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# Intermediate Logics and Polyhedra

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# Contents

*Dedicated to my Grandparents and to Carlotta*

# Introduction

### Spatial Logic Reloaded

The present work is framed within the field of spatial logic. According to the definition contained in the influential collective work that traced the boundaries of this field, detaching it from other declinations of logic, *spatial logic* means «any formal language interpreted over a class of structures featuring geometrical entities and relations, broadly construed» (Aiello, van Benthem, and Pratt-Hartmann 2007). The formal language in question can utilize any logical syntax: for example, one may employ the language of first order logic or some of its fragments. By contrast, the structures on which those languages are interpreted can inhabit any class of "geometrical" spaces: for example, classes of topological spaces, affine spaces, metric spaces, even single spaces such as the sphere or the three-dimensional Euclidean space can be considered. The non-logical primitives of the language can be interpreted as geometrical properties or relations defined on specific domains. For example, they can be interpreted in terms of connection between spatial regions, parallelism of lines or equidistance of two points from a third.

The essential feature of these logics lies in the fact that the notion of validity depends on the underlying geometry of the structures over which their spatial primitives are interpreted. According to the just outlined perspective, spatial logic is therefore simply *the study of the family of spatial logics* so defined (see Aiello, van Benthem, and Pratt-Hartmann 2007). This definition is clearly modeled on the example of temporal logic. In fact, a temporal logic is a formal language interpreted on a class of structures based on frameworks of temporal relations. However, even if the definition of spatial logic so conceived is insightful, it only tells us half of the story.

The examination of some early historical examples of logical treatment of space that are considered, according to Aiello, van Benthem, and Pratt-Hartmann (2007), founding moments for this field problematizes this perspective. We refer here not only to Alfred Tarski's work on the formalization of elementary and solid geometry, but above all to the one that constitutes the starting point of the present thesis, that is, Tarski's paper on the topological interpretation of intuitionistic and classical logic (Tarski 1938).

As we shall see in the following, according to the underlying perspective of this groundbreaking work, spatial logic becomes more properly *the study of the interrelationship that exists between geometric structures and logical languages*, including the relation between logical languages and geometric structures which "interpret" them. Logic *of* space is therefore properly understood both as a genitive-object case and a genitive-subject case. On the one hand, a *formalization of space* as suggested by the above definition of Aiello, van Benthem, and Pratt-Hartmann (2007); on the other hand, a *geometrization of logic* – a movement that can hardly be framed in the explanatory prism of the "linguistic turn" adopted by Aiello, van Benthem, and Pratt-Hartmann (2007).[1] Actually, in light of the developments of logic *within* the burgeoning of twentieth-century mathematics, the linguistic turn co-occured with a "spatial turn".[2]

As is well documented[3], the historical period in which Tarski's work flourished saw the rapid development of the methods of general topology, universal algebra, and lattice theory, and witnessed an increasingly active interaction between these disciplines. The mathematical climate and the general attitude of the 1930s may be summarized by Marshall Stone's aphorism that we find in his survey article – deeply connected to the one of Tarski[4] – on topological representation of Boolean algebras: «a cardinal principle of modern mathematical research may be stated as a maxim: one must always topologize» (Stone 1938).

The present work fits into this broader perspective on spatial logic that has its roots in Tarski's and Stone's work mentioned above as well as in Garrett Birkhoff's pioneering work on universal algebra and lattice theory and the more recent generalisations of Stone's representation to other classes of lattices by Leo Esakia and Hilary Priestley.[5]

These researches are all pervaded by the idea of a deep duality between geometry and logic that manifests itself as dualities between classes of topological spaces or their generalizations, such as ordered topological spaces, and logics appropriately algebraized. This reconfiguration of spatial logic in this broader perspective is not an idle question, but rather a meaningful one. In fact, starting from this more general

[1] «Tarski's discovery illustrates the most distinctive feature of logic in the wake of the model-theoretic revolution of the previous century: its fundamentally linguistic orientation» (Aiello, van Benthem, and Pratt-Hartmann 2007). The turning point of model theory (and universal algebra) in the previous century and its influence on logic could hardly be grasped solely in terms of "linguistic orientation". For more on the contemporary connection between logic and algebraic or real semi-algebraic geometry, see Bouscaren (1998) and van den Dries (1998).

[2] The overlooking of this aspect in the book of Aiello, van Benthem, and Pratt-Hartmann (2007) is further showed by the emphasis on areas such as mereology (or mereotopology), characterized by a top-down approach that seeks to find a general logic of parthood relation (or connection relation) suitable for any geometric shape. A well-known blindspot of this approach is the notion of boundary. By superimposing an implicit boolean framework for abstract reasons disconnected from the consideration of specific spaces, mereology is prevented "by construction" from grasping this notion that, from a cointuitionistic perspective – i.e., from a perspective of a more suitable logic for topological spaces –, is immediately grasped. (For further details, see Mormann 2013. It is worth to note that Mormann himself falls into the wrong path of absolutizing intuitionistic logic as a kind of mereology, that is, as a logic of space in general. On the limits of mereology and mereotopology, see Tsai 2005).

[3] See for instance the historical notes in Johnstone (1982), McKenzie *et al.* (1987) and Adams and Dziobiak (1996).

[4] In Tarski (1938) there are several references to Stone's articles devoted to his celebrated topological representation theory.

[5] Birkhoff (1935), Esakia (1974) and Priestley (1970).

framework it is possible to grasp the crucial feature of this field of research: that is, "bridge" results.

### Bridge Theorems

Suppose that a framework has been established in which a general procedure has been adopted to associate to each logic (appropriately algebrized) a class $\mathcal{K}$ of topological spaces, or space-based models, as its geometric counterpart. In such a framework, a *bridge theorem* is a mathematical result stating that for a certain property $P$ concerning a logic and a certain property $P'$ concerning a class of topological spaces, if $\mathsf{L}$ is a logic and $\mathcal{K}$ it is the corresponding geometric counterpart, then

$$\mathsf{L} \text{ satisfies } P \text{ if and only if } \mathcal{K} \text{ satisfies } P'.$$

The term "bridge theorem" can be traced back at least to Andréka *et al.* (2001) in a more algebraic context, but it can be analogously rephrased in terms of geometry. Results of this type establish a bridge between two different realms, that of logic and that of geometry, and allow to transform the problems of a logic into problems related to a class of topological spaces. Then one can use powerful tools from geometry to solve the problem, and then go back (crossing the bridge again) and get a solution to the original logical problem. Of course, it can be just as useful to do the opposite, that is, to transform the problems related to a class of topological spaces into problems of a logic. In a way, this method allows us to approach logical and geometrical problems as a whole.

In the present work, especially from Chapter 3 onwards, we shall see several instances of such bridge theorems.

### Tarski's Topological Interpretation of Intuitionistic Logic

Our starting point is Tarski's topological interpretation of intuitionistic logic, which we state in a different form than the original one.

In his article "Sentential Calculus and Topology" (Tarski 1938), whose main results were obtained by him in 1935, Tarski develops such an interpretation of intuitionistic propositional calculus. On this Tarski's reading, the syntax of propositional logic is interpreted in terms of *open sets* of a given topological space $X$ and operations with these open sets, so a formula $\varphi$ designates a certain open set of the space constructed from other open sets. A formula is true under this interpretation precisely when it evaluates to $X$. Tarski demonstrated that intuitionistic logic is complete with respect to this semantics.

More precisely, given a set $L$ of propositional atoms, an interpretation is a map $[\![-]\!]$ from the set of sentences in intuitionistic propositional logic over $L$ to (the collection of open sets of) a topological space $X$, such that $[\![\alpha]\!]$ is an arbitrary open set of $X$ for each atomic proposition $\alpha$, and which satisfies the following conditions for sentential connectives and propositional constants.[6]

$$
\begin{aligned}
[\![\top]\!] &= X \\
[\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] \\
[\![\bot]\!] &= \emptyset \\
[\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \cup [\![\psi]\!] \\
[\![\varphi \Rightarrow \psi]\!] &= \mathsf{int}(\complement [\![\varphi]\!] \cup [\![\psi]\!]) \\
[\![\neg\varphi]\!] &= \mathsf{int}(\complement [\![\varphi]\!])
\end{aligned}
$$

An interpretation $(X, [\![-]\!])$ models a sentence if the sentence is "true" under $[\![-]\!]$, i.e.:

$$(X, [\![-]\!]) \models \varphi \text{ if and only if } [\![\varphi]\!] = X$$

Thus the correspondence between the rules of the topological operations and the rules of the sentential operators, immediately implies that intuitionistic propositional logic is sound with respect to this topological semantics.

**Theorem** (Tarski 1938, Lemma 4.9)**.** *For any sentence $\varphi$ in intuitionistic propositional logic over L,*

$$\vdash \varphi \textit{ implies every topological interpretation } (X, [\![-]\!]) \textit{ satisfies } [\![\varphi]\!] = X.$$

Tarski showed also that intuitionistic logic is complete with respect to the topological semantics, in the following strong form.

**Theorem** (Tarski 1938, Second Principal Theorem)**.** *For any sentence $\varphi$ in intuitionistic propositional logic over L,*

$$\nvdash \varphi \textit{ implies there exists a topological interpretation } (X, [\![-]\!]) \textit{ such that } [\![\varphi]\!] \neq X.$$

### Tarski's Theorem on Intuitionistic Logic for Polyhedra

The result of Tarski has opened a research area that continues to thrive to this day. The closest descendants of Tarski (1938) are the influential articles McKinsey and Tarski (1944) and (1946), in which they offer a different proof of the Second Principal Theorem in the dual language of closed sets.

In 2015, N. Bezhanishvili *et al.* (2018) returns to Tarski's Second Principal Theorem. At the margins of his main results, Tarski showed that the class $C$ of topological spaces under examination can be considerably reduced without compromising its completeness' result. In particular, one can take $C := \{\mathbb{R}^n\}$, $n \geq 1$, or $C := \{2^{\mathbb{N}}\}$, where

[6] The $\mathsf{int}$ operator that transforms sets into open sets is needed since, in general, the set theoretic complement $\complement U$ of an open set $U$ is not open (for further details, see Chapter 1).

$2^{\mathbb{N}}$ denotes the Cantor space. As pointed out in N. Bezhanishvili *et al.* (2018), Tarski's result shows that in all those cases the corresponding logic is *always* intuitionistic logic – regardless of the fact that $\mathbb{R}^n, \mathbb{R}, 2^{\mathbb{N}}$ have *different* topological dimensions ($n$, 1 and 0, respectively).

Moreover, intuitionistic logic has the finite model property, that is, any non-valid formula $\alpha$ has a *finite* counter-model. In other words, there exists a finite model $Y$ together with a valuation $[\![-]\!]$ such that $[\![\alpha]\!] \neq Y$. (See Jaśkowski 1936. A proof of this result reformulated in algebraic terms can be also found in McKinsey and Tarski 1944.) As observed in N. Bezhanishvili *et al.* (2018), Tarski's Theorem does not expose the finite model property of intuitionistic logic. In fact, McKinsey and Tarski (1946) showed that, for instance, counter-models to formulae that are not intuitionistically provable always exist in $\mathbb{R}$ or $2^{\mathbb{N}}$, but there is no guarantee that they are automatically finite (for further details, see N. Bezhanishvili *et al.* 2018).

Starting from these limitations of Tarski's work and considering the polyhedra rather than topological spaces tout court, the main result of N. Bezhanishvili *et al.* (2018) is to provide a completeness theorem quite similar to that of Tarski, which however manages to capture the topological dimension of spaces logically. This is made possible by the fundamental fact that, as N. Bezhanishvili *et al.* (2018) shows, the logic of the class $P_d$ of polyhedra of a given dimension $d$ is the intermediate logic of bounded depth $d$.

**Theorem** (N. Bezhanishvili *et al.* 2018)**.** *For each $d \in \mathbb{N}$, the logic of the class $P_d$ is intuitionistic logic extended by the axiom schema formulas $\mathsf{bd}_d$ defined as follows:*

$$\mathsf{bd}_0 = p_0 \vee \neg p_0,$$
$$\mathsf{bd}_d = p_d \vee (p_d \Rightarrow \mathsf{bd}_{d-1}).$$

Further, due to the geometric structure of Polyhedra, this Theorem exposes the finite model property of this logic.[7]

### Main Results of the Present Work, Limits and Further Developments

At the last remark of N. Bezhanishvili *et al.* (2018), it is suggested that through polyhedra it is possible to logically express geometric properties of spaces other than their dimension. In fact, polyhedra enjoy another peculiar property: every geometric shape with a certain "regularity" – in specific terms, certain classes of (closed) topological manifolds – can be captured by a polyhedron via triangulation, that is, by subdividing the geometric shapes into appropriate "triangles", called simplices (which, in the 1- and 0-dimensional case, are simply edges and vertices, respectively). Therefore, one might well wonder: what is the intermediate logic of the class of triangulable topological manifolds of a given dimension $d$?

[7] For further details on N. Bezhanishvili *et al.* 2018, see Chapter 5 and 6.

As pointed out in N. Bezhanishvili *et al.* (2018), a clue to how this intermediate logic should be is given by a classical polyhedral geometry theorem stating that, for every triangulation of a given topological manifold of dimension $d$, each simplex of dimension $d-1$ is a face of exactly two simplices of dimension $d$. For example, in the 1-dimensional case this means that, for every inscribed polygon of the circle, each vertex is incident on exactly two edges.

The main result of the present work was to give substance to this intuition. In fact, we provided a completeness theorem in the style of Tarski that allows us to understand logically the above property of triangulations in the case of 1-dimensional manifolds, that is, the circle $S^1$ and the closed interval $[0,1]$.[8]

**Theorem** (Theorem 7.3.3)**.** *The logic of* 1*-dimensional manifolds is intermediate logic of bounded depth* 1 *extended by the axiom schema of bounded branching* $\mathsf{bb}_2$ *defined as follows:*

$$\mathsf{bb}_2 = \bigwedge_{i=0}^{2}((p_i \Rightarrow \bigvee_{i\neq j} p_j) \Rightarrow \bigvee_{i\neq j} p_j) \Rightarrow \bigvee_{i=0}^{2} p_i.$$

This result immediately opens the way to corroborating the intuition above even in the case of dimension greater than one. Moreover, in accordance with the two axes of the perspective delineated at the beginning of this Introduction, namely that of formalization of space and that of geometrization of logic, other possible developments of this work are the following: Given any class of polyhedra $C$, what is the intermediate logic of $C$? Is this logic (finitely, or recorsively) axiomatizable? Viceversa, given any intermediate logic $\mathsf{L}$, is there a class of polyhedra $C$ whose logic is $\mathsf{L}$?

It is also necessary to observe some of the limits arising from this interrelationship between intermediate logics and polyhedra. First of all, the Theorem above immediately shows that it is not possible to grasp the *homotopy class* of topological manifolds. In fact, the closed interval $[0,1]$, which is contractible to a point, has the same logic of the circle $S^1$ which it is not contractible (due to the topologically unavoidable "hole").

Much less evident from the Theorem above, but emerging during and through the proof, is the fact that actually the circle $S^1$ has the same logic as a disjoint union of circles (the same applies to the closed interval $[0,1]$). This means that it is not possible to grasp another important topological property, namely the *connectedness* of spaces.

Naturally, it is not ruled out that these limitations suggest better combinations between intermediate logics (or their generalizations) and polyhedra (or their generalizations).

### Contents

The present thesis consists of seven chapters. In Chapter 1, we introduce the intuitionistic and intermediate logics as sets of formulae closed under inference rules.

[8]For further details see Chapter 7.

In Chapter 2, we deal with the main mathematical objects studied in this work, i.e. ordered sets. Those structures are the most common models of intuitionistic and intermediate logics. They will be studied not only as such but also as categories and topological spaces. On the one hand, this will allow us to deal with important concepts such as adjoint functors and closure (and interior) operators. On the other hand, it will allow us to introduce generalizations of ordered sets (and topological spaces) – i.e., the ordered topological spaces, which are essential for developing the representation theory of lattices.

In Chapter 3, we deal with the main algebraic structures studied in this thesis, that is lattices. In particular, we will focus on Heyting algebras, since they provide the algebraization of intuitionistic and intermediate logics. Lattices will be studied both as ordered sets and algebraic structures. The main representation theorem for distributive lattices and Heyting algebras – the Esakia and Priestley dualities – will also be discussed carefully.

In Chapter 4, our main concern is to reconstruct a suitable model theory for the algebraic structures introduced in the previous chapter that applies to intuitionistic and intermediate logics. After introducing generalities on algebraic languages and their classes of models, the chapter culminates in the discussion of Birkhoff's Theorem, which will allow us to construct appropriate semantics for intuitionistic and intermediate logics. Many facts about the algebraic approach to those logics are well known. However, these results are scattered in the literature. A central point of the Chapter is to give a coherent exposition of this approach.

In Chapters 5 and 6, the main results of N. Bezhanishvili *et al.* (2018) are discussed in depth. In particular, in Chapter 5, after introducing general facts on polyhedra, we shall show that the algebra of sub-polyhedra of a polyhedron is a locally finite Heyting algebra. In Chapter 6, on the other hand, we shall see the main fact of N. Bezhanishvili *et al.* (2018): the logic of the class of polyhedra of a given dimension $d$ is intuitionistic logic extended by the axiom schema of bounded depth $d$.

Finally, in Chapter 7, after characterizing triangulations of the circle $S^1$ and the closed interval $[0,1]$, we shall prove that their logic is given by the intermediate logic of bounded depth 1 extended by the axiom schema of bounded branching $\mathsf{bb}_2$.

# 1 Intermediate Logics

In this chapter, we shall introduce intuitionistic and intermediate logics. In order to characterize those logics, two components are to specify: a language, which contains formal expressions called formulae and built up using various logical operators; and a relation of derivation, which should be a relation between sets of formulae and formulae.

Historically, intuitionistic logic was introduced by Arend Heyting in 1930 as a formalization of Luitzen Brouwer's ideas about intuitionism and constructive mathematics. For a detailed discussion of the relationship between intuitionistic logic and constructive mathematics we refer to Troelstra and van Dalen 1988.

## 1.1 Intuitionistic Logic

Just as classical propositional logic, intuitionistic propositional logic is designed to study a set of simple statements, and the compound statements built up from them. As we shall see in the following, intuitionistic propositional logic are useful for describing subsets of a given structure with particular properties.

We first set up intuitionistic propositional logic as a formal language. The symbols of our language are as follows:

- a set $C$ of connectives of different arities, $\wedge, \vee, \Rightarrow, \top, \bot$, and parentheses (, );
- a nonempty set $L$ of sentence symbols.

Intuitively, the sentence symbols stand for simple statements, and the connectives $\wedge, \vee, \Rightarrow$ stand for the words used to combine simple statements into compound statements. In a more formal way, we can say that, taking $C$ and $L$ together, they give rise to a free monoid $(L \cup C)^*$ whose elements are all the finite sequences (or strings) of zero or more elements from those sets, with string concatenation as the monoid operation and with the unique sequence of zero elements as the identity element. Of course, not every finite sequence of symbols in $(L \cup C)^*$ is a sentence: for instance $(\alpha \wedge \beta)$ is a sentence, but $\wedge\wedge)\gamma$ is not. In order to distinguish a sentence from what is not, the sentences' language $\mathcal{SL}$ is defined inductively as follows:

**Definition 1.1.1.**

(i) $\top, \bot$ is a sentence, $\bot \in \mathcal{SL}$.

(ii) Every sentence symbol $\alpha \in L$ is a sentence, $L \subseteq \mathcal{SL}$.

(iii) If $\alpha, \beta$ are sentences, then $(\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \Rightarrow \beta)$ are sentences.

(iv) A finite sequence of symbols is a sentence only if it can be shown to be a sentence by a finite number of applications of (i)-(iii).

**Remark 1.1.2.** Let us pause briefly for a notational remark on Greek letters. In the above paragraphs we have used the lower case Greek letters $\alpha, \beta, \gamma, \ldots$ as names for arbitrary finite sequences of symbols of $\mathcal{SL}$. The situation is similar to arithmetic, where we study natural numbers $0, 1, 2, 3, \ldots$ but much of the time we write down letters like $m, n, x, y, \ldots$ as names for arbitrary natural numbers. In the following we shall also use capital Greek letters $\Gamma, \Delta, \ldots$ as names for arbitrary set of finite sequences of symbols of $\mathcal{SL}$. It is worth noting that the symbols $\alpha, \beta, \Gamma, \ldots$ are not in our list of formal symbols of our language – they are merely informal symbols which we use to talk more easily about $\mathcal{SL}$.

We shall introduce abbreviations to our language in the usual way, in order to make sentences more readable. The symbols $\neg$ and $\Leftrightarrow$ are abbreviations defined as follows:

- $\neg\alpha$ for $p \Rightarrow \bot$,
- $\alpha \Leftrightarrow \beta$ for $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.

Once fixed $\mathcal{SL}$, we can define a binary relation $\vdash \subseteq 2^{\mathcal{SL}} \times \mathcal{SL}$ on families of formulae $\Gamma \subseteq \mathcal{SL}$ and formulae $\alpha \in \mathcal{SL}$. We can write $(\Gamma, \alpha) \in \vdash$ or, in a customary way, $\Gamma \vdash \alpha$. Syntactically, $\Gamma$ can be seen as a set of hypothesis that we take for granted, as the axioms for the propositional theories that we wish to consider. In order to give a more precise idea of what is meant by a theory, we have to provide an appropriate definition of the binary relation $\Gamma \vdash \alpha$.

Nevertheless, before giving a precise definition of this formal expression, we have to choose a set of axioms $\mathsf{Ax} \subseteq \mathcal{SL}$ for the intuitionistic propositional logic. To make the presentation of this logic more convenient from a theoretical point of view, we use a Hilbert-style calculus for IPL. As is well known, the first Hilbert proof system (Hilbert-style formalization) of the intuitionistic logic is due to Heyting. We present here a Hilbert style proof system that is equivalent to the Heyting's original formalization (see N. Bezhanishvili 2006).

**Definition 1.1.3.** The *intuitionistic propositional logic* is given by the smallest set of formulas containing the axioms ($\mathsf{Ax}$):

(1) $\alpha \Rightarrow (\beta \Rightarrow \alpha)$,

(2) $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$,

(3) $\alpha \wedge \beta \Rightarrow \alpha$,

(4) $\alpha \wedge \beta \Rightarrow \alpha$,

(5) $\alpha \Rightarrow (\beta \Rightarrow (\alpha \wedge \beta))$

(6) $\alpha \Rightarrow \alpha \vee \beta$,

(7) $\beta \Rightarrow \alpha \vee \beta$,

(8) $(\alpha \Rightarrow \gamma) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow ((\alpha \vee \beta) \Rightarrow \gamma))$,

(9) $\perp \Rightarrow \alpha$,

and closed under the inference rules:

$$\frac{\alpha,\ \alpha \Rightarrow \beta}{\beta} \quad (\textit{modus ponens}),$$

and

$$\frac{\varphi(\alpha_1, \ldots, \alpha_n)}{\varphi[\beta_1/\alpha_1, \ldots, \beta_n/\alpha_n]} \ (\textit{substitution}).$$

We are now in a position to give the intended definition to the above formal binary relation and, accordingly, the proper definition of propositional theory.

**Definition 1.1.4.** Given a family of formulae $\Gamma \subseteq \mathcal{SL}$ and a formula $\alpha \in \mathcal{SL}$, there is a $\Gamma$-*derivation of* $\alpha$

$$\Gamma \vdash \alpha$$

if exists a finite sequence of formulae $\alpha_1, \ldots, \alpha_N$ such that:

(i) $\alpha_N = \alpha$

(ii) $\alpha_i \in \mathsf{Ax} \cup \Gamma$ or $\exists \alpha_j, \alpha_k$, with $j, k \leq i$, such that $\alpha_i$ is obtained from $\alpha_j$ and $\alpha_k$ by applying modus ponens.

**Definition 1.1.5.** By a *propositional theory* we mean a family $\Gamma \subseteq \mathcal{SL}$ of formulae, whose elements are called the non-logical *axioms* of $\Gamma$, closed under modus ponens, namely such that $\Gamma \vdash \alpha$ implies $\alpha \in \Gamma$. The set of theories of is denoted by $\mathcal{T}h(\mathsf{INT})$ and a *theorem* of the intuitionistic logic is a formula $\alpha$ such that $\emptyset \vdash \alpha$ (usually written $\vdash \alpha$).

It is easy to see that the set of theorems is the smallest theory of the intuitionistic logic, and that there is always a smallest theory containing a given family $\Gamma$ of formulas, namely the set $\nabla\Gamma := \{\, \alpha \in \mathcal{SL} \mid \Gamma \vdash \alpha \,\}$, called the *theory generated by* $\Gamma$. By definition of the relation of derivation, the set $\mathcal{SL}$ of all formulas is always a theory, the inconsistent one.

**Remark 1.1.6.**

1. Even if we have just ascribed the term "intuitionistic propositional logic" to a set of formulas closed under modus ponens and substitutions, there is always the relation of derivation in the background. This notion of intuitionistic logic actually corresponds to the set of theorems of intuitionistic logic. A more appropriate definition would be as the pair $(\mathcal{SL}, \vdash)$. However, the notion of intuitionistic logic as a set of formulas closed under inference rules is proper enough for our scope.

2. Observe that the relation of derivation $\vdash \subseteq 2^{\mathcal{SL}} \times \mathcal{SL}$ satisfies the following properties, for all $\Gamma \cup \{\alpha\} \subseteq \mathcal{SL}$.

   (a) If $\alpha \in \Gamma$, then $\Gamma \vdash \alpha$ (Identity).

   (b) If $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \alpha$ (monotonicity).

   (c) If $\Gamma \vdash \alpha$ and $\Delta \vdash \beta$ every $\beta \in \Gamma$, then $\Delta \vdash \alpha$ (transitivity)

   (d) If $\Gamma \vdash \alpha$, then $\Gamma[\beta_1/\alpha_1, \ldots, \beta_n/\alpha_n] \vdash \varphi[\beta_1/\alpha_1, \ldots, \beta_n/\alpha_n]$ for every substitutions (structurality).

   The last condition correspond to the closure under substitutions of intuitionistic logic. It is also and more broadly the rendering of the idea that logical consequence is formal, which means that, in some sense, it should depend only on the "form" of the sentences. It allows elements of the set $L$ to behave as real variables, that is, to represent arbitrary formulas in the same sense that variables in real analysis represent arbitrary elements of the domain of a function. For instance, when saying that the consequence $\alpha \wedge \beta \vdash \alpha$ holds, these "$\alpha$" and "$\beta$" may be viewed as representing arbitrary formulas because, by structurality, this implies that also the consequence $\varphi \wedge \psi \vdash \varphi$ holds for any two formulas $\varphi$ and $\psi$.

Intuitionistic logic satisfies the Deduction theorem, namely
**Theorem 1.1.7** (Deduction Theorem)**.** *For all $\Gamma \subseteq \mathcal{SL}$,*

$$\Gamma, \varphi \vdash \psi \quad \textit{if and only if} \quad \Gamma \vdash \varphi \Rightarrow \psi.$$

*Proof.* See Chagrov and Zakharyaschev (1997). ☐

The proof is well known: starting from the assumption that $\Gamma, \varphi \vdash \psi$, it works by induction on the length of a proof of $\psi$ from $\Gamma \cup \{\varphi\}$, showing that for each step $\varphi_i$ in such a proof, there is a proof of $\varphi \Rightarrow \varphi$ from $\Gamma$. For this, one only uses axioms (1) and (2) and the rule of modus ponens.

## 1.2 Intermediate Logics

Let CL denote *classical propositional logic*. It is very well known (see Chagrov and Zakharyaschev 1997) that $\mathsf{CL} \subseteq \mathsf{INT}$: indeed, we have that the formulae $\alpha \vee \neg\alpha$ and $\neg\neg\alpha \Rightarrow \alpha$ belong to CL, but they do not belong to INT. As a consequence, we can have the following equivalent definitions of CL starting from INT:
**Theorem 1.2.1.** *The classical propositional calculus is given by the smallest set of formulae that contains* INT*, the formula $\alpha \vee \neg\alpha$, and is closed under modus ponens and substitutions. Equivalently, it is given by the smallest set of formulae that contains* INT*, the formula $\neg\neg\alpha \Rightarrow \alpha$, and is closed under modus ponens and substitutions.*

*Proof.* See Chagrov and Zakharyaschev (1997). □

**Definition 1.2.2.** A set of formulae $\mathsf{L} \subseteq \mathcal{SL}$ closed under modus ponens and substitutions is called an *intermediate logic* if $\mathsf{CL} \subseteq \mathsf{L} \subseteq \mathsf{INT}$.

Thus, the intermediate logics are "in between" of classical and intuitionistic propositional logics. We briefly introduce now a class containing all the intermediate logics. We shall come back on this class in Subsection 4.3.5.

**Definition 1.2.3.** A set of formulas $\mathsf{L} \subseteq \mathcal{SL}$ closed under modus ponens and substitutions is called a superintuitionistic logic if $\mathsf{L} \supseteq \mathsf{INT}$.

As a consequence of axiom (9) and modus ponens, a superintuitionistic logic $\mathsf{L}$ is inconsistent if and only if $\mathsf{L} = \mathcal{SL}$. The next proposition tells us that not only every intermediate logic is superintuitionistic, but also, for consistent logics, the converse is true.

**Proposition 1.2.4.** *For every consistent superintuitionistic logic* $\mathsf{L} \subsetneq \mathcal{SL}$ *we have* $\mathsf{L} \subseteq \mathsf{CL}$*. Namely,* $\mathsf{L}$ *is intermediate.*

*Proof.* See Chagrov and Zakharyaschev (1997). □

Therefore, every consistent superintuitionistic logic is intermediate and vice versa. However, we refrain to consider only intermediate logics. As we shall see in Subsection 4.3.5, it can be very useful to consider also superintuitionistic logics as a whole.

**Definition 1.2.5.** Let $\mathsf{L}_1$ and $\mathsf{L}_2$ be superintuitionistic logics. We say that $\mathsf{L}_2$ is an *extension* of $\mathsf{L}_1$ if $\mathsf{L}_1 \subseteq \mathsf{L}_2$.

For every intermediate logic $\mathsf{L}$ and a formula $\alpha \in \mathcal{SL}$, we shall adopt the additive notation $\mathsf{L} + \alpha$ for the smallest intermediate logic containing $\mathsf{L} \cup \{\alpha\}$. Hence, we can write, for instance, as a reformulation of Theorem 1.2.1:

$$\mathsf{CL} = \mathsf{INT} + (\alpha \vee \neg\alpha) = \mathsf{INT} + (\neg\neg\alpha \Rightarrow \alpha).$$

The theorem of Deduction is also satisfied by all intermediate logics.

# 2 Categories of Models

In this chapter we introduce some crucial mathematical objects which will be investigated within this work, such as ordered sets and ordered topological spaces. They are structures that not only provide a basis for our subsequent algebraic conceptualization of intuitionistic and intermediate logics but also, and mostly, they are the models for those logics.

## 2.1 Ordered Sets

A partially ordered set (poset, for short) is a set $P$ equipped with a reflexive, transitive and antisymmetric binary relation $\leq$.

**Definition 2.1.1.** Let $P$ be a set. An *order* (or *partial order*) on P is a binary relation on $P$ such that, for all $x, y, z \in P$,

(i) $x \leq x$,

(ii) $x \leq y$ and $y \leq x$ imply $x = y$,

(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$.

These conditions are referred to, respectively, as reflexivity, antisymmetry and transitivity. As we have already said, a set $P$ equipped with an order relation is a poset. UsualIy we say simply "$P$ is an ordered set". Where it is necessary to specify the order relation overtly we write $(P, \leq)$. On any set, = is an order, the *discrete order*. A relation on a set $P$ which is reflexive and transitive but not necessarily antisymmetric is called a *pre-order*. A pre-order relation on $P$ gives rise to an order relation by quotienting the set $P$ by the equivalence relation determined by the pre-order on $P$: if $x \leq y$ and $y \leq x$, then we write $x \simeq y$ and say that x and y are isomorphic elements. Clearly the binary relation $\simeq$ is an equivalence relation. The resulting set ${}^{P}\!/_{\cong} = \{[x] \mid x \in P\}$ of equivalence classes has a well-defined partial order on it given by

$$[x] \leq [y] \text{ if and only if } x \leq y.$$

The poset ${}^{X}\!/_{\cong}$ is called the *poset reflection* of the preorder $X$. Also, an order relation $\leq$ on $P$ gives rise to a relation $<$ of *strict order*: $x < y$ in $P$ if and only if $x \leq y$ and $x \neq y$. It is possible to re-state conditions i-iii above in terms of $<$, and so to regard $<$ rather than $\leq$ as the fundamental relation.

We begin with introducing a basic construction of a new ordered sets from existing ones. Let $P$ be an ordered set and let $Q$ be a subset of $P$. Then $Q$ inherits an order relation from $P$; given $x, y \in Q$, $x \leq y$ in $Q$ if and only if $x \leq y$ in $P$. We say in these circumstances that $Q$ has the induced order, or the order inherited from $P$.

**Definition 2.1.2.** Let $P$ be an ordered set. Then $P$ is a *chain* if , for all $x, y \in P$, either $x \leq y$ or $y \leq x$. At the opposite side from a chain is an antichain. The ordered set $P$ is an *antichain* if $x \leq y$ in $P$ only if $x = y$. Clearly, with the induced order, any subset of a chain (an antichain) is a chain (an antichain). Instead, a chain (an antichain) in $P$ is a subset $C \subseteq P$ that is a chain (an antichain) when equipped with the order inherited from $P$.

With this couple of definitions of chain and antichain in mind we can also introduce the notions of depth and width of a poset. We define the *depth* of P to be

$$\operatorname{dep} P := \sup\{|C| - 1 \mid C \subseteq P \text{ is a chain in } P\} \in \mathbb{N} \cup \{\infty\}$$

In a similar way, the width of a poset can be defined as the size of the largest antichain in it. We define the *width* of P to be

$$\operatorname{wid} P := \sup\{|A| \mid A \subseteq P \text{ is a antichain in } P\} \in \mathbb{N} \cup \{\infty\}$$

Let $P$ be the $n$-element set $\{0, 1, \ldots, n-1\}$. We write $\mathbf{n}$ to denote the chain obtained by giving $P$ the order in which $0 < 1 < \cdots < n-1$ and $\overline{\mathbf{n}}$ for $P$ regarded as an antichain. Any set $S$ may be converted into an antichain $S$ by giving $\overline{S}$ the discrete order.

**Definition 2.1.3.** We say that two ordered sets, $P$ and $Q$, are (*order-*)*isomorphic*, and write $P \simeq Q$, if there exists a map $\phi$ from $P$ onto $Q$ such that $x \leq y$ in $P$ if and only if $\phi(x) \leq \phi(y)$ in $Q$. Then $\phi$ is called an *order-isomorphism*. Such a map $\phi$ faithfully mirrors the order structure. It is necessarily bijective (that is, one-to-one and onto): using reflexivity and antisymmetry of $\leq$ first in $Q$ and then in $P$,

$$\phi(x) = \phi(y) \Leftrightarrow x = y.$$

On the other hand, not every bijective map between ordered sets is an order-isomorphism: consider, for example, $P = Q = 2$ and define $\phi$ by $\phi(0) = 1$ and $\phi(1) = 0$.

We present some important orderings carried by fundamental mathematical structures that will be useful in the following.

**Examples 2.1.4.** Let $X$ be any set. The powerset $\mathcal{P}(X)$, consisting of all subsets of $X$, is ordered by set inclusion: for $A, B \in \mathcal{P}(X)$, we define $A \leq B$ if and only if $A \subseteq B$. Any subset of $\mathcal{P}(X)$ inherits the inclusion order. More commonly, families of sets such as the powerset arise where $X$ carries some additional structure. For instance, $X$ might have an algebraic or a geometrical structure – it might be a group or a topological space. Each of the following is an ordered set under inclusion:

- the set of all subgroups of a group $G$ denoted $\operatorname{Sub} G$;

- the set of all subspaces of a vector space $V$ denoted Sub $V$;
- Let $X$ be a topological space. We may consider either the family of open subsets $\mathcal{O}(X)$ or the family of closed subsets $\mathcal{C}(X)$ as ordered sets under inclusion. For more on this, see section 2.2.

### 2.1.1 Diagrams of Posets

One of the most useful and attractive features of ordered sets is that, in the finite case at least, they can be 'drawn'. To describe how to represent ordered sets diagrammatically, we need the idea of covering.

**Definition 2.1.5.** Let $P$ be an ordered set and let $x, y \in P$. We say $x$ is covered by $y$ (or $y$ covers $x$), and write $x \prec y$, if $x < y$ and $x \leq z < y$ implies $z = x$. The latter condition is demanding that there be no element $z$ of $P$ strictly between $x$ and $y$, that is with $x < z < y$.

Observe that, if $P$ is finite, $x < y$ if and only if there exists a finite sequence of covering relations $x = x_0 \prec x_1 \prec \cdots \prec x_n = y$. Thus, in the finite case, the order relation determines, and is determined by, the covering relation.

**Definition 2.1.6.** Let $P$ be a finite ordered set. We can represent $P$ by a configuration of circles (representing the elements of $P$) and interconnecting lines (indicating the covering relation). The construction goes as follows:

(1) To each point $x \in P$, associate a point $p(x)$ of the Euclidean plane $\mathbb{R}^2$, depicted by a small circle with centre at $p(x)$.

(2) For each covering pair $x \prec y$ in $P$, take a line segment $l(x, y)$ joining the circle at $p(x)$ to the circle at $p(y)$.

(3) Carry out (1) and (2) in such a way that

  (a) if $x \prec y$ , then $p(x)$ is 'lower' than $p(y)$ (that is, in standard Cartesian coordinates, has a strictly smaller second coordinate),

  (b) the circle at $p(z)$ does not intersect the line segment $l(x, y)$ if $z \neq x$ and $z \neq y$.

It is easily proved by induction on the size, $|P|$, of $P$ that (3) can be achieved. A configuration satisfying (1)-(3) is called a diagram (or Hasse diagram) of $P$. In the other direction, a diagram may be used to define a finite ordered set; an example is given below. Of course, the same ordered set may have many different diagrams.

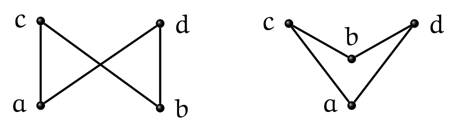


FIGURE 2.1: Two alternative diagrams for the ordered set $P = \{a, b, c, d\}$.

Figure 2.1 shows two alternative diagrams for the ordered set $\{a, b, c, d\}$ in which $a < c, a < d, b < c$ and $b < d$.

The diagrammatic approach to finite ordered sets is made fully legitimate by the following Proposition.

**Proposition 2.1.7.** *Two finite ordered sets $P$ and $Q$ are order-isomorphic if and only if they can be drawn with identical diagrams.*

*Proof.* See Davey and Priestley (2002). □

### 2.1.2 Constructions on Ordered Sets

This section collects together a number of ways of constructing new ordered sets from existing ones. Where we refer to diagrams, it is to be assumed that the ordered sets involved are finite.

Given any ordered set $P$ we can form a new ordered set $P^{\mathrm{op}}$ (the dual of $P$) by defining $x \leq y$ to hold in $P^{\mathrm{op}}$ if and only if $y \leq x$ holds in $P$. For $P$ finite, we obtain a diagram for $P^{\mathrm{op}}$ simply by "turning upside down" a diagram for $P$.

To each statement about the ordered set $P$ there corresponds a statement about $P^{\mathrm{op}}$. In general, given any statement $\Psi$ about ordered sets, we obtain the dual statement $\Psi^{\mathrm{op}}$ by replacing each occurrence of $\leq$ by $\geq$ and vice versa.

The formal basis for this observation is the Duality Principle below.

**Proposition 2.1.8** (Conceptual Duality)**.** *Given a statement $\Psi$ about ordered sets which is true in all ordered sets, the dual statement $\Psi^{op}$ is also true in all ordered sets.*

In what follows we shall make several applications of this principle. By way of illustration,

**Definition 2.1.9.** Let $P$ be an ordered set. We say $P$ has a *bottom* element if there exists $\perp \in P$ with the property that $\perp \leq x$ for all $x \in P$. A bottom element, when it exists, is unique by the anti-symmetric property $\leq$. Dually, $P$ has a *top* element if there exists $\top \in P$ such that $x \leq \top$ for all $x \in P$. By the above Principle of Duality, we can assert immediately that a bottom element, when it exists, is unique.

**Examples 2.1.10.** In $(\mathcal{P}(X); \subseteq)$, we have $\perp = \emptyset$ and $\top = X$. A finite chain always has bottom and top elements, but all infinite chain need not have.

In Chapter 3 we shall introduce other important examples of duality. We present now an important construct for building new ordered sets.

There are several different ways to join two ordered sets together. Here we focus on the sum of two ordered sets. In this construction we require that the sets being joined are disjoint (this is no real restriction since we always find isomorphic copies of the original ordered sets which are disjoint).

**Definition 2.1.11.** Suppose that $P$ and $Q$ are (disjoint) ordered sets. The disjoint union $P \coprod Q$ of $P$ and $Q$ is the ordered set formed by defining $x \leq y$ in $P \coprod Q$ if and only if either $x, y \in P$ and $x \leq y$ in $P$ or $x, y \in Q$ and $x \leq y$ in $Q$. A diagram for $P \coprod Q$ is formed by placing side by side diagrams for $P$ and $Q$.

There is also the (dual) notion of cartesian product among ordered sets.

**Definition 2.1.12.** Let $P_1, \ldots, P_n$ be ordered sets. The Cartesian product $P_1 \times \cdots \times P_n$ can be made into an ordered set by imposing the coordinatewise order defined by

$$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \Leftrightarrow (\forall i) x_i \leq y_i \text{ in } P_i.$$

### 2.1.3 Downsets and Upsets

Associated with any ordered set are two important families of sets. They play a central role in the representation theory of distributive lattices developed in the next Chapter.

**Definition 2.1.13.** Let $P$ be an ordered set and $Q \subseteq P$.

(i) Q is a downset if, whenever $x \in Q$, $y \in P$ and $y \leq x$, we have $y \in Q$.

(ii) Dually, Q is an upset if, whenever $x \in Q$, $y \in P$ and $x \leq y$, we have $y \in Q$.

Given an arbitrary subset $Q$ of $P$ and $x \in P$, we define

$$\begin{aligned} \downarrow Q &:= \{y \in P \mid (\exists x \in Q) y \leq x\} \text{and} & \uparrow Q &:= \{y \in P \mid (\exists x \in Q) y \geq x\}, \\ \downarrow x &:= \{y \in P \mid y \leq x\} \text{and} & \uparrow x &:= \{y \in P \mid y \geq x\}. \end{aligned}$$

It is easily checked that $\downarrow Q$ is the smallest downset containing $Q$ and that $Q$ is a downset if and only if $Q = \downarrow Q$, and dually for $\uparrow Q$. Clearly $\downarrow \{x\} = \downarrow x$, and dually. Downsets (upsets) of the form $\downarrow x$ ($\uparrow x$) are called principal.

The family of all down-sets of $P$ is denoted by $\mathcal{D}o(P)$. It is itself an ordered set, under the inclusion order.

**Examples 2.1.14.**

1. Figure 2.2 shows $\mathcal{D}o(P)$ in a simple case.
2. If $P$ is an antichain, then $\mathcal{D}o(P) = \mathcal{P}(P)$.
3. If $P$ is the chain $\mathbf{n}$, then $\mathcal{D}o(P)$ consists of all the sets $\downarrow x$ for $x \in P$, together with the empty set. Hence $\mathcal{D}o(P)$ is an $(n+1)$-element chain.

The following lemma connects the order relation to down-sets and up-sets.

**Lemma 2.1.15.** *Let $P$ be an ordered set and $x, y \in P$. Then the following are equivalent:*

*(i)* $x \leq y$;

*(ii)* $\downarrow x \subseteq \downarrow y$;

*(iii)* $\uparrow y \subseteq \uparrow x$;

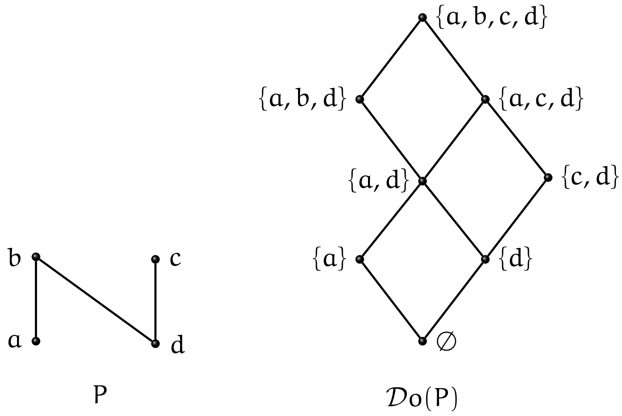


FIGURE 2.2: The ordered sets $P$ and $\mathcal{D}o(P)$.

*(iv)* $(\forall Q \in \mathcal{D}o(P)) y \in Q \Rightarrow x \in Q$;

*(v)* $(\forall Q \in \mathcal{U}p(P)) x \in Q \Rightarrow y \in Q$.

*Proof.* See Davey and Priestley (2002). □

From definition 2.1.13 and lemma 2.1.15 we can easily observe that an up-set of $P$ is nothing more that a down-set of $P^{\mathrm{op}}$ and define $\mathcal{U}p(P)$ to be the set of up-sets of $P$ ordered by reverse inclusion. It follows that

$$\mathcal{U}p(P) = \mathcal{D}o(P^{\mathrm{op}})^{\mathrm{op}}. \tag{2.1}$$

Besides being related by duality, down-set and up-sets are related by complementation: $Q$ is a down-set of $P$ (equivalently, an up-set of $P^{\mathrm{op}}$) if and only if $P \backslash Q$ is an up-set of $P$ (equivalently, a down-set of $P^{\mathrm{op}}$). For subsets $A, B$ of $P$, we have $A \subseteq B$ if and only if $P \backslash A \supseteq P \backslash B$. It follows that

$$\mathcal{D}o(P)^{\mathrm{op}} \simeq \mathcal{D}o(P^{\mathrm{op}}) \quad \text{and} \quad \mathcal{U}p(P)^{\mathrm{op}} \simeq \mathcal{U}p(P^{\mathrm{op}}),$$

the order-isomorphism being the complementation map. The next proposition shows how $\mathcal{D}o(P)$ and $\mathcal{U}p(P)$ can be analysed for compound ordered sets $P$.

**Proposition 2.1.16.** *Let P be an ordered set. Then*

$$\mathcal{D}o(P_1 \coprod P_2) \simeq \mathcal{D}o(P_1) \times \mathcal{D}o(P_2) \quad \textit{and} \quad \mathcal{U}p(P_1 \coprod P_2) \simeq \mathcal{U}p(P_1) \times \mathcal{U}p(P_2).$$

*Proof.* See Davey and Priestley (2002). □

### 2.1.4 Maps Between Ordered Sets

We have already made use of maps of a very special type between ordered sets, namely order isomorphisms. In this section, structure-preserving maps are considered more generally.

**Definition 2.1.17.** Let $P$ and $Q$ be ordered sets. A map $\phi : P \to Q$ is said to be

(i) a *monotone* or order-preserving if $x \leq y$ in $P$ implies $\phi(x) \leq \phi(y)$ in $Q$;

(ii) an *order-embedding* (and we write $\phi : P \hookrightarrow Q$) if $x \leq y$ in $P$ if and only if $\phi(x) \leq \phi(y)$ in $Q$;

(iii) an order-isomorphism if it is an order-embedding which maps $P$ onto $Q$.

Of course, the composite map of order preserving maps is order preserving.

**Examples 2.1.18.**

1. Two basic example of order preserving maps are inverse and direct image functions. Let $f : X \to Y$ be a function between sets, we get a monotone function $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ by taking inverse images of subsets: for $B \subseteq Y$

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Also, we get a monotone function $f[-] : \mathcal{P}(Y) \to \mathcal{P}(X)$ by taking direct images of subsets: for $A \subseteq X$

$$f[A] = \{y \in Y \mid y = f(x) \text{ for some } x \in A\}.$$

2. On the basis of definition ii and 2.1.15, the map $x \mapsto \downarrow x$ sets up an order-embedding from $P$ into $\mathcal{D}o(P)$.

**Remark 2.1.19.** Ordered sets $P$ and $Q$ are order isomorphic if and only if there exist order preserving maps $\phi : P \to Q$ and $\psi : Q \to P$ such that $\phi \circ \psi = 1_Q$ and $\psi \circ \phi = 1_D$ (where $1_S : S \to S$ denotes the identity map on $S$ given by $1_S(x) = x$ for all $x \in S$). This definition is more abstract in the sense that it is given without any reference to elements of the ordered sets. This is the our first example of a category theoretic way of defining important notion. In the following subsections we shall introduce some basic elements of this important subject.

But before we turn on category theory, there is another important kind of order preserving maps between ordered sets to consider for our purpose, i.e. p-morphims.

**Definition 2.1.20.** Let $P$ and $Q$ be ordered sets. A map $f : P \to Q$ is said to be a *p-morphim* if, for all $x \in P$

$$f[\uparrow x] = \uparrow f(x),$$

or, equivalently, for all $A \subseteq P$

$$f[\uparrow A] = \uparrow f[A].$$

We shall see the topological meaning of this peculiar order preserving map in Section 2.2 while, in Section 3.3, we shall recognize its logical relevance.

In conclusion, we can also collect maps in an ordering of maps.

**Definition 2.1.21.** Suppose $X$ is any set and $Y$ an ordered set. We may order the set $Y^X$ of all maps from $X$ to $Y$ as follows. We put $f \leq g$ if and only if $f(x) \leq g(x)$ in $Y$, for all $x \in X$.

Any subset $Q$ of $Y^X$ inherits the pointwise order. When $X$ is itself an ordered set, we may take $Q$ to be the set of all order preserving maps from $X$ to $Y$. Hopefully without generating any confusion, the resulting ordered set is denoted in the same way, i.e. $Y^X$. We write sometimes $(X \to Y)$ in place of $Y^X$. As an example, we note that $(X \to 2) \simeq \mathcal{D}o(X)^{\mathrm{op}}$.

### 2.1.5 Categories of Posets, Posets as Categories

In the previous sections, we introduced a class of structures, partial orders, and the associated class of morphisms that preserve these structures, monotone functions. In Chapter 3 we will see other classes of structures of a certain type such as distributive lattices, Heyting algebras, and their classes of morphisms. These pairs of classes, a class of objects together with its maps that preserve the structures, are those that in modern mathematics are called categories.

So, we are forced to introduce some category-theoretic concepts. Our aim is not to to provide a comprehensive introduction to category theory; rather to delineate the territory we shall regard as familiar, by giving the statements of the major results we shall be assuming later on, and providing references for their proofs. We begin with the definitions of the fundamental concepts.

**Definition 2.1.22.** A *category* $\mathbb{C}$ consists of:

- A class of *objects*: $A, B, C, \ldots$
- A class of *morphisms*: $f, g, h, \ldots$
- For each morphism $f$, there are given objects

  $$\mathsf{dom}(f), \quad \mathsf{cod}(f)$$

  called the *domain* and *codomain* of $f$. which are objects of $\mathbb{C}$; we write

  $$f : A \to B$$

  to indicate that $\mathsf{dom}(f) = A$ and $\mathsf{cod}(f) = B$.
- To each pair of morphisms $f : A \to B$ e $g : B \to C$ with

  $$\mathsf{cod}(f) = \mathsf{dom}(g),$$

  there is given a morphism

  $$g \circ f : A \to C,$$

called the *composite* of $f$ and $g$ such that

$$\mathsf{dom}(f \circ g) = \mathsf{dom}(f) \quad \text{and} \quad \mathsf{cod}(f \circ g) = \mathsf{cod}(g). \tag{2.2}$$

- For each object $A$, there exists a morphism,

$$1_A : A \to A.$$

called the *identity* of $A$ such that

$$\mathsf{dom}(1_A) = A = \mathsf{cod}(1_A). \tag{2.3}$$

Moreover, these data are required to satisfy two axioms:

**Associativity:** for all $f : A \to B, g : B \to C, h : C \to D$,

$$(h \circ g) \circ f = h \circ (g \circ f) \tag{2.4}$$

**Unit:** for all $f : A \to B$,

$$1_A \circ f = f = f \circ 1_B \tag{2.5}$$

We write $\mathsf{Ob}(\mathbb{C})$ for its class of objects and $\mathsf{Mor}(\mathbb{C})$ for its collection of morphisms. For two objects $C$ and $D$, the class of morphisms with domain $C$ and codomain $D$ is denoted by $\mathsf{Hom}_{\mathbb{C}}(C, D)$.

We should consider also the morphisms between categories. A morphism of categories is called functor.

**Definition 2.1.23.** A *functor* between categories $\mathbb{C}$ and $\mathbb{D}$,

$$F : \mathbb{C} \to \mathbb{D}$$

is a mapping of objects to objects and maps to maps, in such a way that:

1. for all $f : A \to B$, $F(f) : F(A) \to F(B)$,
2. $F(1_A) = 1_{F(A)}$,
3. $F(f \circ g) = F(g) \circ F(f)$.

In other words, $F$ preserves domains and codomains, identity arrows and compositions. A functor $F : \mathbb{C} \to \mathbb{D}$ thus gives a sort of "picture" of $\mathbb{C}$ in $\mathbb{D}$. For example, to the diagram in $\mathbb{C}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

corresponds the following one in $\mathbb{D}$

$$\begin{tikzcd} & F(B) \arrow[d, "F(g)"] \\ F(A) \arrow[ur, "F(f)"] \arrow[r, "F(g \circ f)"'] & F(C) \end{tikzcd}$$

Now, we can easily see that functors compose in the expected way, and that every category $\mathbb{C}$ has an identity functor $1_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$.

**Definition 2.1.24.** For categories $\mathbb{C}, \mathbb{D}$ and functors

$$F, G : \mathbb{C} \to \mathbb{D}$$

a *natural transformation* $\theta : F \to G$ is a family of arrows in $\mathbb{D}$

$$(\theta_C : FC \to GC)_{C \in \mathsf{Ob}(\mathbb{C})}$$

such that, for any $f : C \to C'$ in $\mathbb{C}$, one has $\theta'_C \circ F(f) = G(f) \circ \theta_C$, that is, the following square commutes:

$$\begin{tikzcd} FC \arrow[r, "\theta_C"] \arrow[d, "Ff"'] & GC \arrow[d, "Gf'"] \\ FC' \arrow[r, "\theta_{C'}"'] & GC' \end{tikzcd}$$

Given such a natural transformation $\theta : F \to G$, the $\mathbb{D}$-arrow $\theta_C : FC \to GC$ is called the *component* of $\theta$ at $C$. If every component of $\theta$ is an isomorphism, $\theta$ is said to be a *natural isomorphism*. Of course, we say that functors $F$ and $G$ are naturally isomorphic if there exists a natural isomorphism from $F$ to $G$ and we write $F \simeq G$.

**Definition 2.1.25.** Given functors $F, G : \mathbb{C} \to \mathbb{D}$, we say that

$$F(A) \simeq G(A)$$

*naturally in A* if $F$ and $G$ are naturally isomorphic.

If you think of a functor $F : \mathbb{C} \to \mathbb{D}$ as a "picture" of $\mathbb{C}$ in $\mathbb{D}$, then you can think of a natural transformation $\theta_C : FC \to GC$ as a "cylinder" with such a picture at each end.

We shall be concerned mainly with concrete categories whose objects are sets with some kind of structure and whose morphisms are structure-preserving functions, the composition law being the usual composition of functions. The usual convention will be to name such a category by (an abbreviation of) the common name of its objects; thus we have:

- $\mathbb{S}et$, the category of sets and functions;
- $\mathbb{T}op$, the category of topological spaces and continuous maps;
- $\mathbb{P}os$, the category of partially ordered sets and order-preserving maps;
- $\mathbb{D}L$, the category of distributive lattices and homomorphisms;
- $\mathbb{B}ool$, the category of Boolean algebras and homomorphisms;
- $\mathbb{H}eyt$, the category of Heyting algebras and homomorphisms.

Further examples will occur frequently as we go along. We shall have to consider pairs of categories having the same objects but different morphisms; in this case we shall adopt the custom of giving two different names to the objects, which are entirely synonymous when we refer only to objects, but mean different things when applied to morphisms.

We note that a concrete category $\mathbb{C}$ is always *locally small*; that is, for each pair of objects $A, B$, the collection of morphisms $\mathrm{Hom}(A, B)$ in $\mathbb{C}$ form a set rather than a proper class.

The other class of categories which we shall frequently meet is that consisting of categories arising from partially ordered sets themselves. If $(P, \leq)$ is a poset, we can make it into a category $\mathbb{P}$ whose objects are the elements of $P$ and whose morphisms are the instances of the order-relation – i.e. there is just one morphism $a \rightarrow b$ if and only if $a \leq b$. It is clear that transitivity of $\leq$ ensures the existence of a unique composition law for $P$, and reflexivity ensures the existence of identities; thus we can consider posets as a special case of categories. It turns out that many of the basic concepts of category theory become, when we specialize them to posets, concepts already familiar in lattice theory – for example, a functor between posets is just an order-preserving map.

### 2.1.6 Adjoint Functors and Closure Operators on Posets

#### Adjoints Between Categories, Reflections and Equivalences

The central concept of category theory is the notion of adjunction. Consider two functors in opposite directions, $F : \mathbb{C} \rightarrow \mathbb{D}$ and $G : \mathbb{D} \rightarrow \mathbb{C}$. Roughly speaking, $F$ is said to be left adjoint to $G$ if, whenever $C \in \mathbb{C}$ and $D \in \mathbb{D}$, maps $F(C) \rightarrow D$ are essentially the same thing as maps $C \rightarrow G(D)$.

**Definition 2.1.26.** Let $\mathbb{C}$ and $\mathbb{D}$ be two categories and

$$\mathbb{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathbb{D}$$

two functors between them. We say that $F$ is *left adjoint* to $G$, $G$ is *right adjoint* to $F$, and write $F \dashv G$, if

$$\mathrm{Hom}(FC, D) \simeq \mathrm{Hom}(C, GD) \tag{2.6}$$

naturally in $C \in \mathbb{C}$ and $D \in \mathbb{D}$.

Given objects $C \in \mathbb{C}$ and $D \in \mathbb{D}$, the correspondence between maps can be denoted by a horizontal bar, in both directions:

$$\frac{FC \xrightarrow{g} D}{C \xrightarrow{\overline{g}} GD}, \qquad \frac{C \xrightarrow{f} GD}{FC \xrightarrow{\overline{f}} D}.$$

Thus, the naturality axiom satisfied by the specified bijection 2.6 can be split in two parts and stated in the following way:

$$\frac{\overline{FC \xrightarrow{g} D \xrightarrow{q} D'}}{C \xrightarrow{\overline{g}} GD \xrightarrow{G(g)} GD'}, \quad \text{for all } g \text{ and } q,$$

that is, $\overline{q \circ g} = G(q) \circ \overline{g}$, and

$$\frac{\overline{C' \xrightarrow{p} C \xrightarrow{f} GD}}{FC' \xrightarrow{F(p)} FC \xrightarrow{\overline{f}} D}, \quad \text{for all } p \text{ and } f,$$

that is, $\overline{f \circ p} = F(p) \circ \overline{f}$. There is another way of stating the above definition of adjunction that it can be useful for our purposes.

**Definition 2.1.27.** For each $C \in \mathbb{C}$ and, dually, for each $D \in \mathbb{D}$, we have two maps

$$\frac{\overline{FC \xrightarrow{1} FC}}{C \xrightarrow{\eta_C} GFC}, \qquad \frac{\overline{GD \xrightarrow{1} GD}}{FGD \xrightarrow{\epsilon_D} D}.$$

These define natural transformations

$$\eta : 1_{\mathbb{C}} \to G \circ F, \quad \epsilon : F \circ G \to 1_{\mathbb{D}}$$

called the *unit* and *counit* of the adjunction, respectively.

**Lemma 2.1.28.** *Given an adjunction $F \dashv G$ with unit $\eta$ and counit $\eta$, the triangles*

$$\begin{tikzcd} FC \arrow[r, "F(\eta_C)"] \arrow[dr, "1_{FC}"'] & FGFC \arrow[d, "\epsilon_{FC}"] \\ & FC \end{tikzcd} \qquad \begin{tikzcd} GD \arrow[r, "\eta_{GD}"] \arrow[dr, "1_{GD}"'] & GFGD \arrow[d, "G(\epsilon_D)"] \\ & GD \end{tikzcd}$$

*commutes for all* $C \in \mathbb{C}$ *and* $D \in \mathbb{D}$.

*Proof.* See Leinster (2014). □

Amazingly, the unit and counit determine the whole adjunction, even though they appear to know only the transposes of identities. This is the main content of the following pair of results.

**Lemma 2.1.29.** *Let be an adjunction, with unit* $\eta$ *and counit* $\epsilon$. *Then*

$$\overline{g} = G(g) \circ \eta_C$$

*for any* $g : FC \to D$, *and*

$$\overline{f} = \epsilon_D \circ F(f)$$

*for any* $f : C \to GD$.

*Proof.* See Leinster (2014). □

**Theorem 2.1.30.** *Let* $\mathbb{C}$ *and* $\mathbb{D}$ *be two categories and*

$$\mathbb{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathbb{D}$$

*two functors between them. Then* $F \dashv G$ *if and only if there exist natural transformations* $\eta : 1_{\mathbb{C}} \to G \circ F$ *and* $\epsilon : F \circ G \to 1_{\mathbb{D}}$ *such that the above triangles commute.*

*Proof.* See Leinster (2014). □

**Definition 2.1.31.** A *reflection* is an adjunction for which the counit map $\epsilon_D$ is an isomorphism for all $D$. This is equivalent to saying that $G$ induces a bijection between morphisms $D \to D'$ and morphisms $GD \to GD'$ for each pair $(D, D')$ (see Mac Lane 1998). In such case, $\mathbb{D}$ is called reflective subcategory of $\mathbb{C}$. Dually, a *coreflection* is an adjunction for which the unit map $\eta_C$ is an isomorphism for all $C$ (and this is equivalent to saying that $F$ induces a bijection between morphisms $C \to C'$ and morphisms $GC \to GC'$ for each pair $(C, C')$). In such case, $\mathbb{C}$ is called coreflective subcategory of $\mathbb{D}$.

If both $\eta$ and $\epsilon$ are isomorphisms, we call the adjunction an *equivalence* (see Mac Lane 1998), and say that the categories $\mathbb{C}$ and $\mathbb{D}$ are equivalent. We say $\mathbb{C}$ and $\mathbb{D}$ are dual if is equivalent to the opposite category $\mathbb{D}^{\mathrm{op}}$ of $\mathbb{D}$. At first glance, the notion of equivalence might seem weaker than that of isomorphism between categories (which may be regarded as an adjunction for which the unit and counit maps are identities), but instead it is the right notion of sameness between to categories, able to ensure that $\mathbb{C}$ and $\mathbb{D}$ share the same categorical properties.

**Order Adjoints**

As we have said before, If $(P, \leq)$ is a poset, we can make it into a category $\mathbb{P}$ whose objects are the elements of $P$ and whose morphisms are the instances of the order-relation - i.e. there is just one morphism $a \longrightarrow b$ if and only if $a \leq b$. Thus it turns out that we can consider poset as a special case of category: a category in which there is at most one arrow $x \longrightarrow y$ between any two objects. Let $P$ be such a category. Given another such category $Q$, suppose we have adjoint functors:

$$P \underset{G}{\overset{F}{\leftrightarrows}} Q \qquad F \dashv G.$$

Then the correspondence $Hom(Fa, x) \simeq Hom(a, Gx)$ comes down to the simple condition $Fa \leq x$ if and only if $a \leq Gx$. Thus, an adjunction on ordered consists simply of order-preserving maps $F, G$ satisfying the two-way rule or "bicondition":

$$\frac{Fa \leq x}{a \leq Gx}.$$

For each $p \in P$, the unit is therefore an element $p \leq GFp$ that is least among all $x$ with $p \leq Gx$. Dually, for each $q \in Q$ the counit is an element $FGq \leq q$ that is greatest among all $x$ with $Fx \leq q$. Such a setup on ordered sets is sometimes called a Galois connection.

**Examples 2.1.32.**

1. An important example is the adjunction on powersets induced by any function $f : A \longrightarrow B$, between the inverse image operation $f^{-1}$ and the direct image $f[-]$,

$$\mathcal{P}(A) \underset{f[-]}{\overset{f^{-1}}{\leftrightarrows}} \mathcal{P}(B)$$

Here we have an adjunction $f[-] \dashv f^{-1}$ as indicated by the bicondition

$$\frac{f[U] \subseteq V}{U \subseteq f^{-1}(V)}.$$

which is plainly valid for all subsets $U \subseteq A$ and $V \subseteq B$. The inverse image operation $f^{-1} : \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$ also has a right adjoint, sometimes called the dual image, given by

$$f_*(U) = \{b \in B \mid f^{-1}(b) \subseteq U\}.$$

2. Let $\mathbb{1}$ denote the singleton poset $\{*\}$ with $* \leq *$. For each ordered set $X$, the unique map $X \xrightarrow{!} \mathbb{1}$ is clearly order preserving. If it has a right adjoint then it is a map $\top : \mathbb{1} \longrightarrow X$ satisfying, for all $x \in X$, $x \leq \top$. Thus, the existence of a top element in $X$ provides the right adjoint of !. Dually, $X \xrightarrow{!} \mathbb{1}$ has a left adjoint if and only if there is a $\perp : \mathbb{1} \longrightarrow X$ with $\perp \leq x$, for all $x \in x$, which is to say if and only if $X$ has a bottom element.

3. Let $P$ be an ordered set and let $S \subseteq P$. An element $x \in P$ is an upper bound of $S$ if $s \leq x$ for all $s \in S$. A lower bound is defined dually. The set of all upper bounds of $S$ is denoted by $S^u$ and the set of all lower bounds by $S^l$:

$$S^u := \{x \in P \mid (\forall s \in S) s \leq x\} \quad \text{and} \quad S^l := \{x \in P \mid (\forall s \in S) s \geq x\}.$$

   Since $\leq$ is transitive, $S^u$ is always an upset and $S^l$ a downset. It is easy to see that it turns out to be another example of adjunction (Isbell conjugation)

$$\mathcal{P}(A) \underset{u}{\overset{l}{\leftrightarrows}} \mathcal{P}(A)^{\mathrm{op}}$$

   between $\mathcal{P}(A)$ and $\mathcal{P}(A)^{\mathrm{op}}$ for which $(\downarrow x)^u = \uparrow x$ and $(\uparrow x)^l = \downarrow x$.

4. Another characteristic way in which adjoints can arise is the following. Let $\mathcal{R}$ be a binary relation between members of the set $X$ and members of the set $Y$. We define $F : \mathcal{P}(X) \to \mathcal{P}(Y)^{\mathrm{op}}$ by putting

$$F(A) = \{\, b \mid (\forall a \in A) a \mathcal{R} b \,\}$$

   for $A \subseteq X$. Similarly, define $G : \mathcal{P}(Y)^{\mathrm{op}} \to \mathcal{P}(X)$ by putting

$$G(B) = \{\, a \mid (\forall b \in B) a \mathcal{R} b \,\}$$

   for $B \subseteq Y$. It is easy to see that $F \dashv G$. We just have to prove that for any $A \subseteq X$, $B \subseteq Y$, $F(A) \supseteq B$ if and only if $A \subseteq G(B)$. But simply by applying definitions we see $F(A) \supseteq B$ if and only if $(\forall b \in B)(\forall a \in A) aRb$ if and only if $(\forall a \in A)(\forall b \in B) aRb$ if and only if $A \subseteq G(B)$. Let us say that adjoints produced in this way is relation-generated. We shall see in Section 4.2.1 a fundamental example of this kind of adjoints relation-generated.

**Corollary 2.1.33.** *Let*

$$P \underset{G}{\overset{F}{\leftrightarrows}} Q$$

*be order-preserving maps between posets, and regard them as a functors. Then $F \dashv G$ if and only if for all $x \in P$ and $a \in Q$*

$$Fa \leq x \quad \textit{if and only if} \quad a \leq Gx$$

*Equivalently, in terms of unit and counit, $F \dashv G$ if and only if for all $x \in P$ and $a \in Q$*

$$FGx \leq x \quad \textit{if and only if} \quad a \leq GFa$$

*Moreover, if these hold, then $FGF = F$ and $GFG = G$ and $F$ and $G$ restrict to a bijection between the subset $\{a \mid a = GF(a)\}$ and $\{x \mid x = FG(x)\}$ of $Q$ and $P$, respectively.*

*Proof.* See Johnstone (1982), Mac Lane (1998). □

**Remark 2.1.34.**

1. Given ordered sets and order preserving maps
$$P \underset{G}{\overset{F}{\leftrightarrows}} Q \underset{H}{\overset{K}{\leftrightarrows}} Z$$
if $F \dashv G$ and $H \dashv K$, then $H \circ F \dashv G \circ K$.

2. Remember that an order preserving map $F : P \longrightarrow Q$ also determines an order preserving map from $Q^{\text{op}}$ from $P^{\text{op}}$, which we denote by $F^{\text{op}} : Q^{\text{op}} \longrightarrow P^{\text{op}}$. If $F \dashv G$ then $G^{\text{op}} \dashv F^{\text{op}}$ as a consequence of duality principle.

3. If an order preserving map has a right adjoint then it is essentially unique: for $F : P \longrightarrow Q$, if $F \dashv G$ and $F \dashv H$ then $G \simeq H$.

4. Let
$$P \underset{G}{\overset{F}{\leftrightarrows}} Q$$
be a couple of posets and order preserving maps such that $F \dashv G$. Since $GFG = G$, if $G$ is injective, $FG = 1_P$; if $G$ is surjective, $GF = 1_Q$. Since $FGF = F$, if $F$ is injective, $GF = 1_Q$; if $G$ is surjective, $FG = 1_P$.

### Closure Operators and Reflections on Posets

Closely related to adjunction, we have the important concept of closure operator and dually of interior operator. As we have just said, for any adjunction $F \dashv G$ it holds

$$FGF = F \quad \text{and} \quad GFG = G.$$

Therefore, setting

$$\mathsf{cl} := GF : P \longrightarrow P \quad \text{and} \quad \mathsf{int} := FG : Q \longrightarrow Q,$$

we see immediately that, for all $x, y \in P$, we obtain two order preserving maps $\mathsf{cl} : P \longrightarrow P$ and $\mathsf{int} : Q \longrightarrow Q$ such that

$$\mathsf{cl}\,\mathsf{cl} = \mathsf{cl}, \quad 1_P \leq \mathsf{cl} \quad \text{and} \quad \mathsf{int} = \mathsf{int}\,\mathsf{int}, \quad \mathsf{int} \leq 1_Q.$$

Any such map $\mathsf{cl}$ is called a *closure operator* on $X$, and any such $\mathsf{int}$ is called an *interior operator* on $X$. With $\mathsf{cl}$ and $\mathsf{int}$ induced by $f \dashv g$ as discussed, the last part of the above corollary 2.1.33, which is to say that $f$ and $g$ can be restricted to an equivalence between the subsets $\{a \mid a = GF(a)\}$ and $\{x \mid x = FG(x)\}$ of $Q$ and $P$, can be rephrased as an equivalence between subsets

$$\mathsf{Fix}(\mathsf{cl}) := \{x \in X \mid \mathsf{cl}(x) = x\} \quad \text{and} \quad \mathsf{Fix}(\mathsf{int}) := \{y \in Y \mid \mathsf{int}(y) = y\},$$

of closed and open elements, also referred to as fixpoints of $c$ and $d$, respectively. The following diagram summarizes this situation:

$$\begin{tikzcd} X \arrow[r, shift right, "G"'] & Y \arrow[l, shift right, "F"'] \\ \mathsf{Fix}(\mathsf{cl}) \arrow[u, hook] \arrow[r, shift right, "G"'] & \mathsf{Fix}(\mathsf{int}) \arrow[u, hook] \arrow[l, shift right, "F"'] \end{tikzcd}$$

For any closure operator $\mathsf{cl}$ on $X$ we can easily recover an adjunction as well. Let $i : \mathsf{Fix}(\mathsf{cl}) \hookrightarrow X$ be the inclusion and $p$ the factorization of $\mathsf{cl}$ through $\mathsf{Fix}(\mathsf{cl})$. It can be proved that $p \dashv i$:

$$\begin{tikzcd} X \arrow[r, shift left, two heads, "p"] & \mathsf{Fix}(\mathsf{cl}) \arrow[l, shift left, hook, "i"] \end{tikzcd}$$

In particular, since $i$ is injective, it follows from remark 2.1.34(4) that $p \circ i(k) = k$ for all $k \in \mathsf{Fix}(\mathsf{cl})$. Hence, a closure operator induces a reflection (see definition 2.1.31).

We give now a definition and a lemma that delineate the notion of reflection in the specific case of posets and its relation with closure operator.

**Definition 2.1.35.** An order embedding $i : X \hookrightarrow A$ is said to be reflective (coreflective) if the inclusion $i$ has a left (right) adjoint. The left (right) adjoint is called the reflector (coreflector).

**Lemma 2.1.36.** *An order embedding $i : X \hookrightarrow A$ is reflective with reflector $p$,*

$$\begin{tikzcd} A \arrow[r, shift left, two heads, "p"] & X \arrow[l, shift left, hook, "i"] \end{tikzcd}$$

*if and only if $pi = 1_X$ or, equivalently, $ip : A \longrightarrow A$ is a closure operator.*

*Proof.* See Wood (2004). □

Of course, the dual result holds for coreflective order embedding that, instead of being associated with a closure operator, is given by an interior operator.

It can be easily observed that a closure operator is determined by its image, i.e. its fixsets.

**Corollary 2.1.37.** *For any ordered set $X$, there are a bijection between the following sets:*

*(i) The set of closure operators on $X$;*

*(ii) The set of fixsets $Q \subseteq X$, i.e. the set of reflective order embeddings.*

*Proof.* It follows from Lemma 2.1.36. □

In the following, it is convenient to suppress the difference of notation between reflector and closure operator insofar the reflector is basically the same map, just restricted to the fixset on the codomain.

**Examples 2.1.38.**

1. A fundamental example of closure operator, which we have already met, is the downward closure operator. For any ordered set $X$, the map

$$x \longmapsto \downarrow x,$$

which is an order embedding, induces a closure operator

$$\downarrow : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$$

where the set $\mathsf{Fix}(\downarrow)$ is nothing but the poset of downsets of $\mathcal{D}o(X)$.

2. Observe that for any ordered set $X$, the order embedding $i : \mathcal{D}o(X) \hookrightarrow \mathcal{P}(X)$ is also coreflective, i.e. provided with an interior operator

$$\Downarrow : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$$

given by $\Downarrow S = \bigcup\{\downarrow a \mid \downarrow a \subseteq S\}$. The set $\mathsf{Fix}(\Downarrow)$ is again the poset of downsets of $\mathcal{D}o(X)$. We can say that the downward closure operator gives us back the smallest downset generated by a set, while the interior operator $\Downarrow$ just defined gives us back the largest downset included (cogenerated) in a set.

3. Dually, another fundamental example of closure operator, which we shall be heavily exploited in the following, is the upward closure operator. For any ordered set $X$, the map
$$x \longmapsto \uparrow x,$$
induces a closure operator

$$\uparrow : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$$

where the set $\mathsf{Fix}(\uparrow)$ is nothing but the poset of upsets of $\mathcal{U}p(X)$.

4. Observe that for any ordered set $X$, the order embedding $i : \mathcal{U}p(X) \hookrightarrow \mathcal{P}(X)$ is also coreflective, i.e. provided with an interior operator

$$\Uparrow : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$$

given by $\Uparrow S = \bigcup\{\uparrow a \mid \uparrow a \subseteq S\}$. The set $\mathsf{Fix}(\Uparrow)$ is again the poset of upsets of $\mathcal{D}o(X)$. We can say that the upward closure operator gives us back the smallest upset generated by a set, while the interior operator $\Uparrow$ just defined gives us back the largest upset included (cogenerated) in a set.

**Remark 2.1.39.** In anticipation of Section 3.3, it can be useful to express the downward and upward interior operators in terms of upward and downward closure operators, respectively.

For the properties of closure operators, $\complement S \subseteq \downarrow \complement S$, for all $S \in \mathcal{P}(X)$. Then $S \supseteq \complement \downarrow \complement S$. As a consequence, $\uparrow y \subseteq \complement \downarrow \complement S$ implies $\uparrow y \subseteq S$. Vice versa, $\uparrow y \subseteq S$ if and only if $\complement\uparrow y \supseteq \complement S$. For the properties of closure operators and for the fact that $\complement\uparrow y$ is a downset, $\complement\uparrow y \supseteq \complement S$ implies $\complement\uparrow y \supseteq\downarrow\complement S$, which implies $\uparrow y \subseteq \complement\downarrow\complement S$. As a consequence, $\Uparrow = \complement\downarrow\complement$. Analogously, it is easy to see that $\Downarrow = \complement\uparrow\complement$.

## 2.2 Topological Spaces, Specialization Order on Topological Spaces, Ordered Set as Topology

Deeply connected with the notion of closure operator, even for historical reasons, there is the notion of topological space. The notion of topological space aims to axiomatize the idea of a space as a collection of points that hang together in a continuous way. As we shall see in the following, topological spaces equipped with extra properties and structures are models for intuitionistic and intermediate logics. We present first the standard definition and then a list of different equivalent definitions in order to connect our discussion with Section 2.1.

**Definition 2.2.1.** A *topological space* is a set $X$ equipped with $\tau$, a collection of subsets of $X$ which are closed under

(1) finite intersections;

(2) arbitrary unions.

The elements of $\tau$ are called *open sets* and the collection $\tau$ is called a *topology* on $X$.

**Remark 2.2.2.** Since $X$ itself is the intersection of zero subsets, it is open, and since the empty set $\emptyset$ is the union of zero subsets, it is also open. Moreover, every open subset $U$ of $X$ contains the empty set and is contained in $X$

$$\emptyset \subset U \subset X\,,$$

so that the topology of $X$ is determined by a poset of open subsets $\mathcal{O}(X)$ with bottom element $\bot = \emptyset$ and top element $\top = X$.

**Definition 2.2.3.** A morphisms between topological spaces $f : X \to Y$ is a *continuous function*: a function $f : X \to Y$ of the underlying sets such that the inverse image of every open set of $Y$ is an open set of $X$. We can also have the notion of *open morphism*: that is, a function $f : X \to Y$ such that the direct image of every open set of $X$ is an open set of $Y$. Likewise, a *closed morphism* is a function which maps closed sets to closed sets.

Topological spaces with continuous maps between them form a category, usually denoted $\mathbb{T}op$.

**Remark 2.2.4.** The definition of continuous function $f : X \to Y$ is such that it induces a morphism of the corresponding collections of opens the other way around

$$f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X).$$

And this is not just a morphism of posets but even of (complete) lattices. For more on this see in Chapter 3.

There are many equivalent ways to define a topological space. A non-exhaustive list follows:

- A set $X$ with a collections of closed sets (the complements of the open sets) satisfying dual axioms.
- A set $X$ with any collection of subsets whatsoever, to be thought of as a subbase for a topology.
- A pair $(X, \mathsf{int})$, where $\mathsf{int}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is an interior operator on the power set of $X$. The open sets are exactly the fixed points of $\mathsf{int}$.
- A pair $(X, \mathsf{cl})$ where $\mathsf{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator satisfying axioms dual to those of $\mathsf{int}$. The closed sets are the fixed points of $\mathsf{cl}$.

Let us introduce the the fundamental topological concept of compactness that may be regarded as a substitute for finiteness. All topological spaces we shall use in the Chapter 3 are compact.

**Definition 2.2.5.** Let $(X, \tau)$ be a topological space and let $U := \{U_i\}_{i\in I} \subseteq \tau$. The family $U$ is called an *open cover* of $Y \subseteq X$ if $Y \subseteq \bigcup_{i\in I} U_i$. A finite subset of $U$ whose union still contains $Y$ is a *finite subcover*. We say $Y$ is *compact* if every open cover of $Y$ has a finite subcover.

In Example 1 we have seen that for any map $f : A \to B$, there are the induced left and right adjoints of the inverse image $f^{-1}$ on powersets, the direct image $f[-]$ and the dual image $f_*$,

$$\mathcal{P}(B) \underset{f^{-1}}{\overset{f[-]}{\leftrightarrows}} \mathcal{P}(A) \qquad \mathcal{P}(A) \underset{f_*}{\overset{f^{-1}}{\leftrightarrows}} \mathcal{P}(B)$$

Note that if $A$ and $B$ are topological spaces and $f : A \to B$ is continuous, then $f^{-1}$ restricts to the open sets $f^{-1} : \mathcal{O}(B) \to \mathcal{O}(A)$. Now the left adjoint $f[-]$ need not exist on open sets, but the right adjoint $f_*$ still does.

$$\mathcal{O}(A) \underset{f_*}{\overset{f^{-1}}{\leftrightarrows}} \mathcal{O}(B)$$

Also, the condition of continuity can be equivalently written in terms of closure operator in the following ways. A map $f : X \to Y$ is continuous if

$$f[\mathsf{cl}(A)] \subseteq \mathsf{cl}(f[A]), \quad \forall A \subseteq X \qquad \text{(closed if } f[\mathsf{cl}(A)] \supseteq \mathsf{cl}(f[A]))$$
$$\mathsf{cl}(f^{-1}(B)) \subseteq f^{-1}[\mathsf{cl}(B)], \quad \forall B \subseteq Y \qquad \text{(open if } \mathsf{cl}(f^{-1}(B)) \supseteq f^{-1}[\mathsf{cl}(B)])$$

Moreover, any closure operator $\mathsf{cl}$ on $\mathcal{P}(X)$ gives rise to an interior operator $\mathsf{int}$ on $\mathcal{P}(X)$ by complementation
$$\complement\,\mathsf{cl}A = \mathsf{int}\,\complement A.$$

Therefore the continuity condition can be equivalently written via interior operator:

$$\begin{array}{lll} \mathsf{int}(f[A]) \subseteq f[\mathsf{int}(A)] & \forall A \subseteq X & (\text{open if } \mathsf{int}(f[A]) \supseteq f[\mathsf{int}(A)]) \\ f^{-1}[\mathsf{int}(B)] \supseteq \mathsf{int}(f^{-1}(B)) & \forall B \subseteq Y & (\text{open if } f^{-1}[\mathsf{int}(B)] \supseteq \mathsf{int}(f^{-1}(B))) \end{array}$$

**Separation Axioms, Specialization Order and Alexandrov Topologies**

The plain definition of topological space happens to allow examples where distinct points or distinct subsets of the underlying set appear as more-or-less unseparable as seen by the topology on that set. In many circumstances, it can be useful to exclude at least some of such degenerate examples from the discussion. The relevant conditions to be imposed on top of the plain axioms of a topological space are hence known as *separation axioms*.

These axioms are all of the form of saying that two subsets (of certain forms) in the topological space are 'separated' from each other in one sense if they are 'separated' in a (generally) weaker sense. For example the weakest axiom (called $T_0$) demands that if two points are distinct as elements of the underlying set of points, then there exists at least one open subset that contains one but not the other.

In this fashion, one may impose a hierarchy of stronger axioms. For example demanding that given two distinct points, then each of them is contained in some open subset not containing the other ($T_1$) or that such a pair of open subsets around two distinct points may in addition be chosen to be disjoint ($T_2$). This last condition, $T_2$, also called the *Hausdorff condition* is the most common among all separation axioms. Often (but by far not always) this is considered by default. We focus our attention on these separation axioms (see table 2.1).

Table 2.1: Separation Axioms

| | |
|---|---|
| $T_0$-axiom | If $a, b \in X$, there exists an open set $V \in \mathcal{O}(X)$ such that either $a \in V$ and $b \notin V$, or $a \notin V$ and $b \in V$. |
| $T_1$-axiom | If $a, b \in X$, there exists open sets $V, W \in \mathcal{O}(X)$, containing $a$ and $b$ respectively, such that $b \notin V$ and $a \notin W$. |
| $T_2$-axiom | If $a, b \in X$, there exists disjoint open sets $V, W \in \mathcal{O}(X)$, containing $a$ and $b$ respectively. |

We may define a partial order on the points of any $T_0$ topological space $X$ by $x \leq y$ iff $x$ is in the closure of $\{y\}$ (equivalently, $cl\{x\} \subseteq cl\{y\}$). If this relation holds we say $x$ is a specialization of $y$. It is clear that the relation $\leq$ is reflexive and transitive; its antisymmetry is precisely the $T_0$ axiom. Note also that any continuous map between

$T_0$ topological spaces is necessarily order-preserving, and that the order is discrete (i.e. satisfies $x \leq y$ iff $x = y$) if and only if $X$ is a $T_1$ topological space.

In the converse direction, suppose we are given a poset $(X, \leq)$. Can we find a topology on $X$ for which $\leq$ is the specialization ordering? We define the Alexandrov topology to be just $\mathcal{U}p(X)$, the collection of all upper sets in $X$; this is clearly a topology, since it is closed under arbitrary unions and intersections. Thus a map between two ordered sets is monotone if and only if it is a continuous map according to the corresponding Alexandrov topologies. Note also that every finite topological space is a topological space with such a topology.

In category-theoretic terms, Let $\mathbb{P}os$ be the category of orders and monotone maps between them and let $\mathbb{T}op_0$ be the category of $T_0$ topological spaces and monotone maps between them. We have a functor $L_{\mathbb{P}os} : \mathbb{P}os \to \mathbb{T}op_0$ sending an order $(P, \leq)$ to the couple $(P, \mathcal{U}p(P))$, where $\mathcal{U}p(P)$ is the Alexandrov topology on $P$, and sending a monotone map $f : (P, \leq_P) \to (Q, \leq_Q)$ to the continuous function $f : (P, \mathcal{U}p(P)) \to (Q, \mathcal{U}p(Q))$ of $\mathbb{T}op_0$. It can be possible to define a functor $R_{\mathbb{P}os} : \mathbb{T}op_0 \to \mathbb{P}os$ which is right adjoint to $L_{\mathbb{P}os}$ as follows. $R_{\mathbb{P}os}$ sends an object $(X, \tau_X)$ of $\mathbb{T}op_0$ to $(X, \leq)$, where $\leq$ is the order on $X$ given by the specialization order on $X$ induced by the topology $\tau_X$, and an arrow $f : (X, \tau_X) \to (Y, \tau_Y)$ to the monotone map $f : X \to Y$.

**Proposition 2.2.6.** *The functor $L_{\mathbb{P}os} : \mathbb{P}os \to \mathbb{T}op_0$ is left adjoint to the functor $R_{\mathbb{P}os} : \mathbb{T}op_0 \to \mathbb{P}os$, and identifies $\mathbb{P}os$ with a full coreflective subcategory of $\mathbb{T}op_0$.*

*Proof.* See Caramello (2016). □

**Corollary 2.2.7.** *If we restrict to a finite underlying set, the coreflection between the categories $\mathbb{P}os_f$ and $\mathbb{T}op_{0_f}$ of finite posets and finite $T_0$-topological spaces becomes an equivalence of categories.*

As a consequence of the above proposition, if we restrict our consideration on the Alexandrov topology, the previous continuity conditions given in terms of closure and interior operators simply assert the monotonicity of a given map. Since we have defined Alexandrov topology by choosing the open sets to be the upper sets, we obtain the monotonicity condition stated via upward map. A map $f : X \to Y$ is monotone if

$$\begin{array}{lll} \uparrow f[A] \subseteq f[\uparrow A] & \forall A \subseteq X & (\text{open if } \uparrow f[A] \supseteq f[\uparrow A]) \\ f^{-1}[\uparrow B] \subseteq \uparrow f^{-1}(B) & \forall B \subseteq Y & (\text{open if } f^{-1}[\uparrow B] \supseteq \uparrow f^{-1}(B)). \end{array}$$

Hence we can infer that, from a topological point of view, a p-morphism is an open continuous map with respect to Alexandrov topology.

### 2.2.1 The Category of Ordered $T_0$ Topological Spaces

In the light of giving a glance to the general case, we introduce the category of ordered $T_0$ topological spaces that will be essential for the representation theory of Heyting

Algebras. They are a sort of generalization of the previous examples in which, however, the order relation and the topology are not inextricably intertwined.

**Definition 2.2.8.** We define a ordered $T_0$ topological space as a triple $(X, \tau, \leq)$, where $X$ is a set, $\tau$ is a topology on $X$ and $\leq$ is a order relation on $X$. Ordered $T_0$ topological spaces form a category, which we denote $\mathbb{P}top_0$, whose arrows $f : (X, \tau_X, \leq_X) \to (Y, \tau_Y, \leq_Y)$ are the maps $f : X \to Y$ which are order-preserving and continuous; composition and identities in $\mathbb{P}top_0$ are defined by composing the underlying functions set-theoretically.

Let $\mathbb{P}os$ be the category of orders and monotone maps between them. We have a functor $L_{\mathbb{P}os} : \mathbb{P}os \to \mathbb{P}top_0$ sending a order $(P, \leq)$ to the triple $(P, \mathcal{U}p(P), \leq)$, where $\mathcal{U}p(P)$ is the Alexandrov topology on $P$, and sending a monotone map $f : (P, \leq_P) \to (Q, \leq_Q)$ to the arrow $f : (P, \mathcal{U}p(P), \leq_P) \to (Q, \mathcal{U}p(Q), \leq_Q)$ of $\mathbb{P}top_0$. It can be possible to define a functor $R_{\mathbb{P}os} : \mathbb{P}top_0 \to \mathbb{P}os$ which is right adjoint to $L_{\mathbb{P}os}$ as follows. $R_{\mathbb{P}os}$ sends an object $(X, \tau, \leq)$ of $\mathbb{P}top_0$ to $(X, \leq)$, where $\leq$ is the order on $X$ given by the intersection between $\leq$ and the specialization order on $X$ induced by the topology $\tau$, and an arrow $f : (X, \tau_X, \leq_X) \to (Y, \tau_Y, \leq_Y)$ to the monotone map $f : (X, \leq_X) \to (Y, \leq_Y)$.

**Proposition 2.2.9.** *The functor $L_{\mathbb{P}os} : \mathbb{P}os \to \mathbb{P}top_0$ is left adjoint to the functor $R_{\mathbb{P}os} : \mathbb{P}top_0 \to \mathbb{P}os$, and identifies $\mathbb{P}os$ with a full coreflective subcategory of $\mathbb{P}top_0$.*

*Proof.* See Caramello (2016). □

**Corollary 2.2.10.** *If we restrict to a finite underlying set, the coreflection between the categories $\mathbb{P}os_f$ and $\mathbb{P}top_{0_f}$ of finite posets and finite $T_0$-topological spaces becomes an equivalence of categories.*

Among ordered $T_0$ topological spaces, we are especially interested in ones satisfying a further separation condition guaranteeing that the spaces in question are extremely scattered, contrary to topological spaces encountered in elementary analysis and geometry.

**Definition 2.2.11.** Suppose that $(X, \leq, \tau)$ is a ordered $T_0$ topological space. It is said to be a totally order-disconnected if, given $x, y \in X$ with $x \not\leq y$, there exists a clopen upset $U$ such that $x \in U$ and $y \notin U$. We call a compact totally order disconnected space a Priestley space.

We shall denote by $\mathcal{CU}(X)$ the family of clopen upset of a Priestley space $X$. When $X$ is finite, $\mathcal{CU}(X)$ coincides with $\mathcal{U}p(X)$. As we shall see in the following these spaces have many nice properties from a logical point of view. This is illustrated by the following lemma.

**Lemma 2.2.12.** *Let $(X, \leq, \tau)$ be a Priestley space.*

*(i) $x \leq y$ in $X$ if and only if $x \in U$ implies $y \in U$ for every $U \in \mathcal{CU}(X)$.*

*(ii) (a) Let $Y$ be a closed upset in $X$ and let $x \notin Y$. Then there exists a clopen upset $U$ such that $Y \subseteq U$ and $x \notin U$.*

*(b) Let $Y$ and $Z$ be disjoint closed subsets of $X$ such that $Y$ is a upset and $Z$ is an downset. Then there exists a clopen upset $U$ such that $Y \subseteq U$ and $Z \cap U = \emptyset$.*

*Proof.* See Davey and Priestley (2002). □

We shall consider the category $\mathbb{P}S$ of Pristeley spaces, where maps are continuous and order preserving maps, and also the category $\mathbb{E}S$ of Esakia spaces (see Section 3.3.6).

**Examples 2.2.13.**

1. Denote by $\mathbb{N}_\infty$ the set of natural numbers with an additional point, $\infty$, adjoined. We define $\tau$ as follows: a subset $U$ of $\mathbb{N}_\infty$ belongs to $\tau$ if either

   (a) $\infty \notin U$, or

   (b) $\infty \in U$ and $\mathbb{N}_\infty \backslash U$ is finite.

   It is easy to see that $\tau$ is a topology and a subset $V$ of $\mathbb{N}_\infty$ is clopen if and only if both $V$ and $\mathbb{N}_\infty \backslash V$ are open. It follows that the clopen subsets of $\mathbb{N}_\infty$ are the finite sets not containing $\infty$ and their complements. It ican be proved that $\mathbb{N}_\infty$ is compact and totally disconnected. It is also Hausdorff because, given distinct points $x, y \in \mathbb{N}_\infty$, we may assume without loss of generality that $x \neq \infty$; then $\{x\}$ is clopen and contains $x$ but not $y$.

   Also, we define an order as follows: order $\mathbb{N}_\infty$ as the chain $\mathbb{N}^{\mathrm{op}}$ with $\infty$ adjoined as bottom element, as in Figure 2.3(i). Take $x \nleq y$. Then $y < x$ and $\uparrow x$, which is clopen because it is finite and does not contain $\infty$, contains $x$ but not $y$. Hence we have a Priestley space; its collection of clopen up-sets is isomorphic to the chain $\mathbb{1} + \mathbb{N}^{\mathrm{op}}$.

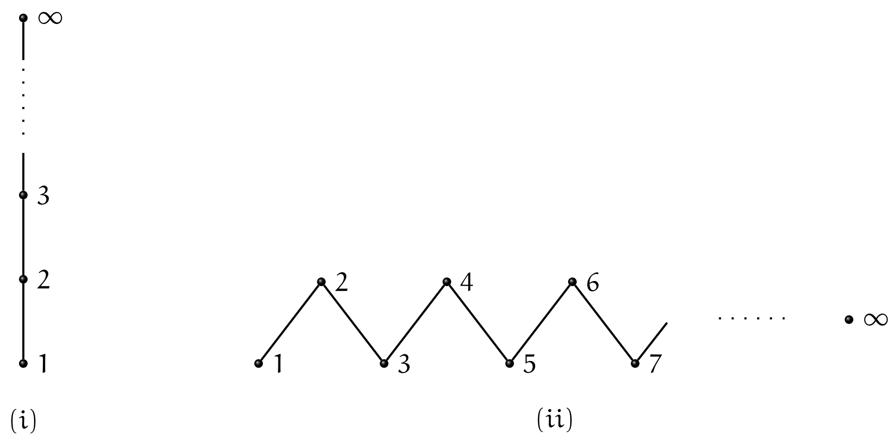


FIGURE 2.3: Priestley spaces obtained from $\mathbb{N}_\infty$.

2. Alternatively, consider the ordered space $Y$ obtained by equipping $\mathbb{N}_\infty$ with the order depicted in Figure 2.3(ii). We have $n - 1 \prec n$ and $n + 1 \prec n$ for each even $n$.

For each $n \in \mathbb{N}$, the upset $\uparrow n$ is finite and does not contain $\infty$ and so is clopen. Given $x \nleq y$ in $Y$, we claim that there exists $U \in \mathcal{CU}(Y)$ such that $x \in U$ and $y \notin U$. Either $x \neq \infty$, in which case $y \notin\uparrow x$ and we may take $U =\uparrow x$, or $x = \infty$, in which case we may take $U = Y \backslash \{1, 2, \ldots, 2y + 1\}$. Hence $Y$ is a Priestley space.

3. Let $C$ be the Cantor set, regarded as a subset of $[0, 1]$. Then $C$ is compact, since $C$ is obtained from $[0, 1]$ by removing open intervals. Also, if $x \nleq y$ in $C$, i.e. $y < x$, there exists $u$ such that $y < u < x$ and $u \notin C$. Then $C \cap \uparrow u$ is clopen because $C \bigcap \uparrow u = [u, 1] \bigcap C = (u, 1] \bigcap C$, and contains $x$ without containing $y$. Hence, with the order inherited from $[0, 1]$, $C$ it is a Priestley space.

# 3 Lattices and Heyting Algebras

In this Chapter we shall introduce the main algebraic structures which will be investigated within this work. Lattices and Heyting algebras, namely those lattices that provide the algebraic conceptualization of intermediate logics, are obtained by enriching posets by some algebraic operations. We shall also deal with representation theorems for distributive lattices and Heyting algebras which reveal the deeply connection of those algebraic structures with Priestley spaces.

## 3.1 Lattices

Many important properties of an ordered set $P$ are expressed in terms of the existence of certain upper bounds or lower bounds of subsets of P. Two of the most important classes of ordered sets defined in this way are lattices and complete lattices. Here we present the basic theory of such ordered sets, and also consider lattices as algebraic structures.

### 3.1.1 Lattices as Ordered Sets

**Definition 3.1.1.** Let $P$ be an ordered set and let $S \subseteq P$. An element $x \in P$ is an *upper bound* of $S$ if $s \leq x$ for all $s \in S$. A lower bound is defined dually: an element $x \in P$ is an *lower bound* of $S$ if $x \leq s$ for all $s \in S$.

Moreover, $x$ is the *least upper bound* of S if

(i) $x$ is an upper bound of $S$, and

(ii) $x \leq y$ for all upper bounds $y$ of $S$.

The least upper bound of $S$ exists if and only if there exists $x \in P$ such that

$$(\forall y \in P)[((\forall s \in S)s \leq y) \Leftrightarrow x \leq y],$$

and this characterizes the least upper bound of $S$.

Dually, $x$ is the *greatest lower bound* of $S$ if

(i) $x$ is a lower bound of $S$, and

(ii) $y \leq x$ for all lower bounds $y$ of $S$.

The antisymmetry axiom in the poset definition ensures that the least upper bounds and greatest lower bounds are unique when they exists. The least upper bound of $S$ is also called the *supremum* of $S$ and is denoted by $\bigvee S$; the greatest lower bound of $S$ is also called the *infimum* of $S$ and is denoted by $\bigwedge S$.

In the two extreme cases, where $S$ is empty or $S$ is $P$ itself, it is easily seen that if $P$ has a top element, then $\bigvee P = \top$. By duality, $\bigwedge P = \bot$ whenever $P$ has a bottom element. Now let $S$ be the empty subset of $P$. Then every element $x \in P$ satisfies $s \leq x$ for all $s \in S$. Thus, if $P$ has a bottom element $\bigwedge \emptyset = \bot$. Dually, $\bigvee \emptyset = \top$ whenever $P$ has a top element. Also, we write $x \vee y$ in place of $\bigvee\{x, y\}$ when it exists and $x \vee y$ in place of $\bigwedge\{x, y\}$ when it exists.

We shall focus on ordered sets in which $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$.

Let $P$ be a non empty ordered set.

**Definition 3.1.2.**

(i) If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then $P$ is called a *lattice*.

(ii) If $\bigvee$ and $\bigwedge S$ exist for all $S \subseteq P$, then $P$ is called a *complete lattice*.

**Proposition 3.1.3.** *Let $P$ be a lattice. Then for all $a, b, c, d \in P$,*

*(i) $a \leq b$ implies $a \vee c \leq b \vee c$ and $a \wedge c \leq b \wedge c$,*

*(ii) $a \leq b$ and $c \leq d$ imply $a \vee c \leq b \vee d$ and $a \wedge c \leq b \wedge d$.*

*Proof.* □

**Examples 3.1.4.**

1. For any set $X$, the ordered set $(\mathcal{P}(X); \subseteq)$ is a complete lattice in which

$$\bigvee_{i \in I} A_i = \bigcup_{i \in I} A_i,$$
$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i.$$

   We verify the assertion about meets; that about joins is proved dually. Let $\{A_i\}_{i \in I}$ be a family of elements of $\mathcal{P}(X)$. Since $\bigcap_{i \in I} A_i \subseteq A_j$ for all $j \in I$, it follows that $\bigcap_{i \in I} A_i$ is a lower bound for $\{A_i\}_{i \in I}$. Also, if $B \in \mathcal{P}(X)$ is a lower bound of $\{A_i\}_{i \in I}$, then $B \subseteq A_i$ for all $i \in I$ and hence $B \subseteq \bigcap_{i \in I} A_i$. Thus $\bigcap_{i \in I} A_i$ is indeed the greatest lower bound of $\{A_i\}_{i \in I}$ in $\mathcal{P}(X)$.

2. Let $\emptyset \neq \mathcal{K} \subseteq \mathcal{P}(X)$. Then $\mathcal{K}$ is a lattice of subsets if it is closed under finite unions and intersections and a complete lattice of subsets if it is closed under arbitrary unions and intersections. If $\mathcal{K}$ is a lattice of subsets, then $(\mathcal{K}; \subseteq)$ is a lattice in which $A \vee B = A \cup B$ and $A \wedge B = A \cap B$. Similarly, if $L$ is a complete lattice of subsets, then $(\mathcal{K}; \subseteq)$ is a complete lattice with join given by set union and meet given by set intersection.

Let $P$ be an ordered set and consider the ordered set $\mathcal{D}o(X)$ of all down-sets of $P$ introduced in . If $\{A_i\}_{i\in I} \subseteq \mathcal{D}o(X)$, then $\bigcup_{i\in I} A_i$ and $\bigcap_{i\in I} A_i$, both belong to $\mathcal{D}o(X)$. Hence $\mathcal{D}o(X)$ is a complete lattice of subsets, called the down-set lattice of $P$.

### 3.1.2 Lattices as Algebraic Structures

In the previous subsection we view a lattice as an ordered set. In this one we view it as an algebraic structure. Given a lattice $L$, we may define binary operations *join* and *meet* on the non-empty set $L$ by

$$a \vee b := \bigvee \{a, b\} \quad \text{and} \quad a \wedge b := \bigwedge \{a, b\} \quad (a, b \in L).$$

Observe that Proposition 3.1.3 ii says precisely that the operations $\vee : L^2 \longrightarrow L$ and $\wedge : L^2 \longrightarrow L$ are order-preserving. In the following we explore the properties of these binary operations. We first emphasize the connection between $\vee, \wedge$ and $\leq$.

**Lemma 3.1.5.** *Let $L$ be a lattice and let $a, b \in L$. Then the following are equivalent:*

*(i)* $a \leq b$;

*(ii)* $a \vee b = b$;

*(iii)* $a \wedge b = a$.

*Proof.* See Davey and Priestley (2002). □

**Theorem 3.1.6.** *Let $L$ be a lattice. Then $\vee$ and $\wedge$ satisfy, for all $a, b, c \in L$,*

$$(a \vee b) \vee c = a \vee (b \vee c) \tag{3.1}$$

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \tag{3.2}$$

$$a \vee b = b \vee a \tag{3.3}$$

$$a \wedge b = b \wedge a \tag{3.4}$$

$$a \vee a = a \tag{3.5}$$

$$a \wedge a = a \tag{3.6}$$

$$a \vee (a \wedge b) = a \tag{3.7}$$

$$a \wedge (a \vee b) = a \tag{3.8}$$

*Proof.* See Davey and Priestley (2002). □

Note that the dual of a statement about lattices phrased in terms of $\vee$ and $\wedge$ is obtained simply by interchanging $\vee$ and $\wedge$ (this is the Duality Principle for lattices).

We now turn things round and start from a set carrying operations $\vee$ and $\wedge$ which satisfy the identities given in the preceding theorem.

**Theorem 3.1.7.** *Let $(L; \vee, \wedge)$ be a non-empty set equipped with two binary operations which satisfy 3.2-3.8 from 3.1.6.*

1. *For all $a, b \in L$, we have $a \vee b = b$ if and only if $a \wedge b = a$.*
2. *Define $\leq$ on L by $a \leq b$ if $a \vee b = b$. Then $\leq$ is an order relation.*
3. *With $\leq$ as in (ii), $(L; \leq)$ is a lattice in which the original operations agree with the induced operations, that is, for all $a, b \in L$,*

$$a \vee b = sup\{a, b\} \quad \text{and} \quad a \wedge b = inf\{a, b\}.$$

*Proof.* See Davey and Priestley (2002). □

**Remark 3.1.8.**

1. We have shown that lattices can be completely characterized in terms of the join and meet operations. The above theorems say that the notion of lattice can be defined either in terms of the order relation or in terms of the join and meet operations.
2. Also, associativity of $\vee$ and $\wedge$ allows us to write iterated joins and meets unambiguously without brackets. An easy induction shows that these correspond to sups and infs in the expected way:

$$\bigvee\{x_1, \ldots, x_n\} = a_1 \vee \cdots \vee a_n,$$

for $a_1, \ldots, a_n \in L$, and dually. Consequently, $\bigvee F$ and $\bigwedge F$ exist for any finite, non-empty subset $F$ of a lattice.

**Definition 3.1.9.** A lattice $(L; \vee, \wedge)$ possessing $\perp$ and $\top$ is called *bounded*. A finite lattice is automatically bounded, with $\top = \bigvee L$ and $\perp = \bigwedge L$.

**Remark 3.1.10.** We can also define meet and join operations entirely in terms of order adjoints in the following way.

For $X$ and $Y$ posets, define $(x, y) \leq (u, v)$ if and only if $x \leq u$ and $y \leq v$. It follows that this is an order on $X \times Y$, that the projection functions are order-preserving and that $X \times Y$ together with the projections is a product of posets. It then follows that the diagonal function $\Delta_x : X \longrightarrow X \times X$ is order-preserving too. Assume that $\Delta_x$ has a right adjoint. Call it $\wedge$ and write $x \wedge y$ for $\wedge(x, y)$. The defining property is this: $z \leq x \wedge y$ if and only if $z \leq x$ and $z \leq y$. Taking $z = x \wedge y$ we see that $x \wedge y < x$ and $x \wedge y < y$, so that $x \wedge y$ is a 'lower bound' for the set that one would like to write as $\{x, y\}$. The defining property says that among all such lower bounds, $z$, $x \wedge y$ is the 'greatest'. So $\wedge$ is a greatest lower bound operation, often called 'meet'. Any other such greatest lower bound operation is isomorphic to $\wedge$.

Dually, $\Delta_x : X \longrightarrow X \times X$ has a left adjoint if and only if for each pair of elements $(x, y)$ there is prescribed an element $x \vee y$ with the property that, for all $z$, $x \vee y \leq z$ if and only if $x \leq z$ and $y \leq z$. Thus $\vee$ is a least upper bound operation, often called 'join'.

### 3.1.3 Sublattices, Products and Homomorphisms

This Subsection presents methods for deriving new lattices.

**Definition 3.1.11.** Let $L$ be a lattice and $\emptyset \neq M \subseteq L$. Then $M$ is a *sublattice* of $L$ if

$$a, b \in M \quad \text{implies} \quad a \vee b \in M \quad \text{and} \quad a \wedge b \in M$$

We denote the collection of all sublattices of $L$ by $\mathsf{Sub}\, L$ and let $\mathsf{Sub}_0\, L = \mathsf{Sub}\, L \cup \{\emptyset\}$. They are both ordered by inclusion.

**Definition 3.1.12.** Let $L$ and $K$ be lattices. Define $\vee$ and $\wedge$ coordinate-wise on $L \times K$, as follows:

$$(l_1, k_1) \vee (l_2, k_2) = (l_1 \vee l_2, k_1 \vee k_2),$$
$$(l_1, k_1) \wedge (l_2, k_2) = (l_1 \wedge l_2, k_1 \wedge k_2).$$

It is easy to check that $L \times K$ satisfies the identities 3.2-3.8 and therefore is a lattice. Also

$$(l_1, k_1) \vee (l_2, k_2) = (l_2, k_2) \Leftrightarrow (l_1, k_1) \leq (l_2, k_2).$$

with respect to the order on $L \times K$ defined in 2.1.12. Hence the lattice formed by taking the ordered set product of lattices $L$ and $K$ is the same as that obtained by defining $\vee$ and $\wedge$ coordinatewise on $L \times K$.

Iterated products and powers are defined in the obvious way.

From the viewpoint of lattices as algebraic structures it is natural to regard as canonical those maps between lattices which preserve the operations join and meet. Since lattices are also ordered sets, we need to discuss the relationship between these classes of maps and order-preserving maps. We begin with some definitions.

**Definition 3.1.13.** Let $L$ and $K$ be lattices. A map $f : L \to K$ is said to be a lattice *homomorphism* if $f$ is join-preserving and meet-preserving, that is, for all $a, b \in L$,

$$f(a \vee b) = f(a) \vee f(b) \quad \text{and} \quad f(a \wedge b) = f(a) \wedge f(b).$$

A bijective homomorphism is a (lattice) *isomorphism*. If $f : L \to K$ is a one-to-one homomorphism, then the sublattice $f(L)$ of $K$ is isomorphic to $L$. We refer to $f$ as an *embedding* (of $L$ into $K$) and we write $L \rightarrowtail K$.

**Remark 3.1.14.** For bounded lattices $L$ and $K$ it is often appropriate to consider homomorphisms $f : L \to K$ such that $f(\bot) = \bot$ and $f(\top) = \top$. Such maps are called $\{\bot, \top\}$-homomorphisms.

In general an order-preserving map may not be a homomorphism. However, as the proposition below shows, there is no demarcation between order-isomorphism and lattice isomorphism.

**Proposition 3.1.15.** *Let $L$ and $K$ be lattices and $f : L \to K$ a map.*

*(i) The following are equivalent:*

*(a) $f$ is order-preserving;*

*(b) $(\forall a, b \in L) f(a \vee b) \geq f(a) \vee f(b)$;*

*(c) $(\forall a, b \in L) f(a \wedge b) \leq f(a) \wedge f(b)$.*

*In particular, if $f$ is a homomorphism, then $f$ is order-preserving.*

*(ii) $f$ is a lattice isomorphism if and only if it is an order-isomorphism.*

*Proof.* See Davey and Priestley (2002). □

Also, It follows from this proposition that a lattice embedding $L \rightarrowtail K$ implies an order embedding $L \hookrightarrow K$. In Section 3.1.5, we shall discuss a condition for which the converse is true.

**Remark 3.1.16.** Let $f : L \to K$ be a lattice homomorphism. If $M \in \mathsf{Sub}\, L$ then $f(M) \in \mathsf{Sub}\, K$. Also, if $N \in \mathsf{Sub}\, K$ then $f^{-1}(N) \in \mathsf{Sub}_0\, L$.

### 3.1.4 Ideals and Filters

Ideals and Filters are of fundamental importance in algebra, logic and topology. Filters, specifically prime filters, which we consider after, form the basis for the representation theory that we are going to present in the following sections. We start with the notion of ideal.

**Definition 3.1.17.** Let $L$ be a lattice. A non-empty subset $J$ of $L$ is called an *ideal* if

(i) $a, b \in J$ implies $a \vee b \in J$,

(ii) $a \in L, b \in J$ and $a \leq b$ imply $a \in J$.

The definition can be also stated by declaring an ideal to be a non-empty down-set closed under join. Clearly, every ideal $J$ of a lattice $L$ is a sublattice, since $a \wedge b \leq a$ for any $a, b \in L$. A dual ideal is called a filter.

**Definition 3.1.18.** Specifically, a non-empty subset $G$ of $L$ is called a *filter* if

(i) $a, b \in G$ implies $a \wedge b \in G$,

(ii) $a \in L, b \in G$ and $a \geq b$ imply $a \in G$.

The set of all ideals of $L$ is denoted by $\mathcal{I}(L)$ and carries the usual inclusion order; while the set of all filters of $L$ is denoted by $\mathcal{F}(L)$ and carries the opposite order.

An ideal or filter is called *proper* if it does not coincide with $L$. It can be easily shown that an ideal $J$ of a lattice with $\top$ is proper if and only if $\top \notin J$, and dually, a filter $G$ of a lattice with $\bot$ is proper if and only if $\bot \notin G$. For each $a \in L$, the set $\downarrow a$ is an ideal; it is known as the *principal ideal* generated by $a$. Dually, $\uparrow a$ is a *principal filter*.

**Examples 3.1.19.**

(1) In a finite lattice, every ideal or filter is principal: the ideal $J$ equals $\downarrow \bigvee J$, and dually for a filter.

(2) Let $L$ and $K$ be bounded lattices and $f : L \to K$ a $\{\bot, \top\}$-homomorphism. Then $f^{-1}(\bot)$ is an ideal and $f^{-1}(\top)$ is a filter in $L$ (see Johnstone 1982).

(3) The following are ideals in $\mathcal{P}(X)$:

(a) all subsets not containing a fixed element of $X$;

(b) all finite subsets (this ideal is non-principal if $X$ is infinite).

(4) Let $(X, \tau)$ be a topological space and let $x \in X$. Then the set $\{V \subseteq X \mid (\exists U \in T)x \in U \subseteq V\}$ is a filter in $\mathcal{P}(X)$.

### 3.1.5 Complete Lattices

We now return to complete lattices, which were briefly introduced at the start of this section. Recall from Definition 3.1.2 that a complete lattice is defined to be a non-empty, ordered set $P$ such that the join (supremum), $\bigwedge S$, and the meet (infimum), $\bigvee S$, exist for every subset $S$ of $P$.

We first collect together in a sequence of elementary lemmas useful information for computing with arbitrary joins and meets, extending the results for binary joins and meets presented earlier. The first lists some immediate consequences of the definitions of least upper bound and greatest lower bound.

**Lemma 3.1.20.** *Let $P$ be an ordered set, let $S, T \subseteq P$ and assume that $\bigvee S$, $\bigvee T$, $\bigwedge S$ and $\bigwedge T$ exist in $P$.*

*(i) $s \leq \bigvee S$ and $s \geq \bigwedge S$ for all $s \in S$.*

*(ii) Let $x \in P$; then $x \geq \bigvee S$ if and only if $x \geq s$ for all $s \in S$.*

*(iii) Let $x \in P$; then $x \leq \bigwedge S$ if and only if $x \leq s$ for all $s \in S$.*

*(iv) $\bigvee S \leq \bigwedge T$ if and only if $s \leq t$ for all $s \in S$ and all $t \in T$.*

*(v) If $S \subseteq T$, then $\bigvee S \leq \bigvee T$ and $\bigwedge S \geq \bigwedge T$.*

A straightforward application of Lemma 3.1.20 yields the next one, which shows that join and meet behave well with respect to set union.

**Lemma 3.1.21.** *Let $P$ be a lattice, let $S, T \subseteq P$ and assume that $\bigvee S$, $\bigvee T$, $\bigwedge S$ and $\bigwedge T$ exist in $P$. Then*

$$\bigvee(S \cup T) = (\bigvee S) \vee (\bigvee T) \quad \textit{and} \quad \bigwedge(S \cup T) = (\bigwedge S) \wedge (\bigwedge T)$$

An easy induction now yields the following results, previously noted in remark 3.1.8(2), but worth reiterating. The corollary follows easily from the definition of top and bottom elements.

**Lemma 3.1.22.** *Let $P$ be a lattice. Then $\bigvee F$ and $\bigwedge F$ exist for every finite, non-empty subset $F$ of $P$.*

**Corollary 3.1.23.** *Every finite lattice is complete.*

To show that an ordered set is a complete lattice requires only half as much work as the definition would have us believe.

**Lemma 3.1.24.** *Let $P$ be an ordered set such that $\bigwedge S$ exists in $P$ for every non-empty subset $S$ of $P$. Then $\bigvee S$ exists in $P$ for every subset $S$ of $P$ which has an upper bound in $P$; indeed, $\bigvee S = \bigwedge S^u$.*

*Proof.* See Johnstone (1982), Davey and Priestley (2002). □

**Theorem 3.1.25.** *An ordered set $P$ is a complete lattice if and only if $\bigwedge S$ exist for every subset $S$ of $P$.*

*Proof.* See Davey and Priestley (2002). □

### Adjoint Functor Theorem on Posets

We now describe how joins and meets interact with order-preserving maps and order-isomorphisms. First we need a definition.

**Definition 3.1.26.** Let $P$ and $Q$ be ordered sets and $\phi : P \to Q$ a map. Then we say that $\phi$ *preserves existing joins* if whenever $\bigvee S$ exists in $P$ then $\bigvee \phi(S)$ exists in $Q$ and $\phi(\bigvee S) = \bigvee \phi(S)$. Preservation of existing meets is defined dually.

**Lemma 3.1.27.** *Let $P$ and $Q$ be an ordered sets and $\phi : P \to Q$ be an order preserving map.*

*(i) Assume that $S \subseteq P$ is such that $\bigvee S$ exists in $P$ and $\bigvee \phi S$ exists in $Q$. Then $\phi \bigvee S \geq \bigvee \phi S$. Dually, $\phi \bigwedge S \leq \bigwedge \phi S$ if both meet exists.*

*(ii) Assume now that $\phi : P \to Q$ is an order isomorphism. Then $\phi$ preserves all existing joins and meets.*

*Proof.* See Davey and Priestley (2002). □

Of course, there is a more general condition for preservation of joins and meets, given in category-theoretic terms by the existence of adjunctions. Conversely, another important result from category theory, the Adjoint Functor Theorem restricted to ordered sets, establishes the preservation of joins and meets as a condition for the existence of adjunctions.

**Theorem 3.1.28.** *Let $f : X \to Y$ be a monotone function between ordered sets.*

*(i) If $f$ has a right adjoint then it preserves all joins which exists in $X$. Dually If $f$ has a left adjoint then it preserves all meets.*

*(ii) Conversely, provided $X$ has joins of all subsets, $f$ has a right adjoint if it preserves them. Dually, $f$ has a left adjoint if $X$ has and $f$ preserves all meets.*

*Proof.* See Johnstone (1982), Pitts (1989). □

As a consequence, given a morphism $f : X \to Y$ such that $f \dashv g$, the right adjoint $g$ is easily seen to be given by the following formula:

$$g(y) = \bigvee \{x \mid f(x) \leq y\}. \tag{3.9}$$

Dually, given a morphism $g : Y \to X$ such that $f \dashv g$, the left adjoint $f$ is given by:

$$f(x) = \bigwedge \{y \mid x \leq g(y)\} \tag{3.10}$$

**Yoneda Embedding for Posets**

Remember that for $X$ any ordered set the map $x \mapsto \downarrow x$ is an order embedding. It is useful to think of $X$ as being contained in $\mathcal{D}o(X)$ and it can be useful to think about adjoints for $\downarrow : X \to \mathcal{D}o(X)$. A right adjoint would provide, for each downset $S$ of $X$, a largest element $x$ with the property that $\downarrow x \subseteq S$. Since this must apply in particular to $S = \emptyset$, the empty subset, and the $\downarrow x$ are not empty, it follows that $\downarrow$ never has a right adjoint.

The possibility of a left adjoint for $\downarrow : X \to \mathcal{D}o(X)$ is a quite different matter. Recall (the dual of) Proposition 3.1.28. Let $S$ be a subset of $X$ and assume that $\bigwedge S$ exists. It is characterized by the requirement 3.1.20(iii), $x \leq \bigwedge S$ if and only if, for all $s$ in $S$, $x \leq s$. We have the comparison inequality $\downarrow(\bigwedge S) \leq \bigwedge\{\downarrow s | s \in S\}$, provided the right side exists, merely because $\downarrow$ is order-preserving.

**Lemma 3.1.29.** *For any subset $S$ of $X$, $\bigwedge\{\downarrow s \mid s \in S\}$ exists in $\mathcal{D}o(X)$ and if $\bigwedge S$ exists in $X$ then the comparison inequality $\downarrow(\bigwedge S) \leq \bigwedge\{\downarrow s | s \in S\}$ is necessarily an equality.*

*Proof.* See Wood (2004). □

Said otherwise, the order embedding $\downarrow : X \to \mathcal{D}o(X)$ preserves any infima that exists and can be useful to rephrasing the definition of completeness in a categorical manner.

**Definition 3.1.30.** An ordered set $X$ is said to be complete if $\downarrow : X \to \mathcal{D}o(X)$ has a left adjoint.

Sometimes the order embedding $\downarrow : X \to \mathcal{D}o(X)$ is called Yoneda embedding for posets in analogy with the categorical one.

Now we can actually determine, with the help of the formulas 3.10, the left adjoint of the order embedding $\downarrow : X \to \mathcal{D}o(X)$. We have to find an order preserving $F$ map such as to satisfy, for every downset $S$ of $X$,

$$F(S) = \bigwedge \{x \mid S \subseteq \downarrow x\}.$$

It follows easily that the expression on the right is equal to $\bigwedge S^u$, i.e. another instance of $\bigvee S$ for $S$ in $\mathcal{D}o(X)$.

**Theorem 3.1.31.** *For an ordered set $X$, the following are equivalent:*

*(1) $X$ is complete.*

*(2) For every downset $S$ of $X$, $\bigwedge\{x \mid S \subseteq \downarrow x\}$ exists.*

*(3) For every downset $S$ of $X$, $\bigvee S$ exists.*

*(4) For every subset $S$ of $X$, $\bigvee S$ exists.*

*Proof.* See Wood (2004). □

### Reflective Embeddings and Completeness, Closure System

In Subsection 3.1.3 we mention to the fact that, if $i : X \hookrightarrow A$ is an order embedding and $A$ is a lattice, it does not follow that $X$ is a sublattice of $A$. However, if the order embedding is reflective (see Lemma 2.1.36), we have the following result:

**Proposition 3.1.32.** *Reflective order embeddings of lattices are sublattices. In particular, the embedding preserves arbitrary meets. Also, reflective order embeddings of complete lattices are complete sublattices.*

*Proof.* See Wood (2004). □

Note that $\perp$ and joins in $X$ are typically different from their counterparts in $A$. It can be easily proved that they are given in terms of the reflector:

$$\bigvee_X := p(\bigvee_A). \tag{3.11}$$

Remember that left adjoints preserve bottom elements and joins but typically do not preserve top elements and meets. For $i : X \hookrightarrow A$ reflective with reflector $p$, one should observe that nevertheless $p$ preserves top elements. It is an important extra property however for $p$ to preserve meets (see Section 3.3).

**Definition 3.1.33.** If $\mathcal{K} \subseteq \mathcal{P}(X)$ is a non-empty family of subsets of $X$ which satisfies the conditions:

1. $X \in \mathcal{K}$, and
2. $\bigcap_{i\in I} A_i \in \mathcal{K}$ for every non-empty family $\{A_i\}_{i\in I} \subseteq \mathcal{K}$,

then $\mathcal{K}$ is called a topped meet structure on $X$. An alternative term is closure system.

It is easy to see that, as a consequence of a closure system gives rise to a complete lattice.

**Proposition 3.1.34.** *Let $X$ be a set and $\mathcal{K}$ be a closed system on $X$. Then $\mathcal{K}$ is a complete lattice in which*

$$\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i \quad \textit{and} \quad \bigvee_{i\in I} A_i = \bigcap\{B \in \mathcal{K} \mid \bigcup_{i\in I} A_i \subseteq B\}.$$

*Proof.* See Davey and Priestley (2002). □

The connection between closed set systems and closure operators is much like the connection between lattice orderings and lattices, discussed in Subsection 3.1.1. Given a closed set system on $X$, one may define a closure operator on $\mathcal{P}(X)$; given a closure operator on $\mathcal{P}(X)$, one may define a closed set system on $X$. Moreover, these two processes are inverses of each other. More precisely, let $\mathcal{K} \subseteq \mathcal{P}(X)$ be a closed set system on $X$. Define the function $C$ on $\mathcal{P}(X)$ by

$$C(Y) = \bigcap\{A \in \mathcal{K} \mid Y \subseteq A\}$$

for all $Y \subseteq X$. $C$ turns out to be a closure operator on $\mathcal{P}(X)$. For the reverse definition, let $C$ be any closure operator on $X$. Define

$$\mathcal{K} = \{Y \mid C(Y) = Y\}.$$

This correspondence also holds in general, not only for complete lattices of sets.

**Proposition 3.1.35.** *For any complete lattice $P$, there is a bijection between the following sets:*

*(i) The set of closure operators on $P$;*

*(ii) The set of fixsets in $P$, i.e. subsets which are closed under arbitrary meets.*

□

**Examples 3.1.36.**

1. Each of the following is a meet-structure and so forms a complete lattice under inclusion:

   - the subgroups, Sub $G$, of a group $G$;
   - the equivalence relations on a set $X$;
   - the subspaces, Sub $V$ of a vector space $V$;
   - the convex subsets of a real vector space;
   - $\mathrm{Sub}_0\, L$, the sublattices of a lattice $L$, with the empty set added (note that Sub$L$ is not closed under intersections, except when $|L| = 1$);
   - the ideals of a lattice $L$ with $\perp$ (or, if $L$ has no zero element, the ideals of $L$ with the empty set added), and dually for filters.

2. The closed subsets of a topological space are closed under finite unions and finite intersections and hence form a lattice of sets in which $A \vee B = A \cup B$ and $A \wedge B = A \cap B$. In fact, the closed sets form a topped meet-structure and consequently the lattice of closed sets is complete. The formulae for arbitrary (rather than finite) joins and meets given in 3.1.34 and 3.11 show that, in general, meet is given by intersection while the join of a family of closed sets is not their union but is obtained by forming the closure of their union.

3. Since the open subsets of a topological space are closed under arbitrary union and include the empty set, the dual of 3.1.31 shows that they form a complete lattice under inclusion. The dual version of 3.1.34 and 3.11 shows that join and meet are given by

$$\bigvee_{i\in I} A_i = \bigcup_{i\in I} A_i \quad \text{and} \quad \bigwedge_{i\in I} A_i = \mathsf{int}(\bigcap_{i\in I} A_i)$$

where $\mathsf{int}(A)$ denotes the interior of $A$.

## 3.2 Distributive Lattices and Prime Filters

In Subsection 3.1.2 we began to introduce the algebraic theory of lattices, armed with enough axioms on $\vee$ and $\wedge$ to ensure that each lattice $(L; \vee, \wedge)$ arose from a lattice $(L; \leq)$ and vice versa. Now we introduce distributivity identities linking join and meet which are not implied by the laws 3.2-3.8 defining lattices (see Theorem 3.1.6). These hold in many of our examples of lattices, in particular in powersets and downsets/uppersets lattices.

Also distributive lattices provide our first example of propositional theory, namely the coherent theory of prime filters.

### 3.2.1 Distributive Lattices

Before formally introducing distributive lattices we prove two lemmas which will put the definition of distributivity into perspective. The import of these lemmas is discussed after.

**Lemma 3.2.1.** *Let $L$ be a lattice and let $a, b, c \in L$. Then $a \wedge (b \vee e) \geq (a \wedge b) \vee (a \wedge e)$, and dually.*

*Proof.* See Davey and Priestley (2002). □

**Lemma 3.2.2.** *Let $L$ be a lattice. Then the following are equivalent:*

$$(\forall a, b, c \in L) \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c); \tag{3.12}$$

$$(\forall a, b, c \in L) \quad p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r). \tag{3.13}$$

*Proof.* See Davey and Priestley (2002). □

**Definition 3.2.3.** Let $L$ be a lattice. $L$ is said to be distributive if it satisfies the distributive law,

$$(\forall a, b, c \in L) \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

**Remark 3.2.4.**

1. Lemma 3.2.1 shows that any lattice is "half-way" to being distributive. To establish distributivity it suffices to check an inequality.

2. Distributivity can be defined either by 3.12 or by 3.13. Thus the apparent asymmetry between join and meet in the above definition is illusory. In other words, $L$ is distributive if and only if $L^{\mathrm{op}}$ is.

**Examples 3.2.5.**

1. Any powerset lattice $\mathcal{P}(X)$ is distributive. More generally, any lattice of sets is distributive. In Section 3.2.3 we prove the striking result that every distributive lattice is isomorphic to a lattice of peculiar sets (see below Theorem 3.2.12).

2. Any chain is distributive.

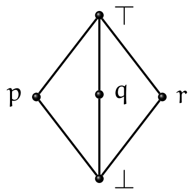


FIGURE 3.1: The lattice $M_3$.

3. Consider the lattice $M_3$ (the diamond) shown in Figure 3.1. It is not distributive. To see this, note that in the diagram of $M_3$

$$p \wedge (q \vee r) = p \wedge \top = p \neq \bot = \bot \vee \bot = (p \wedge q) \vee (p \wedge r).$$

   This example turns out to play a crucial role in the Birkhoff's characterisation of distributive lattices (for further details see Davey and Priestley 2002).

As we saw in Subsection 3.1.3, new lattices can be manufactured by forming sublattices, products and homomorphic images. Distributivity are preserved by these constructions (see Davey and Priestley 2002).

### 3.2.2 Prime Filters

In Subsection 3.1.4, we introduced lattice ideals and filters as part of the development of the algebraic theory of lattices. But we did not take the theory far enough to reveal the importance of ideals, or of their order duals, filters. We now turn our attention to a class of ideals and filters which will serve very well as building blocks for distributive lattices in order to yield a representation theorem. We need Zorn Lemma (ZL) to show that such ideals and filters exist.

**Definition 3.2.6.** Let $L$ be a lattice. Recall from Definition 3.1.17 that a non-empty subset $J$ of $L$ is called an ideal if

(i) $a, b \in J$ implies $a \vee b \in J$,

(ii) $a \in L, b \in J$ and $a \leq b$ imply $a \in J$;

it is proper if $J \neq L$.

A proper ideal $J$ of $L$ is said to be *prime* if $a, b \in L$ and $a \wedge b \in J$ imply $a \in J$ or $b \in J$. The set of prime ideals of $L$ is denoted $\mathcal{I}_p(L)$. It is ordered by set inclusion.

**Definition 3.2.7.** Let $L$ be a lattice. Recall from 3.1.18 that a non-empty subset $G$ of $L$ is called a filter if

(i) $a, b \in G$ implies $a \wedge b \in G$,

(ii) $a \in L, b \in G$ and $a \geq b$ imply $a \in G$.

it is proper if $J \neq L$.

A proper filter $G$ of $L$ is said to be *prime* if $a, b \in L$ and $a \vee b \in G$ imply $a \in G$ or $b \in G$. The set of prime filters of $L$ is denoted $\mathcal{F}_p(L)$. It is ordered by reverse set inclusion.

**Proposition 3.2.8.** *A subset $J$ of a lattice $L$ is a prime ideal if and only if $L \setminus J$ is a prime filter. A subset $J$ of a lattice $L$ is a prime ideal if and only if $J$ is the kernel of a $\{\bot, \top\}$-homomorphism $f : L \to 2$. A subset $J$ of a lattice $L$ is a prime filter if and only if $J$ is the cokernel of a $\{\bot, \top\}$-homomorphism $f : L \to 2$.*

*Proof.* See Johnstone (1982). □

According to the last proposition above, there is a bijection between the set of prime ideals and the set of prime filters, namely $\mathcal{I}_p(L) \simeq \mathcal{F}_p(L)^{\mathrm{op}}$. Thus it is easy to switch between $\mathcal{I}_p(L)$ and $\mathcal{F}_p(L)$. In the sequel we work predominantly with prime filters, for logical reasons.

The question of the existence of prime elements has closer affinities with set theory than with lattice theory. The statements (DFP) introduced below assert the existence of certain prime filters. On one level, (DFP) may be taken as axiom, whose lattice-theoretic implications we pursue. At a deeper level, it can be shown how (DFP) may be derived from (ZL) (See Davey and Priestley 2002).

We consider the following assertion, which embody the existence statements we shall require.

(DFP) Given a distributive lattice $L$ and an ideal $J$ and a filter $G$ of $L$ such that $J \cap G = \emptyset$, there exist $F \in \mathcal{F}_p(L)$ and $I = L \setminus F \in \mathcal{I}_p(L)$ such that $J \subseteq I$ and $G \subseteq F$.

In the remainder of the chapter we employ (ZL). The following result for distributive lattices is often referred to as the Prime Ideal Theorem.

**Theorem 3.2.9.** *(ZL) implies (DFP).*

*Proof.* See Davey and Priestley (2002). □

### 3.2.3 Representation of Distributive Lattices: Priestley Duality

We now move to representation of distributive lattices. Let $L$ be a distributive lattice and let $X = \mathcal{F}_p(L)$ be its set of prime filters ordered, as usual, by order inclusion. We shall obtain representations for $L$ in two cases, finite and infinite. When $L$ is finite, we shall prove that $L$ is isomorphic to the lattice $\mathsf{Up}(X)$ of up-sets of $X$.

In order to represent $L$ in general we should equip $X$ with the inclusion order and a suitable topology. A candidate for a lattice isomorphic to $L$ would then be the lattice of all clopen up-sets of $X$. Our remarks above imply that this lattice coincides with $\mathsf{Up}(X)$ when $L$ is finite. Indeed, we shall prove in theorem 3.2.12 that a bounded distributive lattice $L$ is isomorphic to the lattice of clopen up-sets of $\mathcal{F}_p(L)$, ordered by order inclusion and appropriately topologized.

Let $L$ be a distributive lattice with $\perp$ and $\top$ and for each $a \in L$ let

$$X_a := \{F \in \mathcal{F}_p(L) \mid a \in F\},$$

as before. Let $X := \mathcal{F}_p(L)$. We want a topology $\tau$ on $X$ so that each $X_a$ is clopen. Accordingly, we want every element of

$$S := \{X_b \mid b \in L\} \cup \{X \backslash X_c \mid c \in L\}$$

to be in $\tau$. The family $S$ contains sets of two types and it is also not closed under finite intersections. We let

$$B := \{X_b \cap (X \backslash X_c) \mid b, c \in L\}.$$

Since $L$ has $\perp$ and $\top$, the set $B$ contains $S$. Also $B$ is closed under finite intersections. Finally, we define $\tau$ as follows: $U \in \tau$ if $U$ is a union of members of $B$. Then $\tau$ is the smallest topology containing $S$; in the topological terminology, $S$ is a subbasis for $\tau$ and $B$ a basis.

**Theorem 3.2.10.** *Let L be a bounded distributive lattice. Then the prime filter space* $(\mathcal{F}_p(L), \tau)$ *is compact.*

*Proof.* See Davey and Priestley (2002). □

We can now characterize clopen up-sets in the dual space $(\mathcal{F}_p(L), \supseteq, \tau)$ of a bounded distributive lattice $L$.

**Lemma 3.2.11.** *Let $L$ be a bounded distributive lattice with dual space $(X, \supseteq, \tau)$, where $X = \mathcal{F}_p(L)$. Then $X$ is a Priestley space and the clopen up-sets of $X$ are exactly the sets $X_a$ for $a \in L$.*

*Proof.* See Davey and Priestley (2002), Morandi (2005). □

If $(X, \supseteq, \tau)$ is a Priestley space, we denote by $\mathcal{CU}(X, \supseteq, \tau)$ its lattice of clopen upsets.

**Theorem 3.2.12** (Priestley's representation theorem for distributive lattices)**.** *Let $L$ be a bounded distributive lattice. Then the map*

$$\eta_L : a \longmapsto X_a \in \mathcal{CU}(\mathcal{F}_p(L), \supseteq, \tau)$$

*is an isomorphism of $L$ onto the lattice of clopen up-sets of the dual space $(\mathcal{F}_p(L); \supseteq, \tau)$ of $L$.*

*Proof.* See Davey and Priestley (2002), Morandi (2005). □

Our next task is to give a generalization in categorical terms of this result.

Denote the category of bounded distributive lattices and lattice homomorphisms by $\mathbb{D}L$ and the category of Priestley spaces (compact totally order-disconnected spaces) and continuous order-preserving maps by $\mathbb{P}S$. We now define functors from $\mathbb{D}L$ to $\mathbb{P}S$ and viceversa. Define maps $\mathsf{Spec} : \mathbb{D}L \to \mathbb{P}S$ and $\mathsf{Clup} : \mathbb{P}S \to \mathbb{D}L$ on objects by

$$\mathsf{Spec} : L \longmapsto \mathcal{F}_p(L)(L \in \mathbb{D}L) \quad \text{and} \quad \mathsf{Clup} : X \longmapsto \mathcal{CU}(X)(X \in \mathbb{P}S).$$

On maps, let $f : L \to M$ be a lattice homomorphism. Define $\mathsf{Spec}\,(f) : \mathcal{F}_p(M) \to \mathcal{F}_p(L)$ by $\mathsf{Spec}\,(f)(Q) = f^{-1}(Q)$. It is easy to see that this is a well-defined function, since it is order preserving and the pullback of a prime filter under a lattice homomorphism is a prime filter. Moreover,

$$\mathsf{Spec}\,(f)^{-1}(X_b) = \{Q \in \mathcal{F}_p(M) : b \in f^{-1}(Q)\} = \{Q \in \mathcal{F}_p(M) : f(b) \in Q\} = X_{f(b)}.$$

Thus, $\mathsf{Spec}\,(f)$ is continuous. It is elementary to see that $\mathsf{Spec}$ does define a functor.

Next, we have defined $\mathsf{Clup} : \mathbb{P}S \to \mathbb{D}L$ on objects. On maps, if $g : X \to Y$ is a Priestley morphism, define $\mathsf{Clup}\,(g) : \mathcal{CU}(Y) \to \mathcal{CU}(X)$ by $\mathsf{Clup}\,(g)(V) = g^{-1}(V)$. Since a continuous map pulls back clopen up-sets to clopen up-sets, it is easy to see that this is well-defined, and that $\mathsf{Clup}$ is a functor.

Theorem 3.2.12 asserts that, for all $L \in \mathbb{D}L$, $L \simeq \mathsf{Clup}\,\mathsf{Spec}(L)$. It is also necessary to prove that, for all $X \in \mathbb{P}S$, $\mathsf{Spec}\,\mathsf{Clup}(X) \simeq X$.

**Proposition 3.2.13.** *Let $X$ be a Priestley space. The the map*

$$\epsilon_X : x \longmapsto \epsilon_X(x) := \{U \in \mathcal{CU}(X) \,|\, x \in U\}$$

*is an isomorphism of Priestley spaces between $X$ and $\mathcal{F}_p(\mathcal{CU}(X))$.*

*Proof.* See Davey and Priestley (2002), Morandi (2005). □

This proposition, together with Theorem 3.2.12, can be lifted to a dual equivalence of categories between distributive lattices and Priestley spaces.

**Theorem 3.2.14.** *The functors* Spec *and* Clup *give controvariant equivalence of categories between* $\mathbb{D}L$ *and* $\mathbb{P}S$.

*Proof.* See Davey and Priestley (2002), Morandi (2005). □

Let $X, Y$ be Priestley spaces. It is easy to see that $X \coprod Y$ is also a Priestley space. Hence, we have the following useful "translation rules".

**Corollary 3.2.15.** *If* $X, Y \in \mathbb{P}S$, *then*

$$\mathsf{Clup}(X \coprod Y) \simeq \mathsf{Clup}(X) \times \mathsf{Clup}(Y).$$

*Moreover, if* $L, K \in \mathbb{D}L$, *then*

$$\mathsf{Spec}(L \times K) \simeq \mathsf{Spec}(L) \coprod \mathsf{Spec}(K).$$

In the finite case of interest in the following, topology can be dispensed with. In fact, as a consequence of Proposition 2.2.9, the category of finite Priestley spaces and continuous order-preserving maps is equivalent to the category of finite posets and monotone maps via the equivalence between the categories of finite ordered $T_0$ topological spaces and finite posets (see Corollary 2.2.10). So the clopen up-sets are simply the upper sets, as we have noticed before.

Let now $\mathbb{D}L_f$ and $\mathbb{P}os_f$ denote the categories of finite distributive lattices and their homomorphisms, and of finite posets and monotone maps, respectively.

Let $L$ be a finite distributive lattice. Then $p \in L$ is said to be a join prime if $p \leq a \vee b$ implies that $p \leq a$ or $p \leq b$. Let $\mathcal{J}_p(L)$ be the set of join prime elements of $L$. We order $\mathcal{J}_p(L)$ by $p \sqsubseteq q$ if $q \leq p$. Then $(\mathcal{J}_p(L), \sqsubseteq)$ is a poset. Note that $p \sqsubseteq q$ if and only if $\uparrow p \subseteq \uparrow q$.

**Lemma 3.2.16.** *Let* $L$ *be a finite distributive lattice. If* $P$ *is a filter of* $L$, *then* $P$ *is prime if and only if* $P = \uparrow p$ *for some* $p \in \mathcal{J}_p(L)$.

*Proof.* See Davey and Priestley (2002), Morandi (2005). □

As a consequence, $\mathcal{F}_p(L) = \{\uparrow x \mid x \in \mathcal{J}_p(L)\}$. So, we (re)define functors from $\mathbb{D}L_f$ to $\mathbb{P}os_f$ and viceversa. Define maps $\mathsf{Spec} : \mathbb{D}L_f \to \mathbb{P}os_f$ and $\mathsf{Up} : \mathbb{P}os_f \to \mathbb{D}L_f$ on objects by

$$\mathsf{Spec} : L \longmapsto \mathcal{F}_p(L) (L \in \mathbb{D}L_f) \quad \text{and} \quad \mathsf{Up} : X \longmapsto \mathcal{U}p(X) (X \in \mathbb{P}os_f).$$

On maps, let $f : L \to M$ be a lattice homomorphism. Define $\mathsf{Spec}\,(f) : \mathcal{F}_p(M) \to \mathcal{F}_p(L)$ by $\mathsf{Spec}\,(f)(Q) = f^{-1}(Q)$. It is easy to see that this is a well-defined function, since it is order preserving and the pullback of a prime filter under a lattice homomorphism is a prime filter. Thus, Spec does define a functor.

Next, we have defined $\mathsf{Up} : \mathbb{P}os_f \to \mathbb{D}L_f$ on objects. On maps, if $g : X \to Y$ is a monotone morphism, define $\mathsf{Up}\,(g) : \mathcal{U}p(Y) \to \mathcal{U}p(X)$ by $\mathsf{Up}\,(g)(V) = g^{-1}(V)$. Since a monotone map pulls back up-sets to up-sets, it is easy to see that this is well-defined, and that Up is a functor.

Restricted to the finite case, Theorem 3.2.12 and Proposition 3.2.13 assert that, for all $L \in \mathbb{D}L_f$, $L \simeq \mathsf{Up}\,\mathsf{Spec}(L)$ and, for all $X \in \mathbb{P}os_f$, $\mathsf{Spec}\,\mathsf{Up}(X) \simeq X$. Hence, likewise the general case, these results can be lifted to a dual equivalence of categories between finite distributive lattices and finite posets.

**Theorem 3.2.17** (Birkhoff's representation theorem)**.** *The functors* Spec *and* Up *give controvariant equivalence of categories between $\mathbb{D}L_f$ and $\mathbb{P}os_f$.*

*Proof.* See Davey and Priestley (2002), Morandi (2005). □

## 3.3 Heyting Algebras

For $X$ a lattice and $x$ an element of $X$, observe that $x \wedge - : X \to X$, the function whose value at $y$ is $x \wedge y$, is order-preserving. Suppose that $x \wedge -$ has a right adjoint which we will call $x \Rightarrow -$. Then for any pair of elements $y, z$ the definition of adjunction gives $x \wedge y \leq z$ if and only if $y \leq x \Rightarrow z$. It follows that $x \Rightarrow z$ is a largest element whose meet with $x$ is less than or equal to $z$

$$x \Rightarrow z = \bigvee \{a \mid a \wedge x \leq z\}.$$

Given an arbitrary pair $x, z$ in an arbitrary lattice, an element with this property may or may not exist. Saying that $x \wedge -$ has a right adjoint ensures that $x \Rightarrow z$ exists, for all $z$.

**Definition 3.3.1.** A *Heyting Algebra* is a lattice in which, for each $x$, the order-preserving $x \wedge -$ has a right adjoint (which we will denote by $x \Rightarrow -$).

Since we can define a lattice in terms of operations and equations without reference to a previously given order, it is natural to ask if the same holds for Heyting algebras. It does, as the next lemma shows.

**Lemma 3.3.2.** *For $X$ an lattice with a further binary operation, $- \Rightarrow -$, (not a priori satisfying any order conditions) the resulting structure is a Heyting algebra if and only if for all $x, y, z$ in $X$ we have:*

*(i)* $x \Rightarrow x = \top$;

*(ii)* $x \wedge (x \Rightarrow y) = x \wedge y$;

*(iii)* $y \wedge (x \Rightarrow y) = y$;

*(iv)* $x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z)$.

*Proof.* See Johnstone (1982), Borceux (1994), Wood (2004). □

One should immediately take special note of $x \Rightarrow \bot$, a largest element whose meet with $x$ is less than or equal to $\bot$. Since $\bot$ is less than or equal to all elements we have $x \wedge (x \Rightarrow \bot) = \bot$.

**Proposition 3.3.3.** *In a Heyting algebra, putting* $\neg b = b \Rightarrow \bot$ *yields the greatest element such that* $\neg b \wedge b = \bot$*, i.e.*

*(i)* $\neg b = \bigvee\{a \mid a \wedge b = \bot\}$,

*(ii)* $\neg b \wedge b = \bot$.

*The element* $\neg b$ *is called the pseudo-complement of* $b$.

*Proof.* See Borceux (1994). □

**Proposition 3.3.4.** *In a Heyting algebra, the following conditions hold:*

*(1)* $\neg\bot = \top, \quad \bot = \neg\top$,

*(2)* $a \leq b$ *implies* $\neg b \leq \neg a$,

*(3)* $\neg a = \neg\neg\neg a$,

*(4)* $\neg(a \wedge b) \leq \neg a \vee \neg b$

*(5)* $\neg(a \vee b) = \neg a \wedge \neg b$,

*(6)* $\neg a \vee b \leq a \Rightarrow b$.

*for all elements* $a, b$.

*Proof.* See Borceux (1994), Wood (2004). □

We can also see pseudocomplementation as a special case of the following adjunction.

**Proposition 3.3.5.** *For any* $y$ *in a Heyting algebra* $X$, $- \Rightarrow y : X^{op} \longrightarrow X$ *is order-preserving and* $(- \Rightarrow y)^{op} : X \longrightarrow X^{op}$ *is left adjoint to* $- \Rightarrow y$.

*Proof.* See Wood (2004). □

**Corollary 3.3.6.** *For any* $x, y$ *in a Heyting algebra* $X$, $(- \Rightarrow \bot)^{op} \dashv (- \Rightarrow \bot)$. *In other words,* $x \leq \neg y$ *if and only if* $y \leq \neg x$.

*Proof.* See Wood (2004). □

When the inequality (4) in Proposition 3.3.4 above is an isomorphism we say that de Morgan's law holds. A Heyting algebra satisfying the De Morgan laws, which may be considered a weak form of the law of excluded middle (see the proposition below), is called a *De Morgan* or *Stone algebra*.

**Proposition 3.3.7.** *For a Heyting algebra H the following conditions are equivalent:*

1. *For all* $a, b \in H$*:* $\neg(a \wedge b) = \neg a \vee \neg b$*;*
2. *For all* $a \in H$*:* $\neg a \vee \neg\neg a = \top$*; (weak excluded middle law)*
3. *For all* $a, b \in H$*:* $\neg\neg(a \vee b) = \neg\neg a \vee \neg\neg b$*;*
4. $X$ *is a De Morgan algebra.*

*Proof.* See Borceux (1994). □

Now, in the light of these observations, one is naturally tempted to ask if to be an Heyting algebra implies to be a distributive lattice. That is the case and will be established suddenly. Moreover, finite distributiveness implies Heyting.

**Proposition 3.3.8.** *If* $X$ *is a Heyting algebra then* $X$ *is distributive.*

*Proof.* See Borceux (1994), Wood (2004). □

In a Heyting algebra, since each $x \wedge -$ preserves any suprema that exist we also have $x \wedge (\bigvee S) = \bigvee\{x \wedge s \mid s \in S\}$ whenever $\bigvee S$ exists. As immediate consequence, we have the following.

**Proposition 3.3.9.** *A finite distributive lattice is a Heyting algebra.*

*Proof.* See Wood (2004). □

Next we consider the even more special case of a boolean algebra.

**Proposition 3.3.10.** *For a Heyting algebra* $X$*, the following are equivalent:*

1. $(- \Rightarrow \bot)^{op} : X \to X^{op}$ *is also right adjoint to* $(- \Rightarrow \bot) : X^{op} \to X$*,*
2. *for all* $x$*,* $\neg\neg x = x$*,*
3. *for all* $x$*,* $x \vee \neg x = \top$*,*
4. $X$ *is a Boolean algebra.*

*Proof.* See Wood (2004). □

There are several ways of passing back and forth between Boolean algebras and Heyting algebras, often having to do with the double negation operator. A useful lemma in this regard is the following.

**Proposition 3.3.11.** *Let $H$ be a Heyting algebra. The double negation mapping*

$$\neg\neg : H \to H, \quad a \mapsto \neg\neg a,$$

*satisfies the following conditions:*

*(i)* $a \leq b$ *implies* $\neg\neg a \leq \neg\neg b$;

*(ii)* $a \leq \neg\neg a$;

*(iii)* $\neg\neg\bot = \bot$, $\neg\neg\top = \top$;

*(iv)* $\neg\neg\neg\neg a = \neg\neg a$;

*(v)* $\neg\neg(a \wedge b) = \neg\neg a \wedge \neg\neg b$;

*(vi)* $\neg\neg(a \Rightarrow b) = \neg\neg a \Rightarrow \neg\neg b$.

*Proof.* See Borceux (1994). □

**Corollary 3.3.12.** *The double negation $\neg\neg\colon L \to L$ is a reflector that preserves finite meets.*

Now let $L_{\neg\neg}$ denote the poset of *regular elements* of $L$, namely those elements $x \in L$ such that $\neg\neg x = x$. With the help of the proposition 3.3.11 above, the next one follows easily.

**Proposition 3.3.13.** *the poset $H_{\neg\neg}$ of regular elements of a Heyting algebra $H$ constitutes a boolean algebra (though it is not in general a sublattice of $H$).*

*Proof.* See Borceux (1994), Johnstone (1982). □

### 3.3.1 Nuclei on Heyting Algebras

We have just seen that double negation is an example of reflector that preserves finite meets, so we take now a closer look to the properties of nuclei or local operators, namely closure operators that preserve finite meets as well, that can be always associated to meet-preserving reflectors (see Lemma 2.1.36 and Proposition 3.1.32).

**Definition 3.3.14.** Let $L$ be a lattice. A *nucleus* on $L$ is a map $j\colon L \to L$ that satisfies the following identities:

(1) $a \leq j(a)$,

(2) $j(j(a)) = j(a)$,

(3) $j(a \wedge b) = j(a) \wedge j(b)$.

In other words, a nucleus on $L$ is a meet-preserving closure operator on $L$.

Note that the following properties of a nucleus might be included in the definition, but they follow from the above:

(1′) $j(\top) = \top$,

(2′) $a \leq b$ implies $j(a) \leq j(b)$.

Let $L$ be a lattice. Since a nucleus $j$ on $L$ is a kind of closure operator on a poset, we say that an element $a$ of $L$ is *$j$-closed* if $j(a) = a$. Further, a nucleus $j$ is determined by its image or, equivalently, by (2), its fixset $L_j = \{a \in L \mid j(a) = a\}$, since conditions (2) and (3) plus the fact that $j$ is order-preserving say that the corresponding reflector is left adjoint to the inclusion $i : L_j \hookrightarrow L$.

We begin with a simple but important result.

**Proposition 3.3.15.** *If $i : X \hookrightarrow L$ is a reflective embedding with left adjoint $l$ and $L$ is a Heyting algebra then $l$ preserves finite meets if and only if, for all $x \in X$ and $a \in L$, $a \Rightarrow x$ is in $X$.*

*Proof.* See Wood (2004). □

**Corollary 3.3.16.** *If $i : X \hookrightarrow L$ is a reflective embedding with meet-preserving reflector and $L$ is a Heyting algebra then $X$ is a Heyting algebra.*

Hence, in the context defined by the proposition above, condition (3) on $j$ is equivalent to the assertion that $A_j$ is an *exponential ideal*, namely that $(a \Rightarrow b) \in A_j$ whenever $b \in A_j$, where $\Rightarrow$ is the Heyting implication in $A$. We have thus established the following proposition which is a restatement of Proposition 3.1.35 for (complete) Heyting algebras and nuclei.

**Proposition 3.3.17.** *For any (complete) Heyting algebra $H$, there are bijections between any two of the following sets:*

1. *the set of nuclei on $A$;*
2. *the set of fixsets in $A$, i.e. subsets which are exponential ideals (and closed under arbitrary meets).*

*Proof.* See Johnstone (2002). □

As a consequence, we may equivalently define a nucleus on a complete Heyting algebra $H$ to be a subset $J$ of $H$ that satisfies certain conditions, namely these identities:

(1) $\bigwedge A \in J$ whenever $A \subseteq J$ (using that $H$ is a complete lattice),

(2) $a \Rightarrow b \in J$ whenever $b \in J$ (using that $H$ is a Heyting algebra).

Then we recover $j \colon H \to H$ by

$$j(a) := \bigwedge \{b \in H \mid b \in J,\ a \leq b\}$$

and we have

$$J = \{a \in H \mid j(a) = a\}.$$

### 3.3.2 The Categories $\mathbb{H}eyt$ and $\mathbb{B}ool$

In Subsction 3.2.3 we have introduced the category $\mathbb{D}L$ of distributive lattices and their homomorphisms. At this point we can define another category which will be useful.

Let $\mathbb{H}eyt$ be the category whose objects are Heyting algebras and whose morphisms are lattice homomorphisms which preserve implication; these maps are called Heyting morphisms. The following result concerning Heyting morphims is crucial.

**Proposition 3.3.18.** *Suppose that $f : X \longrightarrow Y$ is a monotone, binary meet preserving function between ordered sets with binary meets and implications. Suppose also that $f$ has a left adjoint $l : Y \longrightarrow X$. Then $f$ preserves implications if and only if $l \dashv f$ satisfies the following condition ("Frobenius Reciprocity"): for all $x \in X$ and $y \in Y$,*

$$l(y \wedge f(x)) = l(y) \wedge x.$$

*Proof.* See Pitts (1989). □

**Corollary 3.3.19.** *Suppose that $f : X \longrightarrow Y$ is a monotone, binary meet preserving function between ordered sets with binary meets and implications. Suppose also that $f$ has a left adjoint $l : Y \longrightarrow X$.*

*(1) If $f$ preserves implications and $l$ preserves $\top$, then $f$ is an order embedding;*

*(2) If $f$ is an order embedding and $l$ preserves finite meets, then $f$ preserves implications.*

*Proof.* See Johnstone (2002). □

Point (2) is basically a restatement of Corollary 3.3.16.

Also, let $\mathbb{B}ool$ be the category whose objects are boolean algebras and whose morphisms are lattice homomorphisms; these maps preserve boolean negation. It can be proved the following theorem.

**Theorem 3.3.20.** *The assignment $H \longrightarrow H_{\neg\neg}$ gives rise to a reflection*

$$\mathbb{H}eyt \underset{i}{\overset{b}{\rightleftarrows}} \mathbb{B}ool$$

*In particular, $\mathbb{B}ool$ is a reflective subcategory of $\mathbb{H}eyt$.*

*Proof.* See Balbes and Dwinger (2011). □

**Examples 3.3.21.**

(1) Any powerset lattice $\mathcal{P}(X)$ is a Heyting algebra, with implication defined by $A \Rightarrow B := \complement A \cup B$. For every element $C \in \mathcal{P}(X)$, we must prove that

$$C \cap A \subseteq B \quad \text{if and only if} \quad C \subseteq \complement A \cup B.$$

Since $\mathcal{P}(X)$ is a distributive lattice $C \cap A \subseteq B$ implies

$$C = C \cap (A \cup \complement A) = (C \cap A) \cup (C \cap \complement A) \subseteq B \cup \complement A.$$

Conversely from $C \subseteq \complement A \cup B$ we deduce

$$C \cap A \subseteq (\complement A \cup B) \cap A = (\complement A \cap A) \cup (B \cap A) = B \cap A \subseteq B.$$

In particular $\mathcal{P}(X)$ is boolean and, for every $A$ in $\mathcal{P}(X)$, $\neg A = \complement A$. Moreover, if $f : X \to Y$ is a function then $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ as a lattice morphism is a boolean algebra morphism as well (since it has both left and right adjoints, hence preserves all meets and joins, and $f^{-1}(\complement A) = \complement f^{-1}(A)$).

(2) Any (bounded) chain $A$ is a Heyting algebra, with implication defined by:

$$a \Rightarrow b = \begin{cases} \top, & \text{if } a \leq b, \\ b, & \text{otherwise.} \end{cases}$$

However $A$ is not boolean, in fact every $a \neq \bot$ in $A$ satisfies $\neg\neg a = \top$. Since $\neg a = \bot$, for every $a \neq \bot$, it turns out that $A$ is a De Morgan algebra.

### 3.3.3 $\mathcal{D}o(X)$, $\mathcal{U}p(X)$ and $\mathcal{O}(X)$ as Heyting Algebras

$\mathcal{D}o(X)$ and $\mathcal{U}p(X)$ are Heyting algebras for any ordered set $X$; however, it does not follow from Corollary 3.3.16 because the reflectors $\downarrow : \mathcal{P}(X) \to \mathcal{D}o(X)$ and $\uparrow : \mathcal{P}(X) \to \mathcal{U}p(X)$ do not preserve meets. For instance, suppose that in $X$ we have distinct $x$ and $y$ with a lower bound $b$. Now $\downarrow(\{x\} \cap \{y\}) = \emptyset$ but $b$ is in $\downarrow\{x\} \cap \downarrow\{y\}$ so that the binary meet comparison inequality for $\downarrow$ is strict.

As we shall see more generally from the next Proposition, for a topological space $X$ we have that $\mathcal{O}(X)$ is a Heyting algebra. Actually, for a topological space $X$, the inclusion $i : \mathcal{O}(X) \hookrightarrow \mathcal{P}(X)$ is a coreflective order embedding with the coreflector given by the interior operator (see Subection 3.1.5), and it is not difficult to see that $i$ preserves finite meets.

**Proposition 3.3.22.** *If $i : X \hookrightarrow L$ is coreflective embedding with coreflector $r$ and $L$ is a Heyting algebra, then $r(ix \Rightarrow iy)$ provides a Heyting implication for $X$ if and only if $i$ preserves meets.*

*Proof.* See Wood (2004). □

So, for a given topological space $X$, the lattice of open sets $\mathcal{O}(X)$ is enriched with a Heyting structure via the interior operator $\mathsf{int} : \mathcal{P}(X) \to \mathcal{O}(X)$, that preserves the top element and meets. For an ordered set $X$, downward and upward interior operators

$\Downarrow, \Uparrow$ provide such interior operators on $\mathcal{P}(X)$, with the extra property that they preserve arbitrary infima (see examples 2.1.38(2) and 2.1.38(4)).

**Corollary 3.3.23.** *For any topological space $X$, $\mathcal{O}(X)$ is a Heyting algebra and, in particular, for any ordered set $X$, $\mathcal{D}o(X)$ and $\mathcal{U}p(X)$ are Heyting algebras.*

**Examples 3.3.24.**

(1) As we have already said, if $X$ is any topological space, $\mathcal{O}(X)$ is a complete distributive lattice, bounded above by $X$ and below by $\emptyset$, with joins given by set-theoretic unions and meets given by

$$\bigwedge_{i\in I} A_i = \mathsf{int}(\bigcap_{i\in I} A_i)$$

for any family $\mathcal{K}$ of open subsets of $X$. Therefore $\mathcal{O}(X)$ has exactly one structure of Heyting algebra compatible with its distributive-lattice structure; namely, for any $U, V \in \mathcal{O}(X)$ the Heyting implication is given by

$$U \Rightarrow V := \mathsf{int}((X \backslash U) \cup V)). \tag{3.14}$$

In particular, the Heyting negation is given by

$$\neg U := \mathsf{int}(X \backslash U).$$

(2) As we have already seen, if $X$ is an ordered set, $\mathcal{D}o(X)$ and $\mathcal{U}p(X)$ are complete distributive lattices, bounded above by $X$ and below by $\emptyset$, with joins given by set-theoretic unions and meets. Therefore each of them have exactly one structure of Heyting algebra compatible with their distributive-lattice structure; namely, for any $U, V \in \mathcal{D}o(X)$ the Heyting implication is given by

$$U \Rightarrow V := \complement \uparrow \complement((X \backslash U) \cup V)) = \complement \uparrow (U \cap \complement V).$$

And for any $U, V \in \mathcal{U}p(X)$ it is given by

$$U \Rightarrow V := \complement \downarrow \complement((X \backslash U) \cup V)) = \complement \downarrow (U \cap \complement V).$$

In particular, their corresponding Heyting negations are given by

$$\neg U := \complement \downarrow U \quad \text{and} \quad \neg U := \complement \uparrow U.$$

(See Remark 2.1.39).

### 3.3.4 Relation Between Openness and Heyting Implication

We shall need the following result from topology.

**Lemma 3.3.25.** *For a continuous map $f : X \to Y$ of topological spaces, the following are equivalent:*

*(1) $f$ is an open map.*

*(2) The lattice homomorphism $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ has a left adjoint $f$, which satisfies the Frobenius reciprocity condition that*

$$f[U \cap f^{-1}(V)] = f[U] \cap V.$$

*for all $U \in \mathcal{O}(X), V \in \mathcal{O}(Y)$.*

*(3) $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a homomorphism of complete Heyting algebras; i.e. it preserves arbitrary meets and the Heyting implication operation.*

*Proof.* See Johnstone (2002). □

**Corollary 3.3.26.** *For a monotone map $f : X \to Y$ of posets, the following are equivalent:*

*(1) $f$ is an p-morphism.*

*(2) The lattice homomorphism $f^{-1} : \mathcal{U}p(Y) \to \mathcal{U}p(X)$ has a left adjoint $f$, which satisfies the Frobenius reciprocity condition that*

$$f[U \cap f^{-1}(V)] = f[U] \cap V.$$

*for all $U \in \mathcal{U}p(X), V \in \mathcal{U}p(Y)$.*

*(3) $f^{-1} : \mathcal{U}p(Y) \to \mathcal{U}p(X)$ is a homomorphism of complete Heyting algebras; i.e. it preserves arbitrary meets and the Heyting implication.*

Note also that the last Corollary can be restated equivalently in terms of $\mathcal{D}o(X)$.

### 3.3.5 Co-Heyting Algebras

Just like the lattice of open subsets of a topological space is the primeval example of a Heyting algebra, its dual lattice of closed subsets is the primeval example of a co-Heyting algebra.

In general, co-Heyting algebras are dual to Heyting algebras and, like them, they are equipped with non-Boolean logical operators that make them very interesting.

**Definition 3.3.27.** A *co-Heyting algebra* is a bounded distributive lattice $L$ equipped with a binary *subtraction* operation $\Leftarrow: L \times L \to L$ such that $x \Leftarrow y \leq z$ iff $x \leq y \vee z$.

Existence of $\Leftarrow$ as left adjoint implies that $y \vee -$ preserves meets, hence the assumption of distributivity in the definition is redundant and has been put in for emphasis only. Co-Heyting algebras were initially called *Brouwerian algebras*.

Also a *bi-Heyting algebra* is a bounded distributive lattice $L$ that carries a Heyting algebra structure with implication $\Rightarrow$ and a co-Heyting algebra structure with subtraction $\Leftarrow$.

Clearly, there is also an alternative equational definition for co-Heyting algebras. It is dual to the equational definition for Heyting algebras given by Lemma 3.3.2 above.

**Examples 3.3.28.**

(1) The lattice of closed subsets of a topological space is a co-Heyting algebra with $X \Leftarrow Y = \mathsf{cl}(X \cap \complement Y)$.

(2) A Boolean algebra provides a (degenerate) example of a bi-Heyting algebra by setting $x \Rightarrow y := \neg x \vee y$ and $x \Leftarrow y := x \wedge \neg y$.

**Remark 3.3.29.**

(a) $a \Leftarrow b = \bot$ iff $a \Leftarrow b \leq \bot$ iff $a \leq b \vee \bot$ iff $a \leq b$. In particular, $a \Leftarrow a = \bot$.

(b) As $- \Leftarrow x$ has a right adjoint it preserves joins hence: $(a \vee b) \Leftarrow x = (a \Leftarrow x) \vee (b \Leftarrow x)$.

(c) $a \Leftarrow \bot \leq a \Leftarrow \bot$ iff $a \leq \bot \vee (a \Leftarrow \bot)$ iff $a \leq a \Leftarrow \bot$. On the other hand, $a \leq \bot \vee a$ and the adjunction yield $a \Leftarrow \bot \leq a$, hence $a \Leftarrow \bot = a$.

(d) Suppose $a \leq b \vee x$ then $a \Leftarrow b \leq x$. As from $a \Leftarrow b \leq a \Leftarrow b$ follows $a \leq b \vee (a \Leftarrow b)$, hence $a \Leftarrow b = \bigwedge\{x | a \leq b \vee x\}$.

**Definition 3.3.30.** The subtraction operation permits to define the *co-Heyting negation* $\ulcorner\colon L \longrightarrow L$ by setting

$$\ulcorner a := \top \Leftarrow a.$$

Co-Heyting nagation $\ulcorner a$ is thus characterized as the smallest element in the lattice for which $\ulcorner a \vee a = \top$. It always holds $\ulcorner (a \wedge b) = \ulcorner a \vee \ulcorner b$, but it can happen that $\ulcorner (a \vee b) \neq \ulcorner a \wedge \ulcorner b$; however, $\ulcorner\ulcorner (a \vee b) = \ulcorner\ulcorner a \vee \ulcorner\ulcorner b$ always holds. More in general, all the conditions obtained by dualizing the ones in Proposition 3.3.4 and 3.3.11 are always valid in a co-Heyting algebras. For instance, $\ulcorner\ulcorner a \leq a$.

**Definition 3.3.31.** Co-Heyting negation operator $\ulcorner$ in turn can then be used to define the *co-Heyting boundary operator* $\partial : L \longrightarrow L$ by

$$\partial a := a \wedge \ulcorner a.$$

That $\partial a$ is not necessary trivial is dual to the non-validity of the *tertium non datur*, or its equivalent conditions given by Proposition 3.3.10, for general Heyting algebras.

A great many useful identities can be proved in general for any co-Heyting algebra (see Lawvere 1986, Reyes *et al.* 2004). For example

$$a = \ulcorner\ulcorner a \vee \partial a,$$

for all $a \in L$, and

$$\partial(a \wedge b) = (\partial a \wedge b) \vee (a \wedge \partial b),$$

for all $a, b \in L$ (Leibniz rule). Also evident and valid in any co-Heyting algebra is

$$\partial(a \vee b) \vee \partial(a \wedge b) = \partial a \vee \partial b.$$

The boundary elements $x = \partial a$ can be characterized as those $x$ with $\partial x = x$ or, equivalently, with $\llcorner\, x = \top$. In particular, $\partial^2 a = \partial a$.

Similarly to Heyting algebras, co-Heyting algebras constitute a category by defining a co-Heyting morphism to be a lattice morphism which preserves subtraction operation $\Leftarrow$.

Let $co\mathbb{H}eyt$ be the category of co-Heyting algebras and their morphisms. By dualizing Proposition 3.3.13 and Theorem 3.3.20, it can be proven that $\mathbb{B}ool$ is a coreflective subcategory of $co\mathbb{H}eyt$.

### 3.3.6 Representation of Heyting Algebras: Esakia Duality

In this section we extend Priestley duality to the category of Heyting algebras. Let $\mathbb{H}eyt$ be the category whose objects are Heyting algebras and their homomorphisms. We wish to restrict Priestley duality to the category $\mathbb{H}eyt$; we thus need to determine which Priestley spaces are duals of Heyting algebras, and which morphisms of such spaces are dual to Heyting morphisms.

An Esakia space is a Priestley space $(X, \leq)$ such that if $U$ is clopen in $X$, then $\downarrow U$ is clopen. Alternatively, $(X, \leq)$ is a Esakia space if for every open set $U$, then downset $\downarrow U$ is open. The equivalence of these conditions follows from the Lemma below. A morphism of Esakia spaces is a continuous p-morphism. The category $\mathbb{E}S$ consists of all Esakia spaces and their morphisms. In this section we will see that Esakia spaces are exactly those Priestley spaces which are dual to Heyting algebras, and that Priestley duality restricts to a duality between $\mathbb{H}eyt$ and $\mathbb{E}S$. We start with some preliminary lemmas.

**Lemma 3.3.32.** *Let $(X, \leq)$ be a Priestley space.*

1. *The set $\leq$ is a closed subset of $X \times X$.*
2. *If $C$ is closed in $X$, then $\uparrow C$ and $\downarrow C$ are closed in $X$.*

*Proof.* See Morandi (2005). □

**Lemma 3.3.33.** *Let $H$ be a Heyting algebra. If $a, b \in H$, then $\downarrow(\eta_H(a) \cap \complement\, \eta_H(b)) = \complement\, \eta(a \Rightarrow b)$, where $\eta$ is given by $\eta_H(a) = \{F \in \mathcal{F}_p(H) \,|\, a \in F\}$, for all $a \in H$.*

*Proof.* See Morandi (2005). □

We now consider the functor $\mathsf{Spec} : \mathbb{D}L \to \mathbb{E}S$, but restricted to $\mathbb{H}eyt$.

**Lemma 3.3.34.** *If $H$ is a Heyting algebra, then $(\mathsf{Spec}(H), \subseteq)$ is an Esakia space.*

*Proof.* See Morandi (2005). □

**Lemma 3.3.35.** *Let $f : H \to H'$ be a Heyting morphism. Then* $\mathsf{Spec}(f) : \mathcal{F}_p(H') \to \mathcal{F}_p(H)$ *is a p-morphism.*

*Proof.* See Morandi (2005). □

The last two lemmas show that $\mathsf{Spec}$ is a functor from $\mathbb{H}eyt$ to $\mathbb{E}S$. We now consider the restriction of the functor $\mathsf{Clup} : \mathbb{E}S \to \mathbb{D}L$ restricted to $\mathbb{E}S$.

**Lemma 3.3.36.** *Let $(X, \leq)$ be an Esakia space. Then $\mathcal{CU}(X, \leq)$ is a Heyting algebra, where implication is defined by $U \Rightarrow V = \complement \downarrow(U \cap \complement V)$.*

*Proof.* See Morandi (2005). □

**Lemma 3.3.37.** *Let $g : (X, \leq) \to (Y, \leq)$ be a morphism of Esakia spaces. Then $g^{-1} : \mathcal{CU}(Y, \leq) \to \mathcal{CU}(X, \leq)$ is a Heyting morphism.*

*Proof.* See Morandi (2005). □

Thus, $\mathsf{Clup}$ is a functor from $\mathbb{E}S$ to $\mathbb{H}eyt$. We now see that these categories are dual to each other. Much of the work involved we did verifying Priestley duality.

**Proposition 3.3.38** (Esakia Representation Theorem)**.** *Let $H$ be a Heyting algebra. Then there is a Heyting isomorphism $\eta_H : H \to \mathcal{CU}(\mathcal{F}_p(H))$, given by $\eta_H(h) = \{P \in \mathcal{F}_p(H) : h \in P\}$.*

*Proof.* See Morandi (2005). □

**Proposition 3.3.39.** *Let $(X, \leq)$ be an Esakia space. Then there is an isomorphism of Esakia spaces $\epsilon_X : (X, \leq) \to \mathcal{F}_p(\mathcal{CU}(X, \leq), \subseteq)$, given by $\epsilon_X(x) = \{U \in \mathcal{CU}(X, \leq) : x \in U\}$.*

*Proof.* See Morandi (2005). □

**Theorem 3.3.40.** *The functors* $\mathsf{Clup}$ *and* $\mathsf{Spec}$ *give a co-equivalence of categories between $\mathbb{H}eyt$ and $\mathbb{E}S$.*

*Proof.* See Morandi (2005). □

As we have noticed before, in the finite case of interest in the following, topology can be dispensed with. In fact, as a consequence of Proposition 2.2.9, the category of finite Esakia spaces and continuous p-morphisms is equivalent to the category of finite posets and p-morphism via the equivalence between the categories of finite ordered $T_0$ topological spaces and finite posets (see Corollary 2.2.10). So the clopen up-sets are again the upper sets as in the case of distributive lattices.

Let now $\mathbb{H}eyt_f$ and $p\mathbb{P}os_f$ denote the categories of finite Heyting algebras and their homomorphisms, and of finite posets and p-morphisms, respectively. Remember from Proposition 3.3.9 that finite Heyting algebras are nothing but finite distributive lattices. As a consequence, given a finite Heyting algebra $H$, we have that $\mathcal{F}_p(H) = \{\uparrow x \mid x \in \mathcal{J}_p(H)\}$, where $\mathcal{J}_p(H)$ is the set of join prime elements of $H$. So, we (re)define functors from $\mathbb{H}eyt_f$ to $p\mathbb{P}os_f$ and viceversa. Define maps $\mathsf{Spec} : \mathbb{H}eyt_f \to p\mathbb{P}os_f$ and $\mathsf{Up} : p\mathbb{P}os_f \to \mathbb{H}eyt_f$ on objects by

$$\mathsf{Spec} : H \longmapsto \mathcal{F}_p(H)(H \in \mathbb{H}eyt_f) \quad \text{and} \quad \mathsf{Up} : X \longmapsto \mathcal{U}p(X)(X \in p\mathbb{P}os_f).$$

On maps, let $f : H \to M$ be a Heyting homomorphism. Lemma 3.3.35 assures once more that $\mathsf{Spec}(f) : \mathcal{F}_p(M) \to \mathcal{F}_p(H)$ is a p-morphism. Thus, $\mathsf{Spec}$ does define a functor.

Next, we have defined $\mathsf{Up} : p\mathbb{P}os_f \to \mathbb{H}eyt_f$ on objects. On maps, if $g : X \to Y$ is a p-morphism, from Lemma 3.3.37 or Corollary 3.3.26 it follows that $\mathsf{Up}(g) : \mathcal{U}p(Y) \to \mathcal{U}p(X)$ is a Heyting morphism and so $\mathsf{Up}$ is a functor.

Restricted to the finite case, Propositions 3.3.38 and 3.3.39 assert that, for all $H \in \mathbb{H}eyt_f$, $H \simeq \mathsf{Up}\,\mathsf{Spec}(H)$ and, for all $X \in p\mathbb{P}os_f$, $\mathsf{Spec}\,\mathsf{Up}(X) \simeq X$. Hence, likewise the general case, these results can be lifted to a dual equivalence between the category of finite Heyting algebras and the category of finite posets and p-morphisms.

**Theorem 3.3.41.** *The functors* $\mathsf{Spec}$ *and* $\mathsf{Up}$ *give controvariant equivalence of categories between* $\mathbb{H}eyt_f$ *and* $p\mathbb{P}os_f$*.*

### 3.3.7 Relevant Consequences of Esakia Duality

A straightforward consequence of Esakia duality is the following result that spells out the connection between homomorphisms, subalgebras and finite products of Heyting algebras with closed up-sets, continuous p-morphisms and finite disjoint unions of Esakia spaces. These “translation rules” between algebras and spaces turn out very useful in the applications to logic that we shall see in Chapter 4.

**Theorem 3.3.42.** *Let $A$ and $B$ be Heyting algebras and $F$ and $G$ be Esakia spaces. Let also $\{A_i\}_{i\in I}$ and $\{F_i\}_{i\in J}$ be finite families of Heyting algebras and Esakia spaces, respectively. Then*

(1) (a) *$A$ is a homomorphic image of $B$ if and only if $\mathsf{Spec}\,A$ is a closed up-set of $\mathsf{Spec}\,B$.*

(b) *$A$ is a subalgebra of $B$ if and only if $\mathsf{Spec}\,A$ is a continuous p-morphic image of $\mathsf{Spec}\,B$.*

(c) *$\mathsf{Spec}\,\prod_{i\in I} A_i$ is isomorphic to the finite disjoint union $\coprod_{i\in I} \mathsf{Spec}\,A_i$.*

(2) (a) *$F$ is a closed up-set of $G$ if and only if $\mathsf{Clup}\,F$ is a homomorphic image of $\mathsf{Clup}\,G$.*

(b) *$F$ is a continuous p-morphic image of $G$ if and only if $\mathsf{Clup}\,F$ is a subalgebra of $\mathsf{Clup}\,G$.*

(c) *$\mathsf{Clup}\,\coprod_{i\in J} F_i$ is isomorphic to the finite product $\prod_{i\in J} \mathsf{Clup}\,F_i$.*

□

**Weak Representation Theorems**

Also of great interest for the applications to logic that we shall see in Chapter 4 are the following weak representation theorems for Heyting algebras due to Tarski, Stone and Kripke. In the light of Esakia duality, they can be seen as the result of the restriction of Esakia's spaces either to their topological side or to their ordered one.

**Theorem 3.3.43** (Tarski-Stone Representation)**.** *Every Heyting algebra can be embedded into the Heyting algebra of open sets of some topological space.*

□

In order to grasp the relationship with Esakia duality, we have just to consider, for every Heyting algebra $H$, the set $X := \mathcal{F}_p(H)$ of prime filters of $H$ and the map $\eta(a) = \{P \in \mathcal{F}_p(H) : a \in P\}$, as usual. Let $\mathcal{O}(X)$ be a topology on $X$ generated by the basis $\mathfrak{B} = \{ \eta(a) \mid a \in H \}$. It easy to see map $\eta : H \to \mathcal{O}(X)$ gives rise to a Heyting algebra embedding.

In a very similar manner, it can be established the following well-known result.

**Theorem 3.3.44** (Kripke Representation)**.** *Every Heyting algebra can be embedded into the Heyting algebra of up-sets of some poset.*

□

In fact, for every Heyting algebra $H$, consider again the set $X$ of prime filters of $H$ and the map $\eta$. Let $\mathcal{U}p(X)$ be the Heyting algebra of up-sets of $X$. Clearly, the map $\eta : H \to \mathcal{U}p(X)$ is a Heyting algebra embedding.

# 4 Model theory for Algebras

In this Chapter we shall investigate a general approach to those algebraic structures, such as lattices, boolean algebras, Heyting algebras, whose basic concepts are being developed in Chapter 3. All those structures are characterized by the existence of several operation which are defined everywhere and satisfy axioms expressed by equations. In the last part of the chapter we shall see how to apply this general framework to intuitionistic and intermediate logics.

## 4.1 $\tau$-Algebras and Varieties

### 4.1.1 Algebraic Language or Similarity Type

We need an appropriate language if we want to describe classes of algebras of a given "type" by logical expressions. This formal language is built up first by choosing a denumerable set of variables $x, y, z, ...$. Also, we need certain non logical symbols denoting basic operations of various kinds. This data constitutes the similarity type of the language.

**Definition 4.1.1.** An *algebraic language* or *similarity type* is a pair $\tau := \langle \mathcal{F}, \mathsf{ar} \rangle$ consisting of a set $\mathcal{F}$ of operation symbols and a function $\mathsf{ar} : \mathcal{F} \to \mathbb{N}$ assigning a non-negative integer, called *arity*, to every operation symbols. We will say that $f \in \mathcal{F}$ is an *n-ary operation symbol* when $\mathsf{ar}(f) = n$. A 0-ary operation symbol is called a *constant symbol*.

**Examples 4.1.2.**

1. The constant symbols that appeared more often in Chapter 3 are $\top$ and $\bot$.
2. Instead, examples of unary operations that we have already met are $\neg$ and $\ulcorner$; among binary operations, $\wedge, \vee, \Rightarrow$ and $\Leftarrow$ are the more commonly used above.

While in general there is no restriction on the cardinality of $\mathcal{F}$, practically all the examples that we shall see use a finite language.

In the specification of particular languages, it is customary to describe a similarity type $\tau$ by a sequence of the symbols actually used together with the sequence of their arities; for instance, in the following we shall often use the language $\langle \wedge, \vee, \Rightarrow, \top, \bot \rangle$ of "type" $\langle 2, 2, 2, 0, 0 \rangle$.

### 4.1.2 Interpretations of the Language: $\tau$-Algebras

**Definition 4.1.3.** We define *$\tau$-algebra* or *algebra of (similarity) type* $\tau$ as an interpretation $\mathfrak{A}$ of $\tau$ in $\mathbb{S}et$, the category of sets and functions. By this we mean:

1. a set $A$, called the *universe* of the algebra; and
2. for each operation symbol $f \in \mathcal{F}$ of arity $k$, a function $f^A : A^k \to A$ from the $n$-fold cartesian product of $A$ into $A$. In particular, constant symbols $c$ are interpreted as functions $c^A : \{*\} \to A$, namely as elements of $A$.

More succintly, the $\tau$-algebra $\mathfrak{A}$ is the tuple $\langle A, \langle f^A : f \in \mathcal{F} \rangle\rangle$ consisting of an underlying set $A$ and the interpretations $f^A$.

**Definition 4.1.4.** If $\mathfrak{X}$ and $\mathfrak{Y}$ are $\tau$-algebras, by *homomorphism* of $\mathfrak{X}$ in $\mathfrak{Y}$ we mean a function $h : \mathfrak{X} \to \mathfrak{Y}$ such that for all $f \in \mathcal{F}$ we have $h \circ f^X = f^Y \circ h^n$, namely the diagram

$$\begin{tikzcd}
\mathfrak{X}^n \arrow[r, "f^X"] \arrow[d, "h^n"'] & \mathfrak{X} \arrow[d, "h"] \\
\mathfrak{Y}^n \arrow[r, "f^Y"'] & \mathfrak{Y}
\end{tikzcd}$$

commutes.

The $\tau$-algebras and their morphisms form a category. We shall indicate that category with $\mathbb{A}lg_\tau$.

If we consider a type $\tau$ that contains only one operation symbol of arity 2, then we have that the $\tau$-algebras are the sets equipped with a binary operation and the morphisms between $\tau$-algebras are the homomorphisms that are usually considered in algebra. Monoids, groups, lattices, boolean algebras, Heyting algebras, etc., they can be seen as particular $\tau$-algebras (which satisfy some axioms) for suitable types $\tau$, and then the concept of morphism is particularized to the usual concept of homomorphism.

We observe that in general a class of algebraic structures, for example groups, does not uniquely identify a similarity type. In fact, groups can be thought of as particular sets with a binary operation, the product, or with three operations: a binary operation, the product, a 0-ary one, the identity element, and a unary one, the inverse.

As we shall see in the following, the c of $\tau$-algebra and morphisms is also useful for logic. In fact the sentences of an intuitionistic propositional language can be seen as $\tau$-algebras in which the symbols of operation are the connectives $\langle \wedge, \vee, \Rightarrow, \top, \bot \rangle$. Hence, It turns out that the concept of interpretation for propositional logic coincides with that of morphism from the algebra of the sentences to an appropriate algebra (or class of algebras).

### 4.1.3 Subalgebras and Generated Subalgebras

For a given algebra $\mathfrak{B}$ of type $\tau$ we obtain new algebras using certain algebraic constructions. The first algebraic construction we want to mention, is the formation of subalgebras.

**Definition 4.1.5.** Let $\mathfrak{A}$ be a $\tau$-algebra. A subset of the universe $A$, which is closed with respect to each fundamental operation of $\mathfrak{A}$, is called a *subuniverse* of $A$. The $\tau$-algebra $\mathfrak{B}$ is said to be a *subalgebra* of $\mathfrak{A}$ if and only if $\mathfrak{A}$ and $\mathfrak{B}$ have the same type, $B$ is a subuniverse of $A$, and $f^B$ is the restriction to $B$ of $f^A$, for each operation symbol $f \in \mathcal{F}$. $\mathsf{Sub}(\mathfrak{A})$ denotes the set of all subuniverses of $\mathfrak{A}$.

It is almost immediate to see that there is a bijective correspondence between subalgebras and non-empty subuniverses.

If $\mathfrak{B}$ is an algebra of type $\tau$ and if $\{\mathfrak{A}_j\}_{j\in J}$ is a family of subalgebras of $\mathfrak{B}$ with the non-empty intersection $A := \bigcap_{j\in J} A_j$ of its universes, then it is easy to see that $A$ is subuniverse of a subalgebra of $\mathfrak{B}$ which is called the intersection of the family $\{\mathfrak{A}_j\}_{j\in J}$ denoted by $\bigcap_{j\in J} \mathfrak{A}_j$. This allows us to consider the subalgebra

$$\langle X\rangle_{\mathfrak{B}} = \bigcap \{\, \mathfrak{A} \mid A \in \mathsf{Sub}(\mathfrak{B}) \text{ and } X \subseteq A \,\}$$

of $\mathfrak{B}$ generated by a subset $X \subseteq B$ of the universe. The set $X$ is called *generating system* or set of generators of this algebra. The process of subalgebra generation is another example of a closure operator, which we have just considered in more detail in the previous Chapters (see Subsections 2.1.6 and 3.1.5). As a consequence, the set of fixsets $\mathsf{Sub}(\mathfrak{B})$ is a complete lattice (see Proposition 3.1.34).

**Definition 4.1.6.** For $X \subseteq A$, we say that $\mathfrak{A}$ is the *subalgebra generated by* $X$ if $\langle X\rangle_{\mathfrak{A}} = \mathfrak{A}$. A $\tau$-algebra $\mathfrak{A}$ is *finitely generated* if it has a finite set of generators. Also, we say that a $\tau$-algebra $\mathfrak{A}$ is *locally finite* if and only if every finitely generated subalgebra of $\mathfrak{A}$ is finite.

If $X \subsetneq A$ is a subset of the universe of a $\tau$-algebra $\mathfrak{A}$, $\langle X\rangle_{\mathfrak{A}}$ is the subalgebra of $\mathfrak{A}$ generated by $X$ and $h : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism, then

$$\langle h(X)\rangle_{\mathfrak{B}} = h(\langle X\rangle_{\mathfrak{A}}).$$

In particular, if $X$ is a generating system of $\mathfrak{A}$ and $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ is surjective homomorphism, then $h(X)$ generates $\mathfrak{B}$.

Clearly, the notion of subalgebra can be also thought of as the generalization of the notion of sublattices that we have encountered in Chapter 3.

### 4.1.4 (Direct) Products

In this section we shall examine another important construction, the formation of product algebras. Subalgebras of a given $\tau$-algebra have cardinalities no larger than its cardinality. The formation of products, however, can lead to algebras with bigger cardinalities than those we started with.

**Definition 4.1.7.** Let $\{\mathfrak{X}_i\}_{i\in I}$ be a family of $\tau$-algebras indexed by a set $I$, we define *product* of the family $\{\mathfrak{X}_i\}_{i\in I}$ a $\tau$-algebra $\mathfrak{X}$ equipped with a family of morphisms $\{\pi_i : \mathfrak{X} \to \mathfrak{X}_i\}_{i\in I}$, called *projections*, such that, for every $\mathfrak{Y}$ and every family of morphisms $\{h_i : \mathfrak{Y} \to \mathfrak{X}_i\}_{i\in I}$, there exists exactly a morphism $h : \mathfrak{Y} \to \mathfrak{X}$ such that for every $i \in I$ we have $\pi_i \circ h = h_i$. Namely, the following diagram

$$\begin{tikzcd} \mathfrak{Y} \arrow[d, dashed, "h"'] \arrow[dr, "h_i"] & \\ \mathfrak{X} \arrow[r, "\pi_i"'] & \mathfrak{X}_i \end{tikzcd}$$

commutes for every $i \in I$.

In the category of $\tau$-algebras $\mathbb{A}lg_\tau$, the product indexed by a set always exists. This product can be defined as follows: an element of the product is a choice $\langle \dots, x_i, \dots \rangle$ of an element $x_i \in X_i$ for each $i \in I$, and the operations between the product elements are defined coordinatewise, namely, if $f \in \mathcal{F}$ is an operation symbol of arity $n$,

$$f^X(\langle \dots, x_i^{(1)}, \dots \rangle, \dots, \langle \dots, x_i^{(n)}, \dots \rangle) = \left\langle \dots, f^{X_i}(x_i^{(1)}, \dots, x_i^{(n)}), \dots \right\rangle$$

Then we have that $\pi_i : \mathfrak{X} \to \mathfrak{X}_i$ is defined by $\langle \dots, x_i, \dots \rangle \mapsto x_i$ and the function $h : \mathfrak{Y} \to \mathfrak{X}$, which is also indicated with $\langle \dots, h_i, \dots \rangle$, is defined by $y \mapsto \langle \dots, h_i(y), \dots \rangle$.

Needless to say, the notion of product can be regarded as the generalization of the notion of product of lattices that we have come across in Chapter 3.

### 4.1.5 Congruences, Quotient Algebras and Homomorphic Images

The last important algebraic construction that we want to mention, is the formation of homomorphic images. Before proceeding to examine the formation of homomorphic images, it is important to recall the notion of equivalence relation and introduce a new one suitable for $\tau$-algebras: the notion of congruence.

Every function $h : A \to B$ from a set $A$ onto a set $B$ defines a partition of $A$ into classes of elements having the same image. Partitions of a set define equivalence relations on that set where two elements are related to each other if and only if they belong to the same block of the partition. Let $A, B$ be the universes of two $\tau$-algebras $\mathfrak{A}$ and $\mathfrak{B}$ and

$h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ is a surjective homomorphism. Then, an equivalence relation $\mathcal{R}$ on $A$ can be well suited with $h$ if satisfies the following compatibility property:

**Definition 4.1.8.** Let $A$ be a $\tau$-algebra. An equivalence relation $\mathcal{R}$ on $A$ is called a *congruence relation* on $\mathfrak{A}$ if for each operation symbols $f \in \mathcal{F}$ and for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$,

$$(a_1, b_1) \in \mathcal{R}, \ldots, (a_n, b_n) \in \mathcal{R} \quad \text{implies} \quad (f^A(a_1, \ldots, a_n), f^A(b_1, \ldots, b_n)) \in \mathcal{R}.$$

We denote by $\mathsf{Con}(\mathfrak{A})$ the set of all congruence relations of the $\tau$-algebra $\mathfrak{A}$.

A congruence of $\mathfrak{A}$ is nothing but an equivalence relation which, as a subset of the product $\mathfrak{A} \times \mathfrak{A}$, is closed with respect to the operations. This means that congruences are the equivalence relations that occur as objects in the category of $\tau$-algebras $\mathbb{A}lg_\tau$.

Similarly to the set of subuniverses of a given $\tau$-algebra, also the set of its congruence relations admits an abstract characterization in terms of closure operator. Let $\mathfrak{A}$ be a $\tau$-algebra, and let $Q$ be a binary relation on $A$. We define the congruence relation $\langle Q \rangle_{\mathsf{Con}(\mathfrak{A})}$ on $\mathfrak{A}$ generated by $Q$ to be the intersection of all congruence relations $\mathcal{R}$ on $\mathfrak{A}$ which contain $Q$:

$$\langle Q \rangle_{\mathsf{Con}(\mathfrak{A})} = \bigcap \{ \mathcal{R} \mid \mathcal{R} \in \mathsf{Con}(\mathfrak{A}) \text{ and } Q \subseteq \mathcal{R} \}$$

It is not difficult to that $\langle Q \rangle_{\mathsf{Con}(\mathfrak{A})}$ is a closure operator and its set of fixsets $\mathsf{Con}(\mathfrak{A})$ is a complete lattice (see Proposition 3.1.34).

If $\mathcal{R}$ is a congruence relation on $\mathfrak{A}$, then we can partition the set $A$ into blocks with respect to $\mathcal{R}$ and obtain the quotient set $A/\mathcal{R}$. In a natural way, for each $n$-ary operation symbol $f \in \mathcal{F}$, we define an $n$-ary operation $f^{A/\mathcal{R}}$ on the quotient set by

$$f^{A/\mathcal{R}} : \left(A/\mathcal{R}\right)^n \to A/\mathcal{R}$$

with

$$([a_1]_\mathcal{R}, \ldots, [a_n]_\mathcal{R}) \longmapsto f^{A/\mathcal{R}}([a_1]_\mathcal{R}, \ldots, [a_n]_\mathcal{R}) := [f^A(a_1, \ldots, a_n)]_\mathcal{R}.$$

Of course, we have to verify that our operations are well-defined, namely that they are independent on the representatives chosen. But this is exactly what the compatibility property of a congruence relation means and so we obtain a new $\tau$-algebra $\mathfrak{A}/\mathcal{R}$, which is called the *quotient algebra* of $\mathfrak{A}$ by $\mathcal{R}$.

Actually, for every congruence relation $\mathcal{R}$ the $\tau$-algebra $\mathfrak{A}/\mathcal{R}$ is a homomorphic image of $\mathfrak{A}$ under the *natural projection* defined by

$$\pi_\mathcal{R} : \mathfrak{A} \twoheadrightarrow \mathfrak{A}/\mathcal{R} \quad \text{with} \quad a \longmapsto [a]_\mathcal{R}$$

for every $a \in A$. It is easy to check that $\pi_\mathcal{R}$ is really a surjective homomorphism. So, for any congruence relation $\mathcal{R} \in \mathsf{Con}(\mathfrak{A})$ we obtain a homomorphism and, finally, it arises the question whether homomorphisms define congruence relations on $\mathfrak{A}$. This is also the case since we have:

**Lemma 4.1.9.** *The kernel*

$$\mathsf{Ker}\, h := \{ (a,b) \in A \times A \mid h(a) = h(b) \}$$

*of any homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is a congruence relation on $\mathfrak{A}$.*

Suppose we have now a homomorphism $h : \mathfrak{A} \to \mathfrak{B}$. We have seen that $\mathsf{Ker}\, h$ is a congruence on $\mathfrak{A}$, so we can form the quotient algebra ${}^{\mathfrak{A}}/_{\mathsf{Ker}\, h}$, along with the natural homomorphism $\pi_{\mathsf{Ker}\, h} : \mathfrak{A} \to {}^{\mathfrak{A}}/_{\mathsf{Ker}\, h}$ which maps the algebra $\mathfrak{A}$ onto this quotient algebra. Now we have two homomorphic images of $\mathfrak{A}$: the original $h(\mathfrak{A})$ and the new quotient ${}^{\mathfrak{A}}/_{\mathsf{Ker}\, h}$. What connection is there between these two homomorphic images? The answer to this question is a consequence of the universal property of quotients.

**Proposition 4.1.10** (Universal property of quotients)**.** *Let $h : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism of $\tau$-algebras, $\pi : \mathfrak{A} \twoheadrightarrow \mathfrak{Q}$ is a surjective homomorphism. If $\mathsf{Ker}\, \pi \subseteq \mathsf{Ker}\, h$, then there exists exactly one homomorphism $h' : \mathfrak{Q} \to \mathfrak{B}$ such that $h' \circ \pi = h$, namely the following diagram commutes.*

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "h"] \arrow[d, two heads, "\pi"'] & \mathfrak{B} \\
\mathfrak{Q} \arrow[ur, dashed, "h'"'] &
\end{tikzcd}$$

*Furthermore,*

*(i) $h'$ is surjective if and only if $h$ is surjective, and*

*(ii) $h'$ is injective if and only if it is $\mathsf{Ker}\, \pi = \mathsf{Ker}\, h$.*

*Proof.* See Burris and Sankappanavar (1981). ☐

**Corollary 4.1.11** (Homomorphic Image Theorem)**.** *Let $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ be a surjective homomorphism of $\tau$-algebras. Then there exists a unique isomorphism $h'$ from ${}^{\mathfrak{A}}/_{\mathsf{Ker}\, h} \to \mathfrak{B}$ with $h' \circ \pi_{\mathsf{Ker}\, h} = h$, where $\pi_{\mathsf{Ker}\, h} : \mathfrak{A} \twoheadrightarrow {}^{\mathfrak{A}}/_{\mathsf{Ker}\, h}$ is the natural projection.*

### Congruence Relations on Heyting Algebras

Having discussed the correspondence between congruence relations and homomorphic images, in the following section we address the close correspondence between congruence relations and filters on Heyting algebras. As we shall see in the Section 4.3, Heyting algebras can be grasp as particular $\tau$-algebras, which satisfy some axioms, for the language $\langle \wedge, \vee, \Rightarrow, \top, \bot \rangle$ of type $\langle 2, 2, 2, 0, 0 \rangle$. Hence, we can define, for each congruence $\mathcal{R}$ on a Heyting algebra $H$, the set

$$\omega^{-1}(\mathcal{R}) = \{ x \in H \mid (x, \top) \in \mathcal{R} \}.$$

$\omega^{-1}(\mathcal{R})$ is a filter of $H$, called the filter determined by a congruence $\mathcal{R}$. Conversely, for each filter $F \subseteq H$,

$$\rho^{-1}(F) = \{ (x, y) \in H \times H \mid x \Leftrightarrow y \in F \}$$

is a congruence, called the congruence determined by a filter $F \subseteq H$. The notational choice to indicate these maps as inverse functions will become clear in the following (see subsection 4.3.5). Interestingly, it can be proved that $\omega^{-1}$ constitute a lattice isomorphism.

**Theorem 4.1.12.** *For all congruences $\mathcal{R}, \mathcal{Q}$ and all filters $F, G$ of a Heyting algebra $H$, the following hold:*

*(1) $\mathcal{R} \subseteq \mathcal{Q}$ implies $\omega^{-1}(\mathcal{R}) \subseteq \omega^{-1}(\mathcal{Q})$.*

*(2) $F \subseteq G$ implies $\rho^{-1}(F) \subseteq \rho^{-1}(G)$.*

*(3) $\rho^{-1}(\omega^{-1}(\mathcal{R})) = \mathcal{R}$ and $\omega^{-1}(\rho^{-1}(F)) = F$.*

*Thus, the map $\omega^{-1}$ is an order-isomorphism, and therefore is a complete lattice isomorphism between the congruence lattice $\mathsf{Con}(H)$ and the lattice of all filters of $H$.*

*Proof.* See Balbes and Dwinger (2011). □

As a consequence of this isomorphism, the quotient algebra ${}^{H}\!/_{\rho^{-1}(F)}$ can be equivalently represented as ${}^{H}\!/_{F}$, for a given filter $F \subseteq H$. Moreover, we can observe that, for each $\mathcal{R} \in \mathsf{Con}(H)$,

$$(a, b) \in \mathcal{R} \quad \text{if and only if} \quad (a \Leftrightarrow b, \top) \in \mathcal{R}.$$

Conversely, for each filter $F \subseteq H$,

$$a \in F \quad \text{if and only if} \quad a \Leftrightarrow \top \in F.$$

Those relations will be useful in the remainder (see subsection 4.3.5).

### 4.1.6 Class Operators on $\tau$-Algebras and Varieties

So far the previous subsections have focused on the various ways in which it can be manufacture a new $\tau$-algebra form a given one. We introduce now the following operators mapping classes of $\tau$-algebras to classes of $\tau$-algebras:

(1) $\mathfrak{A} \in \mathsf{H}(\mathcal{K})$ if and only if $\mathfrak{A}$ is a homomorphic image of some member of $\mathcal{K}$.

(2) $\mathfrak{A} \in \mathsf{S}(\mathcal{K})$ if and only if $\mathfrak{A}$ is isomorphic to a subalgebra of some member of $\mathcal{K}$.

(3) $\mathfrak{A} \in \mathsf{P}(\mathcal{K})$ if and only if $\mathfrak{A}$ is isomorphic to a product of a family of algebras in $\mathcal{K}$.

Each of the class operators just introduced, when restricted to classes of algebras of one similarity type $\tau$, can be regarded as a closure operator on the class of all $\tau$-algebras.

If $\mathsf{Q}_1$ and $\mathsf{Q}_2$ are class operators belonging to $\{\mathsf{H}, \mathsf{S}, \mathsf{P}\}$, we write $\mathsf{Q}_1\mathsf{Q}_2$ for the composition of the operators. We use $\leq$ to denote the following partial ordering of class operators: $\mathsf{Q}_1 \leq \mathsf{Q}_2$ if and only if $\mathsf{Q}_1(\mathcal{K}) \leq \mathsf{Q}_2(\mathcal{K})$ for all classes $\mathcal{K}$ of $\tau$-algebras. Notice that $\mathsf{Q}(\mathcal{K})$ is always an abstract class since, if a $\tau$-algebra $\mathfrak{A}$ belongs to the class, every $\tau$-algebra isomorphic to $\mathfrak{A}$ also belongs to it. Moreover, we can observe that $\mathsf{P}(\mathcal{K})$ always includes the trivial algebras of the $\tau$-algebras in $\mathcal{K}$, since we allow the product of an empty family of $\tau$-algebras.

**Lemma 4.1.13.** *The class operators* $\mathsf{HS}, \mathsf{SP}$ *and* $\mathsf{HP}$ *are closure operators on the class of $\tau$-algebras. the following inequalities hold:* $\mathsf{SH} \leq \mathsf{HS}$, $\mathsf{PS} \leq \mathsf{SP}$, $\mathsf{PH} \leq \mathsf{HP}$.

*Proof.* See McKenzie *et al.* (1987); Burris and Sankappanavar (1981). □

Let $\mathcal{K}$ be a class of $\tau$-algebras. We say that $\mathcal{K}$ is closed under the class operator $\mathsf{Q}$ if and only if $\mathsf{Q}(\mathcal{K}) \subseteq \mathcal{K}$.

**Definition 4.1.14.** A class $\mathcal{K}$ of $\tau$-algebras is called a *variety* if and only if $\mathcal{K}$ is closed under $\mathsf{H}$, $\mathsf{S}$ and $\mathsf{P}$.

**Examples 4.1.15.** As we shall see in Section 4.2, all of the classes considered in Chapter 3, lattices, distributive lattices, boolean algebras and Heyting algebras, are varieties.

Since the class of all $\tau$-algebras is a variety, and since the intersection of any family of varieties of $\tau$-algebras is again a variety, we can conclude that there does exist a smallest variety containing a given class of $\tau$-algebras.

**Definition 4.1.16.** Let $\mathcal{K}$ be a class of $\tau$-algebras. $\mathsf{V}(\mathcal{K})$ denotes the smallest variety containing $\mathcal{K}$, called the *variety generated* by $\mathcal{K}$. If $\mathcal{K}$ consists of a single $\tau$-algebra $A$, or of finitely many $\tau$-algebras $A_1, \ldots, A_n$, then we write $\mathsf{V}(A)$ or $\mathsf{V}(A_1, \ldots, A_n)$, respectively, for this variety.

**Theorem 4.1.17.** $\mathsf{V} = \mathsf{HSP}$.

*Proof.* See McKenzie *et al.* (1987); Burris and Sankappanavar (1981). □

**Definition 4.1.18.** Let $\mathcal{V}$ be a variety of $\tau$-algebras.

(1) $\mathcal{V}$ is *locally finite* if and only if each of its members is a locally finite $\tau$-algebra.

(2) $\mathcal{V}$ is *finitely generated* iff $\mathcal{V} = \mathsf{V}(\mathfrak{A})$ for some finite $\tau$-algebra $\mathfrak{A} \in \mathcal{V}$.

(3) $\mathcal{V}$ is *finitely approximable* iff $\mathcal{V} = \mathsf{V}(\mathcal{G})$ for some set $\mathcal{G} \subseteq \mathcal{V}$ of finite $\tau$-algebras.

**Theorem 4.1.19.** *Let $\mathcal{K}$ be a finite set of finite algebras. Then* $\mathsf{V}(\mathcal{K})$ *is a locally finite variety. In particular, every finitely generated variety is locally finite.*

*Proof.* See McKenzie *et al.* (1987); Burris and Sankappanavar (1981). □

**Examples 4.1.20.**

1. Let us denote the variety of all distributive lattices by $\mathcal{DL}$. As a consequence of Birkhoff's theorem that we shall see in Section 4.2, it can be proved that $\mathcal{DL}$ is finitely generated by $\mathbf{2}$ and so locally finite (see G. Bezhanishvili 2001).

2. Let us denote the variety of all Boolean algebras by $\mathcal{BA}$. As in the case of distributive lattices, it can be proved that $\mathcal{BA}$ is finitely generated by $\mathbf{2}$ and so locally finite (see G. Bezhanishvili 2001).

3. Let us denote the variety of all Heyting algebras by $\mathcal{HA}$. As a consequence of the fact that the free Heyting algebra generated by one element, also called the Rieger-Nishimura lattice, is infinite, $\mathcal{HA}$ is not locally finite (see example 4.2.10). However, it can be proved that $\mathcal{HA}$ is finite approximable (see Theorem 4.3.19).

**Remark 4.1.21.** In general we have:

$$\text{finitely generated} \Rrightarrow \text{locally finite} \Rrightarrow \text{finitely approximable,}$$

with both of the implications being strict.

Finally, we introduce a useful class operator for the study of locally finite varieties.

**Definition 4.1.22.** Let $\mathcal{K}$ be a class of $\tau$-algebras. $\mathsf{P}_f(\mathcal{K})$ is the class of $\tau$-algebras isomorphic to a product of a finite family of members of $\mathcal{K}$. $\mathcal{K}_f$ is the class of finite members of $\mathcal{K}$.

**Lemma 4.1.23.** $(\mathsf{SP}(\mathcal{K}))_f \subseteq \mathsf{SP}_f(\mathcal{K})$.

*Proof.* See McKenzie *et al.* (1987). □

**Corollary 4.1.24.** *Suppose that $\mathcal{K}$ is a finite set of finite $\tau$-algebras and $\mathcal{V} = \mathsf{V}(\mathcal{K})$. Then $\mathcal{V}_f = \mathsf{HSP}_f(\mathcal{K})$.*

□

## 4.2 Terms, Free $\tau$-Algebras and HSP Theorem

As indicated previously, the concept of $\tau$-algebra is too general to describe mathematical objects such as lattices, boolean algebras, Heyting algebras, etc. In all these cases we do not have to deal with the totality of the algebras of a given type $\tau$, but with its subclass defined through certain properties usually called axioms. We limit ourselves here to the study of the equational properties, that is, of the properties that are expressed as the equality of two expressions that technically we will call terms. The latter represent the derivative operations, namely those obtained by composition from the operations that come from the type $\tau$. More precisely, the derived operations will be obtained as interpretations of the terms.

### 4.2.1 Terms and Basic Adjunction

#### Terms and Equations Between Terms

Let us define the terms of our type $\tau$, the "words" of our language.

**Definition 4.2.1.** The *terms* of a type $\tau$ are expressions constructed inductively by the following rules:

(i) every variable $x, y, z, ...$, is a term,

(ii) if $t_1, \ldots, t_k$ are terms and $f \in \mathcal{F}$ is a $k$-ary operation then $f(t_1, \ldots, t_k)$ is a term.

Let $X$ be any set of variables. The set $\mathsf{Tr}_\tau(X)$ of terms of type $\tau$ over $X$ is the smallest set which contains $X$ and closed under finite application of (ii).

**Definition 4.2.2.** An *equation*, that we shall indicate with $u = v$, is simply given by a pair of terms $(u, v) \in \mathsf{Tr}_\tau(X) \times \mathsf{Tr}_\tau(X)$.

### Interpretation of Terms and Validity of Equations

The interpretation of a similarity type $\tau$ in $\mathbb{S}et$ that we have examined in Subsection 4.1.2 can be extended to all terms of the type $\tau$ as follows: a general term $t$ is always interpreted together with a *context* of variables $x_1, \ldots, x_n$, where the variables appearing in $t$ are among the variables appearing in the context. We write $\vec{x}.t$ to indicate that the term $t$ is to be understood in context $\vec{x} = x_1, \ldots, x_n$.

**Definition 4.2.3.** If $\mathfrak{A}$ is an interpretation of $\tau$ in $\mathbb{S}et$, then the corresponding *interpretation of a term $t$* of type $\tau$ is a function $t^A : A^n \to A$, determined by the following specification:

1. the interpretation of a variable $x_i$ is the $i$-th projection $\pi_i : A^n \to A$.

2. A term of the form $f(t_1, \ldots, t_k)$ is interpreted as the composite:

$$A^n \xrightarrow{(t_1^A, \ldots, t_k^A)} A^k \xrightarrow{f^A} A$$

where $t_i^A : A^n \to A$ is the interpretation of the subterm $t_i$, for $i = 1, \ldots, k$, and $f^A$ is the interpretation of the operation symbol $f$.

Several properties of the fundamental operations of an algebra $\mathfrak{A}$ are also valid for term operations. For instance, for a homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ and an arbitrary $n$-ary term $t$ of the corresponding type we have:

$$h \circ t^A = t^B \circ h^n.$$

Similarly, congruence relations and subuniverses of $\mathfrak{A}$ are not only preserved by all its fundamental operations but also by all its term operations.

Suppose $u$ and $v$ are terms in context $x_1, \ldots, x_n$. Then we say that the equation $u = v$ is satisfied by the interpretation $\mathfrak{A}$ if $u^A$ and $v^A$ are the same function in $\mathbb{S}et$. In other terms, if $u = v$ is an equation, and $x_1, \ldots, x_n$ are all the variables appearing in $u$ and $v$, we say that $A$ satisfies the equation $u = v$ if $[\vec{x}.u]^A$ and $[\vec{x}.v]^A$ are the same function,

$$A^n \underset{[\vec{x}.v]^A}{\overset{[\vec{x}.u]^A}{\rightrightarrows}} A$$

which we also write as:

$$\mathfrak{A} \models u = v \quad \text{if and only if} \quad u^A = v^A. \tag{4.1}$$

**Remark 4.2.4.** Of course, the validity of equations is preserved for subalgebras, homomorphic images and products. That is,

(i) if $\mathfrak{A} \models u = v$ and $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then $\mathfrak{B} \models u = v$;

(ii) if $\mathfrak{A} \models u = v$ and $\mathfrak{C}$ is a homomorphic image of $\mathfrak{A}$, then $\mathfrak{C} \models u = v$;

(iii) if $\mathfrak{X}_i \models u = v$ for all $i \in I$, then $\prod_{i \in I} \mathfrak{X}_i \models u = v$.

### The Basic Adjunction Between Equations and τ-Algebras

We shall indicate the set of all equations of type $\tau$, $\mathsf{Tr}_\tau(X) \times \mathsf{Tr}_\tau(X)$, with $\mathsf{Eq}_\tau^X$. The relation of validity, like every binary relation (see example 2.1.32.4), induces the adjunction

$$\left(2^{\mathcal{Alg}_\tau}\right)^{\mathrm{op}} \underset{\mathsf{Id}^X}{\overset{\mathsf{Mod}}{\leftrightarrows}} 2^{\mathsf{Eq}_\tau^X}$$

between $2^{\mathsf{Eq}_\tau^X}$, whose element $\Theta \subseteq \mathsf{Eq}_\tau^X$ are set of equations that will be thought of as a set of axioms of a theory, and the opposite of $2^{\mathsf{Alg}_\tau}$, whose elment $\mathcal{K} \subseteq \mathcal{Alg}_\tau$ is any class of $\tau$-algebras.

With $\mathsf{Mod}\,(\Theta)$ we mean the class of models of the theory having $\Theta$ as a set of axioms, that is to say

$$\mathsf{Mod}\,(\Theta) = \{\, \mathfrak{A} \in \mathcal{Alg}_\tau \mid \forall (u = v) \in \Theta \quad \mathfrak{A} \models u = v \,\}.$$

Dually, $\mathsf{Id}^X(\mathcal{K})$ is the set of equations that are valid in all $\tau$-algebras of $\mathcal{K}$, that is to say

$$\mathsf{Id}^X(\mathcal{K}) = \{\, (u = v) \in \mathsf{Eq}_\tau^X \mid \forall \mathfrak{A} \in \mathcal{K} \quad \mathfrak{A} \models u = v \,\}.$$

Since the maps $\mathsf{Mod}$ and $\mathsf{Id}^X$ constitute the adjunction $\mathsf{Mod} \dashv \mathsf{Id}^X$, they satisfy the following properties:

(1) for all subsets $\Theta, \Theta'$ of $\mathsf{Eq}_\tau$, if $\Theta \subseteq \Theta'$, then $\mathsf{Mod}(\Theta') \subseteq \mathsf{Mod}(\Theta)$;

(2) for all subclasses $\mathcal{K}, \mathcal{K}'$ of $\mathcal{Alg}_\tau$, if $\mathcal{K} \subseteq \mathcal{K}'$, then $\mathsf{Id}^X(\mathcal{K}') \subseteq \mathsf{Id}^X(\mathcal{K})$.

(3) for all subclasses $\mathcal{K}$ of $\mathcal{Alg}_\tau$ and for all subsets $\Theta$ of $\mathsf{Eq}_\tau$, $\mathcal{K} \subseteq \mathsf{Mod}(\Theta)$ if and only if $\Theta \subseteq \mathsf{Id}^X(\mathcal{K})$.

Moreover, the maps $\mathsf{Id}^X\,\mathsf{Mod}$ and $\mathsf{Mod}\,\mathsf{Id}^X$ are closure operators on $\mathsf{Eq}_\tau$ and on $\mathcal{Alg}_\tau$, respectively. The sets closed under $\mathsf{Mod}\,\mathsf{Id}^X$ are exactly the sets of the form $\mathsf{Mod}(\Theta)$, for some $\Theta \subseteq \mathsf{Eq}_\tau$, and the sets closed under $\mathsf{Id}^X\,\mathsf{Mod}$ are exactly the sets of the form $\mathsf{Id}^X(\mathcal{K})$, for some $\mathcal{K} \subseteq \mathcal{Alg}_\tau$ (see Proposition 3.1.34).

Let us now look at some of the meanings of the conditions now estabilished. First of all $\mathsf{Id}^X\,\mathsf{Mod}(\Theta)$ is the set of all equations that are valid in all the models of the theory having $\Theta$ as a set of axioms, namely $\mathsf{Id}^X\,\mathsf{Mod}(\Theta)$ is the set of *semantics consequences* of the set of axioms $\Theta$. It is clear that we often have $\Theta \subsetneq \mathsf{Id}^X\,\mathsf{Mod}(\Theta)$ because the sets of axioms often have, as consequences, equations that are not axioms. Therefore, the fixsets of $\mathsf{Id}^X\,\mathsf{Mod}$ can be interpreted as follows.

**Definition 4.2.5.** A set of equations $\Theta$ is an *equational theory* when there is a class $\mathcal{K}$ of $\tau$-algebras such that $\Theta$ is the totality of equations valid on $\mathcal{K}$, namely $\Theta = \mathsf{Id}^X(\mathcal{K})$.

Since the equational theories are exactly the fixsets with respect to the closure operator $\mathsf{Id}^X\,\mathsf{Mod}$, their collection forms a complete lattice $\Lambda(\mathsf{Eq}_\tau)$ dually isomorphic to $\Lambda(\mathcal{Alg}_\tau)$, the complete lattices of fixsets of $\mathsf{Mod}\,\mathsf{Id}^X$, that can be interpreted as follows.

**Definition 4.2.6.** A class $\mathcal{K}$ of $\tau$-algebras is an *equational class*, or is said to be *equationally definable* when there is a set of equations $\Theta$ such that $\mathcal{K}$ is the totality of the models of $\Theta$, namely $\mathcal{K} = \mathsf{Mod}(\Theta)$.

**Examples 4.2.7.** All of the classes considered in chapter 3, lattices, distributive lattices, boolean algebras and Heyting algebras, are equational classes.

This suggests that equational class and variety are the same, as we are going to prove in the next subsection.

### 4.2.2 Free $\tau$-Algebras and HSP Theorem

In this section we shall introduce free $\tau$-algebras, $\tau$-algebras of terms and prove that every variety of algebras is equationally definable (the HSP Theorem).

#### Free $\tau$-Algebras

We begin with the definition of free $\tau$-algebras.

**Definition 4.2.8.** Let $\mathcal{K}$ be a class of $\tau$-algebras. Let $X$ be a set, $\mathfrak{A}$ a $\tau$-algebra and $\iota : X \to \mathfrak{A}$ an homomorphism. We say that $\iota : X \to \mathfrak{A}$ is *free with respect to* $\mathcal{K}$ if it holds the following *universal mapping property*: for every $\mathfrak{B} \in \mathcal{K}$ and for every mapping $g : X \to \mathfrak{B}$ there is a unique homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ such that $h \circ \iota = g$. Namely, the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[d, "g"'] & \mathfrak{A} \arrow[dl, dashed, "h"] \\ \mathfrak{B} & \end{tikzcd}$$

If $\iota : X \to \mathfrak{A}$ is free with respect to $\mathcal{K}$ over $X$ and $\mathfrak{A} \in \mathcal{K}$, $\mathfrak{A}$ is said to be the *free algebra* in $\mathcal{K}$ generated by $X$.

**Remark 4.2.9.** Observe that $\iota$ is not necessarily injective.

**Examples 4.2.10** (Rieger-Nishimura lattice)**.** The free Heyting algebra generate by one single element $p$ is infinite and is given recursively as in Figure 4.1.

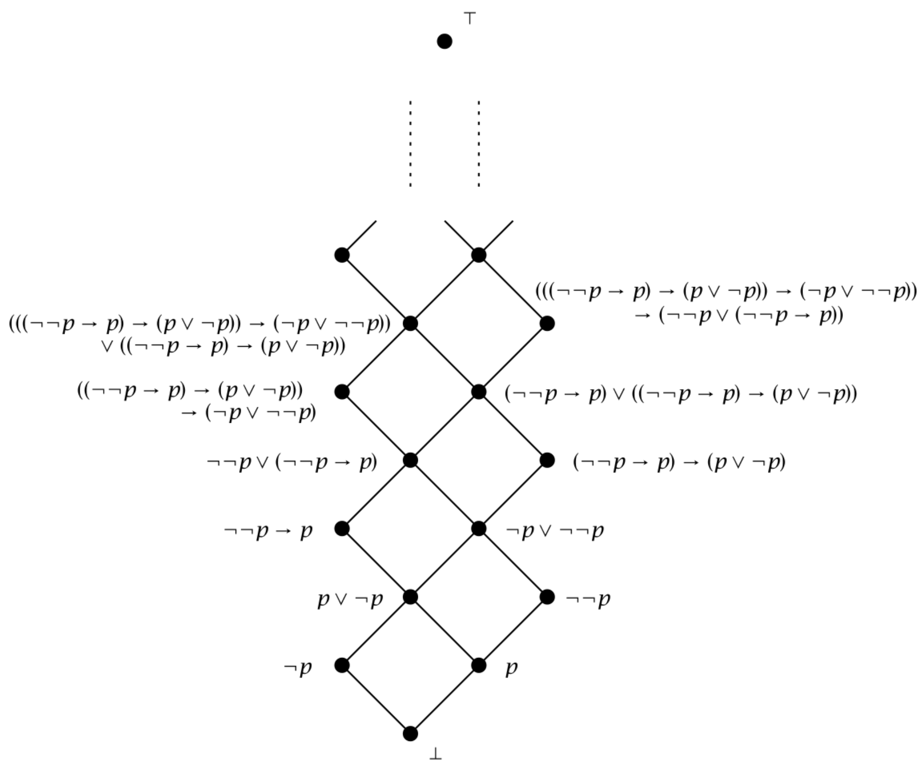


FIGURE 4.1: The Rieger-Nishimura lattice.

Unfortunately, not every class $\mathcal{K}$ contains algebras with the universal mapping property for $\mathcal{K}$. Nonetheless, if we consider the class of all $\tau$-algebras $\mathcal{A}lg_\tau$, then free algebras always exist. In Section 4.2, we will be able to show that any class closed under $\mathsf{H}, \mathsf{S}$ and $\mathsf{P}$ contains free algebras.

Consider now the set $\mathsf{Tr}_\tau(X)$. Clearly, it can be turned into a $\tau$-algebra. Given $\tau$ and $X$, $\mathfrak{Tr}_\tau(X)$ is a $\tau$-algebra of type $\tau$ over $X$, which has as its universe the set $\mathsf{Tr}_\tau(X)$, and the fundamental operations

$$f^{\mathsf{Tr}_\tau(X)} : \mathsf{Tr}_\tau(X)^n \longrightarrow \mathsf{Tr}_\tau(X)$$

satisfy

$$(t_1, \dots, t_n) \longmapsto f^{\mathsf{Tr}_\tau(X)}(t_1, \dots, t_n) \in \mathsf{Tr}_\tau(X)$$

for $f \in \mathcal{F}$ of arity $n$ and $t_i \in \mathsf{Tr}_\tau(X)$.

Notice that $\mathfrak{Tr}_\tau(X)$ is indeed generated by $X$. Hence, the $\tau$-algebra of terms provides us with the simplest examples of algebras with the universal mapping property.

**Lemma 4.2.11.** *for any set $X$, the morphism $\iota : X \to \mathfrak{Tr}_\tau(X)$ is free with respect to the class of all $\tau$-algebras $\mathcal{A}lg_\tau$.*

*Proof.* See McKenzie *et al.* (1987); Burris and Sankappanavar (1981). □

**Remark 4.2.12.** As a consequence, every homomorphism $h : \mathfrak{Tr}_\tau(X) \to \mathfrak{B}$ is completely determined by its restriction $h|_X = g$. Namely, two homomorphisms from the free algebra $\mathfrak{Tr}_\tau(X)$ to the same $\tau$-algebra $\mathfrak{B}$ are equal if and only if they coincide on variables. Also, for each $\vec{x}.t \in \mathsf{Tr}_\tau(X)$, $h(t)$ depends only on the restriction of $h|_{\vec{x}}$.

**Corollary 4.2.13.** *Every $\tau$-algebra is a quotient of some $\mathfrak{Tr}_\tau(X)$.*

Observe that the morphism $\iota : \emptyset \to \mathfrak{A}$ is free with respect to $\mathcal{A}lg_\tau$ if and only if $\mathfrak{A}$ is initial, since for every $\mathfrak{B}$ there exists a unique morphism $! : \mathfrak{A} \to \mathfrak{B}$. Therefore, $\mathfrak{Tr}_\tau(\emptyset)$, the $\tau$-algebra of constants, is initial in $\mathcal{A}lg_\tau$.

With the help of the free $\tau$-algebra $\mathfrak{Tr}_\tau(X)$, two notions with an undeniably logical flavour can be introduced. One such notion is that of evaluation, which can be thought of as being just one of the functions above.

**Definition 4.2.14.** An *assignement* or *evaluation* into $\mathfrak{B}$ is a function $h : \mathfrak{Tr}_\tau(X) \to \mathfrak{B}$ from the free $\tau$-algebra $\mathfrak{Tr}_\tau(X)$ to $\mathfrak{B}$.

Since any function $g : X \to \mathfrak{B}$ extends uniquely to a homomorphism $h$ from $\mathfrak{Tr}_\tau(X)$ to $\mathfrak{B}$ in a such way that

$$h(f(t_1, \ldots, t_n)) = f^A(h(t_1), \ldots, h(t_n)),$$

for each operation symbol $f \in \mathcal{F}$ with arity $n$, an evaluation of terms in $\mathfrak{B}$ is given by the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[d, "g"'] & \mathfrak{Tr}_\tau(X) \arrow[dl, dashed, "h"] \\ \mathfrak{B} & \end{tikzcd}$$

associating to every terms $t \in \mathsf{Tr}_\tau(X)$ an interpretation $t^B : B^n \to B$.

Another notion is that of *substitution*, which is a particular case of an evaluation, actually an evaluation into the $\tau$-algebra of terms. Or, equivalently:

**Definition 4.2.15.** A *substitution* is any function $\sigma : X \to \mathfrak{Tr}_\tau(Y)$ from a set of variables $X$ into the $\tau$-algebra of terms $\mathfrak{Tr}_\tau(Y)$ generated by a set of variables $Y$ not necessarily being identical to $X$.

Notice that $Y$ can be different from $X$. Since any function $\sigma : X \to \mathfrak{Tr}_\tau(Y)$, extends uniquely to a homomorphism $h$ from $\mathfrak{Tr}_\tau(X)$ to $\mathfrak{Tr}_\tau(Y)$, a substitution of terms is given by the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[d, "\sigma"'] & \mathfrak{Tr}_\tau(X) \arrow[dl, dashed, "h"] \\ \mathfrak{Tr}_\tau(Y) & \end{tikzcd}$$

assigning to each term $t(x_1, \ldots, x_k) \in \mathsf{Tr}_\tau(X)$ a term obtained from $t$ by uniformly replacing occurrences of variables $x_i \in X$ by the $\sigma$-corresponding ones, namely

$$h(t) = t(\sigma(x_1), \ldots, \sigma(x_k)).$$

**Birkhoff's Theorem**

Turning now on a cornerstone of universal algebra due to Birkhoff. Let $\mathcal{K}$ a class of $\tau$-algebras and $\mathfrak{A}$ a $\tau$-algebra which do not belongs to the class. Consider the equivalence relation $\equiv^A_{\mathcal{K}} \subseteq A \times A$ defined by, for all $u, v \in A$,

$$u \equiv^A_{\mathcal{K}} v \quad \text{if and only if} \quad \forall \mathfrak{B} \in \mathcal{K}\ \forall h \in \mathsf{Hom}(\mathfrak{A}, \mathfrak{B})\ h(u) = h(v).$$

In order to prove that the above relation is actually a congruence, it turns out essential recasting the definition. If we put $\mathsf{Hom}(\mathfrak{A}, \mathcal{K}) := \bigcup_{\mathfrak{B} \in \mathcal{K}} \mathsf{Hom}(\mathfrak{A}, \mathfrak{B})$, we can rephrase the previous definition in the following way:

$$\equiv^A_{\mathcal{K}} := \bigcap \{\, \mathsf{Ker}(h) \mid h \in \mathsf{Hom}(\mathfrak{A}, \mathcal{K}) \,\}$$

Since $\mathsf{Ker}(h)$ is a congruence relation and any intersection of congruences is in turn a congruence, $\equiv^A_{\mathcal{K}}$ is a congruence on $A$.

**Lemma 4.2.16.** *The following facts are equivalent:*

1. $\equiv^A_{\mathcal{K}}$ *is a congruence relation.*
2. *There exists a set $I \subseteq \mathcal{K}$ such that $\mathfrak{A}/_{\equiv^A_{\mathcal{K}}}$ is a subalgebra of the product $\prod_{\mathfrak{B} \in I} \mathfrak{B}$, namely*

$$\mathfrak{A}/_{\equiv^A_{\mathcal{K}}} \in \mathsf{SP}(\mathcal{K}).$$

*Proof.* See Meloni (1979). □

Let us now return to the relation of validity (4.1). The equation $u^B = v^B$ means that for every mapping $g : X \to \mathfrak{B}$, we have $h(u) = h(v)$, where $h : \mathfrak{Tr}_\tau(X) \to \mathfrak{B}$ is the uniquely determined extension of $g$. Therefore, $u = v$ is an identity in $\mathfrak{B}$ if and only if $(u, v) \in \mathsf{Ker}(h)$ for all evaluation $h : \mathfrak{Tr}_\tau(X) \to \mathfrak{B}$. That is, the pair $(u, v)$ must belong to the intersection of the kernels of all these mappings $h$. Thus an identity $u = v$ holds in an $\tau$-algebra $\mathfrak{B}$ (or in a class $\mathcal{K}$ of $\tau$-algebras), if and only if $(u, v)$ is in the intersection of the kernels of $h$, for every evaluation $h : \mathfrak{Tr}_\tau(X) \to \mathfrak{B}$ (for every $\tau$-algebra $\mathfrak{B}$ in $\mathcal{K}$).

Hence, for a class $\mathcal{K}$ of algebras of type $\tau$ we have:

**Proposition 4.2.17.** *Let $\mathcal{K} \subseteq \mathcal{A}lg_\tau$ be a class of $\tau$-algebras and let $\mathsf{Id}^X(\mathcal{K})$ be the set of all identities satisfied in each algebra $\mathfrak{B} \in \mathcal{K}$. Then, $\mathsf{Id}^X(\mathcal{K})$ is a congruence relation on $\mathfrak{Tr}_\tau(X)$. In particular,*

$$\equiv_{\mathcal{K}}^{\mathsf{Tr}_\tau(X)} = \mathsf{Id}^X(\mathcal{K}).$$

*Proof.* See McKenzie *et al.* (1987); Burris and Sankappanavar (1981). □

Considered under this light, the set $\mathsf{Id}^X(\mathcal{K})$ has several interesting properties. It is invariant with respect to all substitutions, that is, for any $\sigma : X \to \mathfrak{Tr}_\tau(X)$,

$$u = v \in \mathsf{Id}^X(\mathcal{K}) \quad \text{implies} \quad \sigma u = \sigma v \in \mathsf{Id}^X(\mathcal{K}).$$

We simply have to note that, since $\mathsf{Id}^X(\mathcal{K})$ is equal to the intersection of the kernels of the homomorphisms $h$ for all maps $h : \mathfrak{Tr}_\tau(X) \to \mathfrak{B}$ and all $\tau$-algebras $\mathfrak{B}$ in $\mathcal{K}$ and, for any such $\mathfrak{B}$ and map $h$, the map $h \circ \sigma$ is also a homomorphism from $\mathfrak{Tr}_\tau(X)$ into $\mathfrak{B}$, our pair $u = v$ from $\mathsf{Id}^X(\mathcal{K})$ must also be in the kernel of this new homomorphism $h \circ \sigma$. And this means precisely that the pair $\sigma u = \sigma v$ must be in $\mathsf{Id}^X(\mathcal{K})$.

Also, there is a crucial connection with the identities valid on ${}^{\mathfrak{Tr}_\tau(X)}\!/_{\mathsf{Id}^X(\mathcal{K})}$, namely

**Theorem 4.2.18.** *Given a class $\mathcal{K}$ of $\tau$-algebras, we have*

$$\mathsf{Id}^X(\mathcal{K}) = \mathsf{Id}^X\left({}^{\mathfrak{Tr}_\tau(X)}\!/_{\mathsf{Id}^X(\mathcal{K})}\right). \tag{4.2}$$

*Proof.* See McKenzie *et al.* (1987); Burris and Sankappanavar (1981). □

Finally, as a consequence of Remark 4.2.4, all of the classes $\mathsf{H}(\mathcal{K})$, $\mathsf{S}(\mathcal{K})$, $\mathsf{P}(\mathcal{K})$ and $\mathsf{V}(\mathcal{K})$ satisfy the same identities as $\mathcal{K}$, over any set of variables $X$.

**Lemma 4.2.19.** *For any class $\mathcal{K}$ of $\tau$-algebras and any class operator $\mathsf{Q} \in \{\mathsf{H}, \mathsf{S}, \mathsf{P}, \mathsf{V}\}$*

$$\mathsf{Id}^X(\mathcal{K}) = \mathsf{Id}^X(\mathsf{Q}(\mathcal{K})).$$

*Proof.* See McKenzie *et al.* (1987); Burris and Sankappanavar (1981). □

With these results in the background, it can be proven the main theorem on equational theories, also called Birkhoff's Theorem.

**Theorem 4.2.20** (Birkhoff's Theorem)**.** *Let $|X| \geq \aleph_0$. For any class $\mathcal{K}$ of similar algebras,*

$$\mathsf{HSP}(\mathcal{K}) = \mathsf{Mod}\,\mathsf{Id}^X(\mathcal{K}).$$

*Thus $\mathcal{K}$ is a variety if and only if is an equational class.*

*Proof.* See Meloni (1979). □

Several important results, regarding the relation between free $\tau$-algebras and varieties of $\tau$-algebras, follow easily from the HSP theorem.

**Corollary 4.2.21.** *Let $\mathcal{K}$ be a class of $\tau$-algebras. $X \to \mathfrak{Tr}_\tau(X)/_{\mathsf{Id}^X(\mathcal{K})}$ is free with respect to $\mathsf{HSP}(\mathcal{K})$.*

In particular, if $\mathcal{K}$ is a variety, $\mathfrak{Tr}_\tau(X)/_{\mathsf{Id}^X(\mathcal{K})}$ is the free $\tau$-algebra in $\mathcal{K}$. Moreover, from (4.2) follows that every variety of $\tau$-algebras is generated by its free $\tau$-algebra.

**Corollary 4.2.22.** *Let $\mathcal{V}$ be a variety of $\tau$-algebras. Then,*

$$\mathcal{V} = \mathsf{HSP}\Big(\mathfrak{Tr}_\tau(X)/_{\mathsf{Id}^X(\mathcal{V})}\Big).$$

**Corollary 4.2.23.** *a variety $\mathcal{V}$ of $\tau$-algebras is locally finite if and only if $\mathfrak{Tr}_\tau(X)/_{\mathsf{Id}^X(\mathcal{V})}$ is a locally finite $\tau$-algebra.*

The basic facts about varieties, free algebras, and equations can now be summarized. Let $\mathcal{V}$ be a variety of $\tau$-algebras. $\mathcal{V}$ is generated by its free $\tau$-algebra $\mathfrak{Tr}_\tau(X)/_{\mathsf{Id}^X(\mathcal{V})}$, which can be constructed as a quotient of $\mathfrak{Tr}_\tau(X)$. Two terms $u$ and $v$ are identified by the quotient map if and only if the equation is valid in $\mathcal{V}$, namely $\mathcal{V} \models u = v$. In this way the equational theory of $\mathcal{V}$ determines $\mathfrak{Tr}_\tau(X)/_{\mathsf{Id}^X(\mathcal{V})}$ and $\mathcal{V}$ itself can be defined as the class of all models of its equational theory. This algebraic machinery will turn out to be very useful in the study of the intuitionistic and intermediate logics.

## 4.3 The Algebraic Approach to Intuitionistic Logics

We start by describing the first two steps involved in the algebraic study of intuitionistic propositional logic. Both are needed in order to endow the propositional language with an algebraic conceptualization.

The two steps we are about to expound can be summarized in the slogan: *propositional formulas are terms*.

The first step consist in looking at the formulas of the propositional language $\mathcal{SL}$ as the terms of the algebraic language $\tau = \{\wedge, \vee, \Rightarrow, \top, \bot\}$ with type $\{2, 2, 2, 0, 0\}$. This means that

1. every connective of $\mathcal{SL}$ of a given arity is taken as an operation symbol of the same arity (thus every 0-ary symbol of $\mathcal{SL}$ is taken as a constant), and that
2. the propositional formulas of $\mathcal{SL}$ are taken as the terms of this algebraic language; in particular the sentence symbols are the variables of the algebraic language. From this point of view the definition of formula of $\mathcal{SL}$ is exactly the definition of term of type $\tau$.

As a consequence, the set of formulas of a language $\mathcal{SL}$ turns into a$\tau$-algebra, the $\tau$-algebra of propositions of $\mathcal{SL}$.

The second and fundamental step is to build a specific $\tau$-algebra out of the propositional formulas, modulo provable equivalence, of intuitionistic logic. It thus embodies the "algebraic" structure of this logic, but in syntax invariant way. Moreover, it turns out to be free $\tau$-algebra on the variety of Heyting algebras.

The following part of this chapter is devoted to describe in more detail this process of algebraization of intuitionistic logic.

### 4.3.1 The $\tau$-Algebra of Propositions

Let us fix a set of sentence symbols $L$. As we have seen in the first chapter, we can reconstruct recursively the sentences' language $\mathcal{SL}$. The elements of $L$ are usually called atomic propositions while the elements of $\mathcal{SL}$ are the sentences. Now, according to the first step above, we can think $\mathcal{SL}$ as an algebra with similarity type $\tau := \{\wedge, \vee, \Rightarrow, \top, \bot\}$ and arity $(2,2,2,0,0)$. $\mathcal{SL}$ has an obvious $\tau$-algebra structure given by the following.

$$\begin{array}{rll} \wedge^{\mathcal{SL}} : \mathcal{SL} \times \mathcal{SL} \to \mathcal{SL} & \text{with} & (\alpha,\beta) \mapsto \alpha \wedge \beta \\ \vee^{\mathcal{SL}} : \mathcal{SL} \times \mathcal{SL} \to \mathcal{SL} & \text{with} & (\alpha,\beta) \mapsto \alpha \vee \beta \\ \Rightarrow^{\mathcal{SL}} : \mathcal{SL} \times \mathcal{SL} \to \mathcal{SL} & \text{with} & (\alpha,\beta) \mapsto \alpha \Rightarrow \beta \\ \top^{\mathcal{SL}} : \mathcal{SL} \to \mathcal{SL} & \text{with} & \alpha \mapsto \top \\ \bot^{\mathcal{SL}} : \mathcal{SL} \to \mathcal{SL} & \text{with} & \alpha \mapsto \bot \end{array}$$

In particular, $\mathcal{SL}$ is precisely the set of all $\mathfrak{Tr}_\tau(L)$ terms. In fact, due to the freedom of $\mathfrak{Tr}_\tau(L)$, a morphism $h : \mathfrak{Tr}_\tau(L) \to \mathcal{SL}$ is uniquely determined by the identity function and is an isomorphism of algebras. We can think of $h$ as a translation from a prefix notation to an infixed notation. formulas. Therefore, $\mathcal{SL}$ and $\mathfrak{Tr}_\tau(L)$ are regarded as two names for the same syntactic objects.

As a consequence of this isomorphism, the main property of the $\tau$-algebra of propositions can be summarized in the next result.

**Lemma 4.3.1.** *The algebra $\mathcal{SL}$ is the free $\tau$-algebra in $\mathcal{A}lg_\tau$ generated by $L$.*

□

### 4.3.2 The Class of $\tau$-Algebras for Intuitionistic logic

We need first to identify a class of algebras for which holds a weak form of soundness for the intuitionistic logic.

**Theorem 4.3.2.** *Let $\vdash \subseteq 2^{\mathcal{SL}} \times \mathcal{SL}$ be as above in definition 1.1.4 and $\mathcal{HA}$ the variety of Heyting algebras. For all $\varphi, \psi \in \mathcal{SL}$,*

*(1) if $\vdash \varphi$ and $H \in \mathcal{HA}$, then $h(\varphi) = \top$ for all $h \in \mathsf{Hom}(\mathcal{SL}, H)$;*

(2) *if* $\vdash \varphi$ *and* $\vdash \psi$*, then* $\mathcal{HA} \models \varphi = \psi$*;*

(3) *if* $\vdash \varphi$*, then* $\mathcal{HA} \models \varphi = \top$*.*

*Sketch of proof.* The first point is proved by what is normally qualified as routine checking; that is, by just checking, for an arbitrary $H \in \mathcal{HA}$ and an arbitrary $h \in \mathsf{Hom}(\mathcal{SL}, H)$, that $h(\varphi) = \top$ for all axioms $\varphi$ of our chosen axiomatization $\mathsf{Ax}$, and that modus ponens preserves the property of "being equal to $\top$": If $h(\varphi) = \top$ and $h(\varphi \Rightarrow \psi) = \top$, then $h(\psi) = \top$, this amounts to checking that in $H$, if $\top \Rightarrow a = \top$, then $a = \top$, for all $a \in H$, and this is obvious from the properties of Heyting algebras. Then, induction on the length of proofs completes the demonstration. The other points follows easily in succession. $\square$

The above property suggests using the "transformation" $\varphi \mapsto \varphi = \top$ to turn every formula into an equation. With this trick, it is not difficult to prove:

**Proposition 4.3.3.** *The equational class* $\mathcal{HA}$ *can be presented by the equations that result by applying the transformation* $\varphi \mapsto \varphi = \top$ *to the set of axioms* $\mathsf{Ax}$*.*

$\square$

### 4.3.3 The Lindenbaum-Tarski Algebra of Intuitionistic Logic

Consider the $\tau$-algebra $\mathcal{SL}$. We can identify $\alpha$ and $\beta$ in $\mathcal{SL}$ according to the following binary relation:

**Definition 4.3.4** (Equivalence modulo provability)**.** for all $\alpha, \beta \in \mathcal{SL}$, we define a binary relation $\equiv_\vdash$ on $\mathcal{SL}$ putting

$$\alpha \equiv_\vdash \beta \quad \text{if and only if} \quad \vdash \alpha \Leftrightarrow \beta \tag{4.3}$$

This is clearly well defined on equivalence classes, in the sense that if $\vdash p \Rightarrow q$ and $[p]_\vdash = [p']_\vdash$ then $\vdash p' \Rightarrow q$, and similarly for $q$. We can construct an Heyting algebra $\mathcal{SL}/_{\equiv_\vdash}$, consisting of equivalence classes $[p]$ of formulas $p$, according to the binary relation $\equiv_\vdash$. The operations in $\mathcal{SL}/_{\equiv_\vdash}$ are then induced in the expected way by the logical operations:

$$\top = [\top]_\vdash$$

$$\bot = [\bot]_\vdash$$

$$[p] \wedge [q]_\vdash = [p \wedge q]_\vdash$$

$$[p] \vee [q]_\vdash = [p \vee q]_\vdash$$

$$[p] \Rightarrow [q]_\vdash = [p \Rightarrow q]_\vdash$$

Again, these operations are easily seen to be well defined on equivalence classes. As a consequence, $\equiv_\vdash$ is a congruence. Moreover, these operations satisfy the axioms for a Heyting algebra because the logical axioms evidently imply them, for Proposition 4.3.3.

**Lemma 4.3.5.** *The congruence* $\equiv_\vdash$ *provides the* $\tau$*-algebra* $\mathcal{SL}$ *with the structure of Heyting algebra, namely* ${}^{\mathcal{SL}}\!/_{\equiv_\vdash} \in \mathcal{HA}$.

*Proof.* See Borceux (1994), Awodey (2010). □

So, it is in fact a model of intuitionistic logic but it is also "generic" in the sense that validates only the provable formulas.

**Lemma 4.3.6.** *The Heyting algebra* ${}^{\mathcal{SL}}\!/_{\equiv_\vdash}$ *has the property that the top element* $[\top]$ *is constituted by the set* $\nabla\emptyset$*, which is a single equivalence class, namely, for all formula* $p \in \mathcal{SL}$,

$$\vdash p \quad \textit{if and only if} \quad [p]_\vdash = \top.$$

*Proof.* See Awodey (2010). □

The quotient algebra ${}^{\mathcal{SL}}\!/_{\equiv_\vdash}$ is popularly called the Lindenbaum-Tarski algebra of intuitionistic logic.

### 4.3.4 Algebraic Completeness for Intuitionistic Logic

With the constructions made up to this point, it is clear that the Lindenbaum-Tarski algebra provides a weak completeness theorem for intuitionistic logic. In fact, we now have all the machinery to prove the main general relations linking intuitionistic logic with the variety of Heyting algebras.

**Theorem 4.3.7** (Algebraic completeness for INT)**.**

$$\mathsf{Id}^L({}^{\mathcal{SL}}\!/_{\equiv_\vdash}) = \mathsf{Id}^L(\mathcal{HA})$$

*Proof.* ${}^{\mathcal{SL}}\!/_{\equiv_\vdash} \in \mathcal{HA}$. Then,

$$\mathsf{Id}^L({}^{\mathcal{SL}}\!/_{\equiv_\vdash}) \supseteq \mathsf{Id}^L(\mathcal{HA})$$

The opposite inclusion follows from Theorem 4.3.2. □

This says that the variety $\mathcal{HA}$ is (weakly) complete for intuitionistic logic. In particular, 4.3.2(3) can now be enhanced via Deduction Theorem:

$$\vdash \varphi \quad \text{if and only if} \quad \mathcal{HA} \models \varphi = \top. \tag{4.4}$$

Another crucial consequence of this result is that $\equiv_\vdash$ and $\equiv^{\mathsf{Tr}_\tau(L)}_{\mathcal{HA}}$ are equivalent.

**Corollary 4.3.8.** *The Lindenbaum-Tarski algebra* ${}^{\mathcal{SL}}\!/_{\equiv_\vdash}$ *is the free* $\tau$*-algebra in* $\mathcal{HA}$ *generated by* $L$.

□

Since $\mathcal{SL}/_{\equiv_\vdash}$ is free on the variety of Heyting algebras, it has a universal mapping property and, given any other model $H \in \mathcal{HA}$ in Heyting algebras, there is a unique homomorphism $h : \mathcal{SL}/_{\equiv_\vdash} \to H$. In this sense, the Lindenbaum-Tarski Heyting algebra can be said to contain a "universal model" of the logic. Also every model anywhere else, in a class of ordered sets or topological spaces, is a structure preserving image of $\mathcal{SL}/_{\equiv_\vdash}$ by an essentially unique, logic-preserving function. Such a universal model is then "logically generic", in the sense that it has all and only those logical properties had by all models.

**Remark 4.3.9.**

1. This Lindenbaum-Tarski algebra's construction can be applied to a very large number of logics other than intuitionistic logic, in particular to all its axiomatic extensions. In that case, there is a subclass of Heyting algebras that plays the role of $\mathcal{HA}$ and the properties of these $\tau$-algebras produce properties of the logic, among them enhanced completeness theorems; namely, completeness theorems with respect to restricted classes of Heyting algebras that could eventually carry a more definite structure such as algebras of upper sets on a posets or algebras of specific kind of (open) sets in a topological space. We will illustrate the point by taking the argument further in Subsection 4.3.5.

2. Still in the case of intuitionistic logic, some observations or consequences of the above construction can be highlighted. Any weak representation theorem of Heyting algebras in terms of algebras in a particular subclass (see Theorem 3.3.43 and 3.3.44) yields a corresponding completeness theorem for that class (for instance, algebras of upper sets of posets or algebras of open sets of topological spaces).

The following result gives substance to the last remark above.

**Proposition 4.3.10** (Tarski-Kripke algebraic completeness for $\mathsf{INT}$)**.** *Let $C \subseteq \mathcal{HA}$ be a subclass of Heyting algebras of the form $\{\,\mathsf{Up}\,P \mid P \in \mathcal{P}os\,\}$ or $\{\,\mathcal{O}(X) \mid X \in \mathcal{T}op\,\}$. Then,*

$$\mathsf{Id}^L(\mathcal{SL}/_{\equiv_\vdash}) = \mathsf{Id}^L(C).$$

*Proof.* As a consequence of the basic adjunction, we have

$$\mathsf{Id}^L(\mathcal{SL}/_{\equiv_\vdash}) \subseteq \mathsf{Id}^L(C)$$

In order to prove the opposite inclusion, it suffices to take the contrapositive. Suppose the equation $(\varphi = \psi) \notin \mathsf{Id}^L(\mathcal{SL}/_{\equiv})$. Then, in particular it fails on some $A \in \mathcal{HA}$. By the weak representation theorems 3.3.43 and 3.3.44, there exists $A' \in C$, namely $A' = \mathsf{Up}\,P$ or $A' = \mathcal{O}(X)$ for some $P \in \mathcal{P}os$ and $X \in \mathcal{T}op$, such that $A \in \mathcal{HA}$ can be embedded in $A' \in C$. As a consequence of Remark 4.2.4(i), $\varphi = \psi$ fails on $A'$ and $(\varphi = \psi) \notin \mathsf{Id}^L(C)$. □

### 4.3.5 The Lattice of Superintuitionistic Logics

Until now, we have discussed only intuitionistic logic. In this section, we turn on superintuitionistic logics, namely logics stronger than or equal to intuitionistic logic. The first question should be "what are superintuitionistic logics from an algebraic point of view?"

We fix a set of formulas $\Gamma$, and consider $\nabla_{\vdash}\Gamma = \{ \alpha \in \mathcal{SL} \mid \Gamma \vdash \alpha \}$, the theory generated by $\Gamma$. Comparing the definition of closure operator (see subsection 2.1.6) and Remark 1.1.6(2), it easy to see that $\nabla_{\vdash}$ is a closure operator defined as

$$\varphi \in \nabla_{\vdash}\Gamma \quad \text{if and only if} \quad \Gamma \vdash \varphi.$$

Hence, $\Gamma$ is a theory if $\nabla_{\vdash}\Gamma = \Gamma$, namely if it belongs to the lattices of closed sets associated to $\nabla_{\vdash}$, which is precisely the lattice of theories $\mathcal{T}h(\mathsf{INT})$. The set $\mathcal{SL}$ of all formulas and the set $\mathsf{INT}$ of all formulas which are provable in $\vdash$ are two extreme examples of theories, the top and the bottom of the lattice, respectively.

Obviously, non every theory generated by a set of formulae is a superintuitionistic logic. According to Definition 1.2.2, a set of formulas $\Gamma$ is a superintuitionistic logic (or, an intermidiate logic, if we leave aside $\mathcal{SL}$) if and only if it is closed under both provability *and substitution*. Hence, only those theory invariant under substitutions. This suggests that intermediate logics are in correspondence with the equational theories. We will try to corroborate this intuition.

Consider now more carefully the transformation given by $\varphi \mapsto \varphi = \top$. Naturally, this map induces the adjunction $\omega^{-1} \dashv \omega$.

$$2^{\mathcal{SL}\times\mathcal{SL}} \underset{\omega^{-1}}{\overset{\omega}{\leftrightarrows}} 2^{\mathcal{SL}}$$

where $\omega(\Gamma) = \{ \alpha = \top \mid \alpha \in \Gamma \}$ and $\omega^{-1}(\Theta) = \{ \alpha \mid \alpha = \top \in \Theta \}$ spring from a generalization of one of the maps involved in Theorem 4.1.12. This pair of maps establishes an equivalence between set of formulae and set of equations of the form $\{ \alpha = \top \mid \alpha \in \Gamma \}$, for some set of sentences $\Gamma$. We can combine this equivalence with $\mathsf{Mod} \dashv \mathsf{Id}$ restricted to Heyting algebras.

$$(2^{\mathcal{HA}})^{\mathrm{op}} \underset{\mathsf{Id}^X}{\overset{\mathsf{Mod}}{\leftrightarrows}} 2^{\mathcal{SL}\times\mathcal{SL}} \underset{\omega^{-1}}{\overset{\omega}{\leftrightarrows}} 2^{\mathcal{SL}}$$

For Remark 2.1.34(1), this composite is also an adjunction

$$(2^{\mathcal{HA}})^{\mathrm{op}} \underset{\mathsf{Log}}{\overset{\mathsf{Alg}}{\leftrightarrows}} 2^{\mathcal{SL}} \tag{4.5}$$

For all $\mathcal{K} \subseteq \mathcal{HA}$, we define $\mathsf{Log}(\mathcal{K}) := \omega^{-1}\mathsf{Id}^X(\mathcal{K})$, namely as the set of formulae that are valid in all Heyting algebras of $\mathcal{K}$, that is to say

$$\mathsf{Log}(\mathcal{K}) = \{\, \alpha \in \mathcal{SL} \mid \forall \mathfrak{A} \in \mathcal{K} \quad \mathfrak{A} \models \alpha = \top \,\}.$$

Dually, we define $\mathsf{Alg}\,(\Gamma) := \mathsf{Mod}\,\omega(\Gamma)$ for all $\Gamma$, namely as the class of Heyting algebras of the theory having $\Gamma$ as a set of axioms, that is to say

$$\mathsf{Alg}\,(\Gamma) = \{\, \mathfrak{A} \in \mathcal{HA} \mid \forall \alpha \in \Gamma \quad \mathfrak{A} \models \alpha = \top \,\}.$$

**Remark 4.3.11.** Since $\mathsf{Alg} \dashv \mathsf{Log}$, they satisfy the following properties:

(1) for all subsets $\Gamma, \Gamma'$ of $\mathcal{SL}$, if $\Gamma \subseteq \Gamma'$, then $\mathsf{Alg}(\Gamma') \subseteq \mathsf{Alg}(\Gamma)$;

(2) for all subclasses $\mathcal{K}, \mathcal{K}'$ of $\mathcal{HA}$, if $\mathcal{K} \subseteq \mathcal{K}'$, then $\mathsf{Log}(\mathcal{K}') \subseteq \mathsf{Log}(\mathcal{K})$.

(3) for all subclasses $\mathcal{K}$ of $\mathcal{HA}$ and for all subsets $\Gamma$ of $\mathcal{SL}$, $\mathcal{K} \subseteq \mathsf{Alg}(\Gamma)$ if and only if $\Gamma \subseteq \mathsf{Log}(\mathcal{K})$.

Moreover, the maps $\mathsf{Log}\,\mathsf{Alg}$ and $\mathsf{Alg}\,\mathsf{Log}$ are closure operators on $\mathcal{SL}$ and on $\mathcal{HA}$, respectively. The sets closed under $\mathsf{Alg}\,\mathsf{Log}$ are exactly the sets of the form $\mathsf{Alg}(\Gamma)$, for some $\Gamma \subseteq \mathcal{SL}$, and the sets closed under $\mathsf{Log}\,\mathsf{Alg}$ are exactly the sets of the form $\mathsf{Log}(\mathcal{K})$, for some $\mathcal{K} \subseteq \mathcal{HA}$.

Let us now look at some of the meanings of the conditions now estabilished.

First consider the significance of the composite map $\mathsf{Alg}\,\mathsf{Log}$. Since $\omega\omega^{-1} = 1_{\mathcal{SL}\times\mathcal{SL}}$, this closure operator is nothing but $\mathsf{Mod}\,\mathsf{Id}$. Its fixsets are therefore varieties of Heyting algebras, axiomatized by equations of the form $\{\, \alpha = \top \mid \alpha \in \Gamma \,\}$, for some set of sentences $\Gamma$, or more simply by $\Gamma$ itself. The collection of fixsets of $\mathsf{Alg}\,\mathsf{Log}$ forms the complete lattice $\Lambda(\mathcal{HA})$ dually isomorphic to the complete lattice of fixsets of $\mathsf{Log}\,\mathsf{Alg}$, that we shall indicate $\Lambda(\mathsf{INT})$

$$\Lambda^{\mathrm{op}}(\mathcal{HA}) \underset{\mathsf{Log}}{\overset{\mathsf{Alg}}{\rightleftarrows}} \Lambda(\mathsf{INT})$$

By definition, $\mathsf{Log}\,\mathsf{Alg}(\Gamma)$ is the most inclusive set of sentences which are valid on all those Heyting algebras which are models for $\Gamma$. In more familiar terms, $\varphi \in \mathsf{Log}\,\mathsf{Alg}(\Gamma)$ if and only if every interpretation which makes all of $\Gamma$ valid makes $\varphi$ valid. Hence $\mathsf{Log}\,\mathsf{Alg}(\Gamma)$ is just the set of semantic consequences of the set of axioms $\Gamma$. More precisely, $\mathsf{Log}\,\mathsf{Alg}$ generates the smallest closure under semantical consequence.

Moreover, since $\alpha \in \mathsf{Log}\,\mathsf{Alg}(\Gamma)$ if and only if $\omega(\alpha) \in \mathsf{Id}^X\mathsf{Alg}(\Gamma)$ and the latter is invariant under substitutions (see subsection 4.2.2), we have that $\mathsf{Log}\mathsf{Alg}(\Gamma)$ is invariant under substitutions as well. Therefore, the fixsets of $\mathsf{Log}\,\mathsf{Alg}$ can be interpreted as follows.

**Proposition 4.3.12.** *Let $\mathcal{K} \subseteq \mathcal{HA}$ be a class of Heyting algebras, then $\mathsf{Log}(\mathcal{K})$, the totality of formulae valid on $\mathcal{K}$, is a superintutionistic logic.*

□

For each superintuitionistic logic $\mathsf{L}$, we can perform the Lindenbaum-Tarski construction as before (see subsection 4.3.3). More precisely, we define a binary relation $\equiv_{\mathsf{L}}$ on the set $\mathcal{SL}$ of formulas, putting (equivalence modulo $\mathsf{L}$)

$$\alpha \equiv_{\mathsf{L}} \beta \quad \text{if and only if} \quad \vdash_{\mathsf{L}} \alpha \Leftrightarrow \beta.$$

By construction, $\mathsf{Log}(\mathcal{SL}/_{\equiv_{\mathsf{L}}}) = \mathsf{L}$. Since we have

$$\mathsf{Log}(\mathcal{SL}/_{\equiv_{\mathsf{L}}}) = \mathsf{Log}(\mathsf{V}(\mathcal{SL}/_{\equiv_{\mathsf{L}}}))$$

by Lemma 4.2.19, every intermediate logics is of the form $\mathsf{Log}(\mathcal{K})$, for $\mathcal{K} = \mathsf{V}(\mathcal{SL}/_{\equiv_{\mathsf{L}}})$. Moreover, $\mathcal{SL}/_{\equiv_{\mathsf{L}}}$ generates the variety $\mathsf{Alg}(\mathsf{L})$ and it is also the free Heyting algebra in $\mathsf{Alg}(\mathsf{L})$.

**Theorem 4.3.13.** *Every intermidiate logic $\mathsf{L}$ is complete with respect to $\mathsf{Alg}(\mathsf{L})$.*

*Proof.* See Chagrov and Zakharyaschev (1997). □

Finally, let us take a closer look at the operations on both lattices of intermediate logics and subvarieties of $\mathcal{HA}$. Suppose $\{\mathsf{L}_i\}_{i\in I}$ is a collection of superintuitionistic logics. Then the intersection $\bigwedge_{i\in I} \mathsf{L}_i := \bigcap_{i\in I} \mathsf{L}_i$ is also a superintuitionistic logic. Instead, we define $\bigvee_{i\in I} \mathsf{L}_i := \mathsf{Log} \bigcap_{i\in I} \mathsf{Alg}\, \mathsf{L}_i$ as the smallest intermediate logic containing $\{\mathsf{L}_i\}_{i\in I}$.

Dually, suppose that $\{\mathcal{V}_i\}_{i\in I}$ is a collection of subvarieties of $\mathcal{HA}$. Then the intersection $\bigwedge_{i\in I} \mathcal{V}_i := \bigcap_{i\in I} \mathcal{V}_i$ is also a variety and $\bigvee_{i\in I} \mathcal{V}_i$, defined by $\mathsf{Alg} \bigcap_{i\in I} \mathsf{Log}(\mathcal{V}_i)$, is the smallest variety containing that collection $\{\mathcal{V}_i\}_{i\in I}$.

Among all of its subvarieties, $\mathcal{HA}$ is the greatest while the variety $\mathcal{BA}$ of Boolean algebras is the atom of the lattice $\Lambda(\mathcal{HA})$, namely it is greater than the smallest (trivial) variety and there are no subvarieties between them.

### 4.3.6 Semantic Universes

We can now regard specific classes of Heyting algebras as semantic universes. More precisely:

(1) Classes of Heyting algebras of the kind $\mathsf{Clup}\,(X)$, for an Esakia space $X$, are called Esakia semantics.

(2) Classes of Heyting algebras of the kind $\mathcal{O}(X)$, for a topological space $X$, are called Tarski-Stone semantics.

(3) Classes of Heyting algebras of the kind $\mathsf{Up}\,(P)$, for a poset $P$, are called Kripke semantics;

The representation theorems 3.3.38,3.3.43 and 3.3.44 allow us to express all these algebraic semantics in terms of Esakia spaces, topological spaces and order sets, respectively.

Observe that, in a semantical context, Esakia spaces and ordered sets are called Esakia frames and Kripke frames, respectively. It is worth recalling that, if we limit ourself to the finite case, they are the same (see subsection 3.3.6).

### Esakia Semantics

Consider the mapping $\mathsf{Clup}\,(\mathcal{K}) = \{\, \mathsf{Clup}\, X \mid X \in \mathcal{K}\,\}$, for all $\mathcal{K} \subseteq \mathcal{ES}$. Naturally, this map induces the adjunction $\mathsf{Clup}^{-1} \dashv \mathsf{Clup}$.

$$\left(2^{\mathcal{ES}}\right)^{\mathrm{op}} \underset{\mathsf{Clup}}{\overset{\mathsf{Clup}^{-1}}{\rightleftarrows}} \left(2^{\mathcal{HA}}\right)^{\mathrm{op}} \underset{\mathsf{Log}}{\overset{\mathsf{Alg}}{\rightleftarrows}} 2^{\mathcal{SL}}$$

For all $\Gamma \subseteq \mathcal{SL}$, we define $\mathsf{Fr}_e\,(\Gamma) := \mathsf{Clup}^{-1}\mathsf{Alg}\,(\Gamma)$, namely the class of Esakia frames that validates $\Gamma$, that is to say

$$\mathsf{Fr}_e\,(\Gamma) = \{\, X \in \mathcal{ES} \mid \forall \alpha \in \Gamma \quad \mathsf{Clup}(X) \models \alpha = \top \,\}\,.$$

Dually, we define $\mathsf{Log}_e(\mathcal{K}) := \mathsf{Log}\,\mathsf{Clup}(\mathcal{K})$, namely as the set of formulae that are valid in all Esakia frames of $\mathcal{K}$, that is to say

$$\mathsf{Log}_e(\mathcal{K}) = \{\, \alpha \in \mathcal{SL} \mid \forall X \in \mathcal{K} \quad \mathsf{Clup}(X) \models \alpha = \top \,\}\,.$$

For Remark 2.1.34(1), they can be encapsulated in the following adjunction.

$$\left(2^{\mathcal{ES}}\right)^{\mathrm{op}} \underset{\mathsf{Log}_e}{\overset{\mathsf{Fr}_e}{\rightleftarrows}} 2^{\mathcal{SL}}$$

This adjunction is exactly the same as (4.5). As a consequence,
**Theorem 4.3.14.** *Every intermidiate logic* $\mathsf{L}$ *is complete with respect to* $\mathsf{Fr}_e(\mathsf{L})$.

□

The only difference with (4.5) lies in the fact that, as a consequence of Theorem 3.3.42, a class of Esakia frames validating some set of formulae is closed under subframes (that is up-sets), continuous p-morphic images and finite disjoint unions rather than finite products, as in the pure algebraic context. Adopting the notation $- \models \varphi$ for $\mathsf{Clup}(-) \models \varphi = \top$, we can summarized the observation as follows.
**Remark 4.3.15.** The validity of formulae is preserved for subframes, continuous p-morphic images and finite disjoint unions. That is,

(i) if $X \models \varphi$ and $Y$ is a subframe of $X$, then $Y \models \varphi$;

(ii) if $X \models \varphi$ and $Z$ is a continuous p-morphic image of $X$, then $Z \models \varphi$;

(iii) if $X_i \models \varphi$ for all $i \in I$ ($I$ finite), then $\coprod_{i \in I} X_i \models \varphi$.

### Tarski-Stone Semantics

In a similar way, considering the map $\mathsf{Op}\,(\mathcal{K}) = \{\, \mathcal{O}(X) \mid X \in \mathcal{K}\,\}$, for all $\mathcal{K} \subseteq \mathcal{T}op$, we obtain

$$(2^{\mathcal{T}op})^{\mathrm{op}} \underset{\mathsf{Log}_s}{\overset{\mathsf{Sp}}{\rightleftarrows}} 2^{\mathcal{SL}}$$

where $\mathsf{Sp}\,(\Gamma) := \mathsf{Op}^{-1}\mathsf{Alg}\,(\Gamma)$ is the class of topological spaces that validates $\Gamma$, namely

$$\mathsf{Sp}\,(\Gamma) = \{\, X \in \mathcal{T}op \mid \forall \alpha \in \Gamma \quad \mathcal{O}(X) \models \alpha = \top\,\}\,.$$

and $\mathsf{Log}_s(\mathcal{K}) := \mathsf{Log}\,\mathsf{Op}(\mathcal{K})$ the set of formulae that are valid in all topological spaces of $\mathcal{K}$, that is to say

$$\mathsf{Log}_s(\mathcal{K}) = \{\, \alpha \in \mathcal{SL} \mid \forall X \in \mathcal{K} \quad \mathcal{O}(X) \models \alpha = \top\,\}\,.$$

The main difference with the previous semantics is crucial: since we have, for any intermediate logic $\mathsf{L}$, $\mathsf{L} \supseteq \mathsf{Log}_s\mathsf{Sp}\,\mathsf{L}$, there is no guarantee that $\mathsf{L}$ is complete with respect to $\mathsf{Sp}(\mathsf{L})$.

As a consequence of Lemma 3.3.25, the closure properties of classes of topological spaces validating some set of formulae can be recap in the following remark.

**Remark 4.3.16.** The validity of formulae is preserved for subspaces, open continuous images and disjoint unions. That is, adopting the notation $- \models \varphi$ for $\mathcal{O}(-) \models \varphi = \top$,

(i) if $X \models \varphi$ and $Y$ is a subspace of $X$, then $Y \models \varphi$;

(ii) if $X \models \varphi$ and $Z$ is an open continuous image of $X$, then $Z \models \varphi$;

(iii) if $X_i \models \varphi$ for all $i \in I$, then $\coprod_{i\in I} X_i \models \varphi$.

### Kripke Semantics

Analogously, considering the mapping $\mathsf{Up}\,(\mathcal{C}) = \{\, \mathsf{Up}(X) \mid X \in \mathcal{C}\,\}$ for all $\mathcal{C} \subseteq \mathcal{P}os$, we obtain

$$(2^{\mathcal{P}os})^{\mathrm{op}} \underset{\mathsf{Log}_k}{\overset{\mathsf{Fr}_k}{\rightleftarrows}} 2^{\mathcal{SL}}$$

where $\mathsf{Fr}_k\,(\Gamma) := \mathsf{Up}^{-1}\mathsf{Alg}\,(\Gamma)$ is the class of Kripke frames that validates $\Gamma$, namely

$$\mathsf{Fr}_k\,(\Gamma) = \{\, X \in \mathcal{P}os \mid \forall \alpha \in \Gamma \quad \mathsf{Up}(X) \models \alpha = \top\,\}\,.$$

and $\mathsf{Log}_k(\mathcal{C}) := \mathsf{Log}\,\mathsf{Up}(\mathcal{C})$ the set of formulae that are valid in all Kripke frames of $\mathcal{C}$, that is to say

$$\mathsf{Log}_k(\mathcal{C}) = \{\, \alpha \in \mathcal{SL} \mid \forall X \in \mathcal{C} \quad \mathsf{Up}(X) \models \alpha = \top\,\}\,.$$

Here too, since we have, for any intermediate logic $\mathsf{L}$, $\mathsf{L} \supseteq \mathsf{Log}_k\mathsf{Fr}_k\mathsf{L}$, there is no guarantee that $\mathsf{L}$ is complete with respect to $\mathsf{Fr}_k\mathsf{L}$.

The closure properties of classes of Kripke frames validating some set of formulae can be summarized as follows.

**Remark 4.3.17.** The validity of formulae is preserved for subframes, p-morphic images and disjoint unions. That is, adopting the notation $- \models \varphi$ for $\mathsf{Up}(-) \models \varphi = \top$,

(i) if $X \models \varphi$ and $Y$ is a subframe of $X$, then $Y \models \varphi$;

(ii) if $X \models \varphi$ and $Z$ is a p-morphic image of $X$, then $Z \models \varphi$;

(iii) if $X_i \models \varphi$ for all $i \in I$, then $\coprod_{i \in I} X_i \models \varphi$.

### 4.3.7 Basic Properties of Intermediate Logics

Next we look at the important properties of intermediate logics that we will be concerned with in the last part of the thesis.

First we recall the definition of the finite model property.

**Definition 4.3.18.** An intermediate logic $\mathsf{L}$ is said to have the *finite model property* if there exists a class $\mathcal{K}$ of finite Heyting algebras such that $\mathsf{L} = \mathsf{Log}(\mathcal{K})$ or, equivalently, if $\mathsf{Alg}(\mathsf{L})$ is finite approximable.

As we have already said, the variety of Heyting algebras is generated by the finite Heyting algebras. This crucial fact is a consequence of the following theorem.

**Theorem 4.3.19.** *Intuitionistic propositional logic has the finite model property. Or, equivalently, the variety of Heyting algebras is generated by the finite Heyting algebras.*

*Proof.* See Chagrov and Zakharyaschev (1997). □

Clearly every logic that has the finite model property is complete. The converse, in general, does not hold.

Let $\mathsf{L}$ be an intermediate logic. If $\mathsf{L}$ has the finite model property, then it is complete with respect to a class $\mathcal{K}$ of finite Heyting algebras. Clearly $\mathcal{K}$ can be very big. Now we define a very restricted notion of the finite model property.

**Definition 4.3.20.** An intermediate logic $\mathsf{L}$ is called *tabular* if there exists a finite Heyting algebra $H$ such that $\mathsf{L} = \mathsf{Log}(H)$ or, equivalently, if $\mathsf{Alg}(\mathsf{L})$ is finitely generated.

Obviously, if $\mathsf{L}$ is tabular, then $\mathsf{L}$ has the finite model property. However, there are logics with the finite model property that are not tabular. In particular, $\mathsf{INT}$ enjoys the finite model preperty but is not tabular (see Chagrov and Zakharyaschev 1997). The best known example of a tabular logic is the classical propositional logic $\mathsf{CL}$ (see example 4.1.20(2)).

**Definition 4.3.21.** An intermediate logic $\mathsf{L}$ is called *locally tabular* if

$$|L| < \aleph_0 \quad \text{implies} \quad {}^{\mathcal{SL}}\!/\!_{\equiv_{\mathsf{L}}} < \aleph_0.$$

This logical property finds its algebraic counterpart in the local finiteness of the corresponding varieties. Of course, every tabular logic is locally tabular. Therefore, $\mathsf{CL}$ is locally tabular (see example 4.1.20(2)). However, there are locally tabular logics that are not tabular.

The following theorem explains this connection between local tabularity and finite model property.

**Theorem 4.3.22.** *If an intermediate logic* $\mathsf{L}$ *is locally tabular, then* $\mathsf{L}$ *enjoys the finite model property.*

*Proof.* See Chagrov and Zakharyaschev (1997). □

The intuitionistic propositional logic provides a counter-example to the converse of the above theorem. As we mentioned above, $\mathsf{INT}$ has the finite model property, but it is not locally tabular (see example 4.1.20(3)).

**Remark 4.3.23.** In general we have:

tabularity $\Rrightarrow$ local tabularity $\Rrightarrow$ finite model property,

with both of the implications being strict.

TABLE 4.1: Equivalences between algebraic and logical properties

| | |
|---|---|
| Tabularity | Finite generation |
| Local tabularity | Local finiteness |
| Finite model property | Finite approximability |

### 4.3.8 The Intermediate Logics of Bounded Depth and Bounded Branching

In this section we introduce some intermediate logics and we report some facts that will be used in the next chapters.

#### The Logics of Bounded Depth

Let us consider the sequence of formulas $\mathsf{BD}_n$ defined as follows:

$$\mathsf{bd}_0 = p_0 \vee \neg p_0,$$
$$\mathsf{bd}_n = p_n \vee (p_n \Rightarrow \mathsf{bd}_{n-1}).$$

The family of logics $\mathsf{BD}_n$ of bounded depth, for $n \in \mathbb{N}$, is defined as follows:

$$\mathsf{BD}_n = \mathsf{INT} + \mathsf{bd}_n.$$

The frames for $\mathsf{BD}_n$ are the frames of depth at most $n$, as asserted in next proposition.

**Proposition 4.3.24.** *$P$ is a frame for the logic $\mathsf{BD}_n$, for $n \in \mathbb{N}$, if and only if every point of $P$ has depth at most $n$.*

*Proof.* See Chagrov and Zakharyaschev (1997). □

For those intermediate logics, there is a completeness result with respect to their relative classes.

**Theorem 4.3.25** (Segerberg's theorem)**.** *Every logic $\mathsf{BD}_n$, for $n \in \mathbb{N}$, is characterized by the class of its finite frames of depth at most $n$.*

*Proof.* See Chagrov and Zakharyaschev (1997). □

They also have the important property that their corresponding variety of algebras is locally finite.

**Theorem 4.3.26.** *Every logic $\mathsf{BD}_n$, for $n \in \mathbb{N}$, is locally tabular.*

*Proof.* See G. Bezhanishvili (2001). □

### The Logics of Bounded Branching

Let us consider the following family of formulas:

$$\mathsf{bb}_n = \bigwedge_{i=0}^{n}((p_i \Rightarrow \bigvee_{i \neq j} p_j) \Rightarrow \bigvee_{i \neq j} p_j) \Rightarrow \bigvee_{i=0}^{n} p_i, \quad n \geq 1.$$

The logics $\mathsf{BB}_n$, for $n \geq 1$ (also known as Gabbay-de Jongh), are defined as follows:

$$\mathsf{BB}_n = \mathsf{INT} + \mathsf{bb}_n.$$

Let $P$ be a finite frame and let $x \in P$. We say that $x$ has *branching* $n$ if $x$ has at most $n$ distinct immediate successors. It is not difficult to prove that:

**Proposition 4.3.27.** *Let $P$ be a finite frame. $P$ is a frame for the logic $\mathsf{BB}_n$, with $n \geq 1$, if and only if every point $x$ of $P$ has branching at most $n$.*

*Proof.* See Chagrov and Zakharyaschev (1997). □

Also for those logics, we have a completeness theorem with respect to their relative classes.

**Theorem 4.3.28.** *Every logic $\mathsf{BB}_n$, for $n \geq 1$, is characterized by the class of its finite frames of branching at most $n$.*

*Proof.* See Chagrov and Zakharyaschev (1997). □

# 5 Polyhedra: Heyting Structure and Local Finiteness

In this chapter we recall all needed definitions and results about Polyhedra. Moreover, following N. Bezhanishvili *et al.* (2018), we shall show that the collection of subpolyhedra of a given polyhedron $P$ is, in fact, a Heyting subalgebra of $\mathcal{O}(P)$; and, unlike $\mathcal{O}(P)$, it is always locally finite. This result provides one of the key insights of N. Bezhanishvili *et al.* (2018): local finiteness reflects algebraically a crucial tameness property of polyhedra as opposed to general compact subsets of $\mathbb{R}^n$.

## 5.1 Basic Notions

We begin by summarizing some notation from affine geometry to clarify the terminology and concepts that we will use. As usual, let $\mathbb{R}$ denote the field of real numbers.
**Definition 5.1.1.** Let $A \subseteq \mathbb{R}^n$. An *affine combination* of points $x_0, \ldots, x_d \in A$ is a linear combination $\sum_{i=0}^{d} \lambda_i x_i \in \mathbb{R}^n$, where $\lambda_i \in \mathbb{R}$ and $\sum_{i=0}^{d} \lambda_i = 1$. The points $x_0, \ldots, x_d \in \mathbb{R}^n$ are *affinely independent* if the vectors $x_1 - x_0, x_2 - x_0, \ldots, x_d - x_0$ are linearly independent, a condition which is invariant under permutations of the index set $\{0, \ldots, d\}$.

For example, three distinct points in the real plane are affinely independent while each set of four or more points are affinely dependent.
**Definition 5.1.2.** Let $A \subseteq \mathbb{R}^n$. A *convex combination* of $A$ is an affine combination $\sum_{i=0}^{d} \lambda_i x_i \in \mathbb{R}^n$ which additionally satisfies $\lambda_i \geq 0$ for each $i \in \{0, \ldots, d\}$. The set $\mathsf{conv}\, A$ of all convex combinations of of finite subsets of $A$ is called the *convex hull* of $A$. A set $A \subseteq R^n$ is called *convex* if $A = \mathsf{conv}\, A$.

The empty set is convex by definition. The simplest non-trivial example of a convex set is the closed interval $[a, b] \subseteq \mathbb{R}$. It is one-dimensional and is the convex hull of its end points. Analogously, for $a, b \in \mathbb{R}^n$ we define:

$$[a, b] := \{\, \lambda a + (1 - \lambda) b \leq \lambda \leq 1 \,\} = \mathsf{conv}\{a, b\}.$$

It is not difficult to see that a set $A \subseteq \mathbb{R}^n$ is convex if and only if, for every two points $x, y \in A$, the segment $[x, y]$ is contained in $A$.

**Definition 5.1.3.** A set $P \subseteq \mathbb{R}^n$ is a *polytope* if it can be expressed as the convex hull of finitely many points, that is, if $P = \mathsf{conv}\, V$ for a finite set $V \subseteq \mathbb{R}^n$. A *polyhedron* in $\mathbb{R}^n$ is any subset that can be written as a finite union of polytopes.

The union over an empty index set is allowed, so that $\emptyset$ is a polyhedron. See Figure 5.1 for a classic example of polyhedra (the Octahedron). From an topological viewpoint, a polyhedron is a closed and bounded, and hence compact, subset of $\mathbb{R}^n$.

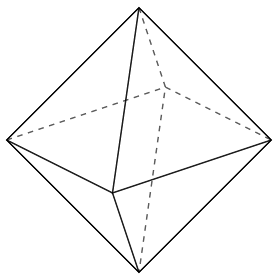

FIGURE 5.1: The Octahedron.

### 5.1.1 Simplices

**Definition 5.1.4.** A non-empty set $\zeta \in \mathbb{R}^n$ is a *simplex* if $\zeta := \mathsf{conv}\, V$, where $V := \{x_0, \dots, x_d\}$ is a set of affinely independent points called *vertices* of the simplex.

It is almost immediate to see that $V$ is the uniquely determined such affinely independent set.

**Proposition 5.1.5.** *A simplex determines its vertices, so that two simplices coincide if and only if they have the same set of vertices.*

*Proof.* See Maunder (1980). □

See Figure 5.2 for some example of simplices. We introduce now the crucial notions of face and dimension of a simplex.

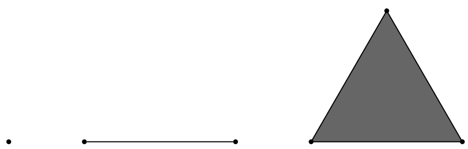

FIGURE 5.2: Some simplices.

**Definition 5.1.6.** A *face* of the simplex $\zeta$ is the convex hull of a non-empty subset of $V$, and thus is itself a simplex for a uniquely determined $V' \subseteq V$.

Hence the 0-faces of $\zeta$ are precisely its vertices. We write

$$\zeta = x_0 \cdots x_d,$$

to indicate that $\zeta$ is the *$d$-simplex* whose vertices are $x_0, \dots, x_d$.

**Definition 5.1.7.** The *(affine) dimension* of a $d$-simplex $\zeta = x_0 \cdots x_d \in \mathbb{R}^n$ is the linear-space dimension of the affine subspace of $\mathbb{R}^n$ spanned by $\zeta$. It is precisely $d$ because of the affine independence of the vertices of $\zeta$.

We write

$$\zeta \leq \xi \quad \text{and} \quad \zeta < \xi$$

to indicate that $\zeta$ is a face of $\xi$, and that $\zeta$ is a proper face of $\xi$, respectively.

Now suppose that $\zeta = x_0 \cdots x_d \in \mathbb{R}^n$ and $\xi = y_0 \cdots y_d \in \mathbb{R}^n$ are $d$-simplices in $\mathbb{R}^n$. Then $\zeta$ and $\xi$ are homeomorphic, in a rather special way.

**Proposition 5.1.8.** *$\zeta$ and $\xi$ are linearly homeomorphic, that is, there exists a homeomorphism $f : \zeta \longrightarrow \xi$, such that*

$$f\Big(\sum_{i=0}^{d} \lambda_i x_i\Big) = \sum_{i=0}^{d} \lambda_i y_i$$

*for all points of $\zeta$.*

*Proof.* See Maunder (1980). □

It follows that a $d$-simplex $\zeta$ is completely characterized, up to homeomorphism, by its dimension.

Notice that, by the affine independence of the vertices of $\zeta$, for each $x \in \zeta$ there exists a unique choice of $\lambda_i \in \mathbb{R}$ with $x = \sum_{i=0}^{d} \lambda_i x_i$ and $\lambda_i \geq 0$, $\sum_{i=0}^{d} \lambda_i = 1$. The $\lambda_i$'s are traditionally called the *barycentric coordinates* of $x$. With this in mind, we introduce the following notion.

**Definition 5.1.9.** Let $\zeta = x_0 \cdots x_d \in \mathbb{R}^n$ be a simplex. The *relative interior* of $\zeta$, denoted $\mathsf{relint}\,\zeta$, is the subset of $\zeta$ of those points $x \in \zeta$ whose barycentric coordinates are strictly positive.

Naturally, it coincides with the the topological interior of $\zeta$ in the affine subspace of $\mathbb{R}^n$ spanned by $\zeta$. The relative interior of a 0-simplex, namely a point, is the point itself. In the following, for any set $S \subseteq \mathbb{R}^n$ we use the notation

$$\mathsf{cl}\, S$$

to denote the closure of $S$ in the ambient Euclidean space $\mathbb{R}^n$. Observe that if $P \subseteq \mathbb{R}^n$ is a polyhedron and $S \subseteq P$, then the closure of $S$ in the subspace $P$ of $\mathbb{R}^n$ agrees with cl $S$, because $P$ is closed in $\mathbb{R}^n$. Note that $\mathsf{cl}\,\mathsf{relint}\zeta = \zeta$ for any simplex $\zeta$.

We recall also the notion of open star of a simplex.

**Definition 5.1.10** (Open star)**.** For $\Sigma$ a triangulation, the open star of $\sigma \in \Sigma$ is the subset of $|\Sigma|$ defined by

$$o(\zeta) := \bigcup_{\zeta \subseteq \xi \in \Sigma} \mathsf{relint}\, \xi. \tag{5.1}$$

We now consider how to combine simplices in order to obtain more complicated spaces.

### 5.1.2 Triangulations

**Definition 5.1.11.** A *triangulation* is a finite set $\Sigma$ of simplices in $\mathbb{R}^n$ satisfying the following conditions.

1. If $\zeta \in \Sigma$ and $\xi$ is a face of $\zeta$, then $\xi \in \Sigma$.
2. If $\zeta, \xi \in \Sigma$, then $\zeta \cap \xi$ is either empty, or a common face of $\zeta$ and $\xi$.

One also says that $\Sigma$ *triangulates* the subset $|\Sigma|$ of $\mathbb{R}^n$. A *subtriangulation* of the triangulation $\Sigma$ is any subset $\Delta \subseteq \Sigma$ that is itself a triangulation. This is equivalent to the condition that $\Delta$ be closed under taking faces – i.e. satisfies just 5.1.11(1). By the vertices of $\Sigma$ we mean the vertices of the simplices in $\Sigma$.

It is important to observe that a triangulation $K$ is not a topological space; it is merely a set whose elements are simplices. However, the set of points of $\mathbb{R}^n$ that lie in at least one of the simplices of $\Sigma$, topologized as a subspace of $\mathbb{R}^n$, is a topological space, called the *support*, or *underlying polyhedron*, of the triangulation $\Sigma$, that is,

$$|\Sigma| := \bigcup \Sigma \subseteq \mathbb{R}^n.$$

if $\Delta$ is a subtriangulation of $\Sigma$, then $|\Delta|$ is called a *subpolyhedron* of $|\Sigma|$. See Figure 5.3 for an example of triangulations.

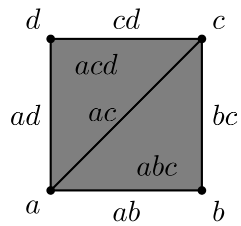


FIGURE 5.3: The triangulation of $[0,1]^2$.

Observe that a triangulation $\Sigma$ can be regarded as a poset under inclusion and a subtriangulation of $\Sigma$ is precisely the same thing as a lower set of $\Sigma$. This fact will be heavily exploited below (see Chapter 6). The following fact makes precise the idea that a triangulation $\Sigma$ provides a finitary description of the triangulated space $|\Sigma|$.

**Lemma 5.1.12.** *If $\Sigma$ is a triangulation, for each $x \in |\Sigma|$ there is exactly one simplex $\zeta^x \in \Sigma$ such that $x \in$ relint $\zeta$.*

*Proof.* See Maunder (1980). □

The simplex $\zeta^x$ is called the *carrier of $x$* in $\Sigma$.

So far we have been concerned exclusively with triangulations and their associated polyhedra, and have said nothing about maps that preserve the simplicial structure. To this end, we make the following definition.

**Definition 5.1.13.** Given triangulations $\Sigma$ and $\Delta$, a *simplicial map $f : \Sigma \to \Delta$* is a function from $\Sigma$ to $\Delta$ with the following properties.

(1) If $x_i$ is a vertex of a simplex of $\Sigma$, then $f(x_i)$ is a vertex of a simplex of $\Delta$.

(2) If $x_0x_1 \cdots x_n$ is a simplex of $\Sigma$, then $f(x_0)f(x_1)\cdots f(x_n)$ span a simplex of $\Delta$ (possibly with repeats).

(3) If $x = \sum_{i=0}^{d} \lambda_i x_i$ is in a simplex $x_0x_1 \cdots x_n$ of $\Sigma$, then $f(x) = \sum_{i=0}^{d} \lambda_i f(x_i)$;

in other words, $f$ is linearly onto on each simplex.

Simplicial maps are then determined by their restriction to the vertices and induce continuous maps between the underlying polyhedra of the triangulations: it suffices to extend linearly using barycentric coordinates.

**Proposition 5.1.14.** *Let $f : \Sigma \to L$ be a simplicial map. Then $f : |\Sigma| \to |\Delta|$ is continuous. Moreover, if $f : \Sigma \to \Delta$ is bijective, $f : |\Sigma| \to |\Delta|$ is an homeomorphism.*

*Proof.* See Maunder (1980). □

We need also to recall the following result.

**Lemma 5.1.15.** *If $\Sigma$ is is a disjoint union of $\Sigma_i$, $i = 1, \ldots, k$, that is*

$$\Sigma = \coprod_{i=0}^{k} \Sigma_i,$$

*then*

$$|\Sigma| = \coprod_{i=0}^{k} |\Sigma_i|.$$

□

### 5.1.3 The Triangulation Lemma and its Consequences

Any subset of $\mathbb{R}^n$ that admits a triangulation, being a finite union of simplices, is evidently a polyhedron. The rather less trivial converse is true, too, in the following strong sense.

**Lemma 5.1.16** (Triangulation Lemma)**.** *Given finitely many polyhedra $P, P_1, \ldots, P_m$ in $\mathbb{R}^n$ with $P_i \subseteq P$ for each $i \in \{1, ..., m\}$, there exists a triangulation $\Sigma$ of $P$ such that, for each $i \in \{1, ..., m\}$, the collection $\Sigma_i := \{\zeta \in \Sigma | \zeta \subseteq P_i\}$ is a triangulation of $P_i$, i.e. $|\Sigma_i| = P_i$.*

*Proof.* See Rourke and Sanderson 1982. □

The Triangulation Lemma is the fundamental tool in the following. In order to better understand its importance we can write, for each polyhedron $P \subseteq \mathbb{R}^n$, $\mathsf{Sub}_c\, P$ for the collection of subpolyhedra of $P$ – i.e., polyhedra in $\mathbb{R}^n$ contained in $P$. We set also

$$\mathsf{Sub}_o\, P := \{O \subseteq P | P \backslash O \in \mathsf{Sub}_c\, P\},$$

whose members are called *open (sub)polyhedra* of $P$.

Here is a first consequence of Lemma 5.1.16.
**Corollary 5.1.17.** *For any polyhedron $P \subseteq \mathbb{R}^n$, both $\mathsf{Sub}_c\, P$ and $\mathsf{Sub}_o\, P$ are distributive lattices (under set-theoretic intersections and unions) bounded above by $P$ and below by $\emptyset$.*

*Proof.* See N. Bezhanishvili *et al.* (2018). □

In Subsection 5.2 we shall see a strengthening of Corollary 5.1.17 to the effect that $\mathsf{Sub}_o\, P$ is a Heyting subalgebra of the Heyting algebra $\mathcal{O}(P)$. Before moving on, we introduce the notion of dimension for polyhedra.
**Definition 5.1.18.** The *(affine) dimension* of a nonempty polyhedron $P$ in $\mathbb{R}^n$ is the maximum of the dimensions of all simplices contained in $P$ ; if $P = \emptyset$, its dimension is $-1$. We write $\dim P$ for the dimension of $P$. Given a triangulation $\Sigma$ in $\mathbb{R}^n$, the (combinatorial) dimension of $\Sigma$ is

$$\dim \Sigma := \max \{\, d \in \mathbb{N} \mid \text{ there exists } \zeta \in \Sigma \text{ such that } \zeta \text{ is a } d\text{-simplex} \,\}$$

Again, the dimension of an empty triangulation is $-1$. Given $d \in \mathbb{N}$, we shall denote by $\mathsf{P}_d$ the set of all polyhedra of dimension less than or equal to $d$.

We recall now the essential notion of Lebesgue covering dimension (for more details, see Pears 1975).
**Definition 5.1.19.** A topological space $X$ is said to have the *Lebesgue covering dimension* $d < \infty$ if $d$ is the smallest non-negative integer with the property that each finite open cover of $X$ has a refinement in which no point of $X$ is included in more than $d + 1$ elements.

With the Triangulation Lemma 5.1.16 available, we have the following equivalent characterizations of polyhedra's dimension.
**Lemma 5.1.20.** *For any polyhedron $\emptyset \neq P \subseteq \mathbb{R}^n$ and every $d \in \mathbb{N}$, the following are equivalent.*

1. $\dim P = d$

2. *There exists a triangulation $\Sigma$ of $P$ such that* $\dim \Sigma = d$.

3. *All triangulations $\Sigma$ of $P$ satisfy* $\dim \Sigma = d$.

4. *The Lebesgue covering dimension of the topological space $P$ is $d$.*

□

## 5.2 The Locally Finite Heyting Algebra of a Polyhedron

Throughout this section we fix $n \in \mathbb{N}$ along with a polyhedron $P \subseteq \mathbb{R}^n$. We shall study the distributive lattice $\mathsf{Sub}_o P$ (see Corollary 5.1.17). We begin by proving that $\mathsf{Sub}_o P$ is in fact a Heyting algebra. We then prove that $\mathsf{Sub}_o P$ is always locally finite. Through this section we essentialy follow N. Bezhanishvili *et al.* (2018).

### 5.2.1 The Heyting Algebra of Open Subpolyhedra

We need the following elementary observation on relative interiors.

**Lemma 5.2.1.** *Let $\Sigma$ be a triangulation in $\mathbb{R}^n$, let $\xi = x_0 \cdots x_d$ be a simplex of $\Sigma$, and let $x \in \mathsf{relint}\, \xi$. Then no proper face $\zeta < \xi$ contains $x$. Hence, in particular, the carrier $\zeta^x$ of $x$ in $\Sigma$ is the inclusion-smallest simplex of $\Sigma$ containing $x$.*

*Proof.* There are $r_0, \ldots, r_d \in (0,1]$ such that $x = \sum_{i=0}^{d} \lambda_i x_i$ and $\sum_{i=0}^{d} \lambda_i = 1$. Let $\rho_i := x_0 \cdots x_{i-1} x_{i+1} \cdots x_d$. Clearly $\rho_i < \xi$ for each $i \in \{0, \ldots, d\}$, and for each $\zeta < \xi$ there exists $i \in \{0, \ldots, d\}$ such that $\zeta \leq \rho_i$. Hence, if we assume by way of contradiction that $x \in \zeta < \xi$, then $x \in \rho_i$ for some $i \in \{0, ..., d\}$; say $x \in \rho_0$. Then $x = \sum_{i=1}^{d} s_i x_i$, for some $s_1, \ldots, s_d \in [0,1]$ such that $\sum_{i=1}^{d} s_i = 1$. It follows that $r_0 = \sum_{i=1}^{d} (s_i - \lambda_i)$, and so

$$0 = x - x = \sum_{i=1}^{d} s_i x_i - \sum_{i=1}^{d} \lambda_i x_i = \sum_{i=1}^{d} (s_i - \lambda_i) x_i - \lambda_0 x_0 = \sum_{i=1}^{d} (s_i - \lambda_i)(x_i - x_0).$$

Since $\lambda_0 > 0$, there must be $i \in \{1, \ldots, d\}$ such that $s_i - \lambda_i \neq 0$, contradicting the affine independence of $x_0, \ldots, x_d$. □

The next lemma is the key fact of this section.

**Lemma 5.2.2.** *Let $P$ and $Q$ be polyhedra in $\mathbb{R}^n$ with $Q \subseteq P$, and suppose $\Sigma$ is a triangulation of $P$ such that*

$$\Sigma_Q := \{\, \zeta \in \Sigma \mid \zeta \subseteq Q \,\},$$

*triangulates $Q$. Define*

- $C := \mathsf{cl}\,(P \backslash Q)$
- $\Sigma_C := \{\, \zeta \in \Sigma \mid \zeta \subseteq C \,\}$
- $\Sigma^* := \{\, \zeta \in \Sigma \mid$ *There exists $\xi \in \Sigma \backslash \Sigma_Q$ such that $\zeta \leq \xi$* $\}$.

*Then*

*(1)* $\Sigma_C = \Sigma^*$*, and*

*(2)* $|\Sigma_C| = |\Sigma^*| = C$.

*In particular, $C$ is a polyhedron.*

*Proof.* We first show that $\Sigma^*$ triangulates $C$, that is:

$$|\Sigma^*| := \bigcup \Sigma^* = C. \tag{5.2}$$

To show $|\Sigma^*| \subseteq C$, let $\zeta \in \Sigma^*$, and pick $\xi \in \Sigma \backslash \Sigma_Q$ such that $\zeta \leq \xi$. We prove that $\mathsf{relint}\, \xi \subseteq P \backslash Q$. For, if $x \in \mathsf{relint}\, \xi$, by Lemma 5.2.1 there are no simplices $\zeta \in \Sigma$ such that $x \in \zeta < \xi$. Then, by definition of triangulation, for any simplex $\rho \in \Sigma$, $x \in \rho$ entails $\xi \leq \rho$. Hence no simplex of $\Sigma_Q$ contains $x$, or equivalently, $x \notin Q$ and therefore $\mathsf{relint}\, \xi \subseteq P \backslash Q$.

Now, it is clear that any simplex $\xi$ satisfies $\xi = \mathsf{cl}\, \mathsf{relint}\, \xi$. It follows that $\zeta \subseteq \xi = \mathsf{cl}\, \mathsf{relint}\, \xi \subseteq \mathsf{cl}\, (P \backslash Q)$, and thus $|\Sigma^*| \subseteq C$ as was to be shown.

Conversely, to show $C \subseteq |\Sigma^*|$, let $x \in C$. Since $C$ is the closure of $P \backslash Q$ in $\mathbb{R}^n$, there exists a sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq P \backslash Q$ that converges to $x$. Clearly the carrier $\zeta^{x_i}$ of $x_i$ in $\Sigma$ lies in $\Sigma \backslash \Sigma_Q$, for all $i \in \mathbb{N}$. Since $\Sigma \backslash \Sigma_Q$ is finite, there must exist a simplex $\xi \in \Sigma \backslash \Sigma_Q$ containing infinitely many elements of $\{x_i\}_{i \in \mathbb{N}}$. Then there exists a subsequence of $\{x_i\}_{i \in \mathbb{N}}$ that is contained in $\xi$ and converges to $x$. Since $\xi$ is closed, $x \in \xi$, and therefore $x \in |\Sigma^*|$ as was to be shown.

This establishes (5.2). It now suffices to prove 5.2.2(1). For the non-trivial inclusion $\Sigma_C \subseteq \Sigma^*$, let $\zeta \in \Sigma$ be such that $\zeta \subseteq C$, and pick $\beta \in \mathsf{relint}\, \zeta$. There is a sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq P \backslash Q$ converging to $\beta \in \zeta$. Since each $x_i$ is in some simplex of $\Sigma \backslash \Sigma_Q$ and $\Sigma$ is finite, there must exist a simplex $\xi \in \Sigma \backslash \Sigma_Q$ containing a subsequence of $\{x_i\}_{i \in \mathbb{N}}$ that converges to $\beta$. Since $\xi$ is closed, $\beta \in \xi$. But by Lemma 5.2.1, $\zeta^\beta = \zeta$, so that $\zeta \subseteq \xi$ and $\zeta \in \Sigma^*$. ◻

**Corollary 5.2.3.** *Given polyhedra $Q_1, Q_2$ in $\mathbb{R}^n$, the set $\mathsf{cl}(Q_2 \backslash Q_1)$ is a polyhedron.*

*Proof.* Observe that $Q_2 \backslash Q_1 = Q_2 \backslash (Q_1 \cap Q_2)$ and apply Corollary 5.1.17 together with Lemma 5.2.2 to the set $P := \mathsf{conv}\, (Q_1 \cup Q_2)$, which clearly is a polyhedron. ◻

**Corollary 5.2.4.** *The lattice $\mathsf{Sub}_c\, P$ of is closed under the co-Heyting implication (see 3.14) of $C(P)$. Dually, the lattice $\mathsf{Sub}_o\, P$ is closed under the Heyting implication (see Example 3.3.28(1)) of $O(P)$.*

◻

### 5.2.2 Local Finiteness of the Heyting Algebra of Open Subpolyhedra

Having established that $\mathsf{Sub}_o P$ is a Heyting subalgebra of $O(P)$, we infer an important structural property of $\mathsf{Sub}_o P$, local finiteness. For this, we first identify the class of subalgebras of $\mathsf{Sub}_o P$ that corresponds to triangulations of $P$. These algebras will have also a central role in Chapter 6.

**Definition 5.2.5** ($\Sigma$-definable polyhedra)**.** For any triangulation $\Sigma$ in $\mathbb{R}^n$, we write $P_c(\Sigma)$ for the sublattice of $O(|\Sigma|)$ generated by $\Sigma$, and $P_o(\Sigma)$ for the sublattice of $O(|\Sigma|)$ generated by $\{\ |\Sigma|\setminus C | C \in P_c(\Sigma)\}$. We call $P_c(\Sigma)$ the set of $\Sigma$-*definable polyedra*, and $P_o(\Sigma)$ the set of $\Sigma$-*definable open polyedra*.

Note that we have

$$P_c(\Sigma) = \{ C \subseteq \mathbb{R}^n \mid C \text{ is the union of some subset of } \Sigma \} .$$

**Lemma 5.2.6.** *For any triangulation $\Sigma$ of $P$, $P_c(\Sigma)$ is a co-Heyting subalgebra of $\mathsf{Sub}_c P$. Dually, $P_o(\Sigma)$ is a Heyting subalgebra of $\mathsf{Sub}_o P$.*

*Proof.* For any $\emptyset \neq C, D \in P_c(\Sigma)$, it follows immediately by the assumptions that $C$ and $D$ are triangulated by the collection of simplices of $\Sigma$ contained in $C$ and $D$, respectively. Hence $C \Rightarrow D := \mathsf{cl}(C \setminus D) = |\Sigma^*| = \Sigma^*$ by Corollary 5.2.3 and Lemma 5.2.2, where $\Sigma^*$ is the appropriate subset of $\Sigma$ as per Lemma 3.2. Thus $C \Rightarrow D \in P_c(\Sigma)$. □

**Corollary 5.2.7.** *Let $H$ be the co-Heyting subalgebra of $\mathsf{Sub}_c P$ generated by finitely many polyhedra $P_1, \ldots, P_m \subseteq P$ . Let further $\Sigma$ be any triangulation of $P$ that triangulates each $P_i$, $i \in \{1, \ldots, m\}$. Then $H$ is a co-Heyting subalgebra of $P_c(\Sigma)$. In particular, $H$ is finite. Dually for the Heyting subalgebra of $\mathsf{Sub}_o P$ generated by $P \setminus P_i$, $i \in \{1, \ldots, m\}$.*

*Proof.* Each $P_i$ is the union of those simplices of $\Sigma$ that are contained in $P_i$, by assumption. It follows that the distributive lattice $L$ generated in $\mathsf{Sub}_c P$ by $\{P_1, \ldots, P_m\}$ is entirely contained in $P_c(\Sigma)$. Now, if $C, D \in L$, $C \Leftarrow D := \mathsf{cl}(C \setminus D) = |\Sigma^*| = \Sigma^*$ by Corollary 5.2.3 and Lemma 5.2.2, where $\Sigma^*$ is the appropriate subset of $\Sigma$ as per Lemma 5.2.2. Hence $C \Leftarrow D \in P_c(\Sigma)$, as was to be shown. □

**Corollary 5.2.8.** *The Heyting algebra $\mathsf{Sub}_o P$ is locally finite, and so is the co-Heyting algebra $\mathsf{Sub}_c P$.*

□

Before proceeding to next Chapter, it is important to introduce the notion of logic for a given family of polyhedra.

**Definition 5.2.9.** If $\mathcal{P}$ is any family of polyhedra, we write $\mathsf{Log}\,\mathcal{P}$ for the extension of intuitionistic logic determined by $\mathcal{P}$, namely

$$\mathsf{Log}\,\mathcal{P} := \{ \alpha \in \mathcal{SL} \mid \forall P \in \mathcal{P} \quad \mathsf{Sub}_o\, P \models \alpha \} ,$$

the unique intermediate logic corresponding to the variety of Heyting algebras generated by the collection of Heyting algebras $\{ \mathsf{Sub}_o P \mid P \in \mathcal{P} \}$.

# 6 Topological Dimension and Bounded Depth

The aim of this chapter is to present the main results of N. Bezhanishvili *et al.* (2018): For each $d \in \mathbb{N}$, the logic of all polyhedra of dimension less than or equal to $d$, $\mathsf{Log}\,\mathsf{P}_d$, is intuitionistic logic extended by the axiom schema $\mathsf{bd}_d$.

We first prove that $\mathsf{Log}\,\mathsf{P}_d$ is contained in intuitionistic logic extended by the axiom schema $\mathsf{bd}_d$. Following to N. Bezhanishvili *et al.* (2018), this result is a consequence of the analysis of frames arising from triangulations and a combinatorial counterpart of the result for triangulations, Lemma 6.2.1 below. The converse is obtained by constructing a polyhedron $P$ of dimension $d$ such that the Heyting algebra of upper sets of $A$ embeds into the Heyting algebra $\mathsf{Sub}_o\,P$. To accomplish the proof, the notion of nerve of a poset will turn out decisive.

## 6.1 Frames of Algebras of Definable Polyhedra

Consider a triangulation $\Sigma$, and the finite Heyting algebra $P_o(\Sigma)$. We shall henceforth regard $\Sigma$ as a poset under the inclusion order. Note that the inclusion order of $\Sigma$ is the same thing as the "face order" $\zeta \leq \xi$ we have been using above: since $\Sigma$ is a triangulation, $\zeta \subseteq \xi$ implies $\zeta \leq \xi$, and the converse implication is obvious. We shall show that the Heyting algebra of upper sets of $\Sigma$ is isomorphic to $P_o(\Sigma)$; or, equivalently, through Esakia duality (see Theorem 3.3.40), that the face poset $\Sigma$ is isomorphic to $\mathsf{Spec}\,P_o(\Sigma)$.

We recall that the open star of any simplex is an open subpolyhedron (see Maunder 1980), that is, for each $\zeta \in \Sigma$

$$o(\zeta) \in P_o(\Sigma). \tag{6.1}$$

Indeed, set

$$K_\zeta := \{\xi \in \Sigma | \zeta \nsubseteq \xi\}.$$

Then $K_\zeta$ is clearly a subtriangulation of $\Sigma$, $|K_\zeta|$ is a subpolyhedron of $|\Sigma|$, and thus $O := |\Sigma| \setminus |K_\zeta| \in P_o(|\Sigma|)$; but one can show using Lemma 5.1.12 that $O = o(\zeta)$, so (6.1) holds. We now define a function

$$\begin{aligned} \gamma^{\uparrow} : \mathsf{Up}\,\Sigma \to & P_o(\Sigma). \\ U \in \mathsf{Up}\,\Sigma \mapsto & \bigcup_{\zeta \in U} \text{relint } \xi. \end{aligned} \tag{6.2}$$

To see that $\gamma^{\uparrow}$ is well-defined, use the fact that $\Sigma$ is a finite poset to list the minimal elements $\zeta_1, ..., \zeta_u$ of the upper set $U$. Then

$$U = \uparrow \zeta_1 \cup \cdots \uparrow \zeta_u,$$

so that

$$\begin{aligned} \gamma^{\uparrow}(U) =& \gamma^{\uparrow}(\uparrow \zeta_1) \cup \cdots \cup \gamma^{\uparrow}(\uparrow \zeta_u), \\ =& \bigcup_{\zeta_1 \subseteq \xi \in \Sigma} \text{relint } \xi \cup \cdots \cup \bigcup_{\zeta_u \subseteq \xi \in \Sigma} \text{relint } \xi \\ =& o(\zeta_1) \cup \cdots \cup o(\zeta_u) \end{aligned}$$

Thus $\gamma^{\uparrow}(U)$ is a union of open stars and hence a member of $\mathrm{P}_o(\Sigma)$.

**Lemma 6.1.1.** *The map $\gamma^{\uparrow}$ of (6.2) is an isomorphism of the finite Heyting algebras $\mathsf{Up}\,\Sigma$ and $P_o(\Sigma)$.*

*Proof.* It suffices to show that $\gamma^{\uparrow}$ is an isomorphism of distributive lattices. It is clear that $\gamma^{\uparrow}$ preserves the top and bottom elements, and that it preserves unions: if $U, V \in \mathsf{Up}\,\Sigma$ then

$$\gamma^{\uparrow}(U \cup V) = \bigcup_{\zeta \in U \cup V} \mathsf{relint}\,\zeta = \bigcup_{\zeta \in U \cup V} \mathsf{relint}\,\zeta \cup \bigcup_{\zeta \in U \cup V} \mathsf{relint}\,\zeta = \gamma^{\uparrow}(U) \cup \gamma^{\uparrow}(V).$$

Concerning intersections,

$$\begin{aligned} \gamma^{\uparrow}(U) \cap \gamma^{\uparrow}(V) &= \bigcup_{\zeta \in U} \mathsf{relint}\,\zeta \cap \bigcup_{\xi \in V} \mathsf{relint}\,\xi \\ &= \bigcup_{\zeta \in U} \bigcup_{\xi \in V} \mathsf{relint}\,\zeta \cap \mathsf{relint}\,\xi \\ &= \bigcup_{\zeta \in U, \xi \in V} \mathsf{relint}\,\zeta \cap \mathsf{relint}\,\xi \end{aligned} \tag{6.3}$$

By Lemma 5.1.12, for any two $\zeta, \xi \in \Sigma$ the intersection $\mathsf{relint}\,\zeta \cap \mathsf{relint}\,\xi$ is empty as soon as $\zeta \neq \xi$. Hence from (6.3) we deduce

$$\gamma^{\uparrow}(U) \cap \gamma^{\uparrow}(V) = \bigcup_{\delta \in U \cap V} \mathsf{relint}\,\delta = \gamma^{\uparrow}(U \cap V),$$

as was to be shown.

To prove $\gamma^{\uparrow}$ is surjective, let $O \in P_o(\Sigma)$ and set $P := |\Sigma| \setminus O \in P_c(\Sigma)$. Then, by definition of $P_c(\Sigma)$, there is exactly one subtriangulation $\Delta$ of $\Sigma$ such that $P = |\Delta|$, and $\Delta$ is a lower set of (the poset) $\Sigma$. Set $U := \Sigma \setminus \Delta$, so that $U$ is an upper set of $\Sigma$. We show:

$$O = \bigcup_{\zeta \in U} \operatorname{relint} \zeta \tag{6.4}$$

To prove (6.4) we use the fact that, since $P$ is a member of $P_c(\Sigma)$, for every $\zeta \in \Sigma$ we have

$$\operatorname{relint} \zeta \cap P \neq \emptyset \text{ if and only if } \zeta \subseteq P \tag{6.5}$$

Only the left-to-right implication in (6.5) is non-trivial, and we prove the contrapositive. Assume $\zeta \not\subseteq P$. If $\zeta \cap P = \emptyset$ obviously $\operatorname{relint} \zeta \cap P = \emptyset$. Otherwise $\xi := \zeta \cap P$ must be a proper face of $\zeta$, and therefore $\operatorname{relint} \zeta \cap \xi = \emptyset$; hence $\operatorname{relint} \zeta \cap P = \emptyset$. This establishes (6.5).

Now, to show (6.4), if $x \in O$ then the carrier $\zeta^x \in \Sigma$ is such that $\operatorname{relint} \zeta \cap P = \emptyset$, so $\zeta^x \not\subseteq P$; equivalently, $\zeta^x \notin \Delta$. Then $\zeta^x \in U$ and hence $x \in \bigcup_{\zeta \in U} \operatorname{relint} \zeta$. Conversely, if $x \notin O$, then $x \in P$, so $\operatorname{relint} \zeta^x \cap P = \emptyset$ and thus $\zeta^x \subseteq P$; equivalently, $\zeta^x \in \Delta$. Then $\zeta^x \in U$ and hence $x \in \bigcup_{\zeta \in U} \operatorname{relint} \zeta$. This proves (6.4).

In light of (6.4) we now have $\gamma^{\uparrow}(U) = O$ so that $\gamma^{\uparrow}$ is surjective.

Finally, to prove injectivity, it suffices to recall that relative interiors of simplices in $\Sigma$ are pairwise-disjoint, so the union in (6.2) is in fact a disjoint one, which makes the injectivity of $\gamma^{\uparrow}$ evident. □

See Figure 6.1 for an example of intuitionistic frame corresponding to a given triangulation.

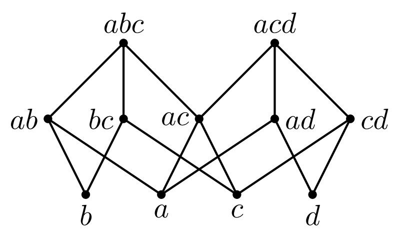


FIGURE 6.1: The intuitionistic frame of the triangulation of $[0, 1]^2$.

## 6.2 Topological Dimension Through Bounded Depth

We can prove now the following Lemma:

**Lemma 6.2.1.** *Let $\Sigma$ be a triangulation in $\mathbb{R}^n$.*

*(1) The join-irreducible elements of $P_c(\Sigma)$ are the simplices of $\Sigma$.*

(2) *The join-irreducible elements of $P_o(\Sigma)$ are the open stars of simplices of $\Sigma$.*

(3) *In both $P_c(\Sigma)$ and $P_o(\Sigma)$ there is a chain of prime filters having cardinality* $\dim \Sigma + 1$. *In neither $P_c(\Sigma)$ nor $P_o(\Sigma)$ is there a chain of prime filters having strictly larger cardinality.*

*Proof.* Item (1) follows from direct inspection of the definitions. Item (2) is an immediate consequence of Lemma 6.1.1 along with Esakia duality (see Theorem 3.3.40). To prove (3), set $d := \mathsf{dim}\,\Sigma$ and note that by definition $\Sigma$ contains at least one $d$-simplex $\zeta = x_0 \cdots x_d \in \Sigma$. By item (1) the chain of simplices $x_0 < x_0x_1 < \cdots < x_0x_1\cdots x_d = \zeta$ is a chain of join-irreducible elements of $P_c(\Sigma)$, and the principal filters generated by these elements yields a chain of prime filters of $P_c(\Sigma)$ of cardinality $d+1$. On the other hand, any chain of prime filters of $P_c(\Sigma)$ must be finite because $P_c(\Sigma)$ is. If $p_1 \subset p_2 \subset \cdots \subset p_l$ is any such chain of prime filters, then each pi is principal – again because $P_c(\Sigma)$ is finite – its unique generator $p_i$ is join-irreducible, and we have $p_l < p_{l-1} < \cdots < p_2 < p_1$ in the order of the lattice $P_c(\Sigma)$. Then $p_i \in \Sigma$, and clearly, since the simplex $p_1$ has $l-1$ proper faces of distinct dimensions, $\mathsf{dim}\, p_1 \geq l-1$. But $d := \mathsf{dim}\, p_1$ by definition of $d := \mathsf{dim}\,\Sigma$, and therefore $d+1 \geq l$, as was to be shown. The proof for $P_o(\Sigma)$ is analogous, using item (2). □

Before finally relating the bounded-depth formulae to topological dimension by giving a proof of the Theorem 6.2.3 below, we recall the following Lemma on Heyting algebras.

**Lemma 6.2.2.** *For any non-trivial Heyting algebra $H$ and each $d \in \mathbb{N}$, the following are equivalent.*

1. *The longest chain of prime filters in $H$ has cardinality $d+1$.*
2. $\mathsf{dep}\,\mathsf{Spec}\, H = d$.
3. *$H$ satisfies the equation $\mathsf{bd}_d = \top$, and fails each equation $\mathsf{bd}_{d'} = \top$ with $1 \leq d' < d$.*

*Proof.* See Chagrov and Zakharyaschev (1997). □

**Theorem 6.2.3.** *For any polyhedron $\emptyset \neq P \subseteq \mathbb{R}^n$ and every $d \in \mathbb{N}$, the following are equivalent.*

*(i)* $\dim P = d$.

*(ii) The Heyting algebra $\mathrm{Sub}_o P$ satisfies the equation $\mathsf{bd}_d = \top$, and fails each equation $\mathsf{bd}_{d'} = \top$ for each integer $0 \leq d' < d$.*

*Proof.* (i) implies (ii): By Lemma 5.1.20, $\mathsf{dim}\,\Sigma = d$ for any triangulation $\Sigma$ of $d$. By Lemmas 6.2.2, 5.2.6, and 6.2.1, the subalgebra $P_o(\Sigma)$ of $\mathsf{Sub}_o P$ satisfies the equation $\mathsf{bd}_d = \top$, and fails each equation $\mathsf{bd}_{d'} = \top$ for each integer $0 \leq d' < d$. To complete the proof it thus suffices to show that any finitely generated subalgebra of $\mathsf{Sub}_o P$ is a subalgebra of $P_o(\Sigma)$ for some triangulation $\Sigma$ of $P$. But this is precisely the content of the Triangulation Lemma 5.1.16.

(ii) implies (i): We prove the contrapositive. Suppose first $\mathsf{dim}\, P > d \geq 0$. Then, by (i) implies (ii), $\mathsf{Sub}_o P$ fails the equation $\mathsf{bd}_d$, so that (ii) does not hold. On the other hand, if $0 \leq d' := \mathsf{dim}\, P < d$, by (i) implies (ii) we know that $\mathsf{Sub}_o P$ satisfies the equation $\mathsf{bd}_{d'} = \top$, so again (ii) does not hold. □

### 6.2.1 Nerves of Posets and the Geometric Finite Model Property

In this section we use a classical construction in polyhedral geometry to realise finite posets geometrically.

**Construction** The nerve (see Alexandrov 1998 and Björner 1995) of a finite poset $A$ is the set

$$\mathcal{N}(A) = \{\, \emptyset \neq C \subseteq A \mid C \text{ is totally ordered by the restriction of } \leq \text{ to } C \times C \,\}$$

In other words, the nerve of $A$ is the collection of all chains of $A$. We always regard the nerve $\mathcal{N}(A)$ as a poset under inclusion order. Let us display the elements of $A$ as $\{a_1, \dots, a_n\}$. Let $e_1, \dots, e_n$ denote the vectors in the standard basis of the linear space $\mathbb{R}^n$. The triangulation induced by the nerve $\mathcal{N}(A)$ is the set of simplices

$$\nabla(\mathcal{N}(A)) := \{\, \mathsf{conv}\, \{e_{i_1}, \dots, e_{i_l}\} \mid \{a_{i_1}, \dots, a_{i_l}\} \in \mathcal{N}(A) \,\}$$

Then it is immediate that $\nabla(\mathcal{N}(A))$ indeed is a triangulation in $\mathbb{R}^n$, and its underlying polyhedron $|\nabla(\mathcal{N}(A))|$ is called the geometric realisation of the poset $A$.

For the proof of Theorem 6.2.4 below, we set $\Sigma := \nabla(\mathcal{N}(A))$. Using the fact that simplices are uniquely determined by their vertices (see Proposition 5.1.5), we see that the map

$$a_{i_1} < a_{i_2} < \cdots < a_{i_l} \in \mathcal{N}(A) \longmapsto \mathsf{conv}\, \{e_{i_1}, \dots, e_{i_l}\} \in \Sigma$$

is an order-isomorphism between $\mathcal{N}(A)$ and $\Sigma$, the latter ordered by inclusion. Therefore,

$$\mathsf{dim}\, \Sigma = \text{cardinality of the longest chain in } A = \mathsf{dep}\, A.$$

To prove Theorem 6.2.4 below it will suffice to construct a p-morphism $\mathcal{N}(A) \twoheadrightarrow A$. To this end, let us define a function

$$f : \mathcal{N}(A) \to A$$
$$C \in \mathcal{N}(A) \longmapsto \mathsf{max}\, C \in A$$

where the maximum is computed in the poset $A$.

**Theorem 6.2.4.** *Let $A$ be a finite, nonempty poset of cardinality $n \in \mathbb{N}$. There exists a triangulation $\Sigma$ in $\mathbb{R}^n$ satisfying the following conditions.*

*(1)* $\mathsf{dep}\, A = \mathsf{dim}\, \Sigma$

(2) *There is a surjective p-morphism $\Sigma \twoheadrightarrow A$, where $\Sigma$ is equipped with the inclusion order.*

*Proof.* To show that $f$ preserves order, just note that $C \subseteq D \in \mathcal{N}(A)$ obviously entails $\max C \leq \max D$ in $A$. To show that $f$ is a p-morphism, for each $C \in \mathcal{N}(A)$ we prove:

$$f[\uparrow C] = \{\, a_k \in A \mid a_k \geq \max C \,\} =: \uparrow \max C =: \uparrow f(C). \tag{6.6}$$

Only the first equality in (6.6) needs proof, and only the right-to-left inclusion is non-trivial. So let $a_k \in A$ be such that $a_k \geq \max C$. Then the set $D := C \cup \{a_k\}$ is a chain in $A$, i.e. a member of $\mathcal{N}(A)$, and $D \in \uparrow C$ because $C \subseteq D$. Further, $\max D = a_k$, because $a_k \geq \max C$, so that $f(D) = a_k$. Hence $a_k \in f[\uparrow C]$, and the proof is complete. □

### 6.2.2 Tarski's Theorem on Intuitionistic Logic for Polyhedra

We have now all ingredients to prove the main result of N. Bezhanishvili *et al.* (2018):

**Theorem 6.2.5.** *For each $d \in \mathbb{N}$, $\mathsf{Log}\,\mathsf{P}_d$ is intuitionistic logic extended by the axiom schema $\mathsf{bd}_d$. Hence, the logic $\mathsf{Log} \bigcup_{d \in \mathbb{N}} \mathsf{P}_d$ of all polyhedra is intuitionistic logic.*

*Proof of Theorem.* By Theorem 6.2.3, $\mathsf{Log}\,\mathsf{P}_d$ is contained in intuitionistic logic extended by the axiom schema $\mathsf{bd}_d$. Conversely, suppose a formula $\alpha$ is not contained in intuitionistic logic extended by the axiom schema $\mathsf{bd}_d$. By Propositions 3.3.38,4.3.19 and Lemma 6.2.2, there exists a finite poset $A$ satisfying $\mathsf{dep}\,A \leq d$ such that there is an evaluation into the frame $A$ that provides a counter-model to $\alpha$; equivalently, the equation $\alpha = \top$ fails in the Heyting algebra $\mathsf{Up}\,A$. By Theorem 6.2.3 there exists a triangulation $\Sigma$ in $\mathbb{R}^{|A|}$ such that $\mathsf{dep}\,A = \dim \Sigma \leq d$, along with a surjective p-morphism

$$p : \Sigma \twoheadrightarrow A \tag{6.7}$$

We set $P := |\Sigma|$ and consider the Heyting algebra $\mathsf{Sub}_o\,P$ and its subalgebra $P_o(\Sigma)$, for Corollary 5.2.4 and Lemma 5.2.6, respectively. Since $\dim P \leq d$, we have $\mathsf{Log}\,P \supseteq \mathsf{Log}\,\mathsf{P}_d$ by Theorem 6.2.3.

By Lemma 6.1.1 there is an isomorphism of (finite) Heyting algebras

$$\gamma^{\uparrow} : \mathsf{Up}\,\Sigma \longrightarrow P_o(\Sigma)$$

defined as in (6.2). By finite Esakia duality (Theorem 3.3.40) we have isomorphisms of posets

$$\Sigma \simeq \mathsf{Spec}\,\mathsf{Up}\,\Sigma \simeq \mathsf{Spec}\,P_o(\Sigma).$$

The Esakia dual $\mathsf{Spec}\,p : \mathsf{Up}\,A \hookrightarrow \mathsf{Up}\,\Sigma$ of the surjective p-morphism (6.7) is an injective homomorphism. We thus have homomorphisms

$$\mathsf{Up}\,A \hookrightarrow \mathsf{Up}\,\Sigma \simeq P_o(\Sigma) \subseteq \mathsf{Sub}_o\,P$$

where the inclusion preserves the Heyting structure by Lemma 5.2.6. Since the equation $\alpha = \top$ fails in $\mathsf{Up}\, A$, it also fails in the larger algebra $\mathsf{Sub}_o\, P$; equivalently, $\alpha \notin \mathsf{Log}\, P \supseteq \mathsf{Log}\, \mathsf{P}_d$, and the proof of the first statement is complete. The second statement follows easily from the first using Proposition 4.3.19. $\square$

# 7 The Intermediate Logic of 1-Dimensional Manifolds

In this Chapter we characterize the triangulations of one dimensional manifolds, namely the circle $S^1$ and the closed interval $I$. This will allow us to study their logic. We shall show that it is given by the logic of all 1-dimensional polyhedra, $\mathsf{INT} + \mathsf{bd}_1$, extended by the axiom schema $\mathsf{bb}_2$.

## 7.1 Characterisation of Triangulations of $S^1$

**Theorem 7.1.1.** *Let be $\Sigma$ a triangulation. $|\Sigma| \cong S^1$ if and only if $\Sigma$ satisfies the following properties:*

*(1)* $\mathsf{dim}\,\Sigma = 1$,

*(2)* $\forall \sigma \in \Sigma(\mathsf{dim}\,\sigma = 0 \Rightarrow \mathsf{deg}\,\sigma = 2)$,

*(3)* *$\Sigma$ is connected.*

*Proof of* $\Rightarrow$). $|\Sigma| \cong S^1$ implies $\Sigma$ satisfies (1): The Lebesgue dimension of $S^1$ is equal to one, $\mathsf{dim}_L\, S^1 = 1$. Since homeomorphism preserves the dimension and the Lebesgue dimension of a triangulation is equal to the customary dimension of a triangulation above defined (see Definition 5.1.18), we obtain the desired result.

$|\Sigma| \cong S^1$ implies $\Sigma$ satisfies (2): Let us suppose there exists a $\sigma$ such that $\mathsf{dim}\,\sigma = 0$ and $\mathsf{deg}\,\sigma > 2$, for instance $\mathsf{deg}\,\sigma = 3$. Given a homeomorphism $h : |\Sigma| \to S^1$, for every $V \in \mathcal{O}(|\Sigma|)$ such that $\sigma \in V$, $V \cong h(V)$ since homeomorphims are also local homeomorphisms. Suppose now to remove $\sigma$: it is easy to check that $V - \{\sigma\}$ and $h(V) - \{h(\sigma)\}$ are still homeomorphic. However, by this very operation, $V - \{\sigma\}$ is divided into three connected components, while $h(V) - \{h(\sigma)\}$ has only two of them, and this contradicts our assumption.

$|\Sigma| \cong S^1$ implies $\Sigma$ satisfies (3): This follows easily from the fact that $S^1$ is connected and homeomorphism preserves connection. □

**Remark 7.1.2.** For a triangulation $\Sigma$ satisfying conditions (1), (2) and (3) of the theorem above, it holds

$$|\{\sigma \in \Sigma \,|\, \mathsf{dim}\,\sigma = 0\}| = |\{\eta \in \Sigma \,|\, \mathsf{dim}\,\eta = 1\}|.$$

This follows easily from the fact that every 1-dimensional triangulation is an undirected simple graph $G = (V, E)$, and we can apply the well known result from graph theory according to which the sum of the degrees is twice the number of edges,

$$\sum_{v \in V} \mathrm{deg}(v) = 2|E|.$$

**Remark 7.1.3.** A triangulation $\Sigma$ satisfying (1), (2) and (3) above must have at least three distinct points, otherwise the degree of each vertex would be lesser than two.

In order to prove $\Leftarrow$) of Theorem 7.1.1, we begin with the following enumeration lemma:

**Lemma 7.1.4.** *Let be $\Sigma$ a triangulation satisfying conditions (1), (2) and (3) of Theorem 7.1.1. There exists an enumeration $x_1, ..., x_n$ of vertices separately such that $x_1$ is exactly adjacent to $(x_2, x_n)$, $x_i$ is exactly adjacent to $(x_{i-1}, x_{i+1})$, $i = 2, ..., n-1$, and $x_n$ is exactly adjacent to $(x_{n-1}, x_1)$.*

*Proof.* We want to show constructively that it is possible to enumerate vertices from $x_1$ to $x_n$ in a separate form so that we have the clockwise (or counterclockwise) orientation on $S^1$. Remember that from Remark 7.1.3 we know that $n \geq 3$:

Step 1) choose $x_1$, it has exactly two adjacents, pick out one randomly and label it $x_2$, distinct from $x_1$ because of Remark 7.1.3.

Step 2) $x_2$ has exactly two adjacents, along with $x_1$ that is already enumerated, there is another one that we can call $x_3$, distinct from $x_1, x_2$ because of Remark 7.1.3.

Step 3) $x_3$ has exactly two adjacents, one being $x_2$ that is already enumerated. We can distinguish two cases:

- if $x_1$ is adjacent to $x_3$, then $n = 3$ and there cannot be other vertices, because $\Sigma$ is connected. In fact, if there is another vertex there is also a path from this one to a previous one, But this would contradict condition (2) of Theorem 7.1.1.
- if $x_1$ is not adjacent to $x_3$, then there will exists $x_4$, adjacent to $x_3$ and distinct from $x_1, x_2, x_3$.

Step 4) $x_4$ has exactly two adjacents, one being $x_3$ that is already enumerated. We can distinguish two cases:

- if $x_1$ is adjacent to $x_4$, then $n = 4$ and there cannot be other vertices, because $\Sigma$ is connected. In fact, if there is another vertex there is also a path from this one to a previous one, But this would contradict condition (2) of Theorem 7.1.1.
- if $x_1$ is not adjacent to $x_4$, then there will exists $x_5$, adjacent to $x_4$, distinct from $x_1, x_3, x_4$ but also from $x_2$ because otherwise the degree of $x_2$ would be more than two.

The Generic Step i) $x_i$ has exactly two adjacents, one being $x_{i-1}$ that is already enumerated. We can distinguish two cases:

- if $x_1$ is adjacent to $x_i$, then $n = i$ and there cannot be other vertices, because $\Sigma$ is connected. In fact, if there is another vertex there is also a path from this one to a previous one, But this would contradict condition (2) of Theorem 7.1.1.
- if $x_1$ is not adjacent to $x_i$, then there will exists $x_{i+1}$, adjacent to $x_i$, distinct from $x_1, x_{i-1}, x_i$ but also from $x_2, \ldots, x_{i-2}$ because otherwise the degree of the latter vertices would be more than two.

Because $\Sigma$ has a finite number of vertices by its own definition, this guarantees that the algorithm ended at some point, and so the proof. □

*Proof of $\Leftarrow$) of Theorem 7.1.1.* Now we can set up the homeomorphism between $|\Sigma|$ and $S^1$ and conclude our proof of Theorem 7.1.1. Consider the function $f$ between the set of vertices of $\Sigma$, $\Sigma^{(0)} = \{x_1, \ldots, x_n\}$, and the set of $n$-th roots of unity $\{z_0, \ldots, z_{n-1}\} \subseteq S^1$, i.e., $f : \Sigma^{(0)} \to \{z_0, \ldots, z_{n-1}\}$ such that $f(x_k) = z_{k-1}$, for $k = 1, \ldots, n$. It is bijective function and can be easily extend to a simplicial map between $\Sigma$ and the boundary of the convex hull spanned by $\{z_0, \ldots, z_{n-1}\}$ that, for the Lemma 5.1.14, is both an isomorphism between triangulations and a homeomorphism between their geometric realizations. So we have $|\Sigma| \cong |\partial \operatorname{conv}\{z_0, \ldots, z_{n-1}\}|$. Now, a radial segment starting from the origin will intersect $S^1$ in one point and $|\partial \operatorname{conv}\{z_0, \ldots, z_{n-1}\}|$ in another. In order to establish a homeomorphic correspondence between $S^1$ and $|\partial \operatorname{conv}\{z_0, \ldots, z_{n-1}\}|$ it suffices to employ these two intersections in the standard way. □

**Corollary 7.1.5.** *Let be $\Sigma$ a triangulation. $|\Sigma| \cong S^1 + \cdots + S^1$ if and only if $\Sigma$ satisfies the following properties:*

*(1)* $\dim \Sigma = 1$,

*(2)* $\forall \sigma \in \Sigma(\dim \sigma = 0 \Rightarrow \deg \sigma = 2)$,

*Proof of $\Rightarrow$).* $|\Sigma| \cong S^1 + \cdots + S^1$ implies $\Sigma$ satisfies (1): The Lebesgue dimension of the disjoint union among $S^1$ is equal to one, $\dim_L S^1 + \cdots + S^1 = 1$ (see Pears 1975). Since homeomorphism preserves the dimension and the Lebesgue dimension of a triangulation is equal to the customary dimension of a triangulation above defined (see Definition 5.1.18), we obtain the desired result.

$|\Sigma| \cong S^1 + \cdots + S^1$ implies $\Sigma$ satisfies (2): Since $|\Sigma| \cong S^1 + \cdots + S^1$, in turn we can see $\Sigma$ as a disjoint union of triangulations $\Sigma_i$, $\coprod \Sigma_i$ such that $|\Sigma_i| \cong S^1$. Let us suppose there exists a $\sigma \in \Sigma_i$, for some $i$, such that $\dim \sigma = 0$ and $\deg \sigma > 2$, for instance $\deg \sigma = 3$. Given a homeomorphism $h : |\Sigma_i| \to S^1$, for every $V \in \mathcal{O}(|\Sigma_i|)$ such that $\sigma \in V$, $V \cong h(V)$ since homeomorphims are also local homeomorphisms. Suppose now to remove $\sigma$: it is easy to check that $V - \{\sigma\}$ and $h(V) - \{h(\sigma)\}$ are still homeomorphic.

However, by this very operation, $V - \{\sigma\}$ is divided into three connected components, while $h(V) - \{h(\sigma)\}$ has only two of them, and this contradicts our assumption. □

*Proof of* $\Leftarrow$) *of Corollary 7.1.5.* Suppose that $\Sigma$ is not connected otherwise we fall back within the previous case of Theorem 7.1.1. Hence, we can see $\Sigma$ as a disjoint union of triangulations $\Sigma_i$, $\coprod \Sigma_i$ such that every $\Sigma_i$ satisfies (1) and (2) of Corollary 7.1.5. Since every $\Sigma_i$ is also connected, from Theorem 7.1.1 we obtain that $|\Sigma_i| \cong S^1$, for every $i$. By the fact that $|\coprod \Sigma_i| = \coprod |\Sigma_i|$ (see Lemma 5.1.15), it follows the statement. □

## 7.2 Characterisation of Triangulations of $I$

**Theorem 7.2.1.** *Let be $\Lambda$ a triangulation. $|\Lambda| \cong I$ if and only if $\Lambda$ satisfies the following properties:*

*(1)* $\dim \Lambda = 1$,

*(2)* $\forall \lambda \in \Lambda(\dim \lambda = 0 \Rightarrow \deg \lambda = 1 \vee \deg \lambda = 2)$,

*(3)* $\exists \sigma, \eta \in \Lambda(\deg \sigma = 1 \wedge \deg \eta = 1 \wedge \sigma \neq \eta)$,

*(4)* *$\Lambda$ is connected.*

*Proof of* $\Rightarrow$). $|\Lambda| \cong I$ implies $\Lambda$ satisfies (1): The Lebesgue dimension of $I$ is equal to one, $\dim_L I = 1$. Since homeomorphism preserves the dimension and the Lebesgue dimension of a triangulation is equal to the customary dimension of a triangulation above defined (see Definition 5.1.18), we obtain the desired result.

$|\Lambda| \cong I$ implies $\Lambda$ satisfies (2): Let us suppose there exists a $\lambda$ such that $\dim \lambda = 0$ and $\deg \lambda \neq 1, 2$, for instance $\deg \lambda = 3$. Given a homeomorphism $h : |\Lambda| \longrightarrow I$, for every $V \in \mathcal{O}(|\Lambda|)$ such that $\lambda \in V$, $V \cong h(V)$ since homeomorphims are also local homeomorphisms. Suppose now to remove $\lambda$: it is easy to check that $V - \{\lambda\}$ and $h(V) - \{h(\lambda)\}$ must be still homeomorphic. However, by this very operation, $V - \{\lambda\}$ is divided into three connected components, while $h(V) - \{h(\lambda)\}$ can have at most two of them, and this contradicts our assumption.

$|\Lambda| \cong I$ implies $\Lambda$ satisfies (3): Given a homeomorphism $g : I \longrightarrow |\Lambda|$, we consider the images $g(0) = x$ and $g(1) = y$. Let us suppose that $\deg x > 1$, for instance $\deg x = 2$. For every $V \in \mathcal{O}(|\Lambda|)$ such that $x \in V$, $V \cong g^{-1}(V)$ since homeomorphims are also local homeomorphisms. Suppose now to remove $x$: it is easy to check that $V - \{x\}$ and $g^{-1}(V) - \{g^{-1}(x)\}$ must be still homeomorphic. However, by this very operation, $V - \{x\}$ is divided into two connected components, while $g^{-1}(V) - \{g^{-1}(x)\}$ can have at most one of them, and this contradicts our assumption. The same line of reasoning can be employed in order to prove that also $\deg y = 1$.

$|\Lambda| \cong I$ implies $\Lambda$ satisfied (4): This follows easily from the fact that $I$ is connected and homeomorphism preserves connection. □

**Remark 7.2.2.** a triangulation $\Lambda$ satisfying (1), (2),(3) and (4) above must have at least two distinct points, otherwise the degree of each vertex would be lesser than one.

In order to prove $\Leftarrow$) of Theorem 7.2.1, we begin with the following enumeration lemma as we did before in the case of $S^1$:

**Lemma 7.2.3.** *Let be $\Sigma$ a triangulation satisfying conditions (1), (2), (3) and (4) of Theorem 7.2.1. There exists an enumeration $x_1, ..., x_n$ of vertices separately such that $x_1$ is exactly adjacent to $x_2$, $x_i$ is exactly adjacent to $(x_{i-1}, x_{i+1})$, $i = 2, ..., n-1$, and $x_n$ is exactly adjacent to $x_{n-1}$.*

*Proof.* We want to show constructively that it is possible to enumerate vertices from $x_1$ to $x_n$ in a separate form so that we have the left (or right) orientation on $I$. Remember that from Remark 7.1.3 we know that $n \geq 2$:

Step 1) from property (3) there exists two vertices of $\Lambda$ whose degree is one, we pick out one randomly and label it $x_1$. It has exactly one adjacent, $x_2$, distinct from $x_1$ because of Remark 7.1.3.

Step 2) $x_2$ has at least one adjacent, $x_1$ that is already enumerated. We can distinguish two cases:

- if $x_1$ is the only adjacent to $x_2$, then $n = 2$ and there cannot be other vertices, because $\Lambda$ is connected. In fact, if there is another vertex, say $y$, there is also a path from $y$ to $x_2$ or $x_1$. However, this would be in contrast with the fact that $\deg x_2 = \deg x_1 = 1$.
- if $x_2$ has exactly two adjacents, one being $x_1$ that is already enumerated, then there will exists $x_3$, adjacent to $x_2$ and distinct from $x_1, x_2$.

Step 3) $x_3$ has at least one adjacent, $x_2$ that is already enumerated. We can distinguish two cases:

- if $x_2$ is the only adjacent to $x_3$, then $n = 3$ and there cannot be other vertices, because $\Lambda$ is connected. In fact, if there is another vertex, say $y$, there is also a path from $y$ to $x_2, x_3$ or $x_1$. However, this would be in contrast with the degree of those vertices.
- if $x_3$ has exactly two adjacents, one being $x_2$ that is already enumerated, then there will exists $x_4$, adjacent to $x_3$, distinct from $x_2, x_3$ but also from $x_1$ because otherwise the degree of $x_1$ would be more than one.

The Generic Step i) $x_i$ has at least one adjacent, $x_{i-1}$ that is already enumerated. We can distinguish two cases:

- if $x_{i-1}$ is the only adjacent to $x_i$, then $n = i$ and there cannot be other vertices, because $\Lambda$ is connected. In fact, if there is another vertex, say $y$, there is also a path from $y$ to $x_2, x_3$ or $x_1$. However, this would be in contrast with the degree of those vertices.

- if $x_i$ has exactly two adjacents, one being $x_{i-1}$ that is already enumerated, then there will exists $x_{i+1}$, adjacent to $x_i$, distinct from $x_{i-1}, x_i$ but also from: $x_1$, because otherwise the degree of $x_1$ would be more than one; and $x_2, \ldots, x_{i-2}$, because otherwise the degree of the latter vertices would be more than two.

Because $\Lambda$ has a finite number of vertices by its own definition, this guarantees that the algorithm ended at some point, and so the proof. □

*Proof of* $\Leftarrow$) *of Theorem 7.2.1.* Now we can set up the homeomorphism between $|\Lambda|$ and $I$ and conclude our proof of Theorem 7.2.1. Consider the function $f$ between the set of vertices of $\Lambda$, $\Lambda^{(0)} = \{x_1, \ldots, x_n\}$, and the set $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\} \subseteq I$, i.e., $f : \Lambda^{(0)} \longrightarrow \{z_0, \ldots, z_{n-1}\}$ such that $f(x_k) = z_{k-1}$, for $k = 1, \ldots, n$. It is bijective function and can be easily extend to a simplicial map between $\Lambda$ and the boundary of the convex hull spanned by $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ that is $I$. By the Lemma 5.1.14, this map is both an isomorphism between triangulations and a homeomorphism between their geometric realizations. So we have $|\Lambda| \cong |\operatorname{conv}\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}| \cong I$. □

**Corollary 7.2.4.** *Let be $\Lambda$ a triangulation. $|\Lambda| \cong I + \cdots + I$ if and only if $\Lambda$ satisfies the following properties:*

*(1)* $\dim \Lambda = 1$,

*(2)* $\forall \lambda \in \Lambda(\dim \lambda = 0 \Rightarrow \deg \lambda = 1 \vee \deg \lambda = 2)$,

*(3)* $\exists \sigma, \eta, \ldots \in \Lambda(\deg \sigma = 1 \wedge \deg \eta = 1 \wedge \cdots \wedge \sigma \neq \eta \neq \ldots)$.

*Proof of* $\Rightarrow$). $|\Lambda| \cong I + \cdots + I$ implies $\Lambda$ satisfies (1): The Lebesgue dimension of the disjoint union among $I$ is equal to one, $\dim_L I + \cdots + I = 1$ (see Pears 1975). Since homeomorphism preserves the dimension and the Lebesgue dimension of a triangulation is equal to the customary dimension of a triangulation above defined (see Definition 5.1.18), we obtain the desired result.

$|\Lambda| \cong I + \cdots + I$ implies $\Lambda$ satisfies (2): Since $|\Lambda| \cong I + \cdots + I$, in turn we can see $\Lambda$ as a disjoint union of triangulations $\Lambda_i$, $\coprod \Lambda_i$ such that $|\Lambda_i| \cong I$. Let us suppose there exists a $\lambda \in \Lambda_i$, for some $i$, such that $\dim \lambda = 0$ and $\deg \lambda \neq 1, 2$, for instance $\deg \lambda = 3$. Given a homeomorphism $h : |\Lambda_i| \longrightarrow I$, for every $V \in \mathcal{O}(|\Lambda_i|)$ such that $\lambda \in V$, $V \cong h(V)$ since homeomorphims are also local homeomorphisms. Suppose now to remove $\lambda$: it is easy to check that $V - \{\lambda\}$ and $h(V) - \{h(\lambda)\}$ are still homeomorphic. However, by this very operation, $V - \{\lambda\}$ is divided into three connected components, while $h(V) - \{h(\lambda)\}$ has only two of them, and this contradicts our assumption.

$|\Lambda| \cong I + \cdots + I$ implies $\Lambda$ satisfies (3): As we have just seen, we can regard $\Lambda$ as a disjoint union of triangulations $\Lambda_i$, $\coprod \Lambda_i$ such that $|\Lambda_i| \cong I$. Given a homeomorphism $g : I \longrightarrow |\Lambda_i|$, we consider the images $g(0) = x$ and $g(1) = y$. Let us suppose that $\deg x > 1$, for instance $\deg x = 2$. For every $V \in \mathcal{O}(|\Lambda_i|)$ such that $x \in V$, $V \cong g^{-1}(V)$ since homeomorphims are also local homeomorphisms. Suppose now to remove $x$: it is easy to check that $V - \{x\}$ and $g^{-1}(V) - \{g^{-1}(x)\}$ must be still homeomorphic.

However, by this very operation, $V - \{x\}$ is divided into two connected components, while $g^{-1}(V) - \{g^{-1}(x)\}$ can have at most one of them, and this contradicts our assumption. The same line of reasoning can be exploited to prove that also $\mathsf{deg}\, y = 1$. It easily follows that $\Lambda$ has an even number of vertices whose degree is equal to one. □

*Proof of* $\Leftarrow$) *of Corollary 7.2.4.* Suppose that $\Lambda$ is not connected otherwise we fall back within the previous case of Theorem 7.2.1. Hence, we can see $\Lambda$ as a disjoint union of triangulations $\Lambda_i$, $\coprod \Lambda_i$ such that every $\Lambda_i$ satisfies (1), (2) and (3) of Corollary 7.2.4. Since every $\Lambda_i$ is also connected, from Theorem 7.2.1 we obtain that $|\Lambda_i| \cong I$, for every $i$. By the fact that $|\coprod \Lambda_i| = \coprod |\Lambda_i|$ (see Lemma 5.1.15), it follows the statement. □

## 7.3 The Logic of 1-Dimensional Manifolds

As a consequence of local finiteness of $\mathsf{Sub}_o\, P$, for any polyhedron $P$ (see Subsection 5.2.2), and the isomorphism $\mathsf{Up}\,\Sigma \cong P_o(\Sigma)$ between $\Sigma$-definable open polyhedra and algebra of upper sets of $\Sigma$ (see Lemma 6.1.1), the logic of a topological manifold $X$ is given by the logic of the class of its triangulations $\mathcal{X} = \{\Sigma \mid |\Sigma| \cong X\}$, namely

$$\mathsf{Log}\, X := \mathsf{Log}\, \mathcal{X} = \{\, \alpha \in \mathcal{SL} \mid \forall \Sigma \in \mathcal{X} \quad \Sigma \models \alpha \,\} .$$

Let us now introduce some classes of frames that are relevant in order to investigate the logic of 1-dimensional topological manifolds.

**Definition 7.3.1.**

- Let $\mathcal{B}$ be the class of all finite Kripke frames such that $\mathsf{Log}\,\mathcal{B} = \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2$, namely the class of all finite frame
  (1) of depth at most 1 (see Theorem 4.3.25) and
  (2) such that every point in the frame has at most 2 distinct immediate successors (see Proposition 4.3.27).
- Let $\mathcal{S}$ be the class of all triangulations $\Sigma$ such that $|\Sigma| \cong S^1$.
- Let $\mathcal{I}$ be the class of all triangulations $\Lambda$ such that $|\Lambda| \cong I$.

Moreover, for each of the last two classes above we can define the class of all finite disjoint unions of its own elements, that is:

**Definition 7.3.2.**

- Let $\widetilde{\mathcal{S}}$ be the class of all disjoint unions of $\Sigma \in \mathcal{S}$.
- Let $\widetilde{\mathcal{I}}$ be the class of all disjoint unions of $\Lambda \in \mathcal{I}$.

So, we can observe that the logic of $S^1$ is given by $\mathsf{Log}\, S^1 = \mathsf{Log}\,\mathcal{S}$, while the logic of $I$ is given by $\mathsf{Log}\, I = \mathsf{Log}\,\mathcal{I}$. In a similar way, the logic of disjoint unions of $S^1$ is given by $\mathsf{Log}\, S^1 + \cdots + S^1 = \mathsf{Log}\,\widetilde{\mathcal{S}}$, while the logic of disjoint unions of $I$ is given by

$\mathsf{Log}\, I + \cdots + I = \mathsf{Log}\, \widetilde{\mathcal{I}}$. Notice that, by Remark 4.3.17, it can be immediately deduced the identity between $\mathsf{Log}\, X$ and $\mathsf{Log}\ X \coprod \cdots \coprod X$. However, in the sequel we will obtain this identity independently from Remark 4.3.17.

We are now able to prove the following theorem.

**Theorem 7.3.3.** *The logic of 1-dimensional manifolds (or, equivalently, of their classes of triangulations) is given by* $\mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2$. *In particular:*

$$\mathsf{Log}\, \mathcal{I} = \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2; \tag{7.1}$$

$$\mathsf{Log}\, \widetilde{\mathcal{I}} = \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2; \tag{7.2}$$

$$\mathsf{Log}\, \mathcal{S} = \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2; \tag{7.3}$$

$$\mathsf{Log}\, \widetilde{\mathcal{S}} = \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2. \tag{7.4}$$

Before starting the proof, we investigate the relations among those classes above.

**Lemma 7.3.4.** *The following facts hold:*

$$\mathcal{I} \subseteq \widetilde{\mathcal{I}} \subseteq \mathcal{B}; \tag{7.5}$$

$$\mathcal{S} \subseteq \widetilde{\mathcal{S}} \subseteq \mathcal{B}. \tag{7.6}$$

*Proof of the Lemma.* $\mathcal{I} \subseteq \mathcal{B}$ is a straightforward consequence of Theorem 7.2.1 while $\mathcal{S} \subseteq \mathcal{B}$ is a straightforward consequence of Theorem 7.1.1. In a similar way, $\widetilde{\mathcal{I}} \subseteq \mathcal{B}$ is a straightforward consequence of Corollary 7.2.4 while $\widetilde{\mathcal{S}} \subseteq \mathcal{B}$ is a straightforward consequence of Corollary 7.1.5. □

*Proof of* $\Leftarrow$) *of Theorem 7.3.3.* From Remark 4.3.11 and Lemma 7.3.4, the following facts hold:

$$\mathsf{Log}\, \mathcal{B} \subseteq \mathsf{Log}\, \widetilde{\mathcal{I}} \subseteq \mathsf{Log}\, \mathcal{I}; \tag{7.7}$$

$$\mathsf{Log}\, \mathcal{B} \subseteq \mathsf{Log}\, \widetilde{\mathcal{S}} \subseteq \mathsf{Log}\, \mathcal{S}. \tag{7.8}$$

Hence, we obtain:

$$\mathsf{Log}\, \mathcal{I} \supseteq \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2;$$

$$\mathsf{Log}\, \widetilde{\mathcal{I}} \supseteq \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2;$$

$$\mathsf{Log}\, \mathcal{S} \supseteq \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2;$$

$$\mathsf{Log}\, \widetilde{\mathcal{S}} \supseteq \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2.$$

□

*Proof of* $\Rightarrow$) *for (7.1) and (7.2) of Theorem 7.3.3.* We first exhibit a proof for $\mathsf{Log}\, \mathcal{I} \subseteq \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2$. We need to prove that $\varphi \in \mathsf{Log}\, \mathcal{I}$ implies $\varphi \in \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2$ or, equivalently, by contrapositive $\varphi \notin \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2$ implies $\varphi \notin \mathsf{Log}\, \mathcal{I}$.

A formula $\varphi \in \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2$ if and only if $\varphi$ is valid on any finite frame (1) of depth at most 1 and (2) such that every point in the frame has at most 2 distinct immediate successors (see Definition 7.3.1). Hence, $\varphi \notin \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2$ means that there exists a finite frame $F \in \mathcal{B}$, namely satisfying (1) and (2), and such that $F \nvDash \varphi$. Similarly, $\varphi \notin \mathsf{Log}\,\mathcal{I}$ means that there exists a finite frame $\Lambda$ satisfying all conditions in Theorem 7.2.1 and such that $\Lambda \nvDash \varphi$.

In order to do the proof, it suffices to show that, given a finite frame $F$ satisfying (1) and (2) and such that $F \nvDash \varphi$, it is possible to exhibit a frame $\Lambda \in \mathfrak{I}$ such that $\Lambda \nvDash \varphi$.

**First step:** let be $F$ a frame satisfying (1) and (2) and such that $F \nvDash \varphi$. We manufacture a surjective p-morphism $f : C \twoheadrightarrow F$, where $C$ is given by the disjoint union of peculiar posets as in Figure 7.1.

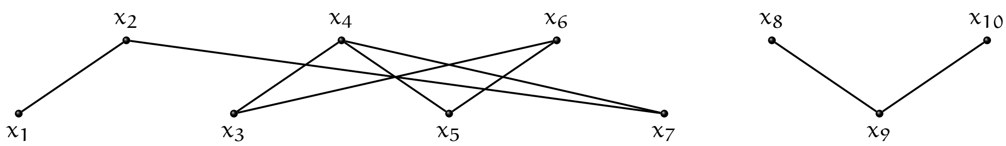


FIGURE 7.1: Example of finite frame $F$ satisfying (1) and (2).

Consider the antichain of $F$, $V = \{y \in F \mid \downarrow y = \{y\}\}$. The cardinality of $V$ is of course finite because $F$ is finite. Suppose $|V| = n$ and set $\mathbf{n} = \{1, \ldots, n\}$ (see Section 2.1), we can also see $V$ as the indexed family of elements in $F$ $(y_i)_{i\in\mathbf{n}}$ such that $\downarrow y_i = \{y_i\}$, for all $i \in \mathbf{n}$. For property (2) we can also say that $0 \leq \deg y_i \leq 2$, for all $i \in \mathbf{n}$. Also, for property (1) we have the useful identity, for $y \in V$, $\deg y = |\uparrow y| - 1$ which in general, for dimension higher than one, is not valid. Hence, $1 \leq |\uparrow y_i| \leq 3$ for all $i \in \mathbf{n}$.

Consider $C = \coprod_{j\in\mathbf{m}} A_j$, the disjoint union of the indexed family of posets $(A_j)_{j\in\mathbf{m}}$ such that:

1. for all $j \in \mathbf{m}$, $\dim A_j \leq 1$,
2. for all $j \in \mathbf{m}$, exists $a \in C$ such that $A_j = \uparrow a$ and,
3. if $S = \{a \in C \mid \uparrow a = A_j,\ \text{for some } j\}, 0 \leq \deg a \leq 2$ or, equivalently, $1 \leq |\uparrow a| \leq 3$, for all $a \in S$.

For Lemma 2.1.15 there is a bijection between $S$ and $(A_j)_{j\in\mathbf{m}}$, so $|S| = m$ and we can also see $S$ as the indexed family of elements in $C$ $(a_j)_{j\in\mathbf{m}}$ such that $\uparrow a_j = A_j$, for all $j \in \mathbf{m}$. Roughly speaking, there are at most three kind of $A_j$ as in Figure 7.2.

Suppose also that $|\{y \in V \mid \deg y = k\}| \leq |\{a \in S \mid \deg a = k\}|$, for all $k = 0, 1, 2$. So, $n \leq m$.

Consider now, the indexed family of morphisms $(f_j : A_j \to F)_{j\in\mathbf{m}} : (A_j)_{j\in\mathbf{m}} \to F$ so defined:

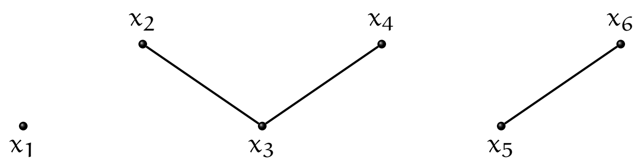


FIGURE 7.2: $A_1, A_2$ and $A_3$

1. If $|\uparrow a_j| = 1$, namely $A_j = \{a_j\}$, $f_j(a_j) \in V$ and $\deg f_j(a_j) = 0$ (or, equivalently, $|\uparrow f_j(a_j)| = 1$).
2. If $|\uparrow a_j| = 2$, namely $A_j = \{a_j, a\}$ with $a \in \uparrow a_j$ and $a \neq a_j$,
   (i) $f_j(a_j) \in V$ and $\deg f_j(a_j) = 1$ (or, equivalently, $|\uparrow f_j(a_j)| = 2$);
   (ii) $f_j(a) \in \uparrow f_j(a_j)$ and $f_j(a) \neq f_j(a_j)$.
3. If $|\uparrow a_j| = 3$, namely $A_j = \{a_j, a, b\}$ with $a, b \in \uparrow a_j$ and $a \neq b, a \neq a_j, b \neq a_j$,
   (i) $f_j(a_j) \in V$ and $\deg f_j(a_j) = 2$ (or, equivalently, $|\uparrow f_j(a_j)| = 3$);
   (ii) $f_j(a) \in \uparrow f_j(a_j)$ and $f_j(a) \neq f_j(a_j), f_j(b)$;
   (iii) $f_j(b) \in \uparrow f_j(a_j)$ and $f_j(b) \neq f_j(a_j), f_j(a)$.
4. $f_j(a_j)_{j \in \mathbf{m}} \supseteq V$.

It is easy to see that, for all $j$, $f_j$ is an order embedding and also a p-morphism.

For instance, for all $a \in A_j$, $a \geq a_j$ if and only if $a \in \uparrow a_j$ and, by constructions (1-4), if and only if $f_j(a) \in \uparrow f_j(a_j)$ if and only if $f_j(a) \geq f_j(a_j)$.

Also, for all $a \in A_j$ such that $a > a_j$, $f_j[\uparrow a] = \uparrow f_j(a)$ is trivially satisfied. In fact, since $\dim A_j \leq 1$ for all $j \in \mathbf{m}$, $\uparrow a = \{a\}$ and so $f_j[\uparrow a] = \{f_j(a)\}$. Similarly, since $\dim F \leq 1$ and $f_j(a) > f_j(a_j)$, $\uparrow f_j(a) = \{f_j(a)\}$.

If $|\uparrow a_j| = 3$, namely $\uparrow a_j = \{a_j, a, b\}$ with $a, b \in \uparrow a_j$ and $a \neq b, a \neq a_j, b \neq a_j$, $f_j[\uparrow a_j] = \{f_j(a_j), f_j(a), f_j(b)\}$. Since also $|\uparrow f_j(a_j)| = 3$, $\uparrow f_j(a_j) = \{f_j(a_j), f_j(a), f_j(b)\}$ by construction (3).

Similarly, if $|\uparrow a_j| = 2$, namely $\uparrow a_j = \{a_j, a\}$ with $a \in \uparrow a_j$ and $a \neq a_j$, $f_j[\uparrow a_j] = \{f_j(a_j), f_j(a)\}$. Since also $|\uparrow f_j(a_j)| = 2$, $\uparrow f_j(a_j) = \{f_j(a_j), f_j(a)\}$ by construction (2).

Trivially, if $|\uparrow a_j| = 1$, namely $\uparrow a_j = \{a_j\}$, $f_j[\uparrow a_j] = \{f_j(a_j)\}$. Since also $|\uparrow f_j(a_j)| = 1$, $\uparrow f_j(a_j) = \{f_j(a_j)\}$ by construction (1).

For universal property of disjoint union, there is one and only one $f = \coprod_{j \in \mathbf{m}} f_j : C \to F$ such that $f_j = f \circ \iota_j$, where $\iota_j : A_j \to C$ are the injections (see Awodey 2010).

It follows easily that $f$ is a p-morphism. In fact, since $f_j[\uparrow a] = \uparrow f_j(a)$ for all $j \in \mathbf{m}$ and $a \in A_j$ and $f_j = f \circ \iota_j$ for all $j \in \mathbf{m}$, we can also write

$$(f \circ \iota_j)[\uparrow a] = \uparrow (f \circ \iota_j)(a) \tag{7.9}$$

for all $j \in \mathbf{m}$ and $a \in A_j$.

Let us consider for a while the injection maps. As usually, the disjoint union is conveniently taken to be $C = \coprod_{j \in \mathbf{m}} A_j = \{(j, a) \mid j \in \mathbf{m} \text{ and } a \in A_j\}$ with the order given by:

$$(j, a) \leq_C (i, b) \quad \text{if and only if} \quad i = j \wedge a \leq_{A_j} b.$$

Now, the injection $\iota_j : A_j \to C$ is given by $\iota_j(a) = (j, a)$, for all $a \in A_j$. Hence $\uparrow \iota_j(a) = \uparrow (j, a) = \{(i, b) \mid i = j \text{ and } a \leq_{A_j} b\} = \{(j, b) \mid b \in \uparrow a\}$. Moreover, if $X \subseteq A_j$, $\iota_j[X] = \{(j, x) \mid x \in X\}$. Hence, $\iota_j[\uparrow a] = \{(j, x) \mid x \in \uparrow a\}$ and $\iota_j[\uparrow a] = \uparrow \iota_j(a)$.

Since the injection $\iota_j : A_j \to C$ is a p-morphism, $(f \circ \iota_j)[\uparrow a] = f[\uparrow \iota_j(a)]$ and we can write the equation (7.9) in the following way:

$$f[\uparrow \iota_j(a)] = \uparrow f(\iota_j(a)) \tag{7.10}$$

for all $j \in \mathbf{m}$ and $a \in A_j$. Or, equivalently,

$$f[\uparrow (j, a)] = \uparrow f(j, a) \tag{7.11}$$

for all $(j, a)$ $j \in \mathbf{m}$ and $a \in A_j$.

Of course, $f$ is not necessarily an order embedding. For instance, we can have $\uparrow f(j, a_j) \bigcap \uparrow f(i, a_i) = \uparrow f_j(a_j) \bigcap \uparrow f_i(a_i) = \{y\}$. As a consequence, there exist $a \in \uparrow a_j$ and $b \in \uparrow a_i$ such that $f_j(a) = f_i(b) = y$ or, in terms of $f$, $f(j, a) = f(i, b) = y$.

Instead, $f$ is definitely surjective. In fact, let be $z \in F$, there exists some $y \in V$ such that $z \in \uparrow y$. Hence, by construction (4), $z \in \uparrow f_j(a_j)$, for some $a_j$ such that $\mathsf{deg}\, a_j = \mathsf{deg}\, f_j(a_j)$ or, equivalently, $|\uparrow a_j| = |\uparrow f_j(a_j)|$. By construction (1-3), $z = f_j(a)$ for some $a \in \uparrow a_j$.

**Second step:** We have just manufactured a surjective p-morphism $f : C \twoheadrightarrow F$ and, according to the Remark 4.3.17(ii), we are allowed to conclude that $C \nvDash \varphi$, given our assumption that $F \nvDash \varphi$. In algebraic terms, we can say equivalently that the equation $\varphi = \top$ is not valid in $\mathsf{Up}\, C$ (see Subsection 4.3.6). Since for Proposition 2.1.16 $\mathsf{Up}\, C = \prod_{j \in \mathbf{m}} \mathsf{Up}\, A_j$, $\varphi = \top$ is not valid in $\mathsf{Up}\, A_j$, for some $j$ (see Remark 4.3.17(iii)). In particular, considering that $(A_j)_{j \in \mathbf{m}}$ can be split up in three classes of isomorphic posets $\{A_j \mid |\uparrow a_j| = 1\}$, $\{A_j \mid |\uparrow a_j| = 2\}$ and $\{A_j \mid |\uparrow a_j| = 3\}$, either $\varphi = \top$ is not valid in $\mathsf{Up}\, A_1$, or is not valid in $\mathsf{Up}\, A_2$, or is not valid in $\mathsf{Up}\, A_3$, where $A_1, A_2, A_3$ are the chosen representing elements for each class, respectively.

Rephrasing the argument in terms of frames, we can say that if $C \nvDash \varphi$ then $A_1 \nvDash \varphi$ or $A_2 \nvDash \varphi$ or $A_3 \nvDash \varphi$.

**Third step:** Consider now the following cases:

1. $|\uparrow a_1| = 1$, namely $A_1 = \{a_1\}$. We can easily construct a surjective p-morphism $g_0 : \Lambda_0 \twoheadrightarrow A_1$, where $\Lambda_0$ is given by $\{c, b, d; c \leq b, d \leq b\}$ as in Figure 7.3,

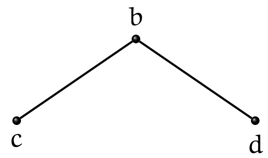


FIGURE 7.3: $\Lambda_0$

namely $g(\lambda) = a_1$, for all $\lambda \in \Lambda_0$. In fact, $g_0[\uparrow\lambda] = \{a_1\} = \uparrow g_0(\lambda)$ for all $\lambda \in \Lambda_0$.

2. $|\uparrow a_2| = 2$, namely $A_2 = \{a_2, a\}$ with $a \in\uparrow a_2$ and $a \neq a_2$. Again, we can easily construct a surjective p-morphism $g_1 : \Lambda_0 \twoheadrightarrow A_2$, where $\Lambda_0$ is given by $\{b, c, d; d \leq b, c \leq b\}$ as in Figure 7.3,

   namely $g_1(c) = g_1(d) = a_2$ and $g_1(b) = a$. In fact, $g_1[\uparrow d] = g_1[\uparrow c] = \{a_2, a\} = \uparrow g_1(d) = \uparrow g_1(c)$ and $g_1[\uparrow b] = \{a\} = \uparrow g_1(b)$.

3. $|\uparrow a_3| = 3$, namely $A_3 = \{a_3, a, e\}$ with $a, e \in\uparrow a_3$ and $a \neq e, a \neq a_3, e \neq a_3$. We can construct a surjective p-morphism $g_2 : \Lambda_1 \twoheadrightarrow A_3$, where $\Lambda_1$ is given by $\{b, c, d, h, l; d \leq b, c \leq b, d \leq h, l \leq h\}$ as in Figure 7.4,

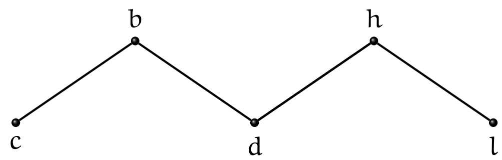


FIGURE 7.4: $\Lambda_1$

namely $g_2(c) = g_2(d) = g_2(l) = a_j$, $g_2(b) = a$ and $g_2(h) = e$. In fact, $g_2[\uparrow d] = \{a_3, a, e\} = \uparrow g_2(d)$, $g_2[\uparrow c] = \{a_3, a\} = \uparrow g_2(c)$, $g_2[\uparrow l] = \{a_3, e\} = \uparrow g_2(l)$, $g_2[\uparrow b] = \{a\} = \uparrow g_2(b)$ and $g_2[\uparrow h] = \{e\} = \uparrow g_2(h)$.

For the Remark 4.3.17(ii), we can conclude that: if $A_1 \nvDash \varphi$ or $A_2 \nvDash \varphi$, then $\Lambda_0 \nvDash \varphi$; if $A_3 \nvDash \varphi$, then $\Lambda_1 \nvDash \varphi$. And so, to summarise, we have just found a certain $\Lambda \in \mathcal{I}$ such that $\Lambda \nvDash \varphi$, given our assumption that $F \nvDash \varphi$.

As a consequence, $\mathsf{Log}\,\mathcal{I} \subseteq \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2$ and, by formula (7.7), $\mathsf{Log}\,\widetilde{\mathcal{I}} = \mathsf{Log}\,\mathcal{I} = \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2$. □

As byproduct of the theorem's proof we have the following result:

**Corollary 7.3.5.** *If $F \nvDash \varphi$ then $A_1 \nvDash \varphi$ or $A_2 \nvDash \varphi$ or $A_3 \nvDash \varphi$.*

We can observe that there is also a surjective p-morphism between $A_3 \longrightarrow A_2$, as well as between $A_2 \longrightarrow A_1$. In fact, it suffices to construct $g : A_3 \twoheadrightarrow A_2$ and $q : A_2 \twoheadrightarrow A_1$, where $A_3$ is given by $\{a_3, a, b\}$, $A_2$ by $\{a_2, c\}$ and $A_1$ by $\{a_1\}$, by placing $g(a_3) = a_2$, $g(a) = g(b) = c$ and $q(a_2) = q(c) = a_1$. So, we have $g[\uparrow a_3] = \{a_2, b\} = \uparrow g(a_3)$, $g[\uparrow a] = \{b\} = \uparrow g(a)$ and $g[\uparrow e] = \{b\} = \uparrow g(e)$, $q[\uparrow a_2] = \{a_1\} = \uparrow q(a_2)$ and $q[\uparrow a] = \{a_1\} = \uparrow q(a)$. According to the Remark 4.3.17(ii), the existence of those p-morphisms implies the fact that, if $A_3 \vDash \varphi$, $A_2 \vDash \varphi$ and $A_1 \vDash \varphi$. Also, we can notice that $A_3 \in \mathcal{B}$ and so $\mathsf{Log}\, A_3 \supseteq \mathsf{Log}\, \mathcal{B}$ (see Remark 4.3.11).

As a consequence, by contrapositive of the previous Corollary, we obtain the following Proposition:

**Proposition 7.3.6.** *If $A_3 \vDash \varphi$ then $F \vDash \varphi$, for all $F \in \mathcal{B}$. Therefore $\mathsf{Log}\, A_3 = \mathsf{Log}\, \mathcal{B}$.*

In algebraic terms, this proposition states that the variety of Heyting algebras generates by $A_3$ is isomorphic to the variety of Heyting algebras generates by the class $\mathcal{B}$ of all finite frames such that $\mathsf{Log}\, \mathcal{B} = \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2$ (see Section 4.3.5).

*Proof of $\Rrightarrow$) for (7.3) and (7.4) of Theorem 7.3.3.* We first exhibit a proof for $\mathsf{Log}\, \mathcal{S} \subseteq \mathsf{Log}\, \mathcal{I}$. we need to prove that $\alpha \in \mathsf{Log}\, \mathcal{S}$ implies $\alpha \in \mathsf{Log}\, \mathcal{I}$ or, equivalently, by contrapositive $\alpha \notin \mathsf{Log}\, \mathcal{I}$ implies $\alpha \notin \mathsf{Log}\, \mathcal{S}$. From Proposition 7.3.6 and the first part of proof of Theorem 7.3.3, it follows that $\mathsf{Log}\, \mathcal{I} = \mathsf{Log}\, A_3$. Hence, it suffices to prove that $\alpha \notin \mathsf{Log}\, A_3$ implies $\alpha \notin \mathsf{Log}\, \mathcal{S}$.

In order to do that, we construct a surjective p-morphism $\Sigma \twoheadrightarrow A_3$, for some $\Sigma$, from which, according to the Remark 4.3.17(ii), follows the result. Consider the frame $\Sigma \in \mathcal{S}$ given by $\Sigma = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ such that $\uparrow x_1 = \{x_1, y_1, y_3\}$, $\uparrow x_2 = \{x_2, y_1, y_2\}$, $\uparrow x_3 = \{x_3, y_2, y_3\}$, $\uparrow y_1 = \{y_1\}$, $\uparrow y_2 = \{y_2\}$ and $\uparrow y_3 = \{y_3\}$ and $A_3 = \{a_3, b, c\}$ as in Figure 7.5.

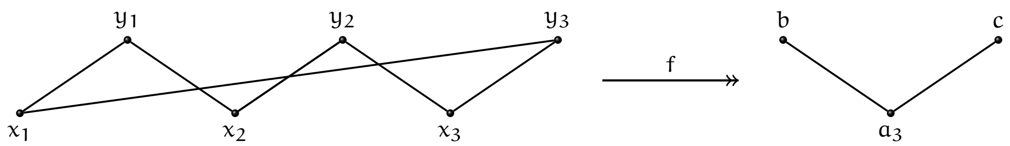


FIGURE 7.5: The p-morphism $f : \Sigma \twoheadrightarrow A_3$.

The map $f : \Sigma \longrightarrow A_3$ given by $f(x_1) = f(x_2) = a_3$, $f(y_2) = f(y_3) = f(x_3) = c$ and $f(y_1) = b$ is the desired p-morphism. In fact, $f[\uparrow x_1] = \{a_3, b, c\} = \uparrow f(x_1)$, $f[\uparrow x_2] = \{a_3, b, c\} = \uparrow f(x_2)$, $f[\uparrow y_1] = \{b\} = \uparrow f(y_1)$, $f[\uparrow y_2] = \{c\} = \uparrow f(y_2)$, $f[\uparrow y_3] = \{c\} = \uparrow f(y_3)$ and $f[\uparrow x_3] = \{c\} = \uparrow f(x_3)$.

For the Remark 4.3.17(ii), we can conclude that $\alpha \notin \mathsf{Log}\, \mathcal{S}$. As a consequence, $\mathsf{Log}\, \mathcal{S} \subseteq \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2$ and, by formula (7.8), $\mathsf{Log}\, \widetilde{\mathcal{S}} = \mathsf{Log}\, \mathcal{S} = \mathsf{INT} + \mathsf{bd}_1 + \mathsf{bb}_2$. $\square$

# Conclusions

The main result of the present work was to prove Theorem 7.3.3.[9] This allows us to understand logically – in the case of 1-dimensional manifolds – a fundamental property of triangulations according to which, for every triangulation of a given topological manifold of dimension $d$, each simplex of dimension $d-1$ is a face of exactly two simplices of dimension $d$.

This result opens the way to corroborating the property above even in the case of dimension greater than one. Moreover, other possible developments of this work are the following: Given any class of polyhedra $C$, what is the intermediate logic of $C$? Is this logic (finitely, or recorsively) axiomatizable? Viceversa, given any intermediate logic $\mathsf{L}$, is there a class of polyhedra $C$ whose logic is $\mathsf{L}$?

There are also some limitations. First of all, the Theorem 7.3.3 shows that it is not possible to grasp the *homotopy class* of topological manifolds. In fact, the closed interval $[0,1]$, which is contractible to a point, has the same logic of the circle $S^1$ which it is not contractible. Furthermore, the proof of Theorem 7.3.3 makes clear the fact that, for instance, the circle $S^1$ has the same logic as a disjoint union of circles. This means that it is not possible to grasp the *connectedness* of spaces. Naturally, it is not ruled out that these limitations suggest better combinations between intermediate logics and polyhedra.

[9]Similar results – albeit for the real line only – occurred in the setting of modal logic: in fact, the modal logic of finite unions of convex subsets of $\mathbb{R}$ is given by a certain normal modal logic over S4. For further details see Aiello, van Benthem, and G. Bezhanishvili 2003 and van Benthem *et al.* 2003.

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