}$, we have the following commutative diagram

$$\begin{tikzcd} \mathcal{V}_0(\mathbb{H}_{\mathcal{E}}^\sharp, s) \arrow[d] \arrow[dr, "\pi"] & & \\ \mathcal{V}_0(\Omega_{\mathcal{E}}, s) \arrow[r, "\pi'"] & \mathfrak{IG}_{n,r,I} \arrow[r, "g_n"] & \mathfrak{X}_{r,I}. \end{tikzcd}$$

Let $\mathcal{T} \subset \mathcal{T}^{\mathrm{ext}}$ be formal groups over $\mathfrak{X}_{r,I}$ defined by

$$\mathcal{T}(S) := 1 + \rho^*(\underline{\beta}_n)\mathcal{O}_S \subset \mathcal{T}^{\mathrm{ext}}(S) := \mathbb{Z}_p^\times(1 + \rho^*(\underline{\beta}_n)\mathcal{O}_S) \subset G_m(S)$$

for any formal scheme $\rho: S \to \mathfrak{X}_{r,I}$. We have natural actions[1] of $\mathcal{T}$ (resp. $\mathcal{T}^{\mathrm{ext}}$) on $\mathcal{V}_0(\mathbb{H}_{\mathcal{E}}^\sharp, s)$ and $\mathcal{V}_0(\Omega_{\mathcal{E}}, s)$ over $\mathfrak{IG}_{n,r,I}$ (resp. $\mathfrak{X}_{r,I}$) defined as follows:

(i) for any $\mathfrak{IG}_{n,r,I}$-formal scheme $\rho: S \to \mathfrak{IG}_{n,r,I}$, a point $v \in \mathcal{V}_0(\mathbb{H}_{\mathcal{E}}^\sharp, s)(S)$ is an $\mathcal{O}_S$-linear map $v: \rho^*(\mathbb{H}_{\mathcal{E}}^\sharp) \to \mathcal{O}_S$ whose reduction $\bar{v}$ modulo $\rho^*(\underline{\beta}_n)$ sends $\rho^*(s)$ to 1. For any $t \in \mathcal{T}(S)$, we define $t * v = t^{-1}v$. The action of $\mathcal{T}$ on $\mathcal{V}_0(\Omega_{\mathcal{E}}, s)$ is similar.

(ii) for any $\mathfrak{X}_{r,I}$-formal scheme $u: S \to \mathfrak{X}_{r,I}$, a point in $\mathcal{V}_0(\mathbb{H}_{\mathcal{E}}^\sharp, s)(S)$ is a pair (ρ, v) consisting of a lift $\rho: S \to \mathfrak{IG}_{n,r,I}$ of u and an $\mathcal{O}_S$-linear map $v: \rho^*(\mathbb{H}_{\mathcal{E}}^\sharp) \to \mathcal{O}_S$ such that $\bar{v}(\rho^*(s)) = 1$. For any $\lambda \in \mathbb{Z}_p^\times$ with image $\bar{\lambda}$ in $(\mathbb{Z}/p^n\mathbb{Z})^\times$, we set $\lambda * (\rho, v) := (\bar{\lambda} \circ \rho, \lambda^{-1}v)$. This makes sense because the automorphism $\bar{\lambda}: \mathfrak{IG}_{n,r,I} \to \mathfrak{IG}_{n,r,I}$ induces isomorphisms

$$\bar{\lambda}^*: \Omega_{\mathcal{E}} \to \Omega_{\mathcal{E}}; \quad \mathbb{H}_{\mathcal{E}}^\sharp \to \mathbb{H}_{\mathcal{E}}^\sharp$$

such that modulo $\underline{\beta}_n$, $\bar{\lambda}^*(s) = \bar{\lambda}s$. The action of $\mathcal{T}^{\mathrm{ext}}$ on $\mathcal{V}_0(\Omega_{\mathcal{E}}, s)$ is similar.

[1] Our group actions on both the points and functions are different from [AI17, § 3.2] by an inverse.

We let $\mathcal{T}^{\mathrm{ext}}$ act on functions via pull-back, i.e. $(t * f)(\rho, v) = f(t * (\rho, v))$.

Definition A.2.1. *Let $\kappa_I : \mathbb{Z}_p^\times \to \Lambda_I^\times$ be the universal character, $\kappa_{I,f}$ be the restriction of κ_I to $(\mathbb{Z}/q\mathbb{Z})^\times$, and $\kappa_I^0 := \kappa_I(\kappa_{I,f})^{-1}$.*

(i) we define $\mathfrak{w}^{\kappa_{I,f}}$ to be the coherent $\mathcal{O}_{\mathfrak{X}_{r,I}}$-module $(g_{i,*}(\mathcal{O}_{\mathfrak{IG}_{i,r,I}}) \otimes_{\Lambda_I^0} \Lambda_I)[(\kappa_{I,f})^{-1}]$ with $i = 2$ if $p = 2$ and $i = 1$ if $p \geq 3$.

(ii) we define $\mathfrak{w}^{\kappa_I,0} := f'_{0,*}(\mathcal{O}_{\mathcal{V}_0(\Omega_{\mathcal{E}},s)})[(\kappa_I^0)^{-1}]$, where $f'_0 := g_n \circ \pi' : \mathcal{V}_0(\Omega_{\mathcal{E}}, s) \to \mathfrak{X}_{r,I}$ is the structure morphism.

(iii) we define $\mathbb{W}^0_{\kappa_I} := f_{0,*}(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{E}},s)})[(\kappa_I^0)^{-1}]$, where $f_0 := g_n \circ \pi : \mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{E}}, s) \to \mathfrak{X}_{r,I}$ is the structure morphism.

(iv) we define $\mathfrak{w}^{\kappa_I} := \mathfrak{w}^{\kappa_I,0} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \mathfrak{w}^{\kappa_{I,f}}, \quad \mathbb{W}_{\kappa_I} := \mathbb{W}^0_{\kappa_I} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \mathfrak{w}^{\kappa_{I,f}}$.

Here the notation $[\chi]$ means taking χ-isotypic components with respect to the $(\mathbb{Z}/p^i\mathbb{Z})^\times$-action for $\mathfrak{w}^{\kappa_{I,f}}$ and the $\mathcal{T}^{\mathrm{ext}}$-action for other sheaves. When the interval I is clear, we omit it in the notation.

Note that the local calculations in Lemma 2.3.3, Lemma 2.3.4, Lemma 2.3.5 and Proposition 2.3.7 carry over verbatim if we replace $\mathcal{G}$ by $\mathcal{E}[p^\infty]$, so as stated in [AI17, Theorem 3.11], we have

Theorem A.2.2. *Let (n, r, I) be an adapted triple. The filtration $\mathrm{Fil}_\bullet \pi_*(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}_{\mathcal{E}},s)})$ induces a filtration $\mathrm{Fil}_\bullet := g_{n,*}\mathrm{Fil}_\bullet \pi_*(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{E}},s)})$ on $f_{0,*}(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{E}},s)})$ which is preserved by the action of $\mathcal{T}^{\mathrm{ext}}$. For every $h \geq 0$, we define*

$$\mathrm{Fil}_h \mathbb{W}^0_\kappa := \mathrm{Fil}_h f_{0,*}(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{E}},s)})[(\kappa^0)^{-1}]; \quad \mathrm{Fil}_h \mathbb{W}_\kappa := \mathrm{Fil}_h \mathbb{W}^0_\kappa \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \mathfrak{w}^{\kappa_f}.$$

Then we obtain an increasing filtration $\mathrm{Fil}_\bullet$ of $\mathbb{W}^{(0)}_\kappa$ such that

(i) for each $h \in \mathbb{N}$, $\mathrm{Fil}_h \mathbb{W}^0_\kappa$ is a locally free $\mathcal{O}_{\mathfrak{X}_{r,I}}$-module of rank $h+1$ for the Zariski topology on $\mathfrak{X}_{r,I}$;

(ii) for each $h \in \mathbb{N}$, we have $\mathrm{Fil}_0 \mathbb{W}^{(0)}_\kappa \cong \mathfrak{w}^{\kappa,(0)}$ and $\mathrm{Gr}_h \mathbb{W}^{(0)}_\kappa \cong \mathfrak{w}^{\kappa,(0)} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \underline{\omega}_{\mathcal{E}}^{-2h}$;

(iii) $\mathbb{W}^{(0)}_\kappa$ is the α-adic completions of $\varinjlim_{h\to\infty} \mathrm{Fil}_h \mathbb{W}^{(0)}_\kappa$.

Moreover, by carrying over the discussion in § 2.3.3, we have the following functoriality results for $\mathbb{W}_\kappa$ in the modular curve case.

Lemma A.2.3. *Let (n, r, I) and (n, r, J) be adapted triples with $J \subset I$, and let $\mathbb{W}_{\kappa,I}$ (resp. $\mathbb{W}_{\kappa,J}$) be the nearly overconvergent modular sheaf defined over $\mathfrak{X}_{r,I}$ (resp. $\mathfrak{X}_{r,J}$). The natural map $\theta_{J,I} : \mathfrak{X}_{r,J} \to \mathfrak{X}_{r,J}$ induces an isomorphism*

$$\theta^0_{J,I} : \theta^*_{J,I}(\mathbb{W}^0_{\kappa,I}) \cong \mathbb{W}^0_{\kappa,J}$$

and a morphism

$$\theta_{J,I} : \theta^*_{J,I}(\mathbb{W}_{\kappa,I}) \cong \mathbb{W}_{\kappa,J}$$

which becomes an isomorphism after inverting α.

