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Almost transitive and almost homogeneous Separable Banach spaces

Mat/05

Cotutore:
Prof. Wiesław Kubiś
Relatore:
Prof. Clemente Zanco
Coordinatore del dottorato:
Prof. Vieri Mastropietro

Tesi di dottorato di
Maria Claudia Viscardi
Matr.R10933

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Contents

Introduction

In this thesis we provide an overview of a problem related to almost homogeneous separable Banach spaces; in particular we focus on the construction of the Gurariĭ space due to J. Garbulińska and W. Kubiś in [2] and some of its consequences, giving a personal contribution to the state of the art (see [14]).

The initial goal was to build a new almost homogeneous separable Banach space. In order to do that, we used the Garbulińska-Kubiś construction, the great advantage of that lying in its abstractness. In particular we tried to produce some new sufficient conditions making such a construction suitable to obtain the desired space. Even if this procedure appears very natural and intuitive, we found some difficulties as we will explain later (see 4). For this reason we did not get the expected result, but just a partial one.

In what follows, X is a real Banach space and all maps are assumed to be linear.

Definition 0.1. *X is $\langle$`almost`$\rangle$ `homogeneous` if for every finite-dimensional subspace A of X, $\langle$for every $\varepsilon > 0\rangle$ and for every isometric embedding $f : A \to X$, there exists a surjective isometry $h : X \to X$ such that $h|_A = f$ $\langle \|f - h|_A\| \leq \varepsilon \rangle$.*

Definition 0.2. *X is $\langle$`almost`$\rangle$ `transitive` if for every one-dimensional subspace A of X, $\langle$for every $\varepsilon > 0\rangle$ and for every isometric embedding $f :$ $A \to X$, there exists a surjective isometry $h : X \to X$ such that $h|_A = f$ $\langle \|f - h|_A\| \leq \varepsilon \rangle$.*

The only known separable homogeneous (and transitive) Banach space is the Hilbert space, and the question, known as the Banach-Mazur rotation

problem, is still open whether a separable Banach space with a transitive norm needs to be isometric to a Hilbert space.
The situation related to almost homogeneous and almost transitive separable Banach spaces is different. In fact some example of such a spaces are available, even if the situation is not yet completely clear.

It is known that spaces $L^p[0,1]$, $1 < p < \infty$, are almost transitive, and Lusky in [8] showed that if $1 \leq p < \infty$ $p \neq 4, 6, 8, \ldots$, for every $A \subseteq L^p[0,1]$ finite-dimensional subspace for every $\varepsilon > 0$ and for every $f : A \to L^p[0,1]$ isometry, there exists $h : X \to X$ surjective $\varepsilon-$isometry such that $f = h|_A$. We investigate whether this property implies these spaces to be almost homogeneous.
Furthermore Lusky proved that every separable Banach space is 1-complemented in some separable almost transitive Banach space ([9]).
A classical example of an almost homogeneous separable Banach space is provided by the Gurariĭ space.
It is the only separable Banach space $\mathbb{G}$ such that given finite-dimensional Banach spaces $X \subseteq Y$, given $\varepsilon > 0$, given an isometry $f : X \to \mathbb{G}$, there exists an ε-isometry $g : Y \to \mathbb{G}$ extending f.

No more almost homogeneous separable Banach spaces are known. Moreover, one of the constructions of the Gurariĭ space that appear in [2] can be generalized. In fact this construction is based on a categorical approach to the class of finite-dimensional Banach spaces as a category (here the objects are the spaces and the arrows are ε-isometries). Hence that approach can be followed when dealing with subclasses of the class of finite-dimensional Banach space in order to construct other almost homogeneous spaces, provided that these subclasses have some analytic and geometrical properties, as it is shown in [4] or in the section 'Final remarks and open problems' in [2] (Problem 7.12).
In this thesis we describe this categorical algorithm as well as the properties that these subclasses must enjoy in order to construct new almost homogeneous spaces with a more analytic approach

In particular let $\mathcal{A}$ be a class of finite-dimensional normed spaces. We will focus on the amalgamation property, i.e. if $X, Y, Z \in \mathcal{A}$, $f : Z \hookrightarrow X$, $g : Z \hookrightarrow Y$ are isometries, then there exist $W \in \mathcal{A}$, $F : Y \hookrightarrow W$ and $G : X \hookrightarrow W$ isometries such that $G \circ f = F \circ g$.
This property, in addition to the hereditarity (i.e. if $Y \subseteq X$ and $X \in \mathcal{A}$ then $Y \in \mathcal{A}$) and the closure under the Banach-Mazur distance of $\mathcal{A}$, is crucial for $\mathcal{A}$ to be a good candidate for the construction of a new almost homogeneous space.
Obviously the class $\mathcal{H}$ of all finite-dimensional Hilbert spaces enjoys all these properties, and the algorithm applied to this class leads to the infinite-dimensional separable Hilbert space. Hence finding new classes, different from $\mathcal{H}$, that enjoy such properties is necessary in order to find some new almost homogeneous spaces.
On the other hand, just one way is known for amalgamate general finite-dimensional normed spaces, that is $W = \frac{X \oplus_1 Y}{\Delta}$, where $\oplus_1$ means the direct sum with the norm defined as the sum of the norms on the two spaces and $\Delta = \{(f(z), -g(z)), z \in Z\}$ (see the Pushout Lemma in [2]).
We prove that the minimal, hereditary and closed class of finite-dimensional Banach spaces with the amalgamation property, that can be constructed with this kind of amalgamation starting from a one dimensional space, is the whole class of finite-dimensional normed spaces (we are preparating a paper where this result is proved).
This result implies that, in order to apply the algorithm to a class $\mathcal{A}$ for the construction of a new almost homogeneous space, it is necessary to find a new way of amalgamating the spaces of $\mathcal{A}$.

Chapter 1

Basic definitions and notions

Let us list here the basic notation which will be used in the thesis; for the reader's convenience some notation will also be recalled during the exposition.
First of all, we will consider only Banach spaces over the real field and all the maps considered are assumed to be linear, even if it will be not specified.

Given a Banach space X with norm $\|\cdot\|$ we denote $B_X := \{x \in X : \|x\| \leq 1\}$ the closed unit ball of X and $S_X := \{x \in X : \|x\| = 1\}$ the unit sphere.

Given a subset A of X, by $\langle A \rangle$ we denote the smallest linear subspace of X that contains A.

Let $1 \leq p \leq \infty$ and $n \in \mathbb{N} \cup \{\infty\}$; we denote by $L^p[0,1]$ the classical Lebesgue space of p-integrable (equivalence classes of) functions defined on the interval $[0,1]$ with the usual norm; ℓ_p^n denotes the space of real sequences of length n equipped with the usual norm.

If X and Y are two Banach spaces, $X \oplus_p Y$ denotes the direct sum of X and Y endowed with the p-norm; that is the vector space of the pairs $\{(x,y) : x \in X, y \in Y\}$ with the norm $\|(x,y)\|_{X \oplus_p Y} := (\|x\|_X^p + \|y\|_Y^p)^{1/p}$, where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms defined on X and Y respectively.

Given a linear map $f : X \to Y$ between two Banach spaces and $0 < \varepsilon < 1$, we will say that:

- f is an *isometry* if $\|f(x)\| = \|x\|$ for every $x \in X$,
- f is an $\varepsilon-$*isometry* if $(1-\varepsilon)\|x\| \leq \|f(x)\| \leq (1+\varepsilon)\|x\|$ for every $x \in X$,
- f is a *strict* $\varepsilon-$*isometry* if $(1-\varepsilon)\|x\| < \|f(x)\| < (1+\varepsilon)\|x\|$ for every $x \in X$, $x \neq 0$.

Note that an isometry is not necessarily surjective; in fact we will always specify when we require the map to have this property.
On the other hand we will say that two Banach spaces are *isometric* if there exists a surjective isometry between these two spaces and we will say that they are *isomorphic* if they are linearly homeomorphic.

For a normed space X, $\hat{X}$ denotes the completion of X, i.e. the smallest Banach space that contains X.

If $(Y, |||\cdot|||)$ is a normed space isomorphic to $(X, \|\cdot\|)$, then the Banach-Mazur distance between the two spaces is defined as

$$d((X, \|\cdot\|), (Y, |||\cdot|||)) =$$

$$= \inf\{\|T\| \cdot \|T^{-1}\| : T \text{ isomorphism between } (X, \|\cdot\|) \text{ and } (Y, |||\cdot|||)\}.$$

If X and Y are isometric then

$$d((X, \|\cdot\|), (Y, |||\cdot|||)) = 1,$$

but it is well known that the converse does not hold true.

Chapter 2

Almost transitive and almost homogeneous normed spaces

Let X be a real normed space. We recall the definitions of homogeneous space and of transitive space.

Definition 2.1. *X is* `homogeneous` *if, for every finite-dimensional subspace A of X and for every isometric embedding $f : A \to X$, there exists a surjective isometry $h : X \to X$ such that $h|_A = f$.*

Definition 2.2. *X is* `transitive` *if for every $x, y \in S_X$ there exists a surjective isometry $h : X \to X$ such that $h(x) = y$.*

Obviously an homogeneous space is also transitive.

If $\dim(X) = n < \infty$ it is well known that the only possibility for X to be transitive is that X is the $n-$dimensional Hilbert space.

As we said in the introduction, the only known separable transitive Banach space is the Hilbert space and the existence of other separable transitive Banach spaces is still unknown. In other words, the following question (Banach-Mazur rotation problem) is still open: "Is every transitive separable Banach space a Hilbert space?"

Concerning the nonseparable Banach spaces, in [10] Pelczynski and Rolewicz showed the first published example of such a non Hilbert space that is transitive.

In this thesis we are interested in the separable case, in particular in separable Banach spaces that enjoy a weaker property, namely the almost transivity.
In particular, there are some weaker definitions concerning the transitivity and there are some separable spaces, different from the Hilbert ones, to which these definitions apply.
We show briefly the situation related to this weaker properties; for a detailed study we refer to [1].

Let $\mathcal{G}$ be the set of all surjective isometries from X to X: then X is transitive if, for every $x, y \in S_X$, we have $y \in \mathcal{G}(x)$, where $\mathcal{G}(x) = \{f(x), f \in \mathcal{G}\} \subseteq S_X$. Using this notation we can give the following definition.

Definition 2.3. *(Almost transitivity) X is* `almost transitive` *if there exists a dense subset $D \subseteq S_X$ such that $\mathcal{G}(x) = D$ for every $x \in D$.*

There are some equivalent ways for defining almost transitivity.

Proposition 2.4. *Let X be a normed space. The following are equivalent:*

(i) *X is almost transitive,*

(ii) *for every $x, y \in S_X$ and for every $\varepsilon > 0$ there exists $f \in \mathcal{G}$ such that $\|f(x) - y\| \leq \varepsilon$,*

(iii) *for every $x \in S_X$ the set $\mathcal{G}(x)$ is dense in S_X.*

Proof. (i)⇒(ii) Fix $x, y \in S_X$ and $\varepsilon > 0$. Let $\tilde{x}, \tilde{y} \in D$ $\varepsilon/2$-close to x and y respectively. The there exists $f \in \mathcal{G}$ such that $f(\tilde{x}) = \tilde{y}$. Moreover

$$\|f(x) - y\| \leq \|f(x) - f(\tilde{x})\| + \|f(\tilde{x}) - y\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

(ii)⇒(iii) Obvious.
(iii)⇒(i) Fix $x \in S_X$ and define $D := \mathcal{G}(x)$. Let $y \in D$: we need to show that $\mathcal{G}(y) = D$.
By definiton of D there exists $h \in \mathcal{G}$ such that $h(x) = y$.
For $z \in D$ there exists $f \in \mathcal{G}$ such that $f(x) = z$. Then $f \circ h^{-1} \in \mathcal{G}$ and

$f(h^{-1}(y)) = z$, that means $z \in \mathcal{G}(y)$.
Now let $z \in \mathcal{G}(y)$: then there exists $f \in \mathcal{G}$ such that $f(y) = z$. Then $h \circ f^{-1} \in \mathcal{G}$ and $h(f^{-1}(z)) = x$, so $z \in D$.

$\square$

Another classical weaker definition is the following.

Definition 2.5. *(Convex transitivity) X is* **convex transitive** *if, for every $x \in S_X$, the closure of the convex hull of $\mathcal{G}(x)$ coincides with B_X.*

From Proposition 2.4 it followos that almost transitivity implies convex transitivity.
On the other hand, convex transitivity does not imply almost transitivity. In fact let $C_0(L)$ the space of all continuous real functions vanishing at infinity, for some locally compact Hausdorff topological space L. Then if $C_0(L)$ is almost transitive, then L reduces to a singleton, while there are some examples of such spaces, with L different from a singleton, that are convex transitive, for example if $L = (0, 1)$.

Classical examples of transitive separable Banach spaces are the spaces $L^p[0, 1]$ with $1 < p < \infty$ as the following argument from [13] shows.
Let $f \in L^p[0, 1]$ a norm-one function such that

$$\text{ess inf}|f| > 0$$

and consider the operator $T_f : L^p[0, 1] \to L^p[0, 1]$ defined as follows:

$$T_f(h) := (h \circ F) \cdot f, \text{ where } F(x) := \int_0^x |f(t)|^p dt.$$

Note that $T_f(1) = f$, where 1 denotes the constant one function on $[0, 1]$.
The operator T_f is an isometry into $L^p[0, 1]$, in fact:

$$\|T_f(h)\|_p^p = \int_0^1 |h(F(t))f(t)|^p dt =$$

$$= \int_0^1 |h(F(t))|^p dF(t) = \|h\|_p^p$$

since F is a strictly increasing function such that $F(0) = 0$ and $F(1) = 1$.
For the proof of surjectivity note that, since ess $\inf|f| \neq 0$, the inverse map of T_f turns out to be

$$T_f^{-1}(h) = \frac{h \circ F^{-1}}{f \circ F^{-1}}$$

and it is well defined on the whole $L^p[0,1]$, hence T_f is a surjective isometry.
Now fix $f, g \in L^p[0,1]$ and $\varepsilon > 0$. Let $f_\varepsilon, g_\varepsilon \in L^p[0,1]$ such that

$$\text{ess inf} f_\varepsilon,\ \text{ess inf} g_\varepsilon > \varepsilon/4$$

and

$$\|f - f_\varepsilon\|, \|g - g_\varepsilon\| < \varepsilon/2.$$

$U := T_{g_\varepsilon} \circ T_{f_\varepsilon}^{-1}$ is a surjective isometry and $U(f_\varepsilon) = g_\varepsilon$. Hence

$$\|U(f) - g\| \leq \|U(f) - U(f_\varepsilon)\| + \|U(f_\varepsilon) - g\| < \varepsilon.$$

This completes the proof.

There are a lot of other examples of separable Banach spaces that are almost transitive, even if their description is not available in the literature. In fact Lusky in [9] proved the following theorem that states that every separable Banach space is 1-complemented in some separable almost transitive Banach space.

Theorem 2.6. *Let X be a separable Banach space. Then there exist a separable almost transitive Banach space $Z \supset X$ and a contractive projection $P : Z \to X$.*

To prove the theorem we need the following lemma.

Lemma 2.7. *Let Y be a separable Banach space. Let $E_n \subseteq Y$ be a sequence of subspaces of Y and let $T_n : E_n \to Y$ be isometries. Furthemore, assume that for every $n \in \mathbb{N}$ there exist contractive projections $P_n : Y \to E_n$ and $Q_n : Y \to T_n(E_n)$. Then for every $n \in \mathbb{N}$ there exist a separable Banach space $\tilde{Y} \supseteq Y$, isometric extensions $\tilde{T}_n : Y \to \tilde{Y}$ of T_n and contractive projections $P : \tilde{Y} \to Y$, $\tilde{Q}_n : \tilde{Y} \to \tilde{T}_n(Y)$.*

Proof. (Lemma 2.7) Consider $(\oplus_{i=1}^{\infty} Y)_1$ (endowed with the norm $\|(y_1, y_2, \ldots)\| = \sum_{i=1}^{\infty} \|y_i\|$, for $y_i \in Y$) and let V be the closed linear span of the set of vectors

$$(-T_n(e), 0, \ldots, 0, \underset{n+1}{e}, 0, \ldots)$$

where $e \in E_n$, $n \in \mathbb{N}$.
Set $\tilde{Y} := (\oplus_{i=1}^{\infty} Y)_1 / V$. An element of $\tilde{Y}$ is a class $[(y_1, y_2, \ldots)] = (y_1, y_2, \ldots)+$

V. Since

$$\|y\| \leq \inf\{\|y - \sum_{n=1}^{\infty} T_n(e_n)\| + \sum_{n=1}^{\infty} \|e_n\|,\ e_n \in E_n, n \in \mathbb{N}\} \leq \|y\|$$

for every $y \in Y$, we can identify Y with the subspace spanned by the elements $[(y, 0, 0, \ldots)]$, $y \in Y$.
Let $i : Y \to \tilde{Y}$ be the isometry such that $i(y) = [(y, 0, 0, \ldots)]$. Then each T_n can be seen as a map from E_n to $\tilde{Y}$ provided it is composed with the embedding i.

Observe that $[(T_n(e), 0, 0, \ldots)] = [(0, 0, \ldots, 0, \underset{n+1}{e}, 0, \ldots)]$ for every $e \in E_n$ and for every $n \in \mathbb{N}$. So if we define for every $y \in Y$ and $n \in \mathbb{N}$

$$\tilde{T}_n(y) := [(0, 0, \ldots, 0, \underset{n+1}{y}, 0, \ldots)].$$

Each $\tilde{T}_n$ is an extension of T_n and for every $y \in Y$ we have

$$\|y\| \geq \inf\{\|y - e_n\| + \sum_{\substack{i=1 \\ i \neq n}}^{\infty} \|e_i\| + \|\sum_{i=1}^{\infty} T_i(e_i)\|,\ e_i \in E_i, i \in \mathbb{N}\}$$

$$\geq \inf\{\|y - e_n\| + \|T_n(e_n)\| + \sum_{\substack{i=1 \\ i \neq n}}^{\infty} (\|e_i\| - \|T_i(e_i)\|),\ e_i \in E_i, i \in \mathbb{N}\} \geq \|y\|.$$

Now for every $y_i \in Y$ and $n \in \mathbb{N}$ set

- $P'([(y_1, y_2, \ldots)]) := [(y_1, P_1(y_2), P_2(y_3), \ldots)]$, then $\sum_{i=1}^{\infty} T_i \circ P_i(y_{i+1})$ converges, since

  $$\sum_{i=1}^{\infty} \|T_i \circ P_i(y_{i+1})\| \leq \sum_{i=2}^{\infty} \|y_i\| < \infty.$$

  So $[(y_1 + \sum_{i=1}^{\infty} T_i \circ P_i(y_{i+1}), 0, 0, \ldots)] = [(y_1, P_1(y_2), P_2(y_3), \ldots)]$, hence $P := i^{-1} \circ P'$ satisfies the following

  $$P([(y_1, y_2, \ldots)]) = y_1 + \sum_{i=1}^{\infty} T_i \circ P_i(y_{i+1}).$$

- $\tilde{Q}_n([(y_1, y_2, \ldots)]) := [(Q_n(y_1 + \sum_{\substack{i=1 \\ i \neq n}}^{\infty} T_i \circ P_i(y_{i+1})), 0, 0, \ldots, 0, \underset{n+1}{y}, 0, \ldots)]$

It followos that

$$\|P\| \leq \sup_{i\in\mathbb{N}} \|P_i\| \leq 1 \text{ and } \|\tilde{Q_n}\| \leq \sup_{i\in\mathbb{N}} \|Q_n\|\|P_i\| \leq 1.$$

□

Proof. (Theorem 2.6) We want to construct a nested sequence of separable Banach spaces $X = Y_0 \subseteq Y_1 \subseteq Y_2 \dots$ with contractive projections $R_n : Y_n \to Y_0$ using the previous lemma, in order to obtain that the completion of the limit $\bigcup_{n\in\mathbb{N}} Y_n$ is almost transitive.

We start with $X = Y_0$. Fix a countable dense subset $D_0 = \{x_i\}_{i\in\mathbb{N}}$ of the unit sphere of X and define:

$$\Omega_0 = \{T : \langle x \rangle \to \langle y \rangle \ , \ x, y \in D_0\}.$$

Obviously Ω_0 is countable, hence we can consider it as a sequence of isometries T^0_n, each one defined from a subspace $\langle x \rangle$ ($x \in D_0$) to X. For every $n \in \mathbb{N}$ let E^0_n the domain of T^0_n.
For every $n \in \mathbb{N}$ the Hahn-Banach theorem provides us with contractive projections $P^0_n : Y_0 \to E^0_n$ and $Q^0_n : Y_0 \to T^0_n(E^0_n)$.
Then, from the previous lemma, there exist $Y_1 \supseteq Y_0$, $\tilde{T}^0_m : Y_0 \to Y_1$ isometries that extend T^0_m, $P^1 : Y_1 \to Y_0$ and $\tilde{Q}^0_m : Y_1 \to \tilde{T}^0_m(Y_0)$ contractive projections. Let $R_1 = P^1$.
Now for every $m \in \mathbb{N}$ put

- $E^1_m := \tilde{T}^0_m(Y_0)$,
- $T^1_m := (\tilde{T}^0_m)^{-1} : E^1_m \to Y_1$,
- $Q^1_m := P_1 : Y_1 \to Y_0$ ($Y_0 = T^1_m(E^1_m)$),
- $P^1_m := \tilde{Q}^0_m : Y_1 \to E^1_m$.

Moreover, as we did for Y_0, let D_1 be a countable dense subset of the unit sphere of Y_1, Ω_1 be the set of all isometries between one-dimensional subspaces generated by the elements of D_1. Consider $\{T^1_m\}_{m\in\mathbb{N}} \cup \Omega_1$ the new set of isometries with domain in $\{E^1_m\}_{m\in\mathbb{N}} \cup \{\langle x \rangle \ , \ x \in D_1\}$ and the relative projections and apply again Lemma 2.7.

At this way, we inductively construct the sequence $\{Y_n\}$, with $\tilde{T}^n_m : Y_n \to Y_{n+1}$ that are isometric extensions of the isometries defined between the one-dimensional subspaces $\{\langle x \rangle\,,\ x \in D_i$ for some $i \le n\}$, and $\bigcup_{i=1}^n D_i$ is dense in the unit sphere of $\bigcup_{i=1}^n Y_i$. Moreover, for every $n \in \mathbb{N}$, $R_n := P^n \circ P^{n-1} \circ \ldots \circ P^1 : Y_n \to Y_0$ is a contractive projection.
Hence, passing to the limit, we obtain that the completion of $\bigcup_{i\in\mathbb{N}} Y_i$ is the desired space. $\square$

No other examples of almost transitive separable space are known right now.

Concerning the approximations of finite-dimensional isometries with surjective isometries on the whole space one can investigate for wich spaces a stronger property holds.

Definition 2.8. *(Almost homogeneity) X is* `almost homogeneous` *if for every finite-dimensional subspace A of X, for every $\varepsilon > 0$ and for every isometric embedding $f : A \to X$, there exists a surjective isometry $h : X \to X$ such that $\|f - h|_A\| \le \varepsilon$.*

Obviously almost homogeneity implies almost transitivity.
Lusky in [8] showed that if $1 \le p < \infty$, $p \ne 4, 6, 8, \ldots$, for every $A \subseteq L^p[0,1]$ finite-dimensional subspace, for every $\varepsilon > 0$ and for every isometry $f : A \to L^p[0,1]$ isometry, there exists $h : X \to X$, h a surjective $\varepsilon-$isometry, such that $f = h|_A$.
In [12] Randrianantoanina showed that this property doesn't hold for $L^p[0,1]$ when p is not an even integer.
We don't know whether this property implies the almost homogeneity, but we can prove the converse.

Proposition 2.9. *Let X be an almost homogeneous Banach space. Then for every finite-dimensional subspace $A \subseteq X$, for every $\varepsilon > 0$ and for every isometry $f : A \to X$, there exists $h : X \to X$, h a surjective $\varepsilon-$isometry, such that $f = h|_A$.*

Before starting the proof we recall the following well known theorem.

Theorem 2.10. *(From Von Neumann series) The set of the invertible operators between two Banach spaces is open in the topology induced by the operator norm.*

Proof. (Proposition 2.9) Fix $A \subset X$ a finite-dimensional subspace. A is complemented, that means that there exist $\tilde{X} \subset X$, $\tilde{X}$ closed, with $A \cap \tilde{X} = \{0\}$ such that for every $x \in X$ there are $\tilde{x} \in \tilde{X}$ and $a \in A$ such that $x = \tilde{x} + a$. Moreover there exists a bounded propjection $P : X \to A$.
Let $f : A \to X$ be isometry, $0 < \varepsilon < 1/\|P\|$ and (from the almost homogeneity) let $F : X \to X$ be an isometry such that $\|F|_A - f\| \leq \varepsilon$.
Define $h : X \to X$ in such a way: for every $x \in X$ let $\tilde{x} \in \tilde{X}$ and $a \in A$ such that $x = \tilde{x} + a$ and $h(x) := F(\tilde{x}) + f(a)$.
h is linear, moreover it is an $\varepsilon\|P\|-$isometry. In fact for every $\tilde{x} + a$ as before we have

$$\|h(\tilde{x} + a)\| = \|F(\tilde{x}) + f(a)\| \leq \|F(\tilde{x}) + F(a)\| + \|f(a) - F(a)\|$$

$$\leq \|\tilde{x} + a\| + \varepsilon\|a\| = \|\tilde{x} + a\| + \varepsilon\|P(\tilde{x} + a)\| \leq \|\tilde{x} + a\|(\varepsilon\|P\| + 1).$$

On the other hand, at the same way we get

$$\|F(\tilde{x}) + f(a)\| \geq \|F(\tilde{x}) + F(a)\| - \|f(a) - F(a)\| \geq (1 - \varepsilon\|P\|)\|\tilde{x} + a\|.$$

Obviously h extends f on X. It remains to prove that h is surjective.
Since h is $\varepsilon-$close to F, if ε is suffecently small from theorem 2.10 we have that h is invertible.
This complete the proof. □

An example of almost homogeneous separable Banach space is the Gurariĭ space, which will be studied in the next chapter.
There are no other examples of almost homogeneous separable Banach spaces, but, as we observed in the introduction, there exists a general algorithm that, when applied to certain subclass of the class $\mathcal{B}$ of all finite-dimensional normed spaces, leads to the construction of spaces with these properties. So the problem related to the research of such spaces can be restricted to the study of some properties of $\mathcal{B}$, as we will show in chapter 4.

Chapter 3

The Gurariĭ space

In [3] Gurariĭ introduces the notions of spaces of universal and almost-universal disposition for a given class $\mathcal{K}$ of Banach spaces as follows.

Definition 3.1. *Let $\mathcal{K}$ a class of Banach spaces.*

- *A Banach space U is said to be of* **almost universal disposition** *for the class $\mathcal{K}$ if, given $A, B \in \mathcal{K}$, $A \subseteq B$, any isometry $f : A \hookrightarrow U$, and any $\varepsilon > 0$, there is an ε-isometry $F : B \to U$ such that $F = f|_A$.*
- *A Banach space U is said to be of* **universal disposition** *for the class $\mathcal{K}$ if, given $A, B \in \mathcal{K}$, $A \subseteq B$, any isometry $f : A \hookrightarrow U$, extends to an isometry $F : B \to U$.*

From now on, let $\mathcal{B}$ the class of all finite-dimensional real normed spaces. A Banach space that turns out to be of almost universal disposition for the class $\mathcal{B}$ is called Gurariĭ space. In other words the following definition is given.

Definition 3.2. *A* **Gurariĭ space** *(constructed by Gurariĭ [3] in 1965) is a separable Banach space $\mathbb{G}$ satisfying the following condition:*

(G) Given finite-dimensional Banach real spaces $X \subseteq Y$, given $\varepsilon > 0$, given an isometry $f : X \hookrightarrow \mathbb{G}$ there exists an $\varepsilon-$isometry $g : Y \to \mathbb{G}$ extending f.

It has been unknown for some time whether the Gurariĭ space is unique up to surjective isometries; the question was answered in the affirmative by

Lusky in [7] in 1976.
Very recently, Solecki and Kubiś in [6] have found a simple and elementary proof of the uniqueness of the Gurariĭ space. We show the arguments of this proof in Section 3.2 below. We will see also that from the theorem of the uniqueness of the Gurariĭ space it follows that this space is almost homogeneous.
In the next Section we show three different constructions of the Gurariĭ space, due to Garbulińska and Kubiś ([2] and [5]). The first has a more analytic approach, while the second and the third are more abstract. In particular the second one could be generalized in order to obtain other almost homogeneous separable Banach space, as we will discuss in the next chapter.

On the other hand the spaces of universal disposition for $\mathcal{B}$ are called *strong Gurariĭ spaces*.
No separable strong Gurariĭ space exist. In fact consider a separable Banach space G, let $\{e_1, e_2\}$ be the canonical basis of $F = \ell_1^2$ and let $E = \langle e_1 \rangle$ its one-dimensional subspace generated from $\{e_1\}$.
Let $x \in S_G$ be a smooth point and $f : E \to G$ be the isometry such that $f(e_1) = x$, then it is obvious that there is no isometric extension $h : F \to G$ since e_1 is not smooth in F.

The situation concerning nonseparable Banach spaces of universal disposition for $\mathcal{B}$ is different: in fact there exist some spaces that satisfy this property. For a depth study of non-separable case we refer to [2].

Before starting with the constructions and the proof of the uniqueness of the Gurariĭ space, we need to show two important properties of the class $\mathcal{B}$.

Lemma 3.3. *(Pushout Lemma) Let Z, X, Y be Banach spaces, let $i : Z \hookrightarrow X$ be an isometry and let $f : Z \to Y$ be an ε-isometry, with $\varepsilon > 0$. Then there exist a Banach space W, an isometry $j : Y \hookrightarrow W$ and an ε-isometry $g : X \to W$ under which the diagram*

$$\begin{tikzcd} Y \arrow[r, "j"] & W \\ Z \arrow[u, "f"] \arrow[r, "i"'] & X \arrow[u, "g"'] \end{tikzcd}$$

commutes.
Furthermore if X, Y are finite-dimensional, so W is finite-dimensional too.

Proof. For simplicity, let us assume that i is the inclusion $Z \subseteq X$. Define $W = (X \oplus Y)/\Delta$, where $X \oplus Y$ is endowed with the weighted ℓ_1 norm

$$\|(x, y)\|_{X\oplus Y} := (1+\varepsilon)\|x\|_X + \|y\|_Y,$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms of X and Y respectively, and $\Delta = \{(z, -f(z)), z \in Z\}$. Let g and j be the quotients under the canonical embeddings, i.e. $g(x) = (x, 0) + \Delta$ and $j(y) = (0, y) + \Delta$ for $x \in X$, $y \in Y$. Obviously $g \circ i = j \circ f$.
Observe that

$$\|g(x)\| = \inf_{z\in Z}\Big((1+\varepsilon)\|x+z\|_X + \|-f(z)\|_Y\Big) \leq (1+\varepsilon)\|x\|_X.$$

Similarly,

$$\|j(y)\| = \inf_{z\in Z}\Big((1+\varepsilon)\|z\|_X + \|y-f(z)\|_Y\Big) \leq \|y\|_Y.$$

It remains to estimate $\|g(x)\|$ and $\|j(y)\|$ from below.
Fix $x \in X$. Given $z \in Z$, we have

$$(1+\varepsilon)\|x+z\|_X + \|-f(z)\|_Y \geq (1-\varepsilon)\Big(\|x+z\|_X + \|-z\|_X\Big) \geq (1-\varepsilon)\|x\|_X.$$

It follows that $\|g(x)\| \geq (1-\varepsilon)\|x\|_X$.
Now fix $y \in Y$. Given $z \in Z$, we have

$$(1+\varepsilon)\|z\|_X + \|y-f(z)\|_Y \geq \|f(z)\|_Y + \|y-f(z)\|_Y \geq \|y\|_Y.$$

Thus $\|j(y)\| \geq \|y\|_Y$. This completes the proof.
The "furthermore" part of the lemma follows from the construction of W, j and g. □

Lemma 3.3 is called Pushout Lemma since it turns out that the amalgamation constructed in the proof is the pushout of i and f in the category of Banach spaces with linear operators of norm less than 1. Specifically, given arbitrary bounded linear operators $T : X \to V$, $S : Y \to V$ such that $T \circ i = S \circ f$, there exists a unique linear operator $h : W \to V$ satisfying $h \circ g = T$ and $h \circ f = S$.

Note that in the Pushout Lemma, if f is an isometry, then g is too. We will refer to this isometric version several times.

Lemma 3.4. *(Small distortion property) Let X and Y be Banach spaces, let $\varepsilon > 0$ and let $f : X \to Y$ be an ε-isometry. Then there exist a Banach space Z and isometries $g : Y \hookrightarrow Z$, $h : X \hookrightarrow Z$, such that $\|g \circ f - h\| \leq \varepsilon$.*
In particular if X and Y are finite-dimensional, then Z is finite-dimensional too.

Proof. Let $Z = X \oplus Y$ be endowed with the following norm:

$$\|(x, y)\| := \inf\{\|\tilde{x}\|_X + \|\tilde{y}\|_Y + \varepsilon\|w\|_X :$$

$$x = \tilde{x} + w \text{ and } y = \tilde{y} - f(w), \tilde{x}, w \in X, y \in Y\}$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms on X and Y respectively.
It is easy to check that $\|\cdot\|$ is a norm on Z, since its unit ball is the convex hull of

$$(B_X \times \{0\}) \cup (B_Y \times \{0\}) \cup \{(w, f(w) : w \in X, \|w\|_X \leq 1/\varepsilon)\}.$$

Let g and h the canonical embeddings of X and Y in Z. We have to show that they are isometries.
Obviously $\|x\|_X \geq \|h(x)\| = \|(x, 0)\|$ and $\|y\|_Y \geq \|g(y)\| = \|(0, y)\|$. On the other and, let $x \in X \setminus \{0\}$, then

$$\|(x, 0)\| = \inf\{\|\tilde{x}\|_X + \|f(x - \tilde{x})\|_Y + \varepsilon\|x - \tilde{x}\|_X, \tilde{x} \in X\}$$

$$\geq \inf\{\|\tilde{x}\|_X + (1 - \varepsilon)\|x - \tilde{x}\|_X + \varepsilon\|x - \tilde{x}\|_X\} \geq \|x\|_X.$$

In a similar way we can prove that $\|g(y)\| \geq \|y\|_Y$, so h and g are isometries.
Now consider $\|g \circ f(x) - h(x)\| = \|(x, -f(x))\|$ and

$$\|(x, -f(x))\| = \inf\{\|\tilde{x}\|_X + \|f(\tilde{x})\|_Y + \varepsilon\|x - \tilde{x}\|_X\} \leq \varepsilon\|x\|_X.$$

This complete the proof. □

3.1 Three constructions

In this section we show three different kind of constructions of the Guraiĭ space from [2]. The first one has an essentially analytic approach, while the second and third ones are more abstract (indeed the second one can be extended to other classes of spaces and needs a simple result of set theory).

The first two constructions are based on the density of the class of finite-dimensional rational spaces in $\mathcal{B}$.

Definition 3.5. *We say that a finite-dimensional normed space E is* ***rational*** *if it is isometric to some $(\mathbb{R}^n, \|\cdot\|)$ whose unit sphere is a polyhedron all vertices of which have rational coordinates.*
Equivalently, E is rational if, up to isometry, $E = \mathbb{R}^n$ with a "maximum norm" $\|\cdot\|$ induced by finitely many functionals $\varphi_1, \dots, \varphi_m$ such that $\varphi_i(\mathbb{Q}^n) \subseteq \mathbb{Q}$ for every $i < m$. More precisely, $\|x\| = \max_i\{\varphi_i(x)\}$ for $x \in \mathbb{R}^n$.

Note that there are continuum many isometric types of finite-dimensional Banach spaces. Thus, to check that a given Banach space is Gurariĭ one should need to show the existence of suitable extensions of continuum many isometries. Of course, that can be relaxed. One way to do it is to consider the subclass of all rational spaces.
It is clear that there are (up to isometry) only countably many rational Banach spaces and for every $\varepsilon > 0$, every finite-dimensional space is ε-isometric to some rational Banach space.
In what follow it is shown how $\mathcal{B}$ can be replaced by the class of rational spaces.

Definition 3.6. *A pair of Banach spaces $E \subseteq F$ is called* ***rational pair*** *if, up to isometry, $F = \mathbb{R}^n$ with a rational norm, and $E \cap \mathbb{Q}^n$ is dense in E.*

Note that, if $E \subseteq F$ is a rational pair, then both E and F are rational Banach spaces.
It is clear that there are, up to isometry, only countably many rational pairs of Banach spaces.

Theorem 3.7. *Let X be a Banach space. Then X is Gurariĭ if and only if X satisfies the following condition.*

(G)' Given $\varepsilon > 0$ and a rational pair of spaces $E \subseteq F$, for every strict ε-isometry $f : E \to X$ there exists an ε-isometry $g : F \to X$ such that $\|g|_E - f\| \leq \varepsilon$.

Furthermore, in condition (G)' it suffices to consider ε from a given set $T \subset (0, +\infty)$ with $\inf T = 0$.

Proof. Every Gurariĭ space satisfies (G)' by definition.
Assume X satisfies (G)'.
Fix two finite-dimensional spaces $E \subseteq F$ and fix an isometry $f : E \hookrightarrow X$ and $\varepsilon > 0$.
Fix a linear basis $\mathcal{B} = \{e_1, \dots, e_m\}$ in F so that $\mathcal{B} \cap E = \{e_1, \dots, e_k\}$ is a basis of E (so E is k-dimensional and F is m-dimensional).
Choose $\delta > 0$ small enough. In particular, δ should have the property that for every linear operators $h, g : F \to X$, if $\max_{i \le m} \|h(e_i) - g(e_i)\| < \delta$ then $\|h - g\| < \varepsilon/3$. In fact, δ depends only on the norm of F; a good estimation is $\delta = \varepsilon/(3M)$, where

$$M = \sup\Big\{\sum_{i \le m} |\lambda_i| : \|\sum_{i \le m} \lambda_i e_i\| = 1\Big\}.$$

Now choose a δ-equivalent norm $\|\cdot\|'$ on F such that $E \subseteq F$ becomes a rational pair (in particular, the basis $\mathcal{B}$ gives a natural coordinate system under which all e_i's have rational coordinates).
The operator f becomes a δ-isometry, therefore by (G)' there exists a δ-isometry $g : F \to X$ such that $\|f - g|_E\|' < \delta$.
Now let $h : F \to X$ be the unique linear operator satisfying $h(e_i) = f(e_i)$ for $i \le k$ and $h(e_i) = g(e_i)$ for $k < i \le m$. Then $h|_{\mathcal{B}}$ is δ-close to $g|_{\mathcal{B}}$ with respect to the original norm, therefore $\|h - g\| < \varepsilon/3$. Clearly, $h|_E = f$.
If δ is small enough, we can be sure that g is an $\varepsilon/3$-isometry with respect to the original norm of F.
Finally, assuming $\varepsilon < 1$, a standard calculation shows that h is an ε-isometry, being $(\varepsilon/3)$-close to g.
The "furthermore" part clearly follows from the arguments above. □

3.1.1 First construction

Now fix:

- a separable Banach space X,
- a countable dense set $D \subseteq X$,
- a rational pair of Banach spaces $E \subseteq F$,
- a linear basis $\mathcal{B}$ in E consisting of vectors with rational coordinates,

- $\varepsilon \in (0,1) \cap \mathbb{Q}$,

such that a strict ε-isometry $f : E \to X$ exists such that $f(\mathcal{B}) \subset D$.
Using the Pushout Lemma, we can find a separable Banach space $X' \supseteq X$ such that f extends to an ε-isometry $g : F \to X'$.
Note that there are only countably many pairs of rational Banach spaces and almost isometries as described above. Thus, there exists a separable Banach space $G(X) \supseteq X$ such that, given a rational pair $E \subseteq F$, for every $\varepsilon \in (0,1) \cap \mathbb{Q}$ and for every strict ε-isometry $f : E \to X$ there exists an ε-isometry $g : F \to X$ such that $g|_E$ is arbitrarily close to f.

Repeat this construction countably many times.
Namely, let $G = \widehat{\bigcup_{n\in\mathbb{N}} X_n}$, where $X_0 = X$ and $X_{n+1} = G(X_n)$ for $n \in \mathbb{N}$.
Clearly, G is a separable Banach space.
By Theorem 3.7, G is the Gurariĭ space.

Since the space X was chosen arbitrarily and the Gurariĭ space is unique up to surjective isometries, we get the following result:

Theorem 3.8. *(Universality) The Gurariĭ space contains an isometric copy of every separable Banach space.*

3.1.2 Second construction

Next we show the more general construction. For this construction we need a simple result of set theory, namely the Rasiowa-Sikorski's lemma.

Given a partially ordered set $\mathbb{P}$, recall that a subset $D \subset \mathbb{P}$ is *cofinal* if for every $p \in \mathbb{P}$ there exists $d \in D$ with $p \leq d$.

Lemma 3.9. *(Rasiowa-Sikorski) Given a directed partially ordered set $\mathbb{P}$, given a countable family $\{D_n\}_{n\in\mathbb{N}}$ of cofinal subsets of $\mathbb{P}$, there exists a sequence $\{p_n\}_{n\in\mathbb{N}} \subset \mathbb{P}$ such that $p_n \in D_n$ for every $n \in \mathbb{N}$ and*

$$p_0 \leq p_1 \leq p_2 \leq \dots .$$

Proof. Let $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$ and fix $p \in \mathbb{P}$. Using the fact that each D_n is cofinal, construct inductively $\{p_n\}_{n\in\mathbb{N}}$ so that $p_n \in D_n$ for $n \in \mathbb{N}$ and

$$p_0 \leq p_1 \leq p_2 \leq \dots$$

□

Recall that c_{00} denotes the linear subspace of $\mathbb{R}^{\mathbb{N}}$ consisting of all vectors with finite support. In other words, $x \in c_{00}$ iff $x \in \mathbb{R}^{\mathbb{N}}$ and $x(n) = 0$ for all but finitely many $n \in \mathbb{N}$. Given a finite set $S \subset \mathbb{N}$, we shall identify each space $\mathbb{R}^S$ with a suitable subset of c_{00}.

Let $\mathbb{P}$ be the following partially ordered set. An element of $\mathbb{P}$ is a pair $p = (\mathbb{R}^{S_p}, \|\cdot\|_{S_p})$, where $S_p \subset \mathbb{N}$ is a finite set and $\|\cdot\|_{S_p}$ is a norm on $\mathbb{R}^{S_p} \subset c_{00}$. We put $p \leq q$ iff $S_p \subset S_q$ and $\|\cdot\|_{S_q}$ extends $\|\cdot\|_{S_p}$.
Clearly, $\mathbb{P}$ is a partially ordered set.

Suppose

$$p_0 \leq p_1 \leq p_2 < \dots$$

is a sequence in $\mathbb{P}$ such that the chain of sets $\bigcup_{n\in\mathbb{N}} S_{p_n} = \mathbb{N}$. Then c_{00} naturally becomes a normed space.
Let X be the completion of c_{00} endowed with this norm. We shall call it the *limit* of $\{p_n\}_{n\in\mathbb{N}}$ and write $X = \lim_{n\to\infty} p_n$. It is rather clear that every separable Banach space is of the form $\lim_{n\to\infty} p_n$ for some sequence $\{p_n\}_{n\in\mathbb{N}}$ in $\mathbb{P}$. We are going to show that, for a "typical" sequence in $\mathbb{P}$, its limit is the Gurariĭ space.

We now define a countable family of open cofinal sets which is good enough for producing the Gurariĭ space.
Namely, fix a rational pair of spaces $E \subseteq F$ and fix a rational embedding $f : E \to c_{00}$, that is, an injective linear operator that maps points of E with rational coordinates into $c_{00} \cap \mathbb{Q}^{\mathbb{N}}$.
The point is that there are only countably many possibilities for E, f.

Let E, F, f as above, $n \in \mathbb{N}$ and $\varepsilon \in (0,1) \cap \mathbb{Q}$. Define $D_{E,F,f,n,\varepsilon}$ as the set of all $p \in \mathbb{P}$ such that $n \in S_p$ and p satisfies the following implication: if f is a ε-isometry into $(\mathbb{R}^{S_p}, \|\cdot\|_{S_p})$, then there exists a ε-isometry $g : F \to (\mathbb{R}^{S_p}, \|\cdot\|_{S_p})$ such that $g|_E = f$.

Fix $D_{E,F,f,n,\varepsilon}$: we want to show that it is cofinal.
Let $p \in \mathbb{P}$; without loss of generality we can suppose that $n \in S_p$ (possiblyS_p

can be enlarged).
Suppose that f is a ε-isometry into $(\mathbb{R}^{S_p}, \|\cdot\|_{S_p})$ (otherwise clearly $p \in D_{E,F,f,n,\varepsilon}$). Using the Pushout Lemma, find a finite-dimensional Banach space W extending $(\mathbb{R}^{S_p}, \|\cdot\|_{S_p})$ and a ε-isometry $g : F \to W$ such that $g|_F = f$.
We may assume that $W = (\mathbb{R}^T, \|\cdot\|_W)$ for some $T \supseteq S_p$, where the norm $\|\cdot\|_W$ extends $\|\cdot\|_{S_p}$. Let $q = (\mathbb{R}^T, \|\cdot\|_W) \in \mathbb{P}$.
Clearly, $p \leq q$ and $q \in D_{E,F,f,n}$.

Let $\mathcal{D}$ consist of all sets of the form $D_{E,F,f,n,\varepsilon}$ as above.
Then $\mathcal{D}$ is countable; therefore applying Lemma 3.9 we obtain a sequence $\{p_n\}_{n\in\mathbb{N}}$ such that for every E, F, f, n, ε as above there exists $n \in \mathbb{N}$ for which $p_n \in D_{E,F,f,n}$ and $p_m \leq p_{m+1}$ for every $m \in \mathbb{N}$.
Moreover, from the definition of $D_{E,F,f,n,\varepsilon}$ we have $\bigcup_{n\in\mathbb{N}} S_{p_n} = \mathbb{N}$.

We want to show that $X = \lim_{n\to\infty} p_n$ has property (G)', that means that it is the Gurariĭ space.

Let $E \subseteq F$ a rational pair and $f : E \to X$ a strict $\varepsilon-$isometry.
We want to show that there exists a ε-isometry $g : F \to X$ such that $\|g|_E - f\| \leq \varepsilon$.
Let $\tilde{\tilde{\varepsilon}} < \tilde{\varepsilon} \leq \varepsilon$ with $\tilde{\varepsilon} \in \mathbb{Q}$ and $\tilde{\tilde{\varepsilon}}$ such that f is a $\tilde{\tilde{\varepsilon}}-$isometry. The key point is that for every $\eta > 0$ there exists a rational embedding $\tilde{f} : E \to X$ that is $\eta-$close to f, i.e. $\|f - \tilde{f}\| \leq \eta$. In particular $f(E) \subset c_{00}$, that means that there $f(E) \subseteq \mathbb{R}^{S_{p_m}}$ for m big enough. Moreover, if x is in the unit sphere of E, then

$$1 - \tilde{\tilde{\varepsilon}} - \eta \leq \|f(x)\| - \eta \leq \|\tilde{f}(x)\| \leq \|f(x)\| + \eta \leq 1 + \tilde{\tilde{\varepsilon}} + \eta.$$

With $\eta \leq \tilde{\varepsilon} - \tilde{\tilde{\varepsilon}}$, it turns out that $\tilde{f}$ is a $\tilde{\varepsilon}-$isometry.

Fix $n \in \mathbb{N}$ and consider $D_{E,F,\tilde{f},n,\tilde{\varepsilon}}$: then $p_m \in D_{E,F,\tilde{f},n,\tilde{\varepsilon}}$ and $\tilde{f} : E \to \mathbb{R}^{S_{p_m}}$ is a $\tilde{\varepsilon}-$isometry for m big enough.
This means that there exists a $\tilde{\varepsilon}$-isometry

$$g : F \to \mathbb{R}^{S_{p_m}} \subset X$$

that extends $\tilde{f}$. Moreover from the construction of $\tilde{f}$ we obtain:

$$\|g|_E - f\| \leq \|g|_E - \tilde{f}\| + \|\tilde{f} - f\| \leq \tilde{\varepsilon} \leq \varepsilon.$$

The construction is done.

3.1.3 Third construction

This last construction, made by Kubiś in [5], is apparently easy and can be understood from everybody, and this is the reason why we want to show it.
On the other hand it has an abstract approach, hence can be used in other situations for the constructions of other spaces.

We consider the following game. Namely, two players (called *Eve* and *Odd*) alternately choose finite-dimensional Banach spaces $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$, with no additional rules. For obvious reasons, Eve should start the game. The result is the completion of the chain $\bigcup_{n\in\mathbb{N}} E_n$.
This game is a special case of an abstract Banach-Mazur game.

$$\begin{tikzcd}
\text{Eve:} & E_0 \arrow[dr, hook] & & E_2 \arrow[dr, hook] & & \cdots & \\
\text{Odd:} & & E_1 \arrow[ur, hook] & & E_3 \arrow[ur, hook] & & \cdots
\end{tikzcd}$$

The main result that we will show is the following:

Theorem 3.10. *There exists a unique, up to linear isometries, separable Banach space G such that Odd has a strategy Σ in the Banach-Mazur game leading to G, namely, the completion of every chain resulting from a play this game is linearly isometric to G, assuming Odd uses strategy Σ, and no matter how Eve plays.*
Furthermore, G is the Gurariĭ space.

For the proof of the theorem we need the following result, that is a corollary of Theorem 3.12 of the next section.

Lemma 3.11. *A separable Banach space G is Gurariĭ if and only if*

(H) *for every* $\varepsilon > 0$*, for every finite-dimensional normed spaces* $A \subseteq B$*, for every isometry* $e : A \to G$ *there exists an isometry* $f : B \to G$ *such that* $\|e - f|_A\| < \varepsilon$.

Proof. (of Theorem 3.10) Odd fixes a separable Banach space G satisfying (H). We do not assume a priori that it is uniquely determined, therefore the arguments below will also show the uniqueness of G (up to bijective isometries).
Odd's strategy Σ in the Banach-Mazur game can be described as follows.

Fix a countable set $\{v_n\}n \in \mathbb{N}$ dense in G. Let E_0 be the first move of Eve.
Odd finds an isometric embedding $f_0 : E_0 \to G$ and finds $E_1 \supseteq E_0$ together with an isometric embedding $f_1 : E_1 \to G$ extending f_0 and such that $v_0 \in f_1(E_1)$.
Suppose now that $n = 2k > 0$ and E_n was the last move of Eve.
We assume that a linear isometric embedding $f_{n-1} : E_{n-1} \to G$ has been fixed.
Using (H) we choose a linear isometric embedding $f_n : E_n \to G$ such that $f_n|_{E_{n-1}}$ is 2^{-k}-close to f_{n-1}.
Extend f_n to a linear isometric embedding $f_{n+1} : E_{n+1} \to G$ so that $E_{n+1} \supseteq E_n$ and $f_{n+1}(E_{n+1})$ contains all the vectors $v_0, \ldots, v_k$. The finite-dimensional space E_{n+1} is Odd's move.
This finishes the description of Odd's strategy Σ.

Let $\{E_n\}n \in \mathbb{N}$ be the chain of finite-dimensional normed spaces resulting from the play, when Odd was using strategy Σ.
In particular, Odd has recorded a sequence $\{f_n : E_n \to G\}_{n\in\mathbb{N}}$ of linear isometric embeddings such that $f_{2n+1}|_{E_{2n-1}}$ is 2^{-n}-close to f_{2n-1} for each $n \in \mathbb{N}$.
Let $E_\infty = \bigcup_{n\in\mathbb{N}} E_n$.
For each $x \in E_\infty$ the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy, therefore we can set $f_\infty(x) = \lim_{n\to\infty} f_n(x)$, thus defining a linear isometric embedding $f_\infty : E_\infty \to G$.
The assumption that $f_{2n+1}(E_{2n+1})$ contains all the vectors $v_0, \ldots, v_n$ ensures that $f_\infty(E_\infty)$ is dense in G.
Finally, f_∞ extends to a linear isometry from the completion of E_∞ onto G.

This completes the proof of the Theorem. □

3.2 Uniqueness and almost homogeneity

In this section we are going to show a proof of the following theorem.

Theorem 3.12. *Let X, Y be separable Gurariĭ spaces and $\varepsilon > 0$. Assume $E \subseteq X$ is a finite dimensional space and $f : E \to Y$ is a strict ε-isometry. Then there exists a bijective isometry $h : X \to Y$ such that $\|h|_E - f\| < \varepsilon$.*

By taking E to be the trivial space, we obtain the following corollary.

Theorem 3.13. *The Gurariĭ space is unique up to a bijective isometry.*

A second important easy consequence of theorem 3.12 is the following.

Theorem 3.14. *(Almost homogeneity) The Gurariĭ space is almost homogeneous.*

For the proof of theorem 3.12 we need the following intermediate result.

Lemma 3.15. *Let X be a Gurariĭ space and let $f : E \to F$ be a strict ε-isometry, where E is a finite-dimensional subspace of X and $\varepsilon > 0$. Then for every $\delta > 0$ there exists a δ-isometry $g : F \to X$ such that $\|g \circ f - Id_X\| < \varepsilon$.*

Proof. Choose $0 < \varepsilon' < \varepsilon$ so that f is an ε'-isometry.
Choose $0 < \delta' < \delta$ such that $(1 + \delta')\varepsilon' < \varepsilon$. By Lemma 3.4, there exist a finite dimensional space Z and isometries $i : E \to Z$ and $j : F \to Z$ satisfying $\|j \circ f - i\| \leq \varepsilon'$. Since X is Gurariĭ there exists a δ'-isometry $h : Z \to X$ such that $h \circ i(x) = x$ for $x \in E$. Let $g = h \circ j$. Clearly, g is a δ-isometry. Finally, given $x \in S_E$, we have

$$\|g\circ f(x)-x\| = \|h\circ j\circ f(x)-h\circ i(x)\| \leq (1+\delta')\|j\circ f(x)-i(x)\| \leq (1+\delta')\varepsilon' < \varepsilon,$$

as required. □

Proof. (Theorem 3.12) Fix a decreasing sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers such that

$$\sum_{n\in\mathbb{N}} \varepsilon_n < \infty,$$

$$2\varepsilon_0\varepsilon_1 + \varepsilon_1 + \sum_{n=1}^{\infty}(\varepsilon_n + 2\varepsilon_n\varepsilon_{n+1} + \varepsilon_{n+1}) < \varepsilon - \varepsilon_0. \tag{3.1}$$

and $0 < \varepsilon_0 < \varepsilon$ so that f is a ε_0-isometry. We define inductively sequences of linear operators $\{f_n\}_{n\in\mathbb{N}}$, $\{g_n\}_{n\in\mathbb{N}}$ and finite-dimensional subspaces $\{X_n\}_{n\in\mathbb{N}}$, $\{Y\}_{n\in\mathbb{N}}$ of X and Y, respectively, so that the following conditions are satisfied:

(0) $X_0 = E$, $Y_0 = f(E)$, and $f_0 = f$;

(1) $f_n : X_n \to Y_n$ is an ε_n-isometry;

(2) $g_n : Y_n \to X_{n+1}$ is an ε_{n+1}-isometry;

(3) $\|g_n f_n(x) - x\| \leq \varepsilon_n \|x\|$ for $x \in X_n$;

(4) $\|f_{n+1} \circ g_n(y) - y\| \leq \varepsilon_{n+1}\|y\|$ for $y \in Y_n$;

(5) $X_n \subseteq X_{n+1}$, $Y_n \subseteq Y_{n+1}$, $\bigcup_{n\in\mathbb{N}} X_n$ and $\bigcup_{n\in\mathbb{N}} Y_n$ are dense in X and Y, respectively.

Condition (0) allows us how to start the inductive construction.

Suppose f_i, X_i, Y_i, for $i \leq n$, and g_i, for $i < n$, have been constructed. We easily find g_n, X_{n+1}, f_{n+1} and Y_{n+1}, in this order, using Lemma 3.15. Condition (5) can be realized by defining X_{n+1} and Y_{n+1} to be suitably enlarged $g_n(Y_n)$ and $f_{n+1}(X_{n+1})$, respectively. Thus, the construction can be carried out.

Fix $n \in \mathbb{N}$ and $x \in X_n$ with $\|x\| = 1$. Using (4) and (1), we get

$$\|f_{n+1} \circ g_n \circ f_n(x) - f_n(x)\| \leq \varepsilon_{n+1}\|f_n(x)\| \leq \varepsilon_{n+1}(1 + \varepsilon_n).$$

Using (1) and (3), we get

$$\|f_{n+1} \circ g_n \circ f_n(x) - f_{n+1}(x)\| \leq \|f_{n+1}\| \cdot \|g_n f_n(x) - x\| \leq (1 + \varepsilon_{n+1}) \cdot \varepsilon_n.$$

These inequalities give

$$\|f_n(x) - f_{n+1}(x)\| \leq \varepsilon_n + 2\varepsilon_n\varepsilon_{n+1} + \varepsilon_{n+1}. \tag{3.2}$$

Now, because of the choice of $\{\varepsilon_n\}_{n\in\mathbb{N}}$, the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy. Given $x \in \bigcup_{n\in\mathbb{N}} X_n$, define $h(x) = \lim_{n\to\infty} f_n(x)$, where $f_n(x)$ is defined for $n \geq m$ where m is such that $x \in X_m$. Then h is an ε_n-isometry for every $n \in \mathbb{N}$, hence it is an isometry.
Consequently, it uniquely extends to an isometry on X, that we denote also by h. Furthermore, (3.2) and (3.1) give

$$\|f(x) - h(x)\| \leq \sum_{n=0}^{\infty} \varepsilon_n + 2\varepsilon_n\varepsilon_{n+1} + \varepsilon_{n+1} < \varepsilon.$$

It remains to see that h is a bijection.
To this end, we check as before that $\{g_n(y)\}n \geq m$ is a Cauchy sequence for every $y \in Y_m$. Once this is done, we obtain an isometry g_∞ defined on F. Conditions (3) and (4) tell us that $g_\infty \circ h = Id_X$ and $h \circ g_\infty = Id_F$, and the proof is complete. $\square$

Chapter 4

A general construction of almost homogeneous spaces

Let $\mathcal{B}$ be the class of all the finite-dimensional real normed spaces. As we saw in the last chapter, we can construct the Gurariĭ space as a kind of limit of a particular sequence of finite-dimensional normed spaces that is in some sense dense in $\mathcal{B}$.

In this chapter we are going to formulate an algorithm that can be applied to a subclass of $\mathcal{B}$, provided that it has some analytic property that we will show, in order to construct different almost homogenous separable Banach spaces.
In fact it is a generalization of the second construction of the Gurariĭ space in the previous chapter.
We will follow the construction made by Kubiś in [4]: the approach used in that paper is based on categorical point of view, but we will never use categorical arguments in this chapter, even if it is easy to find some connection to this branch of Mathematics.

In what follows all the spaces and maps are intended up to surjective isometries.

4.1 The required properties

Let $\mathcal{K}$ be a subclass of $\mathcal{B}$ with $\emptyset \in \mathcal{K}$.

We say that that

- $\mathcal{K}$ is hereditary if for every $X \subseteq Y$ with $Y \in \mathcal{K}$ we have $X \in \mathcal{K}$,
- $\mathcal{K}$ is closed if for every $n \in \mathbb{N}$ the set $\mathcal{K} \cap \{n-$dimensional normed spaces$\}$ is closed under the Banach-Mazur distance.

Definition 4.1. *$\mathcal{K}$ has the* **`small distortion property`** *if for every $X, Y \in \mathcal{K}$ and for every $\varepsilon-$isometry $f : X \to Y$ there is $W \in \mathcal{K}$ and and there are isometries $i : X \hookrightarrow W$, $j : Y \hookrightarrow W$ such that $\|j \circ f - i\| \le \varepsilon$.*

Definition 4.2. *$\mathcal{K}$ has the* **`amalgamation property`** *if for any $Z, X, Y \in \mathcal{K}$ and isometries $i : Z \hookrightarrow X$, $j : Z \hookrightarrow Y$ there exists $W \in \mathcal{K}$ and $J : X \hookrightarrow W$, $I : Y \hookrightarrow W$ such that $I \circ j = J \circ i$, i.e. the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "J"] & W \\ Z \arrow[u, "i"] \arrow[r, "j"'] & Y \arrow[u, "I"'] \end{tikzcd}$$

In the previous chapter it was shown that the class $\mathcal{B}$ has these properties (see the Pushout Lemma 3.3 and 3.4).

It turns out that the amalgamation property can be moved to a bigger class of linear maps, namely:

Proposition 4.3. *Let $\mathcal{K}$ enjoy the amalgamation property and the small distortion property, then for every $Z, X, Y \in \mathcal{K}$, for every $\varepsilon > 0, \delta > 0$ and $f : Z \hookrightarrow X$ $\varepsilon-$isometry, $g : Z \hookrightarrow Y$ $\delta-$isometry there exist $W \in \mathcal{K}$ and isomteries $G : X \hookrightarrow W$, $F : Y \hookrightarrow W$ such that $\|F \circ g - G \circ f\| \le \varepsilon + \delta$, i.e. the following diagram is $(\varepsilon + \delta)-$commutative.*

$$\begin{tikzcd} X \arrow[r, "G"] & W \\ Z \arrow[u, "f"] \arrow[r, "g"'] & Y \arrow[u, "F"'] \end{tikzcd}$$

Proof. Since $\mathcal{K}$ has the small distortion property let $A, B \in \mathcal{K}$ and $i : Z \hookrightarrow A$, $j : X \hookrightarrow A$, $k : Z \hookrightarrow B$, $l : Y \hookrightarrow B$ isometries such that $\|j \circ f - i\| \leq \varepsilon$ and $\|l \circ g - k\| \leq \delta$.
Now consider

$$\begin{tikzcd} A & \\ Z \arrow[u, "i"] \arrow[r, "k"'] & B \end{tikzcd}$$

Using the amalgamation property, find $W \in \mathcal{K}$ and $j' : A \hookrightarrow W$, $l' : B \hookrightarrow W$ such that $l' \circ k = j' \circ i$.
Define $F := l' \circ l$ and $G := j' \circ j$, then $\|F \circ g - G \circ f\| \leq \|j' \circ j \circ f - j' \circ i\| + \|j' \circ i - l' \circ k\| + \|l' \circ k - l' \circ l \circ g\| \leq \varepsilon + \delta$. □

Definition 4.4. *(Directness) $\mathcal{K}$ is* `direct` *if for every $X, Y \in \mathcal{K}$ there exist $W \in \mathcal{K}$ and isometries $i : X \hookrightarrow W$, $j : Y \hookrightarrow W$.*

Note that if $\emptyset \in \mathcal{K}$ and $\mathcal{K}$ has the amalgamation property, then it is direct. In fact we can apply the amalgamation property to the following diagram:

$$\begin{tikzcd} X & \\ \emptyset \arrow[u] \arrow[r] & Y. \end{tikzcd}$$

This is the reason why we will always assume that $\emptyset \in \mathcal{K}$.

4.2 Fraïssé sequences and almost homogeneous spaces

Definition 4.5. *A* `sequence` *in $\mathcal{K}$ is a chain $\vec{U} = \{\vec{U}(n)\}_{n\in\mathbb{N}} \subseteq \mathcal{K}$ with a set of isometries $\{\vec{U}_n^m : \vec{U}(n) \hookrightarrow \vec{U}(m); n \leq m; n, m \in \mathbb{N}\}$, such that if $n_1 \leq n_2 \leq n_3 \in \mathbb{N}$ then $\vec{U}_{n_1}^{n_3} = \vec{U}_{n_2}^{n_3} \circ \vec{U}_{n_1}^{n_2}$.*

Since all the elements that we are considering are defined up to surjective isometries, without loss of generality we can suppose that, if $n < m$, $n, m \in \mathbb{N}$, then $\vec{U}(n) \subseteq \vec{U}(m)$ and $\vec{U}_n^m = Id_{\vec{U}(n)}$, where $Id_{\vec{U}(n)}$ is the identity on $\vec{U}(n)$.

Obviously, for every sequence $\vec{U}$ in $\mathcal{K}$ we can define a unique (up to isometries) separable Banach space U as the completion of the limit of $\vec{U}$, $U := \widehat{\bigcup_{n=1}^{\infty} \vec{U}(n)}$.
For every $n \in \mathbb{N}$ the map $\vec{U}_n^{\infty} := \lim_{m\to\infty} \vec{U}_n^m$ is the inclusion map defined on $\vec{U}(n)$ into U.

Let $\vec{U}, \vec{V}$ be two sequences and let $U = \widehat{\bigcup_{n=1}^{\infty} \vec{U}(n)}$ and $V = \widehat{\bigcup_{n=1}^{\infty} \vec{V}(n)}$. Now consider $\vec{t} = \{t_n\}_{n=1}^{\infty}$ a sequence of linear maps, $t_n : \vec{U}(n) \to \vec{V}(\varphi(n))$ with $\varphi : \mathbb{N} \to \mathbb{N}$ an increasing map, such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that, whenever $n_0 \leq n < m$, all diagrams of the form

$$\begin{tikzcd} \vec{V}(\varphi(n)) \arrow[r, "\vec{V}_{\varphi(n)}^{\varphi(m)}"] & \vec{V}(\varphi(m)) \\ \vec{U}(n) \arrow[u, "t_n"] \arrow[r, "\vec{U}_n^m"'] & \vec{U}(m) \arrow[u, "t_m"'] \end{tikzcd} \tag{4.1}$$

are $\varepsilon-$commutative, i.e. $\|t_m \circ (\vec{U})_n^m - \vec{V}_{\varphi(n)}^{\varphi(m)} \circ t_n\| \leq \varepsilon$.
Then we can define a linear map $T : U \to V$ as the extension of $\lim_{n\to\infty} t_n(x)$ defined on $\bigcup_{n=1}^{\infty} \vec{U}(n)$. In fact for every $x \in \bigcup_{n=1}^{\infty} \vec{U}(n)$ we can consider the sequence $\{t_n(x)\}_{n=\bar{n}}^{\infty}$ for some $\bar{n} \in \mathbb{N}$; this sequence is Cauchy since the diagrams 4.1 are definitively $\varepsilon-$commutative, hence the limit of $\{t_n(x)\}$ exists in V.
Moreover if $\{\varepsilon_n\}_{n\in\mathbb{N}}$ is a positive decreasing sequence, $\varepsilon_n \searrow 0$, and t_n are ε_n-isometries, then T is an isometry.

Definition 4.6. *A sequence $\vec{U}$ of $\mathcal{K}$ is* **Fraïssé** *in $\mathcal{K}$ if*

(U) for every $X \in \mathcal{K}$ and for every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and an $\varepsilon-$isometry $f : X \to \vec{U}(n)$;

(A) for every $\varepsilon > 0$ and every isometry $f : \vec{U}(n) \hookrightarrow X$, with $X \in \mathcal{K}$, there exist $m > n$ and $g : X \to \vec{U}(m)$ $\varepsilon-$isometry such that $\|g \circ f - \vec{U}_n^m\| \leq \varepsilon$.

Now we are going to show that, if $\vec{U}$ is a Fraïssé sequence in $\mathcal{K}$, then $U = \widehat{\bigcup_{n=1}^{\infty} \vec{U}(n)}$ is almost homogeneous. Moreover U is universal for $\mathcal{K}$, $\mathcal{K} = \{X \subset U, X \text{finite-dimensional subspace}\}$ and the sequence $\vec{U}$ is unique in $\mathcal{K}$, that means that if $\vec{V}$ is another Fraïssé sequence in $\mathcal{K}$, then $\widehat{\bigcup_{n=1}^{\infty} \vec{V}(n)}$ is isometric to U.

First of all we have to prove some intermediate results.

Proposition 4.7. *Let $\mathcal{K}$ enjoy the amalgamation property and let $\vec{U}$ be a sequence in $\mathcal{K}$. The following conditions are equivalent.*

(i) $\vec{U}$ is Fraïssé in $\mathcal{K}$,

(ii) $\vec{U}$ has a cofinal subsequence that is Fraïssé in $\mathcal{K}$,

(iii) Every cofinal subsequence of $\vec{U}$ is Fraïssé in $\mathcal{K}$.

Proof. Implications $(i) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$ are obvious, so only $(iii) \Rightarrow (i)$ remains.
Now consider $M \subset \mathbb{N}$ cofinal such that $\{\vec{U}(n)\}_{n\in M}$ is Fraïssé in $\mathcal{K}$. We want to show that $\vec{U}$ is Fraïssé in $\mathcal{K}$, in particular we have to prove that condition (A) in definition 4.6 holds. Fix $n \in \mathbb{N} \setminus M$, fix an isometry $f : \vec{U}(n) \to Y$ and $\varepsilon > 0$. Let $m \in M$, $m > n$: using the amalgamation property we can find isometries $F : \vec{U}(m) \hookrightarrow W$ and $j : Y \hookrightarrow W$ such that $j \circ f = F \circ \vec{U}^m_n$. Since $\{\vec{U}(n)\}_{n\in M}$ is Fraïssé in $\mathcal{K}$, there are $l > m$, $l \in M$ and an $\varepsilon-$isometry $g : W \to \vec{U}(l)$ such that $\|g \circ F - \vec{U}^l_m\| \le \varepsilon$. Finally $g \circ j$ is an $\varepsilon-$isometry and $\|g \circ j \circ f - \vec{U}^l_n\| \le \varepsilon$. □

Proposition 4.8. *Let $\mathcal{K}$ be a subclass of $\mathcal{B}$ enjoying the small distortion property and let $\vec{U}$ a sequence in $\mathcal{K}$ satisfying (U). Then $\vec{U}$ is Fraïssé in $\mathcal{K}$ if and only if it satisfies the following condition:*

(B) given $\eta, \delta > 0$, given $n \in \mathbb{N}$ and a $\delta-$isometry $f : \vec{U}(n) \to Y$ with $Y \in \mathcal{K}$, there exist $m > n$ and an $\eta-$isometry $g : Y \to \vec{U}(m)$ such that $\|g \circ f - \vec{U}^m_n\| \le \eta + \delta$.

Proof. It is obvious that (B)⇒(A).
Suppose that $\vec{U}$ is Fraïssé. Because of the small distortion property there are isometries $i : \vec{U}(n) \hookrightarrow W$ and $j : Y \hookrightarrow W$ such that $\|j \circ f - i\| \le \delta$. Let $0 < \tilde{\eta} \le \eta/(1+\delta)$. Using (A), find $m > n$ and an $\tilde{\eta}-$isometry $k : Y \to \vec{U}(m)$ such that $\|k\circ i - \vec{U}^m_n\| \le \tilde{\eta}$. $g := k \circ j$ is an $\tilde{\eta}-$isometry, so it is an $\eta-$isometry, and $\|g \circ f - \vec{U}^m_n\| \le \|k \circ j \circ f - k \circ i\| + \|k \circ i - \vec{U}^m_n\| \le (1+\tilde{\eta})\delta + \tilde{\eta} \le \eta + \delta$. □

Proposition 4.9. *Let $\mathcal{K}$ be a subclass of $\mathcal{B}$ with the small distortion property and let $\vec{U}$ and $\vec{V}$ be Fraïssé sequences in $\mathcal{K}$. Furthermore, let $\varepsilon > 0$ and let $h : \vec{U}(0) \to \vec{V}(0)$ be a strict $\varepsilon-$isometry. Then there exists a surjective isometry*

$$F : \widehat{\bigcup_{n=1}^{\infty} \vec{U}(n)} \to \widehat{\bigcup_{n=1}^{\infty} \vec{V}(n)}$$

such that $\|F|_{\vec{U}(0)} - h\| \leq \varepsilon$.

Proof. Let $0 < \delta < \varepsilon$ such that h is a $\delta-$isometry. Fix a decreasing sequence of positive reals $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that

$$\delta < \varepsilon_0 < \varepsilon \text{ and } 2\sum_{n=1}^{\infty} \varepsilon_n \leq \varepsilon - \varepsilon_0.$$

We define inductively sequences of linear maps $f_n : \vec{U}(\varphi(n)) \to \vec{V}(\psi(n))$, $g_n : \vec{V}(\psi(n)) \to \vec{U}(\varphi(n+1))$ such that

(1) $\varphi(n) \leq \psi(n) < \varphi(n+1)$,

(2) $\|g_n \circ f_n - \vec{U}_{\varphi(n)}^{\varphi(n+1)}\| \leq \varepsilon_n$,

(3) $\|f_n \circ g_{n-1} - \vec{V}_{\psi(n-1)}^{\psi(n)}\| \leq \varepsilon_n$,

(4) f_n is an ε_n-isometry, g_n is an $\varepsilon_{n+1}-$isometry and $\|f_n\|, \|g_n\| \leq 1$.

We start by setting $\varphi(0) = \psi(0) = 0$ and $f_0 = h$. We find g_0 and $\varphi(1)$ by using condition *(B)* of Proposition 4.8 with an appropriate value of $\eta > 0$ (if necessary, we can normalize g_0 in order to obtain $\|g_0\| \leq 1$).
We continue repeatedly using condition *(B)* for both sequences . More precisely, having defined f_{n-1} and g_{n-1}, we first use property *(B)* of the sequence $\vec{V}$, constructing f_n satisfying (3) and (4); next we use the fact that $\vec{U}$ satisfies *(B)* in order to find g_n satisfying (2) and (4). Now we check that for every $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that, whenever $n_0 \leq n < m$, all diagrams of the form

$$\begin{tikzcd}
\vec{V}(\psi(n)) \arrow[r, "\vec{V}_{\psi(n)}^{\psi(m)}"] & \vec{V}(\psi(m)) \\
\vec{U}(\varphi(n)) \arrow[u, "f_n"] \arrow[r, "\vec{U}_{\varphi(n)}^{\varphi(m)}"'] & \vec{U}(\varphi(m)) \arrow[u, "f_m"']
\end{tikzcd}$$

and

$$\begin{tikzcd} \vec{U}(\varphi(n)) \arrow[r, "\vec{U}^{\varphi(m)}_{\varphi(n)}"] & \vec{U}(\varphi(m)) \\ \vec{V}(\psi(n)) \arrow[r, "\vec{V}^{\psi(m)}_{\psi(n)}"'] \arrow[u, "g_n"] & \vec{V}(\psi(m)) \arrow[u, "g_m"'] \end{tikzcd}$$

are $\varepsilon-$commutative. Fix $n \in \mathbb{N}$ and observe that

$$\|\vec{V}^{\psi(n+1)}_{\psi(n)} \circ f_n - f_{n+1} \circ (\vec{U})^{\varphi(n+1)}_{\varphi(n)}\|$$

$$\le \|\vec{V}^{\psi(n+1)}_{\psi(n)} \circ f_n - f_{n+1} \circ g_n \circ f_n\| + \|f_{n+1} \circ g_n \circ f_n - f_{n+1} \circ (\vec{U})^{\varphi(n+1)}_{\varphi(n)}\|$$

$$\le \|\vec{V}^{\psi(n+1)}_{\psi(n)} - f_{n+1} \circ g_n \circ \| + \|g_n \circ f_n - (\vec{U})^{\varphi(n+1)}_{\varphi(n)}\|$$

$$\le \varepsilon_{n+1} + \varepsilon_n.$$

Since $\sum_{n\in\mathbb{N}} \varepsilon_n$ is convergent, for every $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ big enough in order to make the first diagram $\varepsilon-$commutative for every $n, m \ge n_0$. By symmetry we deduce the same for the second diagram.
Let F and G the limits of $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ respectively. Then conditions (2) and (3) force the compositions $F \circ G$ and $G \circ F$ to be equivalent to the identities, while condition (4) guarantees that F and G are isometries.
Finally, recalling that $h = f_0$, we obtain

$$\|F|_{\vec{U}(0)} - h\| = \|F \circ \vec{U}_0^{\infty} - \vec{V}_0^{\infty} \circ h\| = \lim_{n\to\infty} \|f_n \circ \vec{U}_0^{\varphi(n)} - \vec{V}_0^{\psi(n)} \circ f_0\|$$

$$\le \sum_{n=1}^{\infty} \|f_{n+1} \circ \vec{U}^{\varphi(n+1)}_{\varphi(n)} - \vec{V}^{\psi(n+1)}_{\psi(n)} \circ f_n\| \le \sum_{n=1}^{\infty} (\varepsilon_n + \varepsilon_{n+1})$$

$$= \varepsilon_0 + 2\sum_{n=1}^{\infty} \varepsilon_n \le \varepsilon.$$

□

Theorem 4.10. *(Uniqueness) Let $\mathcal{K}$ be a subclass of $\mathcal{B}$ with the small distortion property and let $\vec{U}$ and $\vec{V}$ be Fraïssé sequences in $\mathcal{K}$. Then $\widehat{\bigcup_{n=1}^{\infty} \vec{U}(n)}$ and $\widehat{\bigcup_{n=1}^{\infty} \vec{V}(n)}$ are isometric.*

Proof. Consider $\vec{U}(0)$ and let $\varepsilon > 0$. Using (U) applied to $\vec{V}$, for some $\bar{n} \in \mathbb{N}$ we can find an $\varepsilon-$isometry $h : \vec{U}(0) \to \vec{V}(\bar{n})$. Since $\{\vec{V}(n)\}_{n\ge\bar{n}}$ still is Fraïssé, Proposition 4.9 gives the required isometry. □

Theorem 4.11. *(Almost homogeneity) Let $\mathcal{K}$ as usual. Suppose that $\mathcal{K}$ has the amalgamation property and the small distortion property and contains a Fraïssé sequence $\vec{U}$, and let $U = \widehat{\bigcup_{n=1}^{\infty} \vec{U}(n)}$. Then for every $X \subset U$, for every isometry $f : X \hookrightarrow U$ and for every $\varepsilon > 0$ there exists a surjective isometry $F : U \to U$ such that $\|F|_X - f\| \leq \varepsilon$.*

Proof. Fix $\varepsilon > 0$ and let $\delta > 0$ such that $6\delta + \delta^2 \leq \varepsilon$.
Considering X and $f(X)$ in U, we can find $n, m \in \mathbb{N}$ big enough and $i : X \to \vec{U}(n)$, $j : f(X) \to \vec{U}(m)$ $\delta-$isometries such that $\|x - i(x)\| \leq \delta$ and $\|f(x) - j \circ f(x)\| \leq \delta$ for every $x \in X$ $\|x\| = 1$.
Define $f_1 := j \circ f$: f_1 is a $\delta-$isometry. Using proposition 4.3 we find two isometries $f_2 : \vec{U}(n) \hookrightarrow W$ and $g_1 : \vec{U}(m) \hookrightarrow W$ such that $\|f_2 \circ i - g_1 \circ f_1\| \leq 2\delta$.
Using the fact that $\vec{U}$ is Fraïssé, we find $l > m$ and $g_2 : W \to \vec{U}(l)$ $\delta-$isometry such that $\|g_2 \circ g_1 - \vec{U}_m^l\| \leq \delta$.
Define $g := g_2 \circ f_2$. Then g is a $\delta-$isometry and the sequences $\{\vec{U}(j)\}_{j \geq n}$ $\{\vec{U}(j)\}_{j \geq l}$ are Fraïssé. Therefore by Proposition 4.9 there exists $F : U \to U$ such that $\|F|_{\vec{U}(n)} - g\| \leq \delta$.

$$\begin{tikzcd}
X \arrow[r, "i"'] \arrow[dd, "f"'] \arrow[ddr, "f_1"] & \vec{U}(n) \arrow[rrr] \arrow[dr, "f_2"] & & & \ldots \arrow[r] & U \arrow[dd, "F"] \\
& & W \arrow[dr, "g_2"] & & & \\
f(X) \arrow[r, "j"'] & \vec{U}(m) \arrow[rr] \arrow[ur, "g_1"] & & \vec{U}(l) \arrow[r] & \ldots \arrow[r] & U
\end{tikzcd}$$

Applying the properties of the diagram and of the maps that have been defined we obtain

$$\begin{aligned}
\|F|_X - f\| &\leq \|F|_X - F \circ i\| + \|F \circ i - f\| \\
&\leq \delta + \|F \circ i - \vec{U}_m^l \circ j \circ f\| + \|\vec{U}_m^l \circ j \circ f - f\| \\
&\leq 2\delta + \|F \circ i - g \circ i\| + \|g \circ i - \vec{U}_m^l \circ j \circ f\| \\
&\leq 2\delta + (1 + \delta)\delta + 3\delta = 6\delta + \delta^2 \leq \varepsilon.
\end{aligned}$$

This completes the proof. □

Now we want to show that the almost homogeneous space we have constructed is in some sense universal for the class $\mathcal{K}$.

Theorem 4.12. *(Universality) Let $\mathcal{K}$ as usual. Suppose that $\mathcal{K}$ has the amalgamation property and the small distortion property and contains a Fraïssé sequence $\vec{U}$, and let $U = \widehat{\bigcup_{n=1}^{\infty} \vec{U}(n)}$. Then for every sequence $\vec{X}$ of $\mathcal{K}$ there exists an isometry $F : \widehat{\bigcup_{n=1}^{\infty} \vec{X}(n)} \to U$.*

Proof. We construct a strictly increasing sequence $\{\varphi(n)\}$ of natural numbers and a sequence of linear maps $f_n : \vec{X}(n) \to \vec{U}(\varphi(n))$ such that, for each $n \in \mathbb{N}$, f_n is a 2^{-n}-isometry and $\|\vec{U}_{\varphi(n)}^{\varphi(n+1)} \circ f_n - f_{n+1} \circ \vec{X}_n^{n+1}\| \leq 3 \cdot 2^{-n}$. Then $F := \lim_{n\to\infty} f_n$ is the desired isometry.
We start by finding f_0 and $\varphi(0)$ using condition *(U)* of Fraïssé sequences.
Fix $n \in \mathbb{N}$ and suppose f_n and $\varphi(n)$ have been already defined. Since f_n is a 2^{-n}-isometry, there exist two isometries $i : \vec{X}(n) \to V$ and $j : \vec{U}(\varphi(n)) \to V$ such that $\|j \circ f_n - i\| \leq 2^{-n}$.
Next using amalgamation property, we find two isometries $k : V \to W$ and $l : \vec{X}(n+1) \to W$ such that $k \circ i = l \circ \vec{X}_n^{n+1}$.
Finally, using the fact that $\vec{U}$ is Fraïssé, find $\varphi(n+1) > \varphi(n)$ and a $2^{-(n+1)}$-isometry $g : W \to \vec{U}(\varphi(n+1))$ such that$\|g \circ k \circ j - \vec{U}_{\varphi(n)}^{\varphi(n+1)}\| \leq 2^{-n}$.

$$\begin{tikzcd}
\vec{U}(\varphi(n)) \arrow[rr] \arrow[dr, "j"] & & \vec{U}(\varphi(n+1)) \arrow[r] & \dots \\
& V \arrow[r, "k"] & W \arrow[u, "g"] & \\
\vec{X}(n) \arrow[uu, "f_n"] \arrow[ur, "i"] \arrow[rr] & & \vec{X}(n+1) \arrow[u, "l"] \arrow[r] & \dots
\end{tikzcd}$$

Define $f_{n+1} := g \circ l$: it is a $2^{-(n+1)}-$isometry and the sequence $\{f_n\}_{n\in\mathbb{N}}$ satisfies the conditions at the beginning of the proof.

□

As a corollary we obtain the following result.

Corollary 4.13. *Let $\mathcal{K}$ a hereditary and closed subclass of $\mathcal{B}$. Suppose that $\mathcal{K}$ has the amalgamation property and the small distortion property and contains a Fraïssé sequence $\vec{U}$, and let $U = \widehat{\bigcup_{n=1}^{\infty} \vec{U}(n)}$. Then $\mathcal{K} = \{X \subset U, X$ finite-dimensional subspace$\}$.*

Proof. Let $X \in \mathcal{K}$ and consider the sequence $\vec{X}$ with $\vec{X}(n) = X$ and $\vec{X}_n^m = Id_X$ for every $n, m \in \mathbb{N}$. Because of the last theorem there exists an isometry

$F : X \to U$, that means that $X \subset U$ up to a biijective isometry.
Now let $X \subset U$ a finite-dimensional subspace. Then for every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and an $\varepsilon-$isometry $i : X \to \vec{U}(n)$. From the hereditarity of $\mathcal{K}$ we have that $i(X) \in \mathcal{K}$ and is $\varepsilon-$close to X. Then $X \in \mathcal{K}$, since $\mathcal{K}$ is closed. □

4.3 The construction of a Fraïssé sequence

In order to find a Fraïssé sequence in $\mathcal{K}$ we have to require that $\mathcal{K}$ contains a countable subclass that is dense in some sense. We are going to specify what we mean.

Definition 4.14. *A subclass $\mathcal{F}$ of $\mathcal{K}$ with a set of linear isometries $A = \{f : A \to B, A, B \in \mathcal{F}\}$ is **dominating** in $\mathcal{K}$ if*

(D1) for every $X \in \mathcal{K}$ and for every $\varepsilon > 0$ there exist $Y \in \mathcal{F}$ and an $\varepsilon-$isometry $f : X \to Y$;

(D2) for every $\varepsilon > 0$ and every isometry $f : Y \hookrightarrow X$, with $Y \in \mathcal{F}$ and $X \in \mathcal{K}$, there exist $W \in \mathcal{F}$, $g : X \to W$ $\varepsilon-$isometry and $u : Y \hookrightarrow W$ in $\mathcal{A}$ such that $\|g \circ f - u\| \leq \varepsilon$;

(D3) for every $X \in \mathcal{F}$, the identity map Id_Y is in $\mathcal{A}$.

Note that if we consider $\mathcal{B}$ as a metric space under the Banach-Mazur distance, then it is separable. Hence every subset of $\mathcal{B}$ is separable, in particular this means that every $\mathcal{K}$ has a countable subclass $\mathcal{F}$ that satisfies (D1), so the nontrivial part of the last definition is condition (D2).
We say that $\mathcal{K}$ has a countable dominating subclass if there exists $\mathcal{F} \subseteq \mathcal{K}$ such that the subset $\mathcal{A}$ of the isometries in $\mathcal{F}$ is dominating in $\mathcal{K}$ and such that both $\mathcal{F}$ and $\mathcal{A}$ are countable.
Observe that a Fraïssé sequence is a countable dominating subclass.

The following result shows that if $\mathcal{K}$ has the amalgamation property and contains a countable dominating subclass, then it contains a Fraïssé sequence.

Theorem 4.15. *Let $\mathcal{K}$ be a subclass of $\mathcal{B}$ with the amalgamation property. The following are equivalent:*

(i) $\mathcal{K}$ contains a countable dominating subclass,

(ii) $\mathcal{K}$ contains a Fraïssé sequence.

Proof. (ii)$\Rightarrow$(i) is obvious.
Let $\mathcal{K}$ with a countable dominating subclass $\mathcal{F}$; $\mathcal{A}$ will denote the set of its isometries. We are going to construct a Fraïssé sequence in $\mathcal{K}$.
Let $\mathbb{P}$ the set of all finite sequences $\vec{X}$ in $\mathcal{F}$, $\vec{X} = \{\vec{X}(n)\}_{n=1}^{\mathrm{dom}(\vec{X})}$, $\mathrm{dom}(\vec{X}) < \infty$ and $\vec{X}_n^m \in \mathcal{A}$ for every $n, m \leq \mathrm{dom}(\vec{X})$. Define on $\mathbb{P}$ the following partial order $\vec{X} \leq \vec{Y}$ if $\mathrm{dom}(\vec{X}) \leq \mathrm{dom}(\vec{Y})$ and $\{\vec{Y}(n)\}_{n=1}^{\mathrm{dom}(\vec{X})} = \vec{X}$.

Now fix $f : A \hookrightarrow B$ in $\mathcal{A}$ and $n, k \in \mathbb{N}$. Define

$$D_{f,n,k} := \{\vec{X} \in \mathbb{P} : \mathrm{dom}(\vec{X}) > n,$$

$$\text{(i) } \exists l < \mathrm{dom}(\vec{X}) \text{ s.t. } \exists\, f : A \to \vec{X}(l) \text{ is a } \frac{1}{k} - \text{isometry}$$

$$\text{(ii) if } A = \vec{X}(n), \text{ then } \exists m > n \text{ and } g : B \to \vec{X}(m) \text{ is a } \frac{1}{k} - \text{isometry}$$

$$\text{s.t. } \|g \circ f - \vec{X}_n^m\| \leq \frac{1}{k}\}.$$

Note that there are countably many $D_{f,n,k}$.
From the amalgamation property of $\mathcal{K}$ (hence its directness) and the properties of the dominating class $\mathcal{F}$, it follows that each $D_{f,n,k}$ is cofinal.
Then we can apply the Sikorski Lemma 3.9 to obtain an increasing sequence $\{\vec{U}_r\}_{r\in\mathbb{N}} \subset \mathbb{P}$ such that, for every $f \in \mathcal{A}$ and $n, k \in \mathbb{N}$, there exists $r \in \mathbb{N}$ such that $\vec{U}_r \in D_{f,n,k}$.
Since $\{\vec{U}_r\}$ is increasing we can define without misunderstanding the following sequence:

$$\vec{U} := \bigcup_{r\in\mathbb{N}} \vec{U}_r.$$

It is easy to see that it is Fraïssé in $\mathcal{K}$. □

Finally if $\mathcal{K}$ is a subclass of $\mathcal{K}$ with the amalgamation property, the small distorsion property and admitting a countable dominating subclass, then it is possible to construct a unique (up to surjective isometry) almost homogeneous separable Banach space U; moreover, if $\mathcal{K}$ is closed and hereditary, then $\mathcal{K}$ agrees with the set of all finite-dimensional subspaces of U.

Remark 4.16

We want to point out that for the algorithm it is enough to assume that the class $\mathcal{K}$ is direct and has the almost amalgamation property instead of the amalgamation property. Namely,

Definition 4.17. *$\mathcal{K}$ has the* ***almost amalgamation property*** *if for every $X, Y, Z \in \mathcal{K}$, for every $f : Z \to X$, $g : Z \to Y$ with $\|f\|, \|g\| \leq 1$ and for every $\varepsilon > 0$ there exist $W \in \mathcal{K}$, $F : X \to W$, $G : Y \to W$ with $\|G\|, \|F\| \leq 1$ and such that $\|F \circ f - G \circ g\| \leq \varepsilon$.*

As we saw before, the amalgamation property implies the directness and (obviously) the almost amalgamation property.

On the other hand we have no examples of a class with the almost amalgamation property that would not have the amalgamation property. So we don't know if the request of the almost amalgamation property and directness instead of the amalgamation property is really advantageous.

Chapter 5

Looking for a new way for amalgamation of subspaces

As we saw in the last chapter, if we want to generate a new separable almost homogeneous Banach space it is enough to find a subclass of $\mathcal{B}$ that enjoys some properties.
The main property we have focused on during the PhD program, is the amalgamation property.
Apparently it seems not difficult to find a class of finite-dimensional normed spaces, not dense in $\mathcal{B}$ (otherwise, from Corollary 4.13, we obtain the Gurariĭ space) and different from the class of all finite-dimensional Hilbert spaces, for which the amalgamation property holds.
In fact it is possible to investigate this problem in two different ways:

- Using the amalgamation defined in the Pushout Lemma 3.3, namely finding a subclass $\mathcal{K}$ of $\mathcal{B}$ such that, if $X, Y, Z \in \mathcal{K}$ and $Z \subseteq X$, $Z \subseteq Y$, then the space W constructed with the Pushout Lemma still is in $\mathcal{K}$.

- Finding a new way to amalgamate finite-dimensional normed spaces such that some properties are preserved and defining $\mathcal{K}$ as the class of all the finite-dimensional normed spaces with those properties.

In this chapter we show that the first way is not possible and this is our contributions to the development of the theory.
In fact we prove that the minimal, hereditary and closed class $\mathcal{K}$ of finite-dimensional Banach spaces that can be constructed with the amalgamation

shown in 3.3 is the whole class $\mathcal{B}$.
This result implies that, in order to apply the algorithm of the last chapter to a subclass of $\mathcal{B}$ for the construction of a new almost homogeneous space, it is necessary to find a new way for amalgamating finite-dimensional spaces. We still don't know if there exists such a new way of amalgamating spaces, so the algorithm constructed in Chapter 4 actually can be applied just to the class $\mathcal{B}$.

For simplicity we call the amalgamation made in the Pushout Lemma *pushout's amalgamation*.

In order to show our construction of $\mathcal{B}$ with the pushout's amalgamation we need to recall a result concerning equilateral sets.

Definition 5.1. *Let $C > 0$, let X be a normed space. A subset $E \subseteq X$ is called $C-$* ***equilateral*** *if for every $x, y \in E$, $x \neq y$, we have $\|x - y\| = C$. A set is* ***equilateral****, if it is C-equilateral for some $C > 0$.*

Let $e(X)$ the maximal cardinality an equilateral set in a given normed space X can have. Obviously this value depends on the dimension and the norm of the space X. There is a lot of literature about this parameter as well as about its approximation both in finite and infinite-dimensional spaces.
An important result for X finite-dimensional about upper and lower bounds for $e(X)$, proved by Petty in [11], is the following.

Theorem 5.2. *(Petty) Let X be a normed space with $dim(X) = n \in \mathbb{N}$. Then*

$$\min(4, n+1) \leq e(X) \leq 2^n$$

where the equality $e(X) = 2^n$ holds iff X is isometric to ℓ_∞^n. In this case any equilateral set of size 2^n is the set of extreme points of some ball.

We are going to prove the following result.

Proposition 5.3. *Let $\mathcal{K}$ be the minimal nonempty class of finite-dimensional normed spaces that enjoys the following properties:*

- *$\mathcal{K}$ is hereditiary;*

- *$\mathcal{K}$ is closed under the Banach-Mazur distance;*
- *if $X, Y, Z \in \mathcal{K}$ and $f : Z \hookrightarrow X$, $g : Z \hookrightarrow Y$ are isometries, then* $\dfrac{X \oplus_1 Y}{\{(f(z), -g(z)), z \in Z\}} \in \mathcal{K}$

Then $\mathcal{K} = \mathcal{B}$.

Proof. We want to show that $\mathcal{K}$ is the class of all finite-dimensional normed spaces.
In particular we prove that $\ell_\infty^n \in \mathcal{K}$ for every $n \in \mathbb{N}$. If all these spaces are in $\mathcal{K}$, then $\mathcal{K}$ contains all the finite-dimensional normed spaces since it is closed.

Since $\mathcal{K}$ is nonempty and hereditary, it contains a 1-dimensional space $B = (\mathbb{R}, \|\cdot\|)$.
Then the space $\ell_\infty(2)$ is in $\mathcal{K}$ (take $Z = \emptyset$ and $X = Y = B$ and use the pushout's amalgamation to obtain $X \oplus_1 X$ that is isometric to ℓ_∞^2).

By induction we want to show that, for every $n \in \mathbb{N}$, $\ell_\infty^n \in \mathcal{K}$.

Suppose that $\ell_\infty^{n-1} \in \mathcal{K}$ and define

$$W := \frac{\ell_\infty^{n-1} \oplus_1 \ell_\infty^{n-1}}{\{((\alpha_1, \ldots, \alpha_{n-2}, 0), (-\alpha_1, \ldots, -\alpha_{n-2}, 0)), \alpha_j \in \mathbb{R}\}}.$$

For $x = (a_, \ldots, a_{n-1})$, $y = (b_1, \ldots, b_{n-1}) \in \ell_\infty(n-1)$ let $[(x, y)] \in W$ be the class containing (x, y).
Then

$$[((a_1, \ldots, a_{n-1}), (b_1, \ldots, b_{n-1}))] = [((a_1+b_1, \ldots, a_{n-2}+b_{n-2}, a_{n-1}), (0, \ldots, 0, b_{n-1}))].$$

So W is linearly isomorphic to $\mathbb{R}^n$.
We want to find an equilateral set of 2^n points in W in order to prove that W is isometric to ℓ_∞^n.
Consider the set

$$\{P_i\}_{i=1}^{2^{n-1}} = \{[((\pm 1, \pm 1, \ldots, \pm 1), (0, \ldots, 0))]\} \in W,$$

that is the image by the inclusion of the set of the extreme points of the unit ball of the first ℓ_∞^{n-1} of the direct sum; and consider

$$\{V_i\}_{i=1}^{2^{n-1}} = \{[((\pm 1, \pm 1, \ldots, \pm 1, 0), (0, \ldots, 0, \pm 1))]\} \in W,$$

that is the image by the inclusion of the set of the extreme points of the unit ball of the second ℓ_∞^{n-1} of the direct sum. Both these sets are $2-$equilateral in their original spaces.

CLAIM: The set $\{P_i\} \cup \{V_j\}$ is $2-$equilateral in W.

proof of the CLAIM:
For every $[(x,y)] \in W$ let

$$\|[(x,y)]\|_W = \inf\{\|\tilde{x}\|_\infty + \|\tilde{y}\|_\infty, \tilde{x}, \tilde{y} \in l_\infty^{n-1}, (\tilde{x}, \tilde{y}) \in [(x,y)]\}$$

the norm in W. The following are true.

- $\|V_i - V_j\|_W = 2 = \|P_i - P_j\|_W$ for every $i,j = 1,\dots,2^n$, $i \neq j$, since both the embeddings from ℓ_∞^{n-1} to W are isometries.
- For every $i,j = 1,\dots,2^n$ we have $P_i - V_j = [((a_1,\dots,a_{n-2},\pm 1),(0,\dots,0,\pm 1))]$ with $a_i \in \{\pm 2, 0\}$.
  Moreover $\|V_i - P_j\|_W \leq \|V_i\|_W + \|P_j\|_W = 2$, so it is enough to show that $\|V_i - P_j\| \geq 2$.
  $\|V_i - P_j\|_W = \|[((a_1,\dots,a_{n-2},\pm 1),(0,\dots,0,\pm 1))]\|_W =$
  $= \inf\{\|(a_1+\alpha_1,\dots,a_{n-2}+\alpha_{n-2},\pm 1)\|_\infty + \|(-\alpha_1,\dots,-\alpha_{n-2},\pm 1)\|_\infty, \alpha_j \in \mathbb{R}\}$. But for every $\alpha_j \in \mathbb{R}$, $j = 1,\dots,n-2$, $\|(a_1+\alpha_1,\dots,a_{n-2}+\alpha_{n-2},\pm 1)\|_\infty \geq |\pm 1|$ and $\|(-\alpha_1,\dots,-\alpha_{n-2},\pm 1)\|_\infty \geq |\pm 1|$.
  Hence $\|V_i - P_j\| \geq 2$.

So the set $\{P_i\} \cup \{V_j\}$ is $2-$equilateral and, from Petty's theorem, W is isometric to $\ell_\infty(n)$.

$\square$

Chapter 6

Open problems

In this last part of the thesis we want to summarise the main open problems that we have found in the context of almost homogeneous separable spaces.

- The most important problem is related to the amalgamation property. In particular, as we explained in chapters 5 and 4, the main question is whether there exists a subclass of $\mathcal{B}$, not dense in $\mathcal{B}$ (under the Banach-Mazur distance), different from the class of all finite-dimensional Hilbert spaces, that enjoys the amalgamation property.
  An idea could be to define a subclass $\mathcal{K} \subseteq \mathcal{B}$ such that all the spaces in $\mathcal{K}$ have a fixed geometrical property.
  For example, fix some function $f : [0,2] \to [0,1]$ and let $\mathcal{K}_f$ the subclass of $\mathcal{B}$ whose spaces have modulus of convexity bigger than f. If f is good enough, then $\mathcal{K}_f$ is not dense in $\mathcal{B}$, since it is not possible to approximate polyhedral spaces. Intuitively it seems that such a class has the amalgamation property, but we were not able to find a construction for the amalgamation of these spaces.

  Remember that, once we find a class $\mathcal{K}$ not dense in $\mathcal{B}$ with the amalgamation property, then we have also to require that $\mathcal{K}$ has the Small distortion property (Definition 4.1).

- In order to find (almost) homogeneous separable Banach spaces, one can try to change the definition of almost homogeneity and try to find spaces with weaker/stronger properties.

Definition 6.1. *X is* `strongly almost homogeneous` *(SAO) if for every $A \leq X$ finite-dimensional subspace, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\delta-$isometry $f : A \to X$, there exists $h : X \to X$ surjective isometry such that $\|f - h|_A\| \leq \varepsilon$.*

Definition 6.2. *X is* `quasi almost homogeneous` *(QAO) if for every $A \leq X$ finite-dimensional subspace, for every $\varepsilon > 0$ and for every isometry $f : A \to X$, there exists a surjective $\varepsilon-$isometry $h : X \to X$ such that $f = h|_A$.*

Definition 6.3. *X is* `strongly quasi almost homogeneous` *(SQAO) if for every $A \leq X$ finite-dimensional subspace, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $\delta-$isometry $f : A \to X$, there exists a surjective $\varepsilon-$isometry $h : X \to X$ such that $f = h|_A$.*

We know that: (SAO) $\Rightarrow$ almost homogeneity, (SQAO) $\Rightarrow$ (QAO) and almost homogeneity $\Rightarrow$ (QAO) (see Proposition 2.9).
The first question that arises is whether the converse arrows are true.
We know that for $1 \leq p < \infty$, $p \neq 4, 6, 8, \ldots$, the spaces $L^p[0,1]$ is (QAO), so another question is whether (QAO) implies almost homogeneity in these particular cases.
Another question could be whether there are other spaces that enjoy some of these properties.

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