Forward Countermodel Construction in Modal Logic K*

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Abstract. The inverse method is a saturation based theorem proving technique; it relies on a forward proof-search strategy and can be applied to cut-free calculi enjoying the subformula property. Here we apply this method to derive the unprovability of a formula in the modal logic **K**. To this aim, we design a forward calculus to check the **K**-satisfiability of a set of modal formulas. From a derivation of Ξ, we can extract a Kripke model of Ξ.

1 Introduction

The inverse method, introduced by Maslov [8], is a saturation based theorem proving technique closely related to (hyper)resolution [3]; it relies on a forward proof-search strategy and can be applied to cut-free calculi enjoying the subformula property. Given a goal, a set of instances of the rules of the calculus at hand is selected; such specialized rules are repeatedly applied in the forward direction, starting from the axioms (i.e., the rules without premises). Proof-search terminates if either the goal is obtained or the set collecting the proved facts saturates (nothing new can be added). The inverse method has been originally applied to Classical Logic and successively extended to some non-classical logics [2,3,4,7].

In all of the mentioned papers, the inverse method has been exploited to prove the validity of a formula in a specific logic. In [6] we launched a new perspective designing a forward calculus to derive the unprovability of a goal formula in Intuitionistic Propositional Logic. In this paper we begin to study the applicability of such an approach to modal logics considering the case of the basic modal logic **K**, semantically characterized by finite intransitive trees [1].

We design a forward refutation calculus to check the **K**-satisfiability of a set of modal formulas Ξ (namely, the non-validity of the formula $\neg(\bigwedge \Xi)$ in **K**); to constructively ascertain this we show how to extract from a derivation τ asserting the satisfiability of Ξ, a model $\mathrm{Mod}(\tau)$ of Ξ. In forward reasoning, it is crucial to bound the application of rules, so that the naive saturation forward procedure eventually terminates (see the Finite Rule Property [3]). To achieve this, the rules of the calculus are determined by the goal set Ξ; we call the resulting calculus $\mathbf{RK}(\Xi)$ (Forward Refutation in **K** with goal Ξ). Each node of

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an $\mathbf{RK}(\Xi)$-derivation is a set containing formulas constructed using the ones in Ξ. Axioms of $\mathbf{RK}(\Xi)$ are maximal consistent sets of the set of literals determined by Ξ. Rules of $\mathbf{RK}(\Xi)$ are inspired by semantics, so that proof-search for a derivation of Ξ mirrors the construction of a model of Ξ. A remarkable result is that the models extracted from $\mathbf{RK}(\Xi)$-derivations are in general "small". This is mainly due to the fact that in forward derivations we can re-use the same premise many times, without duplicating it, and this reduces the generation of redundant worlds. Actually, for every satisfiable set Ξ, we can build a derivation of Ξ in $\mathbf{RK}(\Xi)$ such that the height of the extracted model is minimal.

2 The calculus RK(Ξ)

We consider the language $\mathcal{L}$ built from a denumerable set $\mathcal{V}$ of propositional variables and the connectives $\wedge$, $\neg$ and $\Box$. We denote formulas by α, β, ... and set of formulas by Γ, Δ, ...; $\Box\Gamma$ is the set $\{\Box\alpha \mid \alpha \in \Gamma\}$. By $\mathcal{L}_{\mathrm{CL}}$ we denote the set of *classical formulas* of $\mathcal{L}$, namely the formulas not containing $\Box$; with $\mathcal{H}$, $\mathcal{X}$, ... sets of classical formulas, we call *cl-sets*. A *literal* is a formula of the kind p or $\neg p$, with $p \in \mathcal{V}$. A set of literals $\mathcal{X}$ is *lit-consistent* iff there is no $p \in \mathcal{V}$ such that $\{p, \neg p\} \subseteq \mathcal{X}$. The set $\mathrm{Sf}^+(\Gamma)$ is the smallest set of formulas such that $\Gamma \subseteq \mathrm{Sf}^+(\Gamma)$ and:

- $\alpha \wedge \beta \in \mathrm{Sf}^+(\Gamma)$ implies $\{\alpha, \beta\} \subseteq \mathrm{Sf}^+(\Gamma)$; $\neg\neg\alpha \in \mathrm{Sf}^+(\Gamma)$ implies $\alpha \in \mathrm{Sf}^+(\Gamma)$;
- $\neg(\alpha \wedge \beta) \in \mathrm{Sf}^+(\Gamma)$ implies $\{\neg\alpha, \neg\beta\} \subseteq \mathrm{Sf}^+(\Gamma)$;
- $\Box\alpha \in \mathrm{Sf}^+(\Gamma)$ implies $\alpha \in \mathrm{Sf}^+(\Gamma)$; $\neg\Box\alpha \in \mathrm{Sf}^+(\Gamma)$ implies $\neg\alpha \in \mathrm{Sf}^+(\Gamma)$.

Let W be a non-empty finite set of worlds and R be an intransitive *successor* relation on W (i.e., $\forall x, y, z(xRy \wedge yRz \rightarrow \neg xRz)$; note that R is irreflexive as well). An *intransitive tree with root* ρ is a triple (W, R, ρ) such that the reflexive and transitive closure of R is a tree partial order on W with root ρ [1]. A *model* for $\mathcal{L}$ is a structure $\mathcal{M} = \langle W, R, \rho, V\rangle$, where (W, R, ρ) is an intransitive tree with root ρ and V, the *evaluation* function, maps each $p \in \mathcal{V}$ to a subset of W (the set of worlds where p is true). A *(classical) interpretation* is a subset of $\mathcal{V}$; for $w \in W$, by $\mathcal{I}(w)$ we denote the interpretation defined by the propositional variables true at w (i.e., $p \in \mathcal{I}(w)$ iff $w \in V(p)$). The relation $(\mathcal{M}, w) \models \alpha$ (the formula α is *valid* in $\mathcal{M}$ at world w) is defined as usual by induction on α (see e.g. [1]):

- $(\mathcal{M}, w) \models p$, with $p \in \mathcal{V}$, iff $w \in V(p)$;
- $(\mathcal{M}, w) \models \Box\alpha$ iff, for every $v \in W$, wRv implies $(\mathcal{M}, v) \models \alpha$.

The interpretation of $\neg$ and $\wedge$ is classical. Note that the validity of a classical formula at a world w only depends on $\mathcal{I}(w)$. Expressions of the kind Γ, α and Γ, Δ denote the sets $\Gamma \cup \{\alpha\}$ and $\Gamma \cup \Delta$ respectively. By $(\mathcal{M}, w) \models \Gamma$ we mean that, for every $\alpha \in \Gamma$, $(\mathcal{M}, w) \models \alpha$. A set Γ is *(*$\mathbf{K}$*-)satisfiable* iff there exists a model $\mathcal{M}$ and a world w of $\mathcal{M}$ such that $(\mathcal{M}, w) \models \Gamma$. It is well-known that the modal logic $\mathbf{K}$ is the set of formulas α of $\mathcal{L}$ such that, for every model $\mathcal{M}$ and every world w of $\mathcal{M}$, $(\mathcal{M}, w) \models \alpha$ (see [1]); accordingly, α is valid in $\mathbf{K}$ iff

$$\frac{}{\mathcal{X}}\ \mathrm{Lit} \qquad \mathcal{X} \text{ is a maximal lit-consistent subset of } \mathrm{Sf}^+(\Xi)$$

$$\frac{\Gamma, \alpha, \beta}{\Gamma, \alpha, \beta, \alpha \wedge \beta}\ \wedge \qquad \frac{\Gamma, \neg\alpha_k}{\Gamma, \neg\alpha_k, \neg(\alpha_1 \wedge \alpha_2)}\ \neg\wedge \qquad \frac{\Gamma, \alpha}{\Gamma, \alpha, \neg\neg\alpha}\ \neg\neg$$

Cl-rules

Applied only if the formula introduced in the conclusion belongs to $\mathrm{Sf}^+(\Xi)$

$$\frac{\Gamma_1 \quad \cdots \quad \Gamma_n \quad \mathcal{H}}{[\![\, \Box(\bigcap_{i=1}^n \Gamma_i)\,,\ \{\, \neg\Box\alpha \mid \neg\alpha \in \bigcup_{i=1}^n \Gamma_i \,\}]\!],\ \mathcal{H}}\ \bowtie \quad n \geq 1$$

$$\frac{\mathcal{H}}{\{\Box\alpha \mid \Box\alpha \in \mathrm{Sf}^+(\Xi)\},\ \mathcal{H}}\ \bowtie_0$$

Join rules

$[\![\, \Gamma \,]\!] = \Gamma \cap \mathrm{Sf}^+(\Xi)$

Fig. 1. The calculus $\mathbf{RK}(\Xi)$.

the set $\{\neg\alpha\}$ is not satisfiable. In this paper we present a forward calculus to constructively prove the satisfiability of a set of formulas.

The calculus $\mathbf{RK}(\Xi)$ is a forward refutation calculus to prove that a finite nonempty set Ξ of formulas of $\mathcal{L}$ is satisfiable, meaning that the formula $\neg(\bigwedge \Xi)$ is not valid in $\mathbf{K}$. In forward calculi proof-search starts from axioms and rules are applied from premises to the conclusion (forward direction). The rules are displayed in Fig. 1. Axioms of $\mathbf{RK}(\Xi)$, introduced by the axiom rule Lit, are the cl-sets $\mathcal{X}$ such that $\mathcal{X}$ is a *maximal* lit-consistent subset of $\mathrm{Sf}^+(\Xi)$, namely: there is no lit-consistent set $\mathcal{Y}$ such that $\mathcal{X} \subset \mathcal{Y} \subseteq \mathrm{Sf}^+(\Xi)$. Cl-rules are standard refutation rules for classical connectives; they introduce a formula of the kind $\alpha \wedge \beta$, $\neg(\alpha \wedge \beta)$ and $\neg\neg\alpha$. To get termination, we only admit rule applications generating formulas in $\mathrm{Sf}^+(\Xi)$: this motivates the side condition on the application of cl-rules. Join rules $\bowtie$ and $\bowtie_0$ allow one to introduce formulas of the kind $\Box\alpha$ and $\neg\Box\alpha$. The rule $\bowtie$ has $n + 1 \geq 2$ premises, where at least one premise $\mathcal{H}$ is a cl-set. Rule $\bowtie_0$ is the degenerated instance of $\bowtie$ only having the premise $\mathcal{H}$. In both cases, the formulas introduced by the rule applications must belong to $\mathrm{Sf}^+(\Xi)$ (see the definition of the $[\![\,.\,]\!]$ operator). An $\mathbf{RK}(\Xi)$-*derivation* of Δ is a derivation in $\mathbf{RK}(\Xi)$ with the set Δ as root. A set Γ is *provable in* $\mathbf{RK}(\Xi)$ iff there exists an $\mathbf{RK}(\Xi)$-derivation τ of Δ such that $\Gamma \subseteq \Delta$; we also say that Γ is *proved by* τ.

Soundness

Given an $\mathbf{RK}(\Xi)$-derivation τ of Δ, we can build a $\mathbf{K}$-model satisfying Δ; this proves the soundness $\mathbf{RK}(\Xi)$. The height of τ, denoted $\mathrm{h}(\tau)$, is the maximum distance from the root of τ and an axiom sequent of τ. We define the structure $\mathrm{Mod}(\tau)$ inductively on the height of τ.

- If $\mathrm{h}(\tau) = 0$, then τ only consists of an application of Lit and Δ is a maximal lit-consistent subset of $\mathrm{Sf}^+(\Xi)$. We set $\mathrm{Mod}(\tau) = \langle \{\rho\}, \emptyset, \rho, V \rangle$, where $\emptyset$ is the empty successor relation and $V(p) = \{\rho\}$ if $p \in \Delta$ and $V(p) = \emptyset$ otherwise.

– If $\mathrm{h}(\tau) > 0$, let $\mathcal{R}$ be the rule applied at the root of τ.
  (1) If $\mathcal{R}$ is a cl-rule or $\mathcal{R} = \bowtie_0$, let τ' be the immediate subderivation of τ. Then $\mathrm{Mod}(\tau) = \mathrm{Mod}(\tau')$. Note that, if τ is a derivation of a cl-set $\mathcal{H}$, τ only consists of one instance of Lit followed by instances of cl-rules, accordingly $\mathrm{Mod}(\tau)$ consists of a single world.
  (2) Otherwise, $\mathcal{R} = \bowtie$. Let τ_i be the subderivations with root Γ_i and τ_{n+1} the one with root $\mathcal{H}$; let $\mathrm{Mod}(\tau_i) = \langle W_i, R_i, \rho_i, V_i \rangle$ $(1 \leq i \leq n+1)$. Since $\mathcal{H}$ is a cl-set, as discussed in Point (1), we have $W_{n+1} = \{\rho\}$. Without loss of generality, we assume that the W_j's are pairwise disjoint. We set $\mathrm{Mod}(\tau) = \langle W, R, \rho, V \rangle$ where:

$$W = \bigcup_{i=1}^{n+1} W_i, \qquad V = \bigcup_{i=1}^{n+1} V_i, \qquad R = \bigcup_{i=1}^{n} (\, R_i \cup \{\, (\rho, \rho_i) \,\}\,)$$

It is easy to check that $\mathrm{Mod}(\tau)$ is a well-defined model. Moreover, proceeding by induction on the height of τ one can prove:

Theorem 1 (Soundness of $\mathbf{RK}(\Xi)$). *Let τ be an $\mathbf{RK}(\Xi)$-derivation of Δ and ρ be the root of $\mathrm{Mod}(\tau)$. Then $(\mathrm{Mod}(\tau), \rho) \models \Delta$.*

Completeness

We prove that the calculus $\mathbf{RK}(\Xi)$ is complete, namely: if Ξ is a finite satisfiable set, then Ξ is provable in $\mathbf{RK}(\Xi)$. We give a constructive proof by exhibiting how to build an $\mathbf{RK}(\Xi)$-derivation τ of $\Delta \supseteq \Xi$ starting from a model $\mathcal{M} = \langle W, R, \rho, V \rangle$ of Ξ. The *height* of a world $w_0 \in W$, denoted by $\mathrm{h}(w_0)$, is the maximal length of an R-chain $w_0 R w_1 R w_2 \ldots$; since $\langle W, R, \rho \rangle$ is a finite tree, the definition is well-found. The height of $\mathcal{M}$, denoted by $\mathrm{h}(\mathcal{M})$, coincides with $\mathrm{h}(\rho)$. We show that $\mathrm{h}(\mathrm{Mod}(\tau)) \leq \mathrm{h}(\mathcal{M})$; accordingly, we can use $\mathbf{RK}(\Xi)$ to generate "small" models of Ξ. To formalize this, we introduce the following definitions:

– Ξ is *h-satisfiable* iff there is a model $\mathcal{M}$ of Ξ such that $\mathrm{h}(\mathcal{M}) \leq h$.
– If Ξ is satisfiable, $\mathrm{h}(\Xi)$ is the minimum h such that Ξ is h-satisfiable.

The *rank* $\mathrm{Rn}(\tau)$ of an $\mathbf{RK}(\Xi)$-derivation τ is the maximum number of applications of rule $\bowtie$ along a branch of τ. Formally, let r be the root rule of τ and let $\tau_1, \ldots, \tau_n$ be the immediate subderivations of τ. Then:

$$\mathrm{Rn}(\tau) = \begin{cases} 0 & \text{if } r \text{ is the axiom rule Lit} \\ c + \max\{\, \mathrm{Rn}(\tau_1), \ldots, \mathrm{Rn}(\tau_n) \,\} & \text{otherw.} \end{cases} \qquad c = \begin{cases} 1 & \text{if } r = \bowtie \\ 0 & \text{otherw.} \end{cases}$$

Note that $\mathrm{h}(\mathrm{Mod}(\tau)) = \mathrm{Rn}(\tau)$. In the proof of next lemma we show how to build a derivation of a satisfiable set. Note that Theorem 2 immediately follows from Lemma 1. By $|\Gamma|$ we denote the number of symbols occurring in Γ.

Lemma 1. *Let Ξ be a finite set of formulas and let $\Gamma \subseteq \mathrm{Sf}^+(\Xi)$ be a h-satisfiable set. Then, there exists an $\mathbf{RK}(\Xi)$-derivation τ of Δ such that $\Gamma \subseteq \Delta$ and $\mathrm{Rn}(\tau) \leq h$. Moreover, $\Gamma \subseteq \mathcal{L}_{\mathrm{CL}}$ implies $\Delta \subseteq \mathcal{L}_{\mathrm{CL}}$.*

Proof. We prove the assertion by induction hypothesis (IH) on $|\Gamma|$. Since Γ is h-satisfiable, there exists a model $\mathcal{M} = \langle W, R, \rho, V\rangle$ and $w \in W$ such that $(\mathcal{M}, w) \models \Gamma$ and $\mathrm{h}(w) \leq h$. We proceed through a case analysis, only detailing some representative cases.

Case 1: Γ only contains literals.
Let $\mathcal{X}$ be the set of literals $l \in \mathrm{Sf}^+(\Xi)$ such that $(\mathcal{M}, w) \models l$. Then, $\mathcal{X}$ is a maximal lit-consistent subset of $\mathrm{Sf}^+(\Xi)$, hence $\mathcal{X}$ is an axiom of $\mathbf{RK}(\Xi)$. Since $\Gamma \subseteq \mathcal{X} \subseteq \mathcal{L}_{\mathrm{CL}}$, the assertion holds.

Case 2: $\Gamma = \neg(\alpha_1 \wedge \alpha_2), \Gamma_0$, with $\neg(\alpha_1 \wedge \alpha_2) \notin \Gamma_0$.
Since $(\mathcal{M}, w) \models \neg(\alpha_1 \wedge \alpha_2)$, there exists $k \in \{1, 2\}$ such that $(\mathcal{M}, w) \models \neg\alpha_k$. Let $\Gamma_k = \Gamma_0 \cup \{\neg\alpha_k\}$; note that $(\mathcal{M}, w) \models \Gamma_k$, hence Γ_k is h-satisfiable. Since $|\Gamma_k| < |\Gamma|$ and $\Gamma_k \subseteq \mathrm{Sf}^+(\Xi)$, by (IH) there exists an $\mathbf{RK}(\Xi)$-derivation τ_k of Δ_k such that $\Gamma_k \subseteq \Delta_k$ and $\mathrm{Rn}(\tau_k) \leq h$. Let τ be:

$$\dfrac{\begin{array}{c}\vdots\ \tau_k \\ \neg\alpha_k,\ \Gamma_0,\ \Theta_k\end{array}}{\Delta\ =\ \neg(\alpha_1 \wedge \alpha_2),\ \Gamma_0,\ \Theta_k}\ \neg\wedge \qquad \Theta_k\ =\ \Delta_k \setminus \Gamma_k$$

Since $\Gamma \subseteq \Delta$ and $\mathrm{Rn}(\tau) = \mathrm{Rn}(\tau_k) \leq h$, the assertion holds. Moreover, if $\Gamma \subseteq \mathcal{L}_{\mathrm{CL}}$, by (IH) we get $\Delta_k \subseteq \mathcal{L}_{\mathrm{CL}}$, hence $\Delta \subseteq \mathcal{L}_{\mathrm{CL}}$.

Case 3: $\Gamma = \Box\Theta,\ \neg\Box\alpha_1, \dots, \neg\Box\alpha_n,\ \mathcal{H}$, with $n \geq 1$.
Let $j \in \{1, \dots, n\}$ and $\Gamma_j = \Theta \cup \{\neg\alpha_j\}$. Since $(\mathcal{M}, w) \models \neg\Box\alpha_j$, there exists $w_j \in W$ such that wRw_j and $(\mathcal{M}, w_j) \models \neg\alpha_j$. We also have $(\mathcal{M}, w_j) \models \Theta$, hence Γ_j is $\mathrm{h}(w_j)$-satisfiable. Since $|\Gamma_j| < |\Gamma|$ and $\Gamma_j \subseteq \mathrm{Sf}^+(\Xi)$, by (IH) there exists an $\mathbf{RK}(\Xi)$-derivation τ_j of Δ_j such that $\Gamma_j \subseteq \Delta_j$ and $\mathrm{Rn}(\tau_j) \leq \mathrm{h}(w_j)$, hence $\mathrm{Rn}(\tau_j) < h$. Since $\mathcal{H}$ is h-satisfiable (indeed, $(\mathcal{M}, w) \models \mathcal{H}$) and $\mathcal{H} \subseteq \mathcal{L}_{\mathrm{CL}}$ and $|\mathcal{H}| < |\Gamma|$, by (IH) there exists an $\mathbf{RK}(\Xi)$-derivation τ_c of $\mathcal{G}$ such that $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{L}_{\mathrm{CL}}$. We can define τ as follows:

$$\dfrac{\dots \quad \begin{array}{c}\vdots\ \tau_j \\ \Theta,\ \neg\alpha_j,\ \Upsilon_j\end{array} \quad \dots \quad \begin{array}{c}\vdots\ \tau_c \\ \mathcal{H},\ \mathcal{U}\end{array}}{\Delta\ =\ \Box\Theta,\ \neg\Box\alpha_1, \dots, \neg\Box\alpha_n,\ \mathcal{H},\ \Sigma}\ \bowtie \qquad \begin{array}{l} 1 \leq j \leq n \\ \Upsilon_j\ =\ \Delta_j \setminus \Gamma_j \\ \mathcal{U}\ =\ \mathcal{G} \setminus \mathcal{H}\end{array}$$

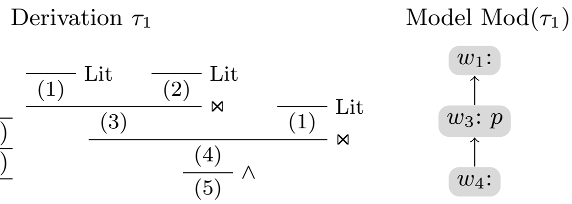
where we leave understood the formulas in the (possibly empty) set Σ. We have $\Gamma \subseteq \Delta$. Note that τ_c cannot contain applications of rule $\bowtie$, hence $\mathrm{Rn}(\tau_c) = 0$. This implies that $\mathrm{Rn}(\tau) = m+1$, where m is the maximum $\mathrm{Rn}(\tau_j)$. Since $m < h$, we get $\mathrm{Rn}(\tau) \leq h$, and this concludes the proof of this case. □

As a consequence of the previous lemma we get:

Theorem 2 (Completeness of $\mathbf{RK}(\Xi)$). *Let Ξ be a finite set of satisfiable formulas. Then, there exists an $\mathbf{RK}(\Xi)$-derivation τ of $\Delta \subseteq \Xi$ such that such that* $\mathrm{h}(\mathrm{Mod}(\tau)) \leq \mathrm{h}(\Xi)$.

DB of proved sets

(1)	$\neg p$	Lit
(2)	p	Lit
(3)	$\neg\Box p\,,\, p$	$\bowtie$ (1) (2)
(4)	$\Box p, \neg\Box\Box p\,,\, \neg p$	$\bowtie$ (3) (1)
(5)	$\Box p \wedge \neg\Box\Box p\,,\, \neg p$	$\wedge$(4)



$$\mathrm{Sf}^{+}(\Xi_1) \;=\; \{\, \Box p \wedge \neg\Box\Box p\,,\, \Box p\,,\, \neg\Box\Box p\,,\, p\,,\, \neg\Box p\,,\, \neg p \,\}$$

Fig. 2. Example 1

Examples

Let $\alpha = \Box p \wedge \neg\Box\Box p$. Note that α corresponds to the negation of the transitivity axiom (4) $\Box p \to \Box\Box p$. It is well-known that (4) is not valid in **K**, hence $\Xi_1 = \{\alpha\}$ is satisfiable. We populate the database DB of sets provable in $\mathbf{RK}(\Xi_1)$ according with the naive recipe of [3]; the obtained DB is displayed in Fig 2 (for the sake of conciseness, we only show the sets needed to get the goal Ξ_1). We start by inserting the axioms ((1) and (2)); then we enter a loop where, at each iteration, we apply the rules to the set collected in previous steps. The iterations stop when, either a superset of the goal is generated (as in our example) or the database saturates (no more new sequents can be generated). The tree-like structure is given by derivation τ_1. As for the model Mod(τ_1), the world w_i is generated by the rule applied to get the sequent (i) in the database, an arrow from w_i to w_j indicates that $w_i R w_j$, while the interpretation $\mathcal{I}(w_i)$ associated with w_i is displayed after the colon (hence $\mathcal{I}(w_1) = \mathcal{I}(w_4) = \emptyset$ and $\mathcal{I}(w_3) = \{p\}$). It is easy to check that $(\mathrm{Mod}(\tau_1), w_4) \models \alpha$. We conclude that Mod($\tau_1$) is a model of $\{\alpha\}$ and a countermodel for the transitivity axiom. We remark that $\mathrm{h}(\mathrm{Mod}(\tau_1)) = 2$, which is the minimal height of a model of Ξ_1.

A more significant example is given in Fig. 3, where we show an $\mathbf{RK}(\Xi_2)$-derivation of a satisfiable set Ξ_2 (read $\Diamond$ as $\neg\Box\neg$) and the related model.

3 Future work

We have presented the naive forward proof-search strategy; we leave as future work the investigation of clever strategies (e.g., the use of subsumption to reduce redundancies) and the implementation of the calculus exploiting the full-fledged Java Framework JTabWb [5]. In this paper we only focus on **K**; we plan to extend the techniques here introduced to other modal logics.

References

1. A. Chagrov and M. Zakharyaschev. *Modal Logic*. Oxford Univ. Press, 1997.
2. K. Chaudhuri, F. Pfenning, and G. Price. A logical characterization of forward and backward chaining in the inverse method. In U. Furbach et al., editor, *IJCAR 2006*, volume 4130 of *LNCS*, pages 97–111. Springer, 2006.

$$\Xi_2 \;=\; \{\, A\,,\ \Diamond(B \,\wedge\, \Diamond(A \wedge \Diamond\Box D))\,,\ \Diamond(C \,\wedge\, \Diamond(A \wedge \Diamond\Box E))\,\}$$

$$A \,=\, \neg p \wedge \neg q \qquad B \,=\, p \wedge \neg q \qquad C \,=\, \neg p \wedge q \qquad D \,=\, p \wedge \neg p \qquad E \,=\, q \wedge \neg q$$

(1)	$\neg p\,,\ \neg q$	Lit
(2)	$p\,,\ \neg q$	Lit
(3)	$\neg p\,,\ q$	Lit
(4)	$A\,,\ \neg p\,,\ \neg q$	$\wedge$ (1)
(5)	$B\,,\ p\,,\ \neg q$	$\wedge$ (2)
(6)	$C\,,\ \neg p\,,\ q$	$\wedge$ (3)
(7)	$\Box D\,,\ \Box E\,,\ \neg p\,,\ \neg q$	$\bowtie_0$ (1)
(8)	$\neg\neg\Box D\,,\ \neg\neg\Box E\,,\ \Box D\,,\ \Box E\,,\ \neg p\,,\ \neg q$	$\neg\neg$ (7) (two times)
(9)	$\Diamond\Box D\,,\ \Diamond\Box E\,,\ A\,,\ \neg p\,,\ \neg q$	$\bowtie$ (8) (4)
(10)	$A \wedge \Diamond\Box D\,,\ A \wedge \Diamond\Box E\,,\ \Diamond\Box D\,,\ \Diamond\Box E\,,\ A\,,\ \neg p\,,\ \neg q$	$\wedge$ (9) (two times)
(11)	$\neg\neg(A \wedge \Diamond\Box D)\,,\ \neg\neg(A \wedge \Diamond\Box E)\,,\ \ldots$	$\neg\neg$(10) (two times)
(12)	$\Diamond(A \wedge \Diamond\Box D)\,,\ \Diamond(A \wedge \Diamond\Box E)\,,\ B\,,\ p\,,\ \neg q$	$\bowtie$ (11) (5)
(13)	$\Diamond(A \wedge \Diamond\Box D)\,,\ \Diamond(A \wedge \Diamond\Box E)\,,\ C\,,\ \neg p\,,\ q$	$\bowtie$ (11) (6)
(14)	$\neg\neg(B \wedge \Diamond(A \wedge \Diamond\Box D))\,,\ B \wedge \Diamond(A \wedge \Diamond\Box D)\,,\ \ldots$	$\wedge$ (12) followed by $\neg\neg$
(15)	$\neg\neg(C \wedge \Diamond(A \wedge \Diamond\Box E))\,,\ C \wedge \Diamond(A \wedge \Diamond\Box E)\,,\ \ldots$	$\wedge$ (13) followed by $\neg\neg$
(16)	$\Diamond(B \wedge \Diamond(A \wedge \Diamond\Box D))\,,\ \Diamond(C \wedge \Diamond(A \wedge \Diamond\Box E))\,,\ A\,,\ \neg p\,,\ \neg q$	$\bowtie$ (14)(15)(4)

$$\mathrm{Sf}^+(\Xi_2) \;=\; \left\{ \begin{array}{l} q, p, \neg p, \neg q, A, B, C, D, E, \Box D, \Box E, \neg\neg\Box D, \neg\neg\Box E \\ \Diamond\Box D, \Diamond(A \wedge \Diamond\Box D), \neg\neg(A \wedge \Diamond\Box D), (A \wedge \Diamond\Box D), \\ \Diamond\Box E, \Diamond(A \wedge \Diamond\Box E), \neg\neg(A \wedge \Diamond\Box E), (A \wedge \Diamond\Box E), \\ \Diamond(B \wedge \Diamond(A \wedge \Diamond\Box D)), \neg\neg(B \wedge \Diamond(A \wedge \Diamond\Box D)), \\ B \wedge \Diamond(A \wedge \Diamond\Box D), \Diamond(C \wedge \Diamond(A \wedge \Diamond\Box E)), \\ \neg\neg(C \wedge \Diamond(A \wedge \Diamond\Box E)), C \wedge \Diamond(A \wedge \Diamond\Box E), \end{array} \right\}$$

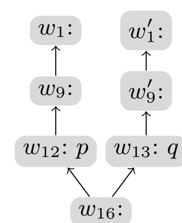


Fig. 3. Example 2

3. A. Degtyarev and A. Voronkov. The inverse method. In J.A. Robinson et al., editor, *Handbook of Automated Reasoning*, pages 179–272. Elsevier and MIT Press, 2001.
4. K. Donnelly, T. Gibson, N. Krishnaswami, S. Magill, and S. Park. The inverse method for the logic of bunched implications. In F. Baader et al., editor, *LPAR 2004*, volume 3452 of *LNCS*, pages 466–480. Springer, 2004.
5. M. Ferrari, C. Fiorentini, and G. Fiorino. JTabWb: a Java framework for implementing terminating sequent and tableau calculi. *Fundamenta Informaticae*, 150:119–142, 2017.
6. Camillo Fiorentini and Mauro Ferrari. A forward unprovability calculus for intuitionistic propositional logic. In R. A. Schmidt et al., editor, *TABLEAUX 2017*, volume 10501 of *LNCS*, pages 114–130. Springer, 2017.
7. L. Kovács, A. Mantsivoda, and A. Voronkov. The inverse method for many-valued logics. In F. Castro-Espinoza et al., editor, *MICAI 2013*, volume 8265 of *LNCS*, pages 12–23. Springer, 2013.
8. S. Ju. Maslov. An invertible sequential version of the constructive predicate calculus. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 4:96–111, 1967.