Lemma A.2.4. *Let (n, r, I) and (n', r, J) be adapted triples such that $n \leq n'$, and let $\mathbb{W}_{\kappa,n}$ (resp. $\mathbb{W}_{\kappa,n'}$) be the nearly overconvergent modular sheaf defined with respect to $\mathfrak{IG}_{n,r,I}$ (resp. $\mathfrak{IG}_{n',r,I}$). Then the natural map $\theta_{n',n} : \mathfrak{IG}_{n',r,I} \to \mathfrak{IG}_{n,r,I}$ induces isomorphisms*

$$\theta^0_{n',n} : \theta^*_{n',n}(\mathbb{W}^0_{\kappa,n}) \cong \mathbb{W}^0_{\kappa,n'}, \quad \theta_{n',n} : \theta^*_{n',n}(\mathbb{W}_{\kappa,n}) \cong \mathbb{W}_{\kappa,n'}.$$

Lemma A.2.5. *Let (n, r, I) and (n, r', J) be adapted triples such $r \leq r'$, and let $\mathbb{W}_{\kappa,r}$ (resp. $\mathbb{W}_{\kappa,r'}$) be the nearly overconvergent modular sheaf defined over $\mathfrak{X}_{r,I}$ (resp. $\mathfrak{X}_{r',I}$). Then the natural map $\theta_{r',r} : \mathfrak{X}_{r',I} \to \mathfrak{X}_{r,I}$ induces an isomorphism*

$$\theta^0_{r',r} : \theta^*_{r',r}(\mathbb{W}^0_{\kappa,r}) \cong \mathbb{W}^0_{\kappa,r'}$$

and a morphism

$$\theta_{r',r} : \theta^*_{r',r}(\mathbb{W}_{\kappa,r}) \cong \mathbb{W}_{\kappa,r'}$$

which becomes an isomorphism after inverting α.

A.2.1 The Gauss-Manin connection on $\mathbb{W}_\kappa$

Let $\mathfrak{Y}_{r,I} \subset \mathfrak{X}_{r,I}$ be the inverse image of $\hat{Y} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spf}(\Lambda^0_I)$ under the structure map $\mathfrak{X}_{r,I} \to \mathfrak{X}_I$, $\mathcal{Y}_{r,I}$ be the adic generic fiber of $\mathfrak{Y}_{r,I}$, and let $\mathfrak{IG}^\circ_{n,r,I} := \mathfrak{Y}_{r,I} \times_{\mathfrak{X}_{r,I}} \mathfrak{IG}_{n,r,I}$. Similar to Lemma 2.2.14, for any pre-adapted triple (n, r, I), there is a finite étale Galois cover $\mathcal{IG}'^{,\circ}_{n,r,I}$ of $\mathcal{Y}_{r,I}$ parameterizing all compatible trivializations

$$(\mathbb{Z}/p^n\mathbb{Z})^2 \cong \mathcal{E}[p^n]^D; \quad \mathbb{Z}/p^n\mathbb{Z} \cong H^D_n$$

over $\mathcal{Y}_{r,I}$, and the normalization $\mathfrak{IG}'^{,\circ}_{n,r,I}$ of $\mathfrak{Y}_{r,I}$ in $\mathcal{IG}'^{,\circ}_{n,r,I}$ is well-defined and finite over $\mathfrak{Y}_{r,I}$. Note that by normality, we have

$$\mathcal{E}[p^n]^D(\mathfrak{IG}'^{,\circ}_{n,r,I}) \cong (\mathbb{Z}/p^n\mathbb{Z})^2, \quad H^D_n(\mathfrak{IG}'^{,\circ}_{n,r,I}) \cong \mathbb{Z}/p^n\mathbb{Z},$$

so by Proposition 1.6.9, the Gauss-Manin connection on $\mathbb{H}_\mathcal{E}$ induces a connection

$$\nabla^\sharp : \mathbb{H}^\sharp_\mathcal{E} \to \mathbb{H}^\sharp_\mathcal{E} \hat{\otimes}_{\mathcal{O}_{\mathfrak{IG}'^{,\circ}_{n,r,I}}} \frac{1}{\underline{\delta}} \Omega^1_{\mathfrak{IG}'^{,\circ}_{n,r,I}/\Lambda^0_I}$$

such that the marked section s is horizontal with respect to $\nabla^\sharp$ modulo $\underline{\beta}_n$. By Lemma 1.6.6, we have a flat connection

$$\nabla^\sharp : \pi_*(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_\mathcal{E}, s)}) \to \pi_*(\mathcal{O}_{\mathcal{V}_0(\mathbb{H}^\sharp_\mathcal{E}, s)}) \otimes \frac{1}{\underline{\delta}} \Omega^1_{\mathfrak{IG}'^{,\circ}_{n,r,I}/\Lambda^0_I}.$$

Following the argument in Lemma 2.3.9, we can show that $\nabla^\sharp$ descends to a flat connection

$$\nabla^0_\kappa : \tilde{\mathbb{W}}^0_\kappa \to \tilde{\mathbb{W}}^0_\kappa \hat{\otimes}_{\mathcal{O}_{\mathfrak{IG}^\circ_{n,r,I}}} \frac{1}{\mathrm{Hdg}} \Omega^1_{\mathfrak{IG}^\circ_{n,r,I}/\Lambda^0_I},$$

which is compatible with the flat connection over $\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}$

$$\nabla^0_\kappa : \tilde{\mathbb{W}}^0_\kappa \to \tilde{\mathbb{W}}^0_\kappa \hat{\otimes}_{\mathcal{O}_{\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}}} \Omega^1_{\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}/\Lambda^0_I}(\log(\mathrm{cusps})),$$

so we have a flat connection over $\mathfrak{IG}_{n,r,I}$, referred to as the Gauss-Manin connection,

$$\nabla^0_\kappa : \quad \tilde{\mathbb{W}}^0_\kappa \to \tilde{\mathbb{W}}^0_\kappa \hat{\otimes}_{\mathcal{O}_{\mathfrak{IG}_{n,r,I}}} \frac{1}{\mathrm{Hdg}} \Omega^1_{\mathfrak{IG}_{n,r,I}/\Lambda^0_I}(\log(\mathrm{cusps}))$$

which satisfies Griffiths' transversality.

Convention A.2.6. *To simplify notation, we omit the the log poles in the notation when discussing the Gauss-Manin connections in the rest of this chapter.*

By similar arguments as those used in the proofs of Lemma 2.3.10, Lemma 2.3.11 and Theorem 2.3.8, as stated in [AI17, Theorem 3.18], we have

Theorem A.2.7. *Let $i = 2$ if $p = 2$ and $i = 1$ if $p \geq 3$, and assume (n, r, I) is an adapted triple.*

(i) The connection $\nabla_\kappa^\sharp$ on $\tilde{\mathbb{W}}_\kappa^0$ over $\mathfrak{IG}_{n,r,I}$ descends to a connection

$$\nabla_\kappa^0 : \ \mathbb{W}_\kappa^0 \to \mathbb{W}_\kappa^0 \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}[1/\alpha]$$

which satisfies Griffiths' transversality, and the induced $\mathcal{O}_{\mathfrak{X}_{r,I}}$-linear map

$$\mathrm{Gr}_h(\nabla_\kappa^0) : \mathrm{Gr}_h(\mathbb{W}_\kappa^0)[1/\alpha] \to \mathrm{Gr}_{h+1}(\mathbb{W}_\kappa^0) \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}[1/\alpha]$$

is an isomorphism times with multiplication by $\mu_\kappa - h$.

(ii) Tensoring ∇_κ^0 with the connection ∇^f induced by the derivation $d : \ \mathcal{O}_{\mathfrak{IG}_{i,r,I}} \to \Omega^1_{\mathfrak{IG}_{i,r,I}/\Lambda_I^0}$, we get a connection

$$\nabla_\kappa := \nabla_\kappa^0 \otimes \nabla^f : \ \mathbb{W}_\kappa[1/\alpha] \to \mathbb{W}_\kappa \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}[1/\alpha]$$

which satisfies Griffiths transversality, and the induced $\mathcal{O}_{\mathfrak{X}_{r,I}}$-linear map

$$\mathrm{Gr}_h(\nabla_\kappa) : \ \mathrm{Gr}_h(\mathbb{W}_\kappa)[1/\alpha] \to \mathrm{Gr}_{h+1}(\mathbb{W}_\kappa) \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}[1/\alpha]$$

is an isomorphism times with multiplication by $\mu_\kappa - h$.

(iii) Let $k \in \mathbb{N}$ be a specialization of κ and denote the specialization of an object along k by a subscript k. Then we have a canonical identification

$$(\mathrm{Sym}^k(\mathbb{H}_\mathcal{E}) \otimes_{\Lambda_I^0} \Lambda_I[1/\alpha])_k \cong (\mathrm{Fil}_k(\mathbb{W}_\kappa)[1/\alpha])_k$$

which is compatible with connection and filtration. Here on the left, we consider the Gauss-Manin connection and the Hodge filtration.

Finally, like the discussion before Definition 2.3.12, with the help of Proposition A.1.11, the Gauss-Manin connection ∇_κ on $\mathbb{W}_\kappa$ induces a map, which is also referred to as the Gauss-Manin connection,

$$\nabla_\kappa : \ \mathbb{W}_\kappa \to \frac{1}{\mathrm{Hdg}^{c(n)}} \mathbb{W}_{\kappa+2}$$

for some constant $c(n)$.

Parallel to Definition 2.3.12, we define

Definition A.2.8. *Let $f_0 : \ \mathcal{V}_0(\mathbb{H}_\mathcal{E}^\sharp, s) \to \mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I}$ be the structure map. We define*

$$\mathbb{W}^0 := f_{0,*}\mathcal{O}_{\mathcal{V}_0(\mathbb{H}_\mathcal{E}^\sharp, s)}; \quad \mathbb{W} := \mathbb{W}^0 \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \mathfrak{w}^{\kappa_f};$$
$$\mathbb{W}_\kappa^{0,\prime} := \sum_{m \in \mathbb{Z}} \mathbb{W}^0_{\kappa+2m} \subset \mathbb{W}^0; \quad \mathbb{W}_\kappa' := \mathbb{W}_\kappa^{0,\prime} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \mathfrak{w}^{\kappa_f},$$

Note that for $m \neq m'$, $\mathbb{W}_{\kappa+2m} \cap \mathbb{W}_{\kappa+2m'} = (0)$, so we have a well-defined connection

$$\nabla_\kappa : \mathbb{W}'_\kappa \to (\frac{1}{\mathrm{Hdg}^{c(n)}})\mathbb{W}'_\kappa; \quad \nabla_\kappa |_{\mathbb{W}_{\kappa+2m}} = \nabla_{\kappa+2m}.$$

For each $M \geq 0$, we denote the iteration

$$\mathbb{W}_\kappa \xrightarrow{\nabla_\kappa} \frac{1}{\mathrm{Hdg}^{c(n)}}\mathbb{W}_{\kappa+2} \xrightarrow{\nabla_{\kappa+2}} \frac{1}{\mathrm{Hdg}^{2c(n)}}\mathbb{W}_{\kappa+4} \cdots \xrightarrow{\nabla_{\kappa+2M-2}} \frac{1}{\mathrm{Hdg}^{Mc(n)}}\mathbb{W}_{\kappa+2M}$$

by ∇_κ^M. When the character κ is clear from context, we simply write ∇_κ as ∇.

A.2.2 Hecke operators

In this subsection, we discuss Hecke operators on the nearly overconvergent sheaf $\mathbb{W}_\kappa$ generalizing [AI17, § 3.6-3.7]. The Shimura curve counterparts are considered in § 2.3.4.

The Frobenius and the U_p-operator

Let (n, r, I) be a pre-adapted triple and let $H_1 \subset \mathcal{E}$ be the level-1 canonical subgroup of the universal semi-abelian scheme $\mathcal{E}$. According to Proposition A.1.9, the quotient isogeny $f : \mathcal{E} \to \mathcal{E}' := \mathcal{E}/H_1$ induces a $(\mathbb{Z}/p^n\mathbb{Z})^\times$-equivariant commutative diagram

$$\begin{tikzcd} \mathfrak{IG}_{n,r+1,I} \arrow[r, "\Phi"] \arrow[d] & \mathfrak{IG}_{n,r,I} \arrow[d] \\ \mathfrak{X}_{r+1,I} \arrow[r, "\phi"] & \mathfrak{X}_{r,I} \end{tikzcd} \tag{A.2.1}$$

where Φ is characterized by $\Phi^*(\mathcal{E}_{/\mathfrak{IG}_{n,r,I}}) = \mathcal{E}'_{/\mathfrak{IG}_{n,r+1,I}}$. The argument in Lemma 2.3.16 implies

Lemma A.2.9. *Let $i = 2$ if $p = 2$ and $i = 1$ if $p \geq 3$. The diagram A.2.1 induces a morphism of $\mathcal{O}_{\mathfrak{X}_{r+1,I}}$-modules*

$$\mathcal{V}^f : \phi^*(\mathfrak{w}^{\kappa_f}) \to \mathfrak{w}^{\kappa_f}$$

which is compatible with connection and becomes an isomorphism after inverting α.

Let $H_{n+1} \subset \mathcal{E}$ be the level-$n+1$ canonical subgroup. Over $\mathcal{IG}_{n+1,r+1,I}$, we have the universal trivialization $\gamma_{n+1} : \mathbb{Z}/p^{n+1}\mathbb{Z} \to H_{n+1}^D$. Moreover, $H'_n := H_{n+1}/H_1$ is the level-n canonical subgroup of $\mathcal{E}'$ and over $\mathfrak{IG}^\circ_{n+1,r+1,I}$, the dual isogeny $f' : \mathcal{E}' \to \mathcal{E}$ induces an isomorphism $H'_n \cong H_n$, so the commutative diagram

$$\begin{tikzcd} \mathbb{Z}/p^n\mathbb{Z} \arrow[r, "\mathrm{Id}"] \arrow[d, "\gamma_n"] & \mathbb{Z}/p^n\mathbb{Z} \arrow[r, "1\mapsto p"] \arrow[d, "\gamma'_n"] & \mathbb{Z}/p^{n+1}\mathbb{Z} \arrow[d, "\gamma_{n+1}"] \\ H_n^D \arrow[r, "(f')^D"] & (H'_n)^D \arrow[r, "f^D"] & H_{n+1}^D \end{tikzcd} . \tag{A.2.2}$$

defines a unique trivialization $\gamma'_n : \mathbb{Z}/p^n\mathbb{Z} \cong (H'_n)^D$ over $\mathcal{IG}^\circ_{n+1,r+1,I}$. By explicit calculations using Tate curves (or the result [Con07, Lemma 4.2.3]), we can show γ'_n extends uniquely to $\mathcal{IG}_{n+1,r+1,I}$. Let $p_{n+1,n} : \mathfrak{IG}_{n+1,r+1,I} \to \mathfrak{IG}_{n,r+1,I}$ be the forgetful map. We have that the composition

$$t_1 : \mathfrak{IG}_{n+1,r+1,I} \xrightarrow{p_1} \mathfrak{IG}_{n,r+1,I} \xrightarrow{\Phi} \mathfrak{IG}_{n,r,I}$$

is the normalization of the morphism on adic generic fibers sending $(\mathcal{E}, \gamma_{n+1})$ to $(\mathcal{E}', \gamma'_n)$. We have (for the Shimura curve analogue, see Lemma 2.3.17)

Lemma A.2.10. *Over $\mathfrak{IG}_{n+1,r+1,I}$, the quotient isogeny $f: \mathcal{E} \to \mathcal{E}'$ induces an isomorphism*

$$\frac{f^*}{p}: \Omega_{\mathcal{E}'} \to \Omega_{\mathcal{E}}$$

and a commutative diagram

$$\begin{tikzcd}
p_{n+1,n}^*(\mathcal{V}_0(\Omega_{\mathcal{E}}, s_n)) \arrow[rr] \arrow[dr] & & t_1^*(\mathcal{V}_0(\Omega_{\mathcal{E}}, s_n)) \arrow[dl] \\
& \mathfrak{IG}_{n+1,r+1,I} \arrow[d] & \\
& \mathfrak{X}_{r+1,I} &
\end{tikzcd} \tag{A.2.3}$$

compatible with the action of $\mathbb{Z}_p^\times(1+\underline{\beta}_{n+1}G_a)$. Moreover, if (n,r,I) is adapted, then the induced morphism of $\mathcal{O}_{\mathfrak{X}_{r,I}}$-modules $\mathcal{V}^0: \phi^*(\mathfrak{w}^{\kappa,0}) \to \mathfrak{w}^{\kappa,0}$ is an isomorphism.

Proof. The claimed result holds away from cusps by a similar but simpler argument as for Lemma 2.3.17. By an explicit calculation using Tate curves (see Theorem A.1.4), or using the fancy result [Con07, Lemma 4.2.3], we can extend to the whole $\mathfrak{X}_{r,I}$. □

Definition A.2.11. *We define $\mathcal{V}: \phi^*(\mathfrak{w}^\kappa) \to \mathfrak{w}^\kappa$ to be $\mathcal{V}^0 \otimes \mathcal{V}^f$ and define the V_p-operator on global sections to be*

$$V_p: H^0(\mathfrak{X}_{r,I}, \mathfrak{w}^\kappa) \xrightarrow{\phi^*} H^0(\mathfrak{X}_{r+1,I}, \phi^*(\mathfrak{w}^\kappa)) \xrightarrow{\mathcal{V}} H^0(\mathfrak{X}_{r+1,I}, \mathfrak{w}^\kappa).$$

We can define the U_p-operator over $\mathbb{W}_\kappa$, whose restriction to $\mathfrak{w}^\kappa$ is a left inverse of V_p. Analogous to Lemma 2.3.19, we have

Lemma A.2.12. *Let (n,r,I) be an adapted triple. The dual isogeny $f': \mathcal{E}' \to \mathcal{E}$ over $\mathfrak{IG}^\circ_{n+1,r+1,I}$ induces a morphism $\mathcal{U}^0: \mathbb{W}^0_\kappa \to \phi^*(\mathbb{W}^0_\kappa)$ of $\mathfrak{X}_{r+1,I}$-modules which is compatible with filtration and connection, and the restriction*

$$\mathfrak{w}^{\kappa,0} = \mathrm{Fil}_0\mathbb{W}^0_\kappa \to \phi^*(\mathrm{Fil}_0\mathbb{W}^0_\kappa) = \phi^*(\mathfrak{w}^{\kappa,0})$$

is inverse to $\mathcal{V}^0$.

Proof. The claimed result holds away from cusps by a similar but simpler argument as for Lemma 2.3.19. By an explicit calculation using Tate curves(see Theorem A.1.4), or using the fancy result [Con07, Lemma 4.2.3], we can extend to the whole $\mathfrak{X}_{r,I}$. □

Definition A.2.13. *On global section of $\mathbb{W}^0_\kappa$, the U_p-operator is the composition*

$$H^0(\mathfrak{X}_{r+1,I}, \mathbb{W}^0_\kappa) \xrightarrow{\mathcal{U}^0} H^0(\mathfrak{X}_{r+1,I}, \phi^*(\mathbb{W}^0_\kappa)) \xrightarrow{\frac{1}{p}\mathrm{Tr}_{\mathfrak{X}_{r+1,I}/\mathfrak{X}_{r,I}}} H^0(\mathfrak{X}_{r,I}, \mathbb{W}^0_\kappa)(\to H^0(\mathfrak{X}_{r+1,I}, \mathbb{W}^0_\kappa)).$$

On the generic fiber $\mathcal{X}_{r,I}$, we define the U_p-operator as the composition

$$H^0(\mathcal{X}_{r+1,I}, \mathbb{W}_\kappa) \xrightarrow{\mathcal{U}^0\otimes(\mathcal{V}^f)^{-1}} H^0(\mathcal{X}_{r+1,I}, \phi^*(\mathbb{W}_\kappa)) \xrightarrow{\frac{1}{p}\mathrm{Tr}_{\mathfrak{X}_{r+1,I}/\mathfrak{X}_{r,I}}} H^0(\mathcal{X}_{r,I}, \mathbb{W}_\kappa)(\to H^0(\mathcal{X}_{r+1,I}, \mathbb{W}_\kappa)).$$

A similar but simpler argument as for Proposition 2.3.21 shows

Proposition A.2.14. [AI17, Proposition 3.25] *For each $m \in \mathbb{N}$, we have*

$$U_p(H^0(\mathcal{X}_{r,I}, \mathrm{Fil}_m(\mathbb{W}_\kappa))) \subset H^0(\mathcal{X}_{r,I}, \mathrm{Fil}_m(\mathbb{W}_\kappa)).$$

(i) The induced map on the graded piece $H^0(\mathfrak{X}_{r,I}, \mathrm{Gr}_h(\mathbb{W}^0_\kappa))$ is 0 modulo $(\frac{p}{\mathrm{Hdg}^2})^h$;

(ii) for any specialization $k \in \mathbb{N}$ of κ, the identification $(\mathrm{Sym}^k(\mathbb{H}_\mathcal{E}) \otimes_{\Lambda^0_I} \Lambda_I[1/\alpha])_k \cong (\mathrm{Fil}_k(\mathbb{W}_\kappa)[1/\alpha])_k$ is compatible with U_p-operators. Here the subscript k means the specialization of the object along k.

The argument of Lemma 2.3.22 implies the promised result.

Lemma A.2.15. *The composition*

$$U_p \circ V_p : \; H^0(\mathcal{X}_{r,I}, \mathfrak{w}^\kappa) \to H^0(\mathcal{X}_{r,I}, \mathfrak{w}^\kappa)$$

is the identity map.

Analogous to the Shimura curve case, the U_p operator can be described using the Hecke correspondence.

Definition A.2.16. *For any positive integer d prime to N, we define $\langle d \rangle : \; X_1(N) \to X_1(N)$ to be the diamond operator sending (E, ψ_N) to $(E, d\psi_N)$. Similar definition applies to $\mathfrak{X}_{r,I}$, and by acting on trivializations trivially, we can extend $\langle d \rangle$ to $\mathfrak{IG}_{n,r,I}$.*

Definition A.2.17. *Let $\mathcal{Z}_{r,I}$ be the compactification of the adic space $\mathcal{Z}^\circ_{r,I}$ over $\mathcal{Y}_{r,I}$ which parameterize all subgroups $D \subset \mathcal{E}[p]$ of rank p which intersect the level-1 canonical subgroup $H \subset \mathcal{E}$ trivially.*

Let $p_1 : \; \mathcal{Z}^\circ_{r,I} \to \mathcal{Y}_{r,I}; \; (\mathcal{E}, \mathcal{D}) \to \mathcal{E}$ be the structure morphism. We have p_1 is finite étale of degree p. Generalizing [Sch15, Theorem III. 2.5(iii)], we have (for the Shimura curve analogue, see Lemma 2.3.25)

Lemma A.2.18. *The map*

$$u : \; \mathcal{Y}_{r+1,I} \to \mathcal{Z}^\circ_{r,I}; \quad \mathcal{E} \mapsto (\mathcal{E}/H, \mathcal{E}[p]/H)$$

induces a commutative diagram

$$\begin{tikzcd} \mathcal{X}_{r+1,I} \arrow[r, "u", "\cong"'] \arrow[d, "\phi"] & \mathcal{Z}_{r,I} \arrow[d, "p_1"] \\ \mathcal{X}_{r,I} \arrow[r, "\mathrm{Id}"] & \mathcal{X}_{r,I} \end{tikzcd}$$

where the upper arrow is an isomorphism whose inverse p_2 is the unique extension of

$$\mathcal{Z}^\circ_{r,I} \to \mathcal{Y}_{r+1,I}; \; (\mathcal{E}, \mathcal{D}) \mapsto \langle p^{-1} \rangle (\mathcal{E}/\mathcal{D}).$$

Moreover, the quotient isogeny $g : \; \mathcal{E} \to \mathcal{E}/\mathcal{D}$ on $\mathcal{Z}^\circ_{r,I}$ induces a morphism

$$p_2^*(\mathbb{W}_\kappa) \to p_1^*(\mathbb{W}_\kappa)$$

such that the composition

$$H^0(\mathcal{X}_{r+1,I}, \mathbb{W}_\kappa) \xrightarrow{p_2^*} H^0(\mathcal{X}_{r+1,I}, p_2^*(\mathbb{W}_\kappa)) \to H^0(\mathcal{X}_{r,I}, p_1^*(\mathbb{W}_\kappa)) \xrightarrow{\frac{1}{p}\mathrm{Tr}_{\pi_1}} H^0(\mathcal{X}_{r,I}, \mathbb{W}_\kappa)$$

coincides with U_p.

Correspondences of degree prime to p

Let $\mathcal{E}$ be the universal semi-abelian scheme over $\mathfrak{X}_{r,I}$, and ψ_N be the universal $\Gamma_1(N)$-level structure on $\mathcal{E}$. Let ℓ be a rational prime such that $(\ell, Np) = 1$, and $\mathfrak{C}_{r,I}$ be the compactification of the formal scheme $\mathfrak{C}^\circ_{r,I}$ over $\mathfrak{Y}_{r,I}$ parameterizing rank-ℓ cyclic subgroups $\mathcal{C}$ of the universal elliptic $\mathcal{E}$ such that $\mathcal{C} \cap \mathrm{Im}(\psi_N) = 0$. The structure map $p_1 : \mathfrak{C}^\circ_{r,I} \to \mathfrak{Y}_{r,I}, (\mathcal{E}, \psi_N, \mathcal{C}) \mapsto (\mathcal{E}, \psi_N)$ is finite étale of degree $\ell + 1$ if $(\ell, Np) = 1$, resp. ℓ if $\ell \mid N$. According to [Con07, Lemma 4.2.3], p_1 extends uniquely to a finite flat morphism $p_1 : \mathfrak{C}_{r,I} \to \mathfrak{X}_{r,I}$. Analogous to Lemma 2.3.26, we have

Lemma A.2.19. *Let $\mathcal{C} \subset \mathcal{E}[\ell]$ over $\mathfrak{C}^\circ_{r,I}$ be the universal rank-ℓ cyclic subgroup and let $\mathfrak{C}_{n,r,I} := \mathfrak{IG}_{n,r,I} \times_{\mathfrak{X}_{r,I}, p_1} \mathfrak{C}_{r,I}$. The quotient isogeny $f : \mathcal{E} \to \mathcal{E}/C$ over $\mathfrak{C}^\circ_{r,I}$ induces morphisms $p_2 : \mathfrak{C}_{r,I} \to \mathfrak{X}_{r,I}$, $p_{2,n} : \mathfrak{C}_{n,r,I} \to \mathfrak{IG}_{n,r,I}$ such that we have the following commutative diagram*

$$\begin{array}{ccc} p_1^*(\mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{E}}, s_n)) & \longrightarrow & \mathcal{V}_0(\mathbb{H}^\sharp_{\mathcal{E}}, s_n) \\ \downarrow & & \downarrow \\ p_1^*(\mathcal{V}_0(\Omega_{\mathcal{E}}, s_n)) & \longrightarrow & \mathcal{V}_0(\Omega_{\mathcal{E}}, s_n) \\ \downarrow & & \downarrow \\ \mathfrak{C}_{n,r,I} & \xrightarrow{p_{2,n}} & \mathfrak{IG}_{n,r,I} \\ \downarrow & & \downarrow \\ \mathfrak{C}_{r,I} & \xrightarrow{p_2} & \mathfrak{X}_{r,I}, \end{array}$$

which is compatible with $\mathcal{T}^{\mathrm{ext}}$-actions and induces a morphism of $\mathcal{O}_{\mathfrak{C}_{r,I}}$-modules

$$f^* : p_2^*(\mathbb{W}_\kappa) \to p_1^*(\mathbb{W}_\kappa)$$

compatible with connection and filtration.

Since p_1 is finite flat, the trace operator Tr_{p_1} is well-defined.

Definition A.2.20. *For rational prime $\ell \neq p$, we define the Hecke operator T_ℓ to be the composition*

$$H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa) \xrightarrow{p_2^*} H^0(\mathfrak{C}_{r,I}, p_2^*(\mathbb{W}_\kappa)) \xrightarrow{f^*} H^0(\mathfrak{C}_{r,I}, p_1^*(\mathbb{W}_\kappa)) \xrightarrow{\frac{1}{\ell}\mathrm{Tr}_{p_1}} H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa).$$

Twists by finite characters

Fix a pre-adapted triple (n, r, I). Let $\mathcal{E}$ be the universal generalized elliptic curve and $H_n \subset \mathcal{E}$ be the level-n canonical subgroup. Let $f : \mathcal{E} \to \mathcal{E}' := \mathcal{E}/H_n$ be the quotient isogeny and $f' : \mathcal{E}' \to \mathcal{E}$ be the dual isogeny. Let $H'_n \subset \mathcal{E}'$ be the level-n canonical subgroup. Over $\mathfrak{Y}_{r+n,I}$, we have $H''_n = \ker f'$. Moreover, since f' induces an isomorphism $H'_n \cong H_n$ and $f \circ f' = p^n$, we have $\mathcal{E}'[p^n] = H''_n \times H'_n$ and the Weil pairing induces an isomorphism

$$H''_n \cong H_n^D \cong (H'_n)^D.$$

Over $\mathfrak{IG}^\circ_{2n,r+n,I}$, the section $\gamma' : \mathbb{Z}/p^n\mathbb{Z} \to H_n'^{,D} = H''_n$ defined by the diagram A.2.2 induces maps

$$s : \mathbb{Z}/p^n\mathbb{Z} \to H''_n, \quad s^D : H'_n \to \mu_{p^n}.$$

Let $K/\mathbb{Q}_p$ be any finite extension containing all p^n-th roots of unity with ring of integers in $\mathcal{O}_K$ and base change $\mathfrak{X}_{r,I}$, $\mathfrak{IG}_{n,r,I}$ from $\mathbb{Z}_p$ to $\mathcal{O}_K$ (but we do not change the notation). Note that a fixed p^n-th root of unity ζ determines an isomorphism

$$\mathbb{Z}/p^n\mathbb{Z} \cong \mathrm{Hom}_{\mathfrak{IG}_{2n,r+n,I}}(\mathbb{Z}/p^n\mathbb{Z}, \mu_{p^n}), \quad j \mapsto (j \mapsto \zeta^j).$$

Similar to Lemma 2.3.28, we have

Lemma A.2.21. [AI17, Proposition 3.26] *Over $\mathfrak{IG}^\circ_{2n,r+n,I}$, the map*

$$\eta:\ \mathrm{Hom}(H''_n, H'_n) \to \mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, \mu_{p^n}) \cong \mathbb{Z}/p^n\mathbb{Z},\ g \mapsto s^D \circ g \circ s$$

is a bijection.

Over $\mathfrak{IG}^\circ_{2n,r+n,I}$, for any $j \in \mathbb{Z}/p^n\mathbb{Z}$, let $\rho_j:\ H''_n \to H'_n$ be the morphism such that $\eta(\rho_j) = j$, and let

$$H_{\rho_j} := (\rho_j \times \mathrm{Id})(H''_n) \subset H'_n \times H''_n = \mathcal{E}'[p^n].$$

Clearly we have $H_{\rho_j} \times H'_n = \mathcal{E}'[p^n]$. Set $\mathcal{E}'_j := \mathcal{E}'/H_{\rho_j}$. The quotient map $f'_j:\ \mathcal{E}' \to \mathcal{E}'_j$ induces an isomorphism between H'_n and its image $H'_{n,j} := f'_j(H'_n)$, which implies that $H'_{n,j} \subset \mathcal{E}'_j$ is the canonical subgroup of $\mathcal{E}'_j$. Note that $H'_{n,j}$ is the kernel of the dual isogeny $\mathcal{E}'_j \to \mathcal{E}'$, so we have $\mathrm{Hdg}(\mathcal{E}'_j)^{p^n} = \mathrm{Hdg}(\mathcal{E}') = \mathrm{Hdg}(\mathcal{E})^{p^n}$, which implies $\mathrm{Hdg}(\mathcal{E}'_j) = \mathrm{Hdg}(\mathcal{E})$.

For each $j \in \mathbb{Z}/p^n\mathbb{Z}$, let $t_j:\ \mathfrak{IG}^\circ_{2n,r+n,I} \to \mathfrak{IG}^\circ_{n,r+n,I}$ be the normalization of the morphism $\mathcal{IG}^\circ_{2n,r+n,I} \to \mathcal{IG}^\circ_{n,r+n,I}$ which sends $(\mathcal{E}, \gamma_{2n})$ to $(\mathcal{E}'_j, \gamma'_j)$, where γ_{2n} is the universal trivialization of H^D_{2n} and γ'_j is the unique trivialization of $(H'_{n,j})^D$ such that $f'^{,D}_j(\gamma'_j) = \gamma'$. By a careful analysis of cusps using Tate curves, we can extend $t_j:\ \mathfrak{IG}^\circ_{2n,r+n,I} \to \mathfrak{IG}^\circ_{n,r+n,I}$ uniquely to

$$t_j:\ \mathfrak{IG}_{2n,r+n,I} \to \mathfrak{IG}_{n,r+n,I}.$$

Let $p_k:\ \mathfrak{IG}_{k,r+n,I} \to \mathfrak{X}_{r+n,I}$ be the forgetful morphism for $k \in \mathbb{N}$ (if meaningful) and v_j be the composition

$$\mathfrak{IG}_{2n,r+n,I} \xrightarrow{t_j} \mathfrak{IG}_{n,r+n,I} \xrightarrow{p_n} \mathfrak{X}_{r+n,I}.$$

By a similar argument as for Lemma 2.3.29 and a careful study of cusps using Tate curve (see Theorem A.1.4), we can show

Lemma A.2.22. *For an adapted triple (n, r, I), the isogeny $f_j:\ \mathcal{E} \xrightarrow{f} \mathcal{E}' \xrightarrow{f'_j} \mathcal{E}'_j$ over $\mathfrak{IG}^\circ_{2n,r+n,I}$ defines a morphism of $\mathcal{O}_{\mathfrak{IG}_{2n,r+n,I}}$-modules*

$$f_j^*:\ v_j^*(\mathbb{W}^0_\kappa) \to p_{2n}^*(\mathbb{W}^0_\kappa)$$

which preserves the filtration $\mathrm{Fil}_\bullet\mathbb{W}_\kappa$ and the Gauss-Manin connection ∇^0_κ. After inverting α, f_j induces a morphism

$$f_j^*:\ v_j^*(\mathbb{W}_\kappa)[\frac{1}{\alpha}] \to p_{2n}^*(\mathbb{W}_\kappa)[\frac{1}{\alpha}]$$

which preserves the filtration $\mathrm{Fil}_\bullet\mathbb{W}_\kappa$ and the Gauss-Manin connection ∇_κ.

Let $\chi:\ (\mathbb{Z}/p^n\mathbb{Z})^\times \to (\Lambda_I[\zeta])^\times$ be any primitive character and $g(\chi) := \sum_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j)\zeta^j$ be the associated Gauss sum. Following essentially the argument in the proof of Corollary 2.3.30,

Corollary A.2.23. *By taking global sections, we get a morphism*

$$\begin{aligned}\theta^\chi:\ H^0(\mathcal{X}_{r+n,I}, \mathbb{W}_\kappa) &\xrightarrow{g(\chi^{-1})(\sum_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j) f_j^* v_j^*)} H^0(\mathcal{X}_{r+n,I}, p_{2n,*}p_{2n}^*(\mathbb{W}_{\kappa+2\chi})) \\ &\xrightarrow{\frac{1}{p^{2n}}\mathrm{Tr}} H^0(\mathcal{X}_{r+n,I}, \mathbb{W}_{\kappa+2\chi})\end{aligned}$$

which preserves the filtration $\mathrm{Fil}_\bullet$.

A.2.3 Comparison with other constructions of the overconvergent modular sheaf

Let (n,r,I) be a pre-adapted triple. In [AIP, § 5.2], the authors construct a formal scheme $f_n:\ \mathfrak{F}_{n,r,I}\to\mathfrak{IG}_{n,r,I}$ which represents the functor

$$\mathrm{Spf}(R)\mapsto\{(P,\omega)\in(H_n^D(R)-H_n^D[p^{n-1}](R))\times\underline{\omega}_{\mathcal{E}}(R):\ \mathrm{HT}(P)=\omega\text{ in }\underline{\omega}_{\mathcal{E}}/p^n\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\}$$

from the category of affine formal schemes $\mathrm{Spf}(R)\to\mathfrak{X}_{r,I}$ where R is α-adically complete and α-torsion-free to the category of sets, and define the $\mathcal{T}^{\mathrm{ext}}(R)$-action on $\mathfrak{F}_{n,r,I}(R)$ by

$$(\lambda x)*(P,\omega)=(\lambda P,\lambda x\omega),\quad\forall\,\lambda\in\mathbb{Z}_p^\times,\ x\in\mathcal{T}(R),\ (P,\omega)\in\mathfrak{F}_{n,r,I}(R).$$

The authors also define the modular sheaf

$$\mathfrak{w}_{\kappa_I,0}:=(g_n\circ f_n)_*\mathcal{O}_{\mathfrak{F}_{n,r,I}}[(\kappa^0)^{-1}],\quad\mathfrak{w}_{\kappa_I}:=\mathfrak{w}_{\kappa_I,0}\otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}}\mathfrak{w}^{\kappa_{I,f}}.$$

for the universal character $\kappa_I:\ \mathbb{Z}_p^\times\to\Lambda_I^\times$, where $g_n:\ \mathfrak{IG}_{n,r,I}\to\mathfrak{X}_{r,I}$ is the structure map.

By [AIP, § 6.8], when restricted to $\mathcal{X}_{r,I}$, $\mathfrak{w}_{\kappa_I}$ is a locally free $\mathcal{O}_{\mathcal{X}_{r,I}}$-module sheaf whose global sections are overconvergent p-adic families of modular forms defined in [Pil13, § 5.1]. The Hecke operators away from p on $\mathfrak{w}_\kappa$ are defined in [Pil13, § 4.1] and the V_p- and U_p-operators on $\mathfrak{w}_{\kappa_I}$ are defined in [AIP, § 5.4.4] and in the discussion before [AIP, Proposition 6.10][2] respectively. According to [AI17, Lemma 3.7], we have

Lemma A.2.24. *Let (n,r,I) be a pre-adapted triple. There is a canonical isomorphism between $\mathcal{V}_0(\Omega_{\mathcal{E}},s)$ and $\mathfrak{F}_{n,r,I}$ which is compatible with the $\mathcal{T}^{\mathrm{ext}}$-actions. As a corollary, we have an isomorphism of $\mathcal{O}_{\mathfrak{X}_{r,I}}$-modules $\mathfrak{w}^{\kappa_I}\cong\mathfrak{w}_{\kappa_I}$ which is compatible with Hecke operators.*

Proof. For the isomorphism part, an argument analogous to Lemma A.2.24 works.

For the compatibility of V_p- and U_p-operators, it suffices to note that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{V}_0(\Omega_{\mathcal{E}},s_{n+1}) \arrow[r] \arrow[d, "\cong"] & p^*(\mathcal{V}_0(\Omega_{\mathcal{E}},s_n)) \arrow[r] & t_1^*(\mathcal{V}_0(\Omega_{\mathcal{E}},s_n)) \arrow[d, "\cong"] \\
\mathfrak{F}_{n+1,r+1,I} \arrow[rr] & & t_1^*(\mathfrak{F}_{n,r,I})
\end{tikzcd}$$

where the upper horizontal arrows are defined in Lemma A.2.10 and the lower horizontal arrow is constructed in [AIP, § 5.4.4].

For the compatibility of Hecke operators away from p, using the notation in Lemma A.2.19, it suffices to note that the isogeny $\mathcal{E}\to\mathcal{E}/\mathcal{C}$ over $\mathfrak{C}_{n,r,I}$ induces a commutative diagram

$$\begin{tikzcd}
p_1^*(\mathcal{V}_0(\Omega_{\mathcal{E}},s)) \arrow[r] \arrow[d, "\cong"] & p_2^*(\mathcal{V}_0(\Omega_{\mathcal{E}},s)) \arrow[d, "\cong"] \\
p_1^*(\mathfrak{F}_{n,r,I}) & p_2^*(\mathfrak{F}_{n,r,I}) \arrow[l]
\end{tikzcd}$$

where the two horizontal arrows are isomorphisms when restricted to the generic fibers. ☐

Note that for each $n\in\mathbb{N}$, $\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}$ is a finite étale Galois cover of $\mathfrak{X}_I^{\mathrm{ord}}$ with Galois group $(\mathbb{Z}/p^n\mathbb{Z})^\times$, so $\pi:\ \mathfrak{IG}^{\mathrm{ord}}_{\infty,r,I}:=\varprojlim_{n\geq0}\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}\to\mathfrak{X}_I^{\mathrm{ord}}$ is well-defined and equipped with an action of $\mathbb{Z}_p^\times$. Moreover, the argument in Proposition 2.3.32 implies the following integral version of [Pil13, Proposition 6.1].

[2] It seems that the definitions of U_p-operator in [Pil13] and [AIP] differ by the diamond operator $\langle p\rangle$.

Proposition A.2.25. *There is a canonical isomorphism*

$$\mathfrak{w}^{\kappa_I,0}_{|_{\mathfrak{X}^{\mathrm{ord}}_{r,I}}} \cong \pi_*(\mathcal{O}_{\mathfrak{IG}^{\mathrm{ord}}_{\infty,r,I}})[(\kappa_I^0)^{-1}].$$

We remark that at least another two equivalent constructions of overconvergent modular forms are available, one in [AIS14] in the spirit of test objects, and another one in [CHJ17, Theorem 1.1] using the perfectoid language.

A.3 Expansion in Serre-Tate coordinates: the modular curve case

Let $k = \bar{\mathbb{F}}_p$ and $W = W(k)$ be the associated ring of Witt vectors. Fix an ordinary elliptic curve $\bar{A}/k$ together with a $\Gamma_1(N)$-level structure ψ_N, and a basis $P \in T_p\bar{A} = T_p\bar{A}^D$. Let $\mathcal{R}$ be the deformation space of $\bar{A}/k$ and $\mathcal{A}/\mathcal{R}$ be the universal deformation of $\bar{A}/k$. By setting $t := q(\mathcal{A}/\mathcal{R}; P, P) - 1 \in \mathcal{R}$, we have $\mathcal{R} = W[[t]]$. Let P^t be the dual basis of P and

$$\omega_{\mathrm{can}} := \mathrm{HT}(P) \in \underline{\omega}_{\mathcal{A}}, \quad \eta_{\mathrm{can}} = \eta(P^t) \in \mathbb{H}_{\mathcal{A}}.$$

According to Theorem 1.4.8, $\{\omega_{\mathrm{can}}, \eta_{\mathrm{can}}\}$ is a basis of $\mathbb{H}_{\mathcal{A}}$ and

$$\nabla \begin{pmatrix} \omega_{\mathrm{can}} \\ \eta_{\mathrm{can}} \end{pmatrix} = \begin{pmatrix} 0 & d\log(1+t) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_{\mathrm{can}} \\ \eta_{\mathrm{can}} \end{pmatrix}.$$

Note that the pair $(\bar{A}/k, \psi_N)$ corresponds to a point $x \in X(k)$, so by adapting the argument for Lemma 2.4.3, we have

Lemma A.3.1. *There exists a canonical isomorphism* $\hat{\mathcal{O}}_{X,x}\hat{\otimes}W \cong W[[t]]$.

Convention A.3.2. *In this section, we will base change* $X := X_1(N)$, $\hat{X}$, $\mathfrak{X}_{r,I}$, $\mathfrak{IG}_{n,r,I}$, $\Lambda_I^{(0)}$ *from* $\mathbb{Z}_p$ *to* W, *and denote them using the same notation.*

Equip $W[[t]]$ with the p-adic topology and $R_t = \Lambda_I^0[[t]]$. Thanks to Lemma A.3.1, since $W[[t]]$ is p-adic complete, we have morphisms

$$\rho: \ \mathrm{Spf}(W[[t]]) \to \hat{X}, \quad \mathrm{Spf}(R_t) \to \mathfrak{X}_I^{\mathrm{ord}} (\to \mathfrak{X}_{r,I} \ \forall r \in \mathbb{N})$$

characterized by $\rho^*((\mathcal{E}, \psi_N)) = (\mathcal{A}, \Psi_N)$. Denote $\rho^*(\mathfrak{w}^\kappa)$, $\rho^*(\mathbb{W}_\kappa)(R_t)$, $\rho^*(\mathbb{W})(R_t)$ by $\mathfrak{w}^\kappa(t)$, $\mathbb{W}_\kappa(t)$, $\mathbb{W}(t)$ respectively. Analogous to Lemma 2.4.5, we have

Lemma A.3.3. *The evaluation maps*

$$H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa) \to \mathbb{W}_\kappa(t), \quad H^0(\mathfrak{X}_I^{\mathrm{ord}}, \mathbb{W}) \to \mathbb{W}(t),$$
$$H^0(\mathfrak{X}_{r,I}^{\mathrm{ord}}, \mathbb{W}/p^N\mathbb{W}) \to \mathbb{W}(t)/p^N\mathbb{W}(t), \quad \forall\ N \geq 1.$$

are injective.

Proof. Note that $\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}$ is geometrically irreducible, see for example in [Hid12, Theorem 3.3], so we can use precisely the same reasoning as in the proof of Lemma 2.4.5. □

We will use the following notation in the rest of this section:

- $\bar{A}/k$ will be a fixed ordinary elliptic curve with $\Gamma_1(N)$-level structure ψ_N and universal deformation $(\mathcal{A}/\mathcal{R}, \Psi_N)$;

- $P \in T_p\bar{A}(k)$ will be a fixed generator and P^t will be the dual basis of P;
- $1+t := q(\mathcal{A}/\mathcal{R}; P, P)$, $\omega_{\mathrm{can}} := \mathrm{HT}(P)$, and $\eta_{\mathrm{can}} := \eta(P^t)$;
- $R_t := \Lambda_I^0[[t]]$, $\tilde{R}_t := R_t \otimes_{\Lambda_I^0} \Lambda_I$, and $\rho :\ \mathrm{Spf}(R_t) \to \mathfrak{X}_I^{\mathrm{ord}} \to \mathfrak{X}_{r,I}$ will the morphisms characterized by $\rho^*(\mathcal{E}, \psi_N) = (\mathcal{A}, \Psi_N)$.
- $\mathfrak{w}^\kappa(t) := \rho^*(\mathfrak{w}^\kappa)(R_t)$, $\mathbb{W}_\kappa(t) := \rho^*(\mathbb{W}_\kappa)(R_t)$, $\mathbb{W}(t) := \rho^*(\mathbb{W})(R_t)$.

Recall that $(\mathbb{H}^\sharp_{\mathcal{E}}, \Omega_{\mathcal{E}}, s)$ is the system of vector bundles with marked sections associated to $\mathcal{E}$ over $\mathfrak{IG}_{n,r,I}$ with respect to the ideal $\underline{\beta}_n$. By considering the level-n canonical subgroup $\mathcal{C}_n \subset \mathcal{A}$, we can show that Lemma 2.4.6 and Proposition 2.4.7 hold in the modular curve case using almost the same argument.

Let ℓ be a rational prime such that $(\ell, p) = 1$, and let $a_\ell = \ell$ if $\ell \mid N$, $a_\ell = \ell + 1$ if $\ell \nmid N$. Let $\{D_j\}_{j=1,..,a_\ell}$ be the set of rank ℓ group subschemes of $\bar{A}/k$ which intersect $\mathrm{Im}(\psi_N)$ trivially. Each D_j deforms uniquely to a rank-ℓ group subscheme $\mathcal{D}_j \subset \mathcal{A}$ such that $\mathcal{D}_j \cap \mathrm{Im}(\Psi_N) = 0$. Let $(\bar{A}/D_j, \psi_j)$ be the quotient of $(\bar{A}/k, \psi_N)$ by D_j, and let $(\mathcal{A}_j/\mathcal{R}_j, \Psi_j)$ be the universal deformation of $(\bar{A}/D_j, \psi_j)$. Then $\mathcal{A}/\mathcal{D}_j$ over $\mathcal{R}$ induces a morphism

$$G_j :\ \mathrm{Spf}(\mathcal{R}) \to \mathrm{Spf}(\mathcal{R}_j); \quad G_j^*(\mathcal{A}_j) = \mathcal{A}/\mathcal{D}_j.$$

Then similar to Lemma 2.4.8, we have

Lemma A.3.4. *Let*

$$g_j :\ \mathcal{A} \to \mathcal{A}/\mathcal{D}_j, \quad g_j' :\ \mathcal{A}/\mathcal{D}_j \to \mathcal{A}$$

be the quotient isogeny and its dual over $\mathcal{R}$, let $P_j \in T_p\bar{A}/D_j(k)$ be the unique element such that $g_j'(P_j) = P$ with a dual basis P_j^t, and let

$$\omega_{\mathrm{can},j} := \mathrm{HT}(P_j), \quad \eta_{\mathrm{can},j} := \eta(P_j^t), \quad 1 + t_j := q(\mathcal{A}_j/\mathcal{R}_j; P_j, P_j) \in \mathcal{R}_j.$$

Then we have

$$g_j^*(G_j^*(\omega_{\mathrm{can},j})) = \omega_{\mathrm{can}}, \quad g_j^*(G_j^*(\eta_{\mathrm{can},j})) = \ell\eta_{\mathrm{can}}, \quad G_j^*(1+t_j) = (1+t)^{\frac{1}{\ell}}.$$

Proof. As $g_j' \circ g_j = g_j \circ g_j' = \ell$, both g_j and g_j' induce isomorphisms on Tate modules, so P_j is well-defined and $g_j^*(P_j^t) = \ell P^t$.

By the functoriality of the Hodge-Tate map, we have the following commutative diagram

$$\begin{tikzcd} T_p(\bar{A}/D_j)(k) \arrow[r, "\mathrm{HT}"] \arrow[d, "g_j'"] & \underline{\omega}_{\mathcal{A}/\mathcal{D}_j} \arrow[d, "g_j^*"] \\ T_p\bar{A}(k) \arrow[r, "\mathrm{HT}"] & \underline{\omega}_{\mathcal{A}}, \end{tikzcd}$$

so that we have $g_j^*(\omega_{\mathrm{can},j}) = \omega_{\mathrm{can}}$ and dually, $g_j^*(\eta_{\mathrm{can},j}) = l\eta_{\mathrm{can}}$. By Theorem 1.4.5 (*iii*), we have

$$\begin{aligned} G_j^*(1+t_j) &= q(\mathcal{A}/D_j/\mathcal{R}; P_j, P_j) = q(\mathcal{A}/D_j/\mathcal{R}; \frac{1}{\ell}g_j(P), P_j) \\ &= q(\mathcal{A}/\mathcal{R}; \frac{1}{\ell}P, g_j'(P_j)) = q(\mathcal{A}/\mathcal{R}; \frac{1}{\ell}P, P) \\ &= (1+t)^{\frac{1}{\ell}} \end{aligned}$$

□

Given Lemma A.3.4 and Proposition 2.4.7, by carrying over almost the same argument, we can show Proposition 2.4.9 holds in the modular curve setting.

Let $\mathcal{C} \subset \mathcal{A}$ be the level-1 canonical subgroup. The quotient isogeny

$$f_0 : \mathcal{A} \to \mathcal{A}/\mathcal{C}$$

is a lift of the Frobenius morphism

$$F : \bar{A} \to \bar{A}^{(p)}.$$

Let $(\bar{A}^{(p)}, \psi_N^{(p)})$ be the quotient of $(\bar{A}/k, \psi_N)$ by $\mathrm{Ker}(F)$, and let $(\mathcal{A}^{(p)}/\mathcal{R}^{(p)}, \Psi_N^{(p)})$ be the universal deformation of $(\bar{A}^{(p)}, \psi_N^{(p)})$. Then we have a morphism $\phi : \mathrm{Spf}(\mathcal{R}) \to \mathrm{Spf}(\mathcal{R}^{(p)})$ characterized by $\phi^*(\mathcal{A}^{(p)}) = \mathcal{A}/\mathcal{C}$. Since $\bar{A}/k$ is ordinary, we have an isomorphism $F : T_p\bar{A}(k) \cong T_p\bar{A}^{(p)}(k)$.

Lemma A.3.5. *Let $Q := F(P)$, Q^t be the dual basis of Q and*

$$1 + t^{(p)} := q(\mathcal{A}^{(p)}/\mathcal{R}^{(p)}; Q, Q), \quad \omega_{\mathrm{can}}^{(p)} := \mathrm{HT}(Q), \quad \eta_{\mathrm{can}}^{(p)} := \eta(Q^t).$$

We have

$$\phi^*(1 + t^{(p)}) = (1 + t)^p, \quad f_0^*\phi^*(\omega_{\mathrm{can}}^{(p)}) = p\omega_{\mathrm{can}}, \quad f_0^*\phi^*(\eta_{\mathrm{can}}^{(p)}) = \eta_{\mathrm{can}}.$$

Proof. Let $V : \bar{A}^{(p)} \to \bar{A}$ be the Verschiebung map. By Theorem 1.4.5, we have

$$\begin{aligned}
&\phi^*(1 + t^{(p)}) = \phi^*(q(\mathcal{A}^{(p)}/\mathcal{R}^{(p)}; Q, Q)) = q(\mathcal{A}/\mathcal{C}/\mathcal{R}; F(P), Q) \\
&= q(\mathcal{A}/\mathcal{R}; P, V(Q)) = q(\mathcal{A}/\mathcal{R}; P, pP) = (1 + t)^p; \\
&f_0^*\phi^*(\Omega_{\mathrm{can}}^{(p)}) = \mathrm{HT}(V(Q)) = p\mathrm{HT}(P) = p\omega_{\mathrm{can}}; \\
&f_0^*\phi^*(\eta_{\mathrm{can}}^{(p)}) = \eta(F^*(Q^t)) = \eta(P^t) = \eta_{\mathrm{can}}.
\end{aligned}$$

□

Given Lemma A.3.5 and Proposition 2.4.7, essentially the same argument shows Proposition 2.4.12 holds in the modular curve setting.

Fix a primitive p^n-th root of unity ζ_n, set $W_n := W[\zeta_n]$, and let $(1 + t)^{p^{-n}}$ be a p^n-th root of $1 + t$.

Proposition A.3.6. *Let $\mathcal{C}_n \subset \mathcal{A}$ be the level-n canonical subgroup. Then the connected-étale exact sequence*

$$0 \to \mathcal{C}_n \to \mathcal{A}[p^n] \to \mathcal{C}_n^D \to 0 \tag{A.3.1}$$

admits p^n different splittings over $W[[t]][\zeta_n, (1 + t)^{p^{-n}}]$.

Proof. We sketch an argument here for completeness and refer to [Hid13, Proposition 6.51] for details. Note that for each $n \geq 1$, we have isomorphisms

$$\eta^D : \mathcal{C}_n \cong \mu_p^n, \quad \eta : \mathbb{Z}/p^n\mathbb{Z} \cong \mathcal{C}_n^D.$$

A splitting of the sequence A.3.1 is equivalent to a lift $y \in \mathcal{A}[p^n]$ of $\eta(\frac{1}{p^n})$. Let $x_m \in \mathcal{A}[p^m]$ be a lift of $\eta(\frac{1}{p^m})$ for each $m \geq 1$. By the construction of Serre-Tate coordinates, (up to the isomorphism η^D) we may view $1 + t$ as $\lim_n p^n x_n$, so we have $y^{p^n} = 1 + t$. From this, we know all splittings of sequence A.3.1 happen over $W[[t]][\zeta_n, (1 + t)^{p^{-n}}]$.

Consider the canonical deformation A_{can}/W of $\bar{A}/k$, i.e. the deformation corresponding to $t = 0$. Then A_{can} has complex multiplications, so we have a canonical splitting of the connected-étale sequence

$$0 \to C_{n,\text{can}} \to A_{\text{can}}[p^n] \to C^D_{n,\text{can}} \to 0.$$

By Proposition A.2.21, for each $i \in \mathbb{Z}/p^n\mathbb{Z}$, the morphism

$$\mathbb{Z}/p^n\mathbb{Z} \to \mu_{p^n}; \quad 1 \mapsto \zeta_n^i$$

induces a homomorphism $\rho_i : C^D_{n,\text{can}} \to C_{n,\text{can}}$. Let

$$C_{n,i} := \text{Im}(\text{Id} \times \rho_i : C^D_{n,\text{can}} \to C^D_{n,\text{can}} \times C_{n,\text{can}} = A_{\text{can}}[p^n]).$$

Then $C_{n,i}$ are all the cyclic étale subgroup of $A_{\text{can}}[p^n]$ of rank p^n.

To complete the proof, we note that by Hensel's lemma, the map

$$W[[t]][\zeta, (1+t)^{p^{-n}}] \to W_n; \quad t \mapsto 0$$

induces a $1-1$ correspondence between rank-p^n étale group subschemes of $\mathcal{A}[p^n]$ and $A_{\text{can}}[p^n]$. □

Note that $\bar{A}[p^n] \cong \mu_{p^n} \oplus \mathbb{Z}/p^n\mathbb{Z}$. Let $(\bar{A}', \psi'_N)$ be the quotient of $(\bar{A}, \psi_N)$, and let $(\mathcal{A}'/\mathcal{R}', \Psi')$ be the universal deformation of $(\bar{A}', \psi'_N)$. Let $\mathcal{C}_i \subset \mathcal{A}$ be the étale subgroup corresponding to $C_{n,i} \subset A_{\text{can}}$ and $(\mathcal{A}/\mathcal{C}_i, \Psi_i)$ be the quotient of $(\mathcal{A}, \Psi_N)$ by $\mathcal{C}_i$. The pair $\langle p^{-1}\rangle(\mathcal{A}/\mathcal{C}_i, \Psi_i)$ over $W[[t]][\zeta_n, (1+t)^{p^{-n}}]$ induces a morphism

$$F_i : \text{Spf}(W[[t]][\zeta_n, (1+t)^{p^{-n}}]) \to \text{Spf}(\mathcal{R}')$$

characterized by $F_i^*(\mathcal{A}', \Psi'_N) = \langle p^{-1}\rangle(\mathcal{A}/\mathcal{C}_i, \Psi_i)$. Note that the quotient isogeny $f : \bar{A} \to \bar{A}'$ induces a morphism $f : T_p\bar{A}(k) \to T_p\bar{A}'(k)$ which is an isomorphism times multiplication by p^n.

Lemma A.3.7. *Let $Q = \frac{f(P)}{p^n}$, Q^t be the dual basis, and*

$$1 + t' := q(\mathcal{A}'/\mathcal{R}'; Q, Q), \ \omega'_{\text{can}} := \text{HT}(Q), \ \eta'_{\text{can}} := \eta(Q^t).$$

Then we have

$$F_i^*(1+t')^{p^n} = 1 + t, \quad F_i^*(1+t')|_{t=0} = \zeta_n^{-i}.$$

Let $f_i : \mathcal{A} \to \mathcal{A}/\mathcal{C}_i$ be the quotient isogeny. Then we have

$$f_i^* F_i^*(\omega'_{\text{can}}) = \omega_{\text{can}}; \quad f_i^* F_i^*(\eta'_{\text{can}}) = p^n \eta_{can}.$$

Proof. All can be shown by similar arguments as for Lemma A.3.4 and Lemma A.3.5 except the statement $F_i^*(1+t')|_{t=0} = \zeta^{-i}$, which was first shown in [Bra11, Proposition 7.2]. We sketch an argument here for completeness.

Over W_n, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mu_{p^\infty} \arrow[r] \arrow[d, "="] & A_{\text{can}}[p^\infty] \cong \mu_{p^\infty} \oplus \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] \arrow[d, "f_i"] & \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] \arrow[d, "p^n"] & 0 \\
0 \arrow[r] & \mu_{p^\infty} \arrow[r] & A_{\text{can}}/C_{n,i}[p^\infty] \arrow[r] & \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] & 0,
\end{tikzcd}$$

so $f_i(1/p^{n+m})$ is a lift of $\frac{1}{p^m}$ in $A_{\text{can}}/C_{n,i}[p^m]$. By the definition of Serre-Tate coordinates, we have

$$F_i^*(1+t')|_{t=0} = (1+t')(A_{\text{can}}/H_{n,i}) = \varinjlim_m p^m f_i(1/p^{n+m}) = f_i(1/p^n),$$

which is ζ_n^{-i} because $C_{n,i}$ is generated by $(\zeta_n^i, 1/p^n)$. □

Given Lemma A.3.7, we can show Proposition 2.4.15 and Proposition 2.4.16 in the modular curve setting by essentially the same argument.

With these preparations, by carrying over the arguments for Proposition 2.4.10, Proposition 2.4.15 and Proposition 2.4.17, we can show

Proposition A.3.8. *For any finite character $\chi : \mathbb{Z}_p^\times \to W_n^\times$ and any $f \in H^0(\mathcal{X}_{r,I}, \mathbb{W}_\kappa)$, we have*

$$pU_p(\nabla_\kappa(f)) = \nabla_\kappa \circ U_p(f); \quad \ell T_\ell \circ \nabla_\kappa(f) = \nabla_\kappa \circ T_\ell(f); \quad \theta^\chi \nabla_\kappa(f) = \nabla_{\kappa+2\chi} \circ \theta^\chi(f).$$

Recall that for each $N \geq 0$, $\nabla^N := \nabla_\kappa^N$ is the N-th iteration of ∇_κ. Since Lemma 2.4.19, Lemma 2.4.20 hold in the modular curve setting and by the argument of Proposition 2.4.18, we have

Proposition A.3.9. *Assume $\mu_\kappa \in p\Lambda_I^0$. Then for any $N \in \mathbb{N}_{\geq 1}$ and $f \in H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa)^{U_p}$, we have*

$$(\nabla^{p-1} - \mathrm{Id})^{Np}(\rho^*(f)) \in p^N \mathbb{W}(t) \cap \mathbb{W}'_\kappa(t).$$

At the end, we would like to mention that the behavior of the Gauss-Manin connection on q-expansions of nearly overconvergent modular forms is considered in [AI17, § 3.9] and [Urb14, § 3.5.2], and a detailed discussion of twists of p-adic modular forms by Dirichlet characters in q-expansion is contained in [Gou06, § 3.6].

A.4 p-adic iteration of the Gauss-Manin connection

Slightly generalizing [AI17, Theorem 4.6 & 4.13], we have the following theorem concerning p-adic iteration of the Gauss-Manin connection on nearly overconvergent modular forms. (For the Shimura curve analogue, see Theorem 2.5.1.)

Theorem A.4.1. *Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$. Fix an interval $I_\theta \subset [0,\infty)$ and an interval $I = [0,1]$ or $[p^a, p^b]$ for $a, b \in \mathbb{N}$, and assume that*

$$k : \mathbb{Z}_p^\times \to \Lambda_I; \quad \theta : \mathbb{Z}_p^\times \to \Lambda_{I_\theta,K}^\times, \quad \Lambda_{I_\theta,K} := \Lambda_{I_\theta} \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_K$$

are weights satisfying that

(i) there exists $c(k) \in \mathbb{N}$ such that $\mu_k + 2c(k) \in p\Lambda_I^0$;

(ii) for any $t \in \mathbb{Z}_p^\times$, $\theta(t) = \theta'(t)t^{c(\theta)}\chi(t)$ for a finite character $\chi : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$, an integer $c(\theta) \geq c(k)$ and a weight $\theta' : \mathbb{Z}_p^\times \to (\Lambda_{I_\theta,K}^0)^\times$ such that

$$\mu_{\theta'} \in q\Lambda_{I_\theta,K}^0, \quad \theta'(t) = \exp(\mu_\theta \log(t)) \ \forall\, t \in \mathbb{Z}_p^\times.$$

Then we have positive integers $r' \geq r$ and γ (depending on n, r and p, $c(\theta)$) such that for any $g \in H^0(\mathfrak{X}_{r,I}, \mathbb{W}_\kappa)^{U_p=0}$,

(a) the sequence $A(g,\theta')_m := \sum_{j=1}^m \frac{(-1)^{j-1}}{j}(\nabla_{k+2c(k)}^{(p-1)} - \mathrm{Id})^j \nabla_k^{c(k)}(g)$ and the sequence

$$B(g,\theta')_m := \sum_{i=0}^m \frac{1}{i!}\frac{\mu_{\theta'}^i}{(p-1)^i}\Bigg(\sum_{j_1+\ldots+j_i \leq m}(\prod_{a=1}^i \frac{(-1)^{j_a-1}}{j_a})(\nabla_{k+2c(k)}^{(p-1)} - \mathrm{Id})^{j_1+\ldots+j_a}\Bigg)\nabla_k^{c(k)}(g)$$

converge in $\mathrm{Hdg}^{-\gamma} H^0(\mathfrak{X}_{r',I} \otimes_{\mathbb{Z}_p} \Lambda_{I_\theta,K}, \mathbb{W})$;

(b) the limit

$$\nabla^{\theta'}_{k+2c(k)}\nabla^{c(k)}_k(g) := \exp(\frac{\mu_{\theta'}}{p-1}\log(\nabla^{p-1}_{k+2c(k)}))\nabla^{c(k)}_k(g) := \lim_{m\to\infty} B(g,\theta')_m$$

belongs to $\mathrm{Hdg}^{-\gamma}H^0(\mathfrak{X}_{r',I}\otimes_{\mathbb{Z}_p}\Lambda_{I_\theta,K},\mathbb{W}_{k+2\theta'})$*;*

(c) the element

$$\nabla^{\theta}_k(g) := \theta^{\chi}\nabla^{c(\theta)-c(k)}_{k+2c(k)+2\theta'}\nabla^{\theta'}_{k+2c(k)}\nabla^{c(k)}_k(g)$$

is well-defined in $H^0(\mathcal{X}_{r',I}\otimes_{\mathbb{Z}_p}\Lambda_{I_\theta,K},\mathbb{W}_{k+2\theta})$*.*

Proof. The argument is analogous to that of Theorem A.4.1. Namely, first we combine Lemma A.3.3 with Proposition A.3.9 to show the modular curve analogue [AI17, Proposition 4.3] of Proposition 2.5.3. Secondly, with the help of Lemma A.1.11, we can show that Proposition 2.5.4 and Corollary 2.5.5 hold in the modular curve setting using essentially the same argument. Finally, since the action $\mathcal{T}^{\mathrm{ext}}$ is analytic, with the help of Proposition A.3.8, we complete the proof. □

For any specialization $x\in\mathbb{Z}$ of k, and $y\in\mathbb{Z}$ of θ, and any $f\in H^0(\mathfrak{X}_{r,I},\mathbb{W}_k)^{U_p}$, let us denote the specialization of f at x by f_x, and the specialization of $\nabla^\theta_k(f)$ at the weight $x+2y$ by $\nabla^\theta_k(f)_{x,y}$. Then as Proposition 2.5.2, we have

Proposition A.4.2. *Notations as in Theorem A.4.1. Let* $x\in\mathbb{Z}$ *be any specialization of* k*,* $y'\in(p-1)\mathbb{N}$ *be a specialization of* $\chi\theta'$*, and let* $y := y'+c(\theta)$ *be the corresponding specialization of* θ*. Then for any* $f\in H^0(\mathfrak{X}_{r,I},\mathbb{W}_k)^{U_p}$*, we have*

$$\nabla^\theta_k(f)_{x,y} = \nabla^y_x(f_x).$$

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