UNIVERSITÀ DEGLI STUDI DI MILANO
FACOLTÀ DI SCIENZE E TECNOLOGIE

Dipartimento di Matematica F.Enriques
Dottorato di Ricerca in Scienze Matematiche
XXX Ciclo

Lie symmetry analysis and geometrical methods for finite and infinite dimensional stochastic differential equations

Mat/06

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Anno Accademico 2016 - 2017

Abstract

The main aim of the thesis is a systematic application (via suitable generalizations) of Lie symmetry analysis, or more generally, of the various geometric techniques for differential equations, to the study of finite and infinite dimensional stochastic differential equations (SDEs). The work can be divided in three parts.
In the first part a new geometric approach to finite dimensional SDEs driven by a *multidimensional Brownian motion* is proposed. In particular we introduce the notion of symmetry of a given SDE as the invariance property of the set of solutions to the SDE with respect to a large group of transformations (in the following called *stochastic transformations*). This group is composed of a diffeomorphism of the dependent variables, a suitable rotation of the driving Brownian motion plus an overall stochastic time change. After studying both the geometric properties and the probabilistic foundations of these stochastic transformations we extend the classical theorems of reduction and reconstruction by quadratures from the deterministic to the stochastic setting. Moreover, we provide many applications of previous results to some interesting SDEs among which the two dimensional Brownian motion, the Kolmogorov-Pearson equation, a generalized Langevin equation and the SABR model. Finally, using the previous theorems, we propose a symmetry-adapted numerical scheme whose effectiveness is verified through both theoretical estimates and numerical simulations.
The second part proposes an extension of the results obtained in the first part to finite dimensional SDEs driven by a *general semimartingale taking values in a Lie group*. In order to provide such an extension we use the notion of geometrical SDEs introduced by Choen in [37]. This class of SDEs includes affine type SDEs, Marcus type SDEs, smooth SDEs driven by $\mathbb{R}^n$ valued Levy processes and iterated random maps. We introduce the original notions of gauge and time symmetries of a semimartingale in a Lie group in order to replace the rotation and time rescaling invariance of Brownian motion. A general criterion based on the characteristics triplet of a semimartingale for finding processes with gauge and time symmetries is provided. Using these mathematical tools we generalize the notion of stochastic transformations in this setting and we propose the natural definition of symmetry based on this group of transformations. The formulated theory allows us to analyse in detail an important class of SDEs with possible relevant applications to iterated random maps theory.
In the third part we use the geometry of the infinite jets bundle $J^\infty(M,N)$ to develop a convenient algorithm for the explicit determination of *finite dimensional solutions to stochastic partial differential equations* (SPDEs). In particular, using the notion of semimartingale smoothly depending on a spatial parameter introduced by Kunita in [116], we see an SPDE as an ordinary SDE on the infinite dimensional Frechét space $J^\infty(M,N)$, and a finite dimensional solution to the SPDE as an invariant finite dimensional manifold for the associated SDE in $J^\infty(M,N)$. A comparison of this notion of solution to an SPDE and the more standard martingale theory based on Hilbert space by Da Prato and Zabczyk [48] is given. Thanks to this identification we are able to propose a generalization of Frobenius theorem in $J^\infty(M,N)$ setting, which, exploiting the classical

notion of characteristics of a PDE, allows us to find some sufficient conditions for the existence of finite dimensional solutions to an SPDE and then to explicitly reduce the SPDE to a finite dimensional SDE. These techniques permits to individuate new finite dimensional solutions to interesting SPDEs among which the proportional volatility equation in Heath-Jarrow-Morton framework, a stochastic perturbation of Hunter-Saxton equation and a filtering problem related to affine type processes.

Riassunto

Lo scopo principale del presente lavoro è quello di proporre una applicazione sistematica (attraverso opportune generalizzazioni) della teoria delle simmetrie di Lie, o più in generale, delle tecniche geometriche sviluppate nell'ambito dello studio delle equazioni differenziali, all'analisi delle equazioni differenziali stocastiche finito (SDE) o infinito (SPDE) dimensionali. La tesi è divisa in tre parti.
Nella prima parte viene proposto un nuovo approccio geometrico allo studio delle simmetrie delle SDE guidate dal moto Browniano. In particolare viene introdotta la definizione di simmetria di una SDE come la proprietà di invarianza della SDE considerata rispetto ad un particolare gruppo di trasformazioni (chiamate nel seguito trasformazioni stocastiche). Questo gruppo è formato dalle trasformazioni della variabile dipendente della SDE tramite un diffeomorfismo, da una rotazione stocastica del moto Browniano e da un opportuno cambio di tempo stocastico. Dopo aver studiato le proprietà, sia geometriche che probabilistiche, di questa famiglia di trasformazioni la tesi propone un'estensione di alcuni ben noti teoremi della teoria deterministica delle simmetrie, e precisamente la riduzione di una SDE simmetrica ad una SDE di dimensione inferiore e la ricostruzione della soluzione dell'equazione (di partenza) attraverso la procedura di integrazione per quadrature. Vengono inoltre presentate molte applicazioni dei risultati sopra descritti ad alcune SDE di interesse sia teorico che applicativo. Tra questi esempi si trovano il moto Browniano bidimensionale, l'equazione di Kolmogorov-Pearson, un'opportuna generalizzazione dell'equazioni di Langevin e il modello finanziario a volatilità stocastica SABR. A conclusione di questa prima parte viene introdotto uno schema numerico adattato alle simmetrie di una SDE assegnata, del quale viene provata l'efficacia sia attraverso alcune stime teoriche che attraverso alcune simulazioni numeriche.
Nella seconda parte della tesi viene presentata un'estensione dei risultati ottenuti nella prima parte alle SDE finito dimensionali guidate da una qualsiasi semimartingala càdlàg. La nozione di SDE geometrica proposta da Choen [37] risulta essere la più adatta per questo tipo di estensione. Questa classe di SDE include le SDE di tipo affine, le SDE definite usando l'integrale di Marcus, le SDE lisce guidate da processi di Lévy e le mappe aleatorie iterate. In questa parte vengono inoltre introdotti i nuovi concetti di gruppo di simmetrie di gauge e di simmetrie temporali di semimartingale a valori in un gruppo di Lie. Dopo aver generalizzato il concetto di caratteristiche stocastiche per il caso di semimartingale a valori in gruppi di Lie, si provano alcuni risultati utili per la determinazione esplicita di semimartingale aventi gruppi di simmetrie di gauge e di simmetrie temporali. Usando gli strumenti matematici sopra descritti si estende al caso generale in esame sia la nozione di trasformazione stocastica che il relativo concetto di simmetria di una SDE. Al termine di questa seconda parte vengono studiati nel dettaglio alcuni esempi di SDE simmetriche con applicazioni alla teoria delle mappe aleatorie iterate e all'integrazione numerica delle SDE.
Nella terza parte della tesi viene sviluppato un algoritmo utile a determinare esplicitamente le soluzioni finito dimensionali delle SPDE sfruttando la geometria del fibrato dei getti infiniti $J^\infty(M,N)$. L'idea chiave è quella di usare la nozione di semimartingala dipendente in maniera liscia da un parametro introdotta da Kunita in [116] e di interpretare le SPDE come SDE nella varietà (infinito dimensionale) $J^\infty(M,N)$. In questo contesto le soluzioni finito dimensionali della SPDE

considerata possono esessere identificate con le sottovarietà finito dimensionali in $J^\infty(M, N)$ invarianti rispetto alla SDE associata (nella tesi viene anche proposto un confronto tra questa nozione di soluzione e il più usuale concetto di soluzione proposto da Da Prato e Zabczyk in [48]). Questa identificazione permette di ottenere una generalizzazione del teorema di Frobenius in $J^\infty(M, N)$, sfruttando la classica nozione di caratteristiche di una PDE. Tale risultato da un lato permette di individuare alcune condizioni sufficienti per l'esistenza di soluzioni finito dimensionali di una SPDE e dall'altro di calcolare esplicitamente tali soluzioni. Queste tecniche permettono di trovare alcune nuove famiglie di soluzioni finito dimensionale di alcune SPDE di interesse applicativo tra le quali l'equazione a volatilità proporzionale nella teoria di Heath-Jarrow-Morton, una perturbazione stocastica dell'equazione di Hunter-Saxton e un problema di filtraggio stocastico legato alla importante classe dei processi di tipo affine.

Contents

Introduction

The concept of symmetry of ordinary or partial differential equations (ODEs and PDEs) was introduced by Sophus Lie at the end of the 19th century with the aim of extending the Galois theory from polynomial to differential equations. Actually, all the theory of Lie groups and algebras was developed by Lie himself as well as the principal tools for facing the problem of symmetries of differential equations (see [90] for an historical introduction to the subject and [26, 74, 147, 164] for some modern presentations). The idea of Lie is very simple: let $\mathcal{E}$ be a differential equation and denote by $\mathcal{S}$ the set of its solutions. If one considers a group of transformations T acting on $\mathcal{S}$ and, implicitly, on the equation $\mathcal{E}$, a transformation T is a symmetry of $\mathcal{E}$ if and only if T transforms the set $\mathcal{S}$ into itself, i.e. $T(\mathcal{S}) = \mathcal{S}$. More precisely, the original Lie idea consists in taking T as the smooth coordinate changes transforming both the independent and dependent variables involved in the equation $\mathcal{E}$. The symmetries arising from these transformations are usually called Lie point symmetries. Since the coordinate transformations form a Lie grupoid, it is possible to consider a smooth one-parameter subgroup T_a introducing, in this way, the concept of infinitesimal transformations. The idea of passing from the finite to the infinitesimal setting was one of the Lie's most important breakthrough which permitted to reduce the problem of finding symmetries of a differential equation (usually a non-linear and non-local problem) to the simple task of solving a system of linear PDEs.
The principal Lie's reason for introducing the concept of symmetry in the framework of differential equations was essentially to propose an extension, to the differential case, of the resolution methods based on Galois theory in the polynomial setting. The main result proved by Lie about this subject is the following: if an ODE $\mathcal{E}$ admits a solvable algebra of infinitesimal symmetries it is possible to reduce $\mathcal{E}$ to a lower dimensional equation $\mathcal{E}^R$ (or to an equation of a lower order in the scalar case) so that the solutions to the original equation $\mathcal{E}$ can be recovered from the solutions to the reduced one using only functions compositions and integrals (the well known reduction and reconstruction by quadratures, see [164]). Thanks to this result Lie symmetry analysis permits an unifying view of all explicit integration techniques of ODEs. Starting from this original Lie's idea the concept have found many other applications. Emmy Noether discovered an important relationship between the symmetries of a differential equation admitting a variational formulation and its conserved quantities (the famous Noether theorem of Lagrangian mechanics, see [147]). Garrett Birkhoff and Lev Ovsyannikov successively introduced a new method for using the infinitesimal symmetries of PDEs: in this case the knowledge of a symmetry algebra for a given PDE cannot be exploited to find the general solution to the PDE, but one can use symmetries to reduce a PDE to a system of ODEs by looking for the invariant solutions with respect to the symmetry algebra (see [147]). Around the 70s, taking inspirations from the pioneering works of Lie, Bäcklund and Noether, the definition of generalized symmetries (also called Lie-Bäcklund symmetries) was introduced and this concept turned out to be very useful in studying infinite dimensional integrable Hamiltonian systems (see [6, 133]). Furthermore the concept of invariance of differential equations with respect to finite or infinite dimensional groups of transformations has been discovered to be

extremely important in theoretical physics for the definition of the equations governing the fundamental forces (for example the equations of general relativity, which are invariant with respect to the group of diffeomorphisms of the space-time manifold, or the Yang-Mills equations of quantum fields theory, which are invariant with respect to gauge transformations, see [58]). More recently, the symmetries or the peculiar geometric properties of differential equations has been exploited to develop numerical algorithms for integrating differential equations, which permit an appreciable reduction of the numerical error or the conservation of integrals of the motion (see e.g. [124] for Hamiltonian dynamics or [56, 126, 127] for more general equations).
In conclusion Lie ideas and their more recent extensions has been used for studying and selecting a class of reasonably manageable systems admitting special closed formulas for their solutions as well as for identifying the relevant models in applications.

These ideas, with no or little references to Lie symmetry analysis, can be found also in the field of stochastic analysis.
In the applications of stochastic differential equations (SDEs), partial stochastic differential equations (SPDEs) or, more generally, of stochastic processes to the modelling of physical, biological or social random phenomena, an important role is played by those models possessing some kind of analytical tractability. For example, the importance of Black-Scholes-Merton model in the framework of mathematical finance is also due to the explicit formula for the value of European put and call options (see [25]). Other models, widely used in mathematical finance and enjoying a great analytical tractability, are the Cox-Ingersoll-Ross and Vasicek models for describing the evolution of interest rate (see e.g. [34]) or the Heston and SABR stochastic volatility models used in the evaluation of option prices (see e.g. [86, 94]). The Kalman-Bucy filter and all its non-linear extensions (such as the extended Kalman filter) are very popular in the applications of stochastic filtering since they reduce the infinite dimensional filtering problem to a set of finite dimensional SDEs giving a closed formula for the conditional probability (see e.g. [17]). These examples also suggest that the knowledge of closed-form expressions for some mathematical objects related with SDEs and SPDEs can be useful in order to formulate faster or more stable algorithms for numerical simulation (see e.g. [28, 94, 134]), to propose better estimators for statistical inference (see e.g. [19, 20, 68]) or to reduce the complexity of the models using asymptotic expansions or perturbation theory techniques (see e.g. [85, 132]).
Lie-type techniques are also important from a theoretical point of view, in particular when stochastic processes are discussed in a geometrical framework. Some interesting examples are Lévy processes on Lie groups [4, 130], the geometry of stochastic filtering (see [61], where invariant diffusions on fibred bundles are discussed), and the study of variational stochastic systems ([46, 99, 176]).

The aim of the present work is twofold. First of all we extend the ideas of deterministic Lie symmetry analysis, and the related geometrical methods, from the deterministic differential equations case to the stochastic setting. By developing this project, we do not only extend the theoretical concept of symmetry of a differential equation, but we also generalize to the stochastic framework many useful applications of this concept, such as the reduction and reconstruction by quadratures, the improvement of numerical integration algorithms for equations with symmetries or the reduction of an infinite dimensional equation to a finite dimensional problem. A second goal is to explain and extend some recent results about stochastic processes admitting closed analytical formulas (or, as we call them, integrable stochastic processes) in terms of stochastic Lie symmetry analysis, reproducing in this way the unifying character of deterministic Lie symmetry analysis of ODEs and PDEs with respect to different integration methods.
Since the research project is really wide and the possible applications cover many different fields we propose a compromise between a very general and unifying point of view, able to include as

special cases all the current approaches to the subject, and a more practical approach suitable for generalizing the interesting applications of deterministic Lie symmetry analysis in the stochastic setting. Since the extension of Lie symmetry analysis to the stochastic case is only at its beginning, there are very different levels of development depending on the specific problems: for example, the research is quite well developed in the finite dimensional Brownian-motion case, but it is almost completely absent in the case of càdlàg-semimartingales-driven SDEs. We organize the thesis in three parts, each one developing different aspects of our program: the first one is devoted to the finite dimensional Brownian-motion-driven SDEs, the second one to the finite dimensional SDEs driven by general càdlàg semimartingales and the third one to the SPDEs driven by continuous semimartingales.

There are three main conceptual results in the thesis: the first one is the definition of symmetry of a (finite dimensional) SDE based on a new set of transformations (which we call stochastic transformations). The introduction of this group is a nice example of interaction between the probabilistic and the geometric aspects of this research. Indeed, although the group structure of the set of stochastic transformations arises from its probabilistic action on stochastic processes, and then it has a completely probabilistic nature, the group of stochastic transformations admits a natural interpretation as the group of diffeomorphisms preserving the structure of a suitable principal bundle. The geometric properties of the group of stochastic transformations play a central role in the generalization of reduction and reconstruction theorems of symmetric equations from the deterministic to the stochastic setting (see Chapter 2).
The second result is the introduction of new probabilistic concepts for the study of symmetries of general SDEs. The main examples are the concepts of *gauge symmetry group* and of *time symmetry* of a semimartingale Z taking values in a Lie group (see Chapter 4). Although the idea of studying the invariance properties of a semimartingale with respect to a group of transformations depending on random processes is not new (see [109, 151]), it is the first time, to the best of our knowledge, that these notions are proposed and studied. This is an example of the fact that our generalization of Lie symmetry analysis does not reduce to a straightforward exercise, but induces fruitful interactions between geometry and theory of stochastic processes, giving rise to new interesting problems.
The third new result is the use of the developed stochastic Lie symmetry analysis to study some well known cases of "integrable" stochastic systems. Two interesting examples are the integration formula of linear scalar SDEs, obtained in Section 2.3.2 by exploiting our symmetry techniques and used in Section 3.3 to construct a symmetry-adapted numerical integration scheme, and the search of finite dimensional solutions to SPDEs (see Part III). In particular, since the straightforward generalization of the concept of symmetry to the infinite dimensional setting would not be helpful to study many important classes of integrable SPDEs, we approached this topic using the geometry of infinite dimensional jet bundles and the theory of differential constraints, which provide a powerful extension of deterministic Lie symmetry analysis applied to PDEs.

In the rest of the introduction we propose a more detailed description of the main results of the thesis, which are based on [51, 52] published in *Journal of Mathematical Physics* and on [2, 49, 50, 54] which have been submitted for publication,

0.1 Introduction to Part I

In Part I, based on [51, 52, 54], we extend the techniques and results of Lie symmetry analysis of deterministic ODEs to the stochastic setting, considering only the case of Brownian-motion-driven

SDEs. In particular, we introduce a new notion of symmetry of a Brownian-motion-driven SDEs, we generalize the well-known theorems of reduction and reconstruction by quadratures from the deterministic to the stochastic setting, and we present an application of our results to geometric numerical integration.

The main aim of Chapter 1 is the introduction of a new notion of symmetry of an SDE, based on a large set of random transformations of a stochastic process. It is not the first time that this problem is faced and in the previous literature we can find two different approaches.
The first one exploits the fact that the solutions to a Brownian-motion-driven SDEs are Markov processes associated with a second order differential operator L, depending on the SDE, which is an analytical deterministic object. In this way one can apply the usual notion of symmetry, coming from the deterministic Lie symmetry analysis, to the generator L considering the probabilistic consequences of this deterministic invariance. Such a perspective was proposed for the first time by Glover et al. in [81, 82, 83] and later developed by Cohen de Lara [39, 40] and Liao [129, 131]. The two main advantages of this approach are the very large class of considered transformations (due to the implicit use of a notion of weak solutions to an SDE) and the fact that it is not limited to Brownian-motion-driven SDEs but it can be applied whenever the solution process to the considered SDE is a Markov process. On the other hand this perspective does not permit a simple generalization of the procedure of reduction and reconstruction by quadratures which represents a very powerful result of the deterministic Lie symmetry analysis.
The second line of research consists in restricting the attention to a suitable set of transformations and directly apply a natural notion of symmetry, closely inspired by the ODEs case (see Albeverio et al. [3], Misawa [144], Gaeta et al. [75, 77], Unal [168], Srihirun, Meleshko and Schulz [163], Fredericks and Mahomed [69], Kozlov [112, 113]; see also [76] for a review on this subject and Lázaro-Camí and Ortega [122] for the same approach applied to SDEs driven by general continuous semimartingales).
In Chapter 1 we follow the second approach, but we introduce a new larger family of transformations which allows us to recover all the symmetries found using the first approach. In particular, we consider the pairs (X, W), where X is a continuous stochastic process in a open set $M \subset \mathbb{R}^m$ and W is an n-dimensional Brownian motion, and we define a class of transformations characterized by three geometrical objects: a diffeomorphism $\Phi : M \to M$ describing the transformation of the state variable X, a matrix valued function $B : M \to O(n)$ inducing a general state-dependent rotation of the Brownian motion W and a density function $\eta : M \to \mathbb{R}_+$ representing a random time change of the process (X, W). We call the triad $T = (\Phi, B, \eta)$ a general (weak) stochastic transformation.
The stochastic transformation T induces an action P_T on the set of processes (X, W) and an action E_T on the set of smooth SDEs (which hereafter we identify with their coefficients (μ, σ)). We note that the fundamental object is the action P_T, which transforms a process (X, W) in a new process $(X', W') = P_T(X, W)$, while the action E_T is completely determined by the following property: if the process (X, W) is a solution to the SDE (μ, σ), then the process $P_T(X, W)$ is solution to the SDE $E_T(\mu, \sigma)$.
The set of stochastic transformations forms an infinite dimensional Lie group with respect to a composition defined in the unique compatible way with respect to the composition of the action P_T. This infinite dimensional Lie group can be identified with the group of diffeomorphisms which preserve the structure of the principal bundle $M \times O(n) \times \mathbb{R}_+$. Thanks to this identification we can consider a one-parameter groups T_a of (weak) stochastic transformations generated by an *infinitesimal stochastic transformation* (Y, C, τ) defined by a vector field $Y : M \to \mathbb{R}^m$ and two smooth functions $C : M \to \mathfrak{so}(n)$ and $\tau : M \to \mathbb{R}$. The set of infinitesimal stochastic transformations forms a Lie algebra with a Lie brackets structure given by the previous natural

identification. We introduce also the notion of *strong stochastic transformations* given by a subgroup of the (weak) stochastic transformations of the form $(\Phi, I_n, 1)$. Finally, we provide Theorem 1.16 which guarantees the possibility of transforming any Lie algebra (Y_1, C_1, τ_1) ,..., (Y_r, C_r, τ_r), satisfying a suitable hypothesis of non-degeneration, into a set of strong infinitesimal stochastic transformations $(Y_1, 0, 0)$,..., $(Y_r, 0, 0)$ through the action of a stochastic transformation of the form $T = (Id_M, B, \eta)$. Theorem 1.16, whose proof is deeply based on the geometric analysis of stochastic transformations proposed in this thesis, is very important in the generalization of reduction and reconstruction techniques of symmetric SDEs proposed in Chapter 2.
Once this new class of transformations is introduced, we propose a natural definition of symmetry of an SDE: a stochastic transformation T is a symmetry of the SDE (μ, σ) if, for any solution (X, W) to the SDE (μ, σ), also $P_T(X, W)$ is a solution to (μ, σ). This definition can be easily generalized to the case of infinitesimal stochastic transformations providing the *determining equations* for infinitesimal symmetries: i.e. a set of linear PDEs for $V = (Y, C, \tau)$ which are identically satisfied if and only if V is an infinitesimal symmetry of the SDE (μ, σ).
Finally we propose a formulation of stochastic symmetries of SDEs within the Stratonovich stochastic calculus, proving that this definition is completely equivalent to the previous one based on the Itô calculus.
Although each part of the weak stochastic transformations (Φ, B, η) has been already considered in some of the previous references (random time change has been used for example in [40, 163], rotation of Brownian motion with a constant matrix B is the W-symmetry introduced in [75]), it is the first time, to the best of our knowledge, that they are considered all together and that a geometrical description, essential in the applications to explicit examples, is proposed (see e.g. the important role played by Theorem 1.16 in Chapter 2). Finally we remark that the introduction of weak stochastic transformations of the previous form it is necessary in order to recover all the symmetries of an SDE obtained using the first described Markovian approach.

In Chapter 2 we propose an extension of the reduction and reconstruction by quadratures procedure to symmetric SDEs driven by Brownian motion. Furthermore we apply these results to some concrete examples coming from different applications of stochastic calculus.
Although these basic applications of infinitesimal symmetries have already been discussed in [113, 122, 152, 177], there are several novelties in our approach. First of all we use the weaker notion of infinitesimal symmetry proposed in Chapter 1, including both strong symmetries, used for reduction and reconstruction in [122], and *quasi-strong symmetries* used for reduction in [177]. A further advantage of our method, inspired by the original ideas of Lie, is that we do not need the existence of a Lie group action related to the infinitesimal symmetries as required in [122, 177]. Moreover, the possibility of working with both the global and the local action of a Lie group turns out to be very useful in order to deal with stochastically complete SDEs admitting infinitesimal symmetries which do not generate a globally defined flow of diffeomorphisms (see the example of Section 2.3.3).
Furthermore we provide a notion of reconstruction inspired by the classical idea of reconstruction by quadratures and similar to the one proposed in [113] for strong symmetries. We remark that this concept is different from the one considered in [122] and our corresponding natural notion of integrability, which is in some respects more restrictive than that of [177] by not including higher order symmetries, allows us to exploit (not only Abelian but) general solvable algebras of symmetries. Besides the interesting stochastic quadrature procedure for one-dimensional diffusion processes proposed in [152] cannot be directly related to our results, since it is based on a well-defined variational structure. We discuss some particular examples of SDEs with a variational structure in Section 2.3.4.
It is worth noting that our approach is completely explicit and allows us to compute symmetries

of an SDE by solving an overdetermined system of first order PDEs. In particular, we apply our complete procedure to a class of one-dimensional diffusions reducing to linear SDEs for a particular choice of the parameters. In this case, considering a suitable two-dimensional SDE including the original one, we are able to find the explicit solutions recovering the well known solution formula for one-dimensional linear SDEs, together with the usual change of variables coupled with the associated homogeneous equation.
We also investigate a class of (stochastic) mechanical models which includes the standard perturbations of stochastic Lagrangian systems. In particular, starting from a mechanical system describing a particle subjected to forces depending on the velocities, we look for general stochastic perturbations of the deterministic system preserving the symmetries and we analyze in details a couple of significant examples within this class.

In Chapter 3 we propose a first extension to the stochastic case of the techniques of *geometric numerical integration* (see e.g. [87, 106, 124, 154]), a numerical analysis line of research which exploits some special geometrical structures for the numerical integration of both ordinary and partial differential equations.
Some results in this direction can be found in the literature. In particular there are some papers proposing numerical stochastic integrators which are able to preserve the symplectic structure (see e.g. [7, 143, 165]), some conserved quantities (see e.g. [35, 101, 134]) or the variational structure (see e.g. [29, 30, 100, 171]) of the considered SDEs.
Although the exploitation of Lie symmetries of ODEs and PDEs to obtain better numerical integrators is an active research topic (see e.g. [33, 56, 127, 126] and references therein), to the best of our knowledge the application of the same techniques in the stochastic setting is not yet pursued. In this chapter we introduce two different numerical methods taking advantage of the presence of Lie symmetries for a given SDE in order to provide a more efficient numerical integration scheme. We start by introducing the definition of invariant numerical integrator for a symmetric SDE as a natural generalization of the corresponding concept for an ODE. When trying to construct general invariant numerical methods in the stochastic framework, in fact, a not trivial problem arises. Since both the SDE solution and the driving Brownian motion are continuous but not differentiable processes, the finite differences discretization may not converge to the SDE solution. We give some necessary and sufficient conditions ensuring that the two standard numerical methods for SDEs (the Euler and the Milstein schemes) are also invariant numerical methods. By using this result, in particular, we are able to identify a class of convenient coordinates systems for the discretization procedure.
Our second numerical method, based on a well-defined change of the coordinates system, is inspired by the standard techniques of reduction and reconstruction of an SDE proposed in Chapter 2.
We apply these two numerical techniques to the first non-trivial class of symmetric equations i.e. the general scalar linear SDEs. In this case the two algorithmic methods can be combined in such a way to produce the same simple family of best coordinates systems for the discretization procedure. Interestingly, the coordinate changes obtained in this way are closely related to the explicit solution formula of linear SDEs. Although the integration formula of linear SDEs is widely known, it is certainly original the recognition of the proposed numerical scheme for scalar linear SDEs as a particular implementation of a general procedure for SDEs with Lie symmetries.
Moreover we investigate from a theoretical point of view the advantages of the new numerical schemes for linear SDEs. More precisely we obtain two estimates for the forward numerical error which, in presence of an equilibrium distribution, guarantee that the proposed method is numerically stable for any size of the time step h. This means that, for any $h > 0$, the error does not grow exponentially with the maximum-integration-time T, but it remains finite for $T \to +\infty$. This property is not shared by standard explicit or implicit Euler and Milstein methods. Our estimates

can be considered original results mainly because the coordinate changes involved in the formulation of the numerical scheme are strongly not-Lipschitz, and so the standard convergence theorems can not be applied. We illustrate our theoretical results by means of numerical simulations.

0.2 Introduction to Part II

In this part of the thesis, based on [2], we provide a generalization of the notion of (weak) stochastic transformations and of the definition of symmetry introduced in Part I for the case of finite dimensional SDEs driven by general càdlàg semimartingales.

Contrary to what happens for the Brownian case, the literature in the setting of SDEs driven by general càdlàg semimartingales is very scarce. The only references are the works of Glover [81, 82, 83], Cohen de Lara [39, 40] and Liao [129, 131] (already quoted) which, dealing with the general case of Markov process, cover the setting where the driving process is a Lévy process. Furthermore there is the work [122] of Lázaro-Camí and Ortega considering the case of SDE driven by general *continuous semimartingales*, and proposing a notion of symmetry which is equivalent, in the Brownian setting, to the definition of strong symmetry.

There are two main differences with respect to the Brownian motion setting. The first one is the lack of a natural geometric transformation rule for processes with jumps replacing the Itô transformation rule for continuous processes. This fact makes the action of a diffeomorphism Φ on an SDE more difficult to be described. The second one is the fact that a general semimartingale has not the invariance properties of Brownian motion in the sense that we cannot "rotate" it or make general time changes.
In order to address the first problem we restrict ourselves to a particular family of SDEs (that we call *canonical SDEs*) introduced by Cohen in [37, 38] (see also [10, 36]). In particular, we consider SDEs defined by a map $\Psi : M \times N \to M$, where M is the manifold where the solution lives and N is the Lie group where the driving process takes values. This definition simplifies the description of the transformations of the solutions $(X, Z) \in M \times N$. In fact, if (X, Z) is a solution to the SDE $\Psi(x, z)$ then, for any diffeomorphism Φ, $(\Phi(X), Z)$ is a solution to the SDE $\Phi(\Psi(\Phi^{-1}(x), z))$ (see Theorem 4.3 and Theorem 4.9). We remark that the family of canonical SDEs is not too restrictive: in fact it includes affine types SDEs, Marcus type SDEs, smooth SDEs driven by Lévy processes and a class of iterated random maps (see Section 4.1.3 for further details).
The second problem is faced by introducing two new notions of invariance of a semimartingale defined on a Lie group. These two notions are extensions of predictable transformations which preserve the law of n dimensional Brownian motion and α-stable processes studied for example in [109, Chapter 4]. The first notion, which we call *gauge symmetry*, generalizes the rotation invariance of Brownian motion, while the second one, which we call *time symmetry*, is an extension of the time rescaling invariance of Brownian motion. The concept of gauge symmetry group is based on the action Ξ_g of a Lie group $\mathcal{G}$ (g is an element of $\mathcal{G}$) on the Lie group N which preserves the identity 1_N of N. A semimartingale Z admits $\mathcal{G}$ as gauge symmetry group if, for any locally bounded predictable process $G_t, t \in \mathbb{R}_+$ taking values in $\mathcal{G}$, the well defined transformation $d\tilde{Z} = \Xi_{G_t}(dZ)$ has the same probability law of Z (see Section 4.2). A similar definition is given for the time symmetry, where Ξ_g is replaced by an $\mathbb{R}_+$ action Γ_r and the process G_t is replaced by an absolutely continuous time change β_t (see Section 4.3).
Given an SDE Ψ and a driving process Z with gauge symmetry group Ξ_g and time symmetry $\Gamma_r, r \in \mathbb{R}_+$, we are able to define a stochastic transformation $T = (\Phi, B, \eta)$, where Φ and η are a

diffeomorphism and a density of a time change as in the Brownian setting, while B is a function taking values in $\mathcal{G}$ (in the Brownian setting $\mathcal{G}$ is the group of rotations in $\mathbb{R}^n$). In order to generalize the results of Chapter 1, using the properties of canonical SDEs and of gauge and time symmetries, we define an action E_T of T on the SDE Ψ as well as an action P_T of T on the solutions (X, Z).

This part of the thesis contains three main results. The first one is that, for the first time, the notion of symmetry of an SDE driven by a general càdlàg, in principle non-Markovian, semimartingale is studied in full detail. The analysis is based on the introduction of a group of transformations which includes both the space transformations Φ and the gauge and time transformations Ξ_g, Γ_r. In this way our approach extends the results of [122], where only general continuous semimartingales Z and space transformations Φ are considered. We also generalize the results to the case of a Markovian process on a manifold M and with a regular generator. Indeed, due to the introduction of gauge and time symmetries, we recover all smooth symmetries of a Markovian process which would be lost if we had just considered the space transformations Φ.
The second new result is the introduction of the notions of gauge symmetry group and time symmetry and the careful analysis of their properties. Predictable transformations which preserve the law of a process have already been considered for special classes of processes as the n dimensional Brownian motion, α-stable processes or Poisson processes (see [109, 151]), but it seems the first time that the invariance with respect to transformations depending on general predictable processes is studied for general semimartingales taking values in Lie groups. Furthermore, proving Theorem 4.18 and Theorem 4.35, we translate the notion of gauge and time symmetries into the language of characteristics of a semimartingale (see [108] for the characteristics of a $\mathbb{R}^n$ semimartingale and Theorem 4.16 for our extension to general Lie groups). This translation permits to see gauge and time symmetries as special examples of predictable transformations preserving the characteristics (and so the law) of a process. This new insight is certainly interesting in itself and, in our opinion, deserves a deeper investigation.
The third novelty of this part is given by our explicit approach: indeed, we provide many results which permit to check explicitly whether a semimartingale admits given gauge and time symmetries and to compute stochastic transformations which are symmetries of a given SDE. In particular, Theorem 4.22 and Corollary 4.25 give easily applicable criteria to construct gauge symmetric Lévy processes (see also the corresponding Theorem 4.36 and Theorem 4.38 for time symmetries). Analogously, Theorem 4.28 permits to construct non-Markovian processes with a gauge symmetry group. Finally we obtain the determining equations (5.9) which are satisfied, under some additional hypotheses on the jumps of the driving process Z, by any infinitesimal symmetry. The possibility of providing explicit determining equations is the main reason to restrict our attention to canonical SDEs instead of considering more general classes of SDEs. Indeed, an interesting consequence of our study is that we provide a black-box method, applicable in several different situations, which permits to explicitly compute symmetries of a given SDE or to construct all the canonical SDEs admitting a given symmetry. For these reasons, in order to show the generality and the user-friendliness of our theory, we conclude this part proposing an example inspired by the iterated random maps theory and defining a concept of weak symmetry of numerical schemes for Brownian-motion-driven SDE, extending in this way the strong notion of symmetry of numerical schemes proposed in Chapter 3.

0.3 Introduction to Part III

In this part of the thesis, based on [49, 50], we apply the geometrical methods developed in Lie symmetry analysis of deterministic PDEs to the case of SPDEs. Our first idea was trying to introduce a notion of symmetry for the stochastic case similar to the deterministic one. Anyway, after the first attempts, we realized that this approach would not be fruitful for two reasons. The first one is that the definition of invariance of an SPDEs with respect to Lie point transformations (the first kind of transformations applied in the deterministic setting) is too restrictive: in fact the SPDEs with this kind of invariance are very few and not so useful in the applications (see [43] where this approach is applied to Zakay equation). The second reason is that, assuming a notion of symmetry of an SPDE based on Lie point transformations, the use of the invariant solutions to an SPDE for reducing the considered SPDE to a finite dimensional SDE, turns out to be too restrictive since the dimension of the reduced SDE is fixed by the order of the considered equation and it is often too low for being interesting in the applications. For these two reasons we decided to move to a different approach. Instead of extending the notion of symmetry to the SPDEs case and then using this property for reducing an SPDE to a finite dimensional SDE, we make these two steps at once facing directly the problem of reducing an SPDE to a finite dimensional SDE. The natural tool we chose for addressing this problem is the geometrical analysis of PDEs developed in Lie theory of infinite dimensional integrable systems.

More precisely we consider the SPDE of the form

$$dU^i_t(x) = F^i_\alpha(x, U_t(x), \partial^\sigma(U_t(x))) \circ dS^\alpha_t, \tag{1}$$

where $x \in M = \mathbb{R}^m$, $U_t(x)$ is a semimartingale taking values in $N = \mathbb{R}^n$, $F^i_\alpha(x, u, u_\sigma)$ are smooth functions of the independent coordinates x^i, the dependent coordinates u^i and their derivatives u^i_σ (here $\sigma \in \mathbb{N}^m_0$ is a multi-index denoting the numbers of derivatives with respect to the coordinates x^i), $S^1, ..., S^r$ are r continuous semimartingales and $\circ$ denotes the Stratonovich integration. We want to establish under which conditions the solution process $U_t(x)$ to SPDE (1) can be written in the form

$$U_t(x) = K(x, B^1_t, ..., B^k_t),$$

where $K : M \times \mathbb{R}^k \to N$ is a smooth function of all its variables and $B_t = (B^1_t, ..., B^k_t) \in \mathbb{R}^k$ is a stochastic process satisfying a suitable finite dimensional SDE. In the following we refer to this problem as the problem of finding finite dimensional solutions to SPDEs.

We remark that the problem of reducing an SPDE to a finite dimensional SDE is not new in stochastic analysis and finds interesting applications to mathematical modelling.
The first setting where this problem arises is in stochastic filtering, where only a special form of equation (1) is considered: the case of Zakai equation (see [17]). Also the relation between the Lie algebra generated by the operators F_i and the existence of finite dimensional filters is not new, but can be found in the classical literature on the subject (see [21, 31, 174]). Indeed the necessary conditions obtained in Proposition 7.15 are, in the case of Zakai equation, equivalent to the conditions obtained for the existence of finite dimensional filters.
A second application of finite dimensional solutions to SPDEs of the form (1) is to the case of the Heath-Jarrow-Morton (HJM) equation appearing in the study of interest rate in mathematical finance (see [92, 24, 64]). Of particular importance, about this problem, are the works of Filipovic, Tappe and Teichmann about finite dimensional solutions to HJM equation (see [63, 65, 66, 166]).
A third application is to the study of stochastic soliton equations. In addition to the pioneering work of Wadati on the stochastic KdV equation preserving soliton solutions (see [170, 173]), we have also been inspired by the recent growing interest in the study of variational stochastic systems of hydrodynamic type (see e.g. [13, 99, 46]). In particular we recall [100], where Holm and Tyranowski found many families of finite dimensional soliton type solutions to a physically important stochastic perturbation of Camassa-Holm equation.

In this thesis we face the problem of determining finite dimensional solutions to SPDE (1) using the geometry of the infinite jet bundle $J^\infty(M,N)$ of the functions defined on $M \subset \mathbb{R}^m$ and taking values in $N \subset \mathbb{R}^n$. Jet bundles have been introduced by Charles Ehresmann and provide a very useful framework for a modern approach to Lie symmetry analysis allowing a natural geometric interpretation of deterministic differential equations. The infinite jet bundle $J^\infty(M,N)$ is an infinite dimensional manifold modelled on $\mathbb{R}^\infty$ whose main advantage, with respect to the more usual infinite dimensional (Banach or Fréchet) spaces of functions such as $L^2(M,N)$ or $C^\infty(M,N)$, is a simple coordinate system which can be exploited in explicit computations.
In this setting an SPDE becomes an infinite dimensional (ordinary) SDE in $J^\infty(M,N)$ and the function $K(x,b)$ becomes a finite dimensional submanifold $\mathcal{K}$ of $J^\infty(M,N)$. We prove that this problem is completely geometrical. Indeed, with the functions $F_\alpha = (F^1_\alpha, ..., F^n_\alpha)$ defining the SPDE (1) it is possible to associate a set of vector fields V_{F_α} defined in $J^\infty(M,N)$, and the probabilistic problem of finding finite dimensional solutions to SPDE (1) is equivalent to find a submanifold $\mathcal{K}$ such that the vector fields V_{F_α} are tangent to $\mathcal{K}$.

It is important to note that our geometrical reinterpretation of the problem needs a suitable definition of solutions to the SPDE (1). In order to give such a definition we use the notion of semimartingales depending smoothly by some spatial parameters proposed by Kunita in [116]. With this probabilistic tool we can give a rigorous sense to the intuitive definition of solution to SPDE (1) based on the idea of taking a process $U_t(x)$ depending both on t and x and smooth with respect to x and then verify equation (1) by replacing $U_t(x)$ in (1) for any fixed $x \in M$. This notion of solution is compared with the more usual ones based on the martingale calculus in Hilbert spaces of [48], proving their equivalence under simple common hypotheses on the Hilbert space and on the process $U_t(x)$.

Once the probabilistic problem has been transformed into a geometric one, we can tackle the latter using natural tools developed in the geometric theory of deterministic PDE. First of all we propose a necessary condition on the coefficients F^i_α (see Theorem 7.14) for the existence of finite dimensional solutions to SPDE (1). Furthermore we prove a sufficient condition for the existence of finite dimensional solutions to SPDEs. This sufficient condition requests that the vector fields V_{F_α} form a finite dimensional Lie algebra and they admit characteristic flow (the notion of characteristic flow of a vector field in $J^\infty(M,N)$ is a generalization of the more common definition of characteristics of a first order scalar PDE in the $J^\infty(M,N)$ setting). Under these new hypotheses, in Theorem 6.19 and Theorem 6.21, for any smooth initial condition for equation (1), we are able to construct a finite dimensional manifold $\mathcal{K}$ which guarantees the existence of finite dimensional solutions to the considered SPDE.

Theorem 6.19 and Theorem 6.21 give an explicit construction method which we develop in a practical algorithm (see Section 7.2). This algorithm is applied to three selected examples taken from three different classical fields (HJM theory, hydrodynamic and filtering theory) where the finite dimensional solutions to SPDEs have their principal applications. The first example is a model for HJM theory with proportional volatility, considered by Morton in his thesis [146], for which we give for the first time, to the best of our knowledge, an explicit solution formula. The second example is a stochastic perturbation of the Hunter-Saxton equation which is a simplification of the stochastic Camassa-Holm equation considered in [100]. The third example is inspired by filtering theory, and is an extension of the well known formulas of Fourier transform of affine processes (see [57]).

The methods proposed in this part of the thesis are deeply inspired from the previous works on finite dimensional solutions to SPDEs. In particular our setting can be seen as a non-trivial generalization of the results proposed in [41, 42] by Cohen De Lara for studying Zakai equation to the case of general non-linear SPDEs of the form (1).

Furthermore the works of Filipovic, Tappe and Teichmann about finite dimensional solutions to HJM equation triggered a part of the thesis. In particular Theorem 6.19 and Theorem 6.21 are reformulations of [66], where the use of the convenient setting of global analysis ([115]) is replaced by the infinite jet bundle geometry and the characteristics of Section 6.2.2.

On the other hand our work introduces some novelties. First of all we propose an unified point of view on the subject which provides, for some respects, a generalization of the current literature. Indeed the form of equation (1) is completely general and includes as special cases both the Zakai equation considered by Cohen De Lara and the semilinear SPDEs considered by Filipovic, Tappe and Teichmann. Furthermore, Theorem 6.19 and Theorem 6.21 allows us to construct all the smooth solutions considered by the previous methods. Nevertheless, our perspective should be considered as complementary and not as alternative to the previous ideas. Indeed we consider only smooth solutions to SPDE (1): Theorem 6.19 is proved only in smooth setting, although, restricting the generality of equation (1), it could be extended to the non-smooth framework. Moreover, using Theorem 6.19, we are able to construct *one* solution to equation (1) between the many possible smooth solutions with the same initial data. In fact, if we do not restrict the class of the possible solutions to a suitable space of functions, we have not a uniqueness result for equation (1). For this reason, once we construct the solution with our method we should, a posteriori, prove that the solution belongs to a suitable space of functions where a uniqueness result for SPDE (1) holds. This feature is a consequence of the generality of our methods, indeed if we are interested in finding a result which permits to construct solutions belonging to a given class of functions we should use different (more analytic) methods such as those proposed in the previous literature.

A second novelty of our perspective is that we provide an algorithm for the explicitly computation of finite dimensional solutions to SPDEs which covers all the relevant cases considered in the current literature. Furthermore we propose new examples of interesting SPDEs: among which all the concrete SPDEs considered in Section 7.3, as well as HJM model considered in Section 7.3.1, whose the explicit solution was not known.

Part I

Symmetries of Brownian-motion-driven SDEs and applications

Chapter 1

Symmetries of Brownian-motion-driven SDEs

In this chapter we introduce the family of *stochastic transformations* of stochastic processes and SDEs. A stochastic transformation is composed by a diffeomorphism which changes the dependent variable X of an m dimensional SDE, a random rotation of the considered n dimensional Brownian motion W and a time change which modifies both X and W. After studying the geometrical and group properties of this family of transformations, we propose the notion of symmetry of an SDE as the set of stochastic transformations which leaves the set of the solutions to the given SDE unchanged. We give a sufficient and necessary condition such that a stochastic transformation T or an infinitesimal stochastic transformation V is a symmetry of a given SDE. Finally we study the notion of symmetries of an SDE on a manifold using the Stratonovich formulation of SDEs.

1.1 SDE Transformations: a probabilistic analysis

Let M be an open subset of $\mathbb{R}^m$. We denote by $x = (x^1, ..., x^m)^T$ the standard Cartesian coordinate system on M and by ∂_i the derivative with respect to x^i. In the following $\cdot$ denotes the usual product between matrices.

Let us consider all processes defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by $\mathcal{F}_t \subset \mathcal{F}$ a filtration of Ω. Unless otherwise specified, we assume that the stochastic processes are adapted with respect to the filtration $\mathcal{F}_t$.
If X is a stochastic process on M we denote by X_t the value of the process X at time t and by X^i the real processes defined as $X^i = x^i(X)$.
Let us consider an n-dimensional Brownian motion $W = (W^1, ..., W^n) = (W^\alpha)$ and two smooth functions $\mu : M \to \mathbb{R}^m$ and $\sigma : M \to \mathrm{Mat}(m, n)$.

Definition 1.1 *A stochastic process X on M and a m-dimensional Brownian motion W (in short the process (X, W)) solves (in the weak sense) the SDE with coefficients μ, σ until the stopping time τ (or shortly solves the SDE (μ, σ)) if for any $t \in \mathbb{R}_+$*

$$X^i_{t\wedge\tau} - X^i_0 = \int_0^{t\wedge\tau} \mu^i(X_s)ds + \int_0^{t\wedge\tau} \sigma^i_\alpha(X_s)dW^\alpha_s.$$

If (X,W) solves the SDE (μ,σ) we write, as usual,

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)\cdot dW_t \\ &= \mu dt + \sigma \cdot dW_t. \end{aligned}$$

The stopping time τ is strictly less than the explosion time of the SDE. When not strictly necessary, we omit the stopping time τ from the definition of solution to an SDE.

1.1.1 Space Transformations

If $A : M \to \mathrm{Mat}(n,k)$ we write A^l_r for the l-th row and r-th column component of the matrix A and identify $\mathrm{Mat}(k,1)$ with $\mathbb{R}^k$. Given a function $\Phi : M \to \mathbb{R}^n$ we consider the smooth function $\nabla(\Phi) : M \to \mathrm{Mat}(m,n)$ defined by

$$\nabla(\Phi)^l_i = \partial_i \Phi^l.$$

It is well known that with any SDE (μ,σ) it is possible to associate a second order differential operator

$$L = A^{ij}\partial_i\partial_j + \mu^i\partial_i,$$

where $A = \frac{1}{2}\sigma\cdot\sigma^T$. The operator L is called the infinitesimal generator of the process and appears, for example, in the following important formula.

Theorem 1.2 (Itô formula) *Let (X,W) be a solution to the SDE (μ,σ) and let $f : M \to \mathbb{R}$ be a smooth function. Then $F = f(X)$ satisfies*

$$dF_t = L(f)(X_t)dt + \nabla(f)(X_t)\cdot\sigma(X_t)\cdot dW_t.$$

By using the well-known Itô formula we can prove the following

Proposition 1.3 *Let us consider a diffeomorphism $\Phi : M \to M'$. If (X,W) is a solution to the SDE (μ,σ), then $(\Phi(X),W)$ is a solution to the SDE (μ',σ'), where*

$$\begin{aligned} \mu' &= L(\Phi)\circ\Phi^{-1} \\ \sigma' &= (\nabla(\Phi)\cdot\sigma)\circ\Phi^{-1}. \end{aligned}$$

Proof. By using the Itô formula, if $X' = \Phi(X)$, we have that $X'^i = \Phi^i(X)$ and so

$$\begin{aligned} dX'^i_t &= L(\Phi^i)(X_t)dt + \nabla(\Phi^i)(X_t)\cdot\sigma(X_t)\cdot dW_t \\ &= (L(\Phi^i)\circ\Phi^{-1})(X'_t)dt + (\nabla(\Phi^i)\circ\Phi^{-1})(X'_t)\cdot\sigma(\Phi^{-1}(X'_t))\cdot dW_t. \end{aligned}$$

■

1.1.2 Random Time Transformations

Let β be a positive adapted stochastic process such that, for any $\omega\in\Omega$, the function $\beta(\omega) : t \mapsto \beta_t(\omega)$ is continuous and strictly increasing. Define

$$\alpha_t = \inf\{s|\beta_s > t\},$$

where, as usual, $\inf(\emptyset) = +\infty$. The process α is an adapted process such that

$$\beta_{\alpha_t} = \alpha_{\beta_t} = t.$$

If Y is a continuous stochastic process we define by $H_\beta(Y)$ the continuous stochastic process such that

$$H_\beta(Y)_t = Y_{\alpha_t}.$$

The process $H_\beta(Y)$ is an adapted process with respect to the filtration $\mathcal{F}'_t = \mathcal{F}_{\alpha_t}$. In the following we restrict to absolute continuous time changes. Given a strictly positive smooth function $\eta : M \to \mathbb{R}_+$ and a stochastic process X defined until the stopping time τ, we consider the process

$$\beta_{t\wedge\tau} = \int_0^{t\wedge\tau} \eta(X_s)ds$$

defined until the stopping time τ. Given a stochastic process X and denoting by $H_\eta(X) := H_\beta(X)$, it is easy to prove that

$$d(\alpha_t) \quad = \quad \frac{1}{\eta(H_\eta(X)_t)}dt.$$

We introduce some useful lemmas for proving a time invariance property of Brownian motion.

Lemma 1.4 *Let β_t be a process of the previous form, then the following assertions hold:*

1. *if Z is a real local martingale with respect to $\mathcal{F}_t$ then $H_\beta(Z)$ is a real local martingale with respect to $\mathcal{F}_t$,*
2. *if Z is a continuous semimartingale and K_t is a locally bounded predictable process*

$$\int_0^{\alpha_t} K_s dZ_s = \int_0^t H_\beta(K)_s dH_\beta(H)_s.$$

Proof. The proof can be found, for example, in [155, Proposition 30.10] ■

In the following, if Z^1, Z^2 are two L^2 real semimartingales, we denote by $[Z^1, Z^2]$ the quadratic covariation between Z^1 and Z^2.

Lemma 1.5 (Lévy characterization of Brownian motion) *If $Z^1, ..., Z^n$ are n real continuous local martingales such that $[Z^\alpha, Z^\beta]_t = \delta^{\alpha\beta}t$, then $Z = (Z^1, ..., Z^n)$ is an n-dimensional Brownian motion.*

Proof. The proof can be found, for example, in [155, Theorem 33.1]. ■

Proposition 1.6 *Let $\eta : M \to \mathbb{R}_+$ be a smooth function and (X, W) be a solution to the SDE (μ, σ). Then $(H_\eta(X), H_\eta(W'))$, with*

$$dW'_t = \sqrt{\eta(X_t)}dW_t,$$

is a solution to the SDE (μ', σ'), where

$$\begin{aligned} \mu' &= \frac{1}{\eta}\mu \\ \sigma' &= \frac{1}{\sqrt{\eta}}\sigma \end{aligned}$$

The following theorem expresses an important invariance property of Brownian motion.

Theorem 1.7 *Let X be a continuous stochastic process taking values in M and consider $\eta, \alpha_{t'}, \beta_t$ defined as before. If W is an m-dimensional Brownian motion, denoting by W' the stochastic process such that*

$$dW'_t = \sqrt{\eta(X_t)}dW_t,$$

we have that $H_\eta(W')$ is an m-dimensional Brownian motion.

Proof. Let $W'' = H_\eta(W')$, by Lévy characterization of Brownian motion we have only to prove that W''^α are real local martingales and $[W''^\alpha, W''^\beta]_t = \delta^{\alpha\beta}t$.
Obviously, by point 1 of Lemma 1.4, W''^α are $\mathcal{F}'_t$ local martingales since they are time change of integrals with respect to Brownian motion which are local martingales.
Furthermore, using the second point of Lemma 1.4, we have

$$\begin{aligned}
[W''^\alpha, W''^\beta]_t &= [W'^\alpha, W'^\beta]_{\alpha_t} \\
&= \int_0^{\alpha_t} \eta(X_s)\delta^{\alpha\beta} ds \\
&= \delta^{\alpha\beta} \int_0^t \eta(H_\eta(X)_s) d\alpha_s \\
&= \delta^{\alpha\beta} \int_0^t \eta(H_\eta(X)_s) \frac{1}{\eta(H_\eta(X)_s)} ds \\
&= \delta^{\alpha\beta} t.
\end{aligned}$$

■

Proof of Proposition 1.6. Let τ be the stopping time associated with the solution (X, W) to the SDE (μ, σ). Denoting by $\tau' := \beta_\tau$, we prove that $(H_\eta(X), H_\eta(W'))$ is a solution to the SDE (μ', σ') until the stopping time τ'. In fact by definition $X'_{t'} = X_{\alpha_{t'}}$ (with $t' = \beta_t$) and therefore

$$\begin{aligned}
X'_{\tau' \wedge t'} &= X_{\tau \wedge \alpha_{t'}} \\
&= \int_0^{\tau \wedge \alpha_{t'}} \mu(X_s) ds + \sigma(X_s) \cdot dW_s \\
&= \int_0^{\beta_\tau \wedge \beta_{\alpha_{t'}}} \mu(H_\eta(X)_s) d\alpha_s + \sigma(H_\eta(X)_s) \cdot d(H_\eta(W)_s) \\
&= \int_0^{\tau' \wedge t'} \frac{1}{\eta(X'_s)} \mu(X'_s) ds + \frac{1}{\sqrt{\eta(X'_s)}} \sigma(X'_s) \cdot d(H_\eta(W'))_s,
\end{aligned}$$

being $\beta_{\alpha_{t'}} = t'$. ■

1.1.3 Brownian motion transformations

In in Section we introduce an invariance property of Brownian motion and a suitable class of transformations.

Proposition 1.8 *Let $B : M \to O(n)$ be a smooth function and (X, W) be a solution to an SDE (μ, σ). Then (X, W'), where*

$$dW'_t = B(X_t) \cdot dW_t,$$

is a solution to the SDE (μ', σ')

$$\begin{aligned} \mu' &= \mu, \\ \sigma' &= \sigma \cdot B^{-1}. \end{aligned}$$

Proof. The only thing to prove is that W' is a Brownian motion. Indeed, by the properties of Itô integral, we have

$$dX_t = \mu dt + \sigma \cdot dW_t = \mu dt + \sigma \cdot B^{-1} dW'_t = \mu' dt + \sigma' \cdot dW'_t.$$

We remark that, for any α, W'^{α} is a local martingale, being an Itô integral along the local martingale W^{β}. On the other hand, by the properties of the Itô integral, we have

$$\begin{aligned} [W'^{\alpha}, W'^{\beta}]_t &= \int_0^t B^{\alpha}_{\gamma} B^{\beta}_{\delta} d[W^{\gamma}, W^{\delta}]_s \\ &= \int_0^t B^{\alpha}_{\gamma} B^{\beta}_{\delta} \delta^{\gamma\delta} ds \\ &= \int_0^t \sum_{\gamma} B^{\alpha}_{\gamma} B^{\beta}_{\gamma} ds \\ &= \int_0^t \delta^{\alpha\beta} ds \\ &= \delta^{\alpha\beta} t, \end{aligned}$$

where we use $[W^{\gamma}, W^{\delta}]_s = \delta^{\gamma\delta} s$ and $B \cdot B^T = I$. The Lèvy characterization of Brownian motion ensures that W' is a Brownian motion. ■

1.1.4 Finite Stochastic Transformations

In the following we consider two open subsets M', M'' of $\mathbb{R}^n$ diffeomorphic to M, and we denote by $O(n)$ the Lie group of orthogonal matrices.

Definition 1.9 *Let* $\Phi : M \to M'$ *be a diffeomorphism, and let* $B : M \to O(n)$ *and* $\eta : M \to \mathbb{R}_+$ *be smooth functions. We call the triad* $T := (\Phi, B, \eta)$ *a (finite) stochastic transformation from* M *onto* M' *and we denote by* $S_n(M, M')$ *the set of all stochastic transformations from* M *onto* M'*. If* T *is of the form* $T = (\Phi, I_n, 1)$ *we call* T *a strong stochastic transformation and we denote the set of strong stochastic transformations by* $SS_n(M, M')$*.*

Definition 1.10 *Let* $T = (\Phi, B, \eta)$ *be a stochastic transformation. If the pair* (X, W) *is a continuous stochastic process, with* X *taking values on* M *and* W *being an* m*-dimensional Brownian motion, we define the process* $P_T(X, W) = (P_T(X), P_T(W))$ *where* $P_T(X)$ *takes values on* M'*, given by*

$$\begin{aligned} P_T(X) &= \Phi(H_{\eta}(X)), && (1.1) \\ dW'_t &= \sqrt{\eta(X_t)} B(X_t) \cdot dW_t && (1.2) \\ P_T(W) &= H_{\eta}(W'). && (1.3) \end{aligned}$$

We call the process $P_T(X, W)$ *the transformed process of* (X, W) *with respect to* T*, and we call the map* P_T *the process transformation associated with* T*.*

We remark that if T is a strong stochastic transformation and W is a Brownian motion, then $P_T(W) = W$.

Definition 1.11 *Let $T = (\Phi, B, \eta)$ be a stochastic transformation. If the pair (μ, σ) is an SDE on M, we define $E_T(\mu, \sigma) = (E_T(\mu), E_T(\sigma))$ the SDE on M' given by*

$$\begin{aligned} E_T(\mu) &= \left(\frac{1}{\eta} L(\Phi)\right) \circ \Phi^{-1} \\ E_T(\sigma) &= \left(\frac{1}{\sqrt{\eta}} \nabla(\Phi) \cdot \sigma \cdot B^{-1}\right) \circ \Phi^{-1}. \end{aligned}$$

We call the SDE $E_T(\mu, \sigma)$ the transformed SDE of (μ, σ) with respect to T, and we call the map E_T the SDE transformation associated with T.

We remark that, despite the similarity of Definition 1.10 and 1.11, the roles played by the transformations P_T and E_T are quite different. In fact, P_T accounts for the various elements composing the stochastic transformation T, while E_T is uniquely characterized by the property that, if (X, W) is a solution to the SDE (μ, σ), then $P_T(X, W)$ is a solution to the SDE $E_T(\mu, \sigma)$. Indeed, the following theorem states:

Theorem 1.12 *Let T be a stochastic transformation and E'_T a generic action form the set of smooth SDEs (μ, σ) into the set of smooth SDEs. If, for any solution (X, W) to (μ, σ), $P_T(X, W)$ is a solution to $E'_T(\mu, \sigma)$, then $E'_T = E_T$.*

Before proving Theorem 1.12 we introduce the following useful result.

Lemma 1.13 *Suppose that (X, W) is a solution to both the SDEs (μ, σ) and (μ', σ') such that $\mathbb{P}(X_0 = x_0) > 0$, where $x_0 \in M$. Then $\mu(x_0) = \mu'(x_0)$ and $\sigma(x_0) = \sigma'(x_0)$.*

Proof. Since (X, W) is solution to both (μ, σ) and (μ', σ') until the stopping time $\tau > 0$ almost surely we have

$$\begin{aligned} X^i_{t\wedge\tau} - X^i_0 &= \int_0^{t\wedge\tau} \mu(X_s) ds + \int_0^{t\wedge\tau} \sigma^i_\alpha(X_s) dW_s \\ X^i_{t\wedge\tau} - X^i_0 &= \int_0^{t\wedge\tau} \mu'(X_s) ds + \int_0^{t\wedge\tau} \sigma'^i_\alpha(X_s) dW_s \end{aligned}$$

Making the difference between the two previous expressions we get that

$$K^i_t = \int_0^{t\wedge\tau} (\mu^i(X_s) - \mu'^i(X_s)) ds + \int_0^{t\wedge\tau} (\sigma^i_\alpha(X_s) - \sigma'^i_\alpha(X_s)) dW^\alpha_s$$

is identically zero. In particular K^i_t is a semimartingale with bounded variation part equal to zero, $\int_0^{t\wedge\tau} (\mu^i(X_s) - \mu'^i(X_s)) ds = 0$. Since X_s is continuous, $\mu'(X_s) = \mu'(X_s)$ for any $s < \tau$ almost surely. Being $\tau > 0$, taking the limit as $s \to 0$ and recalling that $X_0 = x_0$ in a set with positive probability we have $\mu(x_0) = \mu'(x_0)$.
Furthermore, since K^i_t is zero, we have

$$[K^i, K^i]_t = \int_0^{t\wedge\tau} \sum_{\alpha=1}^n (\sigma^i_\alpha(X_s))^2 ds = 0.$$

In a similar way, we get $\sigma^i_\alpha(x_0)=\sigma'^i_\alpha(x_0)$ for any $i=1,...,m$ and $\alpha=1,...,n$. ■

Proof of Theorem 1.12. The proof of the fact that if (X,W) is a solution to (μ,σ) then $P_T(X,W)$ is a solution to $E_T(\mu,\sigma)$ is a simple combination of Propositions 1.3, Proposition 1.6 and Proposition 1.8. Indeed if we first apply time and Brownian transformations, i.e. Proposition 1.6 and Proposition 1.8 regardless of the order and then we apply space transformation, i.e. Proposition 1.3, we obtain the thesis.
We now prove the uniqueness of E_T. Suppose that E'_T is another smooth action with the previous property. Let (X^{x_0},W) be a solution to the SDE (μ,σ) such that $X^{x_0}_t=x_0$ (the existence of such a solution until a stopping time τ is proved in [105, Chapter IV, Theorem 2.3]) then $(X',W')=P_T(X,W)$ is such that $X'_0=\Phi(x_0)$ almost surely. By the property of preserving the solution to a smooth SDE we have that (X',W') is a solution to both $E_T(\mu,\sigma)$ and $E'_T(\mu,\sigma)$. By Lemma 1.13 this implies that $E_T(\mu)(\Phi(x_0))=E'_T(\mu)(\Phi(x_0))$ and $E_T(\sigma)(\Phi(x_0))=E'_t(\sigma)(\Phi(x_0))$. Since x_0 can be chosen arbitrarily and Φ is a diffeomorphism, we obtain $E_T(\mu,\sigma)=E'_T(\mu,\sigma)$. ■

1.2 SDE Transformations: a geometric analysis

1.2.1 The geometric description of stochastic transformations

Let us consider the group $G=O(n)\times\mathbb{R}_+$, with the natural product given by $g_1\cdot g_2=(A_1\cdot A_2,\zeta_1\zeta_2)$, where $g_1=(A_1,\zeta_1)$ and $g_2=(A_2,\zeta_2)$.
Since the manifold $M\times G$ is a trivial principal bundle $\pi_M: M\times G\longrightarrow M$ with structure group G, we can consider the following action of the group G on $M\times G$ leaving M invariant

$$\begin{aligned} R_{M,h}: \quad M\times G &\longrightarrow M\times G \\ (x,g) &\longmapsto (x,g\cdot h). \end{aligned}$$

Definition 1.14 *Given two (trivial) principal bundles $M\times G$ and $M'\times G$, an isomorphism $F: M\times G\to M'\times G$ is a diffeomorphism that preserves the structure of principal bundle of $M\times G$ and of $M'\times G$. This means that there exists a diffeomorphism $\Phi: M\to M'$ such that*

$$F\circ\pi_{M'}=\pi_M\circ\Phi,$$

and, for any $h\in G$,

$$F\circ R_{M,h}=R_{M',h}\circ F.$$

We denote by $\mathrm{Iso}(M\times G,M'\times G)$ *the set of isomorphisms between $M\times G$ and $M'\times G$.*

The previous definition ensures that any $F\in\mathrm{Iso}(M\times G,M'\times G)$ is completely determined by its value on (x,e) (where e is the unit of G), i.e. there is a one-to-one correspondence between F and the pair $F(x,e)=(\Phi(x),g)$. Therefore, there exists a natural identification between a stochastic transformation $T=(\Phi,B,\eta)\in S_n(M,M')$ and the isomorphism F_T defined by

$$F_T(x,g)=(\Phi(x),(B(x),\eta(x))\cdot g)$$

and the set $S_n(M,M')$ inherits the properties of the set $\mathrm{Iso}(M\times G,M'\times G)$. In particular, the natural composition of an element of $\mathrm{Iso}(M\times G,M'\times G)$ with an element of $\mathrm{Iso}(M'\times G,M''\times G)$ to give an element of $\mathrm{Iso}(M\times G,M''\times G)$ ensures the existence of a natural composition law between elements of $S_n(M,M')$ and of $S_n(M',M'')$. If $T=(\Phi,B,\eta)\in S_n(M,M')$ and $\tilde{T}=(\tilde{\Phi},\tilde{B},\tilde{\eta})\in S_n(M',M'')$, we have

$$\tilde{T}\circ T=(\tilde{\Phi}\circ\Phi,(\tilde{B}\circ\Phi)\cdot B,(\tilde{\eta}\circ\Phi)\eta).$$

Moreover, since $\mathrm{Iso}(M \times G, M' \times G)$ is a subset of the diffeomorphism between $M \times G$ and $M' \times G$, if $T \in S_n(M, M')$ we can define its inverse $T^{-1} \in S_n(M', M)$ as

$$T^{-1} = (\Phi^{-1}, (B \circ \Phi^{-1})^{-1}, (\eta \circ \Phi^{-1})^{-1}).$$

The set $S_n(M) := S_n(M, M)$ is a group with respect to the composition $\circ$ and the identification of $S_n(M)$ with $\mathrm{Iso}(M \times G, M \times G)$ (which is a closed subgroup of the group of diffeomorphisms of $M \times G$) suggests to consider the corresponding Lie algebra $V_n(M)$.
For later use, in the following we provide a description of the elements of $V_n(M)$.

Given a one parameter group $T_a = (\Phi_a, B_a, \eta_a) \in S_n(M)$, there exist a vector field Y on M, a smooth function $C : M \to \mathfrak{so}(n)$ (where $\mathfrak{so}(n)$ is the Lie algebra of antisymmetric matrices), and a smooth function $\tau : M \to \mathbb{R}$ such that

$$\begin{array}{rcl} Y(x) & := & \partial_a(\Phi_a(x))|_{a=0} \\ C(x) & := & \partial_a(B_a(x))|_{a=0} \\ \tau(x) & := & \partial_a(\eta_a(x))|_{a=0}. \end{array} \tag{1.4}$$

Conversely, considering Y, C, τ as above, the one parameter solution (Φ_a, B_a, η_a) to the equations

$$\begin{array}{rcr} \partial_a(\Phi_a(x)) & = & Y(\Phi_a(x)) \\ \partial_a(B_a(x)) & = & C(\Phi_a(x)) \cdot B_a(x) \\ \partial_a(\eta_a(x)) & = & \tau(\Phi_a(x))\eta_a(x). \end{array} \tag{1.5}$$

with initial condition $\Phi_0 = id_M$, $B_0 = I_n$ and $\eta_0 = 1$, is a one parameter group in $S_n(M)$. For this reason we identify the elements of $V_n(M)$ with the triples (Y, C, τ).

Definition 1.15 *A triad $V = (Y, C, \tau) \in V_n(M)$, where Y is a vector field on M and $C : M \to \mathfrak{so}(n)$ and $\tau : M \to \mathbb{R}$ are smooth functions, is an infinitesimal stochastic transformation. If V is of the form $V = (Y, 0, 0)$ we call V a strong infinitesimal stochastic transformation, as the corresponding one-parameter group is a group of strong stochastic transformations.*

Since $V_n(M)$ is a Lie sub-algebra of the set of vector fields on $M \times G$, the standard Lie brackets between vector fields on $M \times G$ induce some Lie brackets on $V_n(M)$. Indeed, if $V_1 = (Y_1, C_1, \tau_1), V_2 = (Y_2, C_2, \tau_2) \in V_n(M)$ are two infinitesimal stochastic transformations, we have

$$[V_1, V_2] = ([Y_1, Y_2], Y_1(C_2) - Y_2(C_1) - \{C_1, C_2\}, Y_1(\tau_2) - Y_2(\tau_1)), \tag{1.6}$$

where $\{\cdot, \cdot\}$ denotes the usual commutator between matrices.
Furthermore, the identification of $T = (\Phi, B, \eta) \in S_n(M, M')$ with $F_T \in \mathrm{Iso}(M \times G, M' \times G)$ allows us to define the push-forward $T_*(V)$ of $V \in V_m(M)$ as

$$((\nabla(\Phi) \cdot Y) \circ \Phi^{-1}, (B \cdot C \cdot B^{-1} + Y(B) \cdot B^{-1}) \circ \Phi^{-1}, (\tau + Y(\eta)\eta^{-1}) \circ \Phi^{-1}). \tag{1.7}$$

Analogously, given $V' \in V_n(M')$, we can consider the pull-back of V' defined as $T^*(V') = (T^{-1})_*(V')$.

The following theorem shows that any Lie algebra of general infinitesimal stochastic transformations satisfying a non-degeneracy condition, can be locally transformed, by action of the push-forward of a suitable stochastic transformation $T \in S_n(M)$, into a Lie algebra of strong infinitesimal stochastic transformations.

Theorem 1.16 *Let $K = \mathrm{span}\{V_1, ..., V_k\}$ be a Lie algebra of $V_n(M)$ and let $x_0 \in M$ be such that $Y_1(x_0), ..., Y_k(x_0)$ are linearly independent (where $V_i = (Y_i, C_i, \tau_i)$). Then there exist an open* *neighborhood U of x_0 and a stochastic transformation $T \in S_n(U)$ of the form $T = (id_U, B, \eta)$ such that $T_*(V_1), ..., T_*(V_k)$ are strong infinitesimal stochastic transformations in $V_n(U)$. Furthermore the smooth functions B, η are solutions to the equations*

$$\begin{aligned} Y_i(B) &= -B \cdot C_i \\ Y_i(\eta) &= -\tau_i \eta, \end{aligned}$$

for $i = 1, ..., k$.

Proof. By equation (1.7) with $T = (id_U, B, \eta)$ we have

$$T_*(V_i) = (Y_i, Y_i(B) \cdot B^{-1} + B \cdot C_i \cdot B^{-1}, Y_i(\eta)\eta^{-1} + \tau_i),$$

and $T_*(V_i)$ is a strong infinitesimal stochastic transformation if and only if

$$\begin{aligned} Y_i(B) \cdot B^{-1} + B \cdot C_i \cdot B^{-1} &= 0 \qquad (1.8) \\ Y_i(\eta)\eta^{-1} + \tau_i &= 0. \qquad (1.9) \end{aligned}$$

Denote by L_i, N_i the linear operators on $\mathrm{Mat}(m, m)$-valued and $\mathbb{R}_+$-valued smooth functions respectively such that

$$\begin{aligned} L_i(B) : &= Y_i(B) + B \cdot C_i = (Y_i + R_{C_i})(B) \\ N_i(B) : &= Y_i(\eta) + \eta\tau_i = (Y_i + R_{\tau_i})(\eta), \end{aligned}$$

where R_{C_i}, R_{τ_i} are the operators of right multiplication. A sufficient condition for the existence of a non-trivial solution to equations (1.8) and (1.9), is that there exist some real constants $c^k_{i,j}, d^k_{i,j}$ such that

$$\begin{aligned} L_i L_j - L_j L_i &= \sum_k c^k_{i,j} L_k \qquad (1.10) \\ N_i N_j - N_j N_i &= \sum_k d^k_{i,j} N_k. \qquad (1.11) \end{aligned}$$

A simple computation shows that

$$\begin{aligned} L_i L_j - L_j L_i &= [Y_i, Y_j] + R_{Y_i(C_j) - Y_j(C_i) - \{C_i, C_j\}} \qquad (1.12) \\ &= [Y_i, Y_j] + R_{Y_i(\tau_j) - Y_j(\tau_i)}. \qquad (1.13) \end{aligned}$$

Since $V_i = (Y_i, C_i, \tau_i)$ form a Lie algebra, there exist some constants $f^k_{i,j}$ such that

$$\begin{aligned} [V_i, V_j] &= ([Y_i, Y_j], Y_i(C_j) - Y_j(C_i) - \{C_i, C_j\}, Y_i(\tau_j) - Y_j(\tau_i)) \\ &= \left(\sum_k f^k_{i,j} Y_k, \sum_k f^k_{i,j} C_k, \sum_k f^k_{i,j} \tau_k \right). \end{aligned}$$

Comparing the last equality with equations (1.12) and (1.13) we find equations (1.10) and (1.11) and this completes the proof. ■

1.2.2 Probabilistic foundation of the geometric description

In this section we show how the identification of stochastic transformations with the isomorphisms of suitable trivial principal bundles and the resulting natural definition of composition of stochastic transformations has a deep probabilistic counterpart in terms of SDEs and process transformations introduced in Definitions 1.10 and 1.11.

Theorem 1.17 *If $T \in S_n(M, M')$ and $T' \in S_n(M', M'')$ are two stochastic transformations, then*

$$\begin{aligned} P_{T'} \circ P_T &= P_{T' \circ T} \\ E_{T'} \circ E_T &= E_{T' \circ T}. \end{aligned}$$

Proof. We have to prove that, for any stochastic process (X, W) and for any SDE (μ, σ), we have

$$\begin{aligned} P_{T'}(P_T(X, W)) &= P_{T' \circ T}(X, W) \\ E_{T'}(E_T(\mu, \sigma)) &= E_{T' \circ T}(\mu, \sigma). \end{aligned}$$

We give an idea of the proof for $P_{T'} \circ P_T = P_{T' \circ T}$. We prove the proposition for X in the pair (X, W). The proof for W and (μ, σ) is similar. If $T = (\Phi, B, \eta)$ and $T' = (\Phi', B', \eta')$, and we put $\beta_t = \int_0^t \eta(X_s) ds$, using the identification $H_\eta = H_\beta$ given in Section 1.1.2, we find $\beta'_t = \int_0^t \eta'(\Phi(H_\eta(X_s))) ds$. Then we have that

$$\begin{aligned} P_{T'}(P_T(X)) &= \Phi'(H_{\eta'}(\Phi(H_\eta(X)))) \\ &= \Phi'(H_{\beta'}(\Phi(H_\beta(X)))) \\ &= \Phi' \circ \Phi(H_{\beta'}(H_\beta(X))). \end{aligned}$$

We want to calculate the composition of the random time change $H_{\beta'} \circ H_\beta$. Let α_t, α'_t be the inverses of the processes β_t, β'_t respectively. If Y is any continuous process

$$H_{\beta'}(H_\beta(Y))_t = (H_\beta(Y))_{\alpha'_t} = Y_{\alpha_{\alpha'_t}}.$$

Since the inverse of $\alpha_{\alpha'_t}$ is β'_{β_t} we have that

$$H_{\beta'} \circ H_\beta = H_{\beta' \circ \beta}.$$

If we compute the density of the time change β'_{β_t}, we find

$$\begin{aligned} \beta'_{\beta_t} &= \int_0^{\beta_t} \eta'(\Phi(H_\eta(X)_s)) ds \\ &= \int_0^t H_\alpha(\eta'(\Phi(H_\eta(X))))_s d\beta_s \\ &= \int_0^t \eta'(\Phi(H_\alpha(H_\beta(X))))_s \eta(X_s) ds \\ &= \int_0^t (\eta' \circ \Phi)(X_s) \eta(X_s) ds, \end{aligned}$$

and we have

$$H_{\beta'} \circ H_\beta(X) = H_{(\eta' \circ \Phi)\eta}(X).$$

Hence

$$P_{T'}(P_T(X)) = \Phi' \circ \Phi(H_{\beta'}(H_\beta(X))) = \Phi' \circ \Phi(H_{(\eta' \circ \Phi)\eta}(X)) = P_{T' \circ T}(X).$$

by definition of composition between stochastic transformations given in Section 1.2.1.
The proof that $E_{T'} \circ E_T = E_{T' \circ T}$ follows directly from Theorem 1.12. Indeed, since $P_{T'} \circ P_T = P_{T' \circ T}$, both $E_{T'} \circ E_T$ and $E_{T' \circ T}$ are smooth action of $T' \circ T$ such that if (X, W) is a solution to (μ, σ) then also $P_{T' \circ T}(X, W)$ is a solution to both $E_{T'}(E_T(\mu, \sigma))$ and $E_{T' \circ T}(\mu, \sigma)$. Hence, by Theorem 1.12, $E_{T'} \circ E_T = E_{T' \circ T}$. ∎

1.3 Symmetries of an SDE

In analogy with the usual distinction between strong and weak solutions to an SDE we give the following

Definition 1.18 *A strong stochastic transformation $T \in SS_n(M)$ is a strong (finite) symmetry of the SDE (μ, σ) if, for any solution (X, W) to (μ, σ), the transformed process $P_T(X, W) = (P_T(X), W)$ (Brownian motion is unchanged) is also a solution to (μ, σ). A stochastic transformation $T \in S_n(M)$ is called a weak (finite) symmetry of the SDE (μ, σ) if, for any solution (X, W) to (μ, σ), the generic transformed process $P_T(X, W) := (P_T(X), P_T(W))$ is also a solution to (μ, σ) (Brownian motion is changed).*

Theorem 1.19 *A strong stochastic transformation $T = (\Phi, B, \eta) \in SS_n(M)$ is a strong symmetry of an SDE (μ, σ) if and only if*

$$\begin{aligned} L(\Phi) \circ \Phi^{-1} &= \mu \qquad (1.14) \\ (\nabla(\Phi) \cdot \sigma) \circ \Phi^{-1} &= \sigma. \qquad (1.15) \end{aligned}$$

A stochastic transformation $T \in S_n(M)$ is a weak symmetry of an SDE (μ, σ) if and only if

$$\begin{aligned} \left(\frac{1}{\eta} L(\Phi)\right) \circ \Phi^{-1} &= \mu \qquad (1.16) \\ \left(\frac{1}{\sqrt{\eta}} \nabla(\Phi) \cdot \sigma \cdot B^{-1}\right) \circ \Phi^{-1} &= \sigma. \qquad (1.17) \end{aligned}$$

Proof. We prove the proposition for weak symmetries. The proof for strong symmetries is a subcase of the previous one.
If a stochastic transformation T satisfies equations (1.16) and (1.17), then $E_T(\mu, \sigma) = (\mu, \sigma)$. We have to prove that T is a symmetry of the SDE (μ, σ). Let (X, W) be a solution to (μ, σ): Theorem 1.12 ensures that $P_T(X, W)$ is a solution to $E_T(\mu, \sigma) = (\mu, \sigma)$.
Conversely, suppose that, for any solution (X, W) to (μ, σ), also $P_T(X, W)$ is a solution to (μ, σ). Since the coefficients (μ, σ) are smooth on M, for any $x_0 \in M$ there exists a solution (X^{x_0}, W^{x_0}) defined until the stopping time τ^{x_0} with $\mathbb{P}(\tau^{x_0} > 0) = 1$. Moreover, being $T \in S_n(M)$ a symmetry of the SDE (μ, σ), also $P_T(X^{x_0}, W^{x_0})$ is a solution to the SDE (μ, σ) and, by Theorem 1.12, $P_T(X^{x_0}, W^{x_0})$ is also a solution to $E_T(\mu, \sigma)$. Hence, since $P_T(X^{x_0}) = \Phi(x_0)$ almost surely, by Lemma 1.13 we have that $E_T(\mu)(\Phi(x_0)) = \mu(\Phi(x_0))$ and $E_T(\sigma)(\Phi(x_0)) = \sigma(\Phi(x_0))$. Since $x_0 \in M$ is a generic point and Φ is a diffeomorphism, we have $E_T(\mu, \sigma) = (\mu, \sigma)$. ∎

Definition 1.20 *An infinitesimal stochastic transformation V generating a one parameter group T_a is called a strong (or a weak) infinitesimal symmetry of the SDE (μ, σ) if T_a is a finite strong (or weak) symmetry of the SDE (μ, σ).*

The following theorem provides the *determining equations* for the infinitesimal symmetries of an SDE. They differ from those given in [77] and in [140] for the presence of the antisymmetric matrix C and the smooth function τ.
In order to avoid the use of many indices we introduce the following notation: if $A : M \to \mathbb{R}^m$ and $B : M \to \mathrm{Mat}(m,n)$, we denote by $[A,B]$ the smooth function $[A,B] : M \to \mathrm{Mat}(m,n)$ given by

$$[A,B]^i_j = A^k\partial_k(B^i_j) - B^k_j\partial_k(A^i),$$

where we use Einstein summation convention is used
The bracket $[\cdot,\cdot]$ satisfies the following properties:

$$[A,[C,B]] = [[A,C],B] + [C,[A,B]]$$

$$[A,B\cdot D] = [A,B]\cdot D + B\cdot A(D).$$

In the particular case $B : M \to \mathbb{R}^n$, the expression of $[A,B]$ coincides with the usual Lie bracket between the vector fields A,B.

Theorem 1.21 *An infinitesimal stochastic transformation $V = (Y,C,\tau)$ is an infinitesimal symmetry of the SDE (μ,σ) if and only if Y generates a one parameter group on M and*

$$\begin{aligned} Y(\mu) - L(Y) + \tau\mu &= 0 &(1.18)\\ [Y,\sigma] + \frac{1}{2}\tau\sigma + \sigma\cdot C &= 0. &(1.19)\end{aligned}$$

Proof. Let V be an infinitesimal symmetry of (μ,σ) and let $T_a = (\Phi_a, B_a, \eta_a)$ be the one-parameter group generated by V. By Theorem 1.19, we have that

$$\begin{aligned} \left(\frac{1}{\eta_a}L(\Phi_a)\right)\circ\Phi_{-a} &= \mu\\ \left(\frac{1}{\sqrt{\eta_a}}\nabla(\Phi_a)\cdot\sigma\cdot B_a^{-1}\right)\circ\Phi_{-a} &= \sigma.\end{aligned}$$

If we compute the derivatives with respect to a of the previous expressions and take $a = 0$ we obtain equations (1.18) and (1.19).
Conversely, suppose that equations (1.18) and (1.19) hold. If we define μ_a and σ_a as

$$\begin{aligned} \mu_a &= \left(\frac{1}{\eta_a}L(\Phi_a)\right)\circ\Phi_{-a} &(1.20)\\ \sigma_a &= \left(\frac{1}{\sqrt{\eta_a}}\nabla(\Phi_a)\cdot\sigma\cdot B_a^{-1}\right)\circ\Phi_{-a}, &(1.21)\end{aligned}$$

the functions μ_a, σ_a solve the following first order partial differential equations

$$\begin{aligned} \partial_a(\mu_a) &= -[Y,\mu_a] + A(\sigma_a,Y) - \tau\mu_a &(1.22)\\ \partial_a(\sigma_a) &= -[Y,\sigma_a] - \frac{1}{2}\tau\sigma_a - \sigma_a\cdot C, &(1.23)\end{aligned}$$

where

$$A(\sigma_a,Y)^i = \sum_\alpha \sigma^k_{\alpha,a}\sigma^h_{\alpha,a}\partial_{hk}(Y^i).$$

If we consider $\tilde{\sigma}_a = \sigma_a \circ \Phi_a$ and $\tilde{\mu}_a = \mu_a \circ \Phi_a$, equations (1.22) and (1.23) become

$$\begin{aligned}
\partial_a(\tilde{\mu}_a) &= \tilde{\mu}_a(Y) + A(\tilde{\sigma}_a, Y) - \tau\tilde{\mu}_a \\
\partial_a(\tilde{\sigma}_a) &= \sigma_a(Y) - \frac{1}{2}\tau\tilde{\sigma}_a - \tilde{\sigma}_a \cdot C
\end{aligned}$$

that are, for x fixed, ordinary differential equations in a admitting a unique solution for any initial condition (μ_0, σ_0). As a consequence, when

$$\begin{aligned}
[Y, \mu_0] - A(\sigma_0, Y) + \tau\mu_0 &= 0 \\
[Y, \sigma_0] + \tfrac{1}{2}\tau\sigma_0 + \sigma_0 \cdot C &= 0,
\end{aligned} \tag{1.24}$$

we have $\sigma_a = \sigma_0$ and $\mu_a = \mu_0$ for any a and (1.20) and (1.21) ensure that T_a is a symmetry of (μ, σ). ∎

Definition 1.22 *An infinitesimal stochastic transformation $V \in V_n(M)$ is a general infinitesimal symmetry of the SDE (μ, σ) if it satisfies the determining equations (1.18) and (1.19).*

In order to prove that the Lie bracket of two general infinitesimal symmetries of an SDE is a general infinitesimal symmetry of the same SDE we need the following technical lemma:

Lemma 1.23 *Given a general infinitesimal symmetry (Y, C, τ) of the SDE (μ, σ), for any smooth function $f \in C^\infty(M)$ we have*

$$Y(L(f)) - L(Y(f)) = -\tau L(f),$$

where L is the second order differential operator associated with (μ, σ).

Proof. Given $Y = Y^i\partial_i$ and $L = A^{ij}\partial_{ij} + \mu^i\partial_i$, we can write

$$\begin{aligned}
Y(L(f)) - L(Y(f)) &= Y^i\partial_i(A^{jk}\partial_{jk}(f) + \mu^j\partial_j(f)) \\
&\quad -(A^{jk}\partial_{jk}(Y^i\partial_i(f)) + \mu^j\partial_j(Y^i\partial_i(f))) \\
&= (Y^i\partial_i(A^{jk}) - A^{ik}\partial_i(Y^j) - A^{ji}\partial_i(Y^k))\partial_{jk}(f) \\
&\quad +(Y^i\partial_i(\mu^j) - A^{ik}\partial_{ik}(Y^j) - \mu^i\partial_i(Y^j))\partial_j(f)
\end{aligned}$$

and the thesis of the lemma reads

$$(Y^i\partial_i(A^{jk}) - A^{ik}\partial_i(Y^j) - A^{ji}\partial_i(Y^k)) = -\tau(A^{jk}) \tag{1.25}$$

$$(Y^i\partial_i(\mu^j) - A^{ik}\partial_{ik}(Y^j) - \mu^i\partial_i(Y^j)) = -\tau\mu^j. \tag{1.26}$$

Equation (1.26) can be written in the following way

$$Y(\mu) - L(Y) = -\tau\mu$$

and, denoting by A the symmetric matrix of component A^{ij}, equation (1.25) can be rewritten as follows

$$Y(A) - \nabla(Y) \cdot A - A \cdot \nabla(Y)^T = -\tau A. \tag{1.27}$$

Since (Y, C, τ) is a symmetry of the SDE (μ, σ), by equation (1.19) we have

$$\begin{aligned}
[Y, \sigma] \cdot \sigma^T &= -\tfrac{1}{2}\tau\sigma \cdot \sigma^T - \sigma \cdot C \cdot \sigma^T \\
&= -\tau A - \sigma \cdot C \cdot \sigma^T.
\end{aligned} \tag{1.28}$$

If we sum equation (1.28) with its transposed, being C an antisymmetric matrix, we obtain

$$\begin{array}{rcl}[Y,\sigma]\cdot\sigma^T+([Y,\sigma]\cdot\sigma^T)^T & = & -2\tau A-\sigma\cdot C\cdot\sigma^T-\sigma\cdot C^T\cdot\sigma \\ & = & -2\tau A.\end{array} \quad (1.29)$$

Furthermore, since for any function F

$$[Y,F]=Y(F)-\nabla(Y)\cdot F,$$

we have that

$$\begin{array}{rcl}[Y,\sigma]\cdot\sigma^T+([Y,\sigma]\cdot\sigma^T)^T & = & -2(\nabla(Y)\cdot A)+Y(\sigma)\cdot\sigma^T+ \\ & & -2(\nabla(Y)\cdot A)^T+\sigma\cdot Y(\sigma)^T \\ & = & 2(Y(A)-\nabla(Y)\cdot A-A\cdot\nabla(Y)^T).\end{array} \quad (1.30)$$

Using equations (1.29) and (1.30) we obtain (1.27). ∎

Proposition 1.24 *Let $V_1=(Y_1,C_1,\tau_1),V_2=(Y_2,C_2,\tau_2)\in V_n(M)$ be two general infinitesimal symmetries of the SDE (μ,σ), then $[V_1,V_2]$ is a general infinitesimal symmetry of (μ,σ).*

Proof. We start by proving that condition (1.18) holds for $[V_1,V_2]$ defined by equation (1.6), i.e.

$$[Y_1,Y_2](\mu)-L([Y_1,Y_2])+(Y_1(\tau_2)-Y_2(\tau_1))\mu=0$$

If we rewrite the left-hand side of the previous equation as

$$\begin{array}{rl} & Y_1Y_2(\mu)-Y_2Y_1(\mu)-L([Y_1,Y_2])+Y_1(\tau_2\mu)-\tau_2Y_1(\mu)-Y_2(\tau_1\mu)+\tau_1Y_2(\mu) \\ = & Y_1(Y_2(\mu)+\tau_2\mu)-Y_2(Y_1(\mu)+\tau_1\mu)-L([Y_1,Y_2])-\tau_2Y_1(\mu)+\tau_1Y_2(\mu) \\ = & Y_1(L(Y_2))-Y_2(L(Y_1))-L([Y_1,Y_2])-\tau_2Y_1(\mu)+\tau_1Y_2(\mu)\end{array}$$

and we use Lemma 1.23, we get

$$\begin{array}{rl} & [Y_1,Y_2](\mu)-L([Y_1,Y_2])+(Y_1(\tau_2)-Y_2(\tau_1))\mu \\ & \tau_1(Y_2(\mu)-L(Y_2))-\tau_2(Y_1(\mu)-L(Y_1)) \\ = & -\tau_1\tau_2\mu+\tau_2\tau_1\mu=0.\end{array}$$

Moreover we have to prove that also condition (1.19) holds for $[V_1,V_2]$, i.e.

$$[[Y_1,Y_2],\sigma]+\frac{1}{2}(Y_1(\tau_2)-Y_2(\tau_1))\sigma-\sigma\cdot\{C_1,C_2\}+\sigma\cdot Y_1(C_2)+\sigma\cdot Y_2(C_1)=0.$$

By using the properties of the Lie bracket we have

$$\begin{array}{rcl}[[Y_1,Y_2],\sigma] & = & [Y_1,[Y_2,\sigma]]-[Y_2,[Y_1,\sigma]] \\ & = & -[Y_1,\frac{1}{2}\tau_2\sigma+\sigma\cdot C_2]+[Y_2,\frac{1}{2}\tau_1\sigma+\sigma\cdot C_1] \\ & = & -\frac{1}{2}Y_1(\tau_2)\sigma-\frac{1}{2}\tau_2[Y_1,\sigma]-[Y_1,\sigma]\cdot C_2-\sigma\cdot Y_1(C_2) \\ & & +\frac{1}{2}Y_2(\tau_1)\sigma+\frac{1}{2}\tau_1[Y_2,\sigma]+[Y_2,\sigma]\cdot C_1+\sigma\cdot Y_2(C_1) \\ & = & -\frac{1}{2}(Y_1(\tau_2)-Y_2(\tau_1))\sigma+\sigma\cdot\{C_1,C_2\}-\sigma\cdot Y_1(C_2)+\sigma\cdot Y_2(C_1),\end{array}$$

and this concludes the proof. ∎

Proposition 1.25 *Let $V \in V_n(M)$ be an infinitesimal symmetry of the SDE (μ, σ) and let $T \in S_n(M, M')$ be a stochastic transformation. Then $T_*(V)$ is an infinitesimal symmetry of $E_T(\mu, \sigma)$.*

Proof. Given a solution (X', W') to $E_T(\mu, \sigma)$, Theorem 1.19 ensures that $(X, W) = P_T^{-1}(X', W')$ is a solution to (μ, σ). If T_a denotes the one-parameter group generated by the infinitesimal symmetry V then $P_{T_a}(X, W)$ is a solution to (μ, σ). By Theorem 1.19, $P_T(P_{T_a}(X, W))$ is a solution to the SDE $E_T(\mu, \sigma)$ and, by Theorem 1.17, for any (X', W') solution to $E_T(\mu, \sigma)$, the process $P_{(T \circ T_a \circ T^{-1})}(X', W')$ is a solution to $E_T(\mu, \sigma)$. Since the generator of $T \circ T_a \circ T^{-1}$ is $T_*(V)$ we conclude that $T_*(V)$ is an infinitesimal symmetry of $E_T(\mu, \sigma)$. ■

Theorem 1.26 *Let $V_1 = (Y_1, C_1, \tau_1), ..., V_k = (Y_k, C_k, \tau_k)$ be general infinitesimal symmetries of (μ, σ). If $x_0 \in M$ is such that $Y_1(x_0), ..., Y_k(x_0)$ are linearly independent, then there exist a neighborhood U of x_0 and a stochastic transformation $T \in S_n(U, U')$ such that $T_*(V_i)$ are strong infinitesimal symmetries of $E_T(\mu, \sigma)$.*

Proof. The theorem is an application of Theorem 1.16 and Proposition 1.25. ■

1.4 Symmetries of Stratonovich SDEs on manifolds

The aim of this section is the study of symmetries of SDEs defined on differentiable manifolds. There are two natural approaches to the definition of an SDE on a smooth manifold M. The first one is based on second order geometry and Itô integration on manifolds introduced by Meyer and Schwartz (see [141, 159]) and successively developed by Emery (see [62]). The second possibility consists in using Stratonovich differential equations to define an SDE on a smooth manifold (see [60, 102, 105]). We prefer to use the approach based on Stratonovich SDEs, since it is more similar to the one based on $\mathbb{R}^m$ Itô SDEs developed in the previous sections. A definition of symmetry based on second order geometry and Itô integration on manifolds has been proposed in [53].
Since we consider Stratonovich SDEs we start making a natural comparison of our notion of symmetry with the one proposed in [78, 76]. In particular, in [78], the notion of random symmetry is studied separately for Itô and Stratonovich SDEs, producing two (apparently) different sets of determining equations and hence giving different symmetries for the Stratonovich and Itô formulation of the same SDE. In our approach this duality does not appear. Indeed, in Theorem 1.12 we proved that the action E_T of a stochastic transformation T on an SDE is completely determined by the action P_T of T on the process (X, W). Therefore, the definition of symmetry of an SDE involves only the transformations on the process and only in a derived way the transformation of the SDE. These intuitive idea suggests that the concept of symmetry based on stochastic transformations and their action on the pairs of processes does not depend on the kind of integration used in defining the SDEs. In the following, in order to prove this general conjecture, we develop the idea of symmetries of a Stratonovich SDE.

Given $2r$ semimartingales on $\mathbb{R}$ $S^1, ..., S^r$ and $H_1, ..., H_r$ we define the Stratonovich integral I^S of H_α along S^α by the following expression

$$I^S_t = \int_0^t H_{\alpha,s} \circ dS^\alpha_s := \int_0^t H_{\alpha,s} dS^\alpha + \frac{1}{2}[H_\alpha, S^\alpha]_t,$$

where the integral $\int_0^t H_{\alpha,s} dS^\alpha$ is the usual Itô integral. The Stratonovich integral has the following important change rule property.

Proposition 1.27 *Let $X^i_t - X^i_0 = \int_0^t H^i_{\alpha,s} \circ dS^\alpha_s$ (where $i = 1, ..., m$) be the Stratonovich integral of H^i_α along S^α. If $f : \mathbb{R}^m \to \mathbb{R}$ is a C^2 function we have*

$$f(X_t) - f(X_0) = \int_0^t \partial_i(f)(X_s) H^i_{\alpha,s} \circ dS^\alpha_s. \tag{1.31}$$

Proof. The proof is an easy consequence of Itô formula for continuous semimartingales and of definition of Stratonovich integral. ■

The introduction of Stratonovich integral allows us to define SDEs on a smooth manifold. First we introduce the concept of semimartingale on a manifold M.

Definition 1.28 *A stochastic process X on the manifold M is a semimartingale on M if, for any smooth function $f \in C^\infty(M)$, the real process $f(X)$ is a semimartingale.*

Definition 1.29 *Given $n+1$ vector fields on M $\tilde{\mu}$, $\tilde{\sigma}_1, ..., \tilde{\sigma}_n$, we say that a pair (X, W), where X is a semimartingale on M and W is an n dimensional Brownian motion, is a solution to the SDE defined by $(\tilde{\mu}, \tilde{\sigma}_\alpha)$ (or simply the SDE $(\tilde{\mu}, \tilde{\sigma}_\alpha)$) if, for any smooth function $f \in C^\infty(M)$, we have*

$$f(X_t) - f(X_0) = \int_0^t \tilde{\mu}(f)(X_s) ds + \int_0^t \tilde{\sigma}_\alpha(f)(X_s) \circ dW^\alpha_s. \tag{1.32}$$

Remark 1.30 *When $M \subset \mathbb{R}^m$ we can compare Definition 1.29 with Definition 1.1. Supposing that (X, W) is a solution to the Itô SDE (μ, σ) according with Definition 1.1 and defining $\tilde{\mu}, \tilde{\sigma}_\alpha$ as the vector fields such that*

$$\tilde{\mu} = \left(\mu^i - \frac{1}{2} \sum_{\alpha=1}^n \sigma^j_\alpha \partial_j(\sigma^i)_\alpha \right) \partial_{x^i} \tag{1.33}$$

$$\tilde{\sigma}_\alpha = \sigma^i_\alpha \partial_{x^i}, \tag{1.34}$$

we have that (X, W) is a solution to the Stratonovich SDE $(\tilde{\mu}, \tilde{\sigma}_\alpha)$ according with Definition 1.29. Conversely, if (X, W) is a solution to the Stratonovich SDE $(\tilde{\mu}, \tilde{\sigma}_\alpha)$, then (X, W) is a solution to the Itô SDE (μ, σ). If (μ, σ) is an Itô SDE we denote by $IS(\mu, \sigma) = (\tilde{\mu}, \tilde{\sigma}_\alpha)$ and by $SI(\tilde{\mu}, \tilde{\sigma}_\alpha) = (\mu, \sigma)$ the Stratonovich and Itô SDEs related by equation (1.33) and (1.34).

In Stratonovich SDEs the generator assumes the following simple form

$$L(f) = \tilde{\mu}(f) + \frac{1}{2} \sum_\alpha \tilde{\sigma}_\alpha(\tilde{\sigma}_\alpha(f)).$$

In this setting we can define the group of stochastic transformations $S_n(M, M')$ of the form $T = (\Phi, B, \eta)$ from the manifold M into the manifold M' in the natural way by taking $\Phi : M \to M'$ a diffeomorphism between M and M' and $B : M \to O(n)$, $\eta : M \to \mathbb{R}_+$ smooth functions. On this set we can define the composition and the Lie algebra of infinitesimal stochastic transformations $V_n(M)$ as in the previous sections. The action P_T in Definition 1.10 can be extended to the the present case where (X, W) is a pair formed by a semimartingale on the manifold M and by an n dimensional Brownian motion W, using directly equations (1.1), (1.2) and (1.3).

On the other hand the action E^S_T of the transformation T on the Stratonovich SDEs $(\tilde{\mu}, \tilde{\sigma})$ is definitely different from the previous case.

Definition 1.31 *If $T=(\Phi,B,\eta)$ is a stochastic transformation from the manifold M into the manifold M', we define the action of T on the equation $(\tilde{\mu},\tilde{\sigma}_\alpha)$ as $(\tilde{\mu}',\tilde{\sigma}'_\alpha)=E^S_T(\tilde{\mu},\tilde{\sigma}_\alpha)$ where*

$$\tilde{\sigma}'_\alpha=\Phi_*\left(\frac{1}{\sqrt{\eta}}B^{-1,\beta}_\alpha\tilde{\sigma}_\beta\right) \tag{1.35}$$

$$\tilde{\mu}'=\Phi_*\left(\frac{1}{\eta}\tilde{\mu}+\sum_{\alpha=1}^n\frac{1}{2\eta^2}\tilde{\sigma}_\alpha(\eta)-\sum_{\alpha=1}^n\tilde{\sigma}_\alpha(B^{-1,\gamma}_\beta)B^\beta_\alpha\tilde{\sigma}_\gamma\right), \tag{1.36}$$

where Φ_* denotes the push-forward of functions and vector fields.

In Definition 1.31 we have used the definition of push-forward that has local expression

$$\Phi_*(Y)=(\nabla\Phi\cdot Y)\circ\Phi^{-1}.$$

Before proposing an analogous of Theorem 1.12, we prove the following result.

Lemma 1.32 *Suppose that two Stratonovich SDEs $(\tilde{\mu},\tilde{\sigma}_\alpha)$ and $(\tilde{\mu}',\tilde{\sigma}'_\alpha)$ admit a family of common solutions of the form (X^{x_0},W^{x_0}) for any $x_0\in M$ and $X^{x_0}_0=x_0$ almost surely. Then $(\tilde{\mu},\tilde{\sigma}_\alpha)=(\tilde{\mu}',\tilde{\sigma}'_\alpha)$.*

Proof. Possibly stopping the solution X^{x_0} with a stopping time $\tau_{x_0}>0$, we can assume that, for any $x_0\in U$, $X^{x_0}_t\in U$, where $U\subset M$ is an open subset of M diffeomorphic to $\mathbb{R}^m$. For this reason we can suppose, without loss of generality, that $M=U=\mathbb{R}^m$.
Applying Lemma 1.13 to the Itô SDEs $SI(\tilde{\mu},\tilde{\sigma}_\alpha)$ and $SI(\tilde{\mu}',\tilde{\sigma}'_\alpha)$ we get $SI(\tilde{\mu},\tilde{\sigma}_\alpha)=SI(\tilde{\mu}',\tilde{\sigma}'_\alpha)$, and the application of the operator IS to the previous Itô SDEs completes the proof. ■

Theorem 1.33 *Let T be a stochastic transformation from M into M' and let $(\tilde{\mu},\tilde{\sigma}_\alpha)$ be a Stratonovich SDE on M. The action E^S_T is the only smooth action of T on the set of Stratonovich SDEs such that, for any (X,W) solution to the SDE $(\tilde{\mu},\tilde{\sigma}_\alpha)$, then $P_T(X,W)$ is solution to the SDE $E^S_T(\tilde{\mu},\tilde{\sigma}_\alpha)$.*

Proof. We give the proof for the case $T=(\Phi,I,1)$, $T=(id_M,B,1)$ and $T=(id_M,I,\eta)$. The general case can be obtained combining these three subcases.
Considering $T=(\Phi,I,1)$, by Definition 1.29, for any $f\in C^\infty(M')$

$$\begin{aligned}
f(X'_0)-f(X'_0) &= f(\Phi(X_t))-f(\Phi(X_0))=\Phi^*(f)(X_t)-\Phi^*(f)(X_0)\\
&= \int_0^t\tilde{\mu}(\Phi^*(f))(X_s)ds+\int_0^t\tilde{\sigma}_\alpha(\Phi^*(f))(X_s)\circ dW^\alpha_s\\
&= \int_0^t\Phi^*(\Phi_*(\tilde{\mu})(f))(X_s)ds+\int_0^t\Phi^*(\Phi_*(\tilde{\sigma}_\alpha)(f))(X_s)\circ dW^\alpha_s\\
&= \int_0^t\Phi_*(\tilde{\mu})(f)(X'_s)ds+\int_0^t\Phi_*(\tilde{\sigma}_\alpha)(f)(X'_s)\circ dW^\alpha_s.
\end{aligned}$$

Thus $(\Phi(X),W)=P_T(X,W)$ is a solution to the SDE $(\Phi_*(\tilde{\mu}),\Phi_*(\tilde{\sigma}_\alpha))$.
Taking $T=(id_M,B,1)$, then

$$\begin{aligned}
dW^\alpha_t &= B^{-1,\alpha}_\beta(X_t)dW'^\alpha_t\\
&= B^{-1,\alpha}_\beta(X_t)\circ dW'^\beta_t-\frac{1}{2}[B^{-1,\alpha}_\beta(X),W'^\beta]\\
&= B^{-1,\alpha}_\beta(X_t)\circ dW'^\beta_t-\frac{1}{2}\tilde{\sigma}_\gamma(B^{-1,\alpha}_\beta)(X_t)B^\beta_\gamma(X_t)dt,
\end{aligned}$$

where $W' = P_T(W)$, and we have used that

$$\begin{aligned} dB_{\beta}^{-1,\alpha}(X_t) &= \tilde{\mu}(B_{\beta}^{-1,\alpha})(X_t)dt + \tilde{\sigma}_{\gamma}(B_{\beta}^{-1,\alpha})(X_t) \circ dW^{\gamma} \\ &= L(B_{\beta}^{-1,\alpha})(X_t)dt + \tilde{\sigma}_{\gamma}(B_{\beta}^{-1,\alpha})(X_t)dW_t^{\gamma}. \end{aligned}$$

Replacing the expression for $dW_t'^{\alpha}$ in equation (1.32) for any fixed f and using the associativity of Stratonovich integral we obtain the thesis in this situation.
The case $T = (id_M, I, \eta)$ is similar to the previous one.
The fact that E_T is the unique action with the required property can be proved using Lemma 1.32, and following the same line of the proof of Theorem 1.12. ■

Corollary 1.34 *In the hypothesis of Theorem 1.33 we have that* $E_T^S(IS(\mu,\sigma)) = IS(E_T(\mu,\sigma))$, $E_T(IS(\tilde{\mu},\tilde{\sigma}_{\alpha})) = SI(E_T^S(\mu,\sigma))$. *Furthermore if* T', T *are two stochastic transformations, then* $E_{T'}^S \circ E_T^S = E_{T'\circ T}^S$.

Proof. The proof is essentially based on the uniqueness of the action E_T^S given in Theorem 1.33. ■

A stochastic transformation T is a *symmetry of a Stratonovich SDE* $(\tilde{\mu},\tilde{\sigma}_{\alpha})$ if any solution (X,W) to $(\tilde{\mu},\tilde{\sigma}_{\alpha})$ is transformed by $P_T(X,W)$ into an other solution to the SDE $(\tilde{\mu},\tilde{\sigma}_{\alpha})$. An infinitesimal stochastic transformation V is a symmetry of the SDE $(\tilde{\mu},\tilde{\sigma}_{\alpha})$ if it generates a one-parameter group T_a of symmetries of $(\tilde{\mu},\tilde{\sigma}_{\alpha})$.

Finally we provide the determining equations for infinitesimal symmetries of an SDE formulated in terms of Stratonovich integral.

Theorem 1.35 *A stochastic transformation* T *is a symmetry of the Stratonovich SDE* $(\tilde{\mu},\tilde{\sigma}_{\alpha})$ *if and only if* $E_T^S(\tilde{\mu},\tilde{\sigma}_{\alpha}) = (\tilde{\mu},\tilde{\sigma}_{\alpha})$. *The infinitesimal stochastic transformation* $V = (Y, C, \tau)$ *is an infinitesimal stochastic symmetry of* $(\tilde{\mu},\tilde{\sigma}_{\alpha})$ *if and only if the following equations hold*

$$[Y,\tilde{\sigma}_{\alpha}] = -C_{\alpha}^{\beta}\tilde{\sigma}_{\beta} - \frac{\tau}{2}\tilde{\sigma}_{\alpha} \tag{1.37}$$

$$[Y,\tilde{\mu}] = -\tau\tilde{\mu} + \sum_{\alpha=1}^{m}\left(\frac{\tilde{\sigma}_{\alpha}(\tau)}{2}\tilde{\sigma}_{\alpha} + \tilde{\sigma}_{\alpha}(C_{\alpha}^{\beta})\tilde{\sigma}_{\beta}\right). \tag{1.38}$$

Proof. The proof is analogous to the proof of Theorem 1.19, where Lemma 1.13 is replaced by Lemma 1.32. ■

Corollary 1.36 *A stochastic transformation* T *is a symmetry of an Itô SDE* (μ,σ) *if and only if* T *is a symmetry of the Stratonovich SDE* $IS(\mu,\sigma)$.

Proof. The transformation T is a symmetry of the SDE (μ,σ) if and only if $E_T(\mu,\sigma) = (\mu,\sigma)$. Applying the operator IS to the previous equality and using Corollary 1.34 we obtain the thesis. ■

Chapter 2

Reduction and reconstruction of SDEs and applications

In this chapter we generalize the well known theorems of reduction and reconstruction by quadratures of symmetric differential equations from the deterministic to the stochastic setting. After introducing some geometrical tools necessary for our discussion, we propose these generalizations using the notion of stochastic transformations and the related concept of symmetry of an SDE introduced in Chapter 1. This chapter includes many examples of both theoretical and practical interest.

2.1 Some geometric preliminaries

In this section we recall some geometric preliminaries needed in the following. In particular in Section 2.1.1 we introduce a class of foliations which turns out to be useful in order to reduce SDEs, whereas in Section 2.1.2 we describe adapted coordinate systems for solvable Lie algebras of vector fields which are exploited in the reconstruction process.
We use all the conventions introduced in Chapter 1. Furthermore if $\Psi : M \to \mathbb{R}^h$ is a smooth map and $Y \in TM$ is a vector field we define the push forward of Y by Ψ by

$$\Psi_*(Y) = \nabla(\Psi) \cdot Y.$$

It is important to note that we do not require that Ψ is a diffeomorphism. If Ψ is a diffeomorphism the push-forward is differently defined since $\Psi_*(Y) = (\nabla(\Psi) \cdot Y) \circ \Psi^{-1}$.

2.1.1 Foliations and projections

Let $Y_1, ..., Y_k$ be a set of vector fields on M such that the distribution $\Delta = \mathrm{span}\{Y_1, ..., Y_k\}$ is of constant rank r. If Δ is integrable, i.e. $[Y_i, Y_j] \in \Delta$ for every $i, j = 1, ..., k$, then Δ defines a foliation on M. Moreover, if there is a submersion $\Psi : M \to M'$, where (possibly restricting M) M' is an open subset of $\mathbb{R}^{m-r}$ such that

$$\Delta = \ker(\nabla(\Psi))$$

and the level sets of Ψ are connected subsets of M, the foliation defined by Δ can be used for reduction purposes. In fact, under these assumptions, Ψ is a surjective submersion and the level sets of Ψ are connected closed submanifolds of M.

Definition 2.1 *A surjective submersion $\Psi : M \to M'$ is a* reduction map *if Ψ has connected level sets. The vector fields $Y_1, ..., Y_k$ generating an integrable distribution Δ of constant rank r are* reduction vector fields *for the reduction map $\Psi : M \to M'$ if*

$$\mathrm{span}\{Y_1(x), ..., Y_k(x)\} = \ker \nabla(\Psi)(x) \qquad \forall x \in M.$$

We remark that, if $Y_1, ..., Y_k$ are reduction vector fields for the reduction map Ψ, then (M, Ψ, M') is a fibred manifold.

Definition 2.2 *A set of vector fields $Y_1, ..., Y_r$ on M is* regular *on M if, for any $x \in M$, the vectors $Y_1(x), ..., Y_r(x)$ are linearly independent.*

A set of vector fields $Y_1, ..., Y_k$ generating an integrable distribution Δ of constant rank does not admit in general a global reduction map, but the following well known local result holds.

Proposition 2.3 (Frobenius theorem) *Let $Y_1, ..., Y_k$ be a set of vector fields generating a regular integrable distribution Δ of constant rank r. Then, for any $x \in M$, there exist a neighborhood U of x and a reduction map $\Psi : U \to U' \subset \mathbb{R}^{m-r}$ such that $Y_1, ..., Y_k$ are reduction vector fields for Ψ.*

We remark that the classical reduction of a manifold under a Lie group action is included in Definition 2.1. Indeed, given a connected Lie group G acting on M, we can naturally define an equivalence relation and the quotient manifold $M' = M/G := M/\sim$. If the action of G is proper and free, M' admits a natural structure of $(m-r)$-dimensional manifold (see [147]) and the natural projection $\Pi : M \to M'$ is a submersion. Moreover, if G is connected and $\{Y_1, ..., Y_r\}$ are the generators of the corresponding Lie algebra, Π is a reduction map and

$$\mathrm{span}\{Y_1(x), ..., Y_r(x)\} = \ker(\nabla(\Pi)(x)) \qquad \forall x \in M.$$

In this case (M, Ψ, M') is not only a fibred manifold but also a principal bundle with structure group G.

Proposition 2.4 *Let $\Psi : M \to M'$ be a reduction map and suppose that the vector fields $\{Y_1, ..., Y_k\}$ are reduction vector fields for Ψ. If M is connected, for any function $f \in C^\infty(M)$ such that $Y_i(f) = 0$ there exists a unique function $f' \in C^\infty(M')$ such that*

$$f = f' \circ \Psi.$$

Moreover, if $\mathcal{G} = \mathrm{span}\{Y_1, ..., Y_k\}$ and Y is a vector field on M such that

$$[Y, \mathcal{G}] \subset \mathcal{G},$$

there exists an unique vector field Y' such that

$$\Psi_*(Y) = \nabla(\Psi) \cdot Y = Y' \circ \Psi.$$

Proof. See Chapter 4 of [145]. ∎

2.1.2 Solvable algebras and adapted coordinate systems

For later use, in this section we discuss the local existence of a suitable adapted coordinate system on M such that the generators $Y_1, ..., Y_r$ of a solvable Lie algebra have a special form.

Definition 2.5 *Let $Y_1, ..., Y_r$ be a set of regular vector fields on M which are generators of a solvable Lie algebra $\mathcal{G}$. We say that $Y_1, ..., Y_r$ are in* canonical form *if there are $i_1, ..., i_l$ such that $i_1 + ... + i_l = r$ and*

$$(Y_1|...|Y_r) = \left(\begin{array}{c|c|c|c} I_{i_1} & G_1^1(x) & ... & G_l^1(x) \\ \hline 0 & I_{i_2} & ... & G_l^2(x) \\ \hline \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & ... & I_{i_l} \\ \hline 0 & 0 & 0 & 0 \end{array} \right),$$

where $G_k^h : M \to \mathrm{Mat}(i_h, i_k)$ are smooth functions.

Theorem 2.6 *Let $\mathcal{G}$ be an r-dimensional solvable Lie algebra on M such that $\mathcal{G}$ has constant dimension r as a distribution of TM and let Ψ be a reduction map for $\mathcal{G}$. Then, for any $x_0 \in M$, there is a set of generators $Y_1, ..., Y_r$ of $\mathcal{G}$ and a local diffeomorphism $\Phi : U(x_0) \to \tilde{M}$ of the form*

$$\Phi = \begin{pmatrix} \tilde{\Phi} \\ \Psi \end{pmatrix}, \tag{2.1}$$

such that $\Phi_*(Y_1), ..., \Phi_*(Y_r)$ are generators in canonical form for $\Phi_*(\mathcal{G})$.

Proof. Since $\mathcal{G}$ is solvable, denoting by $\mathcal{G}^{(0)} = \mathcal{G}$ and $\mathcal{G}^{(i+1)} = [\mathcal{G}^{(i)}, \mathcal{G}^{(i)}]$, there exists $l \geq 0$ such that $\mathcal{G}^{(l)} \neq 0$ and $\mathcal{G}^{(l+1)} = \{0\}$. Let $Y_1, ..., Y_{i_1}$ be the generators of $\mathcal{G}^{(l)}$, $Y_{i_1+1}, ..., Y_{i_1+i_2}$ be the generators of $\mathcal{G}^{(l-1)} \setminus \mathcal{G}^{(l)}$ and, in general, $Y_{i_1+...+i_{k-1}+1}, ..., Y_{i_1+...+i_k}$ be the generators of $\mathcal{G}^{(l-k+1)} \setminus \mathcal{G}^{(l-k)}$. Since (M, Ψ, M') is a fibred manifold, for any $x_0 \in M$ we can consider a local smooth section $S : V(\Psi(x_0)) \to M$ defined in $V(\Psi(x_0))$ and we can construct a local diffeomorphism on $W \times V$ (where $W \subset \mathbb{R}^r$) transporting $S(x_0)$ along the flows $\Phi^i_{a_i}$ of the vector fields Y_i. In particular, considering the function $F : W \times V \to M$ (where W is a neighborhood of 0 in $\mathbb{R}^r$) defined by

$$F(a_1, ..., a_r, x'^1, ...x'^{n-r}) = \Phi^1_{a_1}(...(\Phi^r_{a_r}(S(x'^1, ..., x'^r)))...)$$

we can define $\Phi = F^{-1}$. Indeed it is easy to prove that F is a local diffeomorphism since $Y_1, ..., Y_r$ form a regular set of vector fields, S is a local section of the foliation (M, Ψ, M') and Ψ is a reduction function for $Y_1, ..., Y_r$. Furthermore, since F is obtained by composing the flows of $Y_1, ..., Y_r$ in the natural order (i.e. respecting the solvable structure of $\mathcal{G}$), it is easy to prove that $\Phi_*(Y_1), ..., \Phi_*(Y_r)$ are in canonical form.

■

Remark 2.7 *In the particular case of a solvable connected Lie group G acting freely and regularly on M, Theorem 2.6 admits a global version. Indeed, under these hypotheses, Φ can be defined in an open set U of the form $U = \Psi^{-1}(V)$ where V is an open set of M' and (M, Ψ, M') turns out to be a principal bundle with structure group G. So for a neighborhood V of $\Psi(x_0)$ the set $U = \Psi^{-1}(V)$ is diffeomorphic to $V \times G$ and the generators $Y_1, ..., Y_r$ of G are vertical vector fields with respect to the bundle structure of M. Furthermore, it is possible to choose a global coordinate system $g^1, ..., g^r$ on G such that $Y_1, ..., Y_r$ are in canonical form (see for example [169], Chapter 2, Section 3.1, Corollary 1) and equation* (2.1) *is given by $\Phi = (g^1, ..., g^r, \Psi^1, ..., \Psi^{n-r})^T$. Obviously, if (M, Ψ, M') is a trivial bundle, the diffeomorphism Φ of Theorem 2.6 can be defined globally.*

2.2 Reduction and reconstruction procedures

In this section we propose a generalization of some well known results of symmetry reduction for ODEs to the stochastic framework. Moreover we provide suitable conditions for a symmetry to be inherited by the reduced equation and we tackle the problem of the reconstruction of the solution to the original SDE starting from the knowledge of the solution to the reduced one.

2.2.1 Reduction

For later use, we start by introduce the following definition.

Definition 2.8 *A stochastic transformation $T \in S_n(M, M'')$ of the for $T = (\Phi, B, 1)$ is a quasi-strong stochastic transformation. An infinitesimal stochastic transformation of the form $V = (Y, C, 0)$ is a quasi-strong stochastic transformation.*

Theorem 2.9 *Let $\{Y_1, ..., Y_k\}$ be a set of reduction vector fields for the reduction map $\Psi : M \to M'$ such that $(Y_1, C_1, 0), ..., (Y_k, C_k, 0)$ are quasi-strong symmetries of the SDE (μ, σ). If $\nabla(\Psi)\cdot\sigma\cdot C_i = 0$ ($\forall i = 1, ..., k$), there exists an unique SDE (μ', σ') on M' such that*

$$
\begin{aligned}
L(\Psi) &= \mu' \circ \Psi \\
\nabla(\Psi) \cdot \sigma &= \sigma' \circ \Psi.
\end{aligned}
$$

Furthermore if (X, W) is a solution to (μ, σ), then $(\Psi(X), W)$ is a solution to (μ', σ').

Proof. Since we are considering quasi-strong symmetries, Lemma 1.23 ensures that $Y_i(L(\Psi)) = L(Y_i(\Psi))$ and, being $Y_i \in \ker \nabla(\Psi)$, we have $Y_i(L(\Psi)) = 0$. Hence Proposition 2.4 guarantees the existence of a function μ' such that $L(\Psi) = \mu' \circ \Psi$.
Moreover, in the case of quasi-strong symmetries, the determining equations (1.19) reduces to $[Y_i, \sigma] = -\sigma \cdot C_i$ and the hypothesis $\nabla(\Psi) \cdot \sigma \cdot C_i = 0$ ensures that $[Y_i, \sigma] \in \ker \nabla(\Psi)$. Hence, denoting by σ_α the α column of σ, we have

$$[Y_i, \sigma_\alpha] \in \text{span}\{Y_1, ..., Y_k\}.$$

Therefore, by Proposition 2.4, there exists an unique vector field σ'_α on M' such that

$$\sigma'_\alpha \circ \Psi = \nabla(\Psi) \cdot \sigma_\alpha$$

and, considering the matrix-valued function σ' with columns σ'_α, the theorem is proved. ■

Remark 2.10 *If $Y_1, ..., Y_k$ are strong symmetries of the SDE (μ, σ), conditions $\nabla(\Psi)\cdot\sigma\cdot C_i = 0$ of Theorem 2.9 are automatically satisfied. However in Section 2.3 we provide interesting examples of SDEs admitting only quasi-strong symmetries. Indeed a consequence of Theorem 2.10 in [177] is that if $V_1 = (Y_1, C_1, 0), ..., V_k = (Y_k, C_k, 0)$ are quasi-strong symmetries of (μ, σ) and $Y_1, ..., Y_k$ generate an integrable distribution of constant rank, there exists a (local) stochastic transformation $T = (id_M, B, 1)$ such that $T_*(V_1), ..., T_*(V_k)$ satisfy the hypotheses of Theorem 2.9 for the transformed SDE $E_T(\mu, \sigma)$. We remark that, if the fibred manifold (M, Ψ, M') is not a trivial fibred manifold, T is only locally defined.*

Theorem 2.11 *In the hypotheses and with the notations of Theorem 2.9, let $V = (Y, C, \tau)$ be a symmetry of the SDE (μ, σ) such that, for any $i = 1, ..., k$,*

$$[Y, Y_i] \in \text{span}\{Y_1, \ldots, Y_k\}, \qquad Y_i(C) = 0, \qquad Y_i(\tau) = 0. \tag{2.2}$$

Then the infinitesimal transformation (Y', C', τ') on M', where $Y' = \Psi_*(Y)$, $C' \circ \Psi = C$ and $\tau' \circ \Psi = \tau$, is a symmetry of the SDE (μ', σ').

Proof. We prove in detail that Y' satisfies the determining equation (1.19) for σ'. With a similar method it is possible to prove that μ' satisfies the determining equation (1.18).
Given $f \in C^\infty(M)$ such that $Y_i(f) = 0$, Proposition 2.4 ensures that there exists a function $f' \in C^\infty(M')$ such that $f = f' \circ \Psi$. Moreover we have

$$\begin{array}{rcl} Y(f) & = & (Y'(f')) \circ \Psi \\ \nabla(f) \cdot \sigma & = & (\nabla'(f') \cdot \sigma') \circ \Psi, \end{array} \tag{2.3}$$

where ∇' denotes the differential with respect to the coordinates x'^i of M'.
The determining equations (1.19) are equivalent to the relations

$$Y'(\sigma'^i) - \nabla'(Y'^i) \cdot \sigma' = -\frac{1}{2}\tau'\sigma'^i - \sigma'^i \cdot C'.$$

Since (Y, C, τ) is a symmetry of the SDE (μ, σ), by Lemma 1.23 for any smooth function f we have

$$Y(\nabla(f) \cdot \sigma) - \nabla(Y(f)) \cdot \sigma = -\frac{1}{2}\tau\nabla(f) \cdot \sigma - \nabla(f) \cdot \sigma \cdot C. \tag{2.4}$$

Applying equations (2.3) and (2.4) we obtain

$$\begin{array}{rcl} \left\{Y'(\sigma'^i) - \nabla'(Y'^i) \cdot \sigma'\right\} \circ \Psi & = & \left\{Y'(\nabla'(x'^i) \cdot \sigma') - \nabla'(Y'(x'^i)) \cdot \sigma'\right\} \circ \Psi \\ & = & Y((\nabla'(x'^i) \cdot \sigma') \circ \Psi) - \nabla(Y'(x'^i) \circ \Psi) \cdot \sigma \\ & = & Y(\nabla(\Psi^i) \cdot \sigma) - \nabla(Y(\Psi^i)) \cdot \sigma \\ & = & -\frac{1}{2}\tau(\nabla(\Psi^i) \cdot \sigma) - \nabla(\Psi^i) \cdot \sigma \cdot C \\ & = & \left(-\frac{1}{2}\tau'\sigma'^i - \sigma'^i \cdot C'\right) \circ \Psi. \end{array}$$

Since Ψ is surjective, the thesis follows. ■

Remark 2.12 *If $V_1 = (Y_1, 0, 0), ..., V_k = (Y_k, 0, 0)$ are strong symmetries, conditions (2.2) of Theorem 2.11 on $V = (Y, C, \tau)$ can be rewritten as*

$$[V, V_i] = \sum_{j=1}^{k} \lambda_j^i(x) V_j, \tag{2.5}$$

where $\lambda_i^j(x)$ are smooth functions (in the particular case of $V_1, ..., V_k, V$ generating a finite dimensional Lie algebra, λ_j^i are constants). In general, if V_i are quasi-strong symmetries satisfying the hypotheses of Theorem 2.9, it is possible to prove an analogous of Theorem 2.11 using only hypotheses (2.5) but C' in the reduced symmetry (Y', C', τ') satisfies $C' \circ \Psi = C + \tilde{C}$ where $\tilde{C}$ is such that $Y_i(C + \tilde{C}) = 0$ for $i = 1, ..., k$ and $\nabla(\Psi) \cdot \sigma \cdot \tilde{C} = 0$.

2.2.2 Reconstruction

In this section we discuss the problem of reconstructing a process starting from the knowledge of the reduced one. In order to do this we need the following definition mainly inspired by ODEs framework.

Definition 2.13 *Let X and Z be two processes on M and M' respectively. We say that X can be reconstructed from Z until the stopping time τ if there exists a smooth function $F : \mathbb{R}^{k(n+1)} \times M' \times M \to M$ such that*

$$X_t^\tau = F\left(\int_0^{t\wedge\tau} f_0(s, Z_s)ds, \int_0^{t\wedge\tau} f_1(s, Z_s)dW_s^1, ..., \int_0^{t\wedge\tau} f_m(s, Z_s)dW_s^m,\ Z_{t\wedge\tau},\ X_0\right)$$

where $W^1, ..., W^n$ are Brownian motions and $f_i : \mathbb{R} \times M' \to \mathbb{R}^k$ are smooth functions. The process X can be progressively reconstructed from Z until the stopping time τ if there are some real processes $Z^1, ..., Z^r$, such that every Z^i can be reconstructed until the stopping time τ starting from the process $(Z^1, ..., Z^{i-1}, Z)$, and X can be reconstructed until the stopping time τ from the process $(Z^1, ..., Z^r, Z)$.

We remark that Definition 2.13 is general enough for our purposes (as we only consider Brownian motion driven SDEs), but can be easily generalized to include integration with respect more general stochastic processes.

Theorem 2.14 *Let $\{Y_1, ..., Y_r\}$ be a set of regular reduction vector fields for the reduction map $\Psi : M \to M'$ such that $Y_1, ..., Y_r$ generate a solvable algebra of strong symmetries for the SDE (μ, σ) and let X^x be the unique solution to the SDE (μ, σ) with Brownian motion W such that $X_0^x = x$ almost surely. Then, for any $x \in M$, there exists a stopping time τ_x almost surely positive such that the process X^x can be progressively reconstructed from $\Psi(X^x)$.*

Proof. Let Φ be the diffeomorphism given by Theorem 2.6, defined in a neighborhood $U(x_0)$, and $T = (\Phi, I_m, 1)$. If $(\tilde{\mu}, \tilde{\sigma}) = E_T(\mu, \sigma)$ then $\Phi_*(Y_1), ..., \Phi_*(Y_r)$ are symmetries of $(\tilde{\mu}, \tilde{\sigma})$ in canonical form and, denoting by $\tilde{x}^i$ the coordinate system on $\tilde{M} = \Phi(U)$, we have

$$\frac{\partial}{\partial \tilde{x}_i}(\tilde{\mu}^j) = \frac{\partial}{\partial \tilde{x}_i}(\tilde{\sigma}^j_\alpha) = 0,$$

for $i \leq r$ and $j \leq i$. This means that the r-th row of the SDE $(\tilde{\mu}, \tilde{\sigma})$ does not depend on $\tilde{x}^1, ..., \tilde{x}^r$, the $(r-1)$-th row does not depend on $\tilde{x}^1, ..., \tilde{x}^{r-1}$ and so on. Hence the process $\tilde{X} = \Phi(X^{x_0})$ can be progressively reconstructed from $\Pi(\tilde{X})$, where Π is the projection of $\tilde{M}$ on the last $n - r$ coordinates. Since by definition of $\tilde{X}$ and Φ we have $\Pi(\tilde{X}) = \Pi(\Phi(X^{x_0})) = \Psi(X^{x_0})$, the process $\tilde{X}$ can be progressively reconstructed from $\Psi(X^{x_0})$. Moreover, being $X^{x_0} = \Phi^{-1}(\tilde{X})$ until the process X^{x_0} exits from the open set U we have that $\tilde{X}^{x_0}$ can be progressively reconstructed from $\Psi(X)$ until the stopping time

$$\tau = \inf_{t\in\mathbb{R}_+} \{X_t^{x_0} \notin U\},$$

that is almost surely positive since U is a neighborhood of x_0. ∎

Corollary 2.15 *In the hypotheses and with the notations of Theorem 2.14 if the Lie algebra generated by $Y_1, ..., Y_r$ is Abelian then X^x can be reconstructed from $\Psi(X^x)$.*

Proof. If $\mathcal{G} = \text{span}\{Y_1, ..., Y_r\}$ is Abelian, the diffeomorphism Φ of Theorem 2.6 rectifies $\mathcal{G}$. ∎

Remark 2.16 *In order to compare our results with the reconstruction method proposed in [122] we consider the case of $Y_1, ..., Y_r$ generating a general Lie group G whose action is free and proper. In this case (M, Ψ, M') is a principal bundle with structure group G and locally diffeomorphic to $U = V \times G$, where V is an open subset of $\mathbb{R}^{n-r}$. If we denote by $\bar{X}$ the reduced process and by g the coordinates on G we have that the process $\bar{G}$ in U satisfies the following Stratonovich equation*

$$d\bar{G}_t = \sum_{i=1}^r f_0^i(\bar{X}_t)Y_i(\bar{G}_t)dt + \sum_{\alpha=1}^n \sum_{i=1}^r f_\alpha^i(\bar{X}_t)Y_i(\bar{G}_t) \circ dW_t^\alpha, \tag{2.6}$$

where f^i_j are smooth real-valued functions. Despite the fact that the knowledge of the reduced process $\bar{G}_t$ formally allows the reconstruction of the initial process X_t, for a general group G this reconstruction cannot be reduced to quadratures.
On the other hand, if G is solvable, it is possible to choose a set of global coordinates on G which reduce equation (2.6) to integration by quadratures as required by Definition 2.13.

The following definition generalizes to the stochastic framework the well known definition of integrability for a system of ODEs.

Definition 2.17 *An SDE (μ, σ) is completely integrable (or simply integrable) if for any $x \in M$ there exists an almost surely positive stopping time $\tau_x > 0$ such that the solution process X^x can be progressively reconstructed until the stopping time τ_x from a deterministic process.*

Theorem 2.18 *Let (μ, σ) be an SDE on $M \subset \mathbb{R}^m$ admitting an m-dimensional solvable Lie algebra $\mathcal{G}$ of strong symmetries which are also a regular set of vector fields. Then (μ, σ) is integrable.*

Proof. Since $\mathcal{G}$ has the same dimension of M, the map $\Psi = 0$, and the transformed SDE (μ', σ') is such that the first row of μ' and σ' does not depend on x'^1, the second row does not depend on x'^1, x'^2 and so on. Therefore the n-th row of μ' and σ' does not depend on any variables $x'^1, ..., x'^n$ and so it is constant. This means that the solution $X' = \Phi(X)$ can be progressively reconstructed from a constant process. ■

We remark that the hypotheses of Theorem 2.18 are only sufficient and not necessary. For example the SDE

$$\begin{pmatrix} dX_t \\ dZ_t \end{pmatrix} = \begin{pmatrix} 0 \\ g(X_t) \end{pmatrix} + \begin{pmatrix} 1 \\ f(X_t) \end{pmatrix} dW_t,$$

is obviously integrable, but it is easy to prove that (for general $g(x), f(x)$) it does not admit other symmetries than

$$V = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right).$$

2.3 Examples

In this section we apply our general reduction procedure to some explicit examples. Following the line of previous discussion, given an SDE (μ, σ), we start by looking for a solvable algebra of symmetries $\mathcal{G} = \{V_1, \dots, V_r\}$ for (μ, σ). Hence we compute a stochastic transformation $T = (\Phi, B, \eta)$ transforming $V_1, \dots, V_r$ into strong symmetries $V'_k = (Y'_k, 0, 0)$ for the transformed SDE $E_T(\mu, \sigma)$ such that the vector fields $Y'_1, \dots, Y'_r$ are in canonical form. Finally we use the results of Section 2.2 to reduce (or integrate) the transformed SDE $E_T(\mu, \sigma)$ and we reconstruct the solution to (μ, σ) by means of the inverse transformation T^{-1}.

2.3.1 Two dimensional Brownian motion

Let us consider the following SDE on $\mathbb{R}^2$

$$\begin{pmatrix} dX_t \\ dZ_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} dW^1_t \\ dW^2_t \end{pmatrix}$$

with $\mu = 0$ and $\sigma = I_2$. The solution to the previous equation is obviously the two dimensional Brownian motion. This equation does not need any procedure of reduction and reconstruction since it is obviously integrable. Nevertheless this example is interesting due to the importance of the process solution to the equation (namely the two dimensional Brownian motion) and furthermore because it is one of the few example of an SDE with an infinite dimensional group of infinitesimal symmetries. By Theorem 1.21, the infinitesimal transformations (Y, C, τ) of the form

$$(Y, C, \tau) = \left(\begin{pmatrix} Y^1(x,z) \\ Y^2(x,z) \end{pmatrix}, \begin{pmatrix} 0 & c(x,z) \\ -c(x,z) & 0 \end{pmatrix}, \tau(x,z) \right)$$

is a (general) symmetry if and only if the following equations hold

$$\begin{gathered} \partial_x(Y^1) = \tfrac{1}{2}\tau \\ \partial_z(Y^1) = c \\ \partial_x(Y^2) = -c \\ \partial_z(Y^2) = \tfrac{1}{2}\tau \\ \tfrac{1}{2}(\partial_{xx}(Y^1) + \partial_{zz}(Y^1)) = 0 \\ \tfrac{1}{2}(\partial_{xx}(Y^2) + \partial_{zz}(Y^2)) = 0. \end{gathered}$$

These equations are satisfied if and only if

$$\begin{aligned} \partial_x(Y^1) &= \partial_z(Y^2) && (2.7) \\ \partial_z(Y^1) &= -\partial_x(Y^2) && (2.8) \\ c &= \partial_z(Y^1) && (2.9) \\ \tau &= 2\partial_x(Y^1). && (2.10) \end{aligned}$$

Therefore the two dimensional Brownian motion admits an infinite number of general infinitesimal symmetries. Indeed Y^1, Y^2 have to satisfy equations (2.7), (2.8) that are the well-known Cauchy-Riemann equations. It is interesting to remark that the symmetry approach introduced before allows us to recover the expected properties for the two dimensional Brownian motion (see [40], Example 4.1 and [129], Example 4).

Let us now discuss the problem of determining the general infinitesimal symmetries of the two dimensional Brownian motion generating a one-parameter group of stochastic transformations. Since the functions Y^1 and Y^2 satisfy the Cauchy-Riemann equations, and we want consider functions Y^1, Y^2 defined on the whole plane $\mathbb{R}^2$, then the function $u = Y^1 + iY^2$ is an entire function. Denoting by $w = x + iz$, the vector field Y is the real part of the holomorphic vector field $Z = u(w)\partial_w$ on $\mathbb{C}$ and Y generates a one-parameter group on $\mathbb{R}$ if and only if Z generates a one-parameter group on $\mathbb{C}$. Since u is an entire function, Z generates a one-parameter group if and only if $u(w)$ is a linear function in w and Y^1, Y^2 must be linear functions in x, z (see also [129], Example 4).
Therefore the general infinitesimal symmetries of Brownian motion generating a one-parameter

group are

$$\begin{aligned} V_1 &= \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right) \\ V_2 &= \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right) \\ V_3 &= \left(\begin{pmatrix} x \\ z \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 2 \right) \\ V_4 &= \left(\begin{pmatrix} z \\ -x \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0 \right). \end{aligned}$$

The infinitesimal stochastic transformations V_1 and V_2 are the x and z translation respectively, V_3 is the dilatation and V_4 is the rotation around the coordinate origin. It is important to note that, although for V_1, V_2, V_3 the matrix C is 0, in the case of the transformation V_4 we have $C \neq 0$. This circumstance shows that the introduction of the anti-symmetric matrix C is necessary in order to have the rotation as a symmetry for the two dimensional Brownian motion.

2.3.2 A class of one-dimensional Kolmogorov-Pearson diffusions

We consider the following class of SDE within the Kolmogorov-Pearson type diffusions

$$dX_t = (\lambda X_t + \nu)dt + \sqrt{\alpha X_t^2 + 2\beta X_t + \gamma}\, dW_t, \tag{2.11}$$

where $\alpha, \beta, \gamma, \lambda, \nu \in \mathbb{R}$, $\alpha \geq 0$ and $\alpha\gamma - \beta^2 \geq 0$.
For $\alpha = \beta = 0$ the class includes the Ornstein-Uhlenbeck process and for $\alpha\gamma - \beta^2 = 0$ the important class of one-dimensional general linear SDEs of the form

$$dX_t = (\lambda X_t + \nu) + \left(\sqrt{\alpha} X_t + \frac{\beta}{\sqrt{\alpha}} \right) dW_t. \tag{2.12}$$

Beyond the large number of applications of Ornstein-Uhlenbeck process and of linear SDEs and their spatial transformations, the Kolmogorov-Pearson class (2.11) has notable applications to finance (see [27, 161]), physics (see [71, 172]) and biology (see [80]). Moreover, there is a growing interest in the study of statistical inference (see [68]), in the analytical and spectral properties of the Kolmogorov equation associated with (2.11) (see [15]) and in the development of efficient numerical algorithms for its numerical simulation (see [28]). Finally the Kolmogorov-Pearson diffusions are examples of "polynomial processes" that are becoming quite popular in financial mathematics ([47]). For many particular values of the parameters $\alpha, \beta, \gamma, \lambda, \nu$ it is well known that equation (2.11) is an integrable SDE (first of all in the standard linear case corresponding to $\alpha\gamma - \beta^2 = 0$). Anyway this integrability property cannot be directly related to the existence of strong symmetries as showed by the following proposition.

Proposition 2.19 *The SDE $(\lambda x + \nu, \sqrt{\alpha x^2 + 2\beta x + \gamma})$ admits strong symmetries if and only if*

$$\begin{aligned} 2\beta\nu - 2\gamma\lambda + \alpha\gamma - \beta^2 &= 0 \\ \alpha\nu - \beta\lambda &= 0. \end{aligned}$$

Proof. The determining equations (1.18) and (1.19) for a strong symmetry $V = (Y, 0, 0)$ of (2.11), with $Y = Y^1\partial_x$, are

$$\frac{(\alpha x+\beta)}{\sqrt{\alpha x^2+2\beta x+\gamma}}Y^1 - \partial_x(Y^1)\sqrt{\alpha x^2+2\beta x+\gamma} = 0 \tag{2.13}$$

$$\lambda Y^1 - \frac{(\alpha x^2+2\beta x+\gamma)}{2}\partial_{xx}(Y^1) - \partial_x(Y^1)(\lambda x+\nu) = 0. \tag{2.14}$$

Equation (2.13) is an ODE in Y^1 with solution

$$Y^1 = Y_0^1\sqrt{\alpha x^2+2\beta x+\gamma}, \tag{2.15}$$

where $Y_0^1 \in \mathbb{R}$. Inserting the expression (2.15) in (2.14) we obtain

$$Y_0^1\frac{(2\beta\nu - 2\gamma\lambda + \alpha\gamma - \beta^2) + 2x(\alpha\nu - \beta\lambda)}{2\sqrt{\alpha x^2+2\beta x+\gamma}} = 0$$

and this concludes the proof. ∎

We remark that a standard linear SDE of the form (2.12) admits a symmetry if and only if $\alpha\nu - \beta\lambda = 0$. Therefore, in spite of their integrability, standard linear SDEs do not have, in general, strong symmetries.
In order to apply a symmetry approach to the study of the integrability of (2.11), we consider the following two-dimensional system:

$$\begin{pmatrix} dX_t \\ dZ_t \end{pmatrix} = \begin{pmatrix} \lambda X_t + \nu \\ \lambda Z_t \end{pmatrix} dt + \begin{pmatrix} \sqrt{\alpha X_t^2+2\beta X_t+\gamma} & 0 \\ \frac{Z_t(\alpha X_t+\beta)}{\sqrt{\alpha X_t^2+2\beta X_t+\gamma}} & \frac{Z_t\sqrt{\alpha\gamma-\beta^2}}{\sqrt{\alpha X_t^2+2\beta X_t+\gamma}} \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix}, \tag{2.16}$$

where $W_t^1 := W_t$. In the standard linear case, system (2.16) consists of SDE (2.12) and of the associated homogeneous one. If we look for the symmetries of system (2.16) of the form $V = (Y, C, \tau)$, where $Y = (Y^1, Y^2)$, $C = \begin{pmatrix} 0 & c(x,z) \\ -c(x,z) & 0 \end{pmatrix}$ and $\tau(x,z)$, the determining equations are:

$$\frac{(\alpha x+\beta)}{\sqrt{\alpha x^2+2\beta x+\gamma}}Y^1 - \partial_x(Y^1)\sqrt{\alpha x^2+2\beta x+\gamma} - \partial_z(Y^1)\frac{z(\alpha x+\beta)}{\sqrt{\alpha x^2+2\beta x+\gamma}} + \\ +\frac{\tau}{2}\sqrt{\alpha x^2+2\beta x+\gamma} = 0$$

$$\frac{z(\alpha\gamma-\beta^2)}{(\sqrt{\alpha x^2+2\beta x+\gamma})^3}Y^1 + \frac{\alpha x+\beta}{\sqrt{\alpha x^2+2\beta x+\gamma}}Y^2 - \partial_x(Y^2)\sqrt{\alpha x^2+2\beta x+\gamma} + \\ -\partial_z(Y^2)\frac{z(\alpha x+\beta)}{\sqrt{\alpha x^2+2\beta x+\gamma}} + \frac{\tau}{2}\frac{z(\alpha x+\beta)}{\sqrt{\alpha x^2+2\beta x+\gamma}} - \frac{cz\sqrt{\alpha\gamma-\beta^2}}{\sqrt{\alpha x^2+2\beta x+\gamma}} = 0$$

$$-\frac{z\sqrt{\alpha\gamma-\beta^2}}{\sqrt{\alpha x^2+2\beta x+\gamma}}\partial_z(Y^1) + c\sqrt{\alpha x^2+2\beta x+\gamma} = 0$$

$$-\frac{z\sqrt{\alpha\gamma-\beta^2}(\alpha x+\beta)}{(\sqrt{\alpha x^2+2\beta x+\gamma})^3}Y^1 + \frac{\sqrt{\alpha\gamma-\beta^2}}{\sqrt{\alpha x^2+2\beta x+\gamma}}Y^2 - \frac{z\sqrt{\alpha\gamma-\beta^2}}{\sqrt{\alpha x^2+2\beta x+\gamma}}\partial_z(Y^2) \\ +c\frac{z(\alpha x+\beta)}{\sqrt{\alpha x^2+2\beta x+\gamma}} + \frac{\tau}{2}\frac{z\sqrt{\alpha\gamma-\beta^2}}{\sqrt{\alpha x^2+2\beta x+\gamma}} = 0$$

$$\lambda Y^1 - (\lambda x+\nu)\partial_x(Y^1) - \lambda z\partial_z(Y^1) - \frac{\alpha x^2+2\beta x+\gamma}{2}\partial_{xx}(Y^1) - \frac{\alpha z^2}{2}\partial_{zz}(Y^1) + \\ -z(\alpha x+\beta)\partial_{xz}(Y^1) + \tau(\lambda x+\nu) = 0$$

$$\lambda Y^2 - (\lambda x+\nu)\partial_x(Y^2) - \lambda z\partial_z(Y^2) - \frac{\alpha x^2+2\beta x+\gamma}{2}\partial_{xx}(Y^2) - \frac{\alpha z^2}{2}\partial_{zz}(Y^2) + \\ -z(\alpha x+\beta)\partial_{xz}(Y^2) + \tau\lambda z = 0.$$

We can easily solve the previous overdetermined system of PDEs by a computer algebra software and we find two quasi-strong symmetries

$$\begin{aligned}
V_1 = (Y_1, C_1, \tau_1) &= \left(\begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{z\sqrt{\alpha\gamma-\beta^2}}{\alpha x^2+2\beta x+\gamma} \\ -\frac{z\sqrt{\alpha\gamma-\beta^2}}{\alpha x^2+2\beta x+\gamma} & 0 \end{pmatrix}, 0 \right) \\
V_2 = (Y_2, C_2, \tau_2) &= \left(\begin{pmatrix} 0 \\ z \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right).
\end{aligned}$$

Therefore the function $\tilde{\Psi} : M \to \mathbb{R}$ given by $\tilde{\Psi}(x, z) = x$ is a reduction function with respect to the strong symmetry Y_2, being $\nabla(\tilde{\Psi}) \cdot Y_2 = Y_2(\tilde{\Psi}) = 0$, and the reduced equation on $M' = \tilde{\Psi}(M) = \mathbb{R}$ is exactly the original SDE (2.11).

This circumstance partially explains Proposition 2.19: since the original SDE (2.11) turns out as the reduction of the integrable system (2.16) with respect to the “wrong” symmetry, it does not inherit any symmetry.

In order to integrate system (2.16) and therefore also the original equation (2.11), we start by looking for a stochastic transformation $T = (\Phi, B, \eta)$ such that $T_*(V_1)$ and $T_*(V_2)$ are strong transformations and $\Phi_*(Y_1)$ and $\Phi_*(Y_2)$ are in canonical form. Since V_1, V_2 are quasi-strong infinitesimal stochastic transformations we can restrict to a quasi-strong transformation $T = (\Phi, B, 1)$.

Following the explicit construction of Theorem 2.6 the function Φ turns out to be both globally defined and globally invertible on M.

The vector fields Y_1, Y_2, whose flows are defined by

$$\begin{aligned}
\Phi^1_{a^1}(x, z) &= \begin{pmatrix} x + a^1 z \\ z \end{pmatrix} \\
\Phi^2_{a^2}(x, z) &= \begin{pmatrix} x \\ e^{a^2} z \end{pmatrix},
\end{aligned}$$

generate a free and proper action of a solvable simply connected non Abelian Lie group on M. Hence, if we consider the point $p = (0, 1)^T$ and the function $F : \mathbb{R}^2 \to M$ given by

$$F(a^1, a^2) = \Phi^1_{a^1}(\Phi^2_{a^2}(p)) = \begin{pmatrix} a^1 e^{a^2} \\ e^{a^2} \end{pmatrix},$$

the function $\Phi : M \to \mathbb{R}^2$, which is the inverse of F, is given by

$$\Phi(x, z) = \begin{pmatrix} \frac{x}{z} \\ \log(z) \end{pmatrix}.$$

By Theorem 1.16 the equations for B are

$$\begin{aligned}
z\partial_x(B) &= -B \cdot C_1 \\
z\partial_z(B) &= 0.
\end{aligned}$$

Writing:

$$B = \begin{pmatrix} b & \sqrt{1-b^2} \\ -\sqrt{1-b^2} & b \end{pmatrix} \tag{2.17}$$

from the second equation we deduce that B does not depend on z and from the first one we obtain that b satisfies the equation

$$\partial_x b = \frac{\sqrt{\alpha\gamma - \beta^2}}{\alpha x^2 + 2\beta x + \gamma} \sqrt{1 - b^2}.$$

The latter equation is an ODE with separable variables admitting the following solution

$$b = \frac{\alpha x + \beta}{\sqrt{\alpha}\sqrt{\alpha x^2 + 2\beta x + \gamma}},$$

and we get

$$B = \begin{pmatrix} \frac{\alpha x+\beta}{\sqrt{\alpha}\sqrt{\alpha x^2+2\beta x+\gamma}} & \frac{\sqrt{\alpha\gamma-\beta^2}}{\sqrt{\alpha}\sqrt{\alpha x^2+2\beta x+\gamma}} \\ -\frac{\sqrt{\alpha\gamma-\beta^2}}{\sqrt{\alpha}\sqrt{\alpha x^2+2\beta x+\gamma}} & \frac{\alpha x+\beta}{\sqrt{\alpha}\sqrt{\alpha x^2+2\beta x+\gamma}} \end{pmatrix}.$$

Putting $(x', z')^T = \Phi(x, z)$ we have

$$\begin{aligned} V_1' = T_*(V_1) &= \left(\Phi_*(Y_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right), \\ V_2' = T_*(V_2) &= \left(\Phi_*(Y_2) = \begin{pmatrix} -x' \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right), \end{aligned}$$

and so Y_1', Y_2' are two generators of $\Phi_*(\mathcal{G})$ in canonical form. Introducing $dW_t' = B(X_t, Z_t) \cdot dW_t$ and applying Itô formula respectively to $\phi_1 = x/z$ and $\phi_2 = \log(z)$ we can write equation (2.16) in the new variables:

$$\begin{aligned} dX_t' &= (\nu - \beta)e^{-Z_t'}dt + \frac{\beta e^{-Z_t'}}{\sqrt{\alpha}}dW_t'^1 - \sqrt{\frac{\alpha\gamma - \beta^2}{\alpha}}e^{-Z_t'}dW_t'^2 \\ dZ_t' &= \left(\lambda - \frac{\alpha}{2}\right)dt + \sqrt{\alpha}dW_t'^1. \end{aligned}$$

Since this is an integrable SDE, the solutions to equation (2.16) can be recovered following our general procedure. In particular, when $\alpha\gamma - \beta^2 = 0$, the solution to equation (2.11) is given by

$$\begin{aligned} Z_t &= Z_0 e^{(\lambda - \alpha/2)t + \sqrt{\alpha}W_t} \\ X_t &= Z_t\left(\frac{X_0}{Z_0} + \int_0^t \frac{\nu - \beta}{Z_s}ds + \int_0^t \frac{\beta}{\sqrt{\alpha}Z_s}dW_s\right) \end{aligned}$$

which is the well known explicit solution to the (general) linear one-dimensional SDE (2.12).

2.3.3 Integrability of a singular SDE

Let us consider the SDE on $M = \mathbb{R}^2 \backslash \{(0,0)^T\}$

$$\begin{pmatrix} dX_t \\ dZ_t \end{pmatrix} = \begin{pmatrix} \frac{\alpha X_t}{X_t^2+Z_t^2} \\ \frac{-\alpha Z_t}{X_t^2+Z_t^2} \end{pmatrix} dt + \begin{pmatrix} \frac{X_t^2-Z_t^2}{\sqrt{X_t^2+Z_t^2}} & 0 \\ 0 & \frac{X_t^2-Z_t^2}{\sqrt{X_t^2+Z_t^2}} \end{pmatrix} \cdot \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix}, \tag{2.18}$$

where $\alpha \in \mathbb{R}$. Despite the coefficients of (μ, σ) have a singularity in $(0,0)^T$, we will prove that the solution to (2.18) is not singular and that the explosion time of (2.18) is $+\infty$ for any deterministic initial condition $X_0 \in M$.

A symmetry $V = (Y, C, \tau)$ of (2.18), with $Y = (Y^1, Y^2)$, $C = \begin{pmatrix} 0 & c(x,z) \\ -c(x,z) & 0 \end{pmatrix}$ and $\tau(x, z)$, has to satisfy the following determining equations

$$\frac{x^3+3xz^2}{(x^2+z^2)^{3/2}}Y^1 - \frac{z^3+3zx^2}{(x^2+z^2)^{3/2}}Y^2 - \frac{x^2-z^2}{\sqrt{x^2+z^2}}\partial_x(Y^1) + \frac{1}{2}\tau\frac{x^2-z^2}{\sqrt{x^2+z^2}} = 0$$

$$-\tfrac{x^2-z^2}{\sqrt{x^2+z^2}}\partial_x(Y^2)-c\tfrac{x^2-z^2}{\sqrt{x^2+z^2}}=0$$

$$-\tfrac{x^2-z^2}{\sqrt{x^2+z^2}}\partial_z(Y^1)+c\tfrac{x^2-z^2}{\sqrt{x^2+z^2}}=0$$

$$\tfrac{x^3+3xz^2}{(x^2+z^2)^{3/2}}Y^1-\tfrac{z^3+3zx^2}{(x^2+z^2)^{3/2}}Y^2-\tfrac{x^2-z^2}{\sqrt{x^2+z^2}}\partial_z(Y^2)+\tfrac{1}{2}\tau\tfrac{x^2-z^2}{\sqrt{x^2+z^2}}=0$$

$$\begin{aligned}&-\tfrac{\alpha(x^2-z^2)}{(x^2+z^2)^2}Y^1-\tfrac{2\alpha xz}{(x^2+z^2)^2}Y^2-\tfrac{\alpha x}{x^2+z^2}\partial_x(Y^1)\\&+\tfrac{\alpha z}{x^2+z^2}\partial_z(Y^1)-\tfrac{1}{2}\tfrac{(x^2-z^2)^2}{x^2+z^2}(\partial_{xx}(Y^1)+\partial_{zz}(Y^1))+\tau\tfrac{\alpha x}{x^2+z^2}=0\end{aligned}$$

$$\begin{aligned}&\tfrac{2\alpha xz}{(x^2+z^2)^2}Y^1-\tfrac{\alpha(x^2-z^2)}{(x^2+z^2)^2}Y^2-\tfrac{\alpha x}{x^2+z^2}\partial_x(Y^2)\\&+\tfrac{\alpha z}{x^2+z^2}\partial_z(Y^2)-\tfrac{1}{2}\tfrac{(x^2-z^2)^2}{x^2+z^2}(\partial_{xx}(Y^2)+\partial_{zz}(Y^2))-\tau\tfrac{\alpha z}{x^2+z^2}=0\end{aligned}$$

Solving this system of PDEs by a computer algebra software we find the unique (quasi-strong) symmetry

$$V=(Y,C,\tau)=\left(\begin{pmatrix}\tfrac{z}{x^2+z^2}\\ \tfrac{x}{x^2+z^2}\end{pmatrix},\begin{pmatrix}0 & \tfrac{x^2-z^2}{(x^2+z^2)^2}\\ -\tfrac{x^2-z^2}{(x^2+z^2)^2} & 0\end{pmatrix},0\right)$$

which unfortunately does not generate a one parameter group of stochastic transformations, as the trajectories of the points of the form (h,h) and $(h,-h)$ (with $h\in\mathbb{R}\backslash\{0\}$), reach $(0,0)$ in a finite time.

Hence, in order to find a stochastic transformation $T=(\Phi,B,\eta)$ such that $T_*(V)$ is a strong transformation and $\Phi_*(Y)$ is in canonical form we have to solve the equations $Y(\Phi)=(1,0)^T$ and $Y(B)=-B\cdot C$. Writing $\Phi=(\tilde{\Phi},\Psi)^T$, the map Ψ has to solve $Y(\Psi)=0$, i.e.

$$\frac{z}{x^2+z^2}\partial_x(\Psi)+\frac{x}{x^2+z^2}\partial_z(\Psi)=0.$$

By using the method of the characteristics, we immediately obtain a particular solution

$$\Psi(x,z)=x^2-z^2.$$

In order to find an adapted coordinate system for the Abelian Lie algebra $\mathcal{G}=\mathrm{span}\{Y\}$ we have to solve the equation $Y(\tilde{\Phi})=1$, i.e.

$$z\partial_x(\tilde{\Phi})+x\partial_z(\tilde{\Phi})=x^2+z^2.$$

Once again, applying the method of the characteristics, we obtain

$$\tilde{\Phi}(x,z)=xz,$$

so that we can consider the local diffeomorphism of $M=\mathbb{R}^2\backslash\{(0,0)^T\}$

$$\Phi(x,z)=\begin{pmatrix}\tilde{\Phi}\\ \Psi\end{pmatrix}=\begin{pmatrix}xz\\ x^2-z^2\end{pmatrix}$$

which is only locally invertible.

To construct the matrix-valued function B of the form (2.17) we solve the equation $Y(B)=-B\cdot C$. In the new coordinates $(x',z')=\Phi(x,z)$ the equation becomes

$$\partial_{x'}b=\frac{z'}{4x'^2+z'^2}\sqrt{1-b^2},$$

whose solution is

$$b = \pm \frac{\sqrt{\sqrt{z'^2 + 4x'^2} - z'}}{\sqrt{2}(z'^2 + 4x'^2)^{1/4}}$$

and, coming back to the original coordinate system, we find

$$B = \begin{pmatrix} \frac{z}{\sqrt{x^2+z^2}} & \frac{x}{\sqrt{x^2+z^2}} \\ \frac{-x}{\sqrt{x^2+z^2}} & \frac{z}{\sqrt{x^2+z^2}} \end{pmatrix}.$$

The transformed SDE $(\mu', \sigma') = E_T(\mu, \sigma)$ has coefficients

$$\begin{aligned} \mu' &= L(\Phi) \circ \Phi^{-1} = \begin{pmatrix} 0 \\ 2\alpha \end{pmatrix} \\ \sigma' &= (\nabla(\Phi) \cdot \sigma) \circ \Phi^{-1} = \begin{pmatrix} z' & 0 \\ 0 & -2z' \end{pmatrix} \end{aligned}$$

and, by applying Itô formula, the original two-dimensional SDE becomes

$$\begin{aligned} dX'_t &= Z'_t dW'^1_t \\ dZ'_t &= 2\alpha dt - 2Z'_t dW'^2_t, \end{aligned} \tag{2.19}$$

where $dW'_t = B(X_t, Z_t) \cdot dW_t$. Since the equation in Z' is linear, the above SDE is integrable and therefore also (2.18) is integrable. Furthermore, since the map $\Phi : M \to \mathbb{R}^2 \backslash \{(0,0)^T\}$ is a double covering map and since the SDE (2.19) has explosion time $\tau = +\infty$, the SDE (2.18), although singular at the origin, has also explosion time $\tau = +\infty$ for any deterministic initial condition $X_0 \in M$.
This example points out the importance of developing a *local* reduction theory for SDEs, since in this case a global approach cannot be successful.

2.3.4 Stochastic perturbation of mechanical equations

In this example we analyze a wide class of models, related to (stochastic) mechanics, of the form

$$\begin{aligned} dX^i_t &= V^i_t dt \\ dV^i_t &= F^i_0(X_t, V_t)dt + \textstyle\sum_\alpha F^i_\alpha(X_t) dW^\alpha_t, \end{aligned} \tag{2.20}$$

i.e. with SDE coefficients:

$$\begin{aligned} \mu &= \begin{pmatrix} v^1 \\ \vdots \\ v^n \\ F^1_0(x,v) \\ \vdots \\ F^n_0(x,v) \end{pmatrix} \\ \sigma &= \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ F^1_1(x) & \cdots & F^1_m(x) \\ \vdots & \vdots & \vdots \\ F^n_1(x) & \cdots & F^n_m(x) \end{pmatrix}, \end{aligned}$$

where (x^i, v^i) is the standard coordinate system of $M = \tilde{M} \times \mathbb{R}^n$ and $\tilde{M}$ is an open set of $\mathbb{R}^n$. This kind of SDEs, representing a stochastic perturbation of the Newton equations for n particles of mass $m_i = 1$ subjected to forces depending on the positions and on the velocities

$$\frac{d^2 X_t^i}{dt^2} = F^i\left(X_t, \frac{dX_t}{dt}\right), \tag{2.21}$$

arise in many contexts of mathematical physics. The class includes the Langevin type equation often used in the framework of Stochastic Thermodynamics (see, e.g., [136, 160]) for $F_\alpha^i = \delta_\alpha^i$ and $F_0^i = -\gamma V^i + \partial_i(U)(x)$, where $U : \mathbb{R}^n \to \mathbb{R}$ is a smooth function and $\gamma \in \mathbb{R}_+$. Furthermore, if the forces F_0^i arise from a Lagrangian L of the form

$$L = \frac{1}{2}\sum_{i,j} g_{i,j}(x) v^i v^j - U(x)$$

(where $g_{i,j}(x)$ is a metric tensor on $\mathbb{R}^n$) and the random perturbations F_α^i are given by

$$F_\alpha^i = \sum_j g^{i,j}(x)\partial_j(U_\alpha),$$

with U_α smooth functions, (2.20) turns out to be a Lagrangian system with the following action functional

$$S = \int_0^t L(X_s, V_s)ds + \sum_\alpha \int_0^t U_\alpha(X_s) dW_s^\alpha$$

(see, e.g.,[23, 121]). There is a growing interest for this kind of stochastic perturbations of Lagrangian and Hamiltonian systems due both to their special mathematical properties and to their applications in mathematical physics (see, e.g., [11, 12, 98, 99, 125, 128, 152]).

In the following we propose a method to obtain an SDE of the form (2.20) which can be interpreted as a symmetric stochastic perturbation of a symmetric ODE of the form (2.21).

Given a vector field $\tilde{Y}_0 = (\tilde{Y}_0^1(x), ..., \tilde{Y}_0^n(x))^T$ on $\tilde{M}$ which is a symmetry of (2.21), the vector field

$$Y = \begin{pmatrix} \tilde{Y}_0^1(x) \\ \vdots \\ \tilde{Y}_0^n(x) \\ \sum_{k=1}^n \partial_{x^k}(\tilde{Y}_0^1)v^k \\ \vdots \\ \sum_{k=1}^n \partial_{x^k}(\tilde{Y}_0^n)v^k \end{pmatrix},$$

is a symmetry of (2.20), when $F_\alpha^i = 0$.

If $\tilde{Y}_\alpha = (\tilde{Y}_\alpha^1(x), ..., \tilde{Y}_\alpha^n(x))$, for $\alpha = 1, ..., m$, are m vector fields on $\tilde{M}$ such that there exists a matrix-valued function $C : \mathbb{R}^n \to so(m)$ satisfying

$$[\tilde{Y}_0, \tilde{Y}_\alpha] = -\sum_{\beta=1}^m C_\alpha^\beta(x)\tilde{Y}_\beta$$

and we set

$$F_\alpha^i(x) = \tilde{Y}_\alpha^i(x), \qquad i = 1, ..., n$$

we find that $V = (Y, C, 0)$ is a quasi-strong symmetry for the system (2.20). Indeed, the first determining equation (1.18) for $i = n+1, ..., 2n$ becomes:

$$\begin{aligned}
[Y,\sigma]^i_\alpha &= \sum_{j=1}^{2n}(Y^j\partial_j(\sigma^i_\alpha) - \sigma^j_\alpha\partial_j(Y^i)) \\
&= \sum_{j=1}^{n}\tilde{Y}^j_0\partial_{x^j}(\tilde{Y}^{i-n}_\alpha) - \sum_{j=n+1}^{2n}\tilde{Y}^{j-n}_\alpha\partial_{v^j}\left(\sum_{k=1}^{n}\partial_{x^k}(\tilde{Y}^i_0)v^k\right) \\
&= \sum_{j=1}^{n}\tilde{Y}^j_0\partial_{x^j}(\tilde{Y}^{i-n}_\alpha) - \sum_{j=1}^{n}\tilde{Y}^j_\alpha\partial_{x^j}(\tilde{Y}^{i-n}_0) \\
&= [\tilde{Y},\tilde{Y}_\alpha]^{i-n} = -\sum_{\beta=1}^{m}C^\beta_\alpha\tilde{Y}^i_\beta = -\sum_{\beta=1}^{m}C^\beta_\alpha\sigma^i_\beta.
\end{aligned}$$

Since $\sigma^i_\alpha = 0$ for $i \leq n$ and Y^i does not depend on v for $i \leq n$, we have

$$[Y,\sigma]^i_\alpha = -\sum_{\beta=1}^{m}C^\beta_\alpha\sigma^i_\beta, \qquad i \leq n.$$

Furthermore

$$Y(\mu^i) - L(Y^i) = Y(\mu^i) - \mu(Y^i) = [Y,\mu]^i = 0$$

because Y is a symmetry of (2.21).

An interesting particular case within this class is given by the following equation

$$\frac{d^2X_t}{dt} = -\gamma\frac{dX_t}{dt},$$

representing (for $\gamma > 0$) the motion of a particle subjected to a linear dissipative force. This equation has the symmetry $\tilde{Y} = x\partial_x$, so that the system

$$\begin{pmatrix} dX_t \\ dV_t \end{pmatrix} = \begin{pmatrix} V_t \\ -\gamma V_t \end{pmatrix}dt + \begin{pmatrix} 0 \\ \alpha X_t \end{pmatrix}dW_t, \tag{2.22}$$

which provides the equation of a dissipative random harmonic oscillator, has the strong symmetry

$$Y = \begin{pmatrix} x \\ v \end{pmatrix}.$$

If we consider the local diffeomorphism

$$\Phi(x,v) = \begin{pmatrix} \frac{1}{2}\log(x^2+v^2) \\ \frac{v}{x} \end{pmatrix},$$

in the coordinates $(x', v')^T = \Phi(x, v)$ equation (2.22) becomes

$$\begin{aligned}
dX'_t &= \left(-\frac{\gamma(V'_t)^2 - V'_t}{1+V'^2_t} + \alpha^2\frac{1-V'^2_t}{2+4V'^2_t+2V'^4_t}\right)dt + \frac{\alpha V'_t}{1+V'^2_t}dW_t \\
dV'_t &= (-\gamma V'_t - V'^2_t)dt + \alpha dW_t.
\end{aligned}$$

This system is not integrable but the equation for V'_t is known in literature (see [79]). Furthermore, as well as its deterministic counterpart, this equation admits a superposition rule [123].
Another interesting example within the class of the equations described by (2.20) is given by the following system

$$\left(\begin{array}{c} dX_t \\ dZ_t \\ dV_t^x \\ dV_t^z \end{array}\right) = \left(\begin{array}{c} V_t^x \\ V_t^z \\ \frac{X_t f(\sqrt{X_t^2+Z_t^2})}{\sqrt{X_t^2+Z_t^2}} - \gamma V_t^x \\ \frac{Z_t f(\sqrt{X_t^2+Z_t^2})}{\sqrt{X_t^2+Z_t^2}} - \gamma V_t^z \end{array}\right) dt + \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ D & 0 \\ 0 & D \end{array}\right) \cdot \left(\begin{array}{c} dW_t^1 \\ dW_t^2 \end{array}\right), \tag{2.23}$$

where $D \in \mathbb{R}_+$ and $f : \mathbb{R}_+ \to \mathbb{R}$ is a smooth function. The SDE (2.23) is a Langevin type equation describing a point particle of unitary mass subjected to the central force $f(\sqrt{x^2+z^2})$, to an isotropic dissipation linear in the velocities and to a space homogeneous random force. Since both the central force and the dissipation are invariant under the rotation group, the vector field

$$\tilde{Y} = \left(\begin{array}{c} z \\ -x \end{array}\right)$$

is a symmetry of (2.23) for $D = 0$. Furthermore we have that

$$\left[\tilde{Y}, \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)\right] = -\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right) \cdot \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right).$$

So putting

$$\begin{aligned} Y &= \left(\begin{array}{c} z \\ -x \\ v^z \\ -v^x \end{array}\right) \\ C &= \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \end{aligned}$$

the infinitesimal stochastic transformation $V = (Y, C, 0)$ is a quasi-strong symmetry for equation (2.23).
In order to reduce (2.23) using the symmetry V, we have to find the stochastic transformation $T = (\Phi, B, 1)$ which puts V in canonical form. Solving the equations for Φ and B we obtain

$$\begin{aligned} \Phi(x, z, v^x, v^z) &= \left(\begin{array}{c} \operatorname{acos}\left(\frac{x}{\sqrt{x^2+z^2}}\right) \\ \sqrt{z^2+x^2} \\ \frac{xv^z - zv^x}{x^2+z^2} \\ \frac{xv^x+zv^z}{\sqrt{x^2+z^2}} \end{array}\right) \\ B(x, z, v^x, v^z) &= \left(\begin{array}{cc} \frac{x}{\sqrt{x^2+x^2}} & \frac{z}{\sqrt{x^2+z^2}} \\ -\frac{z}{\sqrt{x^2+z^2}} & \frac{x}{\sqrt{x^2+z^2}} \end{array}\right). \end{aligned}$$

With the new coordinates $\Phi = (\theta, r, v^\theta, v^r)^T$ and with the new Brownian motion $dW'_t = B \cdot dW_t$

we have

$$\begin{aligned} d\Theta_t &= V_t^\theta dt \\ dR_t &= V_t^r dt \\ dV_t^\theta &= \left(-\frac{2V_t^r V_t^\theta}{R_t} - \gamma V_t^\theta\right) dt + \frac{D}{R_t} dW_t'^2 \\ dV_t^r &= \left(R_t (V_t^\theta)^2 - \gamma V_t^r + f(R_t)\right) dt + D dW_t'^1, \end{aligned}$$

where the solution Θ_t can be reconstructed from (R_t, V_t^r, V_t^θ).

Remark 2.20 *For $\gamma = 0$ equation (2.23) is a Lagrangian SDE with action functional*

$$S = \int_0^t \left(\frac{1}{2}((V_s^x)^2 + (V_s^z)^2) - F(\sqrt{X_s^2 + Z_s^2})\right) ds + \int_0^t X_s dW_s^1 + \int_0^t Z_t dW_s^2,$$

where $F(r) = \int_0^r f(\rho) d\rho$. The flow of the quasi-strong symmetry V leaves the functional S invariant, but equation (2.23) does not admit a conservation law associated with V. Furthermore, as already noted in [177], the reduction of (2.23) along V allows us to reduce by one (and not by two as in the deterministic case) the dimension of the system.

2.3.5 A financial mathematics application: the SABR model

In this section we discuss a stochastic volatility model used in mathematical finance to describe the stock price s with volatility u under an equivalent martingale measure for s (see [86]). The deep geometric properties of this model, related with Brownian motion on the Poincarè plane, are well known and suitably exploited in order to obtain asymptotic expansion formula for options evaluation (see [85]).
The SABR model is a two-dimensional system of the form

$$\begin{pmatrix} dS_t \\ dU_t \end{pmatrix} = \begin{pmatrix} U_t (S_t)^\beta & 0 \\ \alpha U_t \rho & \alpha U_t \sqrt{1-\rho^2} \end{pmatrix} \cdot \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix}, \tag{2.24}$$

where $\beta, \alpha, \rho \in \mathbb{R}$ and $0 < \beta < 1$ and $0 \leq \rho \leq 1$.
This SDE admits two symmetries

$$\begin{aligned} V_1 = (Y_1, C_1, \tau_1) &= \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0, -\frac{2}{u}\right), \\ V_2 = (Y_2, C_2, \tau_2) &= \left(\begin{pmatrix} s \\ (1-\beta) u \end{pmatrix}, 0, 0\right) \end{aligned}$$

and we can find a suitable random time change transforming (2.24) into an integrable SDE as a part of a stochastic transformation $T = (\Phi, B, \eta)$ with $B = I_2$ satisfying

$$\begin{aligned} Y_1(\Phi) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ Y_2(\Phi) &= \begin{pmatrix} 1 \\ f(s,u) \end{pmatrix} \\ Y_1(\eta) &= -\tau_1 \eta \\ Y_2(\eta) &= 0 \end{aligned}$$

(where f is an arbitrary function). A solution to this system is

$$\begin{aligned}\Phi(s,u) &= \begin{pmatrix} \log s \\ u \end{pmatrix}, \\ \eta &= \frac{u^2}{s^{2-2\beta}}\end{aligned}$$

and putting $t' = \int_0^t \eta(S_s, U_s) ds$ and $(s', u') = \Phi(s,u)$ we obtain the following SDE in the new coordinates

$$\begin{pmatrix} dS'_{t'} \\ dU'_{t'} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} dt' + \begin{pmatrix} 1 & 0 \\ \alpha e^{(1-\beta)S'_{t'}}\rho & \alpha e^{(1-\beta)S'_{t'}}\sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} dW'^1_{t'} \\ dW'^2_{t'} \end{pmatrix},$$

that is easily integrable.

In [8] the Authors, discussing the non-correlated SABR model ($\rho = 0$), propose a different random time change

$$\tilde{t} = \int_0^t U_s^2 ds$$

in order to derive an analytic formula for the solutions to (2.24). According to this new time variable, the equation becomes

$$\begin{pmatrix} dS_{\tilde{t}} \\ dU_{\tilde{t}} \end{pmatrix} = \begin{pmatrix} (S_{\tilde{t}})^\beta & 0 \\ \alpha\rho & \alpha\sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} d\tilde{W}^1_{\tilde{t}} \\ d\tilde{W}^2_{\tilde{t}} \end{pmatrix},$$

and its symmetries are

$$\begin{aligned}\tilde{V}_1 &= \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0, 0 \right), \\ \tilde{V}_2 &= \left(\begin{pmatrix} s \\ (1-\beta)u \end{pmatrix}, 0, 2(1-\beta) \right).\end{aligned}$$

Therefore the time change $\tilde{t}$ transforms V_1 into the strong symmetry $\tilde{V}_1$ and V_2 into the symmetry $\tilde{V}_2$, which, since β is a constant, corresponds to a deterministic time change. The symmetry $\tilde{V}_2$, restricted to the s variable, is the symmetry of a Bessel process: indeed the process S solves an equation for a spatial changed Bessel process. The Bessel process is one of the few one-dimensional stochastic processes whose transition probability is explicitly known and is a special case of the general affine processes class (see [57]). We remark that the time change $\tilde{t}$ can be uniquely characterized by the special form of $\tilde{V}_1$ and $\tilde{V}_2$, whose expression can be recovered within our symmetry analysis. Finally this last example suggests the possibility of extending the integrability notion to processes which are not progressively reconstructible from gaussian processes but, more in general, from other processes with notable analytical properties, such as Bessel process, affine processes or other processes.

Chapter 3

A symmetry-adapted numerical scheme for SDEs

In this chapter we study a possible use of symmetries of an SDE for improving the numerical integration methods for the considered SDE. In particular, after recalling the standard numerical schemes for SDEs, we introduce the notion of symmetric numerical scheme and we give necessary and sufficient conditions such that the standard discretizations preserve the symmetries of the given SDE. We apply our symmetric numerical schemes to the general linear SDE and we investigate the behaviour of the forward error for the proposed integration methods.

3.1 Numerical integration of SDEs

For the convenience of the reader, we recall the two main numerical methods for simulating an SDE and a theorem on the strong convergence of these methods (for a detailed description see e.g. [111]).
Consider the SDE with coefficients (μ, σ), driven by the Brownian motion W, and let $\{t_\ell\}_\ell$ be a partition of $[0, T]$. The *Euler scheme* for the equation (μ, σ) with respect to the given partition is provided by the following sequence of random variables $X_\ell \in M$

$$X^i_\ell = X^i_{\ell-1} + \mu^i(X_{\ell-1})\Delta t_\ell + \sum_{\alpha=1}^{n} \sigma^i_\alpha(X_{\ell-1})\Delta W^\alpha_\ell,$$

where $\Delta t_\ell = t_\ell - t_{\ell-1}$ and $\Delta W^\alpha_\ell = W^\alpha_{t_\ell} - W^\alpha_{t_{\ell-1}}$. The *Milstein scheme* for the same equation (μ, σ) is instead constituted by the sequence of random variables $\bar{X}_\ell \in M$ such that

$$\begin{aligned} \bar{X}^i_\ell &= \bar{X}^i_{\ell-1} + \mu^i(\bar{X}_{\ell-1})\Delta t_\ell + \sum_{\alpha=1}^{n} \sigma^i_\alpha(\bar{X}_{\ell-1})\Delta W^\alpha_\ell + \\ &\quad + \frac{1}{2}\sum_{j=1}^{m}\sum_{\alpha,\beta=1}^{n} \sigma^j_\alpha(\bar{X}_{\ell-1})\partial_j(\sigma^i_\beta)(\bar{X}_{\ell-1})\Delta \mathbb{W}^{\alpha,\beta}_\ell, \end{aligned}$$

where $\Delta \mathbb{W}^{\alpha,\beta}_\ell = \int_{t_{\ell-1}}^{t_\ell} (W^\beta_s - W^\beta_{t_{\ell-1}})dW^\alpha_s$. We recall that when $n = 1$ we have that

$$\Delta \mathbb{W}^{1,1}_\ell = \frac{1}{2}((\Delta W_\ell)^2 - \Delta t_\ell).$$

Theorem 3.1 *Let us denote by X_t the exact solution to an SDE (μ, σ) and by X_N and $\bar{X}_N$ the N-step approximations according with Euler and Milstein scheme respectively. Suppose that the coefficients (μ, σ) are C^2 with bounded derivatives and put $t_\ell = \frac{\ell T}{N}$ and $h = \frac{T}{N}$. Then there exists a constant $C(T, \mu, \sigma)$ such that*

$$\epsilon_N = \left(\mathbb{E}[\|X_T - X_N\|^2]\right)^{1/2} \leq C(T, \mu, \sigma) h^{1/2}.$$

Furthermore when the coefficients (μ, σ) are C^3 with bounded derivatives then there exists a constant $\bar{C}(T, \mu, \sigma)$ such that

$$\bar{\epsilon}_N = \left(\mathbb{E}[\|X_T - \bar{X}_N\|^2]\right)^{1/2} \leq \bar{C}(T, \mu, \sigma) h.$$

Proof. See Theorem 10.2.2 and Theorem 10.3.5 in [111]. ■

Theorem 3.1 states that X_N and $\bar{X}_N$ strongly converge in $L^2(\Omega)$ to the exact solution X_T to the SDE (μ, σ), where the order of the convergence with respect to the step size variation $h = \frac{T}{N}$ is $\frac{1}{2}$ in the Euler case and 1 in the Milstein one.
Nevertheless the theorem gives no information on the behaviour of the numerical approximations when we fix the step size h and we vary the final time T. In the standard proof of Theorem 3.1 one estimates the constants $C(T, \mu, \sigma)$ and $\bar{C}(T, \mu, \sigma)$ by proving that there exist two positive constants $K(\mu, \sigma), K'(\mu, \sigma)$ such that $C(T, \mu, \sigma) = \exp(T \cdot K(\mu, \sigma))$ and $\bar{C}(T, \mu, \sigma) = \exp(T \cdot K'(\mu, \sigma))$, by using Gronwall Lemma. In some situations the exponential growth of the error is a correct prediction (see for example [142]).
Of course this fact does not mean that in any case the errors ϵ_n and $\bar{\epsilon}'_n$ exponentially diverge with the time T. Indeed if the SDE (μ, σ) admits an equilibrium distribution it could happen that the two errors remain bounded with respect to the time T. Unfortunately this situation does not happen for any values of the step size h, but only for values within a certain region. The phenomenon just described is known as the *stability* problem for a discretization method of an SDE. This problem, and the corresponding definition, is usually stated and tested for some specific SDEs (see e.g. [95, 167] for the geometric Brownian motion, see e.g. [93, 156] the Ornstein-Uhlenbeck process, see e.g. [96, 97] for non-linear equations with a Dirac delta equilibrium distribution, and see e.g. [175] for more general situation). We will show some numerical examples of the stability phenomenon for general linear SDEs in Section 3.5.

3.2 Numerical integration via symmetries

When a system of ODEs admits Lie-point symmetries, invariant numerical algorithms can be constructed (see e.g. [127, 126, 56, 33]). By completeness we recall the definition of an invariant numerical scheme for a system of ODEs, in the simple case of one-step algorithms. The obvious extension for multi-step numerical schemes is immediate. The discretization of an ODEs system is a function $F : M \times \mathbb{R} \to M$ such that if $x_\ell, x_{\ell-1} \in M$ are the $\ell, \ell - 1$ steps respectively and Δt_ℓ is the step size of our discretization we have that

$$x_\ell = F(x_{\ell-1}, \Delta t_\ell).$$

If $\Phi : M \to M$ is a diffeomorphism we say that the discretization defined by the map F is *invariant* with respect to the map Φ if

$$\Phi(x_\ell) = F(\Phi(x_{\ell-1}), \Delta t_\ell).$$

If we require that the previous property holds for any $x_\ell \in \mathbb{R}^n$ and for any $\Delta t_\ell \in \mathbb{R}_+$ we get

$$\Phi^{-1}(F(\Phi(x), \Delta t)) = F(x, \Delta t) \tag{3.1}$$

for any $x \in M$ and $\Delta t \in \mathbb{R}$. If Φ_a is an one-parameter group generated by the vector field $Y = Y^i(x)\partial_{x^i}$, by deriving the relation $\Phi_{-a}(F(\Phi_a(x), \Delta t)) = F(x, \Delta t)$ with respect to a, we obtain the relation

$$Y^i(F(x, \Delta t)) - Y^k \partial_{x^k}(F)(x, \Delta t) = 0 \tag{3.2}$$

which guarantees that the discretization F is invariant with respect to the flow Φ_a, generated by Y.

In the following we extend the previous definition to the case of an SDE. We discuss only the case of integration schemes which depending on the time Δt and on the Brownian motion $\Delta W^\alpha_\ell, \alpha = 1, \ldots, n$ (as for example the Euler method). The same discussion for integration methods which depend also on $\Delta \mathbb{W}^{\alpha,\beta}_\ell$ or other random variables (as the Milstein method) is immediate. In the stochastic case the discretization is a map $F : M \times \mathbb{R} \times \mathbb{R}^m \to M$ and we have

$$x_\ell = F(x_{\ell-1}, \Delta t_\ell, \Delta W^1_\ell, ..., \Delta W^m_\ell).$$

Equations (3.1) and (3.2) become

$$\begin{aligned} \Phi^{-1}(F(\Phi(x), \Delta t, \Delta W^\alpha)) &= F(x, \Delta t, \Delta W^\alpha), && (3.3)\\ Y^i(F(x, \Delta t, \Delta W^\alpha)) - Y^k \partial_{x^k}(F)(x, \Delta t, \Delta W^\alpha) &= 0. && (3.4) \end{aligned}$$

Since the convergence of Riemann sums in the Itô integration theory strongly depends on the fact that the Riemann sum approximation is backward (and not forward), we stress again that it is not easy to prove that a given discretization X_ℓ converges to the real solution to the SDE (μ, σ). For this reason we give a theorem which provides a sufficient (and necessary) condition in order to ensure that Euler and Milstein discretizations are invariant with respect to a Lie algebra of strong symmetries $Y_1, ..., Y_r$.

Theorem 3.2 *Let $Y_1, ..., Y_r$ be strong symmetries of an SDE (μ, σ). When $Y^i_j = Y_j(x^i)$ are polynomials of first degree in $x^1, ..., x^m$, then the Euler discretization (or the Milstein discretization) of the SDE (μ, σ) is invariant with respect to $Y_1, ..., Y_r$. If for a given $x_0 \in M$, $\mathrm{span}\{\sigma_1(x_0), \ldots, \sigma_n(x_0)\} = \mathbb{R}^m$, also the converse holds.*

Proof. We give the proof for the Euler discretization; the Milstein discretization case is analogous. In the case of Euler discretization we have that

$$F^i(x) = x^i + \mu^i(x)\Delta t + \sigma^i_\alpha(x)\Delta W^\alpha.$$

The discretization is invariant if and only if

$$\begin{aligned} 0 = Y_j(F^i)(x) - Y^i_j(F(x)) &= +Y^k_j \partial_k(F^i)(x) - Y^i_j(F(x))\\ &= Y^i_j(x) + Y^k_j(x)\partial_k(\mu^i)(x)\Delta t + Y^k_j(x)\partial_k(\sigma^i_\alpha)(x)\Delta W^\alpha +\\ &\quad -Y^i_j(x + \mu\Delta t + \sigma_\alpha \Delta W^\alpha). \end{aligned}$$

Recalling that Y_j is a symmetry for the SDE (μ, σ) and therefore it has to satisfy the determining equations (1.18) and (1.19) when $C = 0$ and $\tau = 0$, we have that the Euler discretization is invariant if and only if

$$\begin{aligned} &Y^i_j(x) + \mu^k(x)\partial_k(Y^i_j)(x)\Delta t + \tfrac{1}{2}\textstyle\sum_\alpha \sigma^k_\alpha \sigma^h_\alpha \partial_{kh}(Y^i_j)(x)\Delta t + \sigma^k_\alpha(x)\partial_k(Y^i_j)(x)\Delta W^\alpha =\\ &\quad = Y^i_j(x + \mu\Delta t + \sigma_\alpha \Delta W^\alpha). \end{aligned} \tag{3.5}$$

Suppose that $Y^i_j = B^i_j + C^i_{j,k} x^k$, then

$$\begin{aligned} Y^i_j(x) + \mu^k(x)\partial_k(Y^i_j)(x)\Delta t + \tfrac{1}{2}\textstyle\sum_\alpha \sigma^k_\alpha \sigma^h_\alpha \partial_{kh}(Y^i_j)(x)\Delta t + \sigma^k_\alpha(x)\partial_k(Y^i_j)(x)\Delta W^\alpha = \\ = B^i_j + C^i_{j,k}x^k + C^i_{j,k}\mu^k(x)\Delta t + C^i_{j,k}\sigma^k_\alpha(x)\Delta W^\alpha \\ = B^i_j + C^i_{j,k}(x^k + \mu^k(x)\Delta t + \sigma^k_\alpha(x)\Delta W^\alpha) \\ = Y^i_j(x + \mu\Delta t + \sigma_\alpha \Delta W^\alpha). \end{aligned}$$

Conversely, suppose that the Euler discretization is invariant and so equality (3.5) holds. Let x_0 be as in the hypotheses of the theorem and choose $\Delta t = 0$. Then

$$Y^i_j(x_0 + \sigma_\alpha \Delta W^\alpha) = Y^i_j(x_0) + \sigma^k_\alpha(x_0)\partial_{x^k}(Y^i_j)(x_0)\Delta W^\alpha.$$

Since ΔW^α are arbitrary and $\text{span}\{\sigma_1(x_0), ...\sigma_m(x_0)\} = \mathbb{R}^m$, Y^i_j must be of first degree in $x^1, ..., x^m$. ■

Theorem 3.2 can be fruitfully applied in the following way. If $Y_1, ..., Y_r$ are strong symmetries of an SDE we search a diffeomorphism $\Phi : M \to M' \subset \mathbb{R}^m$ (i.e. a coordinate change) such that $\Phi_*(Y_1), ..., \Phi_*(Y_r)$ have coefficients of first degree in the new coordinates system $x'^1, ..., x'^m$. We discretize the transformed SDE $\Phi(\mu, \sigma)$ using the Euler discretization, obtaining a discretization $\tilde{F}(x', \Delta t, \Delta W^\alpha)$ which is invariant with respect to $\Phi_*(Y_1), ..., \Phi_*(Y_r)$. As a consequence the discretization $F = \Phi(\tilde{F}(\Phi^{-1}(x), \Delta t, \Delta W^\alpha)$ is invariant with respect to $Y_1, ..., Y_r$. It is easy to prove that if the map Φ is Lipschitz we have that the constructed discretization converges in L^1 to the solution, while if the map Φ is only locally Lipschitz, the weaker convergence in probability can be established.
The existence of the diffeomorphism Φ allowing the application of Theorem 3.2 for general $Y_1, ..., Y_r$ is not guaranteed. Furthermore, even when the map Φ exists, unfortunately in general it is not unique. Consider for example the following one-dimensional SDE

$$dX_t = \left(a\tanh(X_t) - \frac{b^2}{2}\tanh^3(X_t)\right)dt + b\tanh(X_t)dW_t, \tag{3.6}$$

which has

$$Y = \tanh(x)\partial_x$$

as a strong symmetry. There are many transformations Φ which are able to put Y with coefficients of first degree, for example the following two transformations:

$$\begin{aligned} \Phi_1(x) &= \sinh(x) \\ \Phi_2(x) &= \log|\sinh(x)|. \end{aligned}$$

Indeed we have that

$$\Phi_{1,*}(Y) = x'_1 \partial_{x'_1}, \quad \Phi_{2,*}(Y) = \partial_{x'_2}.$$

While the map Φ_1 transforms equation (3.6) into a geometrical Brownian motion, the transformation Φ_2 reduces equation (3.6) to a Brownian motion with drift. By applying Euler method by means of Φ_1 we obtain a poor numerical result (in fact Φ_1 is not a Lipschitz function and in this circumstance errors are amplified). By exploiting Φ_2 to make the discretization we obtain instead an exact simulation. The example shows that this first approach strongly depends on the choice of the diffeomorphism Φ (which has to be invertible in terms of elementary functions). So it is better to have another procedure able to individuate the best coordinate system for performing the SDE discretization.

In order to solve the previous problem and choose the best coordinate system for the discretization procedure we use the results on reduction and reconstruction of Chapter 2.
Indeed suppose that an SDE (μ, σ) admit a set $Y_1, ..., Y_r$ of strong symmetries generating a solvable Lie algebra. By Theorem 2.6 there exists a (local) diffeomorphims $\Phi : M \to M'$ such that, eventually relabelling the vector fields $Y_1, ..., Y_r$, $\Phi_*(Y_1), ..., \Phi(Y_r)$ are in canonical form. This means that if $x_1'^1, ..., x_1'^r$ and $x_2'^1, ..., x_2'^{m-r}$ is the natural coordinate system of M' we have that the SDE $(\mu', \sigma') = E_{(\Phi,I,1)}(\mu, \sigma)$ has the form

$$\begin{aligned} dX_{2,t}'^i &= \mu_2'^i(X_{2,t}')dt + \sigma_{2,\alpha}'^i(X_{2,t}')dW_t^\alpha \\ dX_{1,t}'^j &= \mu_1'^j(X_{1,t}'^1, ..., X_{1,t}'^{i-1}, X_{2,t}')dt + \sigma_{1,\alpha}'^j(X_{1,t}'^1, ..., X_{1_t}'^{i-1}, X_{2,t}')dW_t^\alpha, \end{aligned}$$

where $\mu_1^i, \sigma_{1,\alpha}^i$ do not depend on $x_1^i, ..., x_1^r$. The above SDE is triangular in the variables $(x_1^1, ..., x_1^r)$. By discretizing a triangular SDE (μ', σ') we reasonably expect a better behavior than in the general case. Furthermore if $X_{2,t}'^1, ..., X_{2,t}'^{m-r}$ can be exactly simulated with $\sigma_{2,\alpha}'^i, \mu_2'^i$ growing at most polynomially, we can conjecture that the error grows polynomially with respect to the maximal integration time T.

In this way we can formulate another algorithm to discretize a symmetric SDE. We can discretize (μ', σ') according with one of standard methods obtaining a discretization $\tilde{F}$. By composing $\tilde{F}$ with Φ we obtain a discretization $F(x, \Delta t, \Delta W^\alpha) = \Phi^{-1}(\tilde{F}(\Phi(x), \Delta t, \Delta W^\alpha)$ which, when Φ is Lipschitz, has the property of being a more simple triangular discretization scheme.

The discretization scheme F based on the triangularization of the SDE (μ, σ) is not in general an invariant scheme with respect to the vector fields $Y_1, ..., Y_r$. Indeed in general the (local) diffeomorphism Φ, putting $Y_1, ..., Y_r$ in canonical form, does not transform $\Phi_*(Y_1), ..., \Phi_*(Y_r)$ into a set of vector fields with coefficients of first degree in $x'^1, ..., x'^m$. This means that, for Theorem 3.2, $\tilde{F}$, and hence F, is not an invariant numerical scheme. Nevertheless, if we consider solvable Lie algebras satisfying a special relation, it always possible to choose Φ such that $\Phi_*(Y_1), ..., \Phi_*(Y_r)$ have coefficients of first degree in $x'^1, ..., x'^r$.

Proposition 3.3 *Suppose that the Lie algebra $\mathcal{G} = \mathrm{span}\{Y_1, ..., Y_r\}$ is such that $[[\mathcal{G}, \mathcal{G}], [\mathcal{G}, \mathcal{G}]] = 0$. Then the coefficients of $\Phi_*(Y_1), ..., \Phi_*(Y_r)$ are of first degree in $x'^1, ..., x'^r$. Moreover one can choose Φ such that the coefficients of $\Phi_*(Y_1), \dots, \Phi_*(Y_r)$ are of first degree in all the variables $x'^1, ..., x'^m$.*

Proof. Let us suppose that $Y_1, ..., Y_k$ generate $\mathcal{G}^{(1)} = [\mathcal{G}, \mathcal{G}]$. Then $\Phi^*(Y_i) = (\delta_i^l)$ for $i = 1, ..., k$. Using the fact that $[Y_i, \mathcal{G}^{(1)}] \subset \mathcal{G}^{(1)}$ and the fact that $\Phi_*(Y_1), ..., \Phi_*(Y_r)$ are in canonical form, we have that $\Phi_*(Y_{k+1}), ..., \Phi_*(Y_r)$ do not depend on $x'^{k+1}, ..., x'^r$ and their coefficients have to be of first degree in $x'^1, ..., x'^r$.
The second part of the proposition follows from the well known fact that when the vector fields $Y_1, ..., Y_r$ generate an integrable distribution, it is possible to choose a local coordinate system such that the coefficients of $Y_1, ..., Y_r$ do not depend on $x'^{r+1}, ..., x'^m$. ∎

3.3 General one-dimensional linear SDEs

We consider the one-dimensional linear SDE

$$dX_t = (aX_t + b)dt + (cX_t + d)dW_t, \tag{3.7}$$

where $a,b,c,d \in \mathbb{R}$ and we apply the previous procedure in order to obtain a symmetry adapted discretization scheme. As noted in Section 2.3.2, this is a special example of Kolmogorv-Pearson diffusion (2.11).
For Proposition 2.19, equation (3.7) does not admit strong symmetries if $ad-bc \neq 0$. But, as noted in Section 2.3.2, equation (3.7) can be seen as a part of the two dimensional SDE (2.16) admitting two symmetries forming a two dimensional solvable Lie algebra. In the case of equation (3.7) system (2.16) becomes the following

$$\begin{pmatrix} dX_t \\ dZ_t \end{pmatrix} = \begin{pmatrix} aX_t+b \\ aZ_t \end{pmatrix} dt + \begin{pmatrix} cX_t+d \\ cZ_t \end{pmatrix} dW_t, \tag{3.8}$$

on $\mathbb{R}\times\mathbb{R}_+ = M$, consisting of the original linear equation and the associated homogeneous one. The two symmetries of SDE (3.8) are not quasi-strong symmetries, as in the generic case of equation (2.16), but they are strong symmetries having the form

$$\begin{aligned} Y_1 &= \begin{pmatrix} z \\ 0 \end{pmatrix} \\ Y_2 &= \begin{pmatrix} 0 \\ z \end{pmatrix}. \end{aligned}$$

The more general adapted coordinate system system for the symmetries Y_1, Y_2 is given by

$$\Phi(x,z) = \begin{pmatrix} \frac{x}{z} + f(z) \\ \log(z) + l \end{pmatrix},$$

where $l \in \mathbb{R}$ and $f : \mathbb{R}_+ \to \mathbb{R}$ is a smooth function. Indeed in the coordinate system $(x', z')^T = \Phi(x,z)$ we have

$$\begin{aligned} Y_1' &= \Phi_*(Y_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ Y_2' &= \Phi_*(Y_2) = \begin{pmatrix} -x' + e^{z'-l}\partial_z(f)(e^{z'-l}) + f(e^{z'-l}) \\ 1 \end{pmatrix}. \end{aligned}$$

In order to guarantee that the Euler and Milstein discretization schemes are invariant, by Theorem 3.2 it is sufficient to choose $f(z) = -\frac{k}{z}$ for some constant k.
In the new coordinates the original two dimensional SDE becomes

$$dX_t' = \left(\left(b - cd + ak - c^2k\right) e^{-Z_t'+l}\right) dt + (d+ck)e^{-Z_t'+l} dW_t \tag{3.9}$$

$$dZ_t' = \left(a - \frac{c^2}{2}\right) dt + c\,dW_t. \tag{3.10}$$

In the following, for simplicity, we consider the discretization scheme only for $l = 0$. The Euler integration scheme becomes:

$$\begin{aligned} \begin{pmatrix} Z_\ell' \\ X_\ell' \end{pmatrix} &= \begin{pmatrix} Z_{\ell-1}' \\ X_{\ell-1}' \end{pmatrix} + \begin{pmatrix} \left(a - \frac{c^2}{2}\right) \\ \left(b - cd + ak - c^2k\right) e^{-Z_{\ell-1}'} \end{pmatrix} \Delta t_\ell + \\ &\quad + \begin{pmatrix} c \\ (d+ck)e^{-Z_{\ell-1}'} \end{pmatrix} \Delta W_\ell, \end{aligned}$$

and the Milstein scheme:

$$\begin{pmatrix} Z'_\ell \\ X'_\ell \end{pmatrix} = \begin{pmatrix} Z'_{\ell-1} \\ X'_{\ell-1} \end{pmatrix} + \begin{pmatrix} \left(a-\frac{c^2}{2}\right) \\ \left(b-\frac{1}{2}cd+ak-\frac{c^2k}{2}\right)e^{-Z'_{\ell-1}} \end{pmatrix}\Delta t_n + \\ + \begin{pmatrix} c \\ (d+ck)e^{-Z'_{\ell-1}} \end{pmatrix}\Delta W_\ell + \begin{pmatrix} 0 \\ -(cd+c^2k)e^{-Z'_{\ell-1}} \end{pmatrix}\frac{(\Delta W_\ell)^2}{2}$$

We note that when $k=-\frac{d}{c}$ the two discretization schemes coincide.
Coming back to the original problem, in the Euler case we get:

$$X_\ell = \exp\left(\left(a-\frac{c^2}{2}\right)\Delta t_\ell + c\Delta W_\ell\right)\cdot[X_{\ell-1}+(b-cd+ak-c^2k)\Delta t_\ell+(d+ck)\Delta W_\ell-k]+k, \quad (3.11)$$

while in the Milstein case we obtain:

$$\begin{aligned} X_\ell &= \exp\left(\left(a-\tfrac{c^2}{2}\right)\Delta t_\ell + c\Delta W_\ell\right)\cdot\Big[X_{\ell-1}+\left(b+ak-\tfrac{cd+c^2k}{2}\right)\Delta t_\ell + \\ &\quad +(d+ck)\Delta W_\ell - \tfrac{(cd+c^2k)}{2}(\Delta W_\ell)^2-k\Big]+k. \end{aligned} \quad (3.12)$$

Remark 3.4 *There is a deep connection between equations (3.11) and (3.12) and the well-known integration formula for scalar linear SDEs. Indeed, equation (3.7) admits as solution*

$$X_t = \Phi_t\left(X_0+\int_0^t \frac{b-cd}{\Phi_s}ds + \int_0^t \frac{d}{\Phi_s}dW_s\right) \quad (3.13)$$

where

$$\Phi_t = \exp\left(\left(a-\frac{c^2}{2}\right)t + cW_t\right).$$

Equation (3.11) and (3.12) can be viewed as the equations obtained by expanding the integrals in formula (3.13) according with stochastic Taylor's Theorem (see [111]). This fact should not surprise since the adapted coordinates obtained in Section 3.2 has been introduced exactly to obtain formula (3.13) from equation (3.8). Since the discretizations schemes (3.11) and (3.12) are closely linked with the exact solution formula of linear SDEs we call them exact methods *(or exact discretizations) for the numerical simulation of linear SDEs.*

3.4 Theoretical estimation of the numerical forward error for linear SDEs

In this section we provide an explicit estimation of the forward error associated with the exact numerical schemes proposed in the previous section for simulating a general linear SDE. The explicit solution to a one-dimensional linear SDE is well known and the use of the resolutive formula for its simulation is extensively used, but in the literature, to the best of our knowledge, there is no explicit estimation of the forward error.

3.4.1 Main results

Dividing $[0,T]$ in N parts we obtain $N+1$ instants $t_0=0, t_i=ih, t_N=T$, with $h=\frac{T}{N}$. We denote by $X_t^{N,T}$ the approximate solution given by exact Euler method, $\bar{X}_t^{N,T}$ the approximate solution with respect to exact Milstein method and by X_t the exact solution to the linear SDE. In the following, where non confusion arises, we will omit T.

Theorem 3.5 *For all $t, T \in \mathbb{R}, t \in [0, T]$, we have*

$$\epsilon_N = \left(\mathbb{E}[(X_t - X_t^{N,T})^2]\right)^{1/2} \leq f(T)g(h)h^{1/2},$$

where $h = \frac{T}{N}$, g is a continuous function and f is a strictly positive continuous function such that for $x \to +\infty$

$$\begin{aligned} f(x) = O(1) \quad & \text{if} \quad a < -c^2/2 \\ f(x) = O(x) \quad & \text{if} \quad a = -c^2/2 \\ f(x) = O(e^{C(a,c)x}) \quad & \text{if} \quad a > -c^2/2, \end{aligned}$$

with $C(a, c) \in \mathbb{R}_+$.

Theorem 3.6 *For all $t, T \in \mathbb{R}, t \in [0, T]$, we have that*

$$\bar{\epsilon}_N = \mathbb{E}[|X_t - \bar{X}_t^{N,T}|] \leq \bar{f}(T)\bar{g}(h)h^{1/2},$$

where $h = \frac{T}{N}$, $\bar{g}$ is a continuous function and f is a strictly positive continuous function such that for $x \to +\infty$

$$\begin{aligned} \bar{f}(x) = O(1) \quad & \text{if} \quad a < 0 \\ \bar{f}(x) = O(e^{C'(a,c)x}) \quad & \text{if} \quad a \geq 0, \end{aligned}$$

with $C'(a, c) \in \mathbb{R}_+$.

Before giving the proof of the two previous theorems we propose some remarks. We recall that a linear SDE with $ad - bc \neq 0$ has an equilibrium distribution if and only if $a - \frac{c^2}{2} < 0$. Furthermore the equilibrium distribution admits a finite first moment if and only if $a < 0$ and a finite second moment if and only if $a + \frac{c^2}{2} < 0$. Since we approximate the Itô integral up to the order $h^{1/2}$, the three cases in Theorem 3.5 are consequences of the observation that in order to give an estimate of the error in Euler discretization we need a control on the second moment. More precisely we can expect a bounded error with respect to T only when the second moment is finite as $T \to +\infty$. Since in the Milstein case a finite first moment sufficies, in the second theorem we obtain that the error does not grow with T when $a < 0$. We can obtain an analogous estimate for the Euler method when $d = 0$, i.e. in the case in which the Milstein and Euler discretizations coincide (this situation is similar to the additive-noise-SDEs setting). The use of only the first moment finiteness for estimating the error has a price: indeed we obtain an $h^{1/2}$ dependence of the error. We remark that the techniques used in the proof of Theorem 3.6 exploit some ideas from the recent *rough path* integration theory (see e.g. [73]), and in particular this circumstance explains the $\frac{1}{2}$ order of convergence. This fact induces us to conjecture that our proof probably works also in the general rough path framework (for example for fractional Brownian motion, by following [72]). If in Theorem 3.6 we do not require an uniform-in-time estimate, we can apply the methods used in the proof of Theorem 3.5 for obtaining an error convergence of order 1.
Essentially the above theorems prove that for $a + \frac{c^2}{2} < 0$ and for $a < 0$ respectively, our symmetry adapted discretization methods are stable for any value of h. In Section 3.5 we give a comparison between the stability of the adapted-coordinates schemes with respect to the standard Euler and

Milstein ones, via numerical simulations.
We conclude by noting that Theorem 3.5 and Theorem 3.6 cannot be deduced in a trivial way from the standard theorems about the convergence of Euler and Milstein methods (such as Theorem 3.1). Indeed the Euler and Milstein discretizations of equations (3.9) and (3.10) do not have Lipschitz coefficients. Furthermore, even if a given discretization (X'_n, Z'_n) of the system composed by (3.9) and (3.10) should converge to the exact solution in $L^2(\Omega)$, being the coordinate change Φ (introduced in Section 3.3) not globally Lipschitz, it does not imply that the transformed discretization (X_n, Z_n) converges to the exact solution (X, Z) to equation (2.16) in $L^2(\Omega)$. Finally, as pointed out in Section 3.1, Theorem 3.1 does not guarantee an uniform-in-time convergence as Theorem 3.5 and Theorem 3.6 instead state.

For proving the theorems we need the following two lemmas. The second one allows us to avoid very long calculations (see Appendix).

Lemma 3.7 *Let W_t be a Brownian motion, $\alpha, \beta \in \mathbb{R}$ and $n \in \mathbb{N}$ then for any $t \in \mathbb{R}_+$*

$$\mathbb{E}[\exp(\alpha t + \beta W_t) W_t^n],$$

is a continuous function of t and in particular it is locally bounded. Moreover we have that

$$\mathbb{E}[\exp(\alpha t + \beta W_t)] = \exp\left(\alpha + \frac{\beta^2}{2}\right) t.$$

Proof. The proof is based on the fact that W_t is a normal random variable with zero mean and variance equal to t. ■

Lemma 3.8 *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $F(0,0) = 0$ and such that*

$$\mathbb{E}\left[|\partial_t(F)(h, W_h)|^\alpha\right], \mathbb{E}[|\partial_w(F)(h, W_h)|^\alpha], \mathbb{E}[|\partial_{ww}(F)(h, W_h)|^\alpha] < L(h),$$

for some $\alpha \in 2\mathbb{N}$, for any h and for some continuous function $L : \mathbb{R} \to \mathbb{R}_+$. Then there exists an increasing function $C : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}[|F(h, W_h)|^\alpha] \leq C(h) h^{\alpha/2}.$$

If furthermore $\partial_w(F)(0,0) = 0$ and

$$\mathbb{E}\left[|\partial_{www}(F)(h, W_h)|^\alpha\right], \mathbb{E}[|\partial_{tw}(F)(h, W_h)|^\alpha] \leq L(h)$$

there exists an increasing function $C' : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}[|F(h, W_h)|^\alpha] \leq C'(h) h^\alpha.$$

Proof. The two theses of the lemma are some special cases of Lemma 5.6.4 and Lemma 5.6.5 in [111]. ■

3.4.2 Proof of Theorem 3.5

We consider the case $t = T$. In fact we will find that our estimate is uniform for $t \leq T$. Using the notations in Remark 3.4 we can write $X_T = I_1 + I_2$ where

$$\begin{aligned} I_1 &= \Phi_T \int_0^T (b - cd) \Phi_s^{-1} ds \\ I_2 &= \Phi_T \int_0^T (d) \Phi_s^{-1} dW_s. \end{aligned}$$

Also the approximation X_T^N can be written as the sum of two *integrals* of the form $X_T^N = I_1^N + I_2^N$ where

$$I_1^N = (b-cd)\sum_{i=1}^N \Phi_T\Phi_{t_{i-1}}^{-1}\Delta t_i, \quad I_2^N = d\sum_{i=1}^N \Phi_T\Phi_{t_{i-1}}^{-1}\Delta W_i.$$

Obviously the strong error ϵ_N can be estimated by $\|I_1 - I_1^N\|_2 + \|I_2 - I_2^N\|_2$, where hereafter $\|\cdot\|_\alpha = (\mathbb{E}[|\cdot|^\alpha])^{1/\alpha}$.

Estimate of $\|I_1 - I_1^N\|_2$

Setting $\Psi_{s,t} = \Phi_t(\Phi_s)^{-1}$ for any $s < t$, we obtain (with $\Delta t_i = h$)

$$\begin{aligned}
\|I_1 - I_1^N\|_2 &= \mathbb{E}\left[\left|\int_0^T (b-cd)\Psi_{t,T}dt - \sum_{i=1}^N (b-cd)\Psi_{t_{i-1},T}h\right|^2\right]^{1/2} \\
&= \mathbb{E}\left[\left|\sum_{i=1}^N \int_{t_{i-1}}^{t_i} (b-cd)(\Psi_{t,T} - \Psi_{t_{i-1},T})dt\right|^2\right]^{1/2} \\
&\le |b-cd|\left(\sum_{i=1}^N \mathbb{E}\left[\left(\int_{t_{i-1}}^{t_i} |\Psi_{t,T} - \Psi_{t_{i-1},T}|dt\right)^2\right]^{1/2}\right).
\end{aligned}$$

By Jensen's inequality

$$\sum_{i=1}^N \mathbb{E}\left[\left(\int_{t_{i-1}}^{t_i} |\Psi_{t,T} - \Psi_{t_{i-1},T}|dt\right)^2\right]^{1/2} \le h^{1/2}\sum_{i=1}^N \left(E\left[\int_{t_{i-1}}^{t_i} (\Psi_{t,T} - \Psi_{t_{i-1},T})^2 dt\right]\right)^{1/2}.$$

and by Fubini theorem we have to calculate $\mathbb{E}[(\Psi_{t,T} - \Psi_{t_{i-1},T})^2]$. Since

$$\Psi_{s,t} = \exp\left(\left(a - \frac{c^2}{2}\right)(t-s) + c(W_t - W_s)\right).$$

and $\Psi_{s,t} = \Psi_{s,u}\Psi_{u,t}$ for any $s \le u \le t$ we obtain that

$$\mathbb{E}[(\Psi_{t,T} - \Psi_{t_{i-1},T})^2] = \mathbb{E}[(\Psi_{t,T})^2]\mathbb{E}[(1 - \Psi_{t_{i-1},t})^2] \tag{3.14}$$

because $\Psi_{t,T}$ and $\Psi_{t_{i-1},t}$ are independent as a consequence of the Brownian increments independence.
We note that the function

$$F_1(t - t_i, W_t - W_{t_i}) = 1 - e^{(t-t_i)\left(a - \frac{c^2}{2}\right) + c(W_t - W_{t_i})},$$

satisfies $F_1(0,0) = 0$ and, by Lemma 3.7,

$$\mathbb{E}[\partial_t(F_1)(t-t_i, W_t - W_{t_i})], \mathbb{E}[\partial_w(F_1)(t-t_i, W_t - W_{t_i})], \mathbb{E}[\partial_{ww}(F_1)(t-t_i, W_t - W_{t_i})] < +\infty$$

Thus, by Lemma 3.8, there exists an increasing function $C_1(h)$

$$\mathbb{E}\left[(F_1(t-t_i, W_t - W_{t_i}))^2\right] \le C_1(t-t_i)(t-t_i).$$

Using Lemma 3.7 we get

$$\mathbb{E}\left[\Psi_{t,T}^2\right] = \exp((2a+c^2)(T-t)),$$

obtaining

$$\begin{aligned} \|I_1 - I_1^N\|_2 &\le |b-cd|\sqrt{C_1(h)}h^{1/2}\sum_{i=1}^N \exp\left(\left(a+\tfrac{c^2}{2}\right)(T-t_i)\right)h \\ &\le |b-cd|\sqrt{C_1(h)}G_1(T)h^{1/2}, \end{aligned} \tag{3.15}$$

where

$$G_1(T) = \int_0^T \exp\left(\left(a+\frac{c^2}{2}\right)(T-t)\right)dt = \frac{1}{a+\frac{c^2}{2}}(\exp((a+c^2/2)T)-1). \tag{3.16}$$

Estimate of $\|I_2 - I_2^N\|_2$

We first consider $I_2 = (d)\Phi_T\int_0^T(\Phi_t)^{-1}dW_t$. Since Itô integral involves only adapted processes we cannot bring Φ_T under the integral sign. However it is possible to take advantage of the backward integral formulation which allows us to integrate processes that are measurable with respect to the (future) filtration $\mathcal{F}^t = \sigma\{W_s | s\in[t,T]\}$. In particular when X_s is $\mathcal{F}^t$-measurable then

$$\int_0^T X_s d^+W_s = \lim_{n\to+\infty}\left(\sum_{i=1}^n X_{t_i^n}(W_{t_i^n} - W_{t_{i-1}^n})\right),$$

where $\{t_i^n\}|_i$ is a sequence of n points partitions of the interval $[0,T]$, having amplitude decreasing to 0 and the limit is understood in probability.
When F is a regular function, $F(W_t,t)$ is a process which is measurable with respect to both the filtrations $\mathcal{F}_t$ and $\mathcal{F}^t$; therefore one can calculate either $\int_0^T F(W_t,t)dW_t$ and $\int_0^T F(W_t,t)d^+W_t$. The next well-known lemma says that we can write I_2 in terms of a backward integral, which allows us to bring Φ_T under the integral sign.

Lemma 3.9 *Let $F:\mathbb{R}^2\to\mathbb{R}$ be a C^2-function such that*

$$\mathbb{E}[(F(W_t,t))^2] < +\infty.$$

Then

$$\int_0^T F(W_t,t)dW_t = \int_0^T F(W_t,t)d^+W_t - \int_0^T \partial_w(F)(W_t,t)dt.$$

Proof. We report the proof for convenience of the reader (see, e.g., [150]). Setting

$$\tilde{F}(w,t) = \int_0^w F(u,t)du,$$

since F is C^2 then also $\tilde{F}$ is C^2. From this fact one deduces that

$$\begin{aligned} \tilde{F}(W_t,t) - \tilde{F}(W_s,s) &= \int_s^t F(W_\tau,\tau)dW_\tau + \int_s^t \partial_t(\tilde{F})(W_\tau,\tau)d\tau \\ &\quad + \frac{1}{2}\int_s^t \partial_w(F)(W_\tau,\tau)d\tau \\ \tilde{F}(W_t,t) - \tilde{F}(W_s,s) &= \int_s^t F(W_\tau,\tau)d^+W_\tau + \int_s^t \partial_t(\tilde{F})(W_\tau,\tau)d\tau \\ &\quad - \frac{1}{2}\int_s^t \partial_w(F)(W_\tau,\tau)d\tau. \end{aligned}$$

By equating the two expressions one obtains the final formula. ∎

Since
$$(\Phi_t)^{-1} = \exp(-(a - c^2/2)t - cW_t) = F(W_t, t),$$
and $\partial_w(F)(w,t) = -cF(w,t)$, by Lemma 3.9, we can write
$$\begin{aligned} I_2 &= \Phi_T d \int_0^T (\Phi_t)^{-1} dW_t \\ &= \Phi_T d \left(\int_0^T (\Phi_t)^{-1} d^+ W_t + c \int_0^T (\Phi_t)^{-1} dt \right) \\ &= d \left(\int_0^T \Psi_{t,T} d^+ W_t + c \int_0^T \Psi_{t,T} dt \right). \end{aligned}$$
Introducing $\tilde{I}_2 = d \int_0^T \Psi_{t,T} d^+ W_t$ and
$$\tilde{I}_2^N = d \sum_{i=1}^N \Psi_{t_i,T} \Delta W_i,$$
we have that
$$\|I_2 - I_2^N\|_2 \le \|\tilde{I}_2 - \tilde{I}_2^N\|_2 + \left\| (\tilde{I}_2^N - I_2^N) + cd \int_0^T \Psi_{t,T} dt \right\|_2. \tag{3.17}$$
We first consider the term $\|\tilde{I}_2 - \tilde{I}_2^N\|_2$. The process $\tilde{I}_2^N$ can be written as $\int_0^T (d) H_t dW_t^+$ where H_t is the $\mathcal{F}^t-$ measurable process given by
$$H_t = \sum_{i=1}^N \Psi_{t_i,T} 1_{(t_{i-1},t_i]}(t),$$
where $1_{(t_{i-1},t_i]}$ is the characteristic function of the interval $(t_{i-1}, t_i]$. By Itô isometry and Fubini Theorem we obtain
$$\begin{aligned} \|\tilde{I}_2 - \tilde{I}_2^N\|_2^2 &= d^2 \mathbb{E}\left[\left(\int_0^T (\Psi_{t,T} - H_t) dW_t \right)^2 \right] \\ &= d^2 \mathbb{E}\left[\int_0^T (\Psi_{t,T} - H_t)^2 dt \right] \\ &= d^2 \int_0^T \mathbb{E}[(\Psi_{t,T} - H_t)^2] dt \\ &= d^2 \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \mathbb{E}[(\Psi_{t,T} - \Psi_{t_i,T})^2] dt. \end{aligned} \tag{3.18}$$
Since Brownian motion has independent increments, we have that
$$\mathbb{E}[(\Psi_{t,T} - \Psi_{t_i,T})^2] = \mathbb{E}[(\Psi_{t_i,T})^2] \mathbb{E}\left[(1 - \Psi_{t,t_i})^2 \right]. \tag{3.19}$$
Introducing the function:
$$H(t_i - t, W_{t_i} - W_t) = 1 - \Psi_{t,t_i}$$
which satisfies $H(0,0) = 0$, by Lemma 3.8 and Lemma 3.7 we obtain
$$\|\tilde{I}_2 - \tilde{I}_2^N\|_2^2 \le d^2 \sum_{i=1}^N \exp((2a + c^2)(T - t_i)) C_2(h) h^2$$

where $C_2(h)$ is an increasing function and, finally,

$$\|\tilde{I}_2 - \tilde{I}_2^N\|_2 \leq |d|\sqrt{(G_2(T)C_2(h))}h^{1/2} \tag{3.20}$$

where

$$G_2(T) = \int_0^T \exp{(2a+c^2)(T-t)}dt. \tag{3.21}$$

In order to estimate the other term in the right-hand side of (3.17) we note that by introducing

$$K_i(t, W_t) = \exp\left(\left(a - \frac{c^2}{2}\right)(T-t) + c(W_T - W_t)\right)(W_{t_i} - W_t)$$

we have

$$I_2^N \quad = \quad d\sum_{i=1}^{N} K_i(t_{i-1}, W_{t_{i-1}}),$$

and

$$K_i(t_i, W_{t_i}) = 0$$

By applying Lemma 3.9 to $K_i(t_i, W_{t_i})$ we can write

$$\begin{aligned} 0 - K_i(t, W_t) \quad &= \quad \int_t^{t_i} \partial_w(K_i)(s, W_s)d^+W_s + \int_t^{t_i} \partial_s(K_i)(s, W_s)ds + \\ &\quad -c\int_t^{t_i} \Psi_{s,T}ds - \frac{c^2}{2}\int_t^{t_i} K_i(s, W_s)ds. \end{aligned}$$

From the previous equality, by Itô isometry and Minkowski integral inequality we get

$$\begin{aligned} \left\|\tilde{I}_2^N - I_2^N + cd\int_0^T \Psi_{t,T}dt\right\|_2 \quad &= \quad |d|\left\|\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} \Psi_{t_i,T}d^+W_t + \int_{t_{i-1}}^{t_i} \partial_w(K_i)(t, W_t)d^+W_t + \right. \\ &\quad \left. + \int_{t_{i-1}}^{t_i} \partial_t(K_i)(t, W_t)dt - \frac{c^2}{2}\int_{t_{i-1}}^{t_i} K_i(t, W_t)dt\right\|_2 \\ &\leq \quad |d|\left(\left\|\int_0^T R_t d^+W_t\right\|_2 + \left\|\int_0^T M_t dt\right\|_2\right), \\ &\leq \quad |d|\left(\left(\int_0^T \mathbb{E}[R_t^2]dt\right)^{1/2} + \int_0^T \left(\mathbb{E}[M_t^2]\right)^{1/2} dt\right), \end{aligned}$$

where

$$\begin{aligned} R_t \quad &= \quad \sum_{i=1}^{N}(\partial_w(K_i)(t, W_t) + \Psi_{t_i,T})1_{(t_{i\_1},t_i]}(t) \\ M_t \quad &= \quad \sum_{i=1}^{N}\left(\partial_t(K_i)(t, W_t) - \frac{c^2}{2}K_i(t, W_t)\right)1_{[t_{i\_1},t_i]}(t) \end{aligned}$$

When $t_{i-1} < t \leq t_i$, by independence

$$\mathbb{E}[R_t^2] \quad \leq \quad 2\mathbb{E}[\Psi_{t_i,T}^2]\mathbb{E}[(c\Psi_{t,t_i}(W_{t_i} - W_t))^2 + (\Psi_{t,t_i} - 1)^2].$$

Introducing

$$F_2(t_i - t, W_{t_i} - W_t) = c\exp\left(\left(a - \frac{c^2}{2}\right)(t_i - t) + c(W_{t_i} - W_t)\right)(W_{t_i} - W_t)$$
$$F_3(t_i - t, W_{t_i} - W_t) = \exp\left(\left(a - \frac{c^2}{2}\right)(t_i - t) + c(W_{t_i} - W_t)\right) - 1,$$

we have that $F_2(0,0) = F_3(0,0) = 0$ and $\mathbb{E}[|\partial_w(F_i)(t, W_{t_i} - W_t)|^2]$, $\mathbb{E}[|\partial_{ww}(F_i)(t, W_{t_i} - W_t)|^2]$, $\mathbb{E}[|\partial_t(F_i)(t, W_{t_i} - W_t)|^2] \leq L(t_i - t)$ and so, by Lemma 3.8, there exist two continuous increasing functions $C_3(t), C_4(t)$ such that

$$\mathbb{E}[R_t^2] \leq 2\exp\left((2a + c^2)(T - t_i)\right)(C_3(t_i - t) + C_4(t_i - t))|t_i - t|.$$

Exploiting the independence we have

$$\mathbb{E}[M_t^2] = \mathbb{E}[(a\Psi_{t,T}(W_{t_i} - W_t))^2] = \mathbb{E}[(\Psi_{t_i,T})^2]\mathbb{E}[(a\Psi_{t,T}(W_{t_i} - W_t))^2],$$

and, in a similar way, we can prove that there exists an increasing function C_5 such that

$$\mathbb{E}[M_t^2] \leq \exp\left(\left(2a + c^2\right)(T - t_i)\right) C_5(t - t)|t_i - t|.$$

For the second term in the right-hand side of (3.17), we have finally the following estimate

$$\left\| \tilde{I}_2^N - I_2^N + cd\int_0^T \Psi_{t,T}dt \right\|_2 \leq |d| \left\{ \sqrt{G_2(T)}(\sqrt{2(C_3(h) + C_4(h))}) + G_1(T)\sqrt{C_5(h)} \right\} h^{1/2}, \quad (3.22)$$

where $G_1(T)$ and $G_2(T)$ are given by (3.16) and (3.21) respectively.

3.4.3 Proof of Theorem 3.6

We make the proof only for $a < 0$, since in the other case the estimate are equal to the Euler case and can be addressed with the same proof. We introduce the two integrals

$$\begin{aligned}
\bar{I}_1^N &= (b - cd)\sum_{i=1}^N \Phi_T\Phi_{t_{i-1}}^{-1}\Delta t_i, \\
\bar{I}_2^N &= d\sum_{i=1}^N \Phi_T\Phi_{t_{i-1}}^{-1}\Delta W_i - \frac{cd}{2}\sum_{i=1}^N \Phi_T\Phi_{t_{i-1}}^{-1}((\Delta W_i)^2 - (\Delta t_i)).
\end{aligned}$$

Estimate of $\|I_1 - \bar{I}_1^N\|_1$

First we note that (with $\Delta t_i = h$)

$$\begin{aligned}
\|I_1 - \bar{I}_1^N\|_1 &\leq |b - cd|\sum_{i=1}^N \left\| \Phi_T\int_{t_{i-1}}^{t_i} \Phi_t^{-1}dt - \Phi_T\Phi_{t_{i-1}}^{-1}h \right\|_1 \\
&\leq |b - dc|\sum_{i=1}^N \|\Psi_{t_i,T}\|_\alpha \left\| \int_{t_{i-1}}^{t_i} \Psi_{t,t_i}dt - \Psi_{t_{i-1},t_i}h \right\|_{2n} \\
&= |b - dc| \left\| \int_0^h (\Psi_{t,h} - \Psi_{0,h})dt \right\|_{2n} \left(\sum_{i=1}^N \|\Psi_{t_i,T}\|_\alpha \right)
\end{aligned}$$

where we have taken, $n \in \mathbb{N}$, $\frac{1}{2n} + \frac{1}{\alpha} = 1$ and $1 < \alpha < 2$ such that $\alpha a + \alpha(\alpha-1)\frac{c^2}{2} \leq 0$ (the last condition guarantees that when $T \to \infty$ we have $\mathbb{E}[\Psi^{\alpha}_{t_i,T}] \to 0$). By Jensen's inequality and Lemma 3.8 we can derive the following estimate:

$$\begin{aligned}\left\| \int_0^h (\Psi_{t,h} - \Psi_{0,h})dt \right\|_{2n}^{2n} &\leq h^{2n-1} \int_0^h \mathbb{E}[(\Psi_{t,h} - \Psi_{0,h})^{2n}]dt \\ &\leq h^{3n} C_5(h),\end{aligned}$$

where $C_5(h)$ is an increasing function and in the last inequality we use the fact that the function $F_4(t, W_t) = \Psi_{t,h} - \Psi_{0,h}$ is such that $F_4(0,0) = 0$. By Lemma 3.7, we have that

$$\|\Psi_{t_i,T}\|_{\alpha} = \exp\left(\left(a + \frac{c^2}{2}(\alpha-1)\right)(T - t_i)\right),$$

and so

$$\begin{aligned}\|I_1 - \bar{I}_1^N\|_1 &\leq |b - cd| \sum_{i=1}^{N} \exp\left(\left(a + \frac{c^2}{2}(\alpha-1)\right)(T - t_i)\right)(C_5(h))^{1/2n} h^{3/2} \\ &\leq |b - cd| G_4(T)(C_5(h))^{1/2n} h^{1/2}\end{aligned}$$

where

$$G_4(T) = \int_0^T \exp\left(\left(a + \frac{c^2}{2}(\alpha - 1))(T - t)\right)\right) dt. \tag{3.23}$$

Estimate of $\|I_2 - \bar{I}_2^N\|_1$

First we note that

$$\begin{aligned}\|I_2 - \bar{I}_2^N\|_1 &\leq |d| \sum_{i=1}^{N} \Bigg\| \Phi_T \int_{t_{i-1}}^{t_i} \Phi_t^{-1} dW_t - \Phi_T \Phi_{t_{i-1}}^{-1} \Delta W_i + \\ &\quad + \frac{c}{2} \Phi_T \Phi_{t_{i-1}}^{-1}((\Delta W_i)^2 - h) \Bigg\|_1 \\ &\leq |d| \sum_{i=1}^{N} \|\Psi_{t_i,T}\|_{\alpha} \Bigg\| \Phi_{t_i} \int_{t_{i-1}}^{t_i} \Phi_t^{-1} dW_t - \Psi_{t_{i-1},t_i} \Delta W_i + \\ &\quad + \frac{c}{2} \Psi_{t_{i-1},t_i}((\Delta W_i)^2 - h) \Bigg\|_{2n}\end{aligned}$$

where α, n are as in the previous section. We introduce the following notation

$$\begin{aligned}I_{2,t_i} &= \Phi_{t_i} \int_{t_{i-1}}^{t_i} (\Phi_t)^{-1} dW_t \\ &= \Phi_{t_i} \left(\int_{t_{i-1}}^{t_i} (\Phi_t)^{-1} d^+ W_t + c \int_{t_{i-1}}^{t_i} (\Phi_t)^{-1} dt \right) \\ &= \int_{t_{i-1}}^{t_i} \Psi_{t,t_i} d^+ W_t + c \int_{t_{i-1}}^{t_i} \Psi_{t,t_i} dt,\end{aligned}$$

where we have used Lemma 3.9 and the fact that $\Psi_{s,t} = \Phi_t(\Phi_s)^{-1}$. By introducing also $\hat{I}_{2,t_i} = \int_{t_{i-1}}^{t_i} \Psi_{t,t_i} d^+W_t$ and

$$\begin{aligned} \bar{I}^N_{2,t_i} &= \Psi_{t_{i-1},t_i}\Delta W_i - \frac{c}{2}\Psi_{t_{i-1},t_i}((\Delta W_i)^2 - h) \\ \hat{I}^N_{2,t_i} &= \Psi_{t_i,t_i}\Delta W_i + \frac{c}{2}((\Delta W_i)^2 - h), \end{aligned}$$

we have that

$$\|I_{2,t_i} - \bar{I}^N_{2,t_i}\|_{2n} \leq \|\hat{I}_{2,t_i} - \hat{I}^N_{2,t_i}\|_{2n} + \left\|(\hat{I}^N_{2,t_i} - \bar{I}^N_{2,t_i}) + c\int_{t_{i-1}}^{t_i} \Psi_{t,t_i} dt\right\|_{2n}.$$

It is simple to see that the two norms on the right-hand side of the previous expression do not depend on t_i but only on the difference $h = t_i - t_{i-1}$, so we study the functions (with $\Psi_{t_i,t_i} = 1$):

$$\begin{aligned} Z_1(h) &= \|\hat{I}_{2,h} - \hat{I}^N_{2,h}\|^{2n}_{2n} = \left\|\int_0^h (\Psi_{t,h} - 1 - c(W_h - W_t))d^+W_t\right\|^{2n}_{2n} \\ Z_2(h) &= \left\|(\hat{I}^N_{2,t_i} - \bar{I}^N_{2,t_i}) + c\int_{t_{i-1}}^{t_i} \Psi_{t,t_i} dt\right\|^{2n}_{2n} \\ &= \left\|(1 - \Psi_{0,h})W_h + \frac{c}{2}(\Psi_{0,h} + 1)W_h^2 - \frac{c}{2}(\Psi_{0,h} + 1)h + c\int_0^h \Psi_{t,h} dt\right\|^{2n}_{2n} \end{aligned}$$

By a well-known consequence of Itô isometry (see, e.g., [70]) we can estimate the function $Z_1(h)$ as:

$$Z_1(h) \leq D_n h^{n-1} \int_0^h \mathbb{E}[(\Psi_{t,h} - 1 - c(W_h - W_t))^{2n}]dt,$$

where $D_n = (n(2n-1))^n$. Since the function

$$F_5(h - t, W_h - W_t) = \exp\left((a - \frac{c^2}{2})(h - t) + c(W_h - W_t)\right) - 1 - c(W_h - W_t)$$

satisfies $F_5(0,0) = \partial_w(F_5)(0,0) = 0$, by Lemma 3.8 there exists an increasing function $C_6(h)$ such that

$$Z_1(h) \leq C_6(h)h^{3n}.$$

As far as concerned the function $Z_2(h)$, by introducing

$$K(t, W_t) = (1 - \Psi_{t,h})(W_h - W_t) + \frac{c}{2}(\Psi_{t,h} + 1)(W_h - W_t)^2 - \frac{c}{2}(\Psi_{t,h} + 1)(h - t),$$

it is immediate to see that

$$Z_2(h) = \left\|K(0,0) + c\int_0^h \Psi_{t,h} dt\right\|^{2n}_{2n}.$$

By applying Lemma 3.9 to $K(h, W_h)$, and by noting that $K(h, W_h) = 0$, we obtain

$$0 - K(0,0) = \int_0^h (\partial_t(K)(t, W_t) - \frac{1}{2}\partial_{ww}(K)(t, W_t)dt + \int_0^h \partial_w K(t, W_t)d^+W_t$$

Since we have that $-\partial_t(K)(h, W_h)+\partial_{ww}(K)(h, W_h)/2+c\Psi_{0,h} = 0$, and that $K(h, W_h) = \partial_w(K)(h, W_h) = \partial_{ww}(K)(h, W_h) = 0$, by Jensen's inequality, Lemma 3.8 and by applying the same techniques used for obtaining (3.22) we find that

$$Z_2(h)^{1/2n} \leq \left\{ (C_7(h))^{1/2n} + (C_8(h))^{1/2n} \right\} h^{3/2}$$

or, equivalently,

$$Z_2(h) \leq C_9(h)h^{3n},$$

with the obvious definition of the function $C_9(h)$.
Finally we have

$$\begin{aligned} \|I_2^N - \bar{I}_2^N\|_1 &\leq |d|(C_6(h)^{1/2n} + C_9(h)^{1/2n}) \sum_{i=1}^N \exp\left(\left(a + \frac{c^2}{2}(\alpha - 1)\right)(T - t_i)\right) h^{3/2} \\ &\leq |d|(C_6(h)^{1/2n} + C_9(h)^{1/2n}) G_4(T) h^{1/2}, \end{aligned}$$

where $G_4(T)$ is given by (3.23).

3.5 Numerical examples

In this section we show some numerical experiments which confirm the theoretical estimate proved in Section 3.4 and permit to study other properties of the new discretization methods introduced in Section 3.3.
We simulate the linear SDE (2.11) with coefficients $a = -2$, $b = 10$, $c = 10$ e $d = 10$. The coefficients are such that $a + \frac{c^2}{2} > 0$ with $a < 0$. This means that the considered linear equation admits an equilibrium probability density with finite first moment and infinite second moment. The coefficient d has been chosen big enough to put in evidence the noise effect.

We make a comparison between the Euler and Milstein methods applied directly to equation (2.11) and the new exact methods (3.11) and (3.12) with the constants $k = 0$ and $k = \frac{-d}{c} = -1$. In particular we observe that when $k = -1$, the schemes (3.11) and (3.12) coincide. We calculate the following two errors:

- the weak error $E^w = |\mathbb{E}[X_t - X_t^N]|$,
- the strong error $E^s = \mathbb{E}[|X_t - X_t^N|]$.

The weak error is estimated trought the explicit expression

$$\mathbb{E}[X_t] = e^{at},$$

for the first moment of the linear SDE solution, and by using Monte-Carlo method with 10^6 paths for calculating $\mathbb{E}[X_t^N]$. The strong error is estimated by exploiting Monte-Carlo simulation of X_t and X_t^N with 10^6 paths. In order to simulate X_t we apply the Milstein method with a steps-size of $h = 10^{-4}$, for which we have verified that it gives a good approximation of both $\mathbb{E}[X_t]$ and the equilibrium density for $t \to +\infty$. Since we use Monte-Carlo methods for estimating E^w and E^s, the two errors include both the systematic errors of the considered schemes and the statistical errors of the Monte-Carlo estimate procedure.

In Figure 3.1 we report the weak and strong errors with respect to the maximum time of integration t which varies from 0.1 to 1 and stepsize $h = 0.025$. As predicted by Theorem 3.6, the

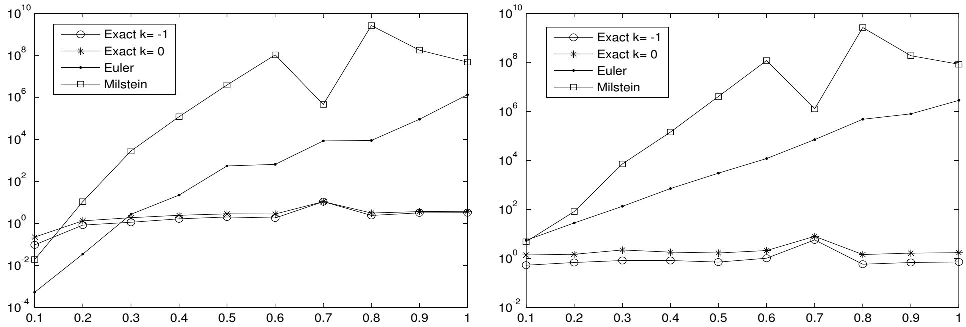

Figure 3.1: Strong and weak errors with $t \in [0.1, 1]$ and stepsize $h = 0.025$

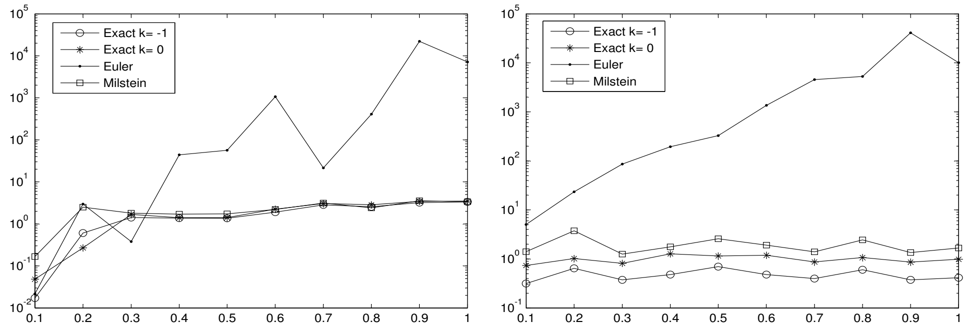

Figure 3.2: Strong and weak errors with $t \in [0.1, 1]$ and stepsize $h = 0.01$

error of the exact method for $k = -1$ remains bounded. It is important to note that for the exact method in the case $k = 0$ (where Theorem 3.5 and Theorem 3.6 do not apply) the errors remains bounded too, while for Euler and Milstein methods the errors grow exponentially with t.

In Figure 3.2 we report the weak and strong errors with respect to the maximum time of integration t, which varies from 0.1 to 1, and stepsize $h = 0.01$. In this situation also the errors of the Mistein method remain bounded. In other words $h = 0.01$ belongs to the stability region of the Milstein method but not to the stability region of the Euler method.

In Figure 3.3 we plot the weak and strong errors with fixed final time $t = 0.5$ and steps number $N = 10, ..., 80$, where the stepsize $h = \frac{t}{N}$. Here we note that the weak and strong errors for the exact methods do not change with the stepsize. This means that with a stepsize of only $h = 0.05$ the exact methods have weak and strong systematic errors less than the statistical errors. Instead for the Milstein scheme the errors grow and only with a stepsize equal to $h = 0.0125$ the systematic errors are comparable with the statistical ones. Equivalently we can say that the stability region is $[0, 0.0125]$. In the Euler case the systematic error is not comparable with the statistical one.

In Figure 3.4 we report the total variation distance between the empirical probabilities of X_t and of X_t^N obtained simulating 10^6 paths. We note that there is a big difference between the exact method for $k = 0$ and for $k = -1$. The discrepancy is due to the fact that the exact method with

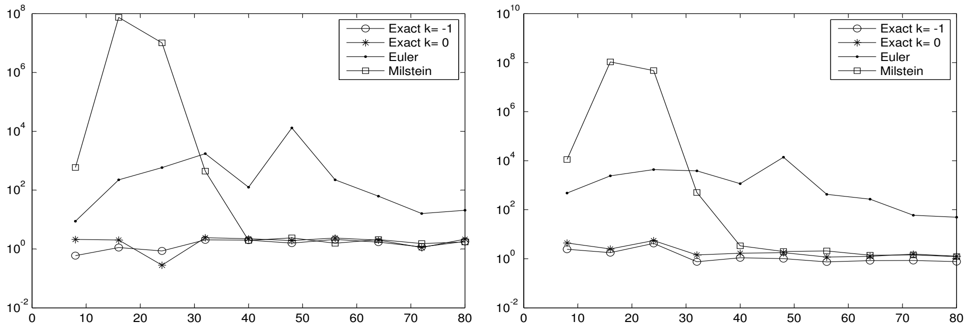

Figure 3.3: Strong and weak errors with $t = 0.5$ and number of steps $N \in [10, 80]$

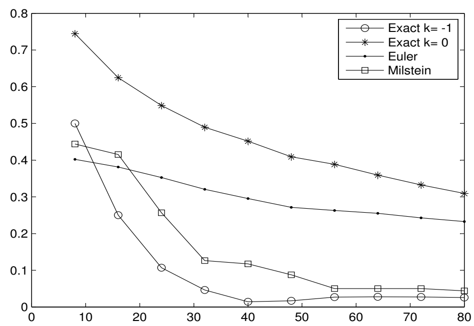

Figure 3.4: Total variation distance with $t = 0.5$ number of steps $N \in [10, 80]$

$k = 0$ tends to overestimate the points with probability less then $\frac{-d}{c}$ more than the Euler scheme does.

3.6 Appendix

In the proof of Theorem 3.5, by using Lemma 3.7 and the independence of Brownian increments, we can estimate the errors in a very explicitely way. In particular without exploiting Lemma 3.8. We show main steps and final expressions.

From (3.14) we obtain that

$$\int_{t_{i-1}}^{t_i} \mathbb{E}[(\Psi_{t,T})^2]\mathbb{E}[(1 - \Psi_{t_{i-1},t})^2]dt =: M_1(h)$$

with

$$
\begin{aligned}
M_1(h) &= \frac{-a-c^2+h\exp\left((2a+c^2)h\right)(c^4+3ac^2+2a^2)+(c^2+3a)\exp\left((2a+c^2)h\right)}{c^4+3ac^2+2a^2}+\\
&\quad +\frac{(2c^2+4a)\exp(ah)}{c^4+3ac^2+2a^2}
\end{aligned}
$$

Since $M_1(0)=\partial_h M_1(0)=0$, then $|M_1(h)|\le M_2(h)h^2$ with $M_2(h):=max_{k\in[0,h]}|\partial_h^2 M_1(k)|$, and, finally,

$$\|I_1-I_1^N\|_2\le |b-cd|h^{1/2}\sqrt{M_2(h)}G_1(T)$$

where $G_1(T)$ is given by (3.16), according with (3.15).
From (3.18) we obtain

$$
\begin{aligned}
\|\tilde{I}_2-\tilde{I}_2^N\|_2^2 &= (d)^2\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\mathbb{E}[(\Psi_{t_i,T})^2]\mathbb{E}\left[(\Psi_{t,t_i})^2+1-2\Psi_{t,t_i}\right]\\
&= (d)^2\sum_{i=1}^N\exp\left((2a+c^2)(T-t_i)\right)M_3(h)
\end{aligned}
$$

where

$$M_3(h)=\frac{3a+2c^2+a\exp(2a+c^2)+h(2a+ac^2)-(4a+2c^2)\exp(ah)}{2a^2+ac^2}$$

Since $M_3(0)=\partial_h M_3(0)=0$, we have that $|M_3(h)|\le M_4(h)h^2$ with $M_4(h):=max_{k\in[0,h]}|\partial_h^2 M_3(k)|$, and

$$\|\tilde{I}_2-\tilde{I}_2^N\|^2\le (d)\sqrt{G_2(T)M_4(h)}h^{1/2},$$

according with (3.20).
The second term on the right-hand side of (3.17) becomes

$$
\begin{aligned}
\left\|\tilde{I}_2^N-I_2^N+cd\int_0^T\Psi_{t,T}dt\right\|_2^2 &= d^2\mathbb{E}\left[\left(\sum_{i=1}^N\Psi_{t_i,T}(1-\Psi_{t_{i-1},t_i})(W_{t_i}-W_{t_{i-1}})\right.\right.\\
&\quad \left.\left.+\sum_{i=1}^N\Psi_{t_i,T}c\int_{t_{i-1}}^{t_i}\Psi_{t,t_i}dt\right)^2\right]\\
&= d^2\left[\sum_{i=1}^N\mathbb{E}[(\Psi_{t_i,T})^2]\mathbb{E}[(K_i+H_i)^2]+\right.\\
&\quad \left.+2\sum_{i<j}\mathbb{E}[(\Psi_{t_j,T})^2]\mathbb{E}[\Psi_{t_{j-1},t_j}(H_j+K_j)]\mathbb{E}[\Psi_{t_i,t_{j-1}}]\mathbb{E}[(H_i+K_i)]\right]
\end{aligned}
$$

where we have used independence and we have set

$$K_i=(1-\Psi_{t_{i-1},t_i})(W_{t_i}-W_{t_{i-1}}),\quad H_i=c\int_{t_{i-1}}^{t_i}\Psi_{t,t_i}dt$$

We can obtain

$$\begin{aligned} M_5(h) &:= \mathbb{E}[(H_i+K_i)^2] = \exp(2a+c^2)(4c^2h^2+h) - 2\exp(ah)(c^2h^2+h) + h \\ &+\frac{c^2(1-\exp((2a+c^2)h)}{a(c^2+2a)} + \frac{c^2(\exp((2a+c^2)h) - \exp(ah)}{a(a+c^2)} \\ &+2\left[-\frac{c^2[(ah-1)\exp(ah)+1]}{a^2} + \frac{2c^2[\exp((2a+c^2)h)(h(a+c^2)-1)+\exp(ah)]}{(a+c^2)^2}\right. \\ &\left.+\frac{c^2[\exp((2a+c^2)h) - \exp(ah)(1+h(a+c^2))]}{(a+c^2)^2}\right] \end{aligned}$$

and, since $M_5(0) = \partial_h M_5(0) = 0$, that $|M_5(h)| \leq M_6(h)h^2$, where $M_6(h) := max_{k\in[0,h]}|\partial_h^2 M_5(k)|$. Being:

$$\begin{aligned} M_7(h) &:= \mathbb{E}[\Psi_{t_{j-1},t_j}(H_j+K_j)] \\ &= \frac{c\exp((2a+c^2)h) - c\exp(ah) + ch(a+c^2)\exp(ah) - 2ch\exp((2a+c^2)h)(a+c^2)}{(a+c^2)} \\ \mathbb{E}[\Psi_{t_i,t_{j-1}}] &= \exp(a(t_{j-1}-t_i)) \\ M_8(h) &:= \mathbb{E}[H_i+K_i] = -ch\exp(ah) + \frac{c(\exp(ah)-1)}{a}, \end{aligned}$$

by putting $M_9(h) = M_7(h)M_8(h)$, one can easily verify that

$$M_9(0) = \partial_h M_9(0) = \partial_h^2 M_9(0) = \partial_h^3 M_9(0) = 0$$

(because $M_7(0) = \partial_h M_7(0) = M_8(0) = \partial_h M_8(0) = 0$) and, therefore, $|M_9(h)| \leq M_{10}(h)h^4$, where $M_{10}(h) := max_{k\in[0,h]}|\partial_h^4 M_9(k)|$. Finally

$$\begin{aligned} \left\|\tilde{I}_2^N - I_2^N + cd\int_0^T \Psi_{t,T}dt\right\|_2^2 &\leq d^2\left[\sum_{i=1}^N \exp((2a+c^2)(T-t_i))M_6(h)h^2 + \right. \\ &\left. +2\sum_{i<j}\exp((2a+c^2)(T-t_j))\exp(a(t_{j-1}-t_i))M_9(h)\right] \\ &\leq d^2\left[G_2(T)M_6(h)h + 2M_{10}(h)[\sum_i \exp((2a+c^2)(T-t_{i+1}))h^4 + \right. \\ &\left. + \sum_{i<j+1}\exp((2a+c^2)(T-t_j))\exp(a(t_{j-1}-t_i))h^4\right] \end{aligned}$$

that is

$$\left\|\tilde{I}_2^N - I_2^N + cd\int_0^T \Psi_{t,T}dt\right\|_2^2 \leq d^2\left[G_2(T)M_6(h)h + 2M_{10}(h)(G_2(T)h^3 + \bar{G}(T)h^2)\right],$$

with

$$\bar{G}(T) = \int_0^T\int_0^t \exp((2a+c^2)(T-t) + a(t-s))dsdt,$$

from which we get:

$$\left\|\tilde{I}_2^N - I_2^N + cd\int_0^T \Psi_{t,T}dt\right\|_2 \leq d\left[\sqrt{G_2(T)M_6(h) + 2M_{10}(h)\bar{G}(T)}h^{1/2} + \sqrt{2M_{10}(h)G_2(T)}h^{3/2}\right],$$

to be compared with (3.22).

Part II

Symmetries of SDEs driven by a general càdlàg semimartingale

Chapter 4

Geometric SDEs, gauge and time symmetries of semimartingales

In this chapter we introduce some fundamental concepts which will be useful in Chapter 5 for extending the notion of symmetry of an SDE from the Brownian motion case to the general càdlàg semimartingales setting. First of all we describe the framework of geometric SDEs, due to Choen, and our definition of canonical SDEs driven by general semimartingales taking values on manifolds. We show that these formulations of SDEs are general enough to include affine-type SDEs, Marcus-type SDEs, smooth SDE driven by Lévy processes and iterated random maps. We introduce the new concepts of *gauge symmetry group* and *time symmetry* of a semimartingale on a Lie group. We study the relationship between gauge and time symmetries with the *characteristic triplet* of a semimartingale Z and we finally propose some methods of construction of gauge and time symmetric semimartingales discussing some specific examples.

4.1 Stochastic differential equations with jumps on manifolds

4.1.1 Geometrical SDEs with jumps

Simplifying the setting of [37], a stochastic differential equation (SDE) defined on a smooth manifold M and driven by a general càdlàg semimartingale on a smooth manifold N can be described in terms of a smooth function

$$\overline{\Psi} : M \times N \times N \to M. \tag{4.1}$$

In particular, let $\overline{\Psi}(x, z', z)$ be a smooth function such that, for any $z \in N$, $\overline{\Psi}(\cdot, z, z) = id_M$ (the identity map on M).

We first consider the case where the manifolds M, N are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ and we take two global coordinate systems x^i and z^α of M and N respectively. The semimartingale X with values in M is a solution to the SDE defined by the map $\overline{\Psi}$ and driven by the semimartingale Z defined on N if, for $t \geq 0$,

$$\begin{aligned} X^i_t - X^i_0 &= \int_0^t \partial_{z'^\alpha}(\overline{\Psi}^i)(X_{s_-}, Z_{s_-}, Z_{s_-}) dZ^\alpha_s + \tfrac{1}{2}\int_0^t \partial_{z'^\alpha z'^\beta}(\overline{\Psi}^i)(X_{s_-}, Z_{s_-}, Z_{s_-}) d[Z^\alpha, Z^\beta]_s \\ &+ \sum_{0 \leq s \leq t} \{\overline{\Psi}^i(X_{s_-}, Z_s, Z_{s_-}) - \overline{\Psi}^i(X_{s_-}, Z_{s_-}, Z_{s_-}) - \partial_{z'^\alpha}(\overline{\Psi}^i)(X_{s_-}, Z_{s_-}, Z_{s_-})\Delta Z^\alpha_s\}, \end{aligned} \tag{4.2}$$

where $\overline{\Psi}^i := x^i(\overline{\Psi})$, the derivation $\partial_{z'^\alpha}$ is the derivative of $\overline{\Psi}^i(x, z', z)$ with respect to the second set z' of variables on N and with respect to the coordinates system z^α, $X^i := x^i(X)$, $Z^\alpha := z^\alpha(Z)$ and $\Delta Z^\alpha_s := Z^\alpha_s - Z^\alpha_{s_-}$.
In order to extend the previous definition to the case of two general smooth manifolds M, N we introduce two embeddings $i_1 : M \to \mathbb{R}^{k_M}$ and $i_2 : N \to \mathbb{R}^{k_N}, k_M, k_N \in \mathbb{N}$, and an extension

$$\tilde{\Psi} : \mathbb{R}^{k_M} \times \mathbb{R}^{k_N} \times \mathbb{R}^{k_N} \to \mathbb{R}^{k_M}$$

of the map $\overline{\Psi}$ such that

$$\tilde{\Psi}(i_1(x), i_2(z'), i_2(z))) = i_1(\overline{\Psi}(x, z', z)).$$

A semimartingale X defined on M solves the SDE defined by $\overline{\Psi}$ with respect to the noise Z defined on N if $i_1(X) \in \mathbb{R}^{k_M}$ solves the integral problem (4.2) where the map $\overline{\Psi}$ is replaced by $\tilde{\Psi}$ and the noise Z is replaced by $i_2(Z)$.

We generalize (4.1) by considering a map $\overline{\Psi}_k$ of the form

$$\overline{\Psi}.(\cdot, \cdot, \cdot) : M \times N \times N \times \mathcal{K} \to M,$$

where $\mathcal{K}$ is a (general) metric space (although in this paper we mostly take $\mathcal{K}$ as a finite dimensional smooth manifold), $\overline{\Psi}_k$ is smooth in the M, N variables, and $\overline{\Psi}_k$ and all its derivatives with respect to the M, N variables are continuous in all their arguments. Let K be a predictable locally bounded process taking values in $\mathcal{K}$. If M, N are two open subsets of $\mathbb{R}^n, \mathbb{R}^m$, we say that (X, Z) solves the SDE $\overline{\Psi}_{K_t}$ if, for $t \geq 0$,

$$\begin{aligned} X^i_t - X^i_0 \;&=\; \int_0^t \partial_{z'^\alpha}(\overline{\Psi}^i_{K_s})(X_{s_-}, Z_{s_-}, Z_{s_-}) dZ^\alpha_s + \tfrac{1}{2}\int_0^t \partial_{z'^\alpha z'^\beta}(\overline{\Psi}^i_{K_s})(X_{s_-}, Z_{s_-}, Z_{s_-}) d[Z^\alpha, Z^\beta]_s \\ &+\; \textstyle\sum_{0\leq s\leq t}\{\overline{\Psi}^i_{K_s}(X_{s_-}, Z_s, Z_{s_-}) - \overline{\Psi}^i_{K_s}(X_{s_-}, Z_{s_-}, Z_{s_-}) - \partial_{z'^\alpha}(\overline{\Psi}^i_{K_s})(X_{s_-}, Z_{s_-}, Z_{s_-})\Delta Z^\alpha_s\}. \end{aligned} \tag{4.3}$$

The extension to the case where M, N are general manifolds can be easily obtained as before by using embeddings i_1, i_2 and an extension $\tilde{\Psi}_k$ of $\overline{\Psi}_k$ which is continuous in the $M, N, \mathcal{K}$ variables and smooth in the N, M variables.

Definition 4.1 *Let M, N be two subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, $\mathcal{K}$ be a metric space and K be a predictable locally bounded process taking values in $\mathcal{K}$. A pair of semimartingales (X, Z) on M and N respectively is a solution to the* geometrical SDE *defined by $\overline{\Psi}_{K_t}$ until the stopping time τ if X and Z, stopped at the stopping time τ, solve the integral equation (4.3). If N, M are two general manifolds, (X, Z) solves the geometrical SDE defined by $\overline{\Psi}_{K_t}$ until the stopping time τ if, for any couple of embeddings i_1, i_2 of M, N in $\mathbb{R}^{k_M}, \mathbb{R}^{k_N}$ respectively and for any extension $\tilde{\Psi}_k$ of $\overline{\Psi}_k$, the pair $(i_1(X), i_2(Z))$ is a solution to the SDE $\tilde{\Psi}_{K_t}$ until the stopping time τ. If (X, Z) is a solution to the SDE $\overline{\Psi}_{K_t}$ until the stopping time τ we write*

$$dX_t = \overline{\Psi}_{K_t}(dZ_t).$$

When not strictly necessary, we omit the stopping time τ from the definition of solution to an SDE.

Theorem 4.2 *Given two open subsets M and N of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, for any semimartingale Z on N and any $x_0 \in M$, there exist a stopping time τ, almost surely strictly positive, and a semimartingale X on M, uniquely defined until τ and such that $X_0 = x_0$ almost surely, such that (X, Z) is a solution to the SDE Ψ_{K_t} until the stopping time τ. Furthermore, if M, N are two general manifolds, Z is a semimartingale on N, i_1, i_2 are two embeddings of N, M in $\mathbb{R}^{k_M}$ and $\mathbb{R}^{k_N}$ and $\tilde{\Psi}_k$ is any extension of $\overline{\Psi}_k$, then the unique solution $(\tilde{X}, i_2(Z))$ to the SDE $\tilde{\Psi}_k$ is of the form $(i_1(X), i_2(Z))$ for a unique semimartingale X on M. Finally, the process X does not depend on the embeddings i_1, i_2 and on the extension $\tilde{\Psi}_k$.*

Proof. Since the process K is locally bounded, the function $\tilde{\Psi}_{K_t}$, up to a sequence of stopping times $\tau_n \to +\infty$, is locally Lipschitz with Lipschitzianity constant uniform with respect to ω. The proof of this fact can be found in [37, Theorem 2]

■

4.1.2 Geometrical SDEs and diffeomorphisms

The notion of geometrical SDE introduced in Definition 4.1 naturally suggests to consider transformations of solutions to an SDE.

Theorem 4.3 *Let $\Phi : M \to M'$ and $\tilde{\Phi} : N \to N'$ be two diffeomorphisms. If (X, Z) is a solution to the geometrical SDE $\overline{\Psi}_{K_t}$, then $(\Phi(X), \tilde{\Phi}(Z))$ is a solution to the geometrical SDE $\overline{\Psi}'_{K_t}$ defined by*

$$\overline{\Psi}'_{K_t}(x, z', z) = \Phi(\overline{\Psi}_{K_t}(\Phi^{-1}(x), \tilde{\Phi}^{-1}(z'), \tilde{\Phi}^{-1}(z))).$$

In order to prove Theorem 4.3 we start by introducing the following lemmas.

Lemma 4.4 (General Itô formula) *Given an $X = (X^1, ..., X^m)$ $\mathbb{R}^m$ real semimartingale and a $C^2(\mathbb{R}^m)$ function $f : \mathbb{R}^m \to \mathbb{R}$ we have*

$$\begin{aligned} f(X_t) - f(X_0) &= \int_0^t \partial_{x^i}(f)(X_{s_-})dX^i_s + \frac{1}{2}\int_0^t \partial_{x^i x^j}(f)(X_{s_-})d[X^i, X^j]^c_s + \\ &+ \sum_{0\le s\le t} \left\{f(X_s) - f(X_{s_-}) - \partial_{x^i}(f)(X_{s_-})\Delta X^i_s\right\}. \end{aligned}$$

Proof. The proof can be found, e.g., in [153, Chapter II, Section 7]. ■

Lemma 4.5 *Given k càdlàg semimartingales $X^1, ..., X^k$, let $H^\alpha_1, ..., H^\alpha_k$, for $\alpha = 1, ..., r$, be predictable processes which can be integrated along $X^1, ..., X^k$ respectively. If $\Phi^\alpha(t, \omega, x^1, x'^1, ..., x^k, x'^k) : \mathbb{R}_+ \times \Omega \times \mathbb{R}^{2k} \to \mathbb{R}$ are some progressively measurable random functions continuous in $x^1, x'^1, ..., x^k, x'^k$ and such that $|\Phi^\alpha(t, \omega, x^1, x'^1, ..., x^k, x'^k)| \le O((x^1 - x'^1)^2 + ... + (x^k - x'^k)^2)$ as $x^i \to x'^i$, for almost every fixed $\omega \in \Omega$ and uniformly on compact subsets of $\mathbb{R}_+ \times \mathbb{R}^{2k}$, the processes*

$$Z^\alpha_t = \int_0^t H^\alpha_{i,s}dX^i_s + \sum_{0\le s\le t} \Phi^\alpha(s, \omega, X^1_{s_-}, X^1_s, ..., X^k_{s_-}, X^k_s)$$

are semimartingales. Furthermore

$$\Delta Z^\alpha_t = H^\alpha_{i,t}\Delta X^i_t + \Phi^\alpha(t, \omega, X^1_{t_-}, X^1_t, ..., X^k_{t_-}, X^k_t), \quad (4.4)$$

$$[Z^\alpha, Z^\beta]^c_t = \textstyle\int_0^t H^\alpha_{i,s}H^\beta_{j,s}d[X^i, X^j]^c_s, \quad (4.5)$$

$$\textstyle\int_0^t K_{\alpha,s}dZ^\alpha_s = \int_0^t K_{\alpha,s}H^\alpha_{i,s}dX^i_s + \sum_{0\le s\le t} K_{\alpha,s}\Phi^\alpha(s, \omega, X^1_{s_-}, X^1_s, ..., X^k_{s_-}, X^k_s). \quad (4.6)$$

Proof. Since $\int_0^t H^\alpha_{i,s}dX^i_s$ are semimartingales, we only need to prove that $\tilde{Z}^\alpha_t = \sum_{0\le s\le t} \Phi^\alpha(s, \omega, X^1_{s_-}, X^1_s, ..., X^k_{s_-}, X^k_s)$ is a càdlàg process of bounded variation.
If $\tilde{Z}^\alpha$ are of bounded variation, then we can prove (4.4), (4.5) and (4.6). In fact, if $\tilde{Z}^\alpha$ are of bounded variation, they do not contribute to the brackets $[Z^\alpha, Z^\beta]^c$. Thus $[Z^\alpha, Z^\beta]^c = [Z^\alpha - \tilde{Z}^\alpha, Z^\beta - \tilde{Z}^\beta]^c$ and we obtain equation (4.5). Furthermore, since $\tilde{Z}^\alpha$ is a sum of pure jumps processes, $\tilde{Z}^\alpha$ is a pure

jump process. Then we get equations (4.4) and (4.6) by using that $\tilde{Z}^\alpha$ are pure jump processes of bounded variation and that the measures $d\tilde{Z}^\alpha$ are pure atomic measures.
The fact that $\tilde{Z}^\alpha$ is of bounded variation can be established by exploiting the standard argument used for proving Itô formula.
Indeed, if $[X^1,X^1]_t(\omega),...,[X^k,X^k]_t(\omega) < +\infty$ for all $t \in \mathbb{R}_+$, then $\sum_i \sum_{0\leq s\leq t}(\Delta X^i_s)^2(\omega) \leq \sum_i [X^i,X^i]_T(\omega) < +\infty$. Since X^i are càdlàg they are locally bounded and so, for all $t<T$ and for almost every $\omega \in \Omega$, there exists a $C(T,\omega)$ such that

$$\begin{aligned}
\mathrm{var}_{[0,T]}(\tilde{Z}^\alpha_t(\omega)) &\leq \sum_{0\leq s\leq t} |\Phi^\alpha(s,\omega,X^1_{s_-},X^1_s,...,X^k_{s_-},X^k_s)| \\
&\leq C(T,\omega)\left(\sum_i \sum_{0\leq s\leq t}(\Delta X^i_s)^2\right) < +\infty.
\end{aligned}$$

■

Remark 4.6 *Let $\mathcal{K}$ be a metric space, $K \in \mathcal{K}$ be a locally bounded predictable process and $\tilde{\Phi}: \mathbb{R}_+ \times \mathcal{K} \times \mathbb{R}^{2k} \to \mathbb{R}$ be a C^2 function in $\mathbb{R}^{2k}$ variables such that $\tilde{\Phi}$ and all its derivatives are continuous in all their arguments. If $\tilde{\Phi}(\cdot,\cdot,x^1,x^1,...,x^k,x^k) = \partial_{x'^i}(\tilde{\Phi})(\cdot,\cdot,x^1,x^1,...,x^k) = 0$ for $i=1,...k$, then $\Phi(t,\omega,...) = \tilde{\Phi}(t,K_t(\omega),...)$ satisfies the hypothesis of Lemma 4.5.*

Proof of Theorem 4.3. The proof is given for $M,M' = \mathbb{R}^m, N,N' = \mathbb{R}^n$ (or more generally M,M',N,N' open subsets of $\mathbb{R}^m,\mathbb{R}^n$). The general case follows exploiting an embedding of M,M',N,N' in $\mathbb{R}^{k_M},\mathbb{R}^{k_N}$ and extending $\Phi,\tilde{\Phi}$ to diffeomorphisms defined in a neighbourhood of the image of M,M',N,N' with respect to these embeddings.
In order to simplify the proof we consider the two special cases $M=M'$, $\Phi = Id_M$ and $N=N'$, $\tilde{\Phi} = Id_N$. The general case can be obtained combining these two cases.
If $M=M'$ and $\Phi = Id_M$, putting $\tilde{Z} = \tilde{\Phi}(Z)$, so that $Z = \tilde{\Phi}^{-1}(\tilde{Z})$, by Itô's formula for semimartingales with jumps, Lemma 4.5 and Remark 4.6 we have

$$\begin{aligned}
Z^\alpha_t - Z^\alpha_0 &= \int_0^t \partial_{\tilde{z}^\beta}(\tilde{\Phi}^{-1})^\alpha(\tilde{Z}_{s_-})d\tilde{Z}^\beta_s + \frac{1}{2}\partial_{\tilde{z}^\beta\tilde{z}^\gamma}(\tilde{\Phi}^{-1})^\alpha(\tilde{Z}_{s_-})d[\tilde{Z}^\beta,\tilde{Z}^\gamma]^c_s + \\
&\quad + \sum_{0\leq s\leq t}((\tilde{\Phi}^{-1})^\alpha(\tilde{Z}_s) - (\tilde{\Phi}^{-1})^\alpha(\tilde{Z}_{s_-}) - \partial_{\tilde{z}^\beta}(\tilde{\Phi}^{-1})^\alpha(\tilde{Z}_{s_-})\Delta\tilde{Z}^\beta_s) \\
d[Z^\alpha,Z^\beta]^c_t &= \partial_{\tilde{z}^\gamma}(\tilde{\Phi}^{-1})^\alpha(Z_{s_-})\partial_{\tilde{z}^\delta}(\tilde{\Phi}^{-1})^\beta(Z_{s_-})d[\tilde{Z}^\gamma,\tilde{Z}^\delta]^c_t \\
\Delta Z^\alpha_t &= (\tilde{\Phi}^{-1})^\alpha(\tilde{Z}_t) - (\tilde{\Phi}^{-1})^\alpha(\tilde{Z}_{t_-}).
\end{aligned}$$

The conclusion of Theorem 4.3 follows using the definition of solution to the geometrical SDE $\overline{\Psi}_{K_t}$, Lemma 4.5 and the chain rule for derivatives.
Suppose now that $N=N'$ and $\tilde{\Phi} = Id_N$. Putting $X' = \Phi(X)$, by Itô's formula we obtain

$$\begin{aligned}
X'^i_t - X'^i_0 &= \int_0^t \partial_{x^j}(\Phi^i)(X_{s_-})dX^j_s + \frac{1}{2}\int_0^t \partial_{x^j x^h}(\Phi^i)(X_{s_-})d[X^j,X^h]^c_s + \\
&\quad + \sum_{0\leq s\leq t}(\Phi^i(X_s) - \Phi^i(X_{s_-}) - \partial_{x^j}(\Phi^i)(X_{s_-})\Delta X^j_s).
\end{aligned}$$

Furthermore, by definition of solutions to the geometrical SDE $\overline{\Psi}$ and by Lemma 4.5 we have

$$\begin{aligned} dX^i_s &= \partial_{z'^\alpha}(\overline{\Psi}^i_{K_s})(X_{s_-},Z_{s_-},Z_{s_-})dZ^\alpha_s + \frac{1}{2}\partial_{z'^\alpha z'^\beta}(\overline{\Psi}^i_{K_s})(X_{s_-},Z_{s_-},Z_{s_-})d[Z^\alpha,Z^\beta]^c_s + \\ &\quad +\overline{\Psi}^i_{K_s}(X_{s_-},Z_s,Z_{s_-}) - \overline{\Psi}^i_{K_s}(X_{s_-},Z_{s_-},Z_{s_-}) - \partial_{z'^\alpha}(\overline{\Psi}^i_{K_s})(X_{s_-},Z_{s_-},Z_{s_-})\Delta Z^\alpha_s, \\ d[X^i,X^j]^c_s &= \partial_{z'^\alpha}(\overline{\Psi}^i_{K_s})(X_{s_-},Z_{s_-},Z_{s_-})\partial_{z'^\beta}(\overline{\Psi}^i_{K_s})(X_{s_-},Z_{s_-},Z_{s_-})d[Z^\alpha,Z^\beta]^c_s \\ \Delta X^i_s &= \overline{\Psi}^i_{K_s}(X_{s_-},Z_s,Z_{s_-}) - \overline{\Psi}^i_{K_s}(X_{s_-},Z_{s_-},Z_{s_-}). \end{aligned}$$

Using the previous relations, the fact that $X = \Phi^{-1}(X')$ and the chain rule for derivatives we get the thesis. ■

4.1.3 A comparison with other approaches

Since the geometrical approach of [37] is not widely known, but nevertheless it is essential in our investigation of symmetries, in this section we compare the definition of geometrical SDEs driven by semimartingales with jumps with some more usual definitions of SDEs driven by càdlàg processes appearing in the literature. We make the comparison with different kinds of SDEs with jumps:

- affine-type SDEs of the type studied in [153, Chapter V] and [22, Chapter 5],
- Marcus-type SDEs (see [120, 137, 138]),
- SDEs driven by Lévy processes with smooth coefficients (see, e.g., [9, 119]),
- smooth iterated random functions (see, e.g., [14, 55]).

In the following we assume, for simplicity, that M and N are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

Affine-type SDEs

We briefly describe the affine type SDEs as proposed, e.g., in [153, Chapter V]. In particular we show how it is possible to rewrite them according to our geometrical setting.
Let $(Z^1,...,Z^n)$ be a semimartingale in N and let $\sigma : M \to \mathrm{Mat}(m,n)$ be a smooth function taking values in the set of $m \times n$ matrices with real elements. We consider the SDE defined by

$$dX^i_t = \sigma^i_\alpha(X_t)dZ^\alpha_t, \tag{4.7}$$

where σ^i_j are the components of the matrix σ. If $Z^1_t = t$ and $Z^2,...,Z^n$ are independent Brownian motions, we have the usual diffusion processes with drift $(\sigma^1_1,...,\sigma^m_1)$ and diffusion matrix $(\sigma^i_\alpha)|_{\substack{i=1,\dots,m\\ \alpha=2,\dots,n}}$.
The previous affine-type SDE can be rewritten as a geometrical SDE defined by the function $\overline{\Psi}$

$$\overline{\Psi}(x,z',z) = x + \sigma(x)\cdot(z'-z),$$

or, in coordinates,

$$\overline{\Psi}^i(x,z',z) = x^i + \sigma^i_\alpha(x)(z'^\alpha - z^\alpha).$$

In fact, by definition of geometrical SDE $\overline{\Psi}$, we have

$$\begin{aligned} X_t^i - X_0^i &= \int_0^t \partial_{z'^\alpha}(\overline{\Psi}^i)(X_{s_-}, Z_{s_-}, Z_{s_-})dZ_s^\alpha + \frac{1}{2}\int_0^t \partial_{z'^\alpha z'^\beta}(\overline{\Psi}^i)(X_{s_-}, Z_{s_-}, Z_{s_-})d[Z^\alpha, Z^\beta]_s \\ &+ \sum_{0\leq s\leq t} \{\overline{\Psi}^i(X_{s_-}, Z_s, Z_{s_-}) - \overline{\Psi}^i(X_{s_-}, Z_{s_-}, Z_{s_-}) - \partial_{z'^\alpha}(\overline{\Psi}^i)(X_{s_-}, Z_{s_-}, Z_{s_-})\Delta Z_s^\alpha\}, \\ &= \int_0^t \sigma_\alpha^i(X_{s_-})dZ_s^\alpha + \sum_{0\leq s\leq t}\{\sigma_\alpha^i(X_{s_-})(Z_s^\alpha - Z_{s_-}^\alpha) - \sigma_\alpha^i(X_{s_-})\Delta Z_s^\alpha\} \\ &= \int_0^t \sigma_\alpha^i(X_{s_-})dZ_s^\alpha. \end{aligned}$$

Marcus-type SDEs

The Marcus-type SDEs with jumps, initially proposed by Marcus in [137, 138] for semimartingales with finitely many jumps in any compact interval, have been extended to the case of general real semimartingales in [120]. The special property of this family of SDEs is their natural behaviour with respect to diffeomorphisms.

Given a manifold M and a global cartesian coordinate system x^i on M, we consider n smooth vector fields $Y_1, ..., Y_n$ on M of the form $Y_\alpha = Y_\alpha^i \partial_{x^i}, \alpha = 1, ..., n$. If the functions Y_α^i grow at most linearly at infinity, the flow of Y_α is defined for any time. Therefore, for any $z = (z^1, ..., z^n) \in \mathbb{R}^n$, we introduce the function

$$\Psi(x, z) = \exp(z^\alpha Y_\alpha)(x),$$

where $\exp(Y)$ is the exponential map with respect to the vector field Y, i.e. the map associating with any $x \in M$ its evolute at time 1 with respect to the flow defined by the vector field Y.

The solution X with values in M (we shall shortly write $X \in M$) to the Marcus-type SDE defined by the vector fields $Y_1, ..., Y_n$ with respect to the semimartingales $(Z^1, ..., Z^n)$ is the unique semimartingale $X \in M$ such that

$$\begin{aligned} X_t^i - X_0^i &= \int_0^t Y_\alpha^i(X_{s_-})dZ_s^\alpha + \frac{1}{4}\int_0^t (Y_\beta(Y_\alpha^i)(X_{s_-}) + Y_\alpha(Y_\beta^i)(X_{s_-}))d[Z^\alpha, Z^\beta]_s + \\ &\sum_{0\leq s\leq t}\{\Psi^i(X_{s_-}, Z_s - Z_{s_-}) - X_{s_-}^i - Y_\alpha^i(X_{s_-})\Delta Z_s^\alpha\}. \end{aligned}$$

We note that the previous equation depends only on $Y_1, ..., Y_n$, which means that if $\Phi : M \to M'$ is a diffeomorphism, the semimartingale $\Phi(X)$ solves the Marcus-type SDE defined by the vector fields $\Phi_*(Y_1), ..., \Phi_*(Y_n)$ (see [120]).

The Marcus-type SDE is a special form of geometrical SDE with defining map given by

$$\overline{\Psi}(x, z', z) = \Psi(x, z' - z).$$

Indeed, by definition of Ψ and $\overline{\Psi}$, we have

$$\begin{aligned} \partial_{z'^\alpha}(\overline{\Psi}^i)(x, z, z) &= \partial_{z^\alpha}(\Psi^i)(x, 0) = Y_\alpha^i \\ \partial_{z'^\alpha z'^\beta}(\overline{\Psi}^i)(x, z, z) &= \partial_{z^\alpha z^\beta}(\Psi^i)(x, 0) = \tfrac{1}{2}(Y_\beta(Y_\alpha^i) + Y_\alpha(Y_\beta^i)). \end{aligned}$$

Smooth SDEs driven by a Lévy process

In this section we describe a particular form of SDEs driven by $\mathbb{R}^n$-valued Lévy processes (see, e.g., [9, 119]). By definition, an $\mathbb{R}^n$-valued Lévy process $(Z^1, ..., Z^n)$ can be decomposed into the sum of Brownian motions and compensated Poisson processes defined on $\mathbb{R}^n$. In particular, a Lévy process on $\mathbb{R}^n$ can be identified by a vector $b_0 = (b_0^1, ..., b_0^n) \in \mathbb{R}^n$, an $n \times n$ matrix $A_0^{\alpha\beta}$ (with real elements) and a positive σ-finite measure ν_0 defined on $\mathbb{R}^n$ (called Lévy measures, see, e.g., [9, 157]) such that

$$\int_{\mathbb{R}^n} \frac{|z|^2}{1+|z|^2} \nu_0(dz) < +\infty.$$

By the Lévy-Itô decomposition, the triplet (b, A, ν) is such that there exist an n dimensional Brownian motion $(W^1, ..., W^n)$ and a Poisson measure $P(dz, dt)$ defined on $\mathbb{R}^n$ such that

$$\begin{aligned} Z_t^\alpha &= b_0^\alpha t + C_\beta^\alpha W_t^\beta + \int_0^t \int_{|z|\leq 1} z^\alpha (P(dz, ds) - \nu_0(dz)ds) + \\ &\quad + \int_0^t \int_{|z|>1} z^\alpha P(dz, ds). \end{aligned}$$

where $A_0^{\alpha\beta} = \sum_\gamma C_\gamma^\alpha C_\gamma^\beta$. Henceforth we suppose for simplicity that $b^1 = 1$ and $b_0^\alpha = 0$ for $\alpha > 1$, that there exists n_1 such that $A_0^{\alpha\beta} = \delta^{\alpha\beta}$ for $1 < \alpha, \beta \leq n_1$ and $A_0^{\alpha\beta} = 0$ for α or β in $\{1, n_1+1, ..., n\}$, and finally that $\int_0^t \int_{|z|\leq 1} z^\alpha (P(dz, ds) - \nu_0(dz)ds) = 0$ and $\int_0^t \int_{|z|>1} z^\alpha P(dz, ds) = 0$ for $\alpha \leq n_1$.

Consider a vector field μ on M, a set of $n_1 - 1$ vector fields $\sigma = (\sigma_2, ..., \sigma_{n_1})$ on M and a smooth (both in x and z) function $F : M \times \mathbb{R}^{n-n_1} \to \mathbb{R}^m$ such that $F(x, 0) = 0$. We say that a semimartingale $X \in M$ is a solution to the smooth SDE (μ, σ, F) driven by the $\mathbb{R}^n$ Lévy process $(Z^1, ..., Z^n)$ if

$$\begin{aligned} X_t^i - X_0^i &= \int_0^t \mu^i(X_{s_-}) dZ_s^1 + \int_0^t \sum_{\alpha=2}^{n_1} \sigma_\alpha^i(X_{s_-}) dZ_s^\alpha + \\ &\quad + \int_0^t \int_{\mathbb{R}^{n-n_1}} F^i(X_{s_-}, z)(P(dz, ds) - I_{|z|\leq 1}\nu_0(dz)ds), \end{aligned}$$

where $I_{|z|\leq 1}$ is the indicator function of the set $\{|z| \leq 1\} \subset \mathbb{R}^{n-n_1}$. Define the function

$$\overline{\Psi}^i(x, z', z) = x^i + \tilde{\mu}^i(x)(z'^1 - z^1) + \sigma_\alpha^i(x)(z'^\alpha - z^\alpha) + F^i(x, z' - z),$$

where

$$\tilde{\mu}^i(x) = \mu^i(x) - \int_{|z|\leq 1} (F^i(x, z) - \partial_{z^\alpha}(F^i)(x, z) z^\alpha) \nu_0(dz).$$

It is easy to see that any solution X to the smooth SDE (μ, σ, F) driven by the Lévy process $(Z^1, ..., Z^n)$ is also solution to the geometrical SDE $\overline{\Psi}$ driven by the $\mathbb{R}^n$ semimartingale $(Z^1, ..., Z^n)$ and conversely.

Remark 4.7 *In the theory of SDEs driven by $\mathbb{R}^n$-valued Lévy processes the usual assumption is that F is Lipschitz in x and measurable in z. Our assumption on smoothness of F in both x, z is thus a stronger requirement. For this reason we say that (μ, σ, F) is a* smooth SDE *driven by a Lévy process.*

Iterated random smooth functions

In the previous sections we have only considered continuous time processes Z_t. Let us now take Z as a discrete time adapted process, i.e. Z is a sequence of random variables $Z_0, Z_1, ..., Z_n, ...$ defined on N. We can consider Z as a càdlàg continuous time process Z_t defined by

$$Z_t = Z_\ell \text{ if } \ell \leq t < \ell + 1.$$

Since the process Z is a pure jump process with a finite number of jumps in any compact interval of $\mathbb{R}_+$, Z is a semimartingale. If (X, Z) is a solution to the geometrical SDE $\overline{\Psi}$, we have that

$$X_\ell = \overline{\Psi}(X_{\ell-1}, Z_\ell, Z_{\ell-1}) \tag{4.8}$$

and $X_t = X_\ell$ if $\ell \leq t < \ell + 1$. The process X can be viewed as a discrete time process defined by the recursive relation (4.8). These processes are special forms of *iterated random functions* (see, e.g., [14, 55, 158]) and this kind of equations is very important in time series analysis (see, e.g., [32, 162]) and in numerical simulation of SDEs (see Chapter 3). In this case we do not need that $\overline{\Psi}$ is smooth in all its variables and that $\overline{\Psi}(x, z, z) = x$ for any $x \in M$ and $z \in N$. In the case of a discrete time semimartingale Z_t these two conditions can be skipped and we can consider more general iterated random functions defined by relation (4.8).
An important example of iterated random functions can be obtained by considering $M = \mathbb{R}^m$, $N = GL(m) \times \mathbb{R}^m$ and the functions

$$\overline{\Psi}(x, z', z) = (z'_1 \cdot z_1^{-1}) \cdot x + (z'_2 - z_2),$$

where $(z_1, z_2) \in GL(m) \times \mathbb{R}^m$. Moreover, taking two sequences of random variables $A_0, ..., A_\ell, ... \in GL(n)$ and $B_0, ..., B_\ell, ... \in \mathbb{R}^m$, we define

$$Z_\ell = (A_\ell \cdot A_{\ell-1} \cdot .... \cdot A_0, B_\ell + B_{\ell-1} + ... + B_0)\,.$$

The iterated random functions associated with the SDE $\overline{\Psi}$ is

$$X_\ell = A_\ell \cdot X_{\ell-1} + B_\ell.$$

This model is very well studied (see, e.g., [14, 16, 110]). In particular the well known ARMA model is of this form (see, e.g., [32, 162]).

4.1.4 Canonical SDEs

In this section, in order to generalize the well known noise change property of affine-type SDEs driven by càdlàg semimartingales, we introduce the concept of canonical SDEs driven by a process on a Lie group N. If $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ and we consider the affine SDE given by

$$dX^i_t = \sigma^i_\alpha(X_{t_-}) dZ^\alpha_t,$$

we can define a new semimartingale on N given by

$$d\tilde{Z}^\alpha_t = B^\alpha_{\beta,t} dZ^\beta_t, \tag{4.9}$$

where $B = (B^\alpha_\beta)$ is a locally bounded predictable process taking values in $GL(n)$, and rewrite the affine SDE in terms of the semimartingale $\tilde{Z}$ in the following way

$$dX^i_t = \sigma^i_\alpha(X_{t_-})(B^{-1})^\alpha_{\beta,t} d\tilde{Z}^\beta_t, \tag{4.10}$$

where B^{-1} is the inverse matrix of B. Since this property, essential in the definition of symmetries of a canonical SDE, has no counterpart for general geometrical SDEs, we restrict our attention to a special class of geometrical SDEs that we call canonical (geometrical) SDEs. The first three kinds of SDEs proposed in Section 4.1.3 are canonical SDEs in the above sense.
Considering now a (general) Lie group N and a semimartingale Z on N, a natural definition of

jump can be given. Indeed, if τ is a stopping time, we define the jump at time τ as the random variable ΔZ_τ taking values on N such that

$$\Delta Z_\tau = Z_\tau \cdot (Z_{\tau_-})^{-1},$$

where $\cdot$ is the multiplication in the group N. In order to define a special class of equations that, in some sense, depends only on the jumps ΔZ_t of a process Z defined on a Lie group, we consider a function Ψ of the form

$$\Psi_\cdot(\cdot,\cdot) : M \times N \times \mathcal{K} \to M,$$

such that $\Psi_k(x, 1_N) = x$ for any k in a metric space $\mathcal{K}$ and $x \in M$, and we introduce the function $\overline{\Psi}_k$ defining the corresponding geometrical SDE as

$$\overline{\Psi}_k(x, z', z) = \Psi_k(x, z' \cdot z^{-1}) = \Psi_k(x, \Delta z).$$

If (X, Z) solves the SDE defined by this $\overline{\Psi}_k$, we write

$$dX_t = \Psi_{K_t}(dZ_t),$$

and we say that (X, Z) is a solution to the canonical SDE Ψ_{K_t}. For canonical SDEs it is possible to consider a sort of generalization of the semimartingales change rule (4.10).
Suppose that $M = \tilde{N}$ for some Lie group $\tilde{N}$ and consider the smooth function

$$\Xi_\cdot(\cdot) : N \times \mathcal{G} \to \tilde{N},$$

where $\mathcal{G}$ is a Lie group, which satisfies the relation $\Xi_g(1_N) = 1_{\tilde{N}}, \forall g \in \mathcal{G}$. We define the map

$$\tilde{\Psi}_g(x, z) = \Xi_g(z) \cdot x.$$

If Z is a semimartingale on N, we define the transformed semimartingale on $\tilde{N}$ by

$$d\tilde{Z}_t = \Xi_{G_t}(dZ_t) \tag{4.11}$$

as the unique solution $(\tilde{Z}, Z)$ to the equation

$$d\tilde{Z}_t = \tilde{\Psi}_{G_t}(dZ_t),$$

with initial condition $\tilde{Z}_0 = 1_{\tilde{N}}$. Before proving further results about transformation (4.11), we show that the semimartingales change (4.9) is a particular case of (4.11). In fact, for $\tilde{N} = N = \mathbb{R}^n$, any map $\Xi_\cdot : \mathbb{R}^n \times \mathcal{G} \to \mathbb{R}^n$ gives the canonical SDE defined by the function

$$\tilde{\Psi}_g(\tilde{z}, z) = \tilde{z} + \Xi_g(z).$$

This means that equation (4.11) is explicitly given by the relation

$$\begin{aligned} \tilde{Z}_t \quad = \quad & \int_0^t \partial_{z^\alpha}(\Xi_{G_s})(0) dZ^\alpha_s + \tfrac{1}{2}\int_0^t \partial_{z^\alpha z^\beta}(\Xi_{G_s})(0) d[Z^\alpha, Z^\beta]^c_s + \\ & + \textstyle\sum_{0 \leq s \leq t} (\Xi_{G_s}(\Delta Z_s) - \partial_{z^\alpha}(\Xi_{G_s})(0)\Delta Z^\alpha_s). \end{aligned} \tag{4.12}$$

If $\mathcal{G} = GL(n)$ and $\Xi_g(z) = \Xi_B(z) = B \cdot z$, since both $\partial_{z^\alpha z^\beta}(\Xi_{B_t})(0)$ and $(\Xi_{G_s}(\Delta Z_s) - \partial_{z^\alpha}(\Xi_{G_s})(0)\Delta Z^\alpha_s)$ are equal to zero, we obtain equation (4.9).

Remark 4.8 *When $N = \mathbb{R}^n$ the right-hand side of equation (4.12) does not depend on $\tilde{Z}$.*

Theorem 4.9 *Let $N, \tilde{N}$ be two Lie groups and suppose that $(X, \tilde{Z})$ (where $\tilde{Z}$ is defined on $\tilde{N}$) is a solution to the canonical SDE Ψ_{K_t}. If $d\tilde{Z}_t = \Xi_{G_t}(dZ_t)$, then (X, Z) is a solution to the canonical SDE defined by*

$$\hat{\Psi}_{k,g}(x,z) = \Psi_k(x, \Xi_g(z)).$$

Proof. We prove the theorem when $N, \tilde{N}, M$ are open subsets of $\mathbb{R}^m, \mathbb{R}^n$. The proof of the general case can be obtained by using suitable embeddings.
Let x^i, z^α and $\tilde{z}^\alpha$ be some global coordinate systems of $M, N, \tilde{N}$ respectively. By definition $\tilde{Z}$ is such that

$$\begin{aligned}
\tilde{Z}^\alpha_t - \tilde{Z}^\alpha_0 &= \int_0^t \partial_{z'^\beta}(\overline{\Xi}^\alpha_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})dZ^\beta_s + \frac{1}{2}\partial_{z'^\beta z'^\gamma}(\overline{\Xi}^\alpha_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})d[Z^\beta, Z^\gamma]^c_s \\
&\quad + \sum_{0\leq s\leq t} \overline{\Xi}^\alpha_{G_s}(\tilde{Z}_{s_-}, Z_s, Z_{s_-}) - \overline{\Xi}^\alpha_{G_s}(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-}) - \partial_{z'^\beta}(\overline{\Xi}^\alpha_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})\Delta Z^\beta_s,
\end{aligned}$$

where $\overline{\Xi}_g(\tilde{z}, z', z) = \Xi_g(z' \cdot z^{-1}) \cdot \tilde{z}$. By the previous equation, Lemma 4.5 and Remark 4.6 we obtain

$$\begin{aligned}
[\tilde{Z}^\alpha, \tilde{Z}^\beta]_t &= \int_0^t \partial_{z'^\gamma}(\overline{\Xi}^\alpha_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})\partial_{z'^\delta}(\overline{\Xi}^\beta_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})d[Z^\gamma, Z^\delta]^c_s \\
\Delta\tilde{Z}^\alpha_t &= \overline{\Xi}^\alpha_{G_t}(\tilde{Z}_{t_-}, Z_t, Z_{t_-}) - \overline{\Xi}^\alpha_{G_t}(\tilde{Z}_{t_-}, Z_{t_-}, Z_{t_-}).
\end{aligned}$$

Therefore, since $(X, \tilde{Z})$ is a solution to the canonical SDE Ψ_{K_t}, using Lemma 4.5 and Remark 4.6, we have

$$\begin{aligned}
X^i_t - X^i_0 &= \int_0^t \partial_{\tilde{z}'^\alpha}(\overline{\Psi}^i_{K_s})(X_{s_-}, \tilde{Z}_{s_-}, \tilde{Z}_{s_-})d\tilde{Z}^\alpha_s + \frac{1}{2}\partial_{\tilde{z}'^\alpha\tilde{z}'^\beta}(\overline{\Psi}^i_{K_s})(X_{s_-}, \tilde{Z}_{s_-}, \tilde{Z}_{s_-})d[\tilde{Z}^\alpha, \tilde{Z}^\beta]_s \\
&\quad + \sum_{0\leq s\leq t}\{\Psi^i_{K_s}(X_{s_-}, \Delta\tilde{Z}_s) - \Psi^i_{K_s}(X_{s_-}, 1_N) - \partial_{\tilde{z}'^\alpha}(\overline{\Psi}^i_{K_s})(X_{s_-}, \tilde{Z}_{s_-}, \tilde{Z}_{s_-})\Delta\tilde{Z}^\alpha_s\} \\
&= \int_0^t \partial_{\tilde{z}'^\alpha}(\overline{\Psi}^i_{K_s})(X_{s_-}, \tilde{Z}_{s_-}, \tilde{Z}_{s_-})\partial_{z'^\beta}(\overline{\Xi}^\alpha_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})dZ^\beta_s \\
&\quad + \frac{1}{2}\int_0^t \partial_{\tilde{z}'^\alpha}(\overline{\Psi}^i_{K_s})(X_{s_-}, \tilde{Z}_{s_-}, \tilde{Z}_{s_-})\partial_{z'^\beta z'^\gamma}(\overline{\Xi}^\alpha_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})d[Z^\beta, Z^\gamma]^c_s + \\
&\quad + \sum_{0\leq s\leq t} \partial_{\tilde{z}'^\alpha}(\overline{\Psi}^i_{K_s})(X_{s_-}, \tilde{Z}_{s_-}, \tilde{Z}_{s_-})(\overline{\Xi}^\alpha_{G_s}(\tilde{Z}_{s_-}, Z_s, Z_{s_-}) - \overline{\Xi}_{G_s}(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-}) + \\
&\quad - \partial_{z'^\beta}(\overline{\Xi}^\alpha_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})\Delta Z^\beta_s) + \\
&\quad + \frac{1}{2}\int_0^t \Big(\partial_{\tilde{z}'^\alpha\tilde{z}'^\delta}(\overline{\Psi}^i_{K_s})(X_{s_-}, \tilde{Z}_{s_-}, \tilde{Z}_{s_-})\partial_{z'^\beta}(\overline{\Xi}^\alpha_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-}) \cdot \\
&\quad \cdot \partial_{z'^\beta}(\overline{\Xi}^\delta_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})\Big)\, d[Z^\beta, Z^\gamma]_s + \sum_{0\leq s\leq t}\Big(\Psi^i(X_{s_-}, \Xi_{G_s}(\Delta Z_s)) - \Psi^i(X_{s_-}, 1_N) + \\
&\quad - \partial_{\tilde{z}'^\alpha}(\overline{\Psi}^i_{K_s})(X_{s_-}, \tilde{Z}_{s_-}, \tilde{Z}_{s_-})(\overline{\Xi}^\alpha_{G_s}(\tilde{Z}_{s_-}, Z_s, Z_{s_-}) - \overline{\Xi}^\alpha_{G_s}(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-}))\Big).
\end{aligned}$$

By the chain rule for derivatives and the fact that $\overline{\Xi}^\alpha(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-}) = \tilde{Z}^\alpha_{s_-}$ we have

$$\partial_{z'^\beta}(\overline{\Psi}^i_{K_s}(x, \overline{\Xi}_{G_s}(\tilde{z}, z', z), \tilde{z}))\Big|_{\substack{x=X_{s_-}, \tilde{z}=\tilde{Z}_{s_-} \\ z=z'=Z_{s_-}}} = \partial_{\tilde{z}'^\alpha}(\overline{\Psi}^i_{K_s})(X_{s_-}, \tilde{Z}_{s_-}, \tilde{Z}_{s_-})\partial_{z'^\beta}(\overline{\Xi}^\alpha_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})$$

$$\begin{aligned}
\partial_{z'^\beta z'^\gamma}(\overline{\Psi}^i_{K_s}(x, \overline{\Xi}_{G_s}(\tilde{z}, z', z), \tilde{z}))\Big|_{\substack{x=X_{s_-}, \tilde{z}=\tilde{Z}_{s_-} \\ z=z'=Z_{s_-}}} &= \partial_{\tilde{z}'^\alpha}(\overline{\Psi}^i_{K_s})(X_{s_-}, \tilde{Z}_{s_-}, \tilde{Z}_{s_-})\partial_{z'^\beta z'^\gamma}(\overline{\Xi}^\alpha_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-}) + \\
&\quad + \partial_{\tilde{z}'^\alpha\tilde{z}'^\delta}(\overline{\Psi}^i_{K_s})(X_{s_-}, \tilde{Z}_{s_-}, \tilde{Z}_{s_-})\partial_{z'^\beta}(\overline{\Xi}^\alpha_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})\partial_{z'^\gamma}(\overline{\Xi}^\delta_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})
\end{aligned}$$

Using the fact that

$$\overline{\Psi}_k(x, \overline{\Xi}_g(\tilde{z}, z', z), \tilde{z}) = \Psi_k(x, (\Xi_g(z' \cdot z^{-1}) \cdot \tilde{z}) \cdot \tilde{z}^{-1}) = \hat{\Psi}_{k,g}(x, z' \cdot z^{-1}) = \Psi_k(x, \Xi_g(z' \cdot z^{-1})) = \overline{\hat{\Psi}}_{k,g}(x, z', z)$$

we obtain

$$\begin{aligned} X^i_t - X^i_0 &= \int_0^t \partial_{z'^\beta}(\overline{\hat{\Psi}}^i_{K_s,G_s})(X_{s_-}, Z_{s_-}, Z_{s_-}) dZ^\beta_s + \frac{1}{2} \partial_{z'^\beta z'^\gamma}(\overline{\hat{\Psi}}^i_{K_s,G_s})(X_{s_-}, Z_{s_-}, Z_{s_-}) d[Z^\beta, Z^\gamma]_s + \\ &+ \sum_{0 \leq s \leq t} \overline{\hat{\Psi}}^i_{K_s,G_s}(X_{s_-}, Z_s, Z_{s_-}) - \hat{\Psi}^i_{K_s,G_s}(X_{s_-}, Z_{s_-}, Z_{s_-}) - \partial_{z'^\beta}(\overline{\hat{\Psi}}^i_{K_s,G_s})(X_{s_-}, Z_{s_-}, Z_{s_-}) \Delta Z^\beta_s, \end{aligned}$$

and so $dX_t = \hat{\Psi}_{K_t,G_t}(dZ_t)$. ■

Corollary 4.10 *Suppose that $\mathcal{G}$ is a Lie group and Ξ is a Lie group action. If (X, Z) is a solution to the canonical SDE Ψ_{K_t}, then $(X, \tilde{Z})$ is a solution to the canonical SDE defined by*

$$\hat{\Psi}_{k,g}(x, z) = \Psi_k(x, \Xi_{g^{-1}}(z)).$$

Proof. The proof is an application of Theorem 4.9 and of the fact that $dZ_t = \Xi_{G_t^{-1}}(d\tilde{Z}_t)$. Indeed, defining $d\hat{Z}_t = \Xi_{G_t^{-1}}(d\tilde{Z}_t)$, by Theorem 4.9 we have that $d\hat{Z}_t = \Xi_{G_t^{-1}} \circ \Xi_{G_t}(dZ_t) = \Xi_{1_\mathcal{G}}(dZ_t) = dZ_t$. The corollary follows directly from Theorem 4.9. ■

4.2 Gauge symmetries of semimartingales on Lie groups

4.2.1 Definition of gauge symmetries

Let us consider the following well known property of Brownian motion. Consider a Brownian motion Z on $\mathbb{R}^n$ and let $B_t : \Omega \times [0, T] \to O(n)$ be a predictable process, with respect to the natural filtration of Z, taking values in the Lie group $O(n)$ of orthogonal matrices. Then the process defined by

$$Z'^\alpha_t = \int_0^t B^\alpha_{\beta,s} dZ^\beta_s \tag{4.13}$$

is a new n dimensional Brownian motion.
We propose a generalization of this property to the case in which Z is a càdlàg semimartingale in a Lie group N (see [151] for a similar result about Poisson measures). In the simple case $N = \mathbb{R}^n$, by replacing the Brownian motion with a general semimartingale, the invariance property (4.13) is no longer true. So we need

- a method to generalize the integral relation to the case where Z is no more a process on $\mathbb{R}^n$ and the Lie group valued process is no more the $O(n)$-valued process B,
- a class of semimartingales on a Lie group N such that the generalization of the integral relation (4.13) holds.

Definition 4.11 *Let Z be a semimartingale on a Lie group N with respect to the filtration $\mathcal{F}_t$. Given a Lie group $\mathcal{G}$ and an element $g \in \mathcal{G}$, we say that Z admits $\mathcal{G}$, with action Ξ_g and with respect to the filtration $\mathcal{F}_t$, as* gauge symmetry group *if, for any $\mathcal{F}_t$-predictable locally bounded process G_t taking values in $\mathcal{G}$, the semimartingale $\tilde{Z}$ solution to the equation $d\tilde{Z}_t = \Xi_{G_t}(dZ_t)$ has the same law as Z.*

In the following we consider that the filtration $\mathcal{F}_t$ of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given and we omit to mention it if it is not strictly necessary.
Since $\tilde{Z}$ in Definition 4.11 solves the canonical equation $d\tilde{Z}_t = \Xi_{G_t}(dZ_t)$, for all times, we are interested in characterizing SDEs of the previous form with explosion time equal to $+\infty$ for any G_t. The following proposition gives us a sufficient condition on the group N such that, for any action Ξ_g, the corresponding canonical SDE has indeed explosion time $+\infty$.

Proposition 4.12 *Suppose that N admits a faithful representation. Then, for any locally bounded process G_t in $\mathcal{G}$, the explosion time of the SDE $d\tilde{Z}_t = \Xi_{G_t}(dZ_t)$ is $+\infty$.*

Proof. Let $K : N \to \mathrm{Mat}(l_N, l_N)$ be a faithful representation of N. In this representation, the geometrical SDEs associated with Ξ_g, is defined by the map $\overline{\Xi}_g$ given by

$$\overline{\Xi}_g(\tilde{z}, z', z) = K(\Xi_g(z' \cdot z^{-1})) \cdot K(\tilde{z}),$$

where $\cdot$ on the right-hand side denotes the usual matrix multiplication. If k^i is the standard cartesian coordinate system in $\mathrm{Mat}(l_N, l_N)$, extending suitably Ξ_g to all $\mathrm{Mat}(l_N, l_N)$, we have that $\overline{\Xi}^i(\tilde{k}, k', k)$, $\partial_{k'^j}(\overline{\Xi}^i_g)(\tilde{k}, k', k)$ and $\partial_{k'^j k'^l}(\overline{\Xi}^i_g)(\tilde{k}, k', k)$ are linear in $\tilde{k}$. So, putting $Z^i = k^i(Z)$ and $\tilde{Z}^i = k^i(\tilde{Z})$, the SDE

$$\begin{aligned} \tilde{Z}^i_t &= K^i_0 + \int_0^t \partial_{k'^j}(\overline{\Xi}^i_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-}) dZ^j_s + \\ &+ \frac{1}{2}\int_0^t \partial_{k'^j k'^l}(\overline{\Xi}^i_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-}) d[Z^j, Z^l]_s + \\ &+ \sum_{0 \leq s \leq t} (\overline{\Xi}^i_{G_s})(\tilde{Z}_{s_-}, Z_s, Z_{s_-}) - \tilde{Z}^i_s - \Delta Z^j_s \partial_{k'^j}(\overline{\Xi}^i_{G_s})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})), \end{aligned}$$

is linear in $\tilde{Z}$ and so, by well known results on SDEs with jumps in $\mathbb{R}^{l_N^2}$ (see, e.g., [22, Chapter 5]) the solution has explosion time $\tau = +\infty$ almost surely. ■

In order to provide a method to construct semimartingales admitting gauge symmetry groups, we start by showing how it is possible to obtain, starting from martingales with gauge symmetries, new semimartingales with different gauge symmetries.

Proposition 4.13 *Given two Lie groups N and N', let Z be a semimartingale on N with gauge symmetry group $\mathcal{G}$ and action Ξ_g. If $\Theta : N \to N'$ is a diffeomorphism from N onto N' such that $\Theta(1_N) = 1_{N'}$, then $d\tilde{Z}_t = \Theta(dZ_t)$ has gauge symmetry group $\mathcal{G}$ with action $\Theta \circ \Xi_g \circ \Theta^{-1}$.*

Proof. By Corollary 4.10 $dZ_t = \Theta^{-1}(d\tilde{Z}_t)$, and since Z has gauge symmetry group $\mathcal{G}$ with action Ξ_g, by Theorem 4.9, $\Xi_{G_t}(dZ_t) = \Xi_{G_t} \circ \Theta^{-1}(d\tilde{Z}_t)$ has the same distribution as Z for any locally bounded predictable process G_t. Moreover, by the uniqueness of the strong solution to a geometrical SDE, we have that $\Theta(\Xi_{G_t} \circ \Theta^{-1}(d\tilde{Z}_t)) = \Theta \circ \Xi_{G_t} \circ \Theta^{-1}(d\tilde{Z}_t)$ has the same distribution as $\tilde{Z}$. ■

In the following, in order to provide some explicit methods to verify that a semimartingale on a Lie group N has the gauge symmetry group $\mathcal{G}$ with action Ξ_g, we introduce the concept of characteristics of a semimartingale on a Lie group. This allows us to formulate a condition, equivalent to Definition 4.11, that can be directly applied to Lévy processes on Lie groups providing a completely deterministic method to verify Definition 4.11 in this case. Then we shall use this reformulation to give some examples of non-Markovian processes admitting gauge symmetry groups.

4.2.2 Characteristics of a Lie group valued semimartingale

In this section we extend the well known concept of semimartingale characteristics from the $\mathbb{R}^n$ setting to the case of a semimartingale defined on a general finite dimensional Lie group N.
Given n generators $Y_1, ..., Y_n$ of right-invariant vector fields on N providing a global trivialization of the tangent bundle TN, the corresponding Hunt functions $h^1, ..., h^n$ are measurable, bounded functions, smooth in a neighbourhood of the identity 1_N, with compact support and such that $h^\alpha(1_N) = 0$ and $Y_\alpha(h^\beta)(1_N) = \delta^\beta_\alpha$ (the existence of these functions is proved, for example, in [103]). Generalizing [108] we give the following

Definition 4.14 *Let b be a predictable semimartingale of bounded variation on $\mathbb{R}^n$, and let A be a predictable continuous semimartingale taking values in the set of semidefinite positive $n \times n$ matrices. Furthermore let ν be a predictable random measure defined on $\mathbb{R}_+ \times N$. If Z is a semimartingale on a Lie group N, we say that Z has characteristics (b, A, ν) with respect to $Y_1, ..., Y_n$ and $h^1, ..., h^n$ if, for any smooth bounded functions $f, g \in C^\infty(N)$ and for any smooth and bounded function p which is identically 0 in a neighbourhood of 1_N, we have that*

$$\textstyle\sum_{0\leq s\leq t} p(\Delta Z_s) - \int_0^t \int_N p(z')\nu(ds, dz'), \tag{4.14}$$

$$[f(Z), g(Z)]^c_t - g(Z_0)f(Z_0) - \textstyle\int_0^t Y_\alpha(f)(Z_{s_-})Y_\beta(g)(Z_{s_-})dA^{\alpha\beta}_s \tag{4.15}$$

$$\begin{aligned} f(Z_t) - f(Z_0) - \textstyle\int_0^t Y_\alpha(f)(Z_{s_-})db^\beta_s - \frac{1}{2}\int_0^t Y_\alpha(Y_\beta(f))(Z_{s_-})dA^{\alpha\beta}_s + \\ - \textstyle\sum_{0\leq s\leq t}(f(Z_s) - f(Z_{s_-}) - h^\alpha(\Delta Z_s)Y_\alpha(f)(Z_{s_-})) \end{aligned} \tag{4.16}$$

are local martingales.

Remark 4.15 *We note that condition (4.15) is redundant, because it can be deduced from (4.14) and (4.16).*

The following theorem states that any semimartingale Z defined on a Lie group N admits (essentially) a unique characteristic triplet (b, A, ν).

Theorem 4.16 *If Z is a semimartingale on a Lie group N, then Z admits a characteristic triplet (b, A, ν) with respect to $Y_1, ..., Y_n$ and $h^1, ...h^n$, which is unique up to $\mathbb{P}$ null sets.*

Proof. We first prove the existence. Given a semimartingale Z on N, we can associate with Z a unique random measure on N given by

$$\mu^Z(\omega, dt, dz) = \sum_{s\geq 0} I_{\Delta Z_s \neq 1_N} \delta_{(s,\Delta Z_s(\omega))}(dt, dz),$$

where δ_a is the Dirac delta with mass in $a \in \mathbb{R}_+ \times N$. The random measure μ^Z is an integer-valued random measure (see, e.g., [108, Chapter II, Proposition 1.16]), hence there exists a unique non-negative predictable random measure $\mu^{Z,p}$, which is the compensator of μ^Z (see, e.g., [108, Chapter II, Theorem 1.8]).
We prove that $\nu = \mu^{Z,p}$. Indeed, by definition of μ^Z, we have $\sum_{0\leq s\leq t} h(\Delta Z_s) = \int_0^t \int_N h(z')\mu^Z(ds, dz')$ and, by definition of compensator, we have that

$$\int_0^t \int_N h(z')\mu^Z(ds, dz') - \int_0^t \int_N h(z')\mu^{Z,p}(ds, dz')$$

is a local martingale.
In order to prove the existence of processes $b^\alpha, A^{\alpha\beta}$ we introduce a Riemannian embedding $K : N \to \mathbb{R}^{k_N}$ with respect to a left invariant metric on N. Put

$$\tilde{Z}^i = k^i(Z),$$

where $K=(k^1,...k^{k_N})$, and write

$$Z^i_t=\tilde{Z}^i_t-\sum_{0\leq s\leq t}\left(\Delta\tilde{Z}^i_s-h^\alpha(\Delta Z_s)Y_\alpha(k^i)(Z_{s_-})\right).$$

Since K is Riemannian, the norms of $K_*(Y_\alpha)(x)$ are constant and so $Y_\alpha(k^i)$ are bounded. Because of

$$\Delta Z^i_t=h^\alpha(\Delta Z_t)Y_\alpha(k^i(Z_{t_-})),$$

and h^α being bounded, Z^i have bounded jumps and so they are special semimartingales. This means that the processes Z^i can be decomposed in a unique way as

$$Z^i=B^i+M^{i,c}+M^{i,d},$$

where B^i is a predictable process having a bounded variation, $M^{i,c}$ is a continuous local martingale and $M^{i,d}$ is a purely discontinuous local martingale. If we consider the matrix

$$P=(Y_\alpha(k^i))|_{\substack{\alpha=1,...,n\\ i=1,...,k_N}},$$

since K is an immersion and $Y_1,...,Y_n$ are point by point linearly independent, P is non-singular. Therefore there exists a pseudoinverse $\tilde{P}=(P^\alpha_i)|_{\substack{i=1,...,k_N\\ \alpha=1,...,n}}$ such that $\tilde{P}\cdot P=I_n$, $P\cdot\tilde{P}=Id|_{Im(P)}$. We can choose, for example, $\tilde{P}=(P^T\cdot P)^{-1}\cdot P^T$. Therefore, we can define

$$\begin{aligned} b^\alpha_t &= \int_0^t \tilde{P}^\alpha_i(Z_{s_-})dB^i_s+Y_\beta(\tilde{P}^\alpha_i)(Z_{s_-})\tilde{P}^\beta_j(Z_{s_-})d[M^{i,c},M^{j,c}]_s\\ A^{\alpha\beta}_t &= \int_0^t \tilde{P}^\alpha_i(Z_{s_-})\tilde{P}^\beta_j(Z_{s_-})d[M^{i,c},M^{j,c}]_s. \end{aligned}$$

Given $f,g\in C^\infty(N)$ let us consider two extensions $\tilde{f},\tilde{g}$ in $\mathbb{R}^{k_N}$ which are constants with respect to a distribution $D\subset T\mathbb{R}^{k_N}|_{K(N)}$ which is transverse to $TK(N)$, i.e., for any $Y,Y'\in D$, $Y(\tilde{f})=Y(\tilde{g})=0$ and $Y(Y'(\tilde{f}))=Y(Y'(\tilde{g}))=0$ (the existence of such kind of extensions is guaranteed by the existence of a tubular neighbourhood of $K(N)$).
By Itô formula we have

$$\begin{aligned} f(Z_t)-f(Z_0) &= \textstyle\int_0^t\partial_{k^i}(\tilde{f})(Z_{s_-})d\tilde{Z}^i_s+\frac{1}{2}\int_0^t\partial_{k^ik^j}(\tilde{f})(Z_{s_-})d[\tilde{Z}^i,\tilde{Z}^j]^c_s+\\ &\quad+\textstyle\sum_{0\leq s\leq t}(f(Z_s)-f(Z_{s_-})-\Delta Z^i_s\partial_{k^i}(\tilde{f})(Z_{s_-})) \end{aligned}\tag{4.17}$$

and the same formula holds for g. Recalling that $[\tilde{Z}^i,\tilde{Z}^j]^c=[Z^i,Z^j]^c=[M^{i,c},M^{j,c}]$ and that, for our choice of the extensions $\tilde{f},\tilde{g}$,

$$\partial_{k^i}(\tilde{f})=\tilde{P}^\alpha_iY_\alpha(f),$$

we have

$$[f(Z),g(Z)]^c_t=\int_0^t Y_\alpha(f)(Z_{s_-})Y_\beta(g)(Z_{s_-})dA^{\alpha\beta}_s.$$

Finally, recalling that

$$\tilde{Z}^i_t=B^i_t+M^{i,c}_t+M^{i,d}_t+\sum_{0\leq s\leq t}(\Delta\tilde{Z}^i_s-h^\alpha(\Delta Z_s)Y_\alpha(k^i)(Z_{s_-}))$$

$$\partial_{k^ik^j}(\tilde{f})=Y_\beta(Y_\alpha(f))\tilde{P}^\alpha_j\tilde{P}^\beta_i+Y_\beta(\tilde{P}^\alpha_j)\tilde{P}^\beta_iY_\alpha(f),$$

and using both equation (4.17) and Lemma 4.5 we obtain that

$$f(Z_t) - f(Z_0) - \int_0^t Y_\alpha(f)(Z_{s_-})db_s^\alpha - \frac{1}{2}\int_0^t Y_\alpha(Y_\beta(f))(Z_{s_-})dA_s^{\alpha\beta} + \\ - \sum_{0\leq s\leq t}(f(Z_s) - f(Z_{s_-}) - h^\alpha(\Delta Z_s)Y_\alpha(f)(Z_{s_-}))$$

is a local martingale.

The uniqueness of ν has already been proved using the uniqueness of the compensator of the random measure μ^Z (see [108, Chapter II, Theorem 1.8]). In order to prove the uniqueness of $b^\alpha, A^{\alpha\beta}$ we use the fact that a predictable martingale of bounded variation is constant (see, e.g., [153, Chapter III, Theorem 12]). Indeed, if (b', A', ν) is another characteristic triplet of Z, we have that, for any $f, g \in C^\infty(M)$,

$$\int_0^t Y_\alpha(f)(Z_{s_-})Y_\beta(g)(Z_{s_-})d(A_s^{\alpha\beta} - A_s'^{\alpha\beta}) \\ \int_0^t Y_\alpha(f)(Z_{s_-})d(b_s^\alpha - b_s'^\alpha) - \int_0^t Y_\alpha(Y_\beta(f))(Z_{s_-})d(A_s^{\alpha\beta} - A_s'^{\alpha\beta})$$

are local martingales. Since the processes involved in the previous integrals are predictable and b, b', A, A' are of bounded variation, they are local martingales having a vanishing bounded variation at the origin and so they are identically equal to 0. Finally, by using the arbitrariness of f, g and the existence of a partition of unity for N, we find that $b - b' = 0$ and $A - A' = 0$ up to $\mathbb{P}$-null sets. ∎

4.2.3 Gauge symmetries and semimartingales characteristics

In this section we provide an equivalent method to verify the conditions in Definition 4.11 using the characteristics introduced in the previous section.

In particular, after introducing suitable geometric and probabilistic tools, we look for conditions to be satisfied by the characteristics of a semimartingale in order to ensure that the semimartingale admits a gauge symmetry group.

First of all we need to study in more detail the role of the filtration $\mathcal{F}_t$ in Definition 4.11. In fact, although the definition of gauge symmetry group apparently concerns only the law of Z and not the chosen filtration, there are examples of semimartingales Z admitting a gauge symmetry group $\mathcal{G}$ with respect to a filtration $\mathcal{F}_t$ but such that $\mathcal{G}$ is no longer a gauge symmetry group for Z if a different filtration $\mathcal{H}_t$ is chosen. For example, let W be a standard n dimensional Brownian motion, let $\mathcal{F}_t$ be its natural filtration and let us put $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(W_T)$. It is well known that W is a semimartingale with respect to both $\mathcal{F}_t$ and $\mathcal{H}_t$, but the rotations are a gauge group only with respect to the filtration $\mathcal{F}_t$ and not with respect to $\mathcal{H}_t$. Indeed, let $B : \mathbb{R}^n \to O(n)$ be a measurable map such that $B(x) \cdot x = (|x|, 0, ..., 0)$. The constant process $B(W_T)$ is predictable with respect to the filtration $\mathcal{H}_t$ and it is not adapted with respect to $\mathcal{F}_t$. On the other hand the semimartingale

$$\tilde{W}_t^\alpha = \int_0^t B_\beta^\alpha(W_T)dW_s^\beta = B_\beta^\alpha(W_T)W_t^\beta,$$

is not a Brownian motion since, for example, $\tilde{W}_T = (|W_T|, 0, ..., 0)$ is not a Gaussian random variable. This phenomenon is due to the fact that the family of the $\mathcal{H}_t$-predictable processes is too large for preserving the invariance property of Brownian motion. In order to avoid this kind of phenomena, and ensuring that a gauge symmetry is a property of the law of the process Z and not of its filtration, we introduce the following definition.

Definition 4.17 *Let Z be a semimartingale with respect to the filtration $\mathcal{F}_t$. We say that the filtration $\mathcal{F}_t$ is a generalized natural filtration if there exists a version of the characteristic triplet*

(b, A, ν) of Z (with respect to the filtration $\mathcal{F}_t$), which is predictable with respect to the natural filtration $\mathcal{F}_t^Z \subset \mathcal{F}_t$ of the semimartingale Z.

It is important to note that if (b, A, ν) are the characteristics of a semimartingale Z with respect to its natural filtration, then they are also the characteristics of Z with respect to any generalized natural filtration for Z. For this reason, hereafter, whenever we consider a generalized natural filtration $\mathcal{F}_t$ for Z we can use the characteristics (b, A, ν) with respect to the natural filtration of Z as the characteristics of Z with respect to $\mathcal{F}_t$.

Let us consider the probability space

$$\Omega^c = \Omega_A \times \Omega_B,$$

where $\Omega_A = \mathcal{D}_{1_N}([0, +\infty), N)$ is the space of càdlàg functions $\omega_A(t)$ taking values on N and such that $\omega_A(0) = 1_N$, and $\Omega_B = L^{\infty}_{loc}([0, +\infty), \mathcal{G})$ is the set of locally bounded and measurable functions taking values in $\mathcal{G}$.
On the set Ω_A we consider the standard filtration $\mathcal{F}_t^A$ of $\mathcal{D}_{1_N}([0, +\infty), N)$ and on Ω_B the filtration $\mathcal{F}_t^B$ generated by the standard filtration of $C^0([0, +\infty), \mathcal{G}) \subset \Omega_B$ (usually called the predictable filtration). We denote by π_A, π_B the projections of Ω on Ω_A and Ω_B respectively and so we define $\mathcal{F}_t^c$ $\mathcal{F}_t^c = \sigma(\pi_A^{-1}(\mathcal{F}_t^A), \pi_B^{-1}(\mathcal{F}_t^B))$. We call Ω^c the canonical probability space and $\mathcal{F}_t^c$ the natural filtration on Ω^c.
We need the space Ω_A in order to define a semimartingale Z on N, and the space Ω_B in order to define a locally bounded predictable process taking values on $\mathcal{G}$. Choosing a particular semimartingale Z on N and a predictable process G_t on $\mathcal{G}$ is equivalent to fixing a probability measure $\mathbb{P}$ on Ω such that $Z_t(\omega) = \pi_A(\omega)(t)$ is a semimartingale on N (the fact that the process $G_t(\omega) = \pi_B(\omega)(t)$ is a locally bounded predictable process is automatically guaranteed by the choices of the space Ω_B and the filtration $\mathcal{F}_t^B$).
Given an N valued semimartingale Z and a generic predictable process G_t taking values in $\mathcal{G}$, both defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, there exist a natural probability measure $\mathbb{P}^c = M_*(\mathbb{P})$ on the canonical probability space Ω^c and a natural map

$$\begin{array}{cccc} M: & \Omega & \longrightarrow & \Omega^c \\ & \omega & \longmapsto & (Z_t(\omega), G_t(\omega)) \end{array}$$

which puts the couple (Z_t, G_t) in canonical form. Thus, fixing the process G_t and the law $\mathbb{P}^Z$ of the semimartingale Z_t is equivalent to fixing the probability law $\mathbb{P}^c$ on Ω^c so that the restriction of $\mathbb{P}^c$ to the Ω_A measurable subsets, $\mathbb{P} = \mathbb{P}^c|_{\mathcal{F}^A}$, is exactly $\mathbb{P}^Z$. As a consequence, proving a statement involving only the measurable objects Z_t, G_t which is independent from the choice of a specific predictable process G_t, is equivalent to proving the same statement on the probability space Ω^c with respect to the canonical processes $\omega_A(t), \omega_B(t)$ and for a suitable subset of probability laws $\mathbb{P}^c$ on Ω^c such that $\mathbb{P}|_{\mathcal{F}^A} = \mathbb{P}^Z$. This subset depends on the filtration $\mathcal{F}_t$ of the probability space chosen. In particular if $\mathcal{F}_t$ is a generalized natural filtration for Z, then $\tilde{\mathcal{F}}_t^c$ is a generalized natural filtration for $\omega_A(t)$ (where $\tilde{\mathcal{F}}_t^c$ is the completion of $\mathcal{F}_t^c$ with respect to $\mathbb{P}^c$). Since we consider only generalized natural filtrations for the semimartingale Z, we suppose that $\mathbb{P}^c$ is such that $\tilde{\mathcal{F}}_t^c$ is a generalized natural filtration.
For this reason, in the following we shall only consider the canonical probability space Ω^c with law $\mathbb{P} = \mathbb{P}^c$ and denote by Z_t the canonical semimartingale $\omega_A(t)$ and by G_t the canonical predictable process $\omega_B(t)$.
In the same way, we identify the solution $\tilde{Z}$ to the SDE $d\tilde{Z}_t = \Xi_{G_t}(dZ_t)$ with the measurable map $\Lambda_A : \Omega \to \Omega_A$ such that $\tilde{Z}_t(\omega) = \Lambda_A(\omega)(t)$. We can extend the map Λ_A to a map $\Lambda : \Omega \to \Omega$ given

by

$$\Lambda(\omega) = (\Lambda_A(\omega), \pi_2(\omega)),$$

defining a new probability measure $\mathbb{P}' = \Lambda_*(\mathbb{P})$. The map Λ is $\mathbb{P}$ invertible, i.e. there exists a map Λ' such that $\Lambda \circ \Lambda'$ is equal to the identity map up to $\mathbb{P}'$ null sets and the map $\Lambda' \circ \Lambda$ is equal to the identity up to $\mathbb{P}$ null sets. The construction of the map Λ' is similar to the construction of Λ starting from the stochastic differential equation $dZ_t = \Xi_{(G_t)^{-1}}(d\tilde{Z}_t)$ and the measure $\mathbb{P}'$. The proof of the fact that Λ' is the $\mathbb{P}'$ inverse of Λ and hence Λ is the $\mathbb{P}$ inverse of Λ', is based on Theorem 4.9. It is important to note that $\hat{\mathcal{F}}^c_t$, i.e. the completion of $\mathcal{F}^c_t$ with respect to the probability $\mathbb{P}'$, could not be a generalized natural filtration for $\omega_A(t)$ even if $\tilde{\mathcal{F}}^c_t$ is a generalized natural filtration for Z_t under $\mathbb{P}$.
Given the probability law $\mathbb{P}^Z$ on Ω_A, by Theorem 4.16 there exist some measurable and predictable functions $b^\alpha, A^{\alpha\beta} : \Omega_A \times \mathbb{R}_+ \to \mathbb{R}$ and a random predictable measure $\nu : \Omega_A \to \mathcal{M}(\mathbb{R}_+ \times N)$ which are the characteristics of the canonical process $Z_t(\omega) = \pi_A(\omega(t))$ and are uniquely defined up to $\mathbb{P}^Z$ null-sets. The characteristic triplets (b, A, ν), seen as $\mathcal{F}^A$ measurable objects, are uniquely determined by the probability measure $\mathbb{P}^Z$. The converse, namely the fact that the $\mathcal{F}^A$ measurable objects (b, A, ν) uniquely individuate the probability law $\mathbb{P}^Z$, is in general not true (the reader can think, for example, to diffusion processes whose martingale problem admits multiple solutions). In the case in which the triplet (b, A, ν) uniquely determines the probability law $\mathbb{P}^Z$ on Ω_A we say that the triplet (b, A, ν) uniquely individuates the law of Z. Examples of this situation are, e.g., the $\mathbb{R}^n$ Brownian motion, $\mathbb{R}^n$ Lévy processes, diffusion processes with a unique solution to the associated martingale problem, and point processes. If the law $\mathbb{P}$ on Ω^c is such that $\mathbb{P}|_{\mathcal{F}^A} = \mathbb{P}^Z$ and the filtration $\tilde{\mathcal{F}}^c_t$ is a generalized natural filtration for $\omega_A(t)$, then the same $\mathcal{F}^A$ measurable characteristics (b, A, ν), viewed as Ω^c semimartingales, are characteristics of Z with respect to $\tilde{\mathcal{F}}^c_t$ as well. Obviously it is possible to define other characteristic triplets $(\bar{b}, \bar{A}, \bar{\nu})$ of Z_t on Ω^c which are only $\tilde{\mathcal{F}}^c$ adapted and not $\mathcal{F}^A_t$ adapted. The characteristics $(\bar{b}, \bar{A}, \bar{\nu})$ are equal to (b, A, ν) up to $\mathbb{P}$ null sets (and not only up to $\mathbb{P}^Z$ null sets).

Let us now consider a map $\Xi_g : N \to N$ such that $\Xi_g(1_N) = 1_N$. This means that the tangent map $T\Xi_g$ of Ξ_g sends the tangent space of the identity $TN|_{1_N}$ into itself. Recalling that the Lie algebra $\mathfrak{n}$ associated with N is exactly the tangent space to the identity, we have that there exists a map

$$\Upsilon_g = T\Xi_g|_{1_N} : \mathfrak{n} \to \mathfrak{n}.$$

The map Υ has the following property: if Y is any right invariant vector field on N and $\hat{\Xi}_g(\tilde{z}, z) = \Xi_g(z) \cdot \tilde{z}$, then, by definition of right invariant vector fields, for any smooth function $f \in C^\infty(N)$, we have

$$Y^z(f \circ \hat{\Xi}_g)(\tilde{z}, 1_N) = \Upsilon_g(Y)(f)(\tilde{z}),$$

where Y^z denotes the vector fields Y applied to the z^α variables. Going further along in this way, instead of working with first derivatives we can work with second derivatives and we can define a linear map

$$O_g : \mathfrak{n} \otimes \mathfrak{n} \to \mathfrak{n}$$

such that, for any two right invariant vector fields Y, Y' defined on N, we have

$$Y'^z(Y^z(f \circ \hat{\Xi}_g))(\tilde{z}, 1_N) = \Upsilon_g(Y')(\Upsilon_g(Y)(f))(\tilde{z}) + O_g(Y, Y')(f)(\tilde{z}).$$

If we fix a basis $Y_1, ..., Y_n$ of $\mathfrak{n}$ (and so of right-invariant vector fields on N), the linear maps Υ_g, O_g become matrices $\Upsilon^\alpha_{g,\beta}$ and $O^\alpha_{g,\beta\gamma}$, where

$$\Upsilon_g(Y_\beta) = \Upsilon^\alpha_{g,\beta} Y_\alpha$$
$$O_g(Y_\beta, Y_\gamma) = O^\alpha_{g,\beta\gamma} Y_\alpha.$$

Theorem 4.18 *Let Z be a semimartingale on a Lie group N with characteristic triplet $(b(\omega_A), A(\omega_A), \nu(\omega_A))$. Suppose that Z admits $\mathcal{G}$ with action Ξ_g as gauge symmetry group with respect to any generalized natural filtration, then if $\mathbb{P}$ is a measure on Ω^c such that $\tilde{\mathcal{F}}_t$ is a generalized natural filtration with respect both Z_t and $d\tilde{Z}_t = \Xi_{G_t}(dZ_t)$, we have*

$$\begin{aligned} db^\alpha_t(\omega) &= \Upsilon^\alpha_{g(\omega_B),\beta} db^\beta_t(\pi_A(\Lambda'(\omega))) + \tfrac{1}{2} O^\alpha_{g(\omega_B),\beta\gamma} dA^{\beta\gamma}_t(\pi_A(\Lambda'(\omega))) + \\ &\quad + \int_N (h^\alpha(z') - h^\beta(\Xi_{g^{-1}(\omega_B)}(z'))\Upsilon^\alpha_{g(\omega_B),\beta}) \nu(\pi_A(\Lambda'(\omega)), dt, dz') \end{aligned} \quad (4.18)$$

$$dA^{\alpha\beta}_t(\omega) = \Upsilon^\alpha_{g(\omega_B),\gamma}\Upsilon^\beta_{g(\omega_B),\delta} dA^{\gamma\delta}_t(\pi_A(\Lambda'(\omega))) \quad (4.19)$$

$$\nu(\omega, dt, dz) = \Xi_{g(\omega_B)*}(\nu(\pi_A(\Lambda'(\omega)), dt, dz)), \quad (4.20)$$

up to a $\mathbb{P}' = \Lambda_*(\mathbb{P})$ null set. Furthermore, if $\tilde{b}, \tilde{A}, \tilde{\nu}$ are $\pi_A^{-1}(\mathcal{F}^A)$ measurable, the previous equalities hold with respect to $\mathbb{P}^Z$ null sets. Finally, if (b, A, ν) uniquely determines the law of Z, the previous conditions are also sufficient for the existence of a gauge symmetry group.

Before proving the theorem we study the transformations of the characteristics under (canonical) semimartingale changes.

Lemma 4.19 *If Z is a semimartingale with characteristics (b, A, ν), then $d\tilde{Z} = \Xi_{G_t}(dZ)$ is a semimartingale with characteristics*

$$\begin{aligned} d\tilde{b}^\alpha_t &= \Upsilon^\alpha_{G_t,\beta} db^\beta_t + \frac{1}{2} O^\alpha_{G_t,\beta\gamma} dA^{\beta\gamma}_t + \int_N (h^\alpha(z') - h^\beta(\Xi_{G_t^{-1}}(z'))\Upsilon^\alpha_{G_t,\beta})\nu(dt, dz') \\ d\tilde{A}^{\alpha\beta}_t &= \Upsilon^\alpha_{G_t,\gamma}\Upsilon^\beta_{G_t,\delta} dA^{\gamma\delta}_t \\ \tilde{\nu} &= \Xi^*_{G_t}(\nu). \end{aligned}$$

Proof. We denote by $(\tilde{b}, \tilde{A}, \tilde{\nu})$ the characteristic triplet of $\tilde{Z}$. Since the jumps of $\tilde{Z}$ are $\Delta\tilde{Z}_t = \Xi_{G_t}(\Delta Z_t)$ and the jump times of $\tilde{Z}$ are the same of Z we have, using the notation of Theorem 4.16,

$$\mu^{\tilde{Z}}(dt, dz) = \sum_{s \geq 0} I_{\Delta\tilde{Z}_s \neq 0}(s)\delta_{(s,\Delta\tilde{Z}_s)}(dt, dz) = \sum_{s\geq 0} I_{\Delta Z_s \neq 0}(s)\delta_{(s,\Xi_{G_s}(\Delta Z_s))}(dt, dz).$$

If we identify, with a slight abuse of notation, the push-forward of the map $(s, z) \to (s, \Xi_{G_s}(z))$ with the push-forward of the map $(s, z) \to \Xi_{G_s}(z)$, we have

$$\delta_{(s,\Xi_{G_s}(\Delta Z_s))}(du, dz) = \Xi_{G_s,*}(\delta_{(s,\Delta Z_s)})(du, dz),$$

and so

$$\mu^{\tilde{Z}} = \Xi_{G_t,*}(\mu^Z).$$

If we consider a function $h : N \to \mathbb{R}$ which is identically zero in a neighbourhood of 1_N, by definition of push-forward of a measure we have

$$\int_0^t \int_N h(z)\Xi_{G_s,*}(\mu^Z - \nu)(ds, dz) = \int_0^t \int_N h(\Xi_{G_s}(z))(\mu^Z - \nu)(ds, dz).$$

Furthermore $\int_0^t \int_N h(\Xi_{G_s}(z))(\mu^Z - \nu)(ds, dz)$ is a martingale, since $h(\Xi_{G_s}(z))$ is a predictable function and ν is the predictable projection of the random measure μ^Z. Since $\mu^{\tilde{Z}} = \Xi_{G_t,*}(\mu^Z)$ we have that $\Xi_{G_t,*}(\nu)$ is the predictable projection of the measure $\mu^{\tilde{Z}}$ and $\tilde{\nu} = \Xi_{G_t,*}(\nu)$.

For the formulas of $\tilde{A}$ and $\tilde{b}$ we use the definition of solution to a canonical SDE, Lemma 4.5 and the properties of Υ_g and O_g. We make the proof only for $\tilde{A}$, the proof for $\tilde{b}$ being entirely similar. Fixing an immersion $K : N \to \mathbb{R}^{k_N}$, by definition and Lemma 4.5, for any functions $f, g \in C^\infty(N)$, the properties of Υ_g ensure that

$$\begin{aligned}
[f(\tilde{Z}), g(\tilde{Z})]^c_t &= \int_0^t \partial_{k'^i}(\tilde{f} \circ \overline{\Xi})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})\partial_{k'^j}(\tilde{g} \circ \overline{\Xi})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})d[k^i(Z), k^j(Z)]^c_s \\
&= \int_0^t Y^{z'}_\alpha(\tilde{f} \circ \overline{\Xi})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-})Y^{z'}_\beta(\tilde{g} \circ \overline{\Xi})(\tilde{Z}_{s_-}, Z_{s_-}, Z_{s_-}) \\
&\quad \tilde{P}^\alpha_i(Z_{s_-})\tilde{P}^\beta_j(Z_{s_-})d[k^i(Z), k^j(Z)]^c_s = \\
&= \int_0^t Y_\gamma(f)(\tilde{Z}_{s_-})Y_\delta(g)(\tilde{Z})\Upsilon^\gamma_{G_s,\alpha}\Upsilon^\delta_{G_s,\beta}\tilde{P}^\alpha_i(Z_{s_-})\tilde{P}^\beta_j(Z_{s_-})d[k^i(Z), k^j(Z)]^c_s,
\end{aligned}$$

where $\tilde{g}, \tilde{f}$ are two extensions of f, g on $\mathbb{R}^{k_N}$, and $\tilde{P}$ is a pseudoinverse matrix of $P = (Y_\alpha(k^i))$ (see Theorem 4.16). By definition of characteristics we have that

$$\begin{aligned}
&[k^i(Z), k^j(Z)]^c_t - \textstyle\int_0^t Y_\alpha(k^i)(Z_{s_-})Y_\beta(k^j)(Z_{s_-})dA^{\alpha\beta}_s = \\
&= [k^i(Z), k^j(Z)]^c_t - \textstyle\int_0^t P^i_\alpha(Z_{s_-})P^j_\beta(Z_{s_-})dA^{\alpha\beta}_s
\end{aligned}$$

is a local martingale. This means that

$$[f(\tilde{Z}), g(\tilde{Z})]^c_t - \int_0^t Y_\gamma(f)(\tilde{Z}_{s_-})Y_\delta(g)(\tilde{Z}_{s_-})\Upsilon^\gamma_{G_s,\alpha}\Upsilon^\delta_{G_s,\beta}dA^{\alpha\beta}_s$$

is a local martingale. ■

Proof of Theorem 4.18. We cannot directly use Lemma 4.19 to compare (b, A, ν) with $(\tilde{b}, \tilde{A}, \tilde{\nu})$, since Z and $\tilde{Z}$, where $d\tilde{Z}_t = \Xi_{G_t}(dZ_t)$, are two different processes being two different functions from $\Omega^c \times \mathbb{R}_+$ into N. Indeed $Z_t(\omega) = \pi_A(\omega)(t)$, while $\tilde{Z}_t(\omega) = \pi_A(\Lambda(\omega))(t)$.
Since Λ' is the $\mathbb{P}'$ inverse of Λ, $\tilde{Z}(\Lambda'(\omega))$ is exactly the same process as Z (as functions defined on Ω^c). If $\tilde{Z}(\Lambda')$ and Z have the same law, and since both the filtration $\hat{\mathcal{F}}^c_t$ and $\tilde{\mathcal{F}}^c_t$ are canonical, they necessarily have the same characteristics up to a $\mathbb{P}'$ null set and therefore $b(\omega) = \tilde{b}(\Lambda'(\omega))$, $A(\omega) = \tilde{A}(\Lambda'(\omega))$ and $\nu(\omega) = \tilde{\nu}(\Lambda'(\omega))$. If $\tilde{b}(\Lambda')$, $\tilde{A}(\Lambda'(\omega))$ and $\tilde{\nu}(\Lambda'(\omega))$ are $\pi_A^{-1}(\mathcal{F}^A_t)$ measurable (usually they are only $\hat{\mathcal{F}}^c_t$ measurable) they are then equal to b, a and ν up to a null set with respect to $\pi_{A*}(\mathbb{P}) = \pi_{A*}(\mathbb{P}')$.
Obviously if (b, A, ν) uniquely identifies in Ω_A the law of Z, the condition stated in the theorem is also sufficient. ■

4.2.4 Gauge symmetries of Lévy processes

Generalizing [108] we introduce the following definition.

Definition 4.20 *A càdlàg semimartingale Z on a Lie group N is called an* independent increments process *if its characteristics (b, A, ν) are deterministic.*
The process Z is a Lévy process if $b_t = b_0 t, A_t = A_0 t, \nu(dt, dx) = \nu_0(dx)dt$ for some $b_0 \in \mathbb{R}^n$, A_0 $n \times n$ symmetric positive semidefinite matrix and some σ-finite measure ν_0 on N such that $\int_N (h^\alpha(z))^2\nu_0(dz) < +\infty$ and $\int_N f(z)\nu_0(dz) < +\infty$ for any smooth and bounded function $f \in C^\infty(N)$ which is identically zero in a neighbourhood of 1_N.

It is evident that the definition of independent increments process depends on the filtration $\mathcal{F}_t$ used for defining the characteristics (b, A, ν). Furthermore since (b, A, ν) are deterministic the filtration $\mathcal{F}_t$ should always be canonical.

Remark 4.21 *The characteristics of a Lévy process introduced in Definition 4.20 are the same as those discussed in Section 4.1.3. Furthermore if Z is a Lévy process, then Z is also an homogeneous Markov process. Its generator L has the following form on $f \in C^\infty(N)$*

$$L(f)(z) = b_0^\alpha Y_\alpha(f)(z) + \frac{1}{2} A_0^{\alpha\beta} Y_\alpha(Y_\beta(f))(z) + \int_N (f(z' \cdot z^{-1}) - f(z) - h^\alpha(z') Y_\alpha(f)(z)) \nu_0(dz'),$$

for any $z \in N$.

Theorem 4.22 *If a semimartingale Z is an independent increments process such that its law is uniquely determined by its characteristics, then Z admits $\mathcal{G}$ as gauge symmetry group with action Ξ_g if and only if, for any $g \in \mathcal{G}$,*

$$\begin{aligned} b_t^\alpha &= \Upsilon_{g,\beta}^\alpha b_t^\beta + \frac{1}{2} O_{g,\beta\gamma}^\alpha A^{\beta\gamma} + \int_0^t \int_N (h^\alpha(z') - h^\beta(\Xi_{g^{-1}}(z')) \Upsilon_{g,\beta}^\alpha) \nu(ds, dz') &\quad (4.21) \\ A_t^{\alpha\beta} &= \Upsilon_{g,\gamma}^\alpha \Upsilon_{g,\delta}^\beta A_t^{\gamma\delta} &\quad (4.22) \\ \nu &= \Xi_{g*}(\nu). &\quad (4.23) \end{aligned}$$

Proof. Let us consider the constant process $G_t = g_0$ for some $g_0 \in \mathcal{G}$. Since Ξ_{g_0} is a diffeomorphism and since the constant process $G_t = g_0$ is measurable with respect to both the natural filtrations of Z_t and of $\tilde{Z}_t$, it is simple to prove that, if $\tilde{\mathcal{F}}_t^c$ is a generalized natural filtration for Z_t, then it is a generalized natural filtration also for $d\tilde{Z}_t = \Xi_{g_0}(dZ_t)$. This fact implies that $\hat{\mathcal{F}}_t^c$ is a generalized natural filtration for $\omega_A(t)$ with respect to the law $\mathbb{P}'$. For this reason since (b, A, ν) and the process G_t do not depend on ω, (4.21), (4.22) and (4.23) follow from the necessary condition in Theorem 4.16.
Conversely, if equations (4.21), (4.22) and (4.23) hold, they imply equations (4.18), (4.19) and (4.20) to any elementary process G_t. Using standard techniques we can extend (4.18), (4.19) and (4.20) for any locally bounded predictable process G_t.
Since the law of Z is uniquely determined by its characteristics, the thesis follows by the sufficient condition in Theorem 4.18. ■

Remark 4.23 *It is important to recall that the law of an independent increments semimartingale on the Lie group $N = \mathbb{R}^n$ is always uniquely determined by its characteristics (see, e.g., [108], Chapter II, Theorem 4.15 and the corresponding comments in that reference).*

We now propose a general method for explicitly constructing Lévy processes admitting a gauge symmetry group $\mathcal{G}$ with action Ξ_g.
In order to show that our construction is a generalization of the Brownian motion case, we begin with a standard example. Consider $N = \mathbb{R}^n$ and the Lévy process with generator given by

$$L(f)(z) = \sum_{\alpha=1}^n \frac{D}{2} \partial_{z^\alpha z^\alpha}(f)(z) + \int_N (f(z + z') - f(z) - I_{|z'|<1}(z') z^\alpha \partial_{z^\alpha}(f)) F(|z'|) dz',$$

where $D \in \mathbb{R}_+$, $|\cdot|$ is the standard norm of $\mathbb{R}^n$ and $F : \mathbb{R}_+ \to \mathbb{R}_+$ is a measurable locally bounded function such that $\int_1^\infty F(r) r^{n-1} dr < +\infty$ and $\int_0^1 F(r) r^{n+1} < +\infty$. When $B \in O(n)$ we have

$$\Xi_B(z) = B \cdot z.$$

By definition, B respects the standard metric in $\mathbb{R}^n$ and so

$$\Xi_{B*}(F(|z|)dz) = \det(B)F(|B^T \cdot z|)dz = F(|z|)dz.$$

Furthermore, since $\Upsilon^\alpha_{B,\beta} = B^\alpha_\beta$ we have

$$\begin{aligned}&\int_N (z^\alpha I_{|z|<1}(z) - \Xi^\beta_{B^{-1}}(z)\Upsilon^\alpha_{B,\beta} I_{|\Xi_{B^{-1}}(\cdot)|<1}(z))F(|z|)dz = \\ &= \int_N (z^\alpha I_{|z|<1}(z) - (B^{-1})^\beta_\gamma B^\alpha_\beta I_{|z|<1}(z) z^\gamma)F(|z|)dz = 0.\end{aligned}$$

Hence, by Theorem 4.18, Z admits $O(n)$ as a gauge symmetry group with action Ξ_B.
In this case the equation $dZ'_t = \Xi_{B_t}(dZ_t)$ is simply

$$Z'^\alpha_t = \int_0^t B^\alpha_{\beta,s} dZ^\beta_s.$$

This example can be easily generalized to the case of a group $\mathcal{G} \subset O(n)$ which is a strict subgroup of $O(n)$ with a faithful action. Indeed in this case we can consider the polynomial $k_1(z), ..., k_l(z)$ as $\mathcal{G}$-invariant with respect to the action Ξ_B, where $B \in \mathcal{G}$. If $G : \mathbb{R}^l \to \mathbb{R}$ is a non-negative smooth function such that $\partial_{y^i}(G) \neq 0$ for $i = 1, ..., l$ and F is a measurable, locally bounded function satisfying the previous conditions, then $\nu_G(dz) = F(|z|)G(k_1(z), ..., k_l(z))dz$ is a Lévy measure strictly invariant with respect to $\mathcal{G}$. So the Lévy process with measure ν_G admits $\mathcal{G}$, but not all $O(n)$, as a gauge symmetry group.

In order to extend the above construction to a general Lie group N, we introduce a special set of Hunt functions. Let $Y_1, ..., Y_n$ be a basis of right-invariant vector fields and consider $a^1, ..., a^n \in \mathbb{R}$. It is possible to define the exponential $\exp(a^\alpha Y_\alpha) \in N$, which is a point in N defined as the evolution at time 1 of 1_N with respect to the vector field $a^\alpha Y_\alpha$. The map $\exp : \mathbb{R}^n \to N$ is a local diffeomorphism, so there exist a neighbourhood U of 1_N and n smooth functions $\hat{h}^1, ..., \hat{h}^n$ such that, for any $z \in U$

$$\exp(\hat{h}^\alpha(z)Y_\alpha) = z. \tag{4.24}$$

From equation (4.24) and the implicit function theorem we deduce that $\hat{h}^\alpha$ are smooth and form a set of Hunt functions.
We introduce a special class of Lie group actions Ξ_g on N. Suppose that Ξ_g is a Lie group action of endomorphisms of N, which means that, for any $z, z' \in N$, $\Xi_g(z \cdot z') = \Xi_g(z) \cdot \Xi_g(z')$. Since the derivative map $T\Xi_g$ is an automorphism of the Lie algebra $\mathfrak{g}$ of right-invariant vector fields, there are some functions $\Upsilon^\alpha_{g,\beta}$ from $\mathcal{G}$ into $\mathbb{R}$ such that

$$T\Xi_g(Y_\alpha) = \Upsilon^\beta_{g,\alpha} Y_\beta. \tag{4.25}$$

We remark that the previous equality holds in all N, and not only at 1_N as happens for general group actions. Moreover, in this case, since equality (4.24) holds in all N, the map O_g associated with Ξ_g is identically equal to 0.

Lemma 4.24 *There exists a small enough neighbourhood U of 1_N such that, for any $y \in U$,*

$$\Upsilon^\beta_{g,\alpha} \hat{h}^\alpha(\Xi_{g^{-1}}(z)) = \hat{h}^\beta(z).$$

Proof. Write

$$f(a^1, ..., a^n, z) = \exp(a^\alpha Y_\alpha)(z).$$

Since $f(a,x)$, where $a \in \mathbb{R}^n$, is the flow at time 1 of $a^\alpha Y_\alpha$, $\Xi_g(f(a,\Xi_{g^{-1}}(z)))$ is the flow at time 1 of $\Xi_{g,*}(a^\alpha Y_\alpha)$. Moreover, the fact that Ξ_g is an automorphism of N ensures that

$$\Xi_{g,*}(a^\alpha Y_\alpha) = a^\alpha \Xi_{g,*}(Y_\alpha) = a^\alpha \Upsilon^\beta_{g,\alpha} Y_\beta$$

which means

$$\Xi_g(f(a,\Xi_{g^{-1}}(z))) = f(a^\alpha \Upsilon^\beta_{g,\alpha}, z).$$

Since $\hat{h}^\alpha$ solve equation (4.24), $\hat{h}^\alpha(\Xi_{g^{-1}}(z))$ solve the equation

$$\Xi_g(f(\hat{h}^\alpha(\Xi_{g^{-1}}(z)), \Xi_{g^{-1}}(z))) = z.$$

Using the properties of f, from the previous equation follows that the $\hat{h}^\alpha(\Xi_{g^{-1}}(z))\Upsilon^\beta_{g,\alpha}$ solve equation (4.24). If we choose the neighbourhood U small enough, by uniqueness of the solutions to equation (4.24), we have $\hat{h}^\beta(z) = \hat{h}^\alpha(\Xi_{g^{-1}}(z))\Upsilon^\beta_{g,\alpha}$. ∎

Suppose that there exists a complete symmetric positive definite matrix $K^{\alpha\beta}$ such that

$$\Upsilon^\gamma_{g,\alpha} K^{\alpha\beta} \Upsilon^\delta_{g,\beta} = K^{\gamma\delta} \tag{4.26}$$

for any $g \in \mathcal{G}$ and define

$$U_R = \{\exp(a^\alpha Y_\alpha) | a^\alpha K_{\alpha\beta} a^\beta < R^2\},$$

where $K_{\alpha\beta}$ is the inverse matrix of $K^{\alpha\beta}$. It is simple to verify that the closure of U_R is a compact set. A consequence of Lemma 4.24 is that, for R small enough and for any $g \in \mathcal{G}$, we have $\Xi_g(U_R) = U_R$.

An automorphism Ξ_g and a right-invariant metric K which satisfy equation (4.26) exist for a large class of Lie groups. Indeed the set of endomorphisms of a Lie group N, which we denote by $Aut(N)$, forms a Lie group itself and we can consider $\mathcal{G}$ as a maximal compact subgroup of $Aut(N)$. Since the representation Υ_g of $\mathcal{G}$ is the representation of a compact subgroup in the Lie algebra $\mathfrak{n}$ of N, there exists a metric K on $\mathfrak{n}$ such that $\mathcal{G}$ is a subgroup of $O(\mathfrak{n})$ with respect to K.

Corollary 4.25 *If $(b_0 t, A_0 t, \nu_0 dt)$ are the characteristics of a semimartingale Z with respect to the Hunt functions $\hat{h}^\alpha$ and $\mathcal{G}$ is a subgroup of $Aut(N)$ with an action satisfying the previous hypothesis, then $\mathcal{G}$ is a gauge symmetry group of Z if and only if*

$$\begin{aligned} b_0^\alpha &= b_0^\beta \Upsilon^\alpha_{g,\beta} \\ A_0^{\alpha\beta} &= \Upsilon^\alpha_{g,\gamma} A_0^{\gamma\delta} \Upsilon^\beta_{g,\delta} \\ \nu_0 &= \Xi_{g*}(\nu_0). \end{aligned}$$

Proof. Since $O^\alpha_{g,\beta\gamma} = 0$ the only thing to prove is that

$$\int_N (\hat{h}^\alpha(z') - \Upsilon^\alpha_{g,\beta} \hat{h}^\beta(\Xi_{g^{-1}}(z'))) \nu_0(dz') = 0.$$

But the last equality follows easily from Lemma 4.24. ∎

Remark 4.26 *Although all Lie groups $\mathcal{G}$ constructed with the previous method are compact, not all gauge symmetry groups of a Lévy process are compact. For example, using Hamiltonian actions on $\mathbb{R}^n$, it is possible to construct Lévy processes with gauge symmetry group $\mathcal{G} = \mathbb{R}^l$.*

Remark 4.27 *The construction proposed here for general Lie groups is equivalent to the one considered in [4] for Lévy processes taking values in the matrix Lie groups.*

4.2.5 Gauge symmetries of non-Markovian processes

In this section we propose a method for the explicit construction of non-Markovian semimartingales admitting gauge symmetries. We remark that the class of semimartingales obtained in this way does not exhaust all the possible non-Markovian semimartingales with gauge symmetries.
The main idea of our construction consists in generalizing the following fact: given three independent Brownian motions W^0, W^1, W^2, the non-Markovian process on $\mathbb{R}^2$ defined by the equations

$$\begin{aligned} \tilde{W}^1_t &= \int_0^t G(W^0_{[0,s]}, s) dW^1_s \\ \tilde{W}^2_t &= \int_0^t G(W^0_{[0,s]}, s) dW^2_s, \end{aligned}$$

where G is a continuous predictable functional on $C^0(\mathbb{R}_+)$, admits the gauge symmetry group $SO(2)$ of two dimensional rotations. Indeed, if $B_s = (B^\alpha_{\beta,s}), s \geq 0$, is a predictable process taking values in $SO(2)$, the process $(\hat{W}^1, \hat{W}^2)$ defined by $\hat{W}^\alpha_t = \int_0^t B^\alpha_{\beta,s} d\tilde{W}^\beta_s, t \geq 0$, has the same law as $(\tilde{W}^1, \tilde{W}^2)$. In fact, if we put $W'^\alpha_t = \int_0^t B^\alpha_{\beta,s} dW^\beta_s$, it is easy to prove that

$$\hat{W}^\alpha_t = \int_0^t G(W^0_{[0,s]}, s) dW'^\alpha_s.$$

Since $[W'^\alpha, W^0]_t = 0$, and since W'^α is a Brownian motion, W^0, W'^1, W'^2 are all independent Brownian motions. Since $\hat{W}^1, \hat{W}^2$ are the integrals with respect to two independent Brownian motions of a function of a third independent Brownian motion W^0, we know that $\hat{W}^1$ and $\hat{W}^2$ have the same law as $\tilde{W}^1, \tilde{W}^2$.

Working in a more general setting, we consider the Lie group $N = N_1 \times N_2$, where N_1, N_2 are two Lie groups and the multiplication on N is defined by

$$(z_1, z_2) \cdot (z'_1, z'_2) = (z_1 \cdot_1 z'_1, z_2 \cdot_2 z_2),$$

where $\cdot_1, \cdot_2$ denote the multiplication on N_1, N_2, respectively. Moreover, we introduce the space $\Omega_A = \Omega^1_A \times \Omega^2_A$, where $\Omega^i_A = \mathcal{D}_{1_{N_i}}([0, +\infty), N_i)$, and we denote by ω^1_A, ω^2_A the elements of Ω^1_A, Ω^2_A, respectively.

Theorem 4.28 *Consider $\Xi_g = (\Xi^1_g, id_{N_2})$ and suppose that the characteristics of a semimartingale Z in N depend only on ω^2_A. If the semimartingale Z admits the Lie group $\mathcal{G}$ with action Ξ_g as a gauge symmetry group then, for any $g \in \mathcal{G}$,*

$$\begin{aligned} b^\alpha_t(\omega^2_A) &= \Upsilon^\alpha_{g,\beta} b^\beta_t(\omega^2_A) + \tfrac{1}{2} O^\alpha_{g,\beta\gamma} A^{\beta\gamma}(\omega^2_A) + \\ &\quad + \int_0^t \int_N (h^\alpha(z') - h^\beta(\Xi_{g^{-1}}(z')) \Upsilon^\alpha_{g,\beta}) \nu(\omega^2_A, ds, dz') \end{aligned} \tag{4.27}$$

$$A^{\alpha\beta}_t(\omega^2_A) = \Upsilon^\alpha_{g,\gamma} \Upsilon^\beta_{g,\delta} A^{\gamma\delta}_t(\omega^2_A) \tag{4.28}$$

$$\nu(\omega^2_A, dt, dz) = \Xi_{g*}(\nu(\omega^2_A, dt, dz)). \tag{4.29}$$

Moreover, if the triplet (b, A, ν) uniquely determines the law of Z on Ω_A, then equations (4.27), (4.28) and (4.29) provide a sufficient condition too.

Proof. The proof is based on Theorem 4.18 and on the fact that the map Λ' appearing in Theorem 4.18 has here the form

$$\Lambda' = \left(\begin{array}{c} \Lambda'^1_A \\ id_{\Omega^2_A} \\ id_{\Omega_B} \end{array} \right).$$

In particular, for proving the necessity it is enough to consider the constant process $G_t = g_0$ and apply Theorem 4.18.
The proof of the sufficiency of equations (4.27), (4.28) and (4.29) is similar to the proof of Theorem 4.22. ∎

Let us apply Theorem 4.28 to the example described at the beginning of this Section. In this case $(\tilde{W}^1, \tilde{W}^2, W^0)$, as a semimartingale on $\mathbb{R}^3$, has characteristics

$$\begin{aligned} db_t &= 0 \\ dA_t &= \left(\begin{array}{ccc} (G(W^0_{[0,t]}, t))^2 dt & 0 & 0 \\ 0 & (G(W^0_{[0,t]}, t))^2 dt & 0 \\ 0 & 0 & dt \end{array} \right) \\ \nu &= 0, \end{aligned}$$

where the Hunt functions can be chosen arbitrarily.
Here $\Xi^1_B(z) = B \cdot z$, where $B \in SO(2)$ and so $\Upsilon_B = \left(\begin{array}{cc} B & 0 \\ 0 & 0 \end{array} \right)$ and $O_B = 0$. It is easy to prove that $\Upsilon_B \cdot b = 0 = b$, $\Upsilon_B \cdot A \cdot \Upsilon_B^T = A$ and $\Xi_g(\nu) = 0 = \nu$. For a suitable choice of G, for example by choosing G Lipschitz with respect to the natural seminorms of $C^0(\mathbb{R}_+)$, the triplet (b, A, ν) uniquely determines the law of $(\tilde{W}^1, \tilde{W}^2, W^0)$ and, therefore, we can apply Theorem 4.28.

All the results of Section 4.2.4 can be generalized in many ways which still permit to apply Theorem 4.28, obtaining thus other examples of non-Markovian semimartingales with gauge symmetries.

4.3 Time symmetries of semimartingales on Lie groups

In this section we briefly discuss the time symmetries of a semimartingale on a Lie group. After recalling some properties of the absolutely continuous time change, we introduce the definition of time symmetry of a semimartingale, and we prove some results analogous to those holding for gauge symmetries.
Finally we study time symmetries of Lévy processes, constructing some explicit examples of Lévy processes with non-trivial time symmetry. Our construction mainly follows [117, 118].

4.3.1 Time symmetries of semimartingales

Given a positive adapted stochastic process β such that, for any $\omega \in \Omega$, the function $\beta(\omega) : t \mapsto \beta_t(\omega)$ is absolutely continuous with strictly positive locally bounded derivative, we define

$$\alpha_t = \inf\{s | \beta_s > t\},$$

where, as usual, by convention $\inf(\mathbb{R}_+) = +\infty$. The process α is an adapted process such that

$$\beta_{\alpha_t} = \alpha_{\beta_t} = t.$$

If X is a stochastic process adapted to the filtration $\mathcal{F}_t$, we denote by $H_\beta(X)$ the stochastic process adapted to the filtration $\mathcal{F}'_t = \mathcal{F}_{\alpha_t}$ such that

$$H_\beta(X)_t = X_{\alpha_t}.$$

Since, by assumption, β_t is absolutely continuous and strictly increasing, then also α_t is absolutely continuous and strictly increasing. Furthermore, denoting by α'_t respectively β'_t the time derivative of α_t respectively β_t, we have

$$\alpha'_t = \frac{1}{\beta'_{\alpha_t}}.$$

If μ is a random measure on N adapted to the filtration $\mathcal{F}_t$, we can introduce a time changed random measure $H_\beta(\mu)$ adapted to the filtration $\mathcal{F}'_t$ such that, for any Borel set $E \subset N$,

$$H_\beta(\mu)([0,t] \times E) = \mu([0,\alpha_t] \times E).$$

In order to introduce a good concept of symmetry with respect to time transformations, we have to recall some fundamental properties of absolutely continuous random time changes with a locally bounded derivative.

Theorem 4.29 *Let β_t be the process described above and let Z, Z' be two real semimartingales, K_t be a predictable process which is integrable with respect to Z and μ be a random measure. Then*

1. *$H_\beta(Z)$ is a semimartingale,*
2. *if Z is a local $\mathcal{F}_t$-martingale, then $H_\beta(Z)$ is a local $\mathcal{F}'_t$-martingale,*
3. *$H_\beta([Z,Z']) = [H_\beta(Z), H_\beta(Z')]$*
4. *$H_\beta(K)$ is integrable with respect to $H_\beta(Z)$ and $\int_0^{\alpha_t} K_s dZ_s = \int_0^t H_\beta(K)_s dH_\beta(Z)_s$.*
5. *if μ^p is the compensator of μ, then $H_\beta(\mu^p)$ is the compensator of $H_\beta(\mu)$.*

Proof. Since the random time change β is continuous, β is an *adapted change of time* in the meaning of [107](Chapter X, Section b)).
Thank to this remark the proofs of assertions 1, ..., 5 can be found in [107](Chapter X, Sections b) and c)). ∎

Taking into account Theorem 4.29, a quite natural definition of time symmetry seems at first view to be the following: a semimartingale Z has time symmetries if, for any β satisfying the previous hypotheses, Z and $H_\beta(Z)$ have the same law. Unfortunately, using for example standard deterministic time changes, it is possible to prove that the only process satisfying the previous definition is the process almost surely equal to a constant. For this reason we introduce the following, different, definition, which has the advantage of admitting non-trivial examples.

Definition 4.30 *Let Z be a semimartingale on a Lie group N and let $\Gamma_\cdot : N \times \mathbb{R}_+ \to N$ be an $\mathbb{R}_+$ action such that $\Gamma_r(1_N) = 1_N$ for any $r \in \mathbb{R}_+$. We say that Z has a time symmetry with action Γ_r with respect to the filtration $\mathcal{F}_t$ if*

$$dZ'_t = H_\beta(\Gamma_{\beta'_t}(dZ_t)),$$

has the same law of Z for any β_t satisfying the previous hypotheses and such that β'_t is a $\mathcal{F}_t$-predictable locally bounded process in $\mathbb{R}_+$.

Remark 4.31 *The request that β'_t is a locally bounded process in $\mathbb{R}_+$ ensures that $\beta'_t(\omega) \geq c(\omega) > 0$ for some $c(\omega) \in \mathbb{R}_+$ and for t in compact subsets of $\mathbb{R}_+$.*

Lemma 4.32 *If (X, Z) is a solution to the SDE $\overline{\Psi}_{K_t}$ and β is an absolutely continuous process such that β'_t is locally bounded in $\mathbb{R}_+$, then $(H_\beta(X), H_\beta(Z))$ is a solution to the SDE $\overline{\Psi}_{H_\beta(K)_t}$.*

Proof. The thesis is a simple consequence of Definition 4.1 and Theorem 4.29, point 4. ∎

We now prove the analogue of Proposition 4.13 in the case of time symmetries.

Proposition 4.33 *Given two Lie groups N and $\tilde{N}$, let Z be a semimartingale on N with the time symmetry Γ_r and let $\Theta : N \to \tilde{N}$ be a diffeomorphism such that $\Theta(1_N) = 1_{\tilde{N}}$. Then the process $d\tilde{Z}_t = \Theta(dZ_t)$ is a semimartingale with the time symmetry $\Theta \circ \Gamma_r \circ \Theta^{-1}$.*

Proof. From Corollary 4.10 we have that $dZ_t = \Theta^{-1}(d\tilde{Z}_t)$ and, since Γ_r is a time symmetry for Z, if $dZ'_t = \Gamma_{\beta'_t}(\Theta^{-1}(d\tilde{Z}_t))$, then $H_\beta(Z')$ has the same law as Z. Hence, by the uniqueness of the solution to a geometrical SDE, $d\tilde{Z}'_t = \Theta(dH_\beta(Z')_t)$ has the same law as $\tilde{Z}$. On the other hand from Lemma 4.32, we have $H_\beta(\Theta(dZ'_t)) = \Theta(dH_\beta(Z)_t)$. ∎

Lemma 4.34 *Let Z be a semimartingale with characteristics (b, A, ν). Then $H_\beta(Z)$ has characteristics $(H_\beta(b), H_\beta(A), H_\beta(\nu))$.*

Proof. First we recall that ν is the compensator of the random measure μ^Z defined by

$$\mu^Z(\omega, dt, dz) = \sum_{s \geq 0} I_{\Delta Z_s \neq 1_N} \delta_{(s, \Delta Z_s(\omega))}(dt, dz)$$

(see the proof of Theorem 4.16). This means that the random measure associated with $\bar{Z} = H_\beta(Z)$ is

$$\begin{aligned} \mu^{\bar{Z}}(\omega, dt, dz) &= \sum_{s \geq 0} I_{\Delta Z_{\alpha_s} \neq 1_N} \delta_{(s, \Delta Z_{\alpha_s}(\omega))}(dt, dz) \\ &= H_\beta(\mu^Z). \end{aligned}$$

Since, by Theorem 4.29, $H_\beta(\mu^Z)$ has $H_\beta(\nu)$ as compensator, the characteristic measure of $H_\beta(Z)$ is $H_\beta(\nu)$.
The proof for b and A is similar and follows from the definition of characteristics and points 2, 3 and 4 of Theorem 4.29. ∎

We shall now discuss a version of Theorem 4.18 for time symmetries, considering Ω_B as the set of locally bounded functions from $\mathbb{R}_+$ into itself and the process β_t defined by

$$\beta_t = \int_0^t \omega_B(s) ds.$$

The map $\Lambda : \Omega^c \to \Omega^c$ (see Section 4.2.3) is the composition of two functions: the map Λ_{Γ_r} induced by the solution to the SDE $\Gamma_{\beta'_t}(dZ_t)$, as in Section 4.2.3, and the map H_β, induced by the time transformation, from Ω^c into itself, defined by

$$H_\beta(\omega_A(t), \omega_B(t)) = (\omega_A(\alpha_t), \omega_B(\alpha_t)).$$

Since both Λ_{Γ_r} and H_β are invertible, also Λ is invertible and we denote by Λ' its inverse.
In the same way we introduced the linear maps Υ_g and O_g for the $\mathcal{G}$-action, in the present case we introduce two maps $\gamma_r : \mathfrak{n} \to \mathfrak{n}$ and $Q_r : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$ such that, for any smooth function $f : N \to N$ and for any right invariant vector fields Y, Y',

$$\begin{aligned} Y^z(f(\Gamma_r(z)\cdot\tilde{z}))|_{z=1_N} &= \gamma_r(Y)(f)(\tilde{z}) \\ Y'^z(Y^z(f(\Gamma_r(z)\cdot\tilde{z})))|_{z=1_N} &= \gamma_r(Y')(\gamma_r(Y)(f))(\tilde{z}) + Q_r(Y,Y')(f)(\tilde{z}). \end{aligned}$$

If $Y_1, ..., Y_n$ is a basis of right-invariant vector fields, we denote by $\gamma^\alpha_{r,\beta}, Q^\alpha_{r,\beta\gamma}$ the components of the maps γ_r, Q_r with respect to the basis $Y_1, ..., Y_n$.

Theorem 4.35 *Let Z be a semimartingale on a Lie group N with characteristics $(b(\omega_A), A(\omega_A), \nu(\omega_A))$. If Z has a time symmetry with action Γ_r then, for any probability measure on Ω^c such that $\tilde{\mathcal{F}}_t$ is a generalized natural filtration with respect to both Z_t and $d\tilde{Z}_t = \Xi_{G_t}(dZ_t)$, we have that*

$$\begin{aligned} db^\alpha_t(\omega) &= \gamma^\alpha_{H_\beta(\beta'_t(\omega_B)),\beta} dH_\beta(b)^\beta(\Lambda'(\omega)) + \frac{1}{2} Q^\alpha_{H_\beta(\beta')_t(\omega_B),\beta\gamma} dH_\beta(A^{\beta\gamma})_t(\Lambda'(\omega))) + \\ &\quad + \left(\int_N (h^\alpha(z') - h^\beta(\Gamma_{r^{-1}}(z'))\gamma^\alpha_{\beta'_t(\omega_B),\beta}) H_\beta(\nu(\Lambda'(\omega), dt, dz'))\right) \\ dA^{\alpha\beta}_t(\omega) &= \gamma^\alpha_{H_\beta(\beta')_t(\omega_B),\gamma} \gamma^\beta_{H_\beta(\beta')_t(\omega_B),\delta} dH_\beta(A)^{\gamma\delta}_t(\Lambda'(\omega)) \\ \nu(\omega, dt, dz) &= \Gamma_{H_\beta(\beta')_t(\omega_B)*}(H_\beta(\nu(\Lambda'(\omega), dt, dz))), \end{aligned}$$

up to a $\mathbb{P}' = \Lambda_*(\mathbb{P})$ null set. Furthermore, if $\tilde{b}, \tilde{A}, \tilde{\nu}$ are $\pi_A^{-1}(\mathcal{F}^A)$ measurable, the previous equalities hold with respect to null sets of the law of Z.
Finally, if the triplet (b, A, ν) uniquely determines the law of Z, the previous conditions are also sufficient for the existence of a time symmetry.

Proof. The proof is completely similar to the proof of Theorem 4.18, using Lemma 4.34 in addition to Lemma 4.19. ∎

4.3.2 Lévy processes with time symmetries

In this section we restrict our attention to Lévy processes on N, proving some general results about Lévy processes with time symmetries and providing explicit examples.

Theorem 4.36 *If Z is a Lévy process with characteristic triplet $(b_0 t, A_0 t, \nu_0 dt)$, which uniquely determines the law of Z, then Z admits a time symmetry with action Γ_r if and only if, for any fixed $r \in \mathbb{R}_+$,*

$$\begin{aligned} b^\alpha_0 &= \frac{1}{r}\left(\gamma^\alpha_{r,\beta} b^\beta_0 + Q^\alpha_{r,\beta\gamma} A^{\beta\gamma}_0\right) + \\ &\quad + \frac{1}{r}\int_N (h^\alpha(z') - h^\beta(\Gamma_{r^{-1}}(z))\gamma^\alpha_{r,\beta})\nu_0(dz') \\ A^{\alpha\beta}_0 &= \frac{1}{r}\gamma^\alpha_{r,\gamma}\gamma^\beta_{r,\delta} A^{\gamma\delta}_0 \\ \nu_0(dz) &= \frac{1}{r}\Gamma_{r*}(\nu_0(dz)). \end{aligned}$$

Proof. The proof is similar to the one of Theorem 4.22, where Theorem 4.18 is replaced by Theorem 4.35. ∎

As in the case of gauge symmetries, also in the case of time symmetries the most difficult task is the construction of suitable Hunt functions satisfying the relations in Theorem 4.36. For this reason we start by considering stable processes on nilpotent Lie groups. In the case where $N = \mathbb{R}^n$, α-stable processes are well known since their generator is the fractional Laplacian, and they can be obtained by a subordination from a Brownian motion (see, e.g., [4, 9]).
The homogeneous α-stable processes are Lévy processes in $\mathbb{R}^n$ depending on a parameter $\alpha \in (0,2]$. If the parameter $\alpha = 2$, then Z is an n dimensional Brownian motion with generator

$$L_2 = \frac{1}{2}\sum_{\beta=1}^{n} \partial_{z^\beta z^\beta}.$$

For $\alpha \in (0,2)$ Z is a pure jump Lévy process with Lévy measure

$$\nu_\alpha(dz) = \frac{1}{|z|^{n+\frac{\alpha}{2}}} dz,$$

where $|\cdot|$ is the standard norm of $\mathbb{R}^n$ and dz is the Lebesgue measure.
The generator L_α of an α-stable process is

$$L_\alpha(f)(z) = \int_{\mathbb{R}^n} \left(f(z+z') - f(z) - I_{|z'|<1}(z')\left(z'^\beta \partial_{z^\beta}(f)(z)\right)\right)\nu_\alpha(dz').$$

Given $B \in O(n)$, let Ξ_B be the standard action of B on $\mathbb{R}^n$. Since, by definition, Ξ_B preserves the standard metric on $\mathbb{R}^n$, Corollary 4.25 implies that $O(n)$ is a gauge symmetry of Z with $\alpha = 2$. Using Corollary 4.25 we obtain the same result for $\alpha \in (0,2)$, with $\Xi^*_B(\nu) = \nu$.
Furthermore, the $\mathbb{R}_+$ action

$$\Gamma^\alpha_r(z) = r^{\frac{1}{\alpha}} z,$$

is a time symmetry for Z. For the Brownian motion case, Theorem 4.35 can be applied directly. For $\alpha \in (0,2)$ it is enough to observe that the space homogeneity of $\nu(z)$ ensures that

$$\int_{\mathbb{R}^n} (I_B(z') - I_{\Gamma_{1/r}(B)}) z'^\alpha \nu(dz') = 0.$$

Moreover, it is easy to see that $Q^\alpha_{\beta\gamma} = 0$ and $\Gamma^\alpha_{r*}(\nu(z)) = r\nu$. Hence, as a consequence of Theorem 4.35, the homogeneous α-stable processes have time symmetry with respect to the action Γ^α_r.

In the following we generalize this construction to some nilpotent group which admits dilations. The presence of dilations is essential to construct Lévy measures satisfying the hypotheses of Theorem 4.35. Although the construction proposed is well known and can be found in [117, 118], for the convenience of the reader, in the following we summarize the main steps.
Given a simply connected nilpotent group N and its Lie algebra $\mathfrak{n}$, the exponential map $\exp : \mathfrak{n} \to N$ is a diffeomorphism. Let $\Gamma_r : N \to N$ be a subset of automorphisms of N such that

$$\Gamma_r \circ \Gamma_s = \Gamma_{rs},$$

and $\Gamma_1 = Id_N$. We say that Γ_r is a dilation on N if, for any $n \in N$, $\Gamma_r(n) \to 1_N$ uniformly on compact sets as $r \to 0$.

Remark 4.37 *It is important to note that not all Lie groups admit a dilation. Indeed a necessary condition for N to admit a dilation is that N is simply connected and nilpotent (this condition is only necessary, but not sufficient, see, e.g., [67]).*

Using the properties of composition of Γ_r, we can prove that there exists a linear transformation S of $\mathfrak{n}$ such that

$$\Gamma_r = \exp(\log(r)S).$$

Moreover, S is a derivation of $\mathfrak{n}$, which means

$$S([Y_1, Y_2]) = [S(Y_1), Y_2] + [Y_1, S(Y_2)]$$

and the linear transformation S decomposes in a natural way the Lie algebra $\mathfrak{n}$. Indeed, let g be the minimal polynomial of S and factorize $g = g_1^{n_1}...g_p^{n_p}$, where $g_1, ..., g_p$ are monic irreducible factors of g and n_j are positive integers. If we write $\mathfrak{n}_j = \ker(g_j(S)^{n_j})$, it is simple to prove that $\mathfrak{n}_j$ are invariant subspaces for S and $\mathfrak{n} = \bigoplus_{j=1}^p \mathfrak{n}_j$. Let $\kappa_j = \alpha_j \pm i\beta_j$ (where $\alpha_j, \beta_j \in \mathbb{R}$), be the eigenvalue associated with the space $\mathfrak{n}_j$ and put

$$\begin{aligned}
I &= \{1 \leq j \leq p | \alpha_j = \frac{1}{2}\} \\
J &= \{1 \leq j \leq p | \frac{1}{2} < \alpha_j\} \\
I_1 &= \{1 \leq j \leq p | \alpha_j = 1\} \\
J_1 &= \{1 \leq j \leq p | \frac{1}{2} < \alpha_j < 1\}.
\end{aligned}$$

If $K \subset \{1, ..., p\}$, we write $\mathfrak{n}_K = \bigoplus_{j \in K} \mathfrak{n}_j$. We denote by $P_{\mathfrak{n}_K}$ the projection onto the space $\mathfrak{n}_K$ given by the decomposition of $\mathfrak{n}$ into the subspaces $\mathfrak{n}_j$. If 1 is not eigenvalue of S, then $S - I$ is invertible. If 1 is eigenvalues of S we can suppose that $\kappa_1 = 1$, and we can decompose the space $\mathfrak{n}_1$ into two subspaces $\tilde{\mathfrak{n}}_1 = \{(S - I)(Y) | Y \in \mathfrak{n}_1$ and $\hat{\mathfrak{n}}_1 = \{Y \in \mathfrak{n}_1 | S(Y) = Y\}$. We can define a pseudo-inverse $(S - I)^{-1}$ of $(S - I)$ such that, fixing $K_1, ..., K_m \in \mathfrak{n}_1 \backslash \hat{\mathfrak{n}}_1$ linearly independent such that $\operatorname{span}\{(S - I)(K_1), ..., (S - I)(K_m)\} = \tilde{\mathfrak{n}}_1$ and putting $V = \bigoplus_{j \neq 1} \mathfrak{n}_j$, we have

$$(S - I)^{-1} \circ (Q - I) = P_{V \oplus \operatorname{span}\{K_1, ..., K_m\}}.$$

Choose on $\mathfrak{n}$ a metric $\langle \cdot, \cdot \rangle$ with norm $|\cdot|$ and define

$$K = \{Y \in \mathfrak{n} | |Y| = 1, |r^S(Y)| > 1 \text{ for any } r > 1\}.$$

If $Y \in \mathfrak{n} \backslash \{0\}$, there exist an unique $\theta \in S$ and an unique $r \in \mathbb{R}_+$ such that $r^Q(\theta) = Y$. The relation described above defines two smooth functions $\theta : \mathfrak{n} \backslash \{0\} \to S$ and $r : \mathfrak{n} \backslash \{0\} \to \mathbb{R}_+$.

Theorem 4.38 *A Lévy process Z on a nilpotent Lie group N with dilation Γ_r has the time symmetry with respect to the action Γ_r if and only if, denoting by $(A_0 t, b_0 t, \nu_0 dt)$ the characteristics of Z with respect to the Hunt functions $h^\alpha(z) = \frac{\log^\alpha(z)}{1 + |z^\alpha Y_\alpha|^2}$ and writing $M = \log_*(\nu_0)$ where* $\log = \exp^{-1} : N \to \mathfrak{n}$ *and* $\log^\alpha$ *are the components of* $\log$ *with respect to the basis $Y_1, ..., Y_n$ of* $\mathfrak{n}$, *the following conditions hold*

1. *$P_{\mathfrak{n}_I} \cdot A \cdot P_{\mathfrak{n}_I}^T = A$ where $P_{\mathfrak{n}_I}^T$ is the transpose of $P_{\mathfrak{n}_I}$,*
2. *the support of the measure M is contained in the subspace $\mathfrak{n}_J$ and*

$$dM(Y) = \frac{d\lambda(\theta(Y)) d(r(Y))}{(r(Y))^2},$$

where λ is a measure on the set K,

3. *if* $\kappa_1 \neq 1$ *then* $b^\alpha Y_\alpha = B_1 = \int_{\mathfrak{n}} \frac{\langle S(Y), Y \rangle}{(1+|Y|^2)^2} (S-I)^{-1}(Y) dM(Y)$ *otherwise* $b^i Y_i - B_1 \in \hat{\mathfrak{n}}_1$,
4. *if* $\kappa_1 = 1$
$$\int_{\mathfrak{n}} \frac{\langle S(Y), Y \rangle}{(1+|Y|^2)^2} P_{\mathfrak{n}_1}(Y) dM(Y) \in \tilde{\mathfrak{n}}_1,$$

Proof. Thank to Theorem 4.36, the statements of this theorem are equivalent to the corresponding statements on the stable processes in [117, 118]. ■

It is important to note the big difference between Theorem 4.38 and the construction of Lévy processes with gauge symmetries proposed in Section 4.2.4. In fact, if we have a compact Lie group $\mathcal{G}$ of automorphisms on a Lie group N, we can construct several Lévy processes on N admitting $\mathcal{G}$ as group of gauge symmetries. On the other hand, it is not true that any dilation Γ_r on a nilpotent Lie group N gives rise to Lévy processes with the time symmetry with respect to the action Γ_r. Indeed in this case the spectral decomposition of the linear operator S associated with Γ_r plays an important role.
Furthermore, in the case of gauge symmetries, it is possible to construct Lévy processes on N with a gauge symmetry and continuous and discontinuous parts can be non trivial. On the contrary, in the case of time symmetries this is not possible. Indeed the space $\mathfrak{n}_J$ (where the jumps are supported) and the space $\mathfrak{n}_I$ (where the continuous martingale part is supported) are complementary. The reason is that the behaviour of the transformation Γ_r as $r \to 0$ is essential for the characterization of the kind of Lévy process with the time symmetry Γ_r.
This property of time symmetric Lévy processes seems to keep holding even if we drop the request that Γ_r is an automorphism of N. Indeed, although we are able to construct various Lévy processes with different behaviour at infinity of the measure ν_0, the behaviour of the measure ν_0 at 1_N is similar to the case in which Γ_r is an automorphism of N. This fact gives strong restrictions on the form of Lévy processes admitting time symmetries.

Chapter 5

Symmetries of SDEs driven by a càdlàg semimartingale and applications

In this chapter we introduce the notion of *stochastic transformations* and the related concept of symmetry of an SDE for general càdlàg processes on manifolds and canonical SDEs. After this general discussion we propose an example, inspired by the theory of iterated random maps, which shows in detail how our general theory can be applied to specific cases. Finally we take advantage of the general theory for introducing a new notion of weak symmetries of numerical schemes for Brownian-motion-driven SDE on $\mathbb{R}^m$.

5.1 Symmetries and invariance properties of an SDE with jumps

5.1.1 Stochastic transformations

Let $\mathcal{C}(\mathbb{P}_0)$ (or simply $\mathcal{C}$) be the class of càdlàg semimartingales Z on a Lie group N inducing the same probability measure on $\mathcal{D}([0,T],N)$ (the metric space of càdlàg functions taking values in N). In order to generalize to the semimartingale case the notion of weak solution to an SDE driven by a Brownian motion, we introduce the following definition.

Definition 5.1 *Given a semimartingale X on M and a semimartingale Z on N such that $Z \in \mathcal{C}$, the pair (X,Z) is called a* process of class $\mathcal{C}$ *on M.*
A process (X,Z) of class $\mathcal{C}$ which is a solution to the canonical SDE Ψ is called a solution of class $\mathcal{C}$ *to Ψ.*

We remark that if (X,Z) and (X',Z') are two solutions of class $\mathcal{C}$ and if X_0 and X'_0 have the same law, then also X and X' have the same law. Hereafter we suppose that the filtration $\mathcal{F}_t$, for which X and Z are semimartingales, is a generalized natural filtration for Z. In the usual case considered in the following, where X is a solution to a geometrical SDEs driven by Z, so that X can be chosen adapted with respect to the natural filtration $\mathcal{F}^Z_t$ of Z, the above restriction on $\mathcal{F}_t$ is not relevant.

In this section we define a set of transformations which transform a process of class $\mathcal{C}$ into a new process of class $\mathcal{C}$. This set of transformations depends on the properties of the processes belonging to the class $\mathcal{C}$.
We start by describing the case of processes in $\mathcal{C}$ admitting a gauge symmetry group $\mathcal{G}$ with action Ξ_g and a time symmetry with action Γ_r. Afterwards, we discuss how to extend our approach to more general situations.

Definition 5.2 *A stochastic transformation from M into M' is a triad (Φ, B, η), where Φ is a diffeomorphism of M into M', $B : M \to \mathcal{G}$ is a smooth function and $\eta : M \to \mathbb{R}_+$ is a positive smooth function. We denote the set of stochastic transformations of M into M' by $S_{\mathcal{G}}(M, M')$.*

A stochastic transformation defines a map between the set of stochastic processes of class $\mathcal{C}$ on M into the set of stochastic processes of class $\mathcal{C}$ on M'. The action of the stochastic transformation $T \in S_{\mathcal{G}}(M, M')$ on the stochastic process (X, Z) is denoted by $(X', Z') = P_T(X, Z)$, and is defined as follows:

$$\begin{aligned} X' &= \Phi\left[H_{\beta^\eta}(X)\right] \\ dZ'_t &= H_{\beta^\eta}\left\{\Xi_{B(X_t)}\left[\Gamma_{\eta(X_t)}(dZ_t)\right]\right\}, \end{aligned}$$

where β^η is the random time change given by

$$\beta^\eta_t = \int_0^t \eta(X_s)ds.$$

The second step is to define an action of a stochastic transformation T on the set of canonical SDEs. This action transforms a canonical SDE Ψ on M into the canonical SDE $\Psi' = E_T(\Psi)$ on M' defined by

$$\Psi'(x, z) = \Phi\left\{\Psi\left[\Phi^{-1}(x), (\Gamma_{(\eta(\Phi^{-1}(x)))^{-1}} \circ \Xi_{(B(\Phi^{-1}(x)))^{-1}})(z)\right]\right\}.$$

Theorem 5.3 *If $T \in S_{\mathcal{G}}(M, M')$ is a stochastic transformation and (X, Z) is a class $\mathcal{C}$ solution to the canonical SDE Ψ, then $P_T(X, Z)$ is a class $\mathcal{C}$ solution to the canonical SDE $E_T(\Psi)$.*

Proof. The fact that $P_T(X, Z)$ is a process of class $\mathcal{C}$ follows from the symmetries of Z, which are the gauge symmetry group $\mathcal{G}$ with action Ξ_g and the time symmetry with action Γ_r.
The fact that, if (X, Z) is a solution to Ψ, then $P_T(X, Z)$ is a solution to $E_T(\Psi)$, follows from Theorem 4.3, Theorem 4.9 and Lemma 4.32. ∎

If $\mathcal{C}$ contains semimartingales which have only the gauge symmetry group $\mathcal{G}$ but without time symmetry, the stochastic transformation T reduces to a pair (Φ, B) and the action on processes and SDEs is the same as in the general case with $\Gamma_r = Id_N$. The same argument can be applied in the case of $\mathcal{C}$ containing semimartingales which possess only the time symmetry property. In the case of semimartingales without neither gauge nor time symmetries, the stochastic transformations can be identified with the diffeomorphisms $\Phi : M \to M'$ and the action on the processes is $P_T(X, Z) = (\Phi(X), Z)$. In the theory of symmetries of SDEs driven by general semimartingales these kinds of transformations, which are of the form $(\Phi, 1_N, 1)$ and do not change the driving process Z, play a special role, thus, generalizing the Brownian motion case of Chapter 1, we call them *strong stochastic transformations*.

5.1.2 The geometry of stochastic transformations

In this section we prove that stochastic transformations have some interesting geometric properties, which are an extension to càdlàg-semimartingales-driven SDEs of the same properties illustrated in Chapter 1 for SDEs driven by Brownian motions.
In order to keep holding some crucial geometric properties, in the following we require an additional property on the maps Ξ_g and Γ_r, i.e. the commutation of the two group actions Ξ_g and Γ_r. In particular we suppose that

$$\Xi_g(\Gamma_r(z)) = \Gamma_r(\Xi_g(z)), \tag{5.1}$$

for any $z \in N$, $g \in \mathcal{G}$ and $r \in \mathbb{R}_+$.

Remark 5.4 *Condition* (5.1) *can be weakened by requiring that the set of diffeomorphisms* $\Theta_{(r,g)} = \Gamma_r \circ \Xi_g$ *is an action of the semidirect product* $\mathbb{R}_+ \rtimes \mathcal{G}$. *This means that there exists a smooth action* $h. : \mathbb{R}_+ \times \mathcal{G} \to \mathcal{G}$ *of* $\mathbb{R}_+$ *on* $\mathcal{G}$ *such that*

$$\Gamma_r \circ \Xi_g = \Xi_{h_r(g)} \circ \Gamma_r.$$

The commutative case is included in this general setting by taking $h_r(g) = g$. *Since we are not able to construct any concrete semimartingale with gauge symmetries and time symmetry admitting non trivial* h_r *and, on the other hand, condition* $h_r(g) = g$ *quite simplifies the exposition, we prefer working with the commutativity assumption.*

We can define a composition between two stochastic transformations $T \in S_{\mathcal{G}}(M, M')$ and $T' \in S_{\mathcal{G}}(M', M'')$, where $T = (\Phi, B, \eta)$ and $T' = (\Phi', B', \eta')$, by

$$T' \circ T = (\Phi' \circ \Phi, (B' \circ \Phi) \cdot B, (\eta' \circ \Phi)\eta). \tag{5.2}$$

The above composition has a nice geometrical interpretation. A stochastic transformation from M into M' can be identified with an isomorphism from the trivial right principal bundle $M \times \mathcal{H}$ into the trivial right principal bundle $M' \times \mathcal{H}$, $\mathcal{H} = \mathcal{G} \times \mathbb{R}_+$, which preserves the principal bundle structure. If we exploit this identification and the natural isomorphism composition we obtain formula (5.2) (see Chapter 1 for the case $\mathcal{G} = SO(n)$).
Composition (5.2), for any $T \in S_{\mathcal{G}}(M, M')$, permits to define an inverse $T^{-1} \in S_{\mathcal{G}}(M', M)$ as follows

$$T^{-1} = (\Phi^{-1}, (B \circ \Phi^{-1})^{-1}, (\eta \circ \Phi^{-1})^{-1}).$$

Hence the set $S_{\mathcal{G}}(M) := S_{\mathcal{G}}(M, M)$ is a group with respect to the composition $\circ$ and the identification of $S_{\mathcal{G}}(M)$ with $\mathrm{Iso}(M \times \mathcal{H}, M \times \mathcal{H})$ (which is a closed subgroup of the group of diffeomorphisms of $M \times \mathcal{H}$) suggests to consider the corresponding Lie algebra $V_{\mathcal{G}}(M)$.
Given a one parameter group $T_a = (\Phi_a, B_a, \eta_a) \in S_{\mathcal{G}}(M)$, there exist a vector field Y on M, a smooth function $C : M \to \mathfrak{g}$ (where $\mathfrak{g}$ is the Lie algebra of $\mathcal{G}$), and a smooth function $\tau : M \to \mathbb{R}$ such that

$$\begin{array}{rcl} Y(x) & := & \partial_a(\Phi_a(x))|_{a=0} \\ C(x) & := & \partial_a(B_a(x))|_{a=0} \\ \tau(x) & := & \partial_a(\eta_a(x))|_{a=0}. \end{array} \tag{5.3}$$

So if Y, C, τ are as above, the one parameter solution (Φ_a, B_a, η_a) to the equations

$$\begin{array}{rcl} \partial_a(\Phi_a(x)) & = & Y(\Phi_a(x)) \\ \partial_a(B_a(x)) & = & R_{B_a(x)*}(C(\Phi_a(x))) \\ \partial_a(\eta_a(x)) & = & \tau(\Phi_a(x))\eta_a(x), \end{array} \tag{5.4}$$

with initial condition $\Phi_0 = id_M$, $B_0 = 1_{\mathcal{G}}$ and $\eta_0 = 1$, is a one parameter group in $S_{\mathcal{G}}(M)$. For this reason we identify the elements of $V_{\mathcal{G}}(M)$ with the triads (Y, C, τ).

Definition 5.5 *A triad $V = (Y, C, \tau) \in V_{\mathcal{G}}(M)$, where Y is a vector field on M, $C : M \to \mathfrak{g}$ and $\tau : M \to \mathbb{R}$ are smooth functions, is an infinitesimal stochastic transformation. If V is of the form $V = (Y, 0, 0)$, we call V a strong infinitesimal stochastic transformation, as the corresponding one-parameter group is a group of strong stochastic transformations.*

Since $V_{\mathcal{G}}(M)$ is a Lie subalgebra of the set of vector fields on $M \times \mathcal{H}$, the standard Lie brackets between vector fields on $M \times \mathcal{H}$ induce some Lie brackets on $V_{\mathcal{G}}(M)$. Indeed, if $V_1 = (Y_1, C_1, \tau_1), V_2 = (Y_2, C_2, \tau_2) \in V_m(M)$ are two infinitesimal stochastic transformations, we have

$$[V_1, V_2] = ([Y_1, Y_2], Y_1(C_2) - Y_2(C_2) - \{C_1, C_2\}, Y_1(\tau_2) - Y_2(\tau_1)), \tag{5.5}$$

where $\{\cdot, \cdot\}$ denotes the usual commutator between elements of $\mathfrak{g}$.
Furthermore, the identification of $T = (\Phi, B, \eta) \in S_{\mathcal{G}}(M, M')$ with $F_T \in \mathrm{Iso}(M \times \mathcal{H}, M' \times \mathcal{H})$ allows us to define the push-forward $T_*(V)$ of $V \in V_{\mathcal{G}}(M)$ as

$$(\Phi_*(Y), (Ad_B(C) + R_{B^{-1}*}(Y(B))) \circ \Phi^{-1}, (\tau + Y(\eta)\eta^{-1}) \circ \Phi^{-1}), \tag{5.6}$$

where Ad denotes the adjoint operation and the symbol $Y(B)$ the push-forward of Y with respect to the map $B : M \to \mathcal{G}$.
Analogously, given $V' \in V_{\mathcal{G}}(M')$, we can consider the pull-back of V' defined as $T^*(V') = (T^{-1})_*(V')$. The following theorem shows that any Lie algebra of general infinitesimal stochastic transformations satisfying a non-degeneracy condition, can be locally transformed, by the action of the push-forward of a suitable stochastic transformation $T \in S_{\mathcal{G}}(M)$, into a Lie algebra of strong infinitesimal stochastic transformations.

Theorem 5.6 *Let $K = \mathrm{span}\{V_1, ..., V_k\}$ be a Lie algebra of $V_{\mathcal{G}}(M)$ and let $x_0 \in M$ be such that $Y_1(x_0), ..., Y_k(x_0)$ are linearly independent (where $V_i = (Y_i, C_i, \tau_i)$). Then there exist an open neighbourhood U of x_0 and a stochastic transformation $T \in S_{\mathcal{G}}(U)$ of the form $T = (Id_U, B, \eta)$ such that $T_*(V_1), ..., T_*(V_k)$ are strong infinitesimal stochastic transformations in $V_{\mathcal{G}}(U)$. Furthermore the smooth functions B, η are solutions to the equations*

$$\begin{aligned} Y_i(B) &= -L_{B*}(C_i) \\ Y_i(\eta) &= -\tau_i \eta, \end{aligned}$$

where L_g is the diffeomorphism given by the left multiplication for $g \in \mathcal{G}$ and $i = 1, ..., k$.

Proof. In the case $\mathcal{G} = SO(n)$ this theorem is exactly Theorem 1.16. Since the proof of Theorem 1.16 does not ever use the specific group properties of $SO(n)$ but only the fact that $SO(n)$ is a Lie group, the proof given in Theorem 1.16 holds also in this case. ■

As shown in Chapter 2, Theorem 5.6 plays a very important role in the applications of the symmetry analysis to concrete SDEs. We give an example of application of Theorem 5.6 in Section 5.2.

5.1.3 Symmetries of an SDE with jumps

Definition 5.7 *A stochastic transformation $T \in S_{\mathcal{G}}(M)$ is a symmetry of the SDE Ψ if, for any process (X, Z) of class $\mathcal{C}$ solution to the SDE $\bar{\Psi}$, also $P_T(X, Z)$ is a solution to the SDE Ψ.*
An infinitesimal stochastic transformation $V \in V_{\mathcal{G}}(M)$ is a symmetry of the SDE Ψ if the one-parameter group of stochastic transformations T_a generated by V is a group of symmetry of the SDE Ψ.

Remark 5.8 *We can give also a local version of Definition 5.7: a stochastic transformation $T \in S_{\mathcal{G}}(U, U')$, where (U, U') are two open sets of M, is a symmetry of Ψ if P_T transforms solutions to $\Psi|_U$ into solutions to $\Psi|_{U'}$.*
In this case it is necessary to stop the solution process X and the driving semimartingale Z with respect to a suitably adapted stopping time.

Theorem 5.9 *A sufficient condition for a stochastic transformation $T \in S_{\mathcal{G}}(M)$ to be a symmetry of the SDE Ψ is that $E_T(\Psi) = \Psi$.*

Proof. This is an easy application of Theorem 5.3. ∎

A natural question arising from previous discussion is whether the condition in Theorem 5.9 is also necessary. Unfortunately, there are counterexamples even for Brownian motion driven SDEs. Indeed in Chapter 1, and in particular in Theorem 1.19, we provide a necessary and sufficient condition such that a weak stochastic transformation (Φ, B, η) is a symmetry of a Brownian-motion-driven SDE (μ, σ). If we rewrite the SDE (μ, σ) as an affine canonical SDE Ψ of the form

$$\Psi(x, z) = x + z^1 \mu(x) + z^{\alpha+1} \sigma_\alpha(x),$$

it is simple to see that the sufficient condition in Theorem 5.9 is more restrictive with respect to the necessary and sufficient condition in Theorem 1.19.
The reason for this fact is that, for a general law of the driving semimartingale in the class $\mathcal{C}$, it is possible to find two different canonical SDEs $\Psi \neq \Psi'$ with the same set of solutions of class $\mathcal{C}$, i.e. any solution (X, Z) of Ψ is also a solution to Ψ' and viceversa.
Exploiting this result it is possible to find sufficient conditions in order to prove the converse of Theorem 5.9.
In the following we say that a semimartingale Z in the class $\mathcal{C}$ and with characteristic triplet (b, A, ν) has *jumps of any size* if the support of ν is all $N \times \mathbb{R}_+$ with positive probability.

Lemma 5.10 *Given a semimartingale Z in the class $\mathcal{C}$ with jumps of any size and such that the stopping time τ of the first jump is almost surely strictly positive, if (X, Z) is a solution to both the SDEs Ψ and Ψ' such that $X_0 = x_0 \in M$ almost surely, then $\Psi(x_0, z) = \Psi'(x_0, z)$ for any $z \in N$.*

Proof. Consider the semimartingale $S^f_t = f(X_t)$, where $f \in C^\infty(M)$ is a bounded smooth function. Given a bounded smooth function $h \in C^\infty(\mathbb{R})$ such that $h(x) = 0$ for x in a neighbourhood of 0, we define the (special) semimartingale

$$H^{h,f}_t = \sum_{0 \leq s \leq t} h(\Delta S^f_s).$$

Since the jumps ΔS^f_t of S^f are exactly $\Delta S^f_t = f(\Psi(X_{t_-}, \Delta Z_t)) - f(X_{t_-})$ or, equivalently, $\Delta S^f_t = f(\Psi'(X_{t_-}, \Delta Z_t)) - f(X_{t_-})$, we have that

$$H^{h,f}_t = \int_{N \times [0,t]} h(f(\Psi(X_{s_-}, z)) - f(X_{s_-}))\mu^Z(ds, dz) = \int_{N \times [0,t]} h(f(\Psi'(X_{s_-}, z)) - f(X_{s_-}))\mu^Z(ds, dz).$$

Since $H^{h,f}$ is a special semimartingale there exists a unique (up to $\mathbb{P}$ null sets) predictable process $R^{h,f}$ of bounded variation such that $H^{h,f}_t - R^{h,f}_t$ is a local martingale. By the definition of characteristic measure ν it is simple to prove that

$$R^{h,f}_t = \int_{N \times [0,t]} h(f(\Psi(X_{s_-}, z)) - f(X_{s_-}))\nu(ds, dz) = \int_{N \times [0,t]} h(f(\Psi'(X_{s_-}, z)) - f(X_{s_-}))\nu(ds, dz).$$

This means that

$$\int_{N\times[0,t]}(h(f(\Psi(X_{s_-},z))-f(X_{s_-}))-h(f(\Psi'(X_{s_-},z))-f(X_{s_-})))\nu(ds,dz)$$

is a semimartingale almost surely equal to 0. Since X_{t_-} is a continuous function for $t\leq\tau$ and the support of ν is all $N\times\mathbb{R}_+$, in a set of positive measure, there exists a set of positive probability such that $h(f(\Psi(X_{t_-},z))-f(X_{t_-}))-h(f(\Psi'(X_{t_-},z))-f(X_{t_-}))=0$ for any $z\in N$. Taking the limit $t\to 0$ we obtain $h(f(\Psi(x_0,z))-f(x_0))=h(f(\Psi'(x_0,z))-f(x_0))$. Since h,f are generic functions, we deduce that $\Psi(x_0,z)=\Psi'(x_0,z)$ for any $z\in N$. ■

Theorem 5.11 *Under the same hypotheses of Lemma 5.10, a stochastic transformation $T\in S_{\mathcal{G}}(M)$ is a symmetry of an SDE Ψ if and only if $E_T(\Psi)=\Psi$.*

Proof. The if part is exactly Theorem 5.9.
Conversely, suppose that T is a symmetry of Ψ and put $\Psi'=E_T(\Psi)$. If X^{x_0} denotes the unique solution to the SDE Ψ driven by the semimartingale Z such that $X^{x_0}=x_0$ almost surely, put $(X',Z')=E_T(X^{x_0},Z)$. By definition of symmetry (X',Z') is a solution to Ψ and, by Theorem 5.3, it is a solution to Ψ'. Since $X'_0=\Phi(x_0)$ almost surely, using Lemma 5.10 we obtain that $\Psi(\Phi(x_0),z)=\Psi'(\Phi(x_0),z)$. Since Φ is a diffeomorphism and $x_0\in M$ is a generic point this concludes the proof. ■

Remark 5.12 *We propose here two possible generalizations of Theorem 5.11*
First we can suppose that Z is a purely discontinuous semimartingale and that $b^\alpha_t=A^{\alpha\beta}_t=0,\forall t\geq 0$ with Hunt functions $h^\alpha=0$. In this case, if the support of ν is $J\times\mathbb{R}_+$ almost surely, the stochastic transformation T is a symmetry of the SDE Ψ if and only if $E_T(\Psi)(x,z)=\Psi(x,z)$ for any $z\in J$. The proof of the necessity of the condition is equal to the one in Lemma 5.10 and Theorem 5.11, instead the proof of the sufficiency part is essentially based on the fact that Z is a pure jump process. This case includes, for example, the Poisson process.
The second generalization covers the important case of continuous semimartingales. An example of the theorem which could be obtained in this case is give by Theorem 1.19 that, in the present language, can be reformulated as follows: T is a symmetry of Ψ driven by a Brownian motion $Z^2,...,Z^m$ and by the time $Z^1_t=t$ if and only if $\partial_{z^\alpha}(\Psi)(x,0)=\partial_{z^\alpha}(E_T(\Psi))(x,0)$ for $\alpha=2,...,m$ and $\partial_{z^1}(\Psi)(x,0)+\frac{1}{2}\sum_{\alpha=2}^m\partial_{z^\alpha z^\alpha}(\Psi)(x,0)=\partial_{z^1}(E_T(\Psi))(x,0)+\frac{1}{2}\sum_{\alpha=2}^m\partial_{z^\alpha z^\alpha}(E_T(\Psi))(x,0)$.

In order to provide an explicit formulation of the *determining equations* for the infinitesimal symmetries of an SDE Ψ, we prove the following proposition.

Proposition 5.13 *A sufficient condition for an infinitesimal stochastic transformation V, generating a one-parameter group T_a of stochastic transformations, to be an infinitesimal symmetry of an SDE Ψ is that*

$$\partial_a(E_{T_a}(\Psi))|_{a=0}=0. \tag{5.7}$$

When the hypotheses of Theorem 5.11 hold, condition (5.7) is also necessary.

Proof. We prove that if equation (5.7) holds, then $E_{T_a}(\Psi)=\Psi$ for any $a\in\mathbb{R}$. Defining $\tilde{\Psi}(a,x,z)=E_{T_a}(\Psi)$, the function $\tilde{\Psi}(a,x,z)$ solves a partial differential equation of the form

$$\partial_a(\tilde{\Psi}(a,x,z))=\mathcal{L}(\tilde{\Psi}(a,x,z))+F(\tilde{\Psi}(a,x,z),x,z), \tag{5.8}$$

where $\mathcal{L}$ is a linear first order scalar differential operator in ∂_x,∂_z and F is a smooth function. It is possible to prove, exploiting standard techniques of characteristics for first order PDEs (see Section

6.2), that equation (5.8) admits a unique local solution as evolution PDE in the time parameter a for any smooth initial value $\tilde{\Psi}(0,x,z)$.
Since $\tilde{\Psi}(0,x,z) = E_{T_0}(\Psi)(x,z) = \Psi(x,z)$ and $\mathcal{L}(\Psi(x,z)) + F(\Psi(x,z),x,z) = \partial_a(E_{T_a}(\Psi))|_{a=0} = 0$, we have that $E_{T_a}(\Psi)(x,z) = \tilde{\Psi}(a,x,z) = \Psi(x,z)$.
The necessity of condition (5.7) under the hypotheses of Theorem 5.11 is trivial since, by Theorem 5.11, we must have $E_{T_a}(\Psi) = \Psi$. ■

In the following we use Proposition 5.13 to rewrite equations (5.7) in any given coordinate systems x^i on M and z^α on N. We denote by $K_1,...,K_r$ the vector fields on N generating the action Ξ_g of $\mathcal{G}$ on N and by H the vector field generating the action Γ_r of $\mathbb{R}_+$ on N. Using these notations, with any infinitesimal stochastic transformation $V = (Y,C,\tau)$ we associate a vector field Y on M, a function τ and r functions $C^1(x),...,C^r(x)$ which correspond to the components of C with respect to the basis $K_1,...,K_r$ of generators of the action Ξ_g. In the chosen coordinate systems on M,N the vector fields Y and $K_1,...,K_r,H$ are of the form

$$Y = Y^i(x)\partial_{x^i} \quad K_\ell = K^\alpha_\ell(z)\partial_{z^\alpha} \quad H = H^\alpha(z)\partial_{z^\alpha}.$$

Therefore, we can rewrite (5.7) as

$$Y^i(\Psi(x,z)) - Y^j(x)\partial_{x^j}(\Psi^i)(x,z) - \tau(x)H^\alpha(z)\partial_{z^\alpha}(\Psi^i)(x,z) - C^\ell(x)K^\alpha_\ell(z)\partial_{z^\alpha}(\Psi^i)(x,z) = 0, \quad (5.9)$$

where $\Psi^i(x,z) = x^i \circ \Psi$ and $i = 1,...,m$. Equations (5.9) are the analogous of determining equations for infinitesimal symmetries in deterministic setting (see, e.g., [147, 164]). It is however important to note some differences with respect to the determining equations of ODEs or also of Brownian-motion-driven SDEs (see Chapter 1 Theorem 1.21). Indeed, in the deterministic case and in the Brownian motion case the determining equations are linear and local overdetermined first order differential equations both in the infinitesimal transformation coefficients and in the equation coefficients (see equations (1.18) and (1.19)). Instead equations (5.9) are *linear non-local* differential equations in the coefficients Y^i,τ,C^ℓ of the infinitesimal transformation V, and they are *non-linear local* differential equations in the coefficient Ψ^i of the SDE.

5.2 An example

In order to give an idea of the generality and of the flexibility of our approach, we propose an example of an application of the previous theory. Further examples of SDEs interesting for mathematical applications will be given in a forthcoming paper.
We consider $M = \mathbb{R}^2$, $N = GL(2) \times \mathbb{R}^2$ (with the natural multiplication), and the canonical SDE

$$\Psi(x, z_{(1)}, z_{(2)}) = z_{(1)} \cdot x + z_{(2)}. \quad (5.10)$$

The SDE associated with Ψ is an affine SDE and its solution (X,Z) satisfies the following stochastic differential relation

$$dX^i_t = X^j_{t-}(Z^{-1}_{(1)})^k_{j,t-}dZ^i_{k,(1),t} + dZ^i_{(2),t}, \quad (5.11)$$

where $Z^{-1}_{(1)}$ is the inverse matrix of $Z_{(1)}$ and $GL(2)$ is naturally embedded in the set of the two by two matrices. If we set

$$\overline{Z}^i_{j,t} = \int_0^t (Z^{-1}_{(1)})^k_{j,s-}dZ^i_{k,(1),s}, \quad (5.12)$$

equation (5.11) becomes the most general equation affine both in the noises $\overline{Z}, Z_{(2)}$ and in the unknown process X. Furthermore, if the noises $Z_{(1)}, Z_{(2)}$ are discrete time semimartingales (i.e.

semimartingales with fixed time jumps at times $n \in \mathbb{N}$) equation (5.11) becomes $X_n = Z^{-1}_{(1),n-1} \cdot Z_{(1),n} \cdot X_{n-1} + Z_{(2),n} - Z_{(2),n-1}$, that is an affine type iterated random map (see Section 4.1.3 and references therein).

The SDE Ψ does not have strong symmetries, in the sense that, for general semimartingales $(Z_{(1)}, Z_{(2)})$, equation (5.11) does not admit symmetries.

For this reason we suppose that the semimartingales $Z_{(1)}, Z_{(2)}$ have the gauge symmetry group $O(2)$ with the natural action

$$\Xi_B(z_{(1)}, z_{(2)}) = (B \cdot z_{(1)} \cdot B^T, B \cdot z_{(2)}), \tag{5.13}$$

where $B \in O(2)$.

In order to use the determining equation (5.9) for calculating the infinitesimal symmetries of the SDE Ψ, we need to explicitly write the infinitesimal generator K of the action Ξ_B on N. In the standard coordinate system of N we have that Ξ_B is generated by

$$\begin{aligned} K &= (-z^2_{1,(1)} - z^1_{2,(1)})\partial_{z^1_{1,(1)}} + (z^1_{1,(1)} - z^2_{2,(1)})\partial_{z^1_{2,(1)}} + (z^1_{1,(1)} - z^2_{2,(1)})\partial_{z^2_{1,(1)}} + \\ &\quad + (z^1_{2,(1)} + z^2_{1,(1)})\partial_{z^2_{2,(1)}} - z^2_{(2)}\partial_{z^1_{(2)}} + z^1_{(2)}\partial_{z^2_{(2)}}. \end{aligned}$$

If we set

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we have that

$$\begin{aligned} K(z_{(1)}) &= R \cdot z_{(1)} + z_{(1)} \cdot R^T \\ K(z_{(2)}) &= R \cdot z_{(2)}, \end{aligned}$$

where the vector field K is applied componentwise to the matrix $z_{(1)}$ and the vector $z_{(2)}$. Using this property of K we can easily prove that

$$V = (Y, C) = \left(-x^2\partial_{x^1} + x^1\partial_{x^2}, 1\right),$$

(where $C = 1$ is the component of the gauge symmetry with respect to the generator K) is a symmetry of the equation Ψ. Indeed, recalling that Y is a linear vector field whose components satisfy the relation

$$Y = R \cdot x,$$

we have that, in this case, the determining equations (5.9) read

$$\begin{aligned} Y \circ \Psi - Y(\Psi) - C(x)K(\Psi) &= R \cdot (z_{(1)} \cdot x + z_{(2)}) - z_{(1)} \cdot (R \cdot x) - K(\Psi) \\ &= R \cdot (z_{(1)} \cdot x + z_{(2)}) + z_{(1)} \cdot R^T \cdot x - (R \cdot z_{(1)} + z_{(1)} \cdot R^T) \cdot x + \\ &\quad -R \cdot z_{(2)} = 0. \end{aligned}$$

Since V satisfies the determining equations (5.9), V is an infinitesimal symmetry of Ψ. The infinitesimal stochastic transformation V generates a one-parameter group of symmetries of Ψ given by

$$T_a = (\Phi_a, B_a) = \left(\begin{pmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{pmatrix} \cdot x, \begin{pmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{pmatrix} \right).$$

In other words if the law of $(Z_{(1)}, Z_{(2)})$ is gauge invariant with respect to rotations then the SDE Ψ is invariant with respect to rotations.

Once we have found an infinitesimal symmetry, we can exploit it to transform the SDE Ψ in an equation of a simpler form as done in Chapter 2 for Brownian-motion-driven SDEs.
The first step consists in looking for a stochastic transformation $T = (\Phi, B)$ such that $T_*(V)$ is a strong symmetry (the existence of the transformation T is guaranteed by Theorem 5.6). In this specific case the transformation T has the following form (for $x = (x^1, x^2) \neq (0,0)$)

$$T = (\Phi(x), B(x)) = \left(\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \begin{pmatrix} \frac{x^1}{\sqrt{(x^1)^2+(x^2)^2}} & \frac{x^2}{\sqrt{(x^1)^2+(x^2)^2}} \\ \frac{-x^2}{\sqrt{(x^1)^2+(x^2)^2}} & \frac{x^1}{\sqrt{(x^1)^2+(x^2)^2}} \end{pmatrix} \right) \tag{5.14}$$

and the SDE $\Psi' = E_T(\Psi)$ becomes for such x

$$\Psi'(x, z_{(1)}, z_{(2)}) = \begin{pmatrix} x^1 & -x^2 \\ x^2 & x^1 \end{pmatrix} \cdot \begin{pmatrix} z^1_{1,(1)} \\ z^2_{1,(1)} \end{pmatrix} + \begin{pmatrix} \frac{x^1}{\sqrt{(x^1)^2+(x^2)^2}} & \frac{-x^2}{\sqrt{(x^1)^2+(x^2)^2}} \\ \frac{x^2}{\sqrt{(x^1)^2+(x^2)^2}} & \frac{x^1}{\sqrt{(x^1)^2+(x^2)^2}} \end{pmatrix} \cdot \begin{pmatrix} z^1_{(2)} \\ z^2_{(2)} \end{pmatrix}.$$

Note that Ψ' does not depend on $z^1_{2,(1)}, z^2_{2,(1)}$, which means that the noise has been reduced by the transformation. The transformation T has an effect similar to the reduction of redundant Brownian motions in Brownian-motion-driven SDE (see [59]). Moreover, if we rewrite the transformed SDE in (pseudo)-polar coordinates

$$\begin{aligned} \rho &= (x^1)^2 + (x^2)^2 \\ \theta &= \arg(x^1, x^2), \end{aligned}$$

where $\arg(a,b)$ is the function giving the measure of the angle between $(0,1)$ and (a,b) in $\mathbb{R}^2$, we find

$$\begin{aligned} \Psi'^{\rho}(\rho,\theta,z) &= (\sqrt{\rho} z^1_{1,(1)} + z^1_{(2)})^2 + (\sqrt{\rho} z^2_{1,(1)} + z^2_{(2)})^2 \\ \Psi'^{\theta}(\rho,\theta,z) &= \theta + \arg(\sqrt{\rho} z^1_{1,(1)} + z^1_{(2)}, \sqrt{\rho} z^2_{1,(1)} + z^2_{(2)}), \end{aligned} \tag{5.15}$$

The canonical SDE defined by $(\Psi^{\rho}, \Psi^{\theta})$ is a triangular SDE with respect to the solutions processes (R_t, Θ_t). Indeed we have

$$\begin{aligned} dR_t = {} & \left(d\left[Z'^1_{(2)}, Z'^1_{(2)}\right]^c_t + d\left[Z'^2_{(2)}, Z'^2_{(2)}\right]^c_t + (\Delta Z'^1_{(2),t})^2 + (\Delta Z'^2_{(2),t})^2 \right) + \\ & + \sqrt{R_{t_-}} \left(2dZ'^1_{(2),t} + 2d\left[Z'^1_{(2)}, \overline{Z}'^1_1\right]^c_t + 2d\left[Z'^2_{(2)}, \overline{Z}'^2_1\right]^c_t + 2\Delta \overline{Z}'^1_{1,t} \Delta Z'^1_{(2),t} + 2\Delta \overline{Z}'^2_{1,t} \Delta Z'^2_{(2),t} \right) + \\ & + R_{t_-} \left(2d\overline{Z}'^1_{1,t} + d\left[\overline{Z}'^1_1, \overline{Z}'^1_1\right]^c_t + d\left[\overline{Z}'^2_1, \overline{Z}'^2_1\right]^c_t + (\Delta \overline{Z}^1_{1,t})^2 + (\Delta \overline{Z}^2_{1,t})^2 \right) \end{aligned} \tag{5.16}$$

$$\begin{aligned} d\Theta_t = {} & \left(d\overline{Z}'^2_{1,t} - 2d\left[\overline{Z}'^1_1, \overline{Z}'^2_1\right]^c_t \right) + \\ & + \frac{1}{\sqrt{R_{t_-}}} \left(dZ'^2_{(2),t} - d\left[\overline{Z}'^1_1, Z'^2_{(2)}\right]^c_t - 2d\left[Z'^1_{(2)}, \overline{Z}'^2_1\right]^c_t \right) - \frac{1}{R_{t_-}} d\left[Z'^1_{(2)}, Z'^2_{(2)}\right]^c_t + \\ & + \left(\arg\left(\sqrt{R_{t_-}}(1 + \Delta \overline{Z}'^1_{1,t}) + \Delta Z'^1_{(2),t}, \sqrt{R_{t_-}}(\Delta \overline{Z}'^2_{1,t}) + \Delta Z'^2_{(2),t} \right) - \Delta \overline{Z}'^2_{1,t} - \frac{\Delta Z'^2_{(2),t}}{\sqrt{R_{t_-}}} \right), \end{aligned} \tag{5.17}$$

where

$$\begin{aligned} dZ'^i_{(2),t} &= B^i_j(X_{t_-}) dZ^j_{(2),t} \\ dZ'^i_{j,(1),t} &= Z'^i_{k(1),t_-} B^k_l(X_{t_-}) B^j_r(X_{t_-}) (Z^{-1}_{(1)})^l_{m,t_-} dZ^m_{r,(1),t} \\ d\overline{Z}'^i_{j,t} &= (Z'^{-1}_{(1)})^i_{k,t_-} dZ'^k_{j(1),t} = B^i_k(X_{t_-}) B^j_r(X_{t_-}) d\overline{Z}^k_{r,t}. \end{aligned}$$

Here $B(x)$ is given in (5.14), $X_t^1 = \sqrt{R_t}\cos(\Theta_t)$, $X_t^2 = \sqrt{R_t}\sin(\Theta_t)$ and $\overline{Z}_j^i$ are given by equation (5.12). It is evident that the SDEs (5.16) and (5.17) are in triangular form. Indeed, the equation for R_t depends only on R_t, while the equation for Θ_t is independent from Θ_t itself. This means that the process Θ_t can be reconstructed from the process R_t and the semimartingales $(Z'_{(1)}, Z'_{(2)})$ using only integrations. Furthermore, using the inverse of the stochastic transformation (5.14), we can recover both the solution process X_t^1, X_t^2 and the initial noise $(Z_{(1)}, Z_{(2)})$ using only inversion of functions and Itô integrations. This situation is very similar to what happens in the deterministic setting (see [147, 164]) and in the Brownian motion case, where the presence of a one-parameter symmetry group allows us to split the differential system into a system of lower dimension and an integration (the so called reduction and reconstruction by quadratures). Also the equation for R_t is recognized to have a familiar form. Indeed, in the case where $Z_{(1)} = I_2$ (the two dimensional identity matrix) and $Z_{(2)}$ is a two dimensional Brownian motion, equation (5.16) becomes the equation of the two dimensional Bessel process. This fact should not surprise since the proposed reduction procedure is the usual reduction procedure of a two dimensional Brownian motion with respect to the rotation group. For generic $(Z_{(1)}, Z_{(2)})$ the equation for R_t has the form

$$dR_t = d\mathfrak{z}_t^1 + \sqrt{R_{t_-}}\, d\mathfrak{z}_t^2 + R_{t_-}\, d\mathfrak{z}_t^2,$$

where

$$\begin{aligned}
\mathfrak{z}_t^1 &= \left[Z'^1_{(2)}, Z'^1_{(2)}\right]^c_t + \left[Z'^2_{(2)}, Z'^2_{(2)}\right]^c_t + \sum_{0\leq s\leq t}\left((\Delta Z'^1_{(2),s})^2 + (\Delta Z'^2_{(2),s})^2\right)\\
\mathfrak{z}_t^2 &= 2Z'^1_{(2),t} + 2\left[Z'^1_{(2)}, \overline{Z}'^1_1\right]^c_t + 2\left[Z'^2_{(2)}, \overline{Z}'^2_1\right]^c_t + \sum_{0\leq s\leq t}\left(2\Delta\overline{Z}'^1_{1,s}\Delta Z'^1_{(2),s} + 2\Delta\overline{Z}'^2_{1,s}\Delta Z'^2_{(2),s}\right)\\
\mathfrak{z}_t^3 &= 2\overline{Z}'^1_{1,t} + \left[\overline{Z}'^1_1, \overline{Z}'^1_1\right]^c_t + \left[\overline{Z}'^2_1, \overline{Z}'^2_1\right]^c_t + \sum_{0\leq s\leq t}\left((\Delta\overline{Z}'^1_{1,t})^2 + (\Delta\overline{Z}'^2_{1,t})^2\right).
\end{aligned}$$

Equation (5.16) can be considered as a kind of generalization of affine processes (see [57]): indeed, in the case where $Z_{(1)}$ is deterministic and $Z_{(2)}$ is a two dimensional Brownian motion, equation (5.16) reduces to a CIR model equation (appearing in mathematical finance), with time dependent coefficients.

Remark 5.14 *The gauge symmetry group $O(2)$ with action Ξ_B on the pair $(Z_{(1)}, Z_{(2)})$ has interesting applications in the iterated map theory. Indeed, let $Z_{(1)}, Z_{(2)}$ be discrete-time semimartingales with independent increments, $Z_{(1),\ell} = K_\ell \cdot Z_{(1),\ell-1}$ and $Z_{(2),\ell} = Z_{(2),\ell-1} + H_\ell$, where $K_\ell \in GL(2), H_\ell \in \mathbb{R}^2$ are random variables independent from $Z_{(1),1}, ..., Z_{(1),\ell-1}, Z_{(2),1}, ..., Z_{(2),\ell-1}$. Therefore we have that $(Z_{(1)}, Z_{(2)})$ have $O(2)$ as gauge symmetry group with action Ξ_B if and only if the distribution of $(K_\ell, H_\ell) \in GL(2) \times \mathbb{R}^2$ is invariant with respect to the action Ξ_B. Indeed in the present case the characteristic triplet (b, A, ν) of $(Z_{(1)}, Z_{(2)})$ is*

$$\begin{aligned}
b &= 0\\
A &= 0\\
\nu(dt, dz) &= \sum_{\ell\in\mathbb{N}} \delta_\ell(dt) m_\ell(dz)
\end{aligned}$$

where δ_ℓ is the Dirac delta distribution on $\mathbb{R}$ with the mass concentrated in $\ell \in \mathbb{R}$ and m_n is the probability distribution on $N = GL(2) \times \mathbb{R}^2$ of the pair of random variables (K_ℓ, H_ℓ). By Theorem 4.22, Ξ_B is a gauge symmetry of $(Z_{(1)}, Z_{(2)})$ if and only if, for any $B \in O(2)$, $\Xi_{B*}(\nu) = \nu$ which is equivalent to request that $\Xi_{B*}(m_\ell) = m_\ell$. This implies that the law of (K_ℓ, H_ℓ) is invariant with

respect to the action Ξ_B.
The invariance of the law of $K_\ell \in GL(2)$ with respect to Ξ_B is exactly the invariance of the matrix random variable K_ℓ with respect to orthogonal conjugation, and the law of the $\mathbb{R}^2$ random variable H_ℓ is rotationally invariant. This kind of random variables and related processes are deeply studied in random matrix theory (see, e.g., [5, 139]).

5.3 Weak symmetries of numerical approximations of SDEs driven by Brownian motion

In this section we deal with the symmetries of numerical schemes of Brownian-motion driven SDEs as studied in Chapter 3. Here the approach, taking advantage of the theory developed in this chapter, is more general and permits to extend some results obtained in Chapter 3. Let us consider again an SDE (μ, σ) of the form

$$dX^i_t = \mu^i(X_t)dt + \sigma^i_\alpha(X_t)dW^\alpha_t. \tag{5.18}$$

In the following we will describe a numerical scheme for equation (5.18) as a canonical SDE associated with the function F driven by a semimartingale Z related to the Brownian motion W. Once the gauge symmetries of the semimartingale Z are analysed, we are able to find the symmetries of the numerical schemes for (5.18) as the symmetries of the canonical SDE F.
In particular we focus on the two more common numerical schemes for equation (5.18): the Euler and the Milstein discretization.

5.3.1 Symmetries of the Euler scheme

The Euler scheme for equation (5.18), given a partition $\{t_\ell\}_\ell$ of $[0, T]$, reads

$$X^{i,N}_{t_\ell} = X^{i,N}_{t_{\ell-1}} + \mu(X^N_{t_{\ell-1}})\Delta t_\ell + \sum_{\alpha=1}^{k} \sigma^i_\alpha(X^N_{t_{\ell-1}})\Delta W^\alpha_\ell$$

where $\Delta t_\ell = t_\ell - t_{\ell-1}$ and $\Delta W^\alpha_\ell = W^\alpha_{t_\ell} - W^\alpha_{t_{\ell-1}}$. We define a semimartingale Z taking values in $N = \mathbb{R}^{k+1}$ by putting

$$\begin{aligned} Z^0_t &= \sum_{\ell=0}^{\infty} t_\ell I_{[t_\ell, t_{\ell+1})}(t) \\ Z^\alpha_t &= \sum_{\ell=0}^{\infty} W^\alpha_{t_\ell} I_{[t_\ell, t_{\ell+1})}(t). \end{aligned}$$

Since Z_t is a process with predictable jumps at t_ℓ, Z_t is a semimartingale. Define the maps

$$F^i(x, z) = x^i + \mu(x)z^0 + \sigma^i_\alpha(x)z^\alpha.$$

It is clear that X^N_t is the solution to the canonical SDE defined by F and driven by the semimartingale Z.

Let Ξ_B be an action of $O(k)$ on $\mathbb{R}^{k+1}$ such that $\Xi_B(z) = (z^0, B^\alpha_\beta z^\beta)$. The following theorem holds.

Theorem 5.15 *The group $O(k)$ with action Ξ_B is a gauge symmetry group for Z with respect to its natural filtration.*

Proof. In order to prove the thesis we directly use the definition of gauge symmetry group. Let $\mathcal{F}_t$ be the natural filtration of the k dimensional Brownian motion W. Let B_t be a predictable locally bounded process taking values $O(k)$ with respect to the natural filtration of Z. This means that B_{t_ℓ} is $\mathcal{F}_{t_{\ell-1}}$ measurable. Moreover, the process $\tilde{B}_t$, defined by

$$\tilde{B}_t = \sum_{\ell=0}^{\infty} B_{t_\ell} I_{(t_{\ell-1},t_\ell]}(t).$$

is an $\mathcal{F}_t$-predictable locally bounded process taking values in $O(k)$. Since the rotations are a gauge symmetry group for the k dimensional Brownian motion with respect to its natural filtration, we have that

$$\tilde{W}^\alpha_t = \int_0^t \tilde{B}^\alpha_{\beta,s} dW^\beta_s = \sum_{\ell=1}^{+\infty} B^\alpha_{\beta,t_\ell}(W^\beta_{t_\ell \wedge t} - W^\beta_{t_{\ell-1}\wedge t})$$

is a k dimensional Brownian motion. In particular the following discretization

$$\tilde{Z}^\alpha_t = \sum_{\ell=0}^{\infty} \tilde{W}^\alpha_{t_\ell} I_{[t_\ell,t_{\ell+1})}(t)$$

has the same law as Z. Since

$$\begin{aligned}
\tilde{Z}^\alpha_{t_i} &= \int_0^{t_i} B^\alpha_{\beta,s} dZ^\beta_s = \sum_{\ell=1}^{i} B^\alpha_{\beta,t_\ell}(Z^\beta_{t_{\ell-1}} - Z^\beta_{t_\ell}) \\
&= \sum_{\ell=1}^{i} B^\alpha_{\beta,t_\ell}(W^\beta_{t_{\ell-1}} - W^\beta_{t_\ell}) = \tilde{W}^\alpha_{t_i}
\end{aligned}$$

and B_t is a generic predictable locally bounded process with respect to the natural filtration of Z, the thesis follows. ■

Using the previous discussion and Theorem 5.15 we can introduce the concept of weak symmetry of the Euler discretization scheme $F(x,z)$. Indeed the weak stochastic transformation $T = (\Phi(x), B(x))$, which acts on the solution to the Euler discretization scheme in the following way $(X'^N, Z') = P_T(X,Z)$ and precisely as

$$\begin{aligned}
X'^N_{t_\ell} &= \Phi(X^N_{t_\ell}) \\
\Delta Z'^\alpha_{t_\ell} &= B^\alpha_\beta(X^N_{t_{\ell-1}})\Delta Z^\beta_{t_\ell} = B^\alpha_\beta(X^N_{t_{\ell-1}})\Delta W^\beta_\ell \\
\Delta Z^0_{t_\ell} &= \Delta t_\ell.
\end{aligned}$$

is a symmetry of the Euler discretization scheme if (X', Z') is also a solution to the discretization scheme defined by F. By Theorem 5.9, a sufficient condition to be a symmetry of F is

$$\Phi(F(\Phi^{-1}(x), \Delta t, (B \circ \Phi^{-1}(x))^{-1} \cdot \Delta W) = F(x, \Delta t, \Delta W), \tag{5.19}$$

where $z = (\Delta t, \Delta W)$. For a given weak infinitesimal stochastic transformation (Y, C) the determining equations reads

$$Y^j(x)\partial_{x^j}(F^i)(x,\Delta t,\Delta W) - F^j(x,\Delta t,\Delta W)\partial_{x^j}(Y^i)(x) = -C^\alpha_\beta(x)\Delta W^\beta \partial_{\Delta W^\alpha}(F^i)(x,\Delta t,\Delta W). \tag{5.20}$$

The following theorem proposes a generalization of Theorem 3.2 to the case of weak stochastic transformations.

Theorem 5.16 *Let $V = (Y, C, 0)$ be a quasi strong symmetry of the SDE (μ, σ) in (5.18). When $Y^i_j = Y_j(x^i)$ are polynomials of first degree in $x^1, ..., x^m$, then $\tilde{V} = (Y, C) \in V_{O(n)}(M)$ is a weak symmetry of the Euler discretization scheme F. If, for a given $x_0 \in M$, $\mathrm{span}\{\sigma_1(x_0), \dots, \sigma_m(x_0)\} = \mathbb{R}^n$, also the converse holds.*

Proof. The proof is similar to the one of Theorem 3.2. We recall that the determining equations (see Theorem 1.21) in the case of quasi strong symmetries read

$$\begin{aligned} Y^i \partial_{x^i}(\sigma^j_\alpha) - \sigma^i_\alpha \partial_{x^i}(Y^j) &= -C^\beta_\alpha \sigma^j_\beta & (5.21)\\ Y^i \partial_{x^i}(\mu^j) - L(Y^j) &= 0. & (5.22)\end{aligned}$$

The determining equations for the Euler discretization method are

$$Y^i\left(x + \mu\Delta t + \sigma_\alpha \Delta W^\alpha\right) - Y^i - Y^j\partial_{x^j}(\mu^i)\Delta t - Y^j \partial_{x^j}(\sigma^i_\alpha)\Delta W^\alpha - \sigma^i_\beta C^\beta_\alpha \Delta W^\alpha = 0. \qquad (5.23)$$

Using equations (5.21) and (5.22) in equation (5.23) we obtain

$$Y^i\left(x + \mu\Delta t + \sigma_\alpha \Delta W^\alpha\right) = Y^i + \mu^j \partial_{x^j}(Y^i)\Delta t + A^{lk}\partial_{x^k x^l}(Y^i)\Delta t + \sigma^j_\alpha \partial_{x^j}(Y^i)\Delta W^\alpha. \qquad (5.24)$$

If Y^i is linear in x^j equation (5.24) is satisfied and (Y, C) is a weak symmetry of the Euler discretization scheme. Conversely, if (Y, C) is a symmetry of the Euler discretization scheme and the condition given on σ_α holds, equation (5.24) implies that Y^i is linear in x^j. ■

5.3.2 Symmetries of the Milstein scheme

Recalling that the Milstein scheme has the form

$$\begin{aligned} \bar{X}^i_{t_\ell} &= \bar{X}^i_{t_{\ell-1}} + \mu^i(\bar{X}_{t_{\ell-1}})\Delta t_\ell + \textstyle\sum_{\alpha=1}^k \sigma^i_\alpha(\bar{X}_{t_{\ell-1}})\Delta W^\alpha_\ell + \\ &\quad + \tfrac{1}{2}\textstyle\sum_{j=1}^m \sum_{\alpha,\beta=1}^k \sigma^j_\alpha(\bar{X}_{t_{\ell-1}})\partial_j(\sigma^i_\beta)(\bar{X}_{t_{\ell-1}})\Delta \mathbb{W}^{\alpha,\beta}_\ell, \end{aligned} \qquad (5.25)$$

we cannot consider this scheme as a canonical SDE driven by a semimartingale in $\mathbb{R}^{k+1}$ anymore. Indeed in this case the driving noise is composed by both the discretization of the Brownian motion W and by the iterated integral $\mathbb{W}^{\alpha,\beta}_t = \int_0^t W^\alpha_s dW^\beta_s$. Therefore

$$N = \mathbb{R} \oplus \mathbb{R}^k \oplus (\mathbb{R}^k \otimes \mathbb{R}^k),$$

and the semimartingale $(t, W_t, \mathbb{W}_t)$ lives exactly in N. The vector space N has a natural Lie group structure with composition given by

$$(\alpha_1, a_1, b_1) \circ (\alpha_2, a_2, b_2) = (\alpha_1 + \alpha_2, a_1 + a_2, a_2 \otimes a_1 + b_1 + b_2),$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$, $a_1, a_2 \in \mathbb{R}^k$ and $b_1, b_2 \in \mathbb{R}^k \otimes \mathbb{R}^k$. In this case $1_N = (0, 0, 0 \otimes 0)$ while the inverse operation is given by

$$(\alpha_1, a_1, b_1)^{-1} = (-\alpha_1, -a_1, -b_1 + a_1 \otimes a_1).$$

Let $Z = (Z^0, Z^\alpha_1, Z^{\alpha,\beta}_2)$ be the semimartingale given by the discretization of $(t, W_t, \mathbb{W}_t)$, in other words

$$\begin{aligned} Z^0_t &= t_\ell \text{ if } t_\ell \leq t < t_{\ell+1}\\ Z^\alpha_{1,t} &= W^\alpha_{t_\ell} \text{ if } t_\ell \leq t < t_{\ell+1}\\ Z^{\alpha,\beta}_{2,t} &= \int_0^{t_\ell} W^\alpha_s dW^\beta_s \text{ if } t_\ell \leq t < t_{\ell+1}. \end{aligned}$$

It is simple to see that

$$\begin{aligned} Z_{t_\ell} \circ Z_{t_{\ell_-}}^{-1} &= \left(t_\ell - t_{\ell-1}, W_{t_\ell}^\alpha - W_{t_{\ell-1}}^\alpha, \int_0^{t_\ell} W_s^\alpha dW_s^\beta + \right. \\ &\left. - \int_0^{t_{\ell-1}} W_s^\alpha dW_s^\beta - W_{t_{\ell-1}}^\alpha W_{t_\ell}^\beta + W_{t_{\ell-1}}^\alpha W_{t_{\ell-1}}^\beta \right) \\ &= \left(\Delta t_\ell, \Delta W_\ell^\alpha, \int_{t_{\ell-1}}^{t_\ell} (W_s^\alpha - W_{t_{\ell-1}}^\alpha) dW_s^\beta \right) \\ &= (\Delta t_\ell, \Delta W_\ell^\alpha, \Delta \mathbb{W}_\ell^{\alpha\beta}). \end{aligned}$$

It is possible to define, in a natural way, an action $\Xi_B : N \to N$ of the orthogonal matrices $B \in O(k)$ on N. Indeed if $B \in N$ we define Ξ_B as the unique linear map from N into itself such that

$$\Xi_B((\alpha, a, b \otimes c)) = (\alpha, B \cdot a, (B \cdot b) \otimes (B \cdot c)),$$

for any $a, b, c \in \mathbb{R}^k$.

Theorem 5.17 *The Lie group $O(k)$ with action Ξ_B is a gauge symmetry group for the discretization $Z = (Z^0, Z_1^\alpha, Z_2^{\alpha,\beta})$ of $(t, W_t, \mathbb{W}_t)$, with respect to its natural filtration.*

Proof. The proof follows the scheme of the proof of Theorem 5.15. Let us define

$$W_t' = \sum_{t_k \leq t} B_{t_k} \cdot \Delta W_k + B_{t_\ell}(W_t - W_{t_\ell}),$$

where t_k is the last time lower then t. The thesis of the theorem is equivalent to prove that

$$\mathbb{W}_{t_\ell}'^{\alpha\beta} = \int_0^{t_\ell} W_s'^\alpha dW_s'^\beta = Z_{2,t_\ell}'^{\alpha\beta}$$

where $dZ_t' = \Xi_B(dZ_t)$, that is

$$\begin{aligned} Z_{2,t_\ell}'^{\alpha\beta} &= \sum_{k \leq \ell} B_{\gamma,t_k}^\alpha B_{\delta,t_k}^\beta \Delta Z_{2,t_k}^{\gamma\delta} + \sum_{h \leq k < \ell} B_{\gamma,t_h}^\alpha B_{\delta,t_k}^\beta \Delta Z_{1,t_h}^\gamma \Delta Z_{1,t_k}^\delta \\ &= \sum_{k \leq \ell} B_{\gamma,t_k}^\alpha B_{\delta,t_k}^\beta \Delta \mathbb{W}_k^{\gamma\delta} + \sum_{h \leq k < \ell} B_{\gamma,t_h}^\alpha B_{\delta,t_k}^\beta \Delta W_h^\gamma \Delta W_k^\delta. \end{aligned}$$

We have that

$$\begin{aligned} \mathbb{W}_{t_\ell}'^{\alpha\beta} &= \int_0^{t_\ell} W_s'^\alpha dW_s'^\beta \\ &= \int_0^{t_\ell} \left(\sum_{t_k \leq s} B_{\gamma,s}^\alpha \cdot \Delta W_k^\gamma + B_{\gamma,t_\ell}^\alpha (W_s^\gamma - W_{t_k}^\gamma) \right) dW_s'^\beta \\ &= \sum_{h \leq k < \ell} B_{\gamma,t_h}^\alpha \Delta W_h^\gamma \Delta W_k'^\beta + \sum_{k \leq \ell} B_{\gamma,t_k}^\alpha \int_{t_{k-1}}^{t_k} (W_s^\gamma - W_{t_{k-1}}^\gamma) dW_s'^\beta \\ &= \sum_{h \leq k < \ell} B_{\gamma,t_h}^\alpha B_{\delta,t_k}^\beta \Delta W_h^\gamma \Delta W_k^\delta + \sum_{k \leq \ell} B_{\gamma,t_k}^\alpha B_{\delta,t_k}^\beta \int_{t_{k-1}}^{t_k} (W_s^\gamma - W_{t_{k-1}}^\gamma) dW_s^\delta \\ &= \sum_{h \leq k < \ell} B_{\gamma,t_h}^\alpha B_{\delta,t_k}^\beta \Delta W_h^\gamma \Delta W_k^\delta + \sum_{k \leq \ell} B_{\gamma,t_k}^\alpha B_{\delta,t_k}^\beta \Delta \mathbb{W}_k^{\gamma\delta} = Z_{2,t_\ell}'^{\alpha,\beta}. \end{aligned}$$

■

Thanks to Theorem 5.17 we can introduce the concept of weak symmetry of a Milstein type discretization scheme. We can see the solution $\bar{X}_\ell$ of the Milstein scheme (5.25) as an iterated random map defined by the canonical SDE $F(x,z) = F(x,\Delta t,\Delta W,\Delta\mathbb{W})$ where F has the form

$$\begin{aligned} F^i(x,\Delta t,\Delta W,\Delta\mathbb{W}) &= x^i + \mu^i(x)\Delta t_n + \sum_{\alpha=1}^{n}\sigma^i_\alpha(x)\Delta W^\alpha + \\ &+\frac{1}{2}\sum_{j=1}^{n}\sum_{\alpha,\beta=1}^{n}\sigma^j_\alpha(x)\partial_j(\sigma^i_\beta)(x)\Delta\mathbb{W}^{\alpha,\beta}, \end{aligned}$$

and driven by the semimartingale Z on the Lie group N with its natural product described above. The weak stochastic transformation $T=(\Phi(x),B(x))$ acts on the solution to the Milstein discretization scheme in the following way $(X'^N,Z') = P_T(X,Z)$, that is

$$\begin{aligned} \bar{X}'^N_{t_\ell} &= \Phi(\bar{X}^N_{t_\ell}) \\ \Delta Z'^\alpha_{1,t_\ell} &= B^\alpha_\beta(\bar{X}^N_{t_{\ell-1}})\Delta Z^\beta_{1,t_\ell} = B^\alpha_\beta(\bar{X}^N_{t_{\ell-1}})\Delta W^\beta_\ell \\ \Delta Z'^{\alpha\beta}_{2,t_\ell} &= B^\alpha_\gamma(\bar{X}^N_{t_{\ell-1}})B^\beta_\delta(\bar{X}^N_{t_{\ell-1}})\Delta Z^{\gamma\delta}_{2,t_\ell} = B^\alpha_\gamma(\bar{X}^N_{t_{\ell-1}})B^\beta_\delta(\bar{X}^N_{t_{\ell-1}})\Delta\mathbb{W}^{\gamma\delta}_\ell \\ \Delta Z^0_{t_\ell} &= \Delta t_\ell, \end{aligned}$$

is a symmetry of the discretization scheme if (X',Z') is also a solution to the discretization scheme F. Using Theorem 5.9, a sufficient condition for having such a symmetry is that

$$\Phi(F(\Phi^{-1}(x),\Delta t,(B\circ\Phi^{-1})^{-1}(x)\cdot\Delta W,(B\circ\Phi^{-1})^{-1}(x)\otimes(B\circ\Phi^{-1})^{-1}(x)\cdot\Delta\mathbb{W}) = F(x,\Delta t,\Delta W,\Delta\mathbb{W}). \tag{5.26}$$

For a given infinitesimal stochastic transformation (Y,C) the determining equations (see equation (5.9)) reads

$$\begin{gathered} Y^j(x)\partial_{x^j}(F^i)(x,\Delta t,\Delta W,\Delta\mathbb{W}) - F^j(x,\Delta t,\Delta W,\Delta\mathbb{W})\partial_{x^j}(Y^i)(x)) = \\ = -C^\alpha_\beta(x)\Delta W^\beta\partial_{\Delta W^\alpha}(F^i)(x,\Delta t,\Delta W,\Delta\mathbb{W}) - C^\alpha_\beta(x)\Delta\mathbb{W}^{\beta\gamma}\partial_{\mathbb{W}^{\alpha\gamma}}(F^i)(x,\Delta t,\Delta W,\Delta\mathbb{W}) + \\ -C^\alpha_\beta(x)\Delta\mathbb{W}^{\beta\gamma}\partial_{\mathbb{W}^{\gamma\alpha}}(F^i)(x,\Delta t,\Delta W,\Delta\mathbb{W}). \end{gathered} \tag{5.27}$$

We propose a version of Theorem 5.16 for the Milstein case.

Theorem 5.18 *Suppose that $(Y,C,0)$ is a quasi strong symmetry of the SDE (μ,σ) that Y^i is linear in x^i and that C is a constant matrix. Then (Y,C) is a weak symmetry of the Milstein discretization scheme for (μ,σ).*

Proof. From Theorem 5.16, equation (5.27) and the hypothesis of the present theorem we have that (Y,C) is a symmetry of the Milstein discretization if and only if

$$\partial_{x^j}(Y^k)\sigma^i_\alpha\partial_{x^i}(\sigma^j_\beta)\Delta\mathbb{W}^{\alpha\beta} = Y^j\partial_{x^j}(\sigma^i_\alpha\partial_{x^i}(\sigma^k_\beta))\Delta\mathbb{W}^{\alpha\beta} + \sigma^i_\alpha\partial_{x^i}(\sigma^j_\beta)(C^\alpha_\gamma\Delta\mathbb{W}^{\gamma\beta} + C^\beta_\delta\Delta\mathbb{W}^{\alpha\delta}). \tag{5.28}$$

Furthermore $(Y,C,0)$ is a quasi-strong symmetry of the SDE (μ,σ) if and only if

$$-Y^i\partial_{x^i}(\sigma^j_\alpha) + \sigma^i_\alpha\partial_{x^i}(Y^j) = C^\beta_\alpha\sigma^j_\beta \tag{5.29}$$

If we derive equation (5.29) with respect to $\sigma^i_\gamma\partial_{x^i}$ we obtain

$$\begin{gathered} -\sigma^i_\alpha\partial_{x^i}(Y^j)\partial_{x^j}(\sigma^k_\beta) - Y^j\sigma^i_\alpha\partial_{x^ix^j}(\sigma^k_\beta) + \sigma^i_\alpha\partial_{x^i}(\sigma^j_\beta)\partial_{x^j}(Y^k) + \sigma^i_\alpha\sigma^j_\beta\partial_{x^ix^j}(Y^k) = \\ = \sigma^i_\alpha\partial_{x^i}(C^\gamma_\beta)\sigma^k_\gamma + \sigma^i_\alpha\partial_{x^i}(\sigma^k_\gamma)C^\gamma_\beta. \end{gathered} \tag{5.30}$$

Using the fact that Y^i is almost linear in x^i, and so $\partial_{x^i x^j}(Y^k) = 0$, the fact that C^α_β is constant, and so $\partial_{x^i}(C^\gamma_\beta) = 0$, and replacing equation (5.29) in equation (5.30) we obtain the relation (5.28) which means that (Y, C) is a weak symmetry of the Milstein discretization scheme. ■

Remark 5.19 *There is some important differences between Theorem 5.16 and Theorem 5.18. Indeed Theorem 5.16 gives a necessary and sufficient condition such that a quasi-strong symmetry $(Y, C, 0)$ of the SDE (μ, σ) is a weak symmetry (Y, C) of the Euler discretization scheme, while Theorem 5.18 gives only a sufficient condition. Furthermore the hypotheses of Theorem 5.18 request that C^α_β is a constant, while C^α_β can be any function in Theorem 5.16.*
This last difference can be explained in the following way: the gauge transformation Ξ_B transforms the process $Z = (Z^0, Z^\alpha_1, Z^{\alpha\beta}_2)$ using an Euler approximation of the usual random rotation and not a Milstein approximation. The two approximations of the random rotation coincide only when the rotations B^α_β (or the generator C^α_β of the rotations) are constants. Finally it is important to note that one cannot use the Milstein approximation for transforming the semimartingales Z, because otherwise the transformation would not preserve the law of the process Z.

Part III

Finite dimensional solutions to SPDEs and the geometry of infinite jets bundles

Chapter 6

The geometry of infinite jet bundles and characteristics

In this chapter we introduce some notions about the geometry of $J^\infty(M,N)$ which will be useful for stochastic applications in Chapter 7. We start by presenting a short and practical introduction to jets bundles, they coordinate systems and their differential geometry. Then we introduce the concept of characteristic vector fields and their characteristic flow which are a $J^\infty(M,N)$ generalization of the characteristics of first order scalar PDEs. Finally we exploit the characteristics in order to construct special submanifolds of $J^\infty(M,N)$ using the characteristics which will be essential in the construction of finite dimensional solution to SPDEs.

6.1 The geometry of $J^\infty(M,N)$

In this section we collect some basic facts about (infinite) jet bundles in order to provide the necessary geometric tools for our aims.

6.1.1 An informal introduction to the geometry of $J^\infty(M,N)$

We start with an informal introduction to the geometry of $J^\infty(M,N)$, where M,N are two open subsets of $\mathbb{R}^m,\mathbb{R}^n$ respectively. The main advantage of the infinite jet bundle setting, with respect to the analytic Fréchet spaces approach, relies on the computational aspects which turn out to be definitely simpler.

Let $C^k(M,N)$ be the infinite dimensional Fréchet space of k times differentiable functions defined on M and taking values in N. We can associate with $C^k(M,N)$ the finite dimensional manifold $J^k(M,N)$ identifying $f,g:M\to N$ whenever $g(x_0)=f(x_0)$ and $\partial^\sigma(f)(x_0)=\partial^\sigma(g)(x_0)$, where $\sigma\in\mathbb{N}_0^n$ is a multi-index with $|\sigma|=\sum_r\sigma_r\leq k$. The space $J^k(M,N)$ is called k-jets bundle of functions from M into N and can be endowed with a natural coordinate system. If x^i is the standard coordinate system on M (the space of independent variables) and u^j is standard coordinate system on N (the space of dependent variables), a coordinate system on $J^k(M,N)$ is given by x^i,u^j and all the variables u^i_σ, where $|\sigma|\leq k$, which formally represent the derivative of the functions $u^j(x)$. The smooth manifold $J^k(M,N)$ is a smooth vector bundle on M with projection $\pi_{k,-1} : J^k(M,N) \to M$ given by

$$\pi_{k,-1}(x^i, u^j, u^j_\sigma) = x^i.$$

With any function $f \in C^k(M,N)$ we can associate a continuous section of the bundle $(J^k(M,N), M, \pi_{k,-1})$ in the following way

$$f \longmapsto D^k(f)(x) = (x, u^j = f^j(x), u^j_\sigma = \partial_\sigma(f^j)(x)).$$

Moreover, for any $k, h \in \mathbb{N}$ with $h < k$, there is a natural projection $\pi_{k,h} : J^k(M,N) \to J^h(M,N)$ given by

$$\pi_{k,h}(x^i, u^j, (u^j_\sigma)|_{|\sigma|\leq k}) = (x^i, u^j, (u^j_\sigma)|_{|\sigma|\leq h}).$$

This allows us to consider the space $J^\infty(M,N)$ defined as the inverse limit of the sequence of projections

$$M \overset{\pi_0}{\leftarrow} M \times N = J^0(M,N) \overset{\pi_{1,0}}{\leftarrow} J^1(M,N) \overset{\pi_{2,1}}{\leftarrow} \dots \overset{\pi_{k,k-1}}{\leftarrow} J^k(M,N) \overset{\pi_{k+1,k}}{\leftarrow} \dots$$

Analogously to $J^k(M,N)$, also $J^\infty(M,N)$ has a natural coordinate system given by x^i, u^j and u^j_σ, with no bound on $|\sigma|$.
Since $J^\infty(M,N)$ is not a finite dimensional manifold, but a Fréchet manifold modelled on $\mathbb{R}^\infty$ (see, e.g., [89] for an introduction to the concept), working with spaces of smooth functions defined on $J^\infty(M,N)$ is quite difficult. On the other hand, the explicit coordinate system on $J^\infty(M,N)$ suggests the possibility of restricting to a suitable space of smooth functions on $J^\infty(M,N)$ which permits explicit calculations. In fact, if we consider the space

$$\mathfrak{F} = \bigcup_k \mathfrak{F}_k,$$

where $\mathfrak{F}_k$ is the set of smooth functions defined on $J^k(M,N)$, i.e. $F \in \mathfrak{F}_k$ if it is of the form $F(x^i, u^j, u^j_\sigma)$ with $|\sigma| \leq k$, $\mathfrak{F}$ is the set of functions depending only on a finite subset of coordinates x^i, u^j, u^j_σ. Given any vector field $V \in TJ^\infty(M,N)$ of the form

$$V = \phi^i \partial_{x^i} + \psi^j \partial_{u^j} + \psi^j_\sigma \partial_{u^j_\sigma},$$

where $\phi^i, \psi^j, \psi^j_\sigma$ are smooth functions on $J^\infty(M,N)$, if $\phi^i, \psi^j, \psi^j_\sigma \in \mathfrak{F}$, we have that $V(\mathfrak{F}) \subset \mathfrak{F}$. In the following we only consider vector fields V whith $\phi^i, \psi^j, \psi^j_\sigma \in \mathfrak{F}$.
Therefore, given two vector fields V_1, V_2, we can define a Lie bracket given by

$$[V_1, V_2] = (V_1(\phi^i_2) - V_2(\phi^i_1))\partial_{x^i} + (V_1(\psi^j_2) - V_2(\psi^j_1))\partial_{u^j} + (V_1(\psi^j_{\sigma,2}) - V_2(\psi^j_{\sigma,1}))\partial_{u^j_\sigma}.$$

We recall that in $J^\infty(M,N)$ one can naturally define the formally integrable Cartan distribution $\mathcal{C} = \mathrm{span}\{D_1, ..., D_m\}$ generated by the vector fields

$$D_i = \partial_{x^i} + \sum_{k,\sigma} u^k_{\sigma+1_i} \partial_{u^k_\sigma}$$

satisfying $[D_i, D_j] = 0$. Another important class of vector fields in $J^\infty(M,N)$ is given by the vector fields V^e commuting with all D_i. It is possible to prove that V^e commutes with all D_i if and only if V^e is of the form

$$V^e = F^j \partial_{u^j} + D^\sigma(F^j)\partial_{u^j_\sigma},$$

where $F^j \in \mathfrak{F}$. We say that V^e is an evolution vector field generated by the function $F = (F^1, ..., F^n) \in \mathfrak{F}^n$ and we write $V^e = V_F$. The Lie brackets between two evolution vector fields is a new evolution vector field. This means that, for any $F, G \in \mathfrak{F}^n$, there exists a unique function $H \in \mathfrak{F}^n$ such that $[F_F, V_G] = V_H$. Denoting by $H = [F, G]$, it is simple to prove that the brackets $[\cdot, \cdot]$ make $\mathfrak{F}^n$ an infinite dimensional Lie algebra.

Using the natural projection $\pi_k : J^\infty(M,N) \to J^k(M,N)$ of $J^\infty(M,N)$ on $J^k(M,N)$, it is possible to define a useful notion of smooth submanifold of $J^\infty(M,N)$. A subset $\mathcal{E}$ of $J^\infty(M,N)$ is a submanifold of $J^\infty(M,N)$ if, for any $p \in \mathcal{E}$, there exists a neighborhood U_p of p such that $\pi_h(\mathcal{E} \cap U_p)$ is a submanifold of $J^h(M,N)$ for $h > H_p$.
If, for any $p \in \mathcal{E}$, all the submanifolds $\pi_h(\mathcal{E} \cap U_p)$ with $h > H_p$ have the same finite dimension L, we say that $\mathcal{E}$ is an L-dimensional submanifold of $J^\infty(M,N)$. In particular, given an L-dimensional manifold B and a smooth immersion $K : B \to J^\infty(M,N)$, for any point $y \in B$ there exists a neighborhood V of p such that $K(V)$ is a finite dimensional submanifold of $J^\infty(M,N)$. A vector field $V \in TJ^\infty(M,N)$ is tangent to the submanifold $\mathcal{E}$ if, for any $h \in \mathfrak{F}$ such that $h|_{\mathcal{E}} = 0$, we have $V(h)|_{\mathcal{E}} = 0$. In this case we write $Y \in T\mathcal{E}$.

Definition 6.1 *A submanifold $\mathcal{E}$ of $J^\infty(M,N)$ is a* canonical submanifold *if $\mathcal{C} \subset T\mathcal{E}$. Any canonical submanifold $\mathcal{E}$ can be locally described as the set of zeros of a finite number of smooth independent functions $f_1, ..., f_L$ and of all their differential consequences $D^\sigma(f_i)$.*

A finite dimensional smooth canonical submanifold $\mathcal{K}$ such that $T\mathcal{K} = \mathcal{C}$ is called integral manifold of the Cartan distribution. In order to construct an integral manifold of the Cartan distribution we recall that $J^\infty(M,N)$ is a smooth bundle over M with projection $\pi : J^\infty(M,N) \to M$ such that $\pi(x^i, u^j, u^j_\sigma) = x^i$. Analogously to the case of finite jets spaces, we can define the operator $D^\infty(f)$ associating with any $f \in C^\infty(M,N)$ a smooth section $D^\infty(f)$ of the bundle $(J^\infty(M,N), M, \pi)$ in the natural way. Given $f \in C^\infty(M,N)$, we define

$$\mathcal{K}^f = \bigcup_{x \in M} (x, D^\infty(f)(x)).$$

We have that $\mathcal{K}^f$ is an n dimensional submanifold of $J^\infty(M,N)$ and $D_i \in T\mathcal{K}^f$. In fact, if $F \in \mathfrak{F}$, the vector fields D_i satisfy

$$D_i[F](x, f(x), \partial^\sigma(f)(x)) = \partial_{x^i} \left[F(x, f(x), \partial^\sigma(f)(x)) \right],$$

for any $f \in C^\infty(M)$. On the other hand, if $\mathcal{K}$ is an integral manifold of $\mathcal{C}$, there exist a unique function $f^{\mathcal{K}} \in C^\infty(M,N)$ such that $\mathcal{K}^{f^{\mathcal{K}}} = \mathcal{K}$. In this way we can identify any integral manifold of $\mathcal{C}$ with a smooth function in $C^\infty(M,N)$ or, equivalently, we can describe any smooth function as an integral manifold of $\mathcal{C}$ in $J^\infty(M,N)$.

Remark 6.2 *The previous considerations and the definitions of D_i and D^∞ provide a natural interpretation for evolution vector fields. In particular, if the function $f \in C^\infty(M \times \mathbb{R}, N)$ solves an evolution equation of the form*

$$\partial_t(f)(x,t) = F(x, f(x,t), \partial^\sigma(f)(x,t)),$$

it is easy to prove that, for any $G \in \mathfrak{F}$, we have

$$\partial_t^n \left[G(x, f(x,t), \partial^\sigma(f)(x,t)) \right] = V_F^n[G](x, f(x,t), \partial^\sigma(f)(x,t)).$$

These properties will play an important role in the representation of SPDEs as ordinary SDEs on the infinite dimensional manifold $J^\infty(M,N)$.

In the following, in order to make the previous discussion more explicit, we rewrite the expressions of the principal objects introduced above in the particular case of $J^\infty(\mathbb{R},\mathbb{R})$. In the space $J^\infty(\mathbb{R},\mathbb{R})$ we consider the coordinate system given by $x \in \mathbb{R}$ (the coordinate on M), by $u \in \mathbb{R}$ (the coordinate in N) and by all the (formal) derivatives of u with respect to x which are $u_{(1)}, u_{(2)}, u_{(3)}, \ldots$. Sometimes, in order to simplify the notation and make more clear the meaning of the coordinate system $x, u, u_{(1)}, \ldots$ we write $u_x = u_{(1)}$, $u_{xx} = u_{(2)}$,... .
If $F \in \mathfrak{F}$, then F is a smooth function depending only on x, u and the derivative $u_{(n)}$ for $n < k$, with k an integer great enough. The vector field $D_1 = D_x$ has the form

$$D_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + \ldots + u_{(n+1)}\partial_{u_{(n+1)}} + \ldots$$

and represents the formal derivative with respect to x in $J^\infty(M,N)$, which means that, if $F(x,u,u_x,\ldots) \in \mathfrak{F}$ and $f \in C^\infty(\mathbb{R})$, then

$$D_x(F)(x, f(x), f'(x), \ldots) = \partial_x(F(x, f(x), f'(x), \ldots)).$$

In this case, the evolution vector field V_F has the form

$$V_F = F\partial_u + D_x(F)\partial_{u_x} + \ldots + D_x^n(F)\partial_{u_{(n)}} + \ldots$$

In particular, if for example $F = xu_x$, we have

$$V_F = xu_x\partial_u + (xu_{xx} + u_x)\partial_{u_x} + \ldots + (xu_{(n+1)} + nu_{(n)})\partial_{u_{(n)}} + \ldots$$

In this setting there is a simple way to see finite dimensional canonical submanifolds of $J^\infty(\mathbb{R},\mathbb{R})$ as ordinary differential equations of arbitrary order for the dependent variable u. Consider for example the subamanifold $\mathcal{K}^n$ in $J^n(\mathbb{R},\mathbb{R})$ defined as the set of zeros of the equation

$$u_{(n)} - h(x, u, u_x, \ldots, u_{(n-1)}) = 0, \tag{6.1}$$

where $h \in \mathfrak{F}_{n-1}$. If we want that $\mathcal{K}^n$ is the projection on $J^n(\mathbb{R},\mathbb{R})$ of some canonical submanifold $\mathcal{K}$ of $J^\infty(\mathbb{R},\mathbb{R})$ we need that $D_x \in T\mathcal{K}$ and so

$$\begin{array}{rcl} 0 &=& D_x(u_{(n)} - h(x,\ldots)) = u_{(n+1)} - D_x(h)(x,\ldots) \\ 0 &=& D_x^2(u_{(n)} - h(x,\ldots)) = u_{(n+2)} - D_x^2(h)(x,\ldots) \\ & \ldots & \end{array} \tag{6.2}$$

on $\mathcal{K}$. Equations (6.2) are called differential consequences of equation (6.1) and a finite dimensional canonical submanifold $\mathcal{K}$ is defined by equation (6.1) and its differential consequences (6.2). It is possible to prove that the generic (with respect to a suitable topology) canonical submanifold of $J^\infty(\mathbb{R},\mathbb{R})$ is of the form described above. In particular, in $J^\infty(\mathbb{R},\mathbb{R})$, every canonical submanifold is finite dimensional (we remark that this is no more true when M is of dimension greater than one).

6.1.2 Finite dimensional canonical submanifolds of $J^\infty(M,N)$ and reduction functions

In this section, generalizing the identification between integral manifolds of the Cartan distribution in $J^\infty(M,N)$ and smooth functions, we prove that any $(m+r)$ dimensional canonical submanifold in $J^\infty(M,N)$ can be identified with a smooth function defined on $M \subset \mathbb{R}^m$ taking values in $N \subset \mathbb{R}^n$ and depending on r parameters.
In fact, given an r dimensional smooth manifold B and a smooth function

$$K : M \times B \to N,$$

which we call a *finite dimensional function*, we can consider the function

$$\mathbf{K} : M \times B \to J^\infty(M,N)$$

defined by

$$\mathbf{K}(x,b) = (x, D^\infty(K)(x,b))$$

and the subset

$$\mathcal{K}^K = \bigcup_{x\in M, b\in B} \mathbf{K}(x,b),$$

where the D^∞ operator acts only on the x^i variables of K.

Theorem 6.3 *If $\mathcal{K}^K$ is a finite dimensional submanifold of $J^\infty(M,N)$, then $\mathcal{K}^K$ is a finite dimensional canonical submanifold. Conversely, if $\mathcal{K}$ is a finite dimensional canonical submanifold of $J^\infty(M,N)$ then, suitably restricting M and $\mathcal{K}$, there exists a finite dimensional function K such that $\mathcal{K}^K = \mathcal{K}$.*

Proof. The fact that, for any smooth finite dimensional function K, if $\mathcal{K}^K$ is a finite dimensional submanifold of $J^\infty(M,N)$, then $\mathcal{K}^K$ is a finite dimensional canonical manifold follows from the fact that, for any fixed b, $D_i \in \bigcup_{x\in M}\mathbf{K}(x,b)$ since $\mathbf{K}(x,b) = (x, D^\infty(K)(x,b))$.
Conversely, let $\mathcal{K}$ be a finite dimensional canonical submanifold of $J^\infty(M,N)$. By definition of submanifold of $J^\infty(M,N)$, possibly restricting M, we can describe $\mathcal{K}$ by the set of zeros of some functions of the form

$$u^i_\sigma - f^i_\sigma(x^1,...,x^m,y^1,...,y^r),$$

where $y^1,...,y^r \in \mathfrak{F}_k$ and $f^i_\sigma : \mathbb{R}^{m+r} \to \mathbb{R}$ are smooth functions. Thanks to the previous property we can work in the finite dimensional manifold $J^k(M,N)$, rather than in the infinite dimensional $J^\infty(M,N)$. If we choose an adapted coordinate system $x^1,...,x^n,y^1,...,y^r$ in $\mathcal{K}$, the vector fields D_i restricted to $\mathcal{K}$ will be of the form

$$D_i = \partial_{x^i} + \psi^k_i(y^1,...,y^r)\partial_{y^k},$$

for some functions ψ^k_i. Fixing $x_0 \in M$, since $[D_i, D_j] = 0$, there is only one solution to the following system of overdetermined PDEs

$$\begin{aligned}\partial_{x^i}(Y^k)(x,y^1_0,...,y^r_0) &= \psi^k_i(x,Y^1(x,...),...,Y^r(x,...))\\ Y^k(x_0,y^1_0,...,y^r_0) &= y^k_0,\end{aligned}$$

for $(y^1_0,...,y^r_0)$ in a suitable open subset of $\mathbb{R}^r$. Hence, if we restrict M to a suitable neighborhood of x_0, any integral submanifold of $\mathcal{C}|_{\mathcal{K}}$ will be of the form

$$\bigcup_{x\in M}(x, Y^1(x,y^1_0,...,y^r_0),...,Y^r(x,y^1_0,...,y^r_0)),$$

for some $y^1_0,...,y^r_0$. Since x^i, y^j form a coordinate system of $\mathcal{K}$, the coordinates $u^i \in \mathfrak{F}$ restricted to $\mathcal{K}$ are functions of the form

$$u^j = \mathfrak{U}^j(x^1,...,x^n,y^1,...,y^r).$$

This means that the finite dimensional canonical manifold $\mathcal{K}$ is the canonical manifold generated by the function $K \in C^\infty(M\times B, N)$ defined by

$$K^j(x,y^1_0,...,y^r_0) = \mathfrak{U}^j(x, Y^1(x,y^1_0,...,y^r_0),...,Y^r(x,y^1_0,...,y^r_0)).$$

■

The proof of Theorem 6.3 provides a constructive method to obtain the finite dimensional function associated with a finite dimensional canonical submanifold $\mathcal{K}$. This method is very simple in the case of $J^\infty(\mathbb{R},\mathbb{R})$, and in the following we give the idea of the construction in an explicit case. Given $\lambda \in \mathbb{R}$, let $\mathcal{K}$ be defined by

$$u_{xx} - \lambda u = 0,$$

and all its differential consequences. This means that $\mathcal{K}$ is defined by the equations

$$\begin{aligned} u_{(2n)} - \lambda^n u &= 0 \\ u_{(2n+1)} - \lambda^n u_x &= 0. \end{aligned}$$

If we choose on $\mathcal{K}$ the coordinate system (x, u, u_x), the vector field D_x restricted to $\mathcal{K}$ is given by

$$D_x = \partial_x + u_x \partial_u + \lambda u \partial_{u_x}.$$

In order to construct the function K generating $\mathcal{K}$ we need to solve the differential equations

$$\begin{aligned} \partial_x(U(x, u_0, u_{x,0})) &= U_x(x, u_0, u_{x,0}) \\ \partial_x(U_x(x, u_0, u_{x,0})) &= \lambda U(x, u_0, u_{x,0}) \\ U(0, u_0, u_{x,0}) &= u_0 \\ U_x(0, u_0, u_{x,0}) &= u_{x,0}. \end{aligned}$$

If, for example, $\lambda > 0$, the solution to the previous system is

$$\begin{aligned} U(x, u_0, u_{x,0}) &= \frac{\sqrt{\lambda}u_0 + u_{x,0}}{2\sqrt{\lambda}} e^{\sqrt{\lambda}x} + \frac{\sqrt{\lambda}u_0 - u_{x,0}}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}x} \\ U_x(x, u_0, u_{x,0}) &= \frac{\sqrt{\lambda}u_0 + u_{x,0}}{2} e^{\sqrt{\lambda}x} + \frac{-\sqrt{\lambda}u_0 + u_{x,0}}{2} e^{-\sqrt{\lambda}x}. \end{aligned}$$

Since in this case $\mathfrak{U} = u$, the finite dimensional function K generating $\mathcal{K}$ is exactly

$$K(x, u_0, u_{x,0}) = U(x, u_0, u_{x,0}) = \frac{\sqrt{\lambda}u_0 + u_{x,0}}{2\sqrt{\lambda}} e^{\sqrt{\lambda}x} + \frac{\sqrt{\lambda}u_0 - u_{x,0}}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}x}.$$

We remark that, in the case $M \subset \mathbb{R}$, constructing K is equivalent to finding the fundamental solution to the ODE $u_{(n)} - h(x, ...) = 0$ defining, together with all its differential consequences, the manifold $\mathcal{K}$ and, conversely, the finite dimensional canonical submanifold $\mathcal{K}$ associated with K is the unique ODE for which K is the fundamental solution.

6.2 Characteristic vector fields in $J^\infty(M, N)$

In this section we define the notion of generalized characteristic flow for an evolution vector field and we discuss the connection with the usual characteristic flow for scalar first order evolution PDEs. These results will play a central role in the explicit construction of finite dimensional functions and of finite dimensional canonical submanifolds of $J^\infty(M, N)$ in Section 6.3.

6.2.1 Characteristics of scalar first order evolution PDEs

It is well known that, if $N = \mathbb{R}$ and $F \in \mathfrak{F} \setminus \mathfrak{F}_0$ (where $\mathfrak{F}_0$ is the set of smooth functions defined on $J^0(M,N) = M \times N$), the evolution vector field V_F is not the prolongation of a vector field on $J^0(M,N)$ and does not admit a flow in $J^\infty(M,N)$, which is why the equation

$$\partial_t(u) = F(x, u, u_\sigma) \tag{6.3}$$

may not admit solutions even for smooth initial data, or may admit infinite solutions for any smooth initial data. For this reason the problem of finding solutions to evolution PDEs is usually solved only in specific situations (for example the linear or semilinear cases) where it is possible to use the powerful techniques of analysis.

Anyway, a classical geometric approach to scalar first order evolution PDEs (see, e.g., [45]) shows that something can be done in order to solve equation (6.3) even when V_F does not admit a flow in $J^\infty(M,N)$. Indeed given a first order scalar autonomous PDE

$$\partial_t(u) = F(x^j, u, u_i) \tag{6.4}$$

it is possible to solve (6.4) considering the following system of ODEs on $J^1(M,N)$

$$\begin{aligned}
\frac{dx^i}{da} &= -\partial_{u_i}(F)(x^j, u, u_k)\\
\frac{du}{da} &= F(x^j, u, u_k) - \sum_h u_h \partial_{u_h}(F)(x^j, u, u_k)\\
\frac{du_i}{da} &= \partial_i(F)(x^j, u, u_k) + u_i \partial_u(F)(x^j, u, u_k).
\end{aligned}$$

If Φ_a is the flow of the vector field on $J^1(M,N)$ corresponding to the previous system and we define $\phi^i_a = (\Phi^*_a(x^i))$ and $\eta_a = \Phi^*_a(u)$, the solution $U(x,t)$ to PDE (6.4) with initial data $U(x,0) = f(x)$ is given by

$$U(x,t) = \eta_t(\bar{\phi}_t^{-1}(x), f(\bar{\phi}_t^{-1}(x)), \partial_{x^i}(f)(\bar{\phi}_t^{-1}(x)))$$

where $\bar{\phi}_a(x) := \phi_a(x, f(x), \partial_j(f)(x))$.

Moreover it is possible to uniquely extend the flow Φ_a to $J^k(M,N)$ as the solution to the following system of ODEs

$$\frac{du_\sigma}{da} = D^\sigma(F)(x, u, u_\sigma) - \sum_i u_{\sigma+1_i} \partial_{u_i}(F)(x, u, u_\sigma).$$

Defining $\psi_{\sigma,a} = \Phi^*_a(u_\sigma)$ we have

$$\partial^\sigma(U)(x,t) = \psi_{\sigma,t}(\bar{\phi}_t^{-1}(x), f(\bar{\phi}_t^{-1}(x)), \partial^\sigma(f)(\bar{\phi}_t^{-1}(x))),$$

and the vector field corresponding to the flow Φ_a on $J^\infty(M,N)$ is given by

$$\bar{V}_F := \partial_a(\Phi_a)|_{a=0} = V_F - \sum_i \partial_{u_i}(F) D_i.$$

We call Φ_a the *characteristic flow* of F and $\bar{V}_F$ its *characteristic vector field.*

6.2.2 Characteristics in the general setting

In this section we propose an extension of the notion of characteristic vector field and characteristic flow to multidimensional and higher order case. This extension is based on the geometric analysis of $J^\infty(M,N)$ presented in [114]. We start by recalling the definition of one-parameter group of local diffeomorphisms on $J^\infty(M,N)$ which reduces to the classical one in the finite dimensional setting.

Definition 6.4 *A map $\Phi_a : U_a \to J^\infty(M,N)$ is a one-parameter group of local diffeomorphisms if Φ_a are smooth maps, U_a are open sets $\forall a$ (with $U_0 = J^\infty(M,N)$) and $\forall p \in U_{a+b} \subset U_b \cap \Phi_b^{-1}(U_a)$ (with $ab \geq 0$) we have $\Phi_a \circ \Phi_b(p) = \Phi_{a+b}(p)$.*
The one-parameter group Φ_a of local diffeomorphisms is the flow of the vector field X if

$$\partial_a(\Phi_a^*(f)(p))|_{a=0} = X(f)(p)$$

for any $f \in \mathfrak{F}$.

Definition 6.5 *Given an evolution vector field V_F, we say that V_F (or its generator F)* admits characteristics *if there exist suitable smooth functions $h^1, ..., h^n \in \mathfrak{F}$ such that the vector field*

$$\tilde{V}_F = V_F - \sum_i h^i D_i,$$

admits a flow on $J^\infty(M,N)$.

If we restrict to the scalar case ($N = \mathbb{R}$), discussed in Section 6.2.1, the following Theorem provides a complete characterization of evolution vector fields admitting characteristics.

Theorem 6.6 *An evolution vector field V_F on $J^\infty(M,\mathbb{R})$ with generator F admits characteristics if an only if $F \in \mathfrak{F}_1$.*

Proof. The proof that any $F \in \mathfrak{F}_1$ admits characteristic flow is given in Section 6.2.1. The proof of the converse can be found in [114]. ■

Remark 6.7 *Theorem 6.6 does not hold if, instead of requiring that $\tilde{V}_F$ admits a flow on the whole $J^\infty(M,\mathbb{R})$, we restrict to a submanifold of $J^\infty(M,\mathbb{R})$. For example if we consider $M = \mathbb{R}^2$ with coordinates (x,y) and $F = u_{xy}$, Theorem 6.6 ensures that $F = u_{xy}$ does not admit characteristics on $J^\infty(\mathbb{R}^2,\mathbb{R})$ but, considering the canonical submanifold $\mathcal{E} \subset J^\infty(\mathbb{R}^2,\mathbb{R})$ generated by the equation $u_{yy} = 0$ and its differential consequences, it is easy to prove that $V_F \in T\mathcal{E}$ and that V_F admits characteristics on $\mathcal{E}$.*

If we do not restrict to the scalar case the situation becomes more complex and, to the best of our knowledge, a complete theory of characteristics in $J^\infty(M,N)$ for $N \neq \mathbb{R}$ has not been developed.
Indeed in this case we can find $F \notin \mathfrak{F}_1^n$ such that V_F admits characteristics. For example if we consider $M = \mathbb{R}$ and $N = \mathbb{R}^2$ (with coordinates x and (u,v) respectively) and $F = (v_{xx}, 0) \in \mathfrak{F}_2$,

the flow of the vector field V_F is given by the following transformation

$$\begin{aligned} x_a &= x \\ u_a &= u + a v_{xx} \\ u_{x,a} &= u_x + a v_3 \\ u_{xx,a} &= u_{xx} + a v_4 \\ &\dots \\ v_a &= v \\ v_{x,a} &= v_x \\ &\dots \end{aligned}$$

In this paper, in order to deal with the general case, we propose a stronger definition of characteristics that, although imitating in some respects the scalar case, is weak enough to include many cases of interest.
Given an open subset $U \subset J^\infty(M,N)$ we denote by

$$\mathfrak{F}|_U = \bigcup_k \mathfrak{F}_k|_U$$

the set of smooth functions defined on U, that is the union of the sets of smooth functions defined on $\pi_k(U) \subset J^k(M,N)$.
Given a subalgebra $\mathfrak{G}_0 \subset \mathfrak{F}|_U$, we denote by $\mathfrak{G}_k$ the algebra generated by smooth composition of functions of the form $D^\sigma(f)$, where $f \in \mathfrak{G}_0$ and σ is a multi-index with $|\sigma| \leq k$.

Definition 6.8 *A subalgebra $\mathfrak{G}_0 \subset \mathfrak{F}|_U$ generates $\mathfrak{F}|_U$ if $x^i \in \mathfrak{G}_0$ and*

$$\mathfrak{F}|_U = \bigcup_k \mathfrak{G}_k.$$

Definition 6.9 *An evolution vector field V_F with generator F admits strong characteristics if for any point $p \in J^\infty(M,N)$ there exists an open neighborhood $U \subset J^\infty(M,N)$ of p, a finitely generated subalgebra $\mathfrak{G}_0$ of $\mathfrak{F}|_U$ generating $\mathfrak{F}|_U$ and $g^1, ..., g^n \in \mathfrak{F}$ such that the vector field $\bar{V}_F = V_F - \sum_i g^i D_i$ satisfies*

$$\bar{V}_F(\mathfrak{G}_0) \subset \mathfrak{G}_0.$$

In the scalar case an evolution vector field V_F admits characteristics if and only if V_F admits strong characteristics: indeed in this case $\bar{V}_F(\mathfrak{F}_1) \subset \mathfrak{F}_1$. Moreover, if we consider the evolution vector field V_F of the previous example (with generator $F = (v_{xx}, 0)$), it is easy to check that V_F admits strong characteristics. In fact the subalgebra $\mathfrak{G}_0$ generated by x, u, v, v_x, v_{xx} is such that $V_F(\mathfrak{G}_0) \subset \mathfrak{G}_0$. Actually we do not know any example of evolution vector field admitting characteristics which are not strong characteristics.

Remark 6.10 *In Definition 6.9 we consider a general subalgebra $\mathfrak{G}_0$ generating $\mathfrak{F}$ instead of restricting to the case $\mathfrak{G}_0 = \mathfrak{F}_k$ for some $k \in \mathbb{N}$. This is a crucial point because, if $N = \mathbb{R}^n$ (with $n > 1$), the condition $\bar{V}_F(\mathfrak{F}_k) \subset \mathfrak{F}_k$ implies $\bar{V}_F(\mathfrak{F}_0) \subset \mathfrak{F}_0$ (see [114]) and the vector fields $\bar{V}_F$ satisfying $\bar{V}_F(\mathfrak{F}_k) \subset \mathfrak{F}_k$ turn out to be tangent to the prolongations of infinitesimal transformations in $J^0(M,N)$.*
A well-known consequence of this fact is that, in the vector case, the only infinitesimal symmetries of a PDE which can be defined using finite jet spaces $J^k(M,N)$ are Lie-point symmetries. On the other hand, if we allow $\mathfrak{G}_0$ to be a general subalgebra generating $\mathfrak{F}$, we obtain a larger and non-trivial class of evolution vector fields admitting strong characteristics.

Theorem 6.11 *With the notations of Definition 6.9, if an evolution vector field admits strong characteristics then it admits characteristics, and $\bar{V}_F$ is its characteristic vector field.*

Proof. The vector field $\bar{V}_F$ admits flow on the space of functions $\mathfrak{G}_0$ since $\mathfrak{G}_0$ is finite dimensional. In order to show that $\bar{V}_F$ admits flow on all $\mathfrak{F}|_U$ and so (since U depends on a generic point) on $\mathfrak{F}$, we prove by induction that $\bar{V}_F(\mathfrak{G}_k) \subset \mathfrak{G}_k$.
By hypothesis $\bar{V}_F(\mathfrak{G}_0) \subset \mathfrak{G}_0$. Suppose that $\bar{V}_F(\mathfrak{G}_{k-1}) \subset \mathfrak{G}_{k-1}$. Since $\bar{V}_F$ is a symmetry of the Cartan distribution, there exist some functions $h^j_i \in \mathfrak{F}$ such that

$$[\bar{V}_F, D_i] = \sum_j h^j_i D_j$$

where $h^i_j \in \mathfrak{G}_1$, being $\bar{V}_F(\mathfrak{G}_0) \subset \mathfrak{G}_0$ and $x^i \in \mathfrak{G}_0$.
We recall that $\mathfrak{G}_k$ is generated by functions of the form $D_i(g)$ with $g \in \mathfrak{G}_{k-1}$. So

$$\bar{V}_F(D_i(g)) = D_i(\bar{V}_F(g)) + \sum_j h^j_i D_j(g) \in \mathfrak{G}_k$$

since $\bar{V}_F(g) \in \mathfrak{G}_{k-1}$ and $h^i_j \in \mathfrak{G}_1$. Hence $\bar{V}_F$ admits flow on $\mathfrak{G}_k$ and the flow on $\mathfrak{G}_k$ is compatible with the flow on $\mathfrak{G}_{k-1}$, since $\mathfrak{G}_{k-1} \subset \mathfrak{G}_k$.
The problem of the previous construction is that in general the domain U_k of the flow in $\mathfrak{G}_k$ depends on k. This means that, if we denote with $P_{h,k}$ the natural projection of $\mathfrak{G}_h$ on $\mathfrak{G}_k$ with $h > k$, it might happen that $P^{-1}_{h,k}(U_k) \neq U_h$. But this is not actually the case. Indeed since $\mathfrak{G}_0$ generates $\mathfrak{F}|_U$, then $\mathfrak{F}_0|_U \subset \mathfrak{G}_0$ and so $\mathfrak{F}_k|_U \subset \mathfrak{G}_k$. In particular $u^i_\sigma \in \mathfrak{G}_k$ if $|\sigma| \leq k$. But by Remark 6.25 and Corollary 6.26 (see Appendix) $\Phi_a(u^i_\sigma)$ is polynomial in $u^j_{\sigma'}$ for $|\sigma'|$ sufficiently large. This means that $u^j_{\sigma'}$ can vary in all $\mathbb{R}$ and so the domain of definition of Φ_a in $U \subset J^\infty(M,N)$ is not empty and is of the form $U' = \pi^{-1}_{\infty,k}(U_k)$ for k sufficiently large. Since U' is an open subset of $J^\infty(M,N)$ this concludes the proof. ■

Definition 6.12 *Let $y^1, ..., y^l, ... \in \mathfrak{F}|_U$ be a sequence of functions defined in an open set U. We say that $Y = \{y^i\}|_{i\in\mathbb{N}}$ is a local adapted coordinate system with respect to a subalgebra $\mathfrak{G}_0$ generating $\mathfrak{F}|_U$, if there exists a sequence $k_1, ..., k_l, ... \in \mathbb{N}$, with $k_i < k_{i+1}$, such that $y^1, ..., y^{k_i}$ is a coordinate system for $\mathfrak{G}_i$.*

Remark 6.13 *The flow of a vector field with strong characteristics solves a triangular infinite dimensional system of ODEs. Indeed if we consider an adapted coordinate system with respect to a subalgebra $\mathfrak{G}_0$ we have $\bar{V}_F(y^i) = f(y^1, .., y^{k_1})$ for $i = 1, ..., k_1$, $\bar{V}_F(y^i) = f(y^1, .., y^{k_2})$ for $i = k_1 + 1, ..., k_2$ and so on. So we can start by solving the system for $i = 1, ..., k_1$ solving the system for $i = k_1 + 1, ..., k_2$, since the system is of triangular type.*

The main trouble when working with a family of evolution vector fields admitting characteristic flows is that the sum or the Lie brackets of two of them usually do not admit characteristic flow. In order to overcome this problem we give the following Definition.

Definition 6.14 *A set of evolution vector fields $V_{F_1}, ..., V_{F_s}$ with strong characteristics admits a* common filtration *if $\forall p \in J^\infty(M,N)$ there exist a neighborhood U of p and a subalgebra $\mathfrak{G}_0 \subset \mathfrak{F}|_U$ such that $\mathfrak{G}_0$ is the subalgebra required in Definition 6.9 for $\bar{V}_{F_1}, ..., \bar{V}_{F_s}$.*

If $F_1, ..., F_s$, correspond to evolution vector fields with strong characteristics admitting a common filtration, then also $cF_i + dF_j$ (where $c, d \in \mathbb{R}$) and $[F_i, F_j]$ correspond to vector fields with strong characteristics. Furthermore $cF_i + dF_j$ and $[F_i, F_j]$ admit the same common filtration of $F_1, ..., F_s$.

6.3 Building submanifolds of $J^\infty(M,N)$

In this section we propose a construction of particular submanifolds of $J^\infty(M,N)$ using the characteristics flow introduced before. The idea is the following: we have a set of evolution vector fields $V_{F_1},...,V_{F_r}$ and $V_{G_1},...,V_{G_h}$ and a canonical submanifold $\mathcal{H}$ such that $V_{F_i}\in T\mathcal{H}$. Then we want to construct a canonical submanifold $\mathcal{K}$ such that $\mathcal{H}\subset\mathcal{K}$ and $V_{F_1},...,V_{F_r},V_{G_1},...,V_{G_h}\in T\mathcal{K}$. We provide a sufficient condition for the existence of such a $\mathcal{K}$ when $G_1,...,G_h$ admits characteristics flow and $F_1,...,F_r,G_1,...,G_h$ form a finite dimensional Lie algebra. Furthermore the construction of the manifold $\mathcal{K}$ from the manifold $\mathcal{H}$ is explicit and the method of construction will be useful in the applications to SPDEs.

Definition 6.15 *Let $\mathcal{H}\subset J^\infty(M,N)$ be a submanifold and U be an open neighborhood of $p\in\mathcal{H}$. Given a sequence of independent functions $f^i\in\mathfrak{F}|_U$ $(i\in\mathbb{N})$ such that $\mathcal{H}\cap U$ is the annihilator of f^i, we say that a distributions $\Delta=\operatorname{span}\{V_{G_1},\dots,V_{G_h}\}$ is transversal to $\mathcal{H}$ in U if there exist $r_1,\dots,r_h$ such that the matrix $(\bar{V}_{G_i}(f^{r_j}))|_{i,j=1,\dots,h}$ has maximal rank in U. In the following the sequence f^i will be chosen so that $r_j=j$ and f^i is a local coordinate system adapted with respect to the filtration $\mathfrak{G}_k$ for k sufficiently large.*

Lemma 6.16 *Let $G_1,...,G_h$ be a subalgebra of $\mathfrak{F}^n$ admitting strong characteristics and a common filtration. Let $\Phi^i_{a^i}$ be the characteristic flow of G_i and $\mathcal{H}$ be a canonical finite dimensional submanifold of $J^\infty(M,N)$ such that the distribution $T\mathcal{H}\oplus\operatorname{span}\{V_{G_1},...,V_{G_h}\}$ has constant rank and the distribution $\Delta=\operatorname{span}\{V_{G_1},\dots,V_{G_h}\}$ is transversal to $\mathcal{H}$. Then there exists a suitable neighborhood of the origin $\mathcal{V}\subset\mathbb{R}^h$ such that*

$$\mathcal{K}=\bigcup_{(a^1,\dots,a^h)\in\mathcal{V}}\Phi^h_{a^h}(...(\Phi^1_{a^1}(\mathcal{H}))...)\tag{6.5}$$

is a finite dimensional submanifold of $J^\infty(M,N)$.

Proof. In the following, for the sake of clarity, we write

$$\mathbf{\Phi}^*_\alpha(f)=\Phi^{1*}_{a^1}(...\Phi^{h*}_{a^h}(f)...),$$

where $\alpha=(a^1,...,a^h)\in\mathbb{R}^h$. Given a sequence of independent functions f^i $(i\in\mathbb{N})$ such that $\mathcal{H}$ is the annihilator of f^i, for any point $p\in\mathcal{H}$ there exists a neighborhood U such that the matrix $(\bar{V}_{G_i}(f^j))|_{i,j=1,\dots,h}$ has maximal rank in U. Therefore, considering the submanifold $\tilde{\mathcal{H}}$ defined as the annihilator of the functions $f^i\in\mathfrak{F}|_U$ $(i=1,\dots,h)$, the equations

$$\mathbf{\Phi}^*_\alpha(f^i)=0\qquad i=1,\dots h\tag{6.6}$$

can be solved with respect to α. This means that, possibly restricting the open set U, there exist a smooth function $A(p)=(A^1(p),...,A^h(p))$ defined on U such that $\mathbf{\Phi}^*_{A(p)}(f^i)(p)=0$ (for $i=1,\dots h$), i.e. $\Phi^h_{A^h}(...(\Phi^1_{A^1}(p))...)\in\tilde{\mathcal{H}}$. In the following we prove that $\mathcal{K}$ is the annihilator of the functions

$$K^j(p)=\mathbf{\Phi}^*_{A(p)}(f^j)(p),\qquad j>h\tag{6.7}$$

and, since K^j are independent and adapted with respect to the filtration $\mathfrak{G}_k$ for k sufficiently large, $\mathcal{K}$ is a submanifold of $J^\infty(M,N)$.

We start by proving that if $p_0\in\mathcal{K}\cap U$, then $K^j(p_0)=0$ (for $j>h$). Indeed, if $p_0\in\mathcal{K}\cap U$, the point p_0 can be reached starting from $p\in\mathcal{H}$ by means of composition of suitable flows $\Phi^i_{a^i}$. On the other hand, for any $p_0\in\mathcal{K}\cap U$, there exists $A(p_0)=(A^1,\dots,A^h)$ such that $\Phi^h_{A^h}(...\Phi^1_{A^1}(p_0)...)\in\tilde{\mathcal{H}}$.

Since $\mathcal{H} \subset \tilde{\mathcal{H}}$ and the transversality condition ensures that equation (6.6) has a unique solution, we have $\Phi^h_{A^h}(...\Phi^1_{A^1}(p_0)...) \in \mathcal{H}$. Therefore

$$K^j(p_0) = \mathbf{\Phi}^*_{A(p_0)}(f^j)(p_0) = f^j(\Phi^h_{A^h}(...\Phi^1_{A^1}(p_0)...) = 0$$

for any j and in particular for $j > h$. In order to prove the other inclusion we have to ensure that p_0 can be reached starting from a point $p \in \mathcal{H}$ by means of the flows $\Phi^i_{a^i}$. Given $p \in \tilde{\mathcal{H}}$ such that $\Phi^h_{A^h}(...(\Phi^1_{A^1}(p_0))...) = p$, the definition of $A(p_0)$ ensures that $f^i(p) = 0$ for $i = 1, \dots h$ whereas by hypothesis we have

$$K^j(p_0) = \mathbf{\Phi}^*_{A(p_0)}(f^j)(p_0) = f^j(p) = 0 \qquad j > h.$$

Hence $f^i(p) = 0 \ \forall i \in \mathbb{N}$ and $p \in \mathcal{H}$. ∎

Lemma 6.17 *In the hypotheses and with the notations of Lemma 6.16,* $\bar{V}_{G_j} \in T\mathcal{K}$ *and* $D_i \in T\mathcal{K}$.

Proof. We recall that a vector field $V \in T\mathcal{K}$ if and only if $V(K^j) = 0$, where K^j are given by (6.7). Since for any j (with $j > h$) there exists a suitable k such that $f^1, \dots f^j \in \mathfrak{G}_k$, it is possible to chose as coordinates in $\mathcal{K} \cap U \cap \mathfrak{G}_k$ the functions f^i $(i = 1, \dots h)$ and some functions $y^1, \dots, y^r$ (with $r = \dim(\mathfrak{G}_k) - h$) such that $\bar{V}_{G_i}(y^l) = 0$. In particular, for any $j > h$, there exists a smooth function L^j such that

$$f^j(p) = L^j(f^1(p), ..., f^h(p), y^1(p), ...y^r(p)).$$

Since $f^1, \dots, f^h$ vanish on $\tilde{\mathcal{H}}$ we have

$$K^j(p) = L^j(0, ..., 0, y^1(p), ..., y^r(p)),$$

and so $\bar{V}_{G_i}(K^j) = \bar{V}_{G_i}(L^j(0, ..., 0, y^1(p), ..., y^r(p))) = 0$.
In order to prove that $D_i \in T\mathcal{K}$, we consider

$$D^\alpha_i = \mathbf{\Phi}^*_\alpha(D_i).$$

By definition, since $D_i \in T\mathcal{H}$, we have that $D^\alpha_i \in T\mathcal{K}$ and, by Theorem 6.24 (see Appendix), there exist smooth functions $C^i_j(\alpha, p)$ such that

$$D^\alpha_i = \sum_j C^j_i(\alpha, p) D_j.$$

Moreover, since $\Phi^i_{a^i}$ are diffeomorphisms, span$\{D^\alpha_1, ..., D^\alpha_m\}$ and span$\{D_1, ..., D_m\}$ have the same dimension. Hence the matrix C^i_j is invertible for any α, ensuring that $D_i \in T\mathcal{K}$. ∎

Remark 6.18 *The functions* K^j *defined by* (6.7) *are a set of independent invariants for the vector fields* $\bar{V}_{G_i}$. *Furthermore, since* $\mathcal{K}$ *is finite dimensional, it is possible to add a finite number of functions* z^i *such that* (z^i, K^j) *form an adapted coordinate system with respect to the filtration* $\mathfrak{G}_k$ *for* k *sufficiently large.*

Theorem 6.19 *In the hypotheses and with the notations of Lemma 6.16, let* V_F *be an evolution vector field such that* $V_F \in T\mathcal{H}$, $\dim(\text{span}\{V_F, V_{G_1}, ..., V_{G_h}\}) = h + 1$ *and*

$$[G_i, F] = \mu_i F + \sum_k \lambda^k_i G_k \qquad \mu_i, \lambda^j_i \in \mathbb{R}$$

Then $V_F \in T\mathcal{K}$.

Proof. Given the $(m+h+1)$-dimensional distribution

$$\Delta := \mathrm{span}\{D_1, ..., D_m, V_{G_1}, ..., V_{G_h}, V_F\},$$

we have $\Delta|_{\mathcal{H}} \subseteq T\mathcal{H} \oplus \mathrm{span}\{V_{G_1}, \dots, V_{G_h}\} \subseteq T\mathcal{K}|_{\mathcal{H}}$ and, by hypothesis, $[\bar{V}_{G_i}, \Delta] \subseteq \Delta$. If we prove that

$$\Phi^i_{a*}(\Delta) = \Delta,$$

we have $\Delta|_{\mathcal{K}} \subset T\mathcal{K}$ and, in particular, $V_F \in T\mathcal{K}$.
Considering the coordinate system z^i, K^j of Remark 6.18 we can suppose, possibly relabeling some invariant z^i with K^j for some j, that we have exactly h coordinates z^i. Eliminating some element of the form ∂_{K^j}, the sequence $V_F, \bar{V}_{G_j}, D_k, \partial_{K^l}$ forms a basis of $TJ^\infty(M,N)$ and for any vector field $X \in TJ^\infty(M,N)$ there exist suitable functions b, c^i, d^j, e^l depending on a and $p \in U_a$ such that

$$X_a := \Phi^i_{a*}(X) = b(a,p)V_F + \sum_{j,k,l} c^j(a,p)\bar{V}_{G_j} + d^k(a,p)D_k + e^l(a,p)\partial_{K^l}.$$

From the definition of X_a and using $[\bar{V}_{G_i}, \Delta] \subset \Delta$ and $[\bar{V}_{G_i}, \partial_{K^l}] \in \Delta$, we obtain that the functions e^l solve the equations

$$\partial_a(e^l) = -\bar{V}_{G_i}(e^l).$$

Moreover, since $X_0 = X \in \Delta$, we have $e^l(0,p) = 0$ and, from the previous equation, we get $e^l(a,p) = 0$ for any a, which ensures $X_a \in \Delta$ for any a. ■

Remark 6.20 *Theorem 6.19 still holds if we consider r functions $F_i \in \mathfrak{F}$ such that* $\dim(\mathrm{span}\{V_{F_1}, ..., V_{F_r}, V_{G_1}, ..., V_{G_h}\}) = r + h$*, $V_{F_i} \in T\mathcal{H}$ for any $i = 1, ..., r$ and*

$$[G_i, F_j] = \sum_{k,l} (\mu^k_{i,j} F_k + \lambda^l_{i,j} G_l)$$

for some constants $\lambda^l_{i,j}, \mu^k_{i,j} \in \mathbb{R}$.

Theorem 6.21 *Under the hypotheses and the notations of Lemma 6.16, if F, G_i are real analytic, $\mathcal{H}$ is defined by real analytic functions and, denoting by $L = \langle F, G_1, \dots, G_h \rangle$ the Lie algebra generated by F and G_i, we have*

$$L|_{\mathcal{H}} \subset T\mathcal{H} \oplus \mathrm{span}\{V_{G_1}, ..., V_{G_h}\},$$

then $V_F \in T\mathcal{K}$.

Proof. We note that the functions K^i defined by (6.7) are real analytic if the vector fields $\bar{V}_{G_i}$ and the submanifold $\mathcal{H}$ are real analytic.
The vector field V_F is in $T\mathcal{K}$, if for any $p_0 \in \mathcal{K}$ and any K^i, we have

$$V_F(K^i)(p_0) = 0.$$

We know that if $p_0 \in \mathcal{K}$ there exists $\alpha = (a^1, ..., a^h) \in \mathbb{R}^h$ and $p_1 \in \mathcal{H}$ such that

$$p_0 = \Phi^h_{a^h}(...(\Phi^1_{a^1}(p_1))...).$$

Moreover, since K^i are invariants of $\Phi^j_{a^j}$ we have

$$\begin{aligned} V_F(K^i)(p_0) &= \mathbf{\Phi}^*_\alpha(V_F(K^i))(p_1) \\ &= \mathbf{\Phi}^*_\alpha(V_F)(K^i)(p_1). \end{aligned}$$

Since the previous expression is real analytic it is sufficient to prove that any derivative of any order with respect to a^i evaluated in $(a^1, ..., a^h) = 0$ is zero. It is easy to verify that

$$\partial^{k_1}_{a^1}(...\partial^{k_h}_{a^h}(\tilde{\boldsymbol{\Phi}}^*_\alpha(V_F))...)|_{\alpha=0} = \bar{V}^{k_h}_{G_h}(...(\bar{V}^{k_1}_{G_1}(V_F))...),$$

where we use the notation

$$\bar{V}^k_{G_i}(X) = \underbrace{[\bar{V}_{G_i}, [...[\bar{V}_{G_i}, X]...]]}_{k \text{ times}}.$$

By hypothesis $\bar{V}^{k_h}_{G_h}(...(\bar{V}^{k_1}_{G_1}(V_F))...) \in T\mathcal{K}|_{\mathcal{H}}$ and so for any $p_1 \in \mathcal{H}$ and any K^i

$$\bar{V}^{k_1}_{G_1}(...(\bar{V}^{k_{h'}}_{G_{h'}}(V_F))...)(K^i)(p_1) = 0.$$

■

Remark 6.22 *If $\bar{V}_{G_i}, V_F$ are real analytic and $\mathcal{H}$ is defined by real analytic equations, Theorem 6.19 implies Theorem 6.21. On the other hand Theorem 6.19 turns out to be very useful when we consider smooth (not analytic) invariant manifolds $\mathcal{H}$.*

Remark 6.23 *It is important to note that Theorems 6.19 and 6.21 hold also if $\mathcal{H}$ is a manifold with boundary. In this case if $V_{G_1}, ..., V_{G_h} \in T(\partial\mathcal{H})$ we obtain that $\mathcal{K}$ is also a local manifold with boundary.*

6.4 Appendix

In this section we discuss the behavior of the Cartan distribution $\mathcal{C}$ under the action of the characteristic flow Φ_a associated with an evolution vector field V_G. An important consequence of the following Theorem is that $\Phi^*_a(u^i_\sigma)$ is a polynomial function with respect to the variable $u^k_{\sigma'}$ if $|\sigma'|$ is sufficiently large.

Theorem 6.24 *Let V_G be an evolution vector field admitting characteristics and let Φ_a be the corresponding characteristic flow. If A is the $n \times n$ matrix*

$$A = (A^j_i) := (D_i(\Phi^*_a(x^j)))|_{i,j},$$

and $B = (B^i_j)$ is the inverse matrix of A, then

$$\Phi^*_a(D_i) = \sum_j B^j_i D_j \tag{6.8}$$

and, for any $f \in \mathfrak{F}$, we have

$$\Phi^*_a(D_i(f)) = \sum_j B^j_i D_j(\Phi^*_a(f)). \tag{6.9}$$

Proof. Let $\tilde{V}_G = V_G - \sum_i h^i D_i$ be the characteristic vector field of V_G. Since

$$[\tilde{V}_G, D_i] = \sum_j D_i(h^j) D_j,$$

the vector field $D^a_i = \Phi^*_a(D_i)$ solves the equation

$$\partial_a(D^a_i) = \Phi^*_a([\tilde{V}_G, D_i]) = \sum_j D^a_i(\Phi^*_a(h^j)) D^a_j. \tag{6.10}$$

In order to prove (6.8) we show that the vector field $\tilde{D}_i^a := \sum_j B_i^j D_j$ solves equation (6.10) as well. We start by computing

$$\partial_a(A_i^j) = \partial_a(D_i(\Phi_a^*(x^j))) = D_i(\Phi_a^*(\tilde{V}_G(x^j))) = -D_i(\Phi_a^*(h^j)).$$

Since $B = A^{-1}$ the formula for the derivative of the inverse matrix gives

$$\partial_a(B) = -B \cdot \partial_a(A) \cdot B.$$

This means that

$$\partial_a(B_j^i) = \sum_{k,r} B_j^k(D_k(\Phi_a^*(h^r)))B_r^i$$

and we get

$$\begin{aligned}
\partial_a(\tilde{D}_j^a) &= \sum_i \partial_a(B_j^i)D_i \\
&= \sum_{i,k,r} B_j^k(D_k(\Phi_a^*(h^r)))B_r^i D_i \\
&= \sum_r \tilde{D}_j^a(\Phi_a^*(h^r))\tilde{D}_r^a.
\end{aligned}$$

Hence both $\tilde{D}_i^a$ and D_i^a satisfy equation (6.10) and we have $\tilde{D}_i^a = D_i^a$. ■

Remark 6.25 *It is important to note that equation* (6.8) *holds in all* $J^\infty(M,N)$ *while equation* (6.9) *holds in* $\mathfrak{F}_k$ *for k sufficiently large.*

Corollary 6.26 *Given an evolution vector field* V_G *with corresponding characteristic flow* Φ_a*, the expression* $\Phi_a^*(u_\sigma^i)$ *is a polynomial function with respect to the variable* $u_{\sigma'}^k$ *if* $|\sigma'|$ *is sufficiently large.*

Proof. If we apply Theorem 6.24 to $f = u^i$ we get $\Phi_a^*(D_k(u^i)) = \sum_j B_k^j D_j(\Phi_a^*(u^i))$. Since $D_j(\Phi_a^*(u^i))$ is a linear function with respect to the variable $u_{\sigma'}^i$ if $|\sigma'|$ is sufficiently large and $B_k^j \in \mathfrak{F}_h$ for some $h \in \mathbb{N}$, applying iteratively Theorem 6.24 we obtain the thesis. ■

Chapter 7

Finite dimensional solutions to SPDEs

In this chapter we study the link between finite dimensional solutions to SPDEs and particular finite dimensional submanifolds of $J^\infty(M,N)$. In particular we prove that with any smooth finite dimensional solution to an SPDE one can associate a canonical finite dimensional submanifold of $J^\infty(M,N)$, satisfying a special property involving the evolution vector fields which define the considered SPDE. Furthermore we exploit this equivalence to explicitly construct finite dimensional solutions to SPDEs using the characteristics introduced in Chapter 6, and we develop an algorithm to use the proposed construction. Finally we apply this algorithm to three new important examples of SPDEs coming from mathematical finance, mathematical physics and filtering theory.

7.1 Finite dimensional solutions to SPDEs and finite dimensional canonical manifolds

7.1.1 SPDEs and the geometry of $J^\infty(M,N)$

Given a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with a filtration of subsigmalgebras $\mathcal{F}_t \subset \mathcal{F}$, in the following we consider only (local) martingales which are (local) martingales with respect to the filtration $\mathcal{F}_t$. In this setting the definition of semimartingale with a spatial parameter proposed in [116] can be modified as follows.

Definition 7.1 *Let $(t,x,\omega) \mapsto U_t(x)(\omega) \in N$ be a random variable. We say that U_t is a semimartingale dependent on the parameter $x \in M$ of regularity h (in short $U_t(x)$ is a C^h semimartingale) if, for any $t \in \mathbb{R}_+, \omega \in \Omega$, the function $U_t(\cdot)(\omega) \in C^h(M,N)$ and, for any $x \in M$ and multi-index $|\sigma| \leq h$, the process $\partial^\sigma(U_t)(x)$ is an N valued semimartingale. If U_t is a C^h semimartingale for any $h>0$, we say that U_t is a C^∞ semimartingale.*

Remark 7.2 *If U_t is a C^h semimartingale, then $\partial^\sigma(U_t)$ is a $C^{h-|\sigma|}$ semimartingale for any multi-index $|\sigma| \leq h$.*

Definition 7.3 *Given $F_1,...,F_r \in \mathfrak{F}^n_h$ and r real semimartingales $S^1,...,S^r$, we say that the C^h semimartingale $U_t=(U^i_t,...,U^n_t)$ is a solution to the SPDE*

$$dU_t = F_\alpha(U_t) \circ dS^\alpha_t, \tag{7.1}$$

or simply to the SPDE associated with $F_1, ..., F_r$ and $S_1, ..., S_r$ if and only if, for any $x \in M$,

$$U_t^i(x) - U_0^i(x) = \int_0^t F_\alpha(x, U_s(x), ..., \partial^\sigma(U_s)(x)) \circ dS_s^\alpha. \tag{7.2}$$

Remark 7.4 *We can extend Definition 7.3 to more general SPDEs. For example consider r functionals $\Psi_\alpha : C^\infty(M, N) \to \mathbb{R}$. If we suppose that $\Psi_\alpha(f)$ depend only on the values of the function f in some compact subset $\mathfrak{K} \subset M$ and that Ψ_α are smooth with respect to the norm of $C^l(K, N)$ (for some $l \geq 0$) it is easy to prove, using the Itô formula for Hilbert space valued semimartingales, that $\Psi_\alpha(U_t)$ are real semimartingales. In this setting we can modify equation (7.2) in the following way*

$$U_t^i(x) - U_0^i(x) = \int_0^t \Psi_\alpha(U_s) F_\alpha(x, U_s(x), ..., \partial^\sigma(U_s)(x)) \circ dS_s^\alpha \tag{7.3}$$

and we write

$$dU_t = \Psi_\alpha(U_t) F_\alpha(U_t) \circ dS_t^\alpha.$$

We call the SPDEs of the form (7.3) weakly local SPDEs.

In order to reformulate Definition 7.3 in terms of a standard SDE in $J^\infty(M, N)$, we consider a C^∞ semimartingale depending on the spatial parameter $x \in M$ and we define a semimartingale $\mathbf{U}_t(x)$ taking values in J_x^∞ (the fiber of $x \in M$ in the bundle $J^\infty(M, N)$) in the following way

$$u_\sigma^i(\mathbf{U}_t(x)) = \partial^\sigma(U_t^i)(x).$$

Obviously $\mathbf{U}$ is a semimartingale taking values in $J^\infty(M, N)$ for any $x \in M$, i.e. $f(x, \mathbf{U}_t(x))$ is a real semimartingale for any $f \in \mathcal{F}$. Furthermore, fixing $t \in \mathbb{R}_+$ and $\omega \in \Omega$, the section $x \mapsto \mathbf{U}_t(x)(\omega)$ is an integral section of the Cartan distribution $\mathcal{C}$, so that

$$D_{x^i} \in T\mathcal{K}^{\mathbf{U}_t(\omega)}$$

for any $i = 1, ..., m$. The process $\mathbf{U}_t$ dependent on the parameter $x \in M$ is called the lifting of U_t to $J^\infty(M, N)$.
Conversely, if $\mathbf{P}_t(x)$ is a process dependent on $x \in M$ and taking values in J_x^∞ which is a semimartingale in $J^\infty(M, N)$ and, for $t \in \mathbb{R}_+$ and $\omega \in \Omega$, $\mathbf{P}_t(\omega)$ is an integral manifold of the Cartan distribution, then there exists a function $U^{\mathbf{P}}$ such that the lifting $\mathbf{U}^{\mathbf{P}}$ of $U^{\mathbf{P}}$ to $J^\infty(M, N)$ is exactly $\mathbf{P}$. This assertion can be proved using Theorem 6.3 and the fact that $U_t^{\mathbf{P},i}(x) = u^i \circ \mathbf{P}_t(x)$.

If $\mathcal{M}$ is a smooth manifold and $Y_1, ..., Y_r$ are r vector fields on $\mathcal{M}$, the semimartingale X on $\mathcal{M}$ is a solution to the SDE associated with $Y_1, ..., Y_r$ and the semimartingales $S^1, ..., S^r$ if and only if, for any $f \in C^\infty(\mathcal{M})$,

$$f(X_t) - f(X_0) = \int_0^t Y_\alpha(f)(X_s) \circ dS_s^\alpha.$$

In the following, if X is a solution to the SDE associated with $Y_1, ..., Y_r$ and $S^1, ..., S^r$, we write

$$dX_t = Y_\alpha \circ dS_t^\alpha.$$

Theorem 7.5 *The C^∞ semimartingale U is a solution to the SPDE associated with $F_1, ..., F_r$ and $S^1, ..., S^r$ if and only if, for any $x \in M$, $\mathbf{U}_t(x)$ is a solution to the SDE associated with $V_{F_1}, ..., V_{F_r}$ and $S^1, ..., S^r$, i.e., for any $f \in \mathcal{F}$,*

$$f(x, \mathbf{U}_t(x)) - f(x, \mathbf{U}_0(x)) = \int_0^t V_{F_\alpha}(f)(x, \mathbf{U}_s(x)) \circ dS^\alpha. \tag{7.4}$$

In order to prove Theorem 7.5 we recall the following lemma.

Lemma 7.6 *If $G_t(x)$ is a C^h semimartingale and S is a real valued semimartingale, then*

$$\int_0^t G_s(x) \circ dS_s,$$

is a $C^{h-1}(M,N)$ semimartingale and, for any multi-index $|\sigma| < h$,

$$\partial^\sigma \left(\int_0^t G_s(x) \circ dS_s \right) = \int_0^t \partial^\sigma(G_s)(x) \circ dS_s.$$

Proof. The proof is given in [116] Exercise 3.1.6. ∎

Proof of Theorem 7.5. If $\mathbf{U}_t(x)$ is the solution to the SDE associated with $V_{F_1}, ..., V_{F_r}$ and $S^1, ..., S^r$, then $U_t(x) = u \circ \mathbf{U}_t(x)$ is solution to the SPDE (7.1), since equation (7.4) becomes equation (7.2) if we choose $f = u^i$.
Conversely, if $U_t(x)$ is a solution to the SPDE (7.1), we have

$$\begin{aligned} \partial_{x^i}(F_\alpha(x, U_t(x), U_{\sigma,t}(x))) &= \partial^i_x(F_\alpha)(x, U_t(x), U_{\sigma,t}(x)) + \sum_\sigma \partial^{\sigma+1_i}(U_t)(x) \partial_{u^i_\sigma}(F_\alpha)(x, U_t(x), U_{\sigma,t}(x)) \\ &= D_{x^i}(F_\alpha)(x, U_t(x), U_{\sigma,t}(x)). \end{aligned}$$

By induction it is possible to prove

$$\partial^\sigma(F_\alpha(x, U_t(x), U_{\sigma,t}(x))) = D^\sigma(F_\alpha)(x, U_t(x), U_{\sigma,t}(x))$$

and by Lemma 7.6 we find

$$\partial^\sigma(U_t)(x) = \int_0^t D^\sigma(F_\alpha)(x, U_s(x), U_{\sigma,s}(x)) \circ dS^\alpha_t.$$

Using the Itô formula for x fixed and the previous equation we obtain the thesis. ∎

Remark 7.7 *It is possible to extend Theorem 7.5 to the case of more general SPDEs as described in Remark 7.4. In this case the SDE solved by $\mathbf{U}_t(x)$ is*

$$f(x, \mathbf{U}_t(x)) - f(x, \mathbf{U}_0(x)) = \int_0^t \Psi_\alpha(U_s) V_{F_\alpha}(f)(x, \mathbf{U}_s(x)) \circ dS^\alpha_s.$$

This SDE depends not only on $\mathbf{U}_t(x)$ but also on all the functions U_t, since the functional Ψ_α is, in general, a non-local functional.

In the following we discuss the relationship between the definition of solution to an SPDE introduced by Definition 7.3 and the usual definition given in terms of the theory of martingales taking values in Hilbert (or Banach) spaces (see, e.g. [48]).

We start by considering the Itô reformulation of equation (7.2), which is simpler to use in the Hilbert space setting. By Theorem 7.5 and using the relationship between Stratonovich and Itô integral, we have that U_t solves the SPDE associated with $F_1, ..., F_r$ and $S^1, ..., S^r$ in the sense of Definition 7.3 if and only if

$$\begin{aligned} U^i_t(x) - U^i_0(x) &= \int_0^t F^i_\alpha(x, U_s(x), ..., \partial^\sigma(U_s)(x)) dS^\alpha_s + \\ &+ \frac{1}{2} \int_0^t V_{F_\alpha}(F^i_\beta)(x, U_t(x), ..., \partial^\tau(U_t)(x)) d[S^\alpha, S^\beta]. \end{aligned}$$

Definition 7.8 *Let H be a Hilbert space containing some subset of smooth functions defined on M. If $U^i_t(x) \in H$, we say that $U^1,...,U^n$ is a strong solution to the SPDE associated with $F_1,...,F_r$ and $S^1,...,S^r$ if $F^i_\alpha(x,U_t,...,\partial^\sigma(U_t))$ and $V_{F_\alpha}(F^i_\beta)(x,U_t,...,\partial^\tau(U_t))$ are locally bounded processes in H and*

$$U^i_t - U^i_0 = \int_0^t F^i_\alpha(x,U_s,...,\partial^\sigma(U_s))dS^\alpha_s + \frac{1}{2}\int_0^t V_{F_\alpha}(F^i_\beta)(x,U_t,...,\partial^\tau(U_t))d[S^\alpha,S^\beta]_s, \tag{7.5}$$

where the integrals are Itô integrals in H. We say that U_t is a weak solution to the SPDE associated with $F_1,...,F_r$ and $S^1,...,S^r$ if, for any $\xi \in V \subset H$, where V is a suitable subspace of H which separates the points of H, $\langle \xi, F^i_\alpha(x,U_t,...,\partial^\sigma(U_t))\rangle$ and $\langle \xi, V_{F_\beta}(F^i_\alpha)(x,U_t,...,\partial^\tau(U_t))\rangle$ are real locally bounded processes and the following equality holds

$$\langle \xi, U^i_t - U^i_0\rangle = \int_0^t \langle \xi, F^i_\alpha(x,U_s,...,\partial^\sigma(U_s))\rangle dS^\alpha_s + \frac{1}{2}\int_0^t \langle \xi, V_{F_\alpha}(F^i_\beta)(x,U_t,...,\partial^\tau(U_t))\rangle d[S^\alpha,S^\beta]_s, \tag{7.6}$$

(here the integrals are usual $\mathbb{R}$ Itô integrals).

In general, it is not easy to find the relationship between the two notions of solution proposed in Definition 7.3 and in Definition 7.8. For this reason we need to introduce an additional hypothesis (which is satisfied by the usual Hilbert spaces considered in SPDEs theory). Given a smooth function $f \in C^\infty_0(M)$ with compact support, we define a linear functional

$$\begin{array}{rccc} l_f: & C^0(M) & \longrightarrow & \mathbb{R} \\ & g & \longmapsto & l_f(g) := \int_M f(x)g(x)dx. \end{array}$$

We say that the Hilbert space H satisfies the hypotheses L if

- there exists a subset of $\mathcal{L} \subset C^\infty_0(M)$ such that, for any $f \in \mathcal{L}$, the functional $l_f : H \cap C^\infty(M) \to \mathbb{R}$ can be extended in a unique continuous way to all H;

- the functionals of the form l_f for $f \in \mathcal{L}$ separate the points of H and of the Fréchet space $C^0(M)$.

For example, Hilbert spaces satisfying the hypothesis L are Sobolev spaces $H^r_w(M)$ of function weakly derivable r times and whose weak derivatives are square integrable with respect to the measure $w(x)dx$, where $w(x)$ is a positive continuous function $w : M\backslash\{x_1,...,x_l\} \to \mathbb{R}$ and $x_1,...,x_l \in M$. In this case the set $\mathcal{L}$ is formed by the functions $f \in C^\infty_0(M)$ which are identically zero in some neighborhood of $x_1,...,x_l$.

Proposition 7.9 *Let H be a Hilbert space satisfying the hypothesis L. If U^i_t is a C^∞ semimartingale and U^i_t, $F^i_\alpha(x,U_s,...,\partial^\sigma(U_s))$ and $V_{F_\alpha}(F^i_\beta)(x,U_t,...,\partial^\tau(U_t))$ are locally bounded processes in H, then the definition of solutions to an SPDE given in Definition 7.3 and the two definitions given in Definition 7.8 are equivalent.*

Proof. We prove that Definition 7.3 is equivalent to the definition of weak solution in Definition 7.8. The equivalence between weak and strong solutions under the hypotheses of the proposition is standard.
Suppose that U^i_t is a solution to the SPDE $F_1,...,F_r$ and $S^1,...,S^r$ with respect to Definition 7.3. Fix $f \in C^\infty_0(M)$ and denote by $\mathfrak{K}$ the support of f. Since U^i_t are C^∞ semimartingales, U^i_t, $F^i_\alpha(x,U_s,...,\partial^\sigma(U_s))$ and $V_{F_\beta}(F^i_\alpha)(x,U_s,...,\partial^\sigma(U_s))$ are locally bounded processes in $C^0(\mathfrak{K})$.

Definition 7.3 is equivalent to say that for any Dirac delta distribution δ_x centred in $x \in \mathfrak{K}$, the following equality holds

$$\delta_x(U_t^i - U_0^i) = \int_0^t \delta_x(F_\alpha^i(x, U_s, ..., \partial^\sigma(U_s)))dS_s^\alpha + \int_0^t \delta_x(V_{F_\alpha}(F_\beta^i)(x, U_t, ..., \partial^\tau(U_t)))d[S^\alpha, S^\beta]_s. \tag{7.7}$$

Since Dirac delta functionals are continuous linear functionals in $(C^0(\mathfrak{K}))^*$ which separate the points of $C^0(\mathfrak{K})$ and l_f is a continuous linear functional in $(C^0(\mathfrak{K}))^*$, there exists a succession $l_n \in (C^0(\mathfrak{K}))^*$, made by finite linear combinations of Dirac deltas, which converges weakly* to l_f in $(C^0(\mathfrak{K}), (C^0(\mathfrak{K}))^*)$. Furthermore, since U_t^i, $F_\alpha^i(x, U_s, ..., \partial^\sigma(U_s))$ and $V_{F_\alpha}(F_\beta^i)(x, U_t, ..., \partial^\tau(U_t))$ are locally bounded in $C^0(\mathfrak{K})$ and l_n is strongly bounded in $(C^0(\mathfrak{K}))^*$, there exists a locally bounded process H_t in $\mathbb{R}_+$ such that

$$|l_n(U_t^i)|, |l_n(F_\alpha^i(x, U_s, ..., \partial^\sigma(U_s)))|, |l_n(F_\alpha^i(x, U_s, ..., \partial^\sigma(U_s)))| \leq H_t,$$

almost surely. Equation (7.7) holds with δ_x replaced by l_n since l_n is a finite linear combination of Dirac deltas. Taking the limit for $n \to +\infty$, by the dominate convergence theorem for semimartingales (see [153, Chapter IV, Theorem 32]), we obtain

$$l_f(U_t^i - U_0^i) = \int_0^t l_f(F_\alpha^i(x, U_s, ..., \partial^\sigma(U_s)))dS_s^\alpha + \frac{1}{2}\int_0^t l_f(V_{F_\alpha}(F_\beta^i)(x, U_t, ..., \partial^\tau(U_t)))d[S^\alpha, S^\beta]_s. \tag{7.8}$$

Using a similar reasoning and the fact that the linear space composed by l_f, with $f \in \mathcal{L}$, separates the points of H and that U_t^i, $F_\alpha^i(x, U_s, ..., \partial^\sigma(U_s))$ and $V_{F_\alpha}(F_\beta^i)(x, U_t, ..., \partial^\tau(U_t))$ are locally bounded in H, we obtain that equation (7.6) holds for any $\xi \in H$ and thus U_t^i is a weak solution. Conversely, if U_t^i is a weak solution to the SPDE associated with $F_1, ..., F_r$ and $S^1, ..., S^r$, since the space $V \subset H$ separates the point of H and $U_t^i, ...$ are locally bounded in H, it is possible to prove (7.8) for any $f \in \mathcal{L}$. Since the space composed by l_f, with $f \in \mathcal{L}$, separates the points of $C^0(M)$ and $U_t^i, ...$, are locally bounded in $C^0(\mathfrak{K})$ for any compact set $\mathfrak{K} \subset M$, we can prove (7.7) which is equivalent to Definition 7.3. ■

In general, proving the local boundedness of the C^∞ semimartingales $U_t^i, ...$, $F_\alpha^i(x, U_s, ..., \partial^\sigma(U_s))$ and $V_{F_\alpha}(F_\beta^i)(x, U_t, ..., \partial^\tau(U_t))$ with respect to the norm of H as required by Proposition 7.9 is quite hard. Nevertheless there is a case where verifying this hypothesis is trivial. If the closure $\bar{M}$ of M in $\mathbb{R}^m$ is compact and U_t^i is a C^∞ semimartingale on all $\bar{M}$ (in other words for any x derivatives $\partial^\sigma(U_t^i)(x)$ there exists the finite limit $x \to x_0 \in \partial M$) the processes U_t^i, $F_\alpha^i(x, U_s, ..., \partial^\sigma(U_s))$ and $V_{F_\alpha}(F_\beta^i)(x, U_t, ..., \partial^\tau(U_t))$ are locally bounded in all the Sobolev spaces of the form $H_w^r(M)$, where $w \in L^1(M)$.

7.1.2 Finite dimensional solutions to SPDEs

Definition 7.10 *A smooth function $K : M \times B \to N$ is a finite dimensional solution to the SPDE (7.1) if, for any $b_0 \in B$, there exists a semimartingale B_t taking values in B such that $B_0 = b_0$ and $K(x, B_t)$ is a solution to the SPDE (7.1).*

It is important to note that, if $B_t \in B$ is a semimartingale and K is a smooth function, then $K(x, B_t)$ is a semimartingale dependent on the parameter x in the sense of Definition 7.1. Indeed, if we fix $x \in M$, since the function $K(x, b)$ is smooth in all its arguments, $K(x, B_t)$ is a semimartingale, being obtained transforming the semimartingale $B_t \in B$ with respect to the $C^\infty(B, N)$ function $b \mapsto K(x, b)$.

Remark 7.11 *We can request that K is a finite dimensional solution to the SPDE associated with $F_1, ..., F_r$ and $S^1, ..., S^r$ with respect to the weak and strong definition of solution to an SPDE in a Hilbert space H given in Definition 7.8. Thanks to Proposition 7.9 all these definitions are equivalent whenever the function K is locally bounded in H i.e., for any compact $\mathfrak{K}_B$, there exists a constant $C_{\mathfrak{K}_B}$ such that $\sup_{b\in\mathfrak{K}_B} |K(\cdot, b)| \leq C_{\mathfrak{K}_B}$, where $|\cdot|$ is the norm of H.*

Theorem 7.12 *If, for any $\alpha = 1, ..., r$, $V_{F_\alpha} \in \mathcal{K}^K$, then K is a finite dimensional solution to the SPDE (7.1). Conversely, if K is a finite dimensional solution to (7.1) and $\frac{dA_t^{\alpha\beta}}{dt} = \frac{d[S^\alpha, S^\beta]_t}{dt}$ is nonsingular for all $t \in \mathbb{R}_+$, then $V_{F_\alpha} \in \mathcal{K}^K$.*

Proof. If $V_{F_\alpha} \in \mathcal{K}^K$ there exist r uniquely determined vector fields $Y_1, ..., Y_r$ in the trivial bundle $M \times B$ such that $\mathbf{K}_*(Y_\alpha) = V_{F_\alpha}$. Since V_{F_α} are vertical in $J^\infty(M, N)$, the vector fields Y_α are vertical in the bundle $M \times B$. Furthermore, since $[D_{x^i}, V_{F_\alpha}] = 0$ and $\mathbf{K}_*(\partial_{x^i}) = D_{x^i}$, we have that $[\partial_{x^i}, Y_\alpha] = 0$ and so the vector fields Y_α are independent of x^i. If B_t is the unique solution on B to the SDE associated with $Y_1, ..., Y_r$ and $S^1, ..., S^r$ such that $B_0 = b_0 \in B$, then $U_t(x) = K(x, B_t)$ is a solution to the SPDE (7.1). We prove this fact by showing that $\mathbf{U}_t(x)$ is a solution to the SDE $V_{F_1}, ..., V_{F_r}$ and $S^1, ..., S^r$ and then using Theorem 7.5. In fact, if $f \in \mathcal{F}$, then

$$\begin{aligned} f(x, \mathbf{U}_t(x)) - f(x, \mathbf{U}_0(x)) &= \mathbf{K}^*(f)(x, B_t) - \mathbf{K}^*(f)(x, B_0) \\ &= \int_0^t Y_\alpha(\mathbf{K}^*(f))(x, B_s) \circ dS_s^\alpha \\ &= \int_0^t \mathbf{K}^*(\mathbf{K}_*(Y_\alpha)(f))(x, B_s) \circ dS_s^\alpha \\ &= \int_0^t \mathbf{K}^*(V_\alpha(f))(x, B_s) \circ dS_s^\alpha = \int_0^t V_\alpha(f)(x, \mathbf{U}_s(x)) \circ dS_s^\alpha. \end{aligned}$$

Conversely, suppose that, for any $b_0 \in B$, there exists a semimartingale $B_t \in B$ such that $B_0 = b_0$ and $K(x, B_t)$ is a solution to the SPDE (7.1). If, for any function $f \in \mathcal{F}$ such that $f|_{\mathcal{K}^K} = 0$, we have $V_{F_\alpha}(f)|_{\mathcal{K}^K} = 0$, then $V_{F_\alpha} \in T\mathcal{K}^K$. Let $f \in \mathcal{F}$ be such that $f|_{\mathcal{K}^K} = 0$. By Itô formula we have

$$\begin{aligned} 0 &= f(\mathbf{K}(x, B_t)) - f(\mathbf{K}(x, B_0)) \\ &= \int_0^t V_{F_\alpha}(f)(\mathbf{K}(x, B_s)) \circ dS_s^\alpha, \end{aligned}$$

and this ensures that the quadratic covariation of $\int_0^t V_{F_\alpha}(f)(\mathbf{K}(x, B_s)) \circ dS_s^\alpha$ with any S^β is zero, i.e.

$$\int_0^t V_{F_\alpha}(f)(\mathbf{K}(x, B_s)) dA_s^{\alpha\beta} = 0.$$

Since the matrix A_s is nonsigular for any t, in particular we have that

$$V_{F_\alpha}(f)(\mathbf{K}(x, B_t)) = 0,$$

almost surely and for any $t \in \mathbb{R}_+$. Taking the limit for $t \to 0$ we obtain $V_{F_\alpha}(f)(\mathbf{K}(x, b_0)) = 0$. Since $b_0 \in B$ is a generic point and $\mathbf{K}(x, b)$ is a surjective map from $M \times B$ into $\mathcal{K}^K$ we find $V_{F_\alpha}(f)|_{\mathcal{K}^K} = 0$. ∎

Remark 7.13 *It is possible to generalize Theorem 7.12 in several directions. For example it is possible to state both the sufficient and necessary conditions of Theorem 7.12 for the SPDEs described in Remark 7.4.*

Furthermore, if we consider B as a smooth manifold with boundary, the manifold $\mathcal{K}^K$ turns out to be itself a manifold with boundary and the sufficient condition of Theorem 7.12 is no more true. Indeed we have to add a new condition, i.e. that the SDE solved by the process B_t have a solution for any starting point $b_0 \in B$. We remark that this additional condition is satisfied, for example, when $V_{F_\alpha} \in T(\partial\mathcal{K}^K)$.
Finally, the condition on $A^{\alpha\beta}$ can be relaxed: in particular, if S^α (for $\alpha = 1, ..., l$) are absolutely continuous, $\frac{dS^\alpha}{dt} \neq 0$ almost surely and $\frac{dS^\alpha}{dt}$ are almost surely linearly independent with respect to the time t, we have to ensure that $\frac{dA^{\alpha\beta}}{dt}$ is nonsingular only for $\alpha, \beta > l$. For example, this is the case when $S^1_t = t$ and $S^\alpha_t = W^\alpha_t$ (for $\alpha = 2, ..., r$), where W^α_t are $r-1$ independent Brownian motions.

It is interesting to note that Theorem 7.12 provides an explicit method to construct the process B_t appearing in Definition 7.10 when we do not have the explicit reduction function $K(x, b)$ but only the finite dimensional manifold $\mathcal{K}$. In fact, taking a coordinate system x^i on M and coordinates $z^1, ..., z^p$ on $\mathcal{K}$ (we can use for example some u^i, u^i_σ), there are some functions $\Xi^i_\alpha(x, z)$ such that

$$V_{F_\alpha}|_{\mathcal{K}} = \Xi^i_\alpha(x, z)\partial_{z^i},$$

the vector fields V_{F_α} being tangent to $\mathcal{K}$. On the other hand, the fact that $\mathcal{K}$ is a canonical manifold ensures that also the vector fields D_{x^i} are tangent to $\mathcal{K}$: in particular, there are some functions $\Sigma^j_i(x, z)$ such that

$$D_{x^i} = \partial_{x^i} + \Sigma^j_i(x, z)\partial_{z^j}.$$

We define some processes $Z^i_t(x)$ which solve the following system of SDE in t and PDE in x^i

$$dZ^i_t(x) = \Xi^i_\alpha(x, Z_t(x)) \circ dS^\alpha_t \tag{7.9}$$
$$\partial_{x^i}(Z^j_t(x)) = \Sigma^j_i(x, Z_t(x)). \tag{7.10}$$

The function $u^i \in \mathcal{F}$ can be expressed using the coordinates (x^i, z^j) in $\mathcal{K}$, which means that there exists a function $\mathfrak{U}^i(x, z)$ such that $u^i|_{\mathcal{K}} = \mathfrak{U}^i(x, z)$. With this notation the finite dimensional solution to the SPDE (7.1) is given by

$$U^i_t(x) = \mathfrak{U}^i(x, Z_t(x)).$$

7.1.3 A necessary condition for the existence of finite dimensional solutions to an SPDE

The aim of this section is to generalize the necessary condition of Frobenius Theorem providing suitable hypothesis on the vector fields $V_{F_1}, ..., V_{F_r}$ in order to guarantee the existence of finite dimensional solutions to SPDE defined by $F_1, ..., F_r$ and $S^1, ..., S^r$.

Proposition 7.14 *If the evolution vector fields $V_{F_1}, ..., V_{F_r}$ are in the tangent space of a finite dimensional manifold $\mathcal{K}$, then $V_{F_1}, ..., V_{F_r}$ generate a finite dimensional module on $\mathcal{K}$.*

Proof. Since $V_{F_i}, V_{F_j} \in T\mathcal{K}$, we have that $[V_{F_i}, V_{F_j}] = V_{[F_i,F_j]} \in T\mathcal{K}$. Moreover, since $T\mathcal{K}$ is finite dimensional, $V_{F_1}, ..., V_{F_r}$ and all their Lie brackets form a finite dimensional module on $\mathcal{K}$. ∎

Using Proposition 7.14 and the fact that the commutator of two evolution vector fields is an evolution vector field, we can suppose that $\mathcal{S} = \text{span}\{V_{F_1}, ..., V_{F_r}\}$ is a finite dimensional module on $\mathcal{K}$. Indeed, if this is not the case, we can add to the list of V_{F_i} all their commutators $V_{[F_i,F_j]}, ..., V_{[F_i,[F_j,F_k]]}, ...$ and, since $T\mathcal{K}$ is finite dimensional, we are sure that we are adding a

finite number of vector fields.
In particular, if $V_{F_1},...,V_{F_r} \in T\mathcal{K}$, we can suppose that $\mathcal{S}$ is a finite dimensional formally integrable module on $\mathcal{K}$. Since V_{F_i} are not general vector fields on $J^\infty(M,N)$ but they are evolution vector fields we can prove a stronger proposition.

Proposition 7.15 *Let $V_{F_1},...,V_{F_r}$ be evolution vector fields in $J^\infty(M,N)$ such that $\mathcal{S}$ is an r-dimensional (formally) integrable distribution on a submanifold $\mathcal{K}$ of $J^\infty(M,N)$. Then,*

$$[V_{F_i}, V_{F_j}] = \sum_h \lambda^h_{i,j} V_{F_h},$$

we have that $D_l(\lambda^h_{i,j}) = 0$ on $\mathcal{K}$.

Proof. The proof is given for the case $N = M = \mathbb{R}$ and $\mathcal{K} = J^\infty(M,N)$; the general case is a simple generalization of this one.
Since $\mathcal{S}$ is r-dimensional, for any point $p \in J^\infty(M,N)$ there exist a neighborhood U of p and an integer $h \in \mathbb{N}_0$ such that the matrix $A = (D_x^{h+j-1}(F_i))|_{i,j=1,...,s}$ is non-singular. Moreover, since the commutator of two evolution vector fields is an evolution vector field, there exist some $F_{i,j} \in \mathcal{F}$ such that $[V_{F_i}, V_{F_j}] = V_{F_{i,j}}$ and, by the definition of evolution vector field, we have

$$D^r_x(F_{i,j}) = \sum_h \lambda^h_{i,j} D^r_x(F_h). \tag{7.11}$$

Deriving with respect to x the previous relations we obtain

$$D^{r+1}_x(F_{i,j}) = \sum_h D_x(\lambda^h_{i,j}) D^r_x(F_h) + \sum_h \lambda^h_{i,j} D^{r+1}_x(F_h) \tag{7.12}$$

and combining (7.11) and (7.12) we find

$$\sum_h D_x(\lambda^h_{i,j}) D^r_x(F_h) = 0.$$

Since the matrix A is non-singular, we get $D_x(\lambda^h_{i,j}) = 0$. ■

Proposition 7.15 implies that if an SPDE associated with $(F_1,\dots,F_r)$ admits a finite dimensional solution passing through any point of $J^\infty(M,N)$, the vector fields $V_{F_1},...,V_{F_r}$ have to form not only a module on $J^\infty(M,N)$ but a Lie algebra. For this reason in the following we always suppose that $V_{F_1},...,V_{F_r}$ form a Lie algebra, i.e. there exist some constants $\lambda^i_{j,k} \in \mathbb{R}$ such that

$$[V_{F_i}, V_{F_j}] = \sum_h \lambda^h_{i,j} V_{F_h}.$$

7.2 A general algorithm to compute solutions to SPDEs

In this section, starting from Theorem 6.19 and Theorem 6.21, we provide a general algorithm to explicitly compute the finite dimensional solution to an SPDE. The main tool is the introduction of a special coordinate system on the manifold $\mathcal{K}$ which permits to avoid most of the computational problems in $J^\infty(M,N)$.

Given a canonical submanifold $\mathcal{H}$ such that $V_{F_1},...,V_{F_l} \in T\mathcal{H}$, we have to compute the characteristic flows $\Phi^1_{a^1},...,\Phi^h_{a^h}$ of $G_1,...,G_h$ in order to obtain $\mathcal{K} = \mathbf{\Phi}_{(a^1,...,a^h)}(\mathcal{H})$. Once we have $\mathcal{K}$,

which by Theorem 6.19 and Theorem 6.21 is a finite dimensional solution to the SPDE defined by $F_1, ..., F_l, G_1, ..., G_h$, we can choose a coordinate system on $\mathcal{K}$ of the form $x^1, ..., x^l, y^1, .., y^k$ and compute the explicit expressions for the vector fields $V_{F_1}, ..., V_{F_l}, V_{G_1}, ..., V_{G_h}$ and $D_1, ..., D_m$ in the coordinate system (x, y). Finally, by solving equations (7.9) and (7.10), we obtain the explicit solution to the original SPDE.
In the general case it is not possible to explicitly perform all the described steps, so that it is not possible to explicitly reduce the SPDE to a finite dimensional SDE. Despite this fact, there are at least two cases where this reduction can be done:

- Case 1: the SPDE is defined by some functions $G_1, ..., G_h$ admitting characteristics and forming a finite dimensional Lie algebra
- Case 2: the SPDE is defined by a function F which does not admit characteristics and some functions $G_1, ..., G_r$ which admit characteristics.

Furthermore, in order to explicitly compute the solution, we require two additional hypotheses

- the characteristics of $G_1, ..., G_h$ admit a common filtration $\mathfrak{G}_0$ and the characteristic flow of $G_1, ..., G_h$ can be explicitly computed,
- we are able to solve the equation

$$\partial_a(f(x, a)) = F(x, f(x, a), \partial^\sigma(f(x, a)))$$

  for all $a \geq 0$ and for some initial condition $f(x, 0) = f_0(x) \in C^\infty(M, N)$.

All the previous hypotheses are generally satisfied in the literature of finite dimensional solutions to SPDEs and they hold for all the examples in Section 7.3 (the only exception is the second part of Section 7.3.3, where we consider an SPDE such that Theorem 6.19 and Theorem 6.21 does not apply).
In Case 1 the first step consists in choosing the manifold $\mathcal{H}$ as the zeros of the following functions

$$h^i_\sigma = u^i_\sigma - \partial^\sigma(f^i(x)),$$

where $f^i \in C^\infty(M, N)$. It is easy to check that $\mathcal{H}$ is a canonical submanifold of $J^\infty(M, N)$, since

$$T\mathcal{H} = \mathrm{span}\{D_1, ..., D_m\}.$$

In order to apply Theorem 6.19 we need that, for any $x_0 \in M$, there exists a set of h multi-indices $\sigma^1, ..., \sigma^h$ and of indices $i^1, ..., i^h \in \{1, ..., n\}$ such that

$$\begin{pmatrix} D^{\sigma^1}(G^{i_1}_1)(x_0, f(x_0), \partial^\tau(f)(x_0)) & ... & D^{\sigma^1}(G^{i_1}_h)(x_0, f(x_0), \partial^\tau(f)(x_0)) \\ ... & ... & ... \\ D^{\sigma^h}(G^{i_h}_1)(x_0, f(x_0), \partial^\tau(f)(x_0)) & ... & D^{\sigma^h}(G^{i_h}_h)(x_0, f(x_0), \partial^\tau(f)(x_0)) \end{pmatrix} \tag{7.13}$$

has maximal rank. If $f^1, ..., f^r$ are real analytic functions, it is enough to check that previous condition holds in one point. Anyway (7.13) has maximal rank whenever $f^1, ..., f^r$ are generic smooth functions.
Under this hypothesis, we define the manifold $\mathcal{K}$ as in Theorem 6.19

$$\mathcal{K} = \bigcup_{\mathbf{a} \in \mathcal{V}} \Phi^h_{a^h}(...(\Phi^1_{a^1}(\mathcal{H}))...).$$

This means that, for any $\mathbf{a} = (a_1, ..., a_h)$ in a suitable neighborhood of the origin of $\mathbb{R}^h$, $\mathcal{K}$ is the set of all the points $p = (x, u, u_\sigma) \in J^\infty(M, N)$ such that there exist $(a^1_p, ..., a^h_p) \in \mathbb{R}^h$ satisfying

$$\mathbf{\Phi}^*_{(a^1_p,...,a^h_p)}(u^i_\sigma) - \partial^\sigma(f^j)(\mathbf{\Phi}^x_{(a^1_p,...,a^h_p)}(x, u, u_\sigma)) = 0. \tag{7.14}$$

We define a special set of functions, which we still denote by $a^1(x, u, u_\sigma), ..., a^h(x, u, u_\sigma)$, so that they satisfy

$$\mathbf{\Phi}^*_{(a^1(x,u,u_\sigma),...,a^h(x,u,u_\sigma))}(u^{i_k}_{\sigma^k}) - \partial^{\sigma^k}(f^{i_k})(\mathbf{\Phi}^x_{(a^1(x,u,u_\sigma),...,a^h(x,u,u_\sigma))}(x, u, u_\sigma)) = 0. \tag{7.15}$$

Hereafter, in order to avoid confusion, we write a^i only for the functions defined by equation (7.15), while we use other letters, for example $\mathbf{b} = (b^1, ..., b^h)$ to describe the flow $\mathbf{\Phi}$ evaluated at some fixed $\mathbf{b} \in \mathbb{R}^h$.
Our regularity assumption on the matrix (7.13) ensures that equation (7.14) has a unique local solution in a neighborhood of $\pi^{-1}(x_0)$ and the functions $x^1, ..., x^n, a^1, ..., a^h$ provide a local coordinate system for $\mathcal{K}$ in a neighborhood of $\pi^{-1}(x_0)$. Indeed, using (7.15), we have that $\mathcal{K}$ is defined as the set of zeros of the following functions

$$\mathbf{\Phi}^*_{(a^1(x,u,u_\sigma),...,a^h(x,u,u_\sigma))}(u^i_\sigma) - \partial^\sigma(f^i)(\mathbf{\Phi}^x_{(a^1(x,u,u_\sigma),...,a^h(x,u,u_\sigma))}(x, u, u_\sigma)), \tag{7.16}$$

where σ is, here, a generic multi-index. In the coordinate system (x^i, a^j) the vector fields $V_{G_1}, ..., V_{G_h}$ have a special form, as showed by the following theorem.

Theorem 7.16 *The vector fields $V_{G_1}|_\mathcal{K}, ..., V_{G_h}|_\mathcal{K}$ satisfy the relations*

$$V_{G_i} = \phi^l_i(a^1, ..., a^h)\partial_{a^l},$$

and the smooth functions ϕ^l_i are such that

$$\phi^l_i(a^1, ..., a^h) = -\delta^l_i \quad \text{when } a^1, ..., a^{i-1} = 0 \tag{7.17}$$

$$\partial_{a^k}(\phi^l_i(a^1, ..., a^h)) = -\lambda^p_{k,i}\phi^l_p - \sum_{r>k} \phi^r_i \partial_{a^r}(\phi^l_k(a^1, ..., a^h)) \quad \text{when } a^1, ..., a^{k-1} = 0 \text{ and } k \geq i, \tag{7.18}$$

where

$$[G_i, G_j] = \lambda^k_{i,j} G_k.$$

Lemma 7.17 *If $a^1, ..., a^h$ are defined by (7.15), then*

$$D_i(a^j)|_\mathcal{K} = 0.$$

Proof. Using equations (7.15) defining the functions a^i we have that

$$0 = D_i(\mathbf{\Phi}^*_{(a^1,...,a^h)}(h^{i_k}_{\sigma^k})) = \left[D_i(a^j)\partial_{b^j}(\mathbf{\Phi}^*_{(b^1,...,b^h)}(h^{i_k}_{\sigma^k})) + D_i(\mathbf{\Phi}^*_{(b^1,...,b^k)}(h^{i_k}_{\sigma^k}))\right]\Big|_{b^1=a^1,...,b^h=a^h}, \tag{7.19}$$

where

$$h^{i_k}_{\sigma^k} = u^{i_k}_{\sigma^k} - \partial^{\sigma^k}(f^{i_k})(x).$$

By Theorem 6.24 there exist suitable smooth functions $B^i_j(b^1, ..., b^h, x, u, u_\sigma)$ such that

$$\mathbf{\Phi}_{(b^1,...,b^h),*}(D_j) := \Phi^h_{b^h,*}(...(\Phi^1_{b^1,*}(D_j))...) = B^k_j D_k.$$

So equations (7.19) restricted on $\mathcal{K}$ become

$$\begin{aligned}
0 &= D_i(a^j)\left[\partial_{b^j}(\mathbf{\Phi}^*_{(b^1,...,b^k)}(h^{i_k}_{\sigma^k}))\right]\Big|_{b^1=a^1,...,b^h=a^h}+\\
&\quad+\left\{\mathbf{\Phi}^*_{(b^1,...,b^h)}\left[\mathbf{\Phi}_{(b^1,...,b^h),*}(D_i)(h^{i_k}_{\sigma^k})\right]\right\}\Big|_{b^1=a^1,...,b^h=a^h},\\
&= D_i(a^j)\left[\partial_{b^j}(\mathbf{\Phi}^*_{(b^1,...,b^k)}(h^{i_k}_{\sigma^k}))\right]\Big|_{b^1=a^1,...,b^h=a^h}+\\
&\quad+\left[\mathbf{\Phi}^*_{(b^1,...,b^h)}(B^r_i)\mathbf{\Phi}^*_{(b^1,...,b^h)}(D_r(h^{i_k}_{\sigma^k}))\right]\Big|_{b^1=a^1,...,b^h=a^h}\\
&= D_i(a^j)\left[\partial_{b^j}(\mathbf{\Phi}^*_{(b^1,...,b^k)}(h^{i_k}_{\sigma^k}))\right]\Big|_{b^1=a^1,...,b^h=a^h},
\end{aligned}$$

where we use relations (7.16). Since $\left[\partial_{b^j}(\mathbf{\Phi}^*_{(b^1,...,b^k)}(h^{i_k}_{\sigma^k}))\right]\Big|_{b^1=a^1,...,b^h=a^h}$ is nonsingular in a neighborhood of x_0, we have that $D_i(a^j)|_{\mathcal{K}}=0$.

■

Proof of Theorem 7.16. Since $V_{G_i}(x^j)=0$, then $V_{G_i}|_{\mathcal{K}}=\phi^l_i(x,a^1,...,a^h)\partial_{a^i}$. If equations (7.17) and (7.18) hold, then the functions ϕ^l_i are independent from x. So, in order to prove Theorem 7.16, we need only to prove (7.17) and (7.18).
By Lemma 7.17 we get

$$0=\bar{V}_{G_i}(\mathbf{\Phi}^*_{a^1,...,a^h}(h^{i_k}_{\sigma^k}))|_{\mathcal{K}}=\left[V_{G_i}(a^j)\partial_{b^j}(\mathbf{\Phi}^*_{(b^1,...,b^h)}(h^{i_k}_{\sigma^k}))+\bar{V}_{G_i}(\mathbf{\Phi}^*_{(b^1,...,b^h)}(h^{i_k}_{\sigma^k}))\right]\Big|_{b^1=a^1,...,b^h=a^h,\mathcal{K}}.$$

If $b^1,...,b^{i-1}=0$ we have that

$$\bar{V}_{G_i}(\mathbf{\Phi}^*_{(b^1,...,b^h)}(h^{i_k}_{\sigma^k}))=\partial_{b^i}(\mathbf{\Phi}^*_{(b^1,...,b^h)}(h^{i_k}_{\sigma^k})),$$

and we obtain condition (7.17). Equation (7.18) follows using the definition of Lie brackets between vector fields and the Lie algebra structure of $G_1,...,G_h$ together with equation (7.17). ■

Theorem 7.16 allows us to compute the expressions of $V_{G_1},...,V_{G_h}$ on $\mathcal{K}$ in the coordinate system $(x^1,...,x^m,a^1,...,a^h)$, even if we are not able to compute the explicit expression of $a^1,...,a^h$. Indeed, equations (7.17) and (7.18) not only uniquely determine the functions ϕ^i_j, but also permit to get their explicit expressions. In order to show this last assertion we propose here an example that will also be useful in Section 7.3.
Taking $M=N=\mathbb{R}$, $G_1=1,G_2=u,G_3=u^2$ and considering $\mathcal{H},\mathcal{K}$ as in the previous discussion, we have that

$$[G_1,G_2]=G_1,\qquad [G_1,G_3]=2G_2,\qquad [G_2,G_3]=G_3.$$

The equations for ϕ^i_1 and $\partial_{a^1}(\phi^i_j)$ are

$$\begin{gathered}
\phi^1_1=-1,\qquad \phi^2_1=0,\qquad \phi^3_1=0,\\
\partial_{a^1}(\phi^1_2)=1,\qquad \partial_{a^1}(\phi^2_2)=0,\qquad \partial_{a^1}(\phi^3_2)=0,\\
\partial_{a^1}(\phi^1_3)=-2\phi^1_2,\qquad \partial_{a^1}(\phi^2_3)=-2\phi^2_2,\qquad \partial_{a^1}(\phi^3_3)=-2\phi^3_2,
\end{gathered}$$

from which we obtain

$$\begin{gathered}
\phi^1_2=a^1+\tilde{f}^1(a^2,a^3),\qquad \phi^2_2=\tilde{f}^2(a^2,a^3),\qquad \phi^3_2=\tilde{f}^3(a^2,a^3),\\
\phi^1_3=-(a^1)^2-2a^1\tilde{f}^1+\tilde{g}^1(a^2,a^3),\quad \phi^2_3=-2a^1\tilde{f}^2+\tilde{g}^2(a^2,a^3),\quad \phi^3_3=-2a^1\tilde{f}^3+\tilde{g}^3(a^2,a^3).
\end{gathered}$$

From the equations of ϕ^i_2 on $a^1 = 0$ and $\partial_{a^2}(\phi^i_3)$ we have

$$\begin{gathered} \tilde{f}^1 = 0, \quad \tilde{f}^2 = -1, \quad \tilde{f}^3 = 0 \\ \partial_{a^2}(\tilde{g}^1) = -\tilde{g}^1, \quad \partial_{a^2}(\tilde{g}^2) = -\tilde{g}^2, \quad \partial_{a^2}(\tilde{g}^3) = -\tilde{g}^3. \end{gathered}$$

Solving the previous equations and imposing $\tilde{g}^i = -\delta^i_3$ we get

$$\tilde{g}^1 = 0, \quad \tilde{g}^2 = 0, \quad \tilde{g}^3 = -e^{-a^2},$$

so that we find

$$\begin{aligned} V_{G_1}|_{\mathcal{K}} &= -\partial_{a^1} \\ V_{G_2}|_{\mathcal{K}} &= a^1\partial_{a^1} - \partial_{a^2} \\ V_{G_3}|_{\mathcal{K}} &= -(a^1)^2\partial_{a^1} + 2a^1\partial_{a^2} - e^{-a^2}\partial_{a^3}. \end{aligned}$$

We remark that we have been able to obtain the expressions of V_{G_i} without information on the manifold $\mathcal{K}$. This fact is a strong consequence of the Lie algebra structure of G_i.

Once we have the expressions of ϕ^i_j, we can explicitly compute the finite dimensional SDE related to our SPDE, which is

$$dA^i_t = \phi^i_\alpha(A^1_t, ..., A^h_t) \circ dS^\alpha_t, \tag{7.20}$$

where the semimartingales S^α are the same ones of equation (7.1), and, if we know the processes $A^1, ..., A^h$, we can explicitly compute the solutions $U^1_t(x), ..., U^n_t(x)$. Note that the hypothesis that $G_1, ..., G_h$ admit a common filtration plays an important role. In fact, if we project the manifold $\mathcal{H}$ on the manifold defined by the algebra $\mathfrak{G}_0$, we find a finite set of functions $H_1, ..., H_l$ such that $H_i \in \mathfrak{G}_0$ and $H_i(p) = 0$ if and only if $p \in \mathcal{H}$. Since $\bar{V}_{G_i}(H) \in \mathfrak{G}_0$ we have that, for any $b^1, ..., b^h \in \mathbb{R}^h$, $\mathbf{\Phi}_{(b^1,...,b^h)}(H^i) \in \mathfrak{G}_0$. Therefore, the solution $U_t(x) \in C^\infty(M, N)$ it is the unique smooth function such that

$$\mathbf{\Phi}_{(A^1_t,...,A^h_t)}(H^i)(x, U_t(x), \partial^\sigma(U_t(x))) = 0. \tag{7.21}$$

The previous set of equations completely determines the function $U_t(x)$. In the particular case $\mathfrak{G}_0 = \mathfrak{F}_0$ (the set of smooth functions which depend only on x^i, u^j) we have that $\mathbf{\Phi}^{x^i}_{(b^1,...,b^h)}$ and $\mathbf{\Phi}^{u^j}_{(b^1,...,b^h)}$ depend only on x and u; this means that $U^i_t(x)$ is the unique solution to the equations

$$\mathbf{\Phi}^{u^i}_{(A^1_t,...,A^h_t)}(x, U_t(x)) - f^i(\mathbf{\Phi}^x_{(A^1_t,...,A^h_t)}(x, U_t(x))) = 0. \tag{7.22}$$

In this way we can reduce our infinite dimensional SPDE to the finite dimensional SDE (7.20) and to the algebraic (or, more generally, analytic) relations (7.22).

Let us now consider Case 2, where the SPDE is defined by a function F, which does not admit characteristics, and by the functions $G_1, ..., G_h$ which, as in the previous case, admit characteristics. In this case we can choose a manifold $\mathcal{H}$ defined as the set of zeros of the functions

$$h^i_\sigma(a^0, x, u, u_\sigma) = u^i_\sigma - \partial^\sigma(f)(x, a^0),$$

where

$$\partial_{a^0}(f)(x, a^0) = F(x, f(x, a^0), \partial^\sigma(f)(x, a^0)).$$

In order to obtain the manifold $\mathcal{K}$ as in the previous case, we require that there exist $h+1$ indices $i_0, ...i_h \in \{1, ..., n\}$ and $h+1$ multi-indices $\sigma^0, ..., \sigma^h \in \mathbb{N}$ such that

$$\left(\begin{array}{cccc} D^{\sigma^0}(F^{i_0})(x_0, f_0(x_0), \partial^\tau(f_0)(x_0)) & D^{\sigma^0}(G_1^{i_0})(x_0, f_0(x_0), \partial^\tau(f_0)(x_0)) & ... & D^{\sigma^0}(G_h^{i_0})(x_0, f_0(x_0), \partial^\tau(f_0)(x_0)) \\ ... & ... & ... & ... \\ D^{\sigma^h}(F^{i_h})(x_0, f_0(x_0), \partial^\tau(f_0)(x_0)) & D^{\sigma^h}(G_1^{i_h})(x_0, f_0(x_0), \partial^\tau(f_0)(x_0)) & ... & D^{\sigma^h}(G_h^{i_h})(x_0, f_0(x_0), \partial^\tau(f_0)(x_0)) \end{array}\right), \tag{7.23}$$

is non singular (here $f_0(x) = f(x, 0)$). Therefore, we can define a set of new functions, which we denote by $a^0(x, u, u_\sigma), ..., a^h(x, u, u_\sigma)$, such that

$$\mathbf{\Phi}^*_{(a^1(x,u,u_\sigma),...,a^h(x,u,u_\sigma))}(u^{i_k}_{\sigma_k}) - \partial^{\sigma^k}(f(\mathbf{\Phi}^x_{a^1(x,u,u_\sigma),...,a^h(x,u,u_\sigma)}(x, u, u_{\sigma_k}, a^0(x, u, u_\sigma)) = 0$$

and we can consider the submanifold $\mathcal{K}$ defined as the set of zeros of

$$\mathbf{\Phi}^*_{(a^1(x,u,u_\sigma),...,a^h(x,u,u_\sigma))}(u^i_\sigma) - \partial^\sigma(f^i)(\mathbf{\Phi}^x_{(a^1(x,u,u_\sigma),...,a^h(x,u,u_\sigma))}(x, u, u_\sigma), a^0(x, u, u_\sigma)) .$$

Under these hypotheses for the vector fields V_{G_i} on the manifold $\mathcal{K}$, an analogue of Theorem 7.16 holds.

Theorem 7.18 *In the previous setting $D_i(a^j)|_{\mathcal{K}} = 0$ and furthermore*

$$\begin{array}{rl} \phi^l_i(a^1, ..., a^h) = -\delta^l_i & \text{when } a^0, a^1, ..., a^{i-1} = 0 \\ \partial_{a^k}(\phi^l_i(a^1, ..., a^h)) = -\lambda^p_{k,i}\phi^l_p - \sum_{r>k} \psi^r_i \partial_{a^r}(\psi^l_k(a^1, ..., a^h)) & \text{when } a^0, a^1, ..., a^{k-1} = 0 \text{ and } k \geq i, \end{array}$$

Proof. The proof of this Theorem is completely analogous to the proofs of Lemma 7.17 and Theorem 7.16, exploiting the fact that $\partial_{b^0}(H(x, f(x, b^0), ...)) = V_F(H)(x, f(x, b^0), ...)$ for any function $H \in \mathfrak{F}$ (see Remark 6.2). ■

Thanks to Theorem 7.18, all the machinery developed for Case 1 can be extend to Case 2.

Before concluding this section, we want to spend few words about the proposed algorithm and the non local SPDE considered in Remark 7.4. In this case equation (7.20) does not hold, but can be replaced by

$$dA^i_t = \tilde{\Psi}_\alpha(A^1_t, ..., A^h_t)\phi^i_\alpha(A^1_t, ..., A^h_t) \circ dS^\alpha_t, \tag{7.24}$$

where $\tilde{\Psi}_\alpha(b^1, ..., b^h)$ are given by

$$\tilde{\Psi}_\alpha(b^1, ..., b^h) = \Psi_\alpha(K(x, b^1, ..., b^h)),$$

and K is given by relations of the form (7.21), that in the particular case $\mathfrak{G}_0 = \mathfrak{F}_0$ become

$$\mathbf{\Phi}^{u^i}_{(b^1,...,b^h)}(x, K(x, b^1, ..., b^h)) - f(\mathbf{\Phi}^x_{(b^1,...,b^h)}(x, K(x, b^1, ..., b^h)) = 0. \tag{7.25}$$

We remark that equations (7.20) and (7.22) are not decoupled in the non-local case, and so for solving (7.25) it becomes essential to write equations (7.24).
Anyway, there is one case in which it is possible to get explicitly equation (7.24) without solving equation (7.25): suppose that $\Psi_\alpha(u(x))$ are of the form

$$\Psi_\alpha(u(x)) = R_\alpha(\partial^{\sigma^\alpha_1}(u)(h^\alpha_1), ..., \partial^{\sigma^\alpha_r}(u)(k^\alpha_r)),$$

for some smooth functions R_α. If we introduce the new variables $H^\alpha_i = \partial^{\sigma^\alpha_i}(U_t)(k^\alpha_i)$ we can exploit equations (7.22) in order to prove that H^α_i solve the following A^i_t dependent SDE

$$dH^\alpha_{j,t} = \mathfrak{H}^\alpha_{j,\beta}(H_t, A_t) \circ dS^\beta_t \tag{7.26}$$

Indeed, supposing that $H^{\alpha}_{j,t} = U(h^{\alpha}_r)$, we have

$$\begin{aligned} &\partial_u(\mathbf{\Phi}^{u^i}_{(A^1_t,\dots,A^h_t)}(x,u) - f(\mathbf{\Phi}^{x}_{(A^1_t,\dots,A^h_t)}(x,u)))\Big|_{u=H^{\alpha}_{j,t}} \circ dH^{\alpha}_{j,t} + \\ &+ \partial_{b^i}(\mathbf{\Phi}^{u^i}_{(b^1,\dots,b^h)}(x,H^{\alpha}_t) - f(\mathbf{\Phi}^{x}_{(b^1,\dots,b^h)}(x,H^{\alpha}_t))\Big|_{b^1=A^1_t,\dots,b^h=A^h_t} \circ dA^i_t = 0. \end{aligned} \tag{7.27}$$

Using equation (7.27) and equation (7.24) we obtain the SDE (7.26). In this way, even if we are not able to solve (7.25), we can anyway write explicitly a finite dimensional SDE (given by equations (7.24) and (7.26)), which provides the solution to the initial SPDE.

7.3 Examples

7.3.1 The proportional volatility HJM model

In this section we consider the problem of finding finite dimensional solutions to the SPDE which naturally arises in the Heath, Jarrow and Morton (HJM) model to describe the evolution of the interest rate (see [92]). In this setting, the problem of finding finite dimensional solutions is called *consistency problem* (see [24, 64]) and the studies on this topic, in particular the works of Filipovic, Tappe and Tiechmann [63, 65, 66, 166], gave us great inspiration. In this section we use our method to provide a closed formula for the solutions to a particular case of HJM model. Although this SPDE has already been studied, to the best of our knowledge, this is the first time that an explicit closed formula for its solution is provided.
We consider the following SPDE

$$dU_t(x) = \left(\partial_x(U_t)(x) + \Psi(U_t)^2 U_t(x)\left(\int_0^x U_t(y)dy\right)\right)dt + \Psi(U_t)U_t(x)dW_t, \tag{7.28}$$

where W_t is a Brownian motion and $\Psi : H_w \to \mathbb{R}$ is a smooth functional defined in a suitable Hilbert space $C^{\infty}_0(\mathbb{R}_+) \subset H_w$. Equation (7.28) is closely related to the HJM model. Indeed, if

$$P(t,T) = \exp\left(-\int_t^T f(s,T)ds\right) = \exp\left(\int_t^T U_s(T-s)ds\right),$$

is the random function describing the price of a bound at time t with maturity time $T \geq t$, in the HJM framework the evolution of f (called the *forward curve*) is described by

$$df(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \sum_{\beta=1}^{k}\int_0^t \sigma_\beta(s,T)dW^\beta,$$

where

$$\alpha(t,T) = \sum_{\beta=1}^{r}\sigma_\beta(t,T)\int_0^t \sigma_\beta(s,T)ds$$

and $\sigma_\beta(s,T)$ are stochastic predictable processes with respect to s. The function $U_t(x) = f(t,t+x)$ is the Musiela parametrization of the forward curve and solves an SPDE of the form (7.28). In particular we have equation (7.28) when we choose the volatility of the forward curve $\sigma(t,T)$ proportional to the forward curve itself

$$\sigma(t,T) = \Psi(f(t,t+x))f(t,T),$$

where Ψ is a functional described above. The proportional HJM model was considered for the first time by Morton in the case $\Psi = \Psi_0 \in \mathbb{R}$. In particular, in [146], he proved a result implying that equation (7.28) has explosion time almost surely finite (it is possible to choose Ψ non constant such that equation (7.28) has solution for any time $t > 0$). In this subsection we provide an explicit solution formula for equation (7.28). Although the method used is equivalent to the one proposed in [66] (thus the methods of [66] provides the same solution formula) this one seems to be the first time where an explicit solution formula is given.

In order to explicitly compute the solution to equation (7.28) we consider the functional space H_w given by

$$H_w = \left\{ h \text{ absolutely continuous} \left| \int_0^{+\infty} (h'(x))^2 w(x) dx, \lim_{x \to +\infty} h(x) = 0 \right. \right\}.$$

In [64] it is proved that (7.28) admits a (local in time) unique solution in the Hilbert space H_w when w is an increasing C^1 function such that $\int_0^\infty w(x)^{-1/3} dx < +\infty$.

The first step to apply our methods to equation (7.28) is transforming this non-local equation into a local one introducing a new variable v such that the process V associated with v is

$$V_t(x) = \int_0^x U_t(y) dy.$$

With this variable, equation (7.28) becomes

$$dV_t(x) = \left(\partial_x(V_t)(x) - \partial_x(V_t)(0) + \frac{\Psi(\partial_x(V_t))(V_t(x))^2}{2} \right) dt + \Psi(\partial_x(V_t)) V_t(x) dW_t.$$

If we transform the previous equation into a Stratonovich type equation of the form

$$dV_t(x) = \left(\partial_x(V_t)(x) - \partial_x(V_t)(0) + \frac{\Psi(\partial_x(V_t))(V_t(x))^2}{2} - \frac{\tilde{\Psi}(\partial_x(V_t)) V_t(x)}{2} \right) dt + \Psi(\partial_x(V_t)) V_t(x) \circ dW_t, \tag{7.29}$$

where

$$\tilde{\Psi}(f(x)) = \Psi(f(x))^2 + \Psi(f(x)) \cdot \partial_a(\Psi(e^a f(x)))|_{a=0},$$

we can apply the case 1 of the theory proposed in Section 7.2 with

$$\begin{aligned} G_1 &= v_x \\ G_2 &= 1 \\ G_3 &= v \\ G_4 &= v^2. \end{aligned}$$

It is simple to see that the following commutation relations hold

$$[G_1, G_i] = 0, \quad [G_2, G_3] = G_2, \quad [G_2, G_4] = 2G_3, \quad [G_3, G_4] = G_4.$$

The characteristic vector fields of $G_1, ..., G_4$ are

$$\begin{aligned} \bar{V}_{G_1} &= -\partial_x \\ \bar{V}_{G_2} &= \partial_v \\ \bar{V}_{G_3} &= v\partial_v + ... + v_{(n)} \partial_{v_{(n)}} + ... \\ \bar{V}_{G_4} &= v^2 \partial_v + ... + D_x^n(v^2) \partial_{v_n} + ... \end{aligned}$$

They generate the following flows

$$\begin{array}{rcl} \Phi_a^{1,*}(x) & = & x-a \\ \Phi_a^{1,*}(v) & = & v \\ \Phi_b^{2,*}(v) & = & v+b \\ \Phi_c^{3,*}(v) & = & e^c v \\ \Phi_d^{4,*}(v) & = & \dfrac{v}{1-dv} \\ \Phi_k^{i,*}(x) & = & x, \end{array}$$

for $i = 2, 3, 4$. We take as manifold $\mathcal{H}$ the one dimensional manifold defined by

$$v - f(x) = 0,$$

and by all its differential consequences, where f is smooth, $f(0) = 0$, and $f' \in H_w$.
The submanifold $\mathcal{K}$ is constructed as in Section 7.2. In particular $\mathcal{K}$ is given by the union of the zeros of

$$\mathbf{\Phi}_{(a,b,c,d)}(v - f(x)) = \frac{e^c(v+b)}{1 - de^c(v+b)} - f(x-a),$$

and all its differential consequences. From the particular form of $\mathcal{K}$ we can explicitly compute the associated finite dimensional function

$$K(x, a, b, c, d) = \frac{e^{-c} f(x-a)}{1 + df(x-a)} - b.$$

Since the operators $\mathfrak{G}_i : v(x) \longmapsto \partial_x(G_i(v(x)))$ for $i = 2, 3, 4$ are locally Lipschitz in H_w (see [64]) $\partial_x(K) \in H_w$ and the norm of $\|\partial_x(K)\|_{H_w}$ is bounded for (a, b, c, d) in a suitable neighborhood of the origin. For this reason and for Remark 7.11 the finite dimensional solution obtained from K is the unique solution in H_w to equation (7.28) with initial condition $U_0(x) = f'(x)$.
If on $\mathcal{K}$ we choose the coordinate system (x, a, b, c, d) as in Section 7.2, using Theorem 7.16 we obtain

$$\begin{array}{rcl} V_{G_1} & = & -\partial_a \\ V_{G_2} & = & -\partial_b \\ V_{G_3} & = & -\partial_c + b\partial_b \\ V_{G_4} & = & -e^{-c}\partial_d + 2b\partial_c - b^2\partial_b. \end{array}$$

With this coordinate system the equation for A_t, B_t, C_t, D_t are

$$\begin{array}{rcl} dA_t & = & -dt \\ dB_t & = & \left(e^{-C_t} \frac{\partial_x(f)(-A_t)}{(1+D_t f(-A_t))^2} - \frac{\Psi_0}{2} B_t^2 - \frac{\tilde{\Psi}_0}{2} B_t\right) dt + \Psi_0 B_t \circ dW_t \\ dC_t & = & \left(\Psi_0 B_t + \frac{\tilde{\Psi}_0^2}{2}\right) dt - \Psi_0 \circ dW_t \\ dD_t & = & -\frac{\Psi_0}{2} e^{-C_t} dt \end{array} \tag{7.30}$$

and the solution to (7.28) is given by

$$U_t(x) = \partial_x(V_t)(x) = \frac{e^{-C_t}\partial_x(f)(x - A_t)}{(1 + D_t f(x - A_t))^2}. \tag{7.31}$$

It is evident form the explicit solution (7.31) and from equations (7.30) that the solution $U_t(x)$ has explosion time almost surely finite as proved by Morton.

7.3.2 The stochastic Hunter-Saxton equation

In [100] Holm and Tyranowski propose the following stochastic version of the Camassa-Holm (CH) equation

$$\begin{aligned} dM_t(x) &= (-\partial_x(U_t(x)M_t(x)) - \partial_x(U_t(x))M_t(x))dt - \textstyle\sum_{\beta=1}^r(\partial_x(\xi_\beta(x)M_t(x))+ \\ &\quad +\partial_x(\xi_\beta(x))M_t(x)) \circ dW_t^\beta, \\ M_t(x) &= U_t(x) - \alpha^2\partial_{xx}(U_t(x)). \end{aligned} \tag{7.32}$$

This equation is motivated by the study of stochastic perturbations of variational dynamical equation of hydrodynamic type (see [13, 46, 99]). In particular, in [99] Holm proposes a general method to construct stochastic perturbation which preserves some geometrical and physical properties of the considered hydrodynamic PDE. Applying this general principle to the CH equation in one space dimension we obtain equation (7.32). Furthermore, in [100] Holm and Tyranowski find that this kind of stochastic perturbation of CH equation preserves the soliton solution, i.e. it is possible to find an infinite set of finite dimensional solutions to equation (7.32) which are exactly the stochastic counterpart of the finite dimensional families of soliton solutions to CH equation.
In the following we study this phenomenon in more detail exploiting the methods proposed in the previous section. Since equation (7.32) cannot be directly treated in our framework, being a strongly non local equation, and it is not possible to transform equation (7.32) into a local one using the methods proposed in Section 7.3.1, we consider a new equation, related with (7.32), admitting only finite dimensional solutions. In particular, equation (7.32), in the limit $\alpha >> 1$, can be reduced to the following stochastic version of Hunter-Saxton equation

$$\begin{aligned} d\partial_{xx}(U_t(x)) &= (-\partial_x(U_t(x)\partial_{xx}(U_t(x))) - \partial_x(U_t(x))\partial_{xx}(U_t(x)))dt+ \\ &\quad - \textstyle\sum_{\beta=1}^r(\partial_x(\xi_\beta(x)\partial_{xx}(U_t(x))) + \partial_x(\xi_\beta(x))\partial_{xx}(U_t(x))) \circ dW_t^\beta. \end{aligned} \tag{7.33}$$

Choosing a suitable set of possible solutions and the function $\xi_\beta(x)$, we can reduce equation (7.33) to a weakly local SPDE of the form (7.3). In particular, if we consider $\xi_\beta(x) = K_\beta + H_\beta x$, where K_β, H_β are suitable constants, and we suppose that the semimartingale $U_t(x)$ depending on the parameter x solution to the equation (7.33) satisfies

$$\int_{-\infty}^{+\infty} |x\partial_x(U_t(x))|, \int_{-\infty}^{+\infty} |x\partial_{xx}(U_t(x))|, \int_{-\infty}^{+\infty} |x\partial_{xxx}(U_t(x))| < +\infty. \tag{7.34}$$

Furthermore we suppose that there are constants $-\infty \leq a_1 < ... < a_k \leq +\infty$, for some $k \in \mathbb{N}$, and some constants $C_1, ..., C_k \in \mathbb{R}$ such that

$$\sum_{i=1}^k C_i U_t(a_i) = const.$$

Under these conditions, integrating equation (7.33) first for $\int_{\infty}^{x}$ and then $\int_{a_i}^{x}$ we obtain that equation (7.33) is equivalent to the following set of relations

$$\begin{array}{rcl} dU_t(x) & = & (-U_t(x)\partial_x(U_t(x)) + \frac{1}{2}V_t(x) + \Psi_0(U_t(x)))dt + \\ & & -\sum_{\beta=1}^{r}((K_\beta + H_\beta x)\partial_x(U_t(x)) + \Psi_\beta(U_t(x))) \circ dW_t^\beta \end{array} \tag{7.35}$$

$$\begin{array}{rcl} dV_t(x) & = & (-U_t(x)\partial_x(V_t(x)) + \Xi_0(U_t(x)))dt + \\ & & -\sum_{\beta=1}^{r}((K_\beta + H_\beta x)\partial_x(V_t(x)) + H_\beta V_t(x) + \Xi_\beta(U_t(x))) \circ dW_t^\beta \end{array} \tag{7.36}$$

$$V_t(x) = \sum_{i=1}^{k} C_i \int_{a_i}^{x} (\partial_y(U_t(y)))^2 dy \tag{7.37}$$

$$\sum_{i=1}^{k} C_i dU_t(a_i) = 0, \tag{7.38}$$

where

$$\begin{array}{rcl} \Psi_0(f(x)) & = & \sum_{i=1}^{k} C_i f(a_i)\partial_x(f)(a_i) \\ \Psi_\beta(f(x)) & = & -\sum_{i=1}^{k} C_i(K_\beta + H_\beta a_i)\partial_x(f)(a_i) \\ \Xi_0(f(x)) & = & \sum_{i=1}^{k} C_i(\partial_x(f)(a_i))^2 f(a_i) \\ \Xi_\beta(f(x)) & = & -\sum_{i=1}^{k} C_i(K_\beta + H_\beta a_i)(\partial_x(f)(a_i))^2. \end{array}$$

It is easy to prove that equation (7.35) and equation (7.36) preserve (for $V_t(x), U_t(x)$ smooth in space) the relation (7.37), i.e. if (7.37) is satisfied for $t = 0$ and V_t, U_t are solutions with respect to the definition of Remark 7.4 to SPDEs (7.35) and (7.36), then relation (7.37) is satisfied for any $t > 0$. Furthermore, the three equations (7.35), (7.36) and (7.37) imply (7.38) if U_t, V_t are smooth. Thus, equation (7.33), with solutions satisfying (7.38) and with behaviour at infinity given by (7.34), is equivalent to the two dimensional SPDE (7.35) and (7.36) with initial conditions $U_0(x), V_0(x)$ satisfying (7.37).

We can construct infinite smooth finite dimensional solutions to equation (7.35) and (7.36). In fact, if we consider the following smooth functions in $\mathfrak{F}^2$

$$\begin{array}{rcl} G_1 & = & \begin{pmatrix} xu_x \\ xv_x + v \end{pmatrix}, \\ G_2 & = & \begin{pmatrix} uu_x - \frac{1}{2}v \\ uv_x \end{pmatrix}, \\ G_3 & = & \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \\ G_4 & = & \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ G_5 & = & \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{array}$$

it is easy to see that the functions G_i admit strong characteristics and generate a Lie algebra with commutation relations given by

$[\cdot,\cdot]$	G_1	G_2	G_3	G_4	G_5
G_1	0	G_2	G_3	0	$-G_5$
G_2	$-G_2$	0	0	$-G_3$	$\frac{1}{2}G_4$
G_3	$-G_3$	0	0	0	0
G_4	0	G_3	0	0	0
G_5	G_5	$-\frac{1}{2}G_4$	0	0	0

Furthermore $G_1, ..., G_5$ admit characteristic vector fields which are

$$\begin{aligned}
\bar{V}_{G_1} &= V_{G_1} - xD_x = -x\partial_x + v\partial_v + u_x\partial_{u_x} + 2v_x\partial_{v_x} + ... + nu_{(n)}\partial_{u_{(n)}} + (n+1)v_{(n)}\partial_{v_{(n)}} + ... \\
\bar{V}_{G_2} &= V_{G_2} - uD_x = -u\partial_x - \frac{v}{2}\partial_u + ... + \left(D^n_x(uu_x) - \frac{v_{(n)}}{2} - uu_{(n+1)}\right)\partial_{u_{(n)}} + \\
&\quad + \left(D^n_x(uv_x) - uv_{(n+1)}\right)\partial_{v_{(n)}} + ... \\
\bar{V}_{G_3} &= V_{G_3} - D_x = -\partial_x \\
\bar{V}_{G_4} &= V_{G_4} = \partial_u \\
\bar{V}_{G_5} &= V_{G_5} = \partial_v.
\end{aligned}$$

Using the characteristic flows of the vector fields $\bar{V}_{G_i}$ it is possible to apply the results of previous sections. In order to simplify the treatment of this example we suppose that $k = 2$ and $a_1 = -\infty, a_2 = +\infty$. In this case, since we are looking for solutions satisfying $\partial_x(U_t)(\pm\infty) = 0$, we have that $\Psi_0 = \Psi_\beta = \Xi_0 = \Xi_\beta = 0$. This means that equations (7.35) and (7.36) are local SPDEs. Furthermore, in this case, using the theory of stochastic characteristics it is possible to prove that, for any smooth initial conditions, there exists a unique (local in time) solution. This means that the smooth finite dimensional solutions which we found with our algorithm are the unique solutions to equation (7.33) such that $C_1U_t(-\infty) + C_2U(+\infty) = const$ and equation (7.34) hold.

Since $\Psi_0 = \Psi_\beta = \Xi_0 = \Xi_\beta = 0$, we can consider only the functions G_1, G_2 and G_3. The most general one dimensional submanifold $\mathcal{H}$ in $J^\infty(\mathbb{R}, \mathbb{R}^2)$ is defined by the equations

$$\begin{aligned}
g_1 &:= u - f(x) = 0 \\
g_2 &:= v - g(x) = 0
\end{aligned}$$

together with all their differential consequences $D^n_x(g_i) = 0$. In order to have a manifold $\mathcal{H}$ representing a possible initial condition for our problem, we require that $\lim_{x\to\infty} xf'(x) = \lim_{x\to\infty} xf''(x) = \lim_{x\to\infty} xf'''(x) = 0$ and that

$$g(x) = C_1 \int_{-\infty}^{x} (f'(y))^2 dy + C_2 \int_{+\infty}^{x} (f'(y))^2 dy.$$

The first step of our algorithm is to consider the flows of the characteristic vector fields $\bar{V}_{G_i}$

$(i = 1, 2, 3)$ given by

$$
\begin{aligned}
\Phi_a^{1,*}(x) &= e^{-a}x \\
\Phi_a^{1,*}(u) &= u \\
\Phi_a^{1,*}(v) &= e^{a}v \\
\Phi_b^{2,*}(x) &= x - bu + \frac{b^2}{4}v \\
\Phi_b^{2,*}(u) &= u - \frac{b}{2}v \\
\Phi_b^{2,*}(v) &= v \\
\Phi_c^{3,*}(x) &= x - c \\
\Phi_c^{3,*}(u) &= u \\
\Phi_c^{3,*}(v) &= v.
\end{aligned}
$$

Therefore, the manifold $\mathcal{K}$ is defined by the union on $(a, b, c) \in \mathbb{R}^3$ of the zeros of

$$
\begin{aligned}
\tilde{g}_1(a, b, c, x, u, v) = \mathbf{\Phi}^*_{(a,b,c)}(g_1) &= u - \frac{be^a}{2}v - f\left(e^{-a}x - bu + \frac{b^2}{4}e^a v - c\right) \\
\tilde{g}_2(a, b, c, x, u, v) = \mathbf{\Phi}^*_{(a,b,c)}(g_2) &= e^a v - g\left(e^{-a}x - bu + \frac{b^2}{4}e^a v - c\right),
\end{aligned}
$$

and all their differential consequences with respect to x. If, on the manifold $\mathcal{K}$, we use the coordinate system (x, a, b, c), exploting Theorem 7.16, the three vector fields read

$$
\begin{aligned}
V_{G_1} &= -\partial_a \\
V_{G_2} &= -e^{-a}\partial_b \\
V_{G_3} &= -e^{-a}\partial_c.
\end{aligned}
$$

So the process A_t, B_t, C_t generating the solutions to the considered SPDEs are

$$
\begin{aligned}
dA_t &= -\sum_{\beta=1}^{r} H_\beta dW_t^\beta \\
dB_t &= e^{-A_t}dt \\
dC_t &= \sum_{\beta=1}^{r} K_\beta e^{-A_t} \circ dW_t^\beta.
\end{aligned}
$$

Since the system for A, B, C is triangular, it can be solved explicitly using only iterated Riemann and Itô integrals e.g. when one fixes the initial conditions $A_0 = B_0 = C_0 = 0$. In this way we obtain the solution to the SPDEs (7.35) and (7.36) with initial condition $U_0(x) = f(x)$ and $V_0(x) = g(x)$.
The solutions $U_t(x)$ and $V_t(x)$ to the initial SPDE can be obtained solving the following system of non-linear equations

$$
U_t(x) - \frac{B_t e^{A_t}}{2}V_t(x) - f\left(e^{-A_t}x - B_t U_t(x) + \frac{B_t^2 e^{A_t}}{4}V_t(x) - C_t\right) = 0 \tag{7.39}
$$

$$
e^{A_t}V_t(x) - g\left(e^{-A_t}x - B_t U_t(x) + \frac{B_t^2 e^{A_t}}{4}V_t(x) - C_t\right) = 0. \tag{7.40}
$$

Since f, g are bounded and with bounded derivatives, the system (7.39) and (7.40) admits a unique solution whenever A_t, B_t, C_t are in a suitable neighborhood of the origin. The fact that the system (7.39) and (7.40) admits a solution only if A, B, C are suitably bounded is related to the fact that the solutions to the deterministic Hunter-Saxton equation develