Analysis of an Inverse Problem Arising in Photolithography

Luca Rondi* Fadil Santosa†

Abstract

We consider the inverse problem of determining an optical mask that produces a desired circuit pattern in photolithography. We set the problem as a shape design problem in which the unknown is a two-dimensional domain. The relationship between the target shape and the unknown is modeled through diffractive optics. We develop a variational formulation that is well-posed and propose an approximation that can be shown to have convergence properties. The approximate problem can serve as a foundation to numerical methods.

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1 Introduction

Photolithography is a key process in the production of integrated circuits. It is the process by which circuit patterns are transferred onto silicon wafers. A review of this manufacturing technology is given in [16]. The main step in photolithography is the creation of a circuit image on the photoresist coating which sits on the silicon layer that is to be patterned. The image is formed using ultra-violet (UV) light which is diffracted by a mask, and refracted by a system of lenses. The mask simply consists of cut-outs, and lets light through the holes. The parts of the photoresist that are exposed to the UV light can be removed, leaving openings to the layer to be patterned. The next stage is etching, which removes material in the layer that is unprotected by the photoresist. Once etching is done, the photoresist can be removed, and the etched away "channels" may be filled. The entire process is illustrated schematically in Figure 1.

The problem we address in this work is the inverse problem of determining what mask is needed in order to remove a desired shape in the photoresist. The difficulty of producing a desired shape comes from the fact that the UV light is diffracted at the mask. Moreover, the chemicals in the photoresist reacts nonlinearly to UV exposure – only portions of the photoresist that have been exposed to a certain level of intensity are removed in the bleaching process.

The nature of the present work is analytical. Our goal is to formulate mathematically well-posed problems for photolithography. The methods we use to prove well-posedness are constructive and may serve as a foundation for a computational method.

*Dipartimento di Matematica e Informatica, Università degli Studi di Trieste, via Valerio, 12/1, 34127 Trieste, Italy. `rondi@units.it`

†School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA. `santosa@math.umn.edu`

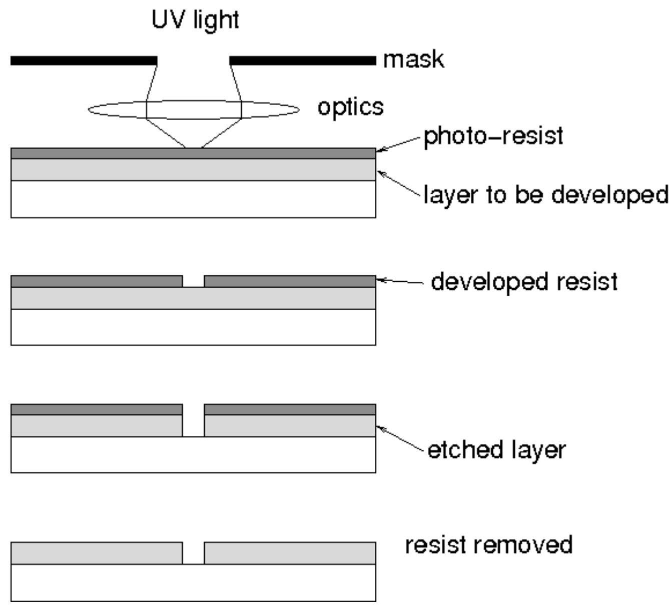

Figure 1: The photolithographic process. Ultraviolet light, diffracted by a mask, forms an image on the photoresist. The exposed portion of the photoresist is removed, leaving openings. Etching removes parts of the layer to be patterned. After etching, the photoresist is removed.

Our investigation into photolithography is inspired by the work of Cobb [3] who was the first to approach this problem from the point of view of optimal design which utilizes a physically-based model. This general approach was further developed by introducing a level set method in [18]. A different computational approach which models the mask as a pixelated binary image can be found in [14].

The plan of the paper is as follows. In the first and preliminary section, Section 2, we develop the most basic model for removal of the exposed photoresist. We describe the inverse problem to be solved. This is followed by a discussion of the approximate problem whose properties we intend to investigate in this work. Section 3 contains mathematical preliminaries needed for our work. We introduce the basic notation and recall various results which will be useful for our analysis. In particular, in Subsection 3.3 we discuss the geometry of masks or circuits and how to measure the distance between two of them. In Section 4 we discuss the properties of the operator which maps the mask into the circuit. Section 5 provides an analysis of the variational approach to the problem of the optimization of the mask and we prove a convergence result for it, Theorem 5.5, in the framework of Γ-convergence.

2 Description of the inverse problem

This section is separated into three subsections. First, we review some basic facts about Fourier transforms and prove a result about approximation of a Gaussian. We follow this with a discussion of the optics involved and a model for photolithography. In the final subsection we describe the inverse problem and its approximation.

2.1 Fourier transform and approximation of Gaussians

We first set some notation and describe a few preliminary results. For every $x \in \mathbb{R}^2$, we shall set $x = (x_1, x_2)$, where x_1 and $x_2 \in \mathbb{R}$. For every $x \in \mathbb{R}^2$ and $r > 0$, we shall denote by $B_r(x)$ the open ball in $\mathbb{R}^2$ centered at x of radius r. Usually we shall write B_r instead of $B_r(0)$. We recall that, for any set $E \subset \mathbb{R}^2$, we denote by χ_E its characteristic function, and for any $r > 0$, $B_r(E) = \bigcup_{x \in E} B_r(x)$.

For any $f \in \mathcal{S}'(\mathbb{R}^2)$, the space of tempered distributions, we denote by $\hat{f}$ its *Fourier transform*, which, if $f \in L^1(\mathbb{R}^2)$, may be written as

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} f(x) \mathrm{e}^{-\mathrm{i}\xi \cdot x} \mathrm{d}x, \quad \xi \in \mathbb{R}^2.$$

We recall that $f(x) = (2\pi)^{-2} \hat{\hat{f}}(-x)$, that is, when also $\hat{f} \in L^1(\mathbb{R}^2)$,

$$f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(\xi) \mathrm{e}^{\mathrm{i}\xi \cdot x} \mathrm{d}\xi, \quad x \in \mathbb{R}^2.$$

If f is a radial function, that is $f(x) = \phi(|x|)$ for any $x \in \mathbb{R}^2$, then

$$\hat{f}(\xi) = 2\pi \mathcal{H}_0(\phi)(|\xi|), \quad \xi \in \mathbb{R}^2,$$

where

$$\mathcal{H}_0(\phi)(s) = \int_0^{+\infty} r J_0(sr) \phi(r) \mathrm{d}r, \quad s \geq 0,$$

is the Hankel transform of order 0, J_0 being the Bessel function of order 0, see for instance [4].

We denote the Gaussian distribution by $G(x) = (2\pi)^{-1} \mathrm{e}^{-|x|^2/2}$, $x \in \mathbb{R}^2$, and let us note that $\hat{G}(\xi) = \mathrm{e}^{-|\xi|^2/2}$, $\xi \in \mathbb{R}^2$. Moreover, $\|G\|_{L^1(\mathbb{R}^2)} = 1$. Furthermore if δ_0 denotes the Dirac delta centered at 0, we have $\widehat{\delta_0} \equiv 1$, therefore $(2\pi)^{-2} \hat{1} = \delta_0$.

For any function f defined on $\mathbb{R}^2$ and any positive constant s, we denote $f_s(x) = s^{-2} f(x/s)$, $x \in \mathbb{R}^2$. We note that $\|f_s\|_{L^1(\mathbb{R}^2)} = \|f\|_{L^1(\mathbb{R}^2)}$ and $\widehat{f_s}(\xi) = \hat{f}(s\xi)$, $\xi \in \mathbb{R}^2$.

We conclude these preliminaries with the following integrability result for the Fourier transform and its applications.

Theorem 2.1 *There exists an absolute constant C such that the following estimate holds*

$$\|\hat{f}\|_{L^1(\mathbb{R}^2)} \leq C \|f\|_{W^{2,1}(\mathbb{R}^2)}.$$

PROOF. This result is contained in Theorem A in [9] and it is based on previous analysis done in [13]. □

We recall that a more detailed analysis on conditions for which integrability of the Fourier transform holds may be found in [17]. However the previous result is simple to use and it is enough for our purposes, in particular for proving the following lemma.

Lemma 2.2 *For any $\tilde{\delta} > 0$ there exist a constant s_0, $0 < s_0 \leq 1$, and a radial function $\hat{T} \in C_0^\infty(\mathbb{R}^2)$ such that $\hat{T} \equiv 1$ on B_{s_0} and, if we call $T = (2\pi)^{-2} \hat{\hat{T}}$, then $T \in W^{2,1}(\mathbb{R}^2)$ and*

$$\|T - G\|_{W^{1,1}(\mathbb{R}^2)} \leq \tilde{\delta}.$$

Proof. We sketch the proof of this result. Let us consider the following cut-off function $\phi \in C^\infty(\mathbb{R})$ such that ϕ is nonincreasing, $\phi \equiv 1$ on$(-\infty, 0]$ and $\phi \equiv 0$ on $[1, +\infty)$.

We define a function $\hat{\tilde{T}}$ as follows

$$\hat{\tilde{T}}(x) = \phi(|x| - 1) + (1 - \phi(|x| - 1))\widehat{G_{s_0}}(x)\phi(|x| - b), \quad x \in \mathbb{R}^2,$$

for suitable constants s_0, $0 < s_0 \leq 1$ and $b \geq 2$. We call $\tilde{T} = (2\pi)^{-2}\hat{\hat{\tilde{T}}}$.

Then lengthy but straightforward computations, with the aid of Theorem 2.1, allow us to prove that for some s_0 small enough and for some $b = s_0^{-1}b_0$, with b_0 large enough, we have

$$\|\tilde{T} - G_{s_0}\|_{W^{1,1}(\mathbb{R}^2)} \leq \tilde{\delta}.$$

Then, let $\hat{T}(x) = \hat{\tilde{T}}(x/s_0)$, $x \in \mathbb{R}^2$, so that $T = \tilde{T}_{1/s_0}$, or equivalently $\tilde{T} = T_{s_0}$. Therefore

$$\|T_{s_0} - G_{s_0}\|_{W^{1,1}(\mathbb{R}^2)} \leq \tilde{\delta}.$$

By a simple rescaling argument we have that $\hat{T}$ satisfies the required properties. Furthermore, by this construction, we may choose $\hat{T}$ such that it is radially nonincreasing, $\hat{T} \equiv 1$ on B_{s_0} and it decays to zero in a suitable smooth, exponential way. □

2.2 A model of image formation

We are now in the position to describe the model we shall use. The current industry standard for modeling the optics is based on Kirchhoff approximation. Under this approximation, the light source at the mask is on where the mask is open, and off otherwise (see Figure 1). Propagation through the lenses can be calculated using Fourier optics. It is further assumed that the image plane, in this case the plane of the photoresist, is at the focal distance of the optical system. If there were no diffraction, a perfect image of the mask would be formed on the image plane. Diffraction, together with partial coherence of the light source, acts to distort the formed image.

The mask, which we mention consists of cut-outs, is represented as a binary function, i.e., it is a characteristic function of the cut-outs. Suppose that D represents the cut-outs, then the mask is given by

$$m(x) = \chi_D(x).$$

The image is the light intensity on the image plane. This is given by [12]

$$I(x) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} m(\xi)K(x - \xi)J(\xi - \eta)K(x - \eta)m(\eta)\mathrm{d}\xi\mathrm{d}\eta, \quad x \in \mathbb{R}^2. \tag{2.1}$$

In the above expression the kernel $K(\cdot)$ is called the *coherent point spread function* and describes the optical system. For an optical system with a circular aperture, once the wavenumber of the light used, $k > 0$, has been chosen, the kernel depends on a single parameter called the Numerical Aperture, NA. Notice that the wavelength is $\lambda = 2\pi/k$. Let us recall that the so-called Jinc function is defined as

$$\mathrm{Jinc}(x) = \frac{J_1(|x|)}{2\pi|x|}, \quad x \in \mathbb{R}^2,$$

where J_1 is the Bessel function of order 1. We notice that in the Fourier space, see for instance [6, page 14],

$$\widehat{\mathrm{Jinc}}(\xi) = \chi_{B_1}(\xi), \quad \xi \in \mathbb{R}^2.$$

If we denote by $s = (k\mathrm{NA})^{-1}$, then the kernel is usually modeled as follows

$$K(x) = \mathrm{Jinc}_s(x) = \frac{k\mathrm{NA}}{2\pi}\frac{J_1(k\mathrm{NA}|x|)}{|x|}, \quad x \in \mathbb{R}^2,$$

therefore

$$\hat{K}(\xi) = \chi_{B_1}(s\xi) = \chi_{B_{1/s}}(s\xi) = \chi_{B_{k\mathrm{NA}}}(\xi), \quad \xi \in \mathbb{R}^2.$$

If NA goes to $+\infty$, that is $s \to 0^+$, then $\hat{K}$ converges pointwise to 1, thus K approximates in a suitable sense the Dirac delta.

For technical reasons, we shall consider a slightly different coherent point spread function K. Let us fix a positive constant $\tilde{\delta}$, to be chosen later. We shall replace the characteristic function χ_{B_1}, the Fourier transform of the Jinc function, with the function $\hat{T}(s_0\xi)$, $\xi \in \mathbb{R}^2$, with $\hat{T}$ and s_0 as in Lemma 2.2. Therefore $\hat{T}(s_0\cdot)$ is a radial function that it is still identically equal to 1 on B_1, it is still compactly supported, it is nonincreasing with respect to the radial variable and it decays to zero in a smooth, exponential way. Its Fourier transform is T_{s_0} and we shall assume that

$$K(x) = (T_{s_0})_s(x) = T_{ss_0}(x), \quad x \in \mathbb{R}^2, \tag{2.2}$$

where again $s = (k\mathrm{NA})^{-1}$. Also in this model, if NA goes to $+\infty$, that is $s \to 0^+$, then $\hat{K}$ converges pointwise to 1, thus K approximates in a suitable sense the Dirac delta.

The function $J(\cdot)$ is called the *mutual intensity function*. If the illumination is fully coherent, $J \equiv 1$. In practice, illumination is never fully coherent and is parametrized by a *coherency coefficient* σ. A typical model for J is

$$J(x) = 2\frac{J_1(k\sigma\mathrm{NA}|x|)}{k\sigma\mathrm{NA}|x|} = \pi\mathrm{Jinc}(k\sigma\mathrm{NA}|x|), \quad x \in \mathbb{R}^2. \tag{2.3}$$

Thus,

$$\frac{1}{(2\pi)^2}\hat{J}(\xi) = \frac{1}{\pi(k\sigma\mathrm{NA})^2}\chi_{B_{k\sigma\mathrm{NA}}}(\xi), \quad \xi \in \mathbb{R}^2,$$

that, as $\sigma \to 0^+$, converges, in a suitable sense, to the Dirac delta. Therefore full coherence is achieved for $\sigma \to 0^+$. In fact, if $\sigma \to 0^+$, J converges to 1 uniformly on any compact subset of $\mathbb{R}^2$. The equation (2.1) is often referred to as the Hopkins areal intensity representation. As it will become apparent from the analysis developed in the paper, the value of s is related to the scale of details that the manufacturing of the mask allows, thus in turn to the scale of details of the desired circuit. Therefore, we typically consider $k\mathrm{NA} \gg 1$, that is $s \ll 1$, and $k\sigma\mathrm{NA} \ll 1$.

2.3 The inverse problem and its approximation

The photoresist material responds to the intensity of the image. When intensity at the photoresist goes over a certain threshold, it is then considered exposed and can be removed. Therefore, the exposed pattern, given a mask $m(x)$, is

$$\Omega = \{x \in \mathbb{R}^2 \, : \, I(x) > h\}, \tag{2.4}$$

where h is the exposure threshold. Clearly, Ω depends on the mask function $m(x)$, which we recall is given by the characteristic function of D representing the cut-outs, that is $\Omega = \Omega(D)$. In photolithography, we have a desired exposed pattern which we wish to achieve. The inverse problem is to find a mask that achieves this desired exposed pattern. Mathematically, this cannot, in general, be done. Therefore, the inverse problem must be posed as an optimal design problem.

Suppose the desired pattern is given by Ω_0. We pose the minimization problem

$$\min_{D\in\mathcal{A}} d(\Omega(D), \Omega_0). \tag{2.5}$$

The distance function $d(\cdot,\cdot)$ will be discussed in detail below. The admissible set $\mathcal{A}$ is our search space, and needs to be defined carefully as well.

Instead of solving (2.5), we pose a variational problem for a function u (instead of the mask D). We will show below that this problem is well-posed and that as the approximation parameter is set to zero, we recover the solution of (2.5) under a perimeter penalization.

Instead of dealing with the characteristic function $\chi_D(x)$ which represents the mask, we will work with a phase-field function u which takes on values of 0 and 1 with smooth transitions. Thus, the intensity in (2.4) is calculated with u instead of $m = \chi_D$ in (2.1), so I is a function of u. At this point, we will not be precise about the space of functions to which u belongs. To force u to take on values of mostly 0 and 1, we introduce the Mordica-Mortola energy

$$P_\varepsilon(u) = \frac{1}{\varepsilon}\int W(u) + \varepsilon\int |\nabla u|^2,$$

where $W(t) = 9t^2(t-1)^2$ is a double-well potential. We will regularize the problem of minimizing the distance between the target pattern and the exposed region by this energy.

Then we relax the hard threshold in defining the exposed region Ω in (2.4). Let $\phi(t)$ be a C^∞ nondecreasing approximate Heaviside function with values $\phi(t \le -1/2) = 0$ and $\phi(t \ge 1/2) = 1$. The function

$$\Phi_\eta(u) = \phi\left(\frac{I(u)-h}{\eta}\right)$$

will be 1 where the intensity $I \ge h + \eta/2$. A sigmoidal threshold function is employed in the computational work in [14].

Now we consider the distance function between Ω and Ω_0 in (2.5). Let

$$d = d(\Omega,\Omega_0) = \int |\chi_\Omega - \chi_{\Omega_0}| + |P(\Omega) - P(\Omega_0)|\,, \tag{2.6}$$

where χ_Ω is the characteristic function of the set Ω and $P(\Omega)$ is the perimeter of the region Ω. To approximate this distance function, we replace it by

$$d_\eta(u,\Omega_0) = \int |\Phi_\eta(u) - \chi_{\Omega_0}| + \left|\int |\nabla(\Phi_\eta(u))| - P(\Omega_0)\right|.$$

The characteristic function of Ω is replaced by the smooth threshold function while its perimeter is replaced by the TV-norm of the function.

The approximate problem we shall solve is

$$F_\varepsilon(u) = d_{\eta(\varepsilon)}(u,\Omega_0) + bP_\varepsilon(u) \to \min\,.$$

The remainder of the paper is an analytical study of this minimization problem. We will show that it is well-posed, and that in the limit $\varepsilon \to 0^+$, we recover the solution of the original problem (2.5) under a perimeter penalization.

3 Mathematical preliminaries

By $\mathcal{H}^1$ we denote the 1-dimensional Hausdorff measure and by $\mathcal{L}^2$ we denote the 2-dimensional Lebesgue measure. We recall that, if $\gamma \subset \mathbb{R}^2$ is a smooth curve, then $\mathcal{H}^1$ restricted to γ coincides with its arclength. For any Borel $E \subset \mathbb{R}^2$ we denote $|E| = \mathcal{L}^2(E)$.

Let $\mathcal{D}$ be a bounded open set contained in $\mathbb{R}^2$, with boundary $\partial\mathcal{D}$. We say that $\mathcal{D}$ has a *Lipschitz boundary* if for every $x = (x_1, x_2) \in \partial\mathcal{D}$ there exist a Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$ and a positive constant r such that for any $y \in B_r(x)$ we have, up to a rigid transformation,

$$y = (y_1, y_2) \in \mathcal{D} \quad \text{if and only if} \quad y_2 < \varphi(y_1).$$

We note that $\mathcal{D}$ has a finite number of connected components, whereas $\partial\mathcal{D}$ is formed by a finite number of rectifiable Jordan curves, therefore $\mathcal{H}^1(\partial\mathcal{D}) = \text{length}(\partial\mathcal{D}) < +\infty$.

We recall some basic notation and properties of functions of bounded variation and sets of finite perimeter. For a more comprehensive treatment of these subjects see, for instance, [1, 7, 8].

Given a bounded open set $\mathcal{D} \subset \mathbb{R}^2$, we denote by $BV(\mathcal{D})$ the Banach space of *functions of bounded variation*. We recall that $u \in BV(\mathcal{D})$ if and only if $u \in L^1(\mathcal{D})$ and its distributional derivative Du is a bounded vector measure. We endow $BV(\mathcal{D})$ with the standard norm as follows. Given $u \in BV(\mathcal{D})$, we denote by $|Du|$ the total variation of its distributional derivative and we set $\|u\|_{BV(\mathcal{D})} = \|u\|_{L^1(\mathcal{D})} + |Du|(\mathcal{D})$. We shall call $P(u, \mathcal{D}) = |Du|(\mathcal{D})$. We recall that whenever $u \in W^{1,1}(\mathcal{D})$, then $u \in BV(\mathcal{D})$ and $|Du|(\mathcal{D}) = \int_{\mathcal{D}} |\nabla u|$, therefore $\|u\|_{BV(\mathcal{D})} = \|u\|_{L^1(\mathcal{D})} + \|\nabla u\|_{L^1(\mathcal{D})} = \|u\|_{W^{1,1}(\mathcal{D})}$.

We say that a sequence of $BV(\mathcal{D})$ functions $\{u_h\}_{h=1}^{\infty}$ *weakly** *converges* in $BV(\mathcal{D})$ to $u \in BV(\mathcal{D})$ if and only if u_h converges to u in $L^1(\mathcal{D})$ and Du_h weakly* converges to Du in $\mathcal{D}$, that is

$$\lim_h \int_{\mathcal{D}} v \mathrm{d}Du_h = \int_{\mathcal{D}} v \mathrm{d}Du \quad \text{for any } v \in C_0(\mathcal{D}). \tag{3.1}$$

By Proposition 3.13 in [1], we have that if a sequence of $BV(\mathcal{D})$ functions $\{u_h\}_{h=1}^{\infty}$ is bounded in $BV(\mathcal{D})$ and converges to u in $L^1(\mathcal{D})$, then $u \in BV(\mathcal{D})$ and u_h converges to u weakly* in $BV(\mathcal{D})$.

We say that a sequence of $BV(\mathcal{D})$ functions $\{u_h\}_{h=1}^{\infty}$ *strictly converges* in $BV(\mathcal{D})$ to $u \in BV(\mathcal{D})$ if and only if u_h converges to u in $L^1(\mathcal{D})$ and $|Du_h|(\mathcal{D})$ converges to $|Du|(\mathcal{D})$. Indeed,

$$d_{st}(u, v) = \int_{\mathcal{D}} |u - v| + \big| |Du|(\mathcal{D}) - |Dv|(\mathcal{D}) \big|$$

is a distance on $BV(\mathcal{D})$ inducing the strict convergence. We also note that strict convergence implies weak* convergence.

Let $\mathcal{D}$ be a bounded open set with Lipschitz boundary. A sequence of $BV(\mathcal{D})$ functions $\{u_h\}_{h=1}^{\infty}$ such that $\sup_h \|u_h\|_{BV(\mathcal{D})} < +\infty$ admits a subsequence converging weakly* in $BV(\mathcal{D})$ to a function $u \in BV(\mathcal{D})$, see for instance Theorem 3.23 in [1]. As a corollary, we infer that for any $C > 0$ the set $\{u \in BV(\mathcal{D}) \; : \; \|u\|_{BV(\mathcal{D})} \leq C\}$ is a compact subset of $L^1(\mathcal{D})$.

For any fixed constant $R > 0$, with a slight abuse of notation, we shall identify $L^1(B_R)$ with the set $\{u \in L^1(\mathbb{R}^2) \; : \; u = 0 \text{ a.e. outside } B_R\}$.

Let E be a bounded Borel set contained in $B_R \subset \mathbb{R}^2$. We shall denote by χ_E its characteristic function. We notice that E is compactly contained in B_{R+1}, which we shall denote by $E \Subset B_{R+1}$. We say that E is a *set of finite perimeter* if χ_E belongs to $BV(B_{R+1})$ and we call the number $P(E) = |D\chi_E|(B_{R+1})$ its *perimeter*. Analogously, for any $u \in L^1(B_R) \cap BV(B_{R+1})$, we shall denote $P(u, B_{R+1}) = |Du|(B_{R+1})$. Obviously, if $u = \chi_E$, then $P(u, B_{R+1}) = P(E)$.

Let us further remark that the intersection of two sets of finite perimeter is still a set of finite perimeter. Moreover, whenever E is open and $\mathcal{H}^1(\partial E)$ is finite, then E is a set of finite perimeter, see for instance [7, Section 5.11, Theorem 1]. Therefore a bounded open set $\mathcal{D}$ with Lipschitz boundary is a set of finite perimeter and its perimeter $P(\mathcal{D})$ coincides with $\mathcal{H}^1(\partial\mathcal{D})$.

3.1 Γ-convergence approximation of the perimeter functional

Let us introduce the following, slightly different, version of a Γ-convergence result due to Modica and Mortola, [11]. We shall follow the notation and proofs contained in [2]. We begin by setting some notation. For the definition and properties of Γ-convergence we refer to [5].

For any bounded open set $\mathcal{D} \subset \mathbb{R}^2$, with a slight abuse of notation, we identify $W_0^{1,p}(\mathcal{D})$, $1 < p < +\infty$, with the subset of $W^{1,p}(\mathbb{R}^2)$ functions u such that u restricted to $\mathcal{D}$ belongs to $W_0^{1,p}(\mathcal{D})$ and u is equal to 0 almost everywhere outside $\mathcal{D}$. Let us assume that for some positive constant R we have $\mathcal{D} \subset B_R$. We recall that any function in $L^1(\mathcal{D})$ is extended to zero outside $\mathcal{D}$ and the same procedure is used for $L^1(B_R)$. Therefore, with this slight abuse of notation, $L^1(\mathcal{D}) \subset L^1(B_R)$. Throughout the paper, for any p, $1 \leq p \leq +\infty$, we shall denote its conjugate exponent by p', that is $p^{-1} + (p')^{-1} = 1$.

Theorem 3.1 *Let $\mathcal{D} \subset B_R \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary. Let us also assume that $\mathcal{D}$ is convex.*

Let $1 < p < +\infty$ and $W : \mathbb{R} \to [0, +\infty)$ be a continuous function such that $W(t) = 0$ if and only if $t \in \{0, 1\}$. Let $c_p = (\int_0^1 (W(s))^{1/p'} \mathrm{d}s)^{-1}$.

For any $\varepsilon > 0$ we define the functional $P_\varepsilon : L^1(\mathbb{R}^2) \to [0, +\infty]$ as follows

$$P_\varepsilon(u) = \begin{cases} \dfrac{c_p}{p'\varepsilon} \displaystyle\int_{\mathcal{D}} W(u) + \dfrac{c_p \varepsilon^{p-1}}{p} \displaystyle\int_{\mathcal{D}} |\nabla u|^p & \text{if } u \in W_0^{1,p}(\mathcal{D}), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.2}$$

Let $P : L^1(\mathbb{R}^2) \to [0, +\infty]$ be such that

$$P(u) = \begin{cases} P(u, B_{R+1}) & \text{if } u \in BV(B_{R+1}),\ u \in \{0,1\} \text{ a.e.,} \\ & \qquad \text{and } u = 0 \text{ a.e. outside } \mathcal{D}, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3}$$

Then $P = \Gamma\text{-}\lim_{\varepsilon \to 0^+} P_\varepsilon$ with respect to the $L^1(\mathbb{R}^2)$ norm.

Remark 3.2 We observe that $P(u) = P(E)$ if $u = \chi_E$ where E is a set of finite perimeter contained in $\overline{\mathcal{D}}$ and $P(u) = +\infty$ otherwise.

Furthermore, we note that the result does not change if in the definition of P_ε we set $P_\varepsilon(u) = +\infty$ whenever u does not satisfy the constraint

$$0 \leq u \leq 1 \text{ a.e. in } \mathcal{D}. \tag{3.4}$$

Proof. We sketch the proof following that of Theorem 4.13 in [2]. In fact, the only difference with respect to that theorem is that we assume $\mathcal{D}$ convex and that we take $W_0^{1,p}(\mathcal{D})$ instead of $W^{1,p}(\mathcal{D})$ in the definition of P_ε.

By Proposition 4.3 in [2], we obtain that $P(u) \leq \Gamma\text{-}\liminf_{\varepsilon\to 0^+} P_\varepsilon(u)$ for any $u \in L^1(\mathbb{R}^2)$. In order to obtain the Γ-lim sup inequality, we follow the procedure described in Section 4.2 of [2]. It would be enough to construct $\mathcal{M} \subset L^1(\mathbb{R}^2)$ such that the following two conditions are satisfied. First, we require that, for any $u \in L^1(\mathbb{R}^2)$ such that $P(u) < +\infty$, there exists a sequence $\{u_j\}_{j=1}^\infty$ such that $u_j \in \mathcal{M}$, for any $j \in \mathbb{N}$, $u_j \to u$ in $L^1(\mathbb{R}^2)$ as $j \to \infty$, and $P(u) = \lim_j P(u_j)$. Second, for any $u \in \mathcal{M}$, $\Gamma\text{-}\limsup_{\varepsilon\to 0^+} P_\varepsilon(u) \leq P(u)$.

We choose $\mathcal{M} = \{u = \chi_E \,:\; E \Subset \mathcal{D},\; E \text{ of class } C^\infty\}$. The second property follows by Proposition 4.10 in [2]. As far as the first property is concerned, this can be obtained by following the proof of Theorem 1.24 in [8]. That theorem states that any bounded set of finite perimeter E can be approximated by a sequence of C^∞ sets $\{E_j\}_{j=1}^\infty$ such that, as $j \to \infty$, $\int_{\mathbb{R}^2} |\chi_{E_j} - \chi_E| \to 0$ and $P(E_j) \to P(E)$. If we assume that $E \subset \overline{\mathcal{D}}$, and that $\mathcal{D}$ is convex, by choosing in the proof of Theorem 1.24 in [8] a value of t satisfying $1/2 < t < 1$, we obtain that the sets E_j are also compactly contained in $\mathcal{D}$, for any $j \in \mathbb{N}$. $\square$

Also the following result, due to Modica, [10], will be useful.

Proposition 3.3 *For any $C > 0$, let us take $1 < p < +\infty$ and any $\varepsilon > 0$, and let us define*

$$A_C = \{u \in L^1(\mathbb{R}^2) \,:\; 0 \leq u \leq 1 \textit{ a.e. and } P_\varepsilon(u) \leq C\}.$$

Then A_C is precompact in $L^1(\mathbb{R}^2)$.

Proof. We repeat, for the reader's convenience, the arguments developed in [10]. Clearly A_C is a bounded subset of $L^1(\mathcal{D})$. Let $\{u_n\}_{n=1}^\infty$ be a sequence in A_C. We need to prove that there exists a subsequence converging in $L^1(\mathcal{D})$. For any t, $0 \leq t \leq 1$, let $\phi(t) = \int_0^t (W(s))^{1/p'} \mathrm{d}s$. For any $n \in \mathbb{N}$, we define $v_n = \phi(u_n)$ and we observe that $0 \leq v_n \leq \phi(1)$ almost everywhere. Therefore, the functions v_n, $n \in \mathbb{N}$, are uniformly bounded in $L^\infty(\mathcal{D})$ and, consequently, in $L^1(\mathcal{D})$. Furthermore, since ϕ is a C^1 function, with bounded C^1 norm, then $Dv_n = \phi'(u_n)Du_n = W^{1/p'}(u_n)Du_n$. Therefore,

$$\int_{\mathcal{D}} |Dv_n| = \int_{\mathcal{D}} |W^{1/p'}(u_n)||Du_n| \leq P_\varepsilon(u_n)/c_p.$$

We infer that there exists a subsequence $\{v_{n_k}\}_{k=1}^\infty$ converging, as $k \to \infty$, to a function v_0 in $L^1(\mathcal{D})$ and almost everywhere. Let ψ be the inverse function of ϕ and let $u_0 = \psi(v_0)$. We observe that ψ is bounded and uniformly continuous on $[0, \phi(1)]$, hence we conclude that, as $k \to \infty$, u_{n_k} converges to u_0 in $L^1(\mathcal{D})$. $\square$

Remark 3.4 With the same proof, we can show the following. Let us consider any family $\{u_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ such that, for some positive constant C and for any ε, $0 < \varepsilon \leq \varepsilon_0$, we have $0 \leq u_\varepsilon \leq 1$ almost everywhere and $P_\varepsilon(u_\varepsilon) \leq C$. Then $\{u_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ is precompact in $L^1(\mathbb{R}^2)$.

3.2 Convolutions

We recall that, for any two functions f and g defined on $\mathbb{R}^2$, we define the *convolution* of f and g, $f * g$, as follows

$$(f * g)(x) = \int_{\mathbb{R}^2} f(x-y)g(y)\mathrm{d}y = \int_{\mathbb{R}^2} f(y)g(x-y)\mathrm{d}y, \quad x \in \mathbb{R}^2,$$

whenever this is well-defined.

The following classical properties of convolutions will be used. First convolution is commutative. Second, as a consequence of Young inequality we have the following result about integrability and regularity of convolutions.

Proposition 3.5 *Let $1 \leq r,\ p,\ q \leq +\infty$ be such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and let $n = 0, 1, 2, \ldots$.*

Let $1 \leq q < +\infty$, let $f \in L^q(\mathbb{R}^2)$ and let $g \in W^{n,p}(\mathbb{R}^2)$. Then $h = f * g \in W^{n,r}(\mathbb{R}^2)$ and there exists a constant C, depending on n, p, q and r only, such that

$$\|h\|_{W^{n,r}(\mathbb{R}^2)} \leq C\|f\|_{L^q(\mathbb{R}^2)}\|g\|_{W^{n,p}(\mathbb{R}^2)}.$$

Let $q = +\infty$ and let $f \in L^\infty(\mathbb{R}^2)$, with compact support. If $g \in W^{n,1}(\mathbb{R}^2)$, then $h = f * g \in W^{n,\infty}(\mathbb{R}^2)$ and there exists a constant C, depending on n only, such that

$$\|h\|_{W^{n,\infty}(\mathbb{R}^2)} \leq C\|f\|_{L^\infty(\mathbb{R}^2)}\|g\|_{W^{n,1}(\mathbb{R}^2)}.$$

If $f \in L^1(\mathbb{R}^2)$ and $g \in L^\infty(\mathbb{R}^2)$, then $h = f * g \in L^\infty(\mathbb{R}^2)$ and it holds $\|h\|_{L^\infty(\mathbb{R}^2)} \leq \|f\|_{L^\infty(\mathbb{R}^2)}\|g\|_{L^1(\mathbb{R}^2)}$. Furthermore, if g is uniformly continuous and ω_g denotes its modulus of continuity, then h is also uniformly continuous and

$$\omega_h \leq \|f\|_{L^1(\mathbb{R}^2)}\omega_g.$$

Finally, let $f \in L^1(\mathbb{R}^2)$ and let $g \in C^{n,\alpha}(\mathbb{R}^2)$, for some α, $0 < \alpha \leq 1$. Then $h \in C^{n,\alpha}(\mathbb{R}^2)$ and there exists a constant C, depending on n and α only, such that

$$\|h\|_{C^{n,\alpha}(\mathbb{R}^2)} \leq C\|f\|_{L^1(\mathbb{R}^2)}\|g\|_{C^{n,\alpha}(\mathbb{R}^2)}.$$

3.3 The geometry of masks and circuits

In this subsection we investigate the following two questions, namely what are reasonable assumptions on the geometry of the mask D and how to measure the distance between the constructed circuit Ω and the desired one Ω_0. We begin with the following definition. During this subsection, in most cases proofs will be omitted and left to the reader.

For given positive constants r and L, we say that a bounded open set $\Omega \subset \mathbb{R}^2$ is *Lipschitz* or $C^{0,1}$ *with constants r and L* if for every $x \in \partial\Omega$ there exists a Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$, with Lipschitz constant bounded by L, such that for any $y \in B_r(x)$, and up to a rigid transformation,

$$y = (y_1, y_2) \in \Omega \quad \text{if and only if} \quad y_2 < \varphi(y_1). \tag{3.5}$$

Without loss of generality, we may always assume that $x = (0,0)$ and $\varphi(0) = 0$. We shall always denote by e_1 and e_2 the vectors of the canonical bases. Clearly the orientation of the canonical bases may vary depending on $x \in \partial\Omega$.

We shall also use the following notation. There exist positive constants $\delta_1 \leq 1/2$, $\delta_2 \leq \delta_1$ and $m_1 \leq 1$, all of them depending on L only, such that the following holds. For any $x \in \partial\Omega$ and for any $\delta > 0$, let $M_\delta(x) = \{y : |y_1| \leq \delta r,\ y_2 = \varphi(y_1)\}$ and $N_\delta(x) = \{y : |y_1| \leq \delta_1 r,\ \varphi(y_1) - \delta r \leq y_2 \leq \varphi(y_1) + \delta r\}$. Then we assume that, for any δ, $0 < \delta \leq \delta_2$, the following properties hold. First, $N_\delta(x) \subset B_{r/2}(x)$ (hence $M_{\delta_1}(x) \subset B_{r/2}(x)$ as well). Clearly $N_\delta(x)$ is contained in $\overline{B}_{\delta r}(\partial\Omega)$, and we assume that $N_\delta(x)$ contains $\overline{B}_{m_1\delta r}(M_{\delta_1/2}(x))$ and that for any $y \in \{y : |y_1| \leq \delta_1 r/2,\ y_2 = \varphi(y_1) \pm \delta r\}$, $y \notin \overline{B}_{m_1\delta r}(\partial\Omega)$.

For any integer $k = 1, 2, \ldots$, any α, $0 < \alpha \leq 1$, and any positive constants r and L, we say that a bounded open set $\Omega \subset \mathbb{R}^2$ is $C^{k,\alpha}$ *with constants* r *and* L if for every $x \in \partial\Omega$ there exists a $C^{k,\alpha}$ function $\varphi : \mathbb{R} \to \mathbb{R}$, with $C^{k,\alpha}$ norm bounded by L, such that for any $y \in B_r(x)$, and up to a rigid transformation, (3.5) holds. Without loss of generality, we may always assume that $x = (0,0)$ and $\varphi(0) = 0$.

Let us fix three positive constants r, L and R. Let $\mathcal{A}^{0,1}(r, L, R)$ be the class of all bounded open sets, contained in $B_R \subset \mathbb{R}^2$, which are Lipschitz with constants r and L. For any integer $k = 1, 2, \ldots$ and any α, $0 < \alpha \leq 1$, we denote with $\mathcal{A}^{k,\alpha}(r, L, R)$ the class of all bounded open sets, contained in $B_R \subset \mathbb{R}^2$, which are $C^{k,\alpha}$ with constants r and L.

Since we shall identify open sets D with their characteristic functions χ_D, if $\mathcal{A} = \mathcal{A}^{0,1}(r, L, R)$, (or $\mathcal{A} = \mathcal{A}^{k,\alpha}(r, L, R)$, respectively) then, with a slight abuse of notation, $\mathcal{A}$ will also denote the subset of functions $u \in L^1(B_R)$ such that $u = \chi_D$ for some $D \in \mathcal{A}$. Moreover, we shall denote

$$A = \{u \in L^1(B_R) : 0 \leq u \leq 1 \text{ a.e. in } B_R\}$$

and, for any $\gamma > 0$,

$$\mathcal{A}_\gamma = \{u \in A : \|u - \chi_D\|_{L^1(B_R)} \leq \gamma \text{ for some } D \in \mathcal{A}\}. \tag{3.6}$$

Let us assume that Ω_1 and Ω_2 belong to $\mathcal{A}^{0,1}(r, L, R)$. There are several ways to define the distance between these two sets. We shall describe four of them and study their relationships. We let

$$d_1 = d_1(\Omega_1, \Omega_2) = d_H(\overline{\Omega}_1, \overline{\Omega}_2); \tag{3.7}$$

$$\tilde{d}_1 = \tilde{d}_1(\Omega_1, \Omega_2) = d_H(\partial\Omega_1, \partial\Omega_2); \tag{3.8}$$

$$d_2 = d_2(\Omega_1, \Omega_2) = |\Omega_1 \Delta \Omega_2| = \|\chi_{\Omega_1} - \chi_{\Omega_2}\|_{L^1(B_{R+1})}; \tag{3.9}$$

$$d_3 = d_3(\Omega_1, \Omega_2) = d_2 + |P(\Omega_1) - P(\Omega_2)| = d_{st}(\chi_{\Omega_1}, \chi_{\Omega_2}). \tag{3.10}$$

Here d_H denotes the Hausdorff distance, whereas we recall that $P(\Omega)$ denotes the perimeter of Ω in B_{R+1} and d_{st} is the distance inducing strict convergence in $BV(B_{R+1})$. First of all, we observe that all of these are distances. We now investigate their relationships.

We begin with the first two, d_1 and $\tilde{d}_1$, and we notice that

$$\text{if } d_1 \leq r/4, \text{ then } d_1 \leq \tilde{d}_1. \tag{3.11}$$

There exists a constant c, $0 < c \leq 1$, depending on L only, such that

$$d_1 \geq c \min\{r, \tilde{d}_1\}.$$

Therefore,

$$\text{if } \tilde{d}_1 \leq r, \text{ then } \tilde{d}_1 \leq C d_1,$$

where $C = 1/c$. Furthermore, if $d_1 \leq (c/2)r$, then $\tilde{d}_1$ must be less than or equal to r, so

$$\text{if } d_1 \leq (c/2)r, \text{ then } \tilde{d}_1 \leq Cd_1.$$

Moreover, we can find a constant c_1, $0 < c_1 \leq 1$, depending on L only, such that

$$\text{if } \tilde{d}_1 \leq c_1 r, \text{ then } d_1 \leq \tilde{d}_1 \leq Cd_1.$$

We conclude that we can find a constant c_1, $0 < c_1 \leq 1$, and a constant $C \geq 1$, both depending on L only, such that

$$\text{if either } d_1 \leq c_1 r \text{ or } \tilde{d}_1 \leq c_1 r, \text{ then } d_1 \leq \tilde{d}_1 \leq Cd_1. \tag{3.12}$$

Since d_1 and $\tilde{d}_1$ are bounded by $2R$, we also have

$$\text{if both } d_1 \geq c_1 r \text{ and } \tilde{d}_1 \geq c_1 r, \text{ then } d_1 \leq \frac{2R}{c_1 r}\tilde{d}_1 \text{ and } \tilde{d}_1 \leq \frac{2R}{c_1 r} d_1. \tag{3.13}$$

We finally observe that the estimates (3.12) and (3.13) are essentially optimal.

Before comparing d_1 (or $\tilde{d}_1$) with d_2 and d_3, let us make the following remark on the lengths of $\partial\Omega_1$ and $\partial\Omega_2$. If Ω is an open set which is Lipschitz with constants r and L, then for any integer $n \geq 0$, we have

$$\mathcal{H}^1(\partial\Omega \cap (\overline{B}_{(n+1)r} \backslash B_{nr})) \leq C(L)r(n+1). \tag{3.14}$$

Here, a simple computation shows that we may choose $C(L) = 48\sqrt{1+L^2}$.

Therefore, if we assume that $\Omega \subset B_R$ and $R \geq 10r$, we may conclude that

$$P(\Omega) \leq C_1(L)R^2/r, \tag{3.15}$$

where $C_1(L) = \frac{1}{2}\left(\frac{11}{9}\right)^2 C(L)$.

Moreover, there exist two constants c_2, $0 < c_2 \leq c_1$, and $C_1 > 0$, depending on L only, such that we have

$$|\overline{B}_d(\partial\Omega)| \leq C_1 \text{length}(\partial\Omega) d \text{ for any } d \leq c_2 r. \tag{3.16}$$

Since

$$\text{if } \tilde{d}_1 \leq c_2 r, \text{ then } d_2 \leq \min\{|\overline{B}_{\tilde{d}_1}(\partial\Omega_2)|, |\overline{B}_{\tilde{d}_1}(\partial\Omega_2)|\},$$

we obtain that

$$\text{if } \tilde{d}_1 \leq c_2 r, \text{ then } d_2 \leq C_1 \min\{\text{length}(\partial\Omega_1), \text{length}(\partial\Omega_2)\}\tilde{d}_1. \tag{3.17}$$

If $\tilde{d}_1 \geq c_2 r$, then $d_2 \leq \pi R^2 \leq \frac{\pi R^2}{c_2 r}\tilde{d}_1$. By (3.15), we may conclude that

$$d_2 \leq \frac{C_2 R^2}{r}\tilde{d}_1. \tag{3.18}$$

Here C_2 depends on L only. Moreover, up to changing the constants c_2, C_1 and C_2, (3.17) and (3.18) still hold if we replace $\tilde{d}_1$ with d_1.

On the other hand, there exists a constant c_3, $0 < c_3 \leq \pi$, depending on L only, such that

$$d_2 \geq c_3 \min\{r^2, d_1^2\}.$$

We infer that either if $d_1 \leq r$ or if $d_2 \leq (c_3/2)r^2$, then $d_1 \leq C_3 d_2^{1/2}$, where $C_3 = 1/c_3^{1/2}$. If $d_2 \geq (c_3/2)r^2$, then $d_1 \leq 2R \leq \frac{4C_3^2}{r^2} R d_2$ or, better, $d_1 \leq \frac{2\sqrt{2}C_3}{r} R d_2^{1/2}$. Summarizing, we have

$$\text{if } d_2 \leq (c_3/2)r^2, \text{ then } d_1 \leq C_3 d_2^{1/2} \tag{3.19}$$

and, finally,

$$\text{if } d_2 \geq (c_3/2)r^2, \text{ then } d_1 \leq \frac{2\sqrt{2}C_3}{r} R d_2^{1/2}. \tag{3.20}$$

Clearly, up to suitably changing the constants c_3 and C_3, the last two estimates still hold if we replace d_1 with $\tilde{d}_1$. We also remark that, as before, the estimates relating d_1, $\tilde{d}_1$ and d_2 are essentially optimal.

We have obtained that d_1, $\tilde{d}_1$ and d_2 are topologically equivalent distances. About d_2 and d_3, obviously $d_2 \leq d_3$, however the two distances are not topologically equivalent. In fact we can find Ω and Ω_i, $i \in \mathbb{N}$, open sets belonging to $\mathcal{A}^{0,1}(r, L, R)$, such that $d_2(\Omega, \Omega_i)$ goes to zero as $i \to \infty$, whereas $d_3(\Omega, \Omega_i) \geq c > 0$ for any $i \in \mathbb{N}$. Therefore d_3 induces a strictly finer topology than the one induced by d_2

An assumption that the mask is a bounded open set which is Lipschitz with given constants r and L is reasonable from the manufacturing point of view as well as from the mathematical point of view, by the following compactness result.

Proposition 3.6 *The set $\mathcal{A}^{0,1}(r, L, R)$ (respectively $\mathcal{A}^{k,\alpha}(r, L, R)$, $k = 1, 2, \ldots$, $0 < \alpha \leq 1$) is compact with respect to the distance d_1.*

We remark that the same result holds with respect to the distances $\tilde{d}_1$ and d_2. Furthermore, we obtain as a corollary that the set $\mathcal{A}_\gamma$ is closed with respect to the L^1 norm, for any $\gamma > 0$.

The previous example shows that compactness fails with respect to the distance d_3, at least for the Lipschitz case. On the other hand, if Ω_1 and Ω_2 belong to $\mathcal{A}^{1,\alpha}(r, L, R)$, with $0 < \alpha < 1$, then, following Lemma 2.1 in [15], we can show that

$$|P(\Omega_1) - P(\Omega_2)| \leq C_4(\tilde{d}_1(\Omega_1, \Omega_2))^{\alpha/(2\alpha+2)}, \tag{3.21}$$

where C_4 depends on r, L, R and α only. We may conclude that in the $C^{k,\alpha}$ case, $k = 1, 2, \ldots$, $0 < \alpha \leq 1$, d_3 is topologically equivalent to the other three distances and that Proposition 3.6 holds also with respect to the distance d_3.

It is worthwhile to observe that, under some circumstances, the estimate (3.21) can be extended to the piecewise $C^{1,\alpha}$ case. For example, typically we may assume that the desired circuit Ω_0 belongs to $\mathcal{A}^{0,1}(r, L, R)$. Moreover, we assume that the boundary of Ω_0 is composed by a finite number of closed segments I_i, $i = 1, \ldots, n$, which are pairwise internally disjoint and whose lengths are greater than or equal to $2r$. Therefore, Ω_0 is actually a piecewise $C^{1,\alpha}$ open set. We shall show in Section 4 that, under suitable assumptions on the mask D, the corresponding constructed circuit Ω belongs to $\mathcal{A}^{1,\alpha}(r_1, L_1, \tilde{R})$, for some suitable positive constants $r_1 \leq r$, $L_1 \geq L$, $\tilde{R} \geq R$ and α, $0 < \alpha < 1$. Then we can find positive constants c_4, $0 < c_4 \leq 1$, C_5 and C_6, depending on r_1, L_1, $\tilde{R}$ and α only, such that if $\tilde{d}_1(\Omega_0, \Omega) \leq c_4 r_1$, then we can subdivide $\partial\Omega$ into smooth curves J_i, $i = 1\ldots, n$, which are pairwise internally disjoint, such that for any $i = 1, \ldots, n$ we have

$$d_H(J_i, I_i) \leq C_5 \tilde{d}_1(\Omega_0, \Omega)$$

and

$$\mathrm{length}(I_i) - 2C_5\tilde{d}_1(\Omega_0, \Omega) \leq \mathrm{length}(J_i) \leq \mathrm{length}(I_i) + C_6(\tilde{d}_1(\Omega_0, \Omega))^{\alpha/(2\alpha+2)}.$$

Therefore,

$$-2nC_5\tilde{d}_1(\Omega_0, \Omega) \leq P(\Omega) - P(\Omega_0) \leq nC_6(\tilde{d}_1(\Omega_0, \Omega))^{\alpha/(2\alpha+2)}.$$

By these reasonings it might seem that we may choose to measure the distance between the desired circuit Ω_0 and the reconstructed one Ω by using any of these distances. However, there are several reasons to prefer the distance d_3, which we actually choose. In fact, it is easier to compute than d_1 and $\tilde{d}_1$, it can be extended in a natural way from characteristic functions to any BV function by using d_{st}, and should provide a better approximation of the desired circuit than d_2, which seems to be too weak for this purpose.

3.4 Convolutions of characteristic functions and Gaussian distributions

We recall that $G(x) = (2\pi)^{-1}\mathrm{e}^{-|x|^2/2}$, $x \in \mathbb{R}^2$, and let us note that $\hat{G}(\xi) = \mathrm{e}^{-|\xi|^2/2}$, $\xi \in \mathbb{R}^2$. Moreover, $\|G\|_{L^1(\mathbb{R}^2)} = 1$. For any positive constant s we denote by $G_s(x) = s^{-2}G(x/s)$, $x \in \mathbb{R}^2$. We note that $\|G_s\|_{L^1(\mathbb{R}^2)} = 1$ and $\widehat{G_s}(\xi) = \hat{G}(s\xi)$, $\xi \in \mathbb{R}^2$.

Let D be a bounded open set which is Lipschitz with constants R_0 and L and let χ_D be its characteristic function. We investigate how χ_D is perturbed if we convolute it with G. We call $v = \chi_D * G$, that is

$$v(x) = \int_{\mathbb{R}^2} \chi_D(x-y)G(y)\mathrm{d}y = \int_{\mathbb{R}^2} \chi_D(y)G(x-y)\mathrm{d}y, \quad x \in \mathbb{R}^2.$$

We recall that we shall use the positive constants δ_1, δ_2 and m_1, and the sets M_{δ_1} and N_δ introduced at the beginning of Subsection 3.3.

Proposition 3.7 *Under the previous notation and assumptions, let us fix δ, $0 < \delta \leq \delta_2/4$. Then there exist constants $R_0 \geq 1$, $\tilde{h}$, $0 < \tilde{h} \leq 1/24$, and $a_1 > 0$, depending on L and δ only, such that the following estimates hold. For any $x \in \mathbb{R}^2$,*

$$\text{(3.22)} \qquad \text{if } \tilde{h} < v(x) < 1 - \tilde{h}, \text{ then } x \in \overline{B}_{m_1\delta R_0}(\partial D),$$

and for any $x \in \partial D$,

$$\text{(3.23)} \qquad \text{if } y \in N_\delta(x), \text{ then } \nabla v(y) \cdot (-e_2) \geq a_1.$$

Proof. If $x \notin \overline{B}_{m_1\delta R_0}(\partial D)$, then we have

$$v(x) \leq \mathrm{e}^{-m_1^2\delta^2R_0^2/2}, \text{ if } x \notin D,$$

and

$$v(x) \geq 1 - \mathrm{e}^{-m_1^2\delta^2R_0^2/2}, \text{ if } x \in D.$$

Consequently, provided $\tilde{h} = \mathrm{e}^{-m_1^2\delta^2 R_0^2/2} \leq 1/24$, we may conclude that (3.22) holds.

Let us take $x \in \partial D$ and $y \in N_\delta(x)$. Then, denoting by ν the exterior unit normal vector to D,

$$\nabla v(y) \cdot (-e_2) = \int_{\partial D} G(y-z)\nu(z) \cdot e_2 \mathrm{d}\mathcal{H}^1(z).$$

Therefore,

$$\begin{aligned}\nabla v(y) \cdot (-e_2) = &\int_{\partial D \cap \overline{B}_{2\delta R_0}(y)} G(y-z)\nu(z) \cdot e_2 \mathrm{d}\mathcal{H}^1(z) + \\ &\int_{\partial D \cap (B_{R_0/2}(y) \setminus \overline{B}_{2\delta R_0}(y))} G(y-z)\nu(z) \cdot e_2 \mathrm{d}\mathcal{H}^1(z) + \\ &\int_{\partial D \setminus B_{R_0/2}(y)} G(y-z)\nu(z) \cdot e_2 \mathrm{d}\mathcal{H}^1(z) = A + B + C.\end{aligned}$$

Since $B_{R_0/2}(y)$ is contained in $B_{R_0}(x)$, for any $z \in \partial D \cap B_{R_0/2}(y)$, we have $\nu(z) \cdot e_2 \geq c_1 > 0$ where c_1 is a constant depending on L only. Moreover, the length of $\partial D \cap \overline{B}_{2\delta R_0}(y)$ is also bounded from below by $c_2 \delta R_0$, $c_2 > 0$ depending on L only. Therefore, we obtain that $A \geq c_1 c_2 \delta R_0 \mathrm{e}^{-2\delta^2 R_0^2}$ and $B \geq 0$.

For what concerns the term C, with the help of (3.14), we can find a constant C_1, depending on L only, such that, for any $R_0 \geq 1$, we have

$$|C| \leq C_1 R_0 \mathrm{e}^{-R_0^2/8}.$$

Therefore, we can find $R_0 \geq 1$, depending on L and δ only, such that $\tilde{h} = \mathrm{e}^{-m_1^2\delta^2 R_0^2/2} \leq 1/24$, and $2C_1 \mathrm{e}^{-R_0^2/8} \leq c_1 c_2 \delta \mathrm{e}^{-2\delta^2 R_0^2}$. We set $a_1 = (1/2) c_1 c_2 \delta R_0 \mathrm{e}^{-2\delta^2 R_0^2}$ and the proof is concluded. □

Remark 3.8 Without loss of generality, we may choose R_0 such that it also satisfies

$$\|\nabla G\|_{L^1(\mathbb{R}^2 \setminus B_{R_0/2})} \leq (1/12) a_1. \tag{3.24}$$

In the sequel, we shall fix $\delta = \delta_2/4$ and R_0 as the corresponding constant in Proposition 3.7 such that (3.24) holds. We note that, in this case, δ and R_0 depend on L only. We shall also fix a constant $R \geq 10 R_0$. We recall that, with a slight abuse of notation, we identify $L^1(B_R)$ with the set of real valued $L^1(\mathbb{R}^2)$ functions that are equal to zero almost everywhere outside B_R. The same proof of Proposition 3.7 allows us to prove this corollary.

Corollary 3.9 *For any s, $0 < s \leq 1$, let $r = sR_0$ and let D be a bounded open set which is Lipschitz with constants r and L. Let $v_s = \chi_D * G_s$. Then, for any $x \in \mathbb{R}^2$,*

$$\text{if } \tilde{h} < v_s(x) < 1 - \tilde{h}, \text{ then } x \in \overline{B}_{m_1\delta r}(\partial D),$$

and for any $x \in \partial D$,

$$\text{if } y \in N_\delta(x), \text{ then } \nabla v_s(y) \cdot (-e_2) \geq a_1 R_0 / r = a_1 / s.$$

We conclude this part with the following perturbation argument. Let us consider a function ψ such that either $\psi \in C^1(\mathbb{R}^2) \cap W^{1,1}(\mathbb{R}^2)$ or $\psi \in W^{2,1}(\mathbb{R}^2)$ and that, for some $\tilde{\delta} > 0$,

$$\|\psi\|_{W^{1,1}(\mathbb{R}^2)} \leq \tilde{\delta}.$$

Let $\tilde{G} = G + \psi$. Then the following result holds.

Corollary 3.10 *Let us assume that $\tilde{\delta} \leq \min\{\tilde{h}, a_1/2\}$.*

For any s, $0 < s \leq 1$, let $r = sR_0$ and let D be a bounded open set which is Lipschitz with constants r and L. Let $v_s = \chi_D * \tilde{G}_s$. Then, for any $x \in \mathbb{R}^2$,

$$\text{if } 2\tilde{h} < v_s(x) < 1 - 2\tilde{h}, \text{ then } x \in \overline{B}_{m_1\delta r}(\partial D), \tag{3.25}$$

and for any $x \in \partial D$,

$$\text{if } y \in N_\delta(x), \text{ then } \nabla v_s(y) \cdot (-e_2) \geq a_1 R_0/2r = a_1/(2s). \tag{3.26}$$

Proof. It follows immediately from the previous corollary and Proposition 3.5. We first notice that in either cases $\chi_D * \tilde{G}_s \in C^1(\mathbb{R}^2)$. Moreover we have

$$\|\chi_D * \tilde{G}_s - \chi_D * G_s\|_{L^\infty(\mathbb{R}^2)} \leq \|\chi_D\|_{L^\infty(\mathbb{R}^2)}\|\psi_s\|_{L^1} \leq \|\psi\|_{L^1(\mathbb{R}^2)}$$

and

$$\|\nabla(\chi_D * \tilde{G}_s - \chi_D * G_s)\|_{L^\infty(\mathbb{R}^2)} = \|\chi_D * (\nabla\psi_s)\|_{L^\infty(\mathbb{R}^2)} \leq \\ \|\chi_D\|_{L^\infty(\mathbb{R}^2)}\|\nabla\psi_s\|_{L^1} \leq \|\nabla\psi\|_{L^1(\mathbb{R}^2)}/s.$$

Thus the conclusion follows. $\square$

4 Relationship between a mask and its image intensity

In this section we study the relationship between a function representing a mask (not necessarily a characteristic function of a domain) and its associated image intensity. We recall the notation used. We fix $\delta = \delta_2/4$ and R_0 as the corresponding constant in Proposition 3.7 such that (3.24) holds. We note that, in this case, δ and R_0 depend on L only. We shall also fix a constant $R \geq 10R_0$. We recall that, with a slight abuse of notation, we identify $L^1(B_R)$ with the set of real valued $L^1(\mathbb{R}^2)$ functions that are equal to zero almost everywhere outside B_R. We recall that $A = \{u \in L^1(B_R) \, : \, 0 \leq u \leq 1 \text{ a.e. in } B_R\}$.

Fixed $\tilde{\delta} > 0$, we assume that $\psi \in W^{2,1}(\mathbb{R}^2)$ and that

$$\|\psi\|_{W^{1,1}(\mathbb{R}^2)} \leq \tilde{\delta}.$$

We denote $\tilde{G} = G + \psi$ and, for any s, $0 < s \leq 1$, we define the operator $\mathcal{T}_s : L^1(B_R) \to W^{2,1}(\mathbb{R}^2)$ as follows

$$\mathcal{T}_s(u) = u * \tilde{G}_s, \quad \text{for any } u \in L^1(B_R).$$

The point spread function we use, T, can be described in general by the function $\tilde{G}$. Therefore a study of properties of convolutions with $\tilde{G}$ will be useful.

We remark that the following continuity properties of the operator $\mathcal{T}_s$ hold. For any p, $1 \leq p \leq +\infty$, and any $u \in L^1(B_R)$, we have, for an absolute constant C,

$$\|\mathcal{T}_s(u)\|_{L^p(\mathbb{R}^2)} \leq \|\tilde{G}\|_{L^1(\mathbb{R}^2)} \|u\|_{L^p(\mathbb{R}^2)}, \tag{4.1}$$

$$\|\nabla \mathcal{T}_s(u)\|_{L^p(\mathbb{R}^2)} \leq (C/s) \|\nabla \tilde{G}\|_{L^1(\mathbb{R}^2)} \|u\|_{L^p(\mathbb{R}^2)}, \tag{4.2}$$

$$\|D^2 \mathcal{T}_s(u)\|_{L^p(\mathbb{R}^2)} \leq (C/s^2) \|D^2 \tilde{G}\|_{L^1(\mathbb{R}^2)} \|u\|_{L^p(\mathbb{R}^2)}. \tag{4.3}$$

Let $J \in C^0(\mathbb{R}^2)$. For any $u \in L^1(B_R)$, we define $U \in L^1(\mathbb{R}^4)$ as follows

$$U(x,y) = u(x)u(y)J(x-y), \quad \text{for any } x, y \in \mathbb{R}^2.$$

Then, for any s, $0 < s \leq 1$, we define $H_s \in W^{2,1}(\mathbb{R}^4)$ in the following way

$$H_s(x,y) = \tilde{G}_s(x)\tilde{G}_s(y), \quad \text{for any } x, y \in \mathbb{R}^2.$$

Therefore, for any p, $1 \leq p \leq +\infty$, and any $u \in L^1(B_R)$, we have, for an absolute constant C,

$$\|U * H_s\|_{L^p(\mathbb{R}^4)} \leq \|\tilde{G}\|^2_{L^1(\mathbb{R}^2)} \|J\|_{L^\infty(B_{2R})} \|u\|^2_{L^p(\mathbb{R}^2)}, \tag{4.4}$$

$$\|\nabla(U * H_s)\|_{L^p(\mathbb{R}^4)} \leq (C/s) \|\tilde{G}\|_{L^1(\mathbb{R}^2)} \|\nabla \tilde{G}\|_{L^1(\mathbb{R}^2)} \|J\|_{L^\infty(B_{2R})} \|u\|^2_{L^p(\mathbb{R}^2)}, \tag{4.5}$$

$$\|D^2(U * H_s)\|_{L^p(\mathbb{R}^4)} \leq (C/s^2) \|\tilde{G}\|^2_{W^{2,1}(\mathbb{R}^2)} \|J\|_{L^\infty(B_{2R})} \|u\|^2_{L^p(\mathbb{R}^2)}. \tag{4.6}$$

Let us fix $p > 4$ and let $\alpha = 1 - 4/p$, $0 < \alpha < 1$. Then, if $u \in A$ we have $U * H_s \in C^{1,\alpha}(\mathbb{R}^4)$ and, for some absolute constant C depending on p,

$$\|U * H_s\|_{C^{1,\alpha}(\mathbb{R}^4)} \leq C \|U * H_s\|_{W^{2,p}(\mathbb{R}^4)}.$$

We define $\mathcal{P}_{J,s} : A \to C^{1,\alpha}(\mathbb{R}^2)$ and $\mathcal{P}_{1,s} : A \to C^{1,\alpha}(\mathbb{R}^2)$ as follows. For any $u \in A$

$$\mathcal{P}_{J,s}(u)(x) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u(\xi)\tilde{G}_s(x-\xi)J(\xi-\eta)\tilde{G}_s(x-\eta)u(\eta)\mathrm{d}\xi\mathrm{d}\eta, \quad x \in \mathbb{R}^2$$

and

$$\mathcal{P}_{1,s}(u) = (\mathcal{T}_s(u))^2.$$

We notice that the two definitions are consistent when $J \equiv 1$ and that

$$\mathcal{P}_{J,s}(u)(x) = (U * H_s)(x,x), \quad x \in \mathbb{R}^2.$$

Putting together the previous estimates we obtain the following result. We recall that we have fixed a number $p > 4$ and that $\alpha = 1 - 4/p$.

Proposition 4.1 *Under the previous notation and assumptions, let* $\varepsilon = \|J - 1\|_{L^\infty(B_{2R})}$. *Then for any* $u \in A$ *and any* s, $0 < s \leq 1$, *we have, for some absolute constant* C *depending on* p,

$$\|\mathcal{P}_{J,s}(u)\|_{C^{0,\alpha}(\mathbb{R}^2)} \leq ((1+\varepsilon)C/s) \|\tilde{G}\|^2_{W^{1,1}(\mathbb{R}^2)} \|u\|^2_{L^p(B_R)} \leq$$
$$((1+\varepsilon)C/s) \|\tilde{G}\|^2_{W^{1,1}(\mathbb{R}^2)} \|u\|^{2/p}_{L^1(B_R)}.$$

The same estimate holds also for the gradient, namely

$$\|\nabla\mathcal{P}_{J,s}(u)\|_{C^{0,\alpha}(\mathbb{R}^2)} \leq ((1+\varepsilon)C/s^2)\|\tilde{G}\|^2_{W^{2,1}(\mathbb{R}^2)}\|u\|^2_{L^p(B_R)} \leq ((1+\varepsilon)C/s^2)\|\tilde{G}\|^2_{W^{2,1}(\mathbb{R}^2)}\|u\|^{2/p}_{L^1(B_R)}.$$

Furthermore, we have

$$\|\mathcal{P}_{J,s}(u)-\mathcal{P}_{1,s}(u)\|_{L^\infty(\mathbb{R}^2)} = \|\mathcal{P}_{J-1,s}(u)\|_{L^\infty(\mathbb{R}^2)} \leq \|\tilde{G}\|^2_{L^1(\mathbb{R}^2)}\varepsilon \tag{4.7}$$

and, for some absolute constant C,

$$\|\nabla(\mathcal{P}_{J,s}(u)-\mathcal{P}_{1,s}(u))\|_{L^\infty(\mathbb{R}^2)} \leq C\|\tilde{G}\|_{L^1(\mathbb{R}^2)}\|\nabla\tilde{G}\|_{L^1(\mathbb{R}^2)}\varepsilon/s. \tag{4.8}$$

Although $\mathcal{P}_{J,s}$ is nonlinear in its argument u, by a simple adaptation of the previous reasonings, we obtain that for any u_1, $u_2 \in A$, and for some absolute constant C depending on p, we have the following corresponding estimates

$$\|\mathcal{P}_{J,s}(u_1)-\mathcal{P}_{J,s}(u_2)\|_{C^{0,\alpha}(\mathbb{R}^2)} \leq ((1+\varepsilon)C/s)\|\tilde{G}\|^2_{W^{1,1}(\mathbb{R}^2)}R^{2/p}\|u_1-u_2\|_{L^p(B_R)} \leq 2((1+\varepsilon)C/s)\|\tilde{G}\|^2_{W^{1,1}(\mathbb{R}^2)}R^{2/p}\|u_1-u_2\|^{1/p}_{L^1(B_R)}$$

and

$$\|\nabla(\mathcal{P}_{J,s}(u_1)-\mathcal{P}_{J,s}(u_2))\|_{C^{0,\alpha}(\mathbb{R}^2)} \leq ((1+\varepsilon)C/s^2)\|\tilde{G}\|^2_{W^{2,1}(\mathbb{R}^2)}R^{2/p}\|u_1-u_2\|_{L^p(B_R)} \leq 2((1+\varepsilon)C/s^2)\|\tilde{G}\|^2_{W^{2,1}(\mathbb{R}^2)}R^{2/p}\|u_1-u_2\|^{1/p}_{L^1(B_R)}.$$

Therefore, $\mathcal{P}_{J,s} : A \to C^{1,\alpha}(\mathbb{R}^2)$ is Lipschitz continuous with respect to the L^p norm and Hölder continuous with exponent $1/p$ with respect to the L^1 norm.

We fix $\tilde{\delta}$ such that $0 < \tilde{\delta} \leq \min\{\tilde{h}, a_1/2\}$, with $\tilde{h}$, $0 < \tilde{h} \leq 1/24$, and $a_1 > 0$ as in Proposition 3.7, thus depending on L only. We define the corresponding T and s_0 as in Lemma 2.2. We finally fix $s = (k\mathrm{NA})^{-1}$, $0 < s \leq 1/s_0$, and σ, $0 < \sigma \leq s$. Then we define

$$I_{s,\sigma}(u) = \mathcal{P}_{J,ss_0}(u), \quad \text{for any } u \in A,$$

where J is given by (2.3). We recall that in (2.2) we defined $K = T_{ss_0}$, therefore for any open set $D \subset B_R$ we have that $I_{s,\sigma}(\chi_D)$ is the light intensity on the image plane corresponding to the mask D, see (2.1).

We denote by $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ the Heaviside function such that $\mathcal{H}(t) = 0$ for any $t \leq 0$ and $\mathcal{H}(t) = 1$ for any $t > 0$. For any constant h we set $\mathcal{H}_h(t) = \mathcal{H}(t-h)$ for any $t \in \mathbb{R}$. Then, for any h, $0 < h < 1$, any s, $0 < s \leq 1/s_0$, and any σ, $0 < \sigma \leq s$, we define the operator $\mathcal{W} : A \to L^\infty(\mathbb{R}^2)$ as follows

$$\mathcal{W}(u) = \mathcal{H}_h(I_{s,\sigma}(u)), \quad \text{for any } u \in A. \tag{4.9}$$

Clearly, for any $u \in A$, $\mathcal{W}(u)$ is the characteristic function of an open set, which we shall call $\Omega(u)$. That is

$$\Omega(u) = \{x \in \mathbb{R}^2 \,:\, I_{s,\sigma}(u)(x) > h\}, \quad \text{for any } u \in A. \tag{4.10}$$

In other words, $\chi_{\Omega(u)} = \mathcal{W}(u) = \mathcal{H}_h(I_{s,\sigma}(u))$. Moreover, whenever $u = \chi_D$, where D is an open set contained in B_R, we shall denote $\Omega(D) = \Omega(\chi_D)$.

The final, and crucial, result of this section is the following.

Theorem 4.2 *Let us fix a positive constant L. Let $\delta = \delta_2/4$ and let R_0 be as in Proposition 3.7 and such that (3.24) holds. Let us also fix $R \geq 10R_0$ and p, $p > 4$, and $\alpha = 1-4/p$.*

We fix $\tilde{\delta}$ such that $0 < \tilde{\delta} \leq \min\{\tilde{h}, a_1/2\}$, with $\tilde{h}$, $0 < \tilde{h} \leq 1/24$, and $a_1 > 0$ as in Proposition 3.7, thus depending on L only. We define the corresponding T and s_0 as in Lemma 2.2. We finally fix $s = (kNA)^{-1}$, $0 < s \leq 1/s_0$, and σ, $0 < \sigma \leq s$. Then, for any $u \in A$ we define

$$I_{s,\sigma}(u) = \mathcal{P}_{J,ss_0}(u)$$

where J is given by (2.3).

Then for any h, $1/3 \leq h \leq 2/3$, and any s, $0 < s \leq 1/s_0$, we can find positive constants $\tilde{\sigma}_0$, $0 < \tilde{\sigma}_0 \leq 1$, and γ_0, depending on L, R, $\|D^2(T-G)\|_{L^1(\mathbb{R}^2)}$, p, and ss_0 only, such that for any σ, $0 < \sigma \leq \tilde{\sigma}_0 s$, and any γ, $0 < \gamma \leq \gamma_0$, the following holds.

Let $\mathcal{A} = \mathcal{A}^{0,1}(r, L, R)$, where $r = ss_0R_0$. Let $\tilde{R} = R + 2m_1\delta R_0$, where m_1, $0 < m_1 \leq 1$, depends on L only. Then, for any $u \in \mathcal{A}_\gamma$, we have that $\Omega(u) \Subset B_{\tilde{R}}$ and $\Omega(u) \in \mathcal{A}^{1,\alpha}(r_1, L_1, \tilde{R})$. Here $r_1 = ss_0\tilde{R}_0 \leq r$, where $\tilde{R}_0 \leq \delta_1 R_0/8$ depends on L only, whereas $L_1 \geq L$ depends on L, R, $\|D^2(T-G)\|_{L^1(\mathbb{R}^2)}$, p and ss_0 only.

Moreover, the map $\mathcal{W} : \mathcal{A}_\gamma \to BV(B_{\tilde{R}})$ is uniformly continuous with respect to the L^1 norm on $\mathcal{A}_\gamma$ and the distance d_{st} on $BV(B_{\tilde{R}})$.

Remark 4.3 We observe that the distance d_{st} in $BV(B_{\tilde{R}})$ between $\mathcal{W}(u_1)$ and $\mathcal{W}(u_2)$ corresponds to the distance d_3 related to $B_{\tilde{R}}$ between $\Omega(u_1)$ and $\Omega(u_2)$.

Proof of Theorem 4.2. The proof is a consequence of the previous analysis. We fix s, $0 < s \leq 1/s_0$, and h, $1/3 \leq h \leq 2/3$.

Let us begin with the following preliminary case. Let $u = \chi_D$, where $D \in \mathcal{A}$, and let $v = \mathcal{T}_{ss_0}(u)$ and $\tilde{W} = \mathcal{H}_h((\mathcal{T}_{ss_0}(u))^2)$. We apply Corollary 3.10 and we obtain the following results.

If $\tilde{\Omega}$ is the open set such that $\tilde{W} = \chi_{\tilde{\Omega}}$, then, by (3.25), we notice that $\partial\tilde{\Omega} \subset \overline{B}_{m_1\delta r}(\partial D)$ and that $(D \backslash \overline{B}_{m_1\delta r}(\partial D)) \subset \tilde{\Omega}$ and $(\mathbb{R}^2 \backslash \overline{B}_{m_1\delta r}(\overline{D})) \cap \tilde{\Omega} = \emptyset$. Therefore $\tilde{\Omega} \Subset B_{\tilde{R}}$.

We take any $x \in \partial D$ and any $y \in M_{\delta_1/2}(x)$, with respect to the coordinate system depending on x. Then we consider the points $y^- = y - \delta r e_2$ and $y^+ = y + \delta r e_2$. We have that $y^\pm \in \partial N_\delta(x) \backslash \overline{B}_{m_1\delta r}(\partial D)$. Moreover, $y^- \in D$ and $v(y^-) \geq 11/12$, whereas $y^+ \notin D$ and $v(y^+) \leq 1/12$. Let us call $\tilde{y}^+ = y + t_0\delta r e_2$, where $t_0 \in (-1, 1]$, t_0 depends on y, and $v(\tilde{y}^+) = 1/12$ whereas $v(y + t\delta r e_2) < 1/12$ for any $t \in (t_0, 1]$. Then we use (3.26) and we obtain that, for any $t \in [-1, t_0]$, $v(y + t\delta r e_2) \geq 1/12$ and

$$-\frac{\mathrm{d}}{\mathrm{d}t}(v(y + t\delta r e_2))^2 \geq \delta r a_1/(12ss_0).$$

We may conclude that there exists a function $\varphi_1 : [-\delta_1 r/2, \delta_1 r/2] \to \mathbb{R}$ such that, for any $y = (y_1, y_2) \in N_\delta(x)$ with $|y_1| \leq \delta_1 r/2$, $(v(y))^2 = h$ if and only if $y_2 = \varphi_1(y_1)$. We recall that

$$\|v\|_{L^\infty(\mathbb{R}^2)} \leq C_1, \quad \|\nabla v\|_{L^\infty(\mathbb{R}^2)} \leq C_1/(ss_0), \tag{4.11}$$

where C_1 is an absolute constant, and

$$\|\nabla v\|_{C^{0,\alpha}(\mathbb{R}^2)} \leq C_2/(ss_0)^2, \tag{4.12}$$

where C_2 depends on R, $\|D^2(T-G)\|_{L^1(\mathbb{R}^2)}$ and p only.

We obtain that v^2 is a $C^{1,\alpha}$ function and, by the implicit function theorem, we conclude that the function φ_1 is actually $C^{1,\alpha}$. We observe that

$$\|\varphi_1'\|_{L^\infty[-\delta_1 r/2,\delta_1 r/2]} \leq C_3,$$

where C_3 depends on L only. Without loss of generality, by a translation we may assume that $\varphi_1(0)=0$, thus $\|\varphi_1\|_{L^\infty[-\delta_1 r/2,\delta_1 r/2]} \leq C_3\delta_1 r/2$. Finally, for any t_1, $t_2 \in [-\delta_1 r/2, \delta_1 r/2]$,

$$|\varphi_1'(t_1)-\varphi_1'(t_2)| \leq (C_4/(ss_0))|t_1-t_2|^\alpha,$$

where C_4 is a constant depending on L, R, $\|D^2(T-G)\|_{L^1(\mathbb{R}^2)}$ and p only.

Then, it is not difficult to prove that for some $r_1 = ss_0\tilde{R}_0 \leq r$, with $\tilde{R}_0 \leq \delta_1 R_0/8$ depending on L only, we can find $L_1 \geq L$, depending on L, R, $\|D^2(T-G)\|_{L^1(\mathbb{R}^2)}$, p and ss_0 only, such that $\tilde{\Omega} \in \mathcal{A}^{1,\alpha}(r_1, L_1, \tilde{R})$. Let us also remark that we have obtained that $\tilde{d}_1(\tilde{\Omega}, D) \leq \delta r$.

Let us call $\varepsilon = \varepsilon(s,\sigma) = \varepsilon(\sigma/s) = \|J-1\|_{L^\infty(B_{2R})}$. We notice that, as $\sigma/s \to 0^+$, we have that ε goes to 0 as well. We also assume, without loss of generality, that ε is increasing with respect to the variable σ/s. Let us recall that, for any $u \in A$, if $w = I_{s,\sigma}(u)$, with $0 < s \leq 1/s_0$ and $0 < \sigma \leq s$, then

$$\|w\|_{L^\infty(\mathbb{R}^2)} \leq C_5, \quad \|\nabla w\|_{L^\infty(\mathbb{R}^2)} \leq C_5/(ss_0), \tag{4.13}$$

and

$$\|\nabla w\|_{C^{0,\alpha}(\mathbb{R}^2)} \leq C_6/(ss_0)^2, \tag{4.14}$$

where C_5 is an absolute constant and C_6 depends on R, $\|D^2(T-G)\|_{L^1(\mathbb{R}^2)}$ and p only.

For positive constants $\tilde{\sigma}_0$, $0 < \tilde{\sigma}_0 \leq 1$, and γ_0, to be precised later, let us fix σ, $0 < \sigma \leq \tilde{\sigma}_0 s$ and γ, $0 < \gamma \leq \gamma_0$. We take $u \in \mathcal{A}_\gamma$, $v = \mathcal{T}_{ss_0}(u)$, $w = I_{s,\sigma}(u)$, and $D \in \mathcal{A}$ such that $\|u-\chi_D\|_{L^1(B_R)} \leq \gamma$. Then we use Proposition 4.1 to infer that

$$\begin{gathered}\left\|I_{\sigma,s}(u)-(\mathcal{T}_{ss_0}(\chi_D))^2\right\|_{L^\infty(\mathbb{R}^2)} \leq \|I_{s,\sigma}(u)-v^2\|_{L^\infty(\mathbb{R}^2)}+\|(\mathcal{T}_{ss_0}(u))^2-(\mathcal{T}_{ss_0}(\chi_D))^2\|_{L^\infty(\mathbb{R}^2)} \leq \\ \|\tilde{G}\|^2_{L^1(\mathbb{R}^2)}\varepsilon + (2C/(ss_0))\|\tilde{G}\|_{L^1(\mathbb{R}^2)}\|\tilde{G}\|_{W^{1,1}(\mathbb{R}^2)}\|u-\chi_D\|_{L^p(B_R)} \leq (C_7/(ss_0))\left(\varepsilon+\gamma^{1/p}\right),\end{gathered}$$

where C is an absolute constant and consequently C_7 depends on p only.

Analogously, we can prove that

$$\left\|\nabla\left(I_{s,\sigma}(u)-(\mathcal{T}_{ss_0}(\chi_D))^2\right)\right\|_{L^\infty(\mathbb{R}^2)} \leq (C_8/(ss_0)^2)\left(\varepsilon+\gamma^{1/p}\right),$$

where the constant C_8 depends on $\|D^2(T-G)\|_{L^1(\mathbb{R}^2)}$ and p only.

We now choose the positive constants $\tilde{\sigma}_0$ and γ_0 in such a way that

$$2(C_7/(ss_0))\left(\varepsilon(\tilde{\sigma}_0)+\gamma_0^{1/p}\right) \leq 1/6$$

and

$$(C_8/(ss_0))\left(\varepsilon(\tilde{\sigma}_0)+\gamma_0^{1/p}\right) \leq a_1/24.$$

Clearly, $\tilde{\sigma}_0$ and γ_0 depends on L, $\|D^2(T-G)\|_{L^1(\mathbb{R}^2)}$, p and ss_0 only.

Then we can apply to $w = I_{s,\sigma}(u)$ and $\Omega = \Omega(u)$ the same analysis we have used for v and $\tilde{\Omega}$ in the first part of this proof. We may therefore conclude that if $u \in \mathcal{A}_\gamma$ and $D \in \mathcal{A}$ is such that $\|u - \chi_D\|_{L^1(B_R)} \leq \gamma$, then $\Omega \Subset B_{\tilde{R}}$, $\tilde{d}_1(\Omega, D) \leq \delta r$ and, taken r_1 as before, possibly with a smaller $\tilde{R}_0$ still depending on L only, we can find $L_1 \geq L$, depending on L, R, $\|D^2(T-G)\|_{L^1(\mathbb{R}^2)}$, p and ss_0 only, such that $\Omega \in \mathcal{A}^{1,\alpha}(r_1, L_1, \tilde{R})$.

This kind of argument leads us also to show that Ω shares the same topological properties of D, that is for example Ω and $\partial\Omega$ have the same number of connected components of D and ∂D, respectively.

It remains to show the uniform continuity property. We recall that the operator $\mathcal{P}_{J,ss_0}$ is Hölder continuous from A, with the $L^1(B_R)$ norm, into $C^{1,\alpha}(\mathbb{R}^2)$, with its usual norm. This means that there exists a constant $\tilde{C}$ such that for any u_1 and $u_2 \in A$, if we call $w_1 = I_{s,\sigma}(u_1)$ and $w_2 = I_{s,\sigma}(u_2)$, then

$$\|w_1 - w_2\|_{C^{1,\alpha}(\mathbb{R}^2)} \leq \tilde{C}\|u_1 - u_2\|^{1/p}_{L^1(B_R)}.$$

A simple application of the previous analysis allows us to prove this claim

Claim 1 There exists a function $g : [0,+\infty) \to [0,+\infty)$, which is continuous, increasing and such that $g(0) = 0$, satisfying the following property. For any $u \in \mathcal{A}_\gamma$, for any $\varepsilon > 0$ and any $x \in \mathbb{R}^2$ we have

$$\text{if } x \notin B_\varepsilon(\partial\Omega(u)), \text{ then } |I_{s,\sigma}(u) - h| > g(\varepsilon). \tag{4.15}$$

Let us now assume that u_1 and u_2 belong to $\mathcal{A}_\gamma$ and let us fix $\varepsilon > 0$. We can find $\eta > 0$ such that if $\|u_1 - u_2\|_{L^1(B_R)} \leq \eta$, then $\|w_1 - w_2\|_{L^\infty}(\mathbb{R}^2) \leq g(\varepsilon)$.

Let us now take $x \in \partial\Omega(u_1)$, that is $x \in \mathbb{R}^2$ such that $w_1(x) = h$. We infer that $|w_2(x) - h| \leq g(\varepsilon)$, therefore by the claim we deduce that $x \in B_\varepsilon(\partial\Omega(u_2))$. That is $\partial\Omega(u_1) \subset B_\varepsilon(\partial\Omega(u_2))$. By symmetry, we conclude that $\tilde{d}_1(\Omega(u_1), \Omega(u_2)) \leq \varepsilon$. In other words, the map which to any $u \in \mathcal{A}_\gamma$ associates the open set $\Omega(u)$ is uniformly continuous with respect to the L^1 norm on $\mathcal{A}_\gamma$ and the distance $\tilde{d}_1$. However, we have shown in Subsection 3.3 that the distances d_1, $\tilde{d}_1$, d_2 and d_3 are topologically equivalent on $\mathcal{A}^{1,\alpha}(r_1, L_1, \tilde{R})$, to which all $\Omega(u)$ belongs, for any $u \in \mathcal{A}_\gamma$. Therefore the map $\mathcal{A}_\gamma \ni u \to \Omega(u)$ is uniformly continuous with respect to the L^1 norm on $\mathcal{A}_\gamma$ and any of the distances d_1, $\tilde{d}_1$, d_2 and d_3 related to $B_{\tilde{R}}$.

We observe that

$$d_2(\Omega(u_1), \Omega(u_2)) = \|\mathcal{W}(u_1) - \mathcal{W}(u_2)\|_{L^1(B_{\tilde{R}+1})} = \|\mathcal{W}(u_1) - \mathcal{W}(u_2)\|_{L^1(B_{\tilde{R}})}$$

whereas

$$d_3(\Omega(u_1), \Omega(u_2)) = d_2(\Omega(u_1), \Omega(u_2)) + |P(\Omega(u_1)) - P(\Omega(u_2))| = d_{st}(\mathcal{W}(u_1), \mathcal{W}(u_2))$$

where d_{st} is here the distance inducing strict convergence in $BV(B_{\tilde{R}})$. Therefore we conclude that $\mathcal{W} : \mathcal{A}_\gamma \to BV(B_{\tilde{R}})$ is uniformly continuous with respect to the L^1 norm on $\mathcal{A}_\gamma$ and, on $BV(B_{\tilde{R}})$, with respect either to the L^1 norm or to the d_{st} distance. $\square$

Remark 4.4 Let us finally remark that if, instead of taking $h \in [1/3, 2/3]$, we simply assume $0 < h < 1$, then the same analysis may still be carried over. Clearly we need to change the values of R_0 and $\tilde{h}_1$ in Proposition 3.7, so that they depend on h as well. As a consequence also the quantities introduced in the above Theorem 4.2 would depend on h.

5 Analysis of the inverse problem

Throughout this section, we shall keep the notation of Theorem 4.2 and we shall also assume that the hypotheses of Theorem 4.2 are satisfied. We shall fix h, $1/3 \leq h \leq 2/3$, and s, $0 < s \leq 1/s_0$, and we shall take σ, $0 < \sigma \leq \tilde{\sigma}_0 s$, and γ, $0 < \gamma \leq \gamma_0$, $\tilde{\sigma}_0$ and γ_0 as in Theorem 4.2.

We call Ω_0 the circuit to be reconstructed and we shall assume that it belongs to $\mathcal{A} = \mathcal{A}^{0,1}(r, L, R)$.

We recall that, by Proposition 3.6, $\mathcal{A}$ is compact with respect to the d_2 distance, which corresponds to the distance induced by the L^1 norm for the corresponding characteristic functions. Then it is an immediate consequence of the last part of Theorem 4.2, see also Remark 4.3, that the problem

$$\min_{D \in \mathcal{A}} d_3(\Omega(D), \Omega_0)$$

admits a solution. We note that $\Omega(D) = \Omega(\chi_D)$ and that here d_3 is the distance defined in (3.10) related to $B_{\tilde{R}}$.

From a numerical point of view, the class $\mathcal{A}$ is rather difficult to handle. We try to reduce this difficulty by enlarging the class $\mathcal{A}$ to a class of characteristic functions of sets with finite perimeter. In order to keep the lower semicontinuity of the functional, we restrict ourselves to characteristic functions of sets with finite perimeter which are contained in $\mathcal{A}_\gamma$. Namely, we define the following functional $F_0 : A \to [0, +\infty]$ such that for any $u \in A$ we have

$$F_0(u) = d_{st}(\mathcal{W}(u), \chi_{\Omega_0}) + bP(u), \tag{5.1}$$

where P is the functional defined in (3.3) with $\mathcal{D}$ chosen to be B_R, b is a positive parameter and d_{st} is the strict convergence distance in $BV(B_{\tilde{R}})$. We recall that, whenever $u \in \{0, 1\}$ almost everywhere in B_R and $u \in BV(B_{R+1})$, then $P(u) = P(u, B_{R+1}) = |Du|(B_{R+1})$. Otherwise, $P(u)$, and consequently also $F_0(u)$, is equal to $+\infty$. Moreover, if $u \in \mathcal{A}_\gamma$, in particular if $u = \chi_D$ for some $D \in \mathcal{A}$, then $d_{st}(\mathcal{W}(u), \chi_{\Omega_0}) = d_3(\Omega(u), \Omega_0)$, where again d_3 is the distance defined in (3.10) related to $B_{\tilde{R}}$.

We look for the solution to the following minimization problem

$$\min\{F_0(u) \ : \ u \in \mathcal{A}_\gamma\}. \tag{5.2}$$

We notice that such a minimization problem admits a solution.

Even if the class $\mathcal{A}_\gamma$ might still be not very satisfactory to handle from a numerical point of view, since it somehow involves handling the class $\mathcal{A}$, we believe that from a practical point of view such a restriction might be dropped and we might use the class $A \subset L^1(B_R)$ instead. In fact, we have a good initial guess, given by the target circuit χ_{Ω_0}, and it is reasonable to assume that the optimal mask will be a rather small perturbation of Ω_0 itself. In fact, under our assumptions, by the arguments developed in the proof of Theorem 4.2, we can show that $\Omega(u)$ has the same topological properties of D, where χ_D is the element of $\mathcal{A}$ which is closest to u. Therefore if we look for a set $\Omega(u)$ as close as possible to Ω_0, then at least we need to require that the set D has the same topological properties of Ω_0. For this reason and since $\Omega_0 \in \mathcal{A}$, it might be essentially the same to perform the minimization in a small neighbourhood of $\mathcal{A}$ or in the whole A. On the other hand, again by our assumptions, we notice that whenever the boundary of Ω_0 presents a corner, and this is often case, as

$\partial\Omega_0$ is often the union of a finite number of segments, then Ω_0 cannot be reconstructed in an exact way, since $\Omega(u)$, for any $u \in \mathcal{A}_\gamma$, is a $C^{1,\alpha}$ set, thus its boundary cannot have any corner.

Besides dealing with the class $\mathcal{A}_\gamma$, there are several other difficulties. In particular, computing $F_0(\chi_E)$ for some $E \subset B_R$ is not an easy task, since it involves at least the computation of the perimeters of E and of $\Omega(\chi_E)$. Furthermore, solving a minimization problem in the class of sets of finite perimeter is not a straightforward task from the numerical point of view.

In order to solve these difficulties, we use the following strategy. We approximate, in the sense of Γ-convergence, the functional F_0 with a family of functional $\{F_\varepsilon\}_{\varepsilon>0}$ which are easier to compute numerically and are defined on a set of smooth functions.

As in Section 2.3, we take a C^∞ function $\phi : \mathbb{R} \to \mathbb{R}$ such that ϕ is nondecreasing, $\phi(t) = 0$ for any $t \leq -1/2$ and $\phi(t) = 1$ for any $t \geq 1/2$. For any $\eta > 0$ and any $\tau \in \mathbb{R}$, let

$$\phi_{\eta,\tau}(t) = \phi\left(\frac{t-\tau}{\eta}\right), \quad \text{for any } t \in \mathbb{R}.$$

Then we have the following result.

Proposition 5.1 *For any $\eta > 0$, let $\Phi_\eta : A \to C^{1,\alpha}(\mathbb{R}^2)$ be defined as*

$$\Phi_\eta(u) = \phi_{\eta,h}(I_{s,\sigma}(u)), \quad \text{for any } u \in A.$$

Then, for any η, $0 < \eta \leq h$, Φ_η is Hölder continuous, with exponent $1/p$, from A, with the $L^1(B_R)$ norm, into $C^{1,\alpha}(\mathbb{R}^2)$, with its usual norm.

Furthermore, as $\eta \to 0^+$, $(\mathcal{W} - \Phi_\eta) : \mathcal{A}_\gamma \to BV(B_{\tilde{R}})$ converges uniformly to zero on $\mathcal{A}_\gamma$ with respect to the distance d_{st} on $BV(B_{\tilde{R}})$.

Proof. The continuity property of Φ_η immediately follows by the continuity of $\mathcal{P}_{J,ss_0}$ and by the properties of ϕ. We just note that the Hölder exponent is fixed, whereas the Hölder constant might depend upon η.

About the convergence result, we begin by recalling that $\mathcal{W}(u) = \mathcal{H}_h(I_{s,\sigma}(u))$, $u \in \mathcal{A}_\gamma$. We use Claim 1 introduced in the proof of Theorem 4.2. We call $t_0 = \min(1, \sup\{g(s) : s \in [0, +\infty)\})$ and s_0 the positive real number such that $g(s_0) = t_0/2$. We call $g^{-1} : [0, t_0/2] \to [0, s_0]$ the continuous, increasing function which is the inverse of g on such intervals. For any η, $0 < \eta \leq t_0$, we infer that $\Phi_\eta(u)(x)$ might be different from $\mathcal{W}(u)(x)$ only if $x \in B_{g^{-1}(\eta/2)}(\partial\Omega(u))$. By estimates like (3.15) and (3.16), which are independent of $u \in \mathcal{A}_\gamma$, we obtain that $\|(\mathcal{W} - \Phi_\eta)(u)\|_{L^1(B_{\tilde{R}})}$ converges to zero, as $\eta \to 0^+$, uniformly for $u \in \mathcal{A}_\gamma$.

For any $t \in \mathbb{R}$ and any $u \in \mathcal{A}_\gamma$, we call

$$P(u,t) = P(\{x \in \mathbb{R}^2 \,:\, I_{s,\sigma}(u)(x) > t\}, B_{\tilde{R}}).$$

It remains to prove that, as $\eta \to 0^+$, $|D(\Phi_\eta(u))|(B_{\tilde{R}}) = \int_{B_{\tilde{R}}} |\nabla(\Phi_\eta(u))|$ converges to $|D(\mathcal{W}(u))|(B_{\tilde{R}}) = P(u,h)$ uniformly for $u \in \mathcal{A}_\gamma$. We argue in the following way. We have that, for any η, $0 < \eta \leq h$,

$$|D(\Phi_\eta(u))|(B_{\tilde{R}}) = \int_{B_{\tilde{R}}} |\nabla(\Phi_\eta(u))| = \int_{B_{\tilde{R}}} |\phi'_{\eta,h}(I_{s,\sigma}(u))||\nabla(I_{s,\sigma}(u))|.$$

Since $\phi'_{\eta,h} \geq 0$, and for η small enough, uniformly with respect to $u \in \mathcal{A}_\gamma$, $\phi'_{\eta,h}(I_{s,\sigma}(u)) = 0$ outside $B_{\tilde{R}}$, without loss of generality, we have that

$$|D(\Phi_\eta(u))|(B_{\tilde{R}}) = \int_{\mathbb{R}^2} \phi'_{\eta,h}(I_{s,\sigma}(u))|\nabla(I_{s,\sigma}(u))|.$$

By the coarea formula,

$$|D(\Phi_\eta(u))|(B_{\tilde{R}}) = \int_{-\infty}^{+\infty} \left(\int_{\{I_{s,\sigma}(u)=t\}} \phi'_{\eta,h}(t) \mathrm{d}\mathcal{H}^1(y) \right) \mathrm{d}t.$$

Therefore,

$$|D(\Phi_\eta(u))|(B_{\tilde{R}}) = \frac{1}{\eta} \int_{-\infty}^{+\infty} \phi' \left(\frac{(t-h)}{\eta} \right) P(u,t) \mathrm{d}t = \int_{-1/2}^{+1/2} \phi'(s) P(u, s\eta + h) \mathrm{d}s.$$

Since $|D(\mathcal{W}(u))|(B_{\tilde{R}}) = P(u,h)$ and $\int_{-1/2}^{+1/2} \phi'(s)\mathrm{d}s = 1$, we obtain that

$$\big| |D(\Phi_\eta(u))|(B_{\tilde{R}}) - |D(\mathcal{W}(u))|(B_{\tilde{R}}) \big| \leq \int_{-1/2}^{+1/2} \phi'(s)|P(u, s\eta + h) - P(u,h)| \mathrm{d}s.$$

It remains to show that, as $\eta \to 0^+$, $\sup\{|P(u, t+h) - P(u,h)| \,:\, t \in [-\eta/2, +\eta/2]\}$ goes to zero uniformly with respect to $u \in \mathcal{A}_\gamma$. Therefore the proof is concluded by using the following claim.

Claim 2 There exist a positive constant η_0 and a continuous, increasing function $g_1 : [0, \eta_0] \to [0, +\infty)$, such that $g_1(0) = 0$, such that for any η, $0 < \eta \leq \eta_0$, and any $u \in \mathcal{A}_\gamma$, we have that

$$\sup\{|P(u, t+h) - P(u,h)| \,:\, t \in [-\eta/2, +\eta/2]\} \leq g_1(\eta).$$

The proof of Claim 2 is a straightforward, although maybe lengthy to describe, consequence of the analysis developed in the proof of Theorem 4.2. We leave the details to the reader. We just notice that Claim 2 is a sort of generalization of Claim 1 and the arguments used to prove the two claims are essentially analogous. □

We are now in the position of describing the approximating functionals and proving the Γ-convergence result. Let us a fix a constant p_1, $1 < p_1 < +\infty$, and a continuous function $W : \mathbb{R} \to [0, +\infty)$ such that $W(t) = 0$ if and only if $t \in \{0, 1\}$. Let us denote by P_ε, $\varepsilon > 0$, the functional defined in (3.2) with $p = p_1$, the function W and $\mathcal{D} = B_R$. We recall that the functional P is defined in (3.3), again with $\mathcal{D} = B_R$.

Then, for any $\varepsilon > 0$, let us define $F_\varepsilon : A \to [0, +\infty]$ such that for any $u \in A$ we have

$$F_\varepsilon(u) = d_{st}(\Phi_{\eta(\varepsilon)}(u), \chi_{\Omega_0}) + bP_\varepsilon(u) \tag{5.3}$$

where $\eta : [0, +\infty) \to [0, +\infty)$ is a continuous, increasing function such that $\eta(0) = 0$.

By the direct method, we can prove that each of the functionals F_ε, $\varepsilon > 0$, admits a minimum either over A or over $\mathcal{A}_\gamma$.

The Γ-convergence result is the following.

Theorem 5.2 *Let us consider the metric space (X, d) where $X = \mathcal{A}_\gamma$ and d is the metric induced by the L^1 norm. Then, as $\varepsilon \to 0^+$, F_ε Γ-converges to F_0 on X with respect to the distance d.*

Proof. Let us fix a sequence $\{\varepsilon_n\}_{n=1}^\infty$ of positive numbers converging to zero as $n \to \infty$. Let, for any $n \in \mathbb{N}$, $F_n = F_{\varepsilon_n}$. We need to prove that Γ-$\lim_n F_n = F$. Let us also remark that we may extend F_n and F over $L^1(\mathbb{R}^2)$ by setting them equal to $+\infty$ outside $\mathcal{A}_\gamma$. Let us define $\tilde{P}_\varepsilon$, $\varepsilon > 0$, and $\tilde{P}$ as the functionals which are equal to the functionals P_ε and P, respectively, on $\mathcal{A}_\gamma$ and $+\infty$ elsewhere. We recall that P_ε, $\varepsilon > 0$, and P are defined in (3.2) and in (3.3), respectively, with $p = p_1$ and $\mathcal{D} = B_R$.

We observe that, as a consequence of Proposition 5.1 and of the stability of Γ-convergence under uniformly converging continuous perturbations, it is enough to show that Γ-$\lim_n \tilde{P}_n = \tilde{P}$, where $\tilde{P}_n = \tilde{P}_{\varepsilon_n}$, $n \in \mathbb{N}$. Let us prove this Γ-convergence result.

The Γ-liminf inequality is an immediate consequence of Theorem 3.1 and of the fact that $\mathcal{A}_\gamma$ is a closed subset of $L^1(B_R)$.

For what concerns the recovery sequence, then we argue in the following way. If $u \in A$ is such that $\|u - \chi_D\|_{L^1(B_R)} < \gamma$, for some $D \in \mathcal{A}$, then we again use Theorem 3.1 to construct a recovery sequence for such a function u, that is a sequence $\{u_n\}_{n=1}^\infty$ contained in $\mathcal{A}_\gamma$ such that, as $n \to \infty$, $u_n \to u$ in $L^1(B_R)$ and $\tilde{P}_n(u_n) \to \tilde{P}(u)$.

It remains to study the case when $u \in \partial\mathcal{A}_\gamma$ and $\tilde{P}(u) < +\infty$. In this case, we have that $u = \chi_E$, where $E \subset B_R$ is a set of finite perimeter, and we pick $D \in \mathcal{A}$ such that $\|\chi_E - \chi_D\|_{L^1(B_R)} = |E \Delta D| = \gamma$. Then at least one of these two cases must be satisfied. Either there exists $x \in B_R \backslash \overline{D}$ such that

$$\lim_{\rho \to 0^+} \frac{|E \cap B_\rho(x)|}{|B_\rho(x)|} = 1$$

or there exists $x \in D$ such that

$$\lim_{\rho \to 0^+} \frac{|E \cap B_\rho(x)|}{|B_\rho(x)|} = 0.$$

We choose an arbitrary sequence $\{\rho_j\}_{j=1}^\infty$ of positive numbers such that $\lim_j \rho_j = 0$. In the first case, for any $j \in \mathbb{N}$, we choose E_j such that $\chi_{E_j} = \chi_E(1 - \chi_{B_{\rho_j}(x)})$. In the second case, we choose E_j such that $\chi_{E_j} = \chi_E(1 - \chi_{B_{\rho_j}(x)}) + \chi_{B_{\rho_j}(x)}$. We notice that, in either cases, for any $j \in \mathbb{N}$, E_j is a set of finite perimeter such that $\|\chi_{E_j} - \chi_D\|_{L^1(B_R)} < \gamma$. Furthermore, as $j \to \infty$ we have that $\chi_{E_j} \to \chi_E$ in $L^1(B_R)$ and $P(E_j) \to P(E)$, that is $\tilde{P}(\chi_{E_j}) \to \tilde{P}(\chi_E)$. Then the proof may be concluded by following the arguments of Section 4.2 in [2] which we have briefly recalled in the proof of Theorem 3.1. $\square$

We remark that $\Omega_0 \in \mathcal{A}$, therefore we may find a family $\{\tilde{u}_\varepsilon\}_{\varepsilon > 0}$ such that, as $\varepsilon \to 0^+$, $\tilde{u}_\varepsilon \to \chi_{\Omega_0}$ in $L^1(B_R)$ and $P_\varepsilon(\tilde{u}_\varepsilon) \to P(\Omega_0)$. Without loss of generality, we may assume that, for any $\varepsilon > 0$, $0 \leq \tilde{u}_\varepsilon \leq 1$ almost everywhere in B_R and that $\tilde{u}_\varepsilon \in \mathcal{A}_\gamma$. By Proposition 5.1, we conclude that $F_\varepsilon(\tilde{u}_\varepsilon) \to F_0(\Omega_0) < +\infty$. We obtain that for any $\varepsilon_0 > 0$ there exists a constant C_1 such that

$$\min_{\mathcal{A}_\gamma} F_\varepsilon \leq C_1 \quad \text{for any } \varepsilon,\ 0 < \varepsilon \leq \varepsilon_0. \tag{5.4}$$

Obviously, the same property is shared by the minimum values of F_ε over A.

It remains to prove that the functionals F_ε are equicoercive over $\mathcal{A}_\gamma$, that is that the following result holds.

Proposition 5.3 *For any $\varepsilon_0 > 0$, there exists a compact subset $\mathcal{K}$ of $\mathcal{A}_\gamma$ such that for any ε, $0 < \varepsilon \leq \varepsilon_0$, we have*

$$\min_{\mathcal{K}} F_\varepsilon = \min_{\mathcal{A}_\gamma} F_\varepsilon.$$

Proof. Let us take the constant C_1 as in (5.4). Let $u_\varepsilon \in \mathcal{A}_\gamma$, $0 < \varepsilon \leq \varepsilon_0$, be such that $F_\varepsilon(u_\varepsilon) = \min_{\mathcal{A}_\gamma} F_\varepsilon$. Then we observe that the set $\{u_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ satisfies the properties of Remark 3.4 for some constant C. Therefore $\{u_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ is precompact in $L^1(B_R)$ and the proof is concluded. $\square$

Remark 5.4 With an analogous proof, the same result of Proposition 5.3 holds if we replace $\mathcal{A}_\gamma$ with A.

By Theorem 5.2 and Proposition 5.3, we can apply the Fundamental Theorem of Γ-convergence to conclude with the following result.

Theorem 5.5 *We have that F_0 admits a minimum over $\mathcal{A}_\gamma$ and*

$$\min_{\mathcal{A}_\gamma} F_0 = \lim_{\varepsilon\to 0^+} \inf_{\mathcal{A}_\gamma} F_\varepsilon = \lim_{\varepsilon\to 0^+} \min_{\mathcal{A}_\gamma} F_\varepsilon.$$

Let ε_n, $n \in \mathbb{N}$, be a sequence of positive numbers converging to 0. For any $n \in \mathbb{N}$, let $F_n = F_{\varepsilon_n}$. If $\{u_n\}_{n=1}^{\infty}$ is a sequence contained in $\mathcal{A}_\gamma$ which converges, as $n \to \infty$, to $u \in \mathcal{A}_\gamma$ in $L^1(B_R)$ and satisfies $\lim_n F_n(u_n) = \lim_n \inf_{\mathcal{A}_\gamma} F_n$, then u is a minimizer for F_0 on $\mathcal{A}_\gamma$, that is u solves the minimization problem (5.2).

We conclude with the following remark. With the notation of Theorem 5.5, if $\{u_n\}_{n=1}^{\infty}$ is a sequence contained in $\mathcal{A}_\gamma$ which satisfies $\lim_n F_n(u_n) = \lim_n \inf_{\mathcal{A}_\gamma} F_n$, then, by Remark 3.4, we have that, up to passing to a subsequence, $\{u_n\}_{n=1}^{\infty}$ actually converges, as $n \to \infty$, to some function $u \in \mathcal{A}_\gamma$ in $L^1(B_R)$.

6 Discussion

We have provided a mathematical study of the inverse problem of photolithography. The approach we propose is to seek an approximate solution by formulating the geometrical problem using a phase-field method. We further relax the hard threshold involved in image exposure with an approximate Heaviside function. We show that the variational problem for the approximate solution is well-posed. This opens a way into designing mathematically rigorous numerical methods. We further show that as the approximation parameter goes to zero, a theoretical limit, the original optimization problem involving geometry is recovered.

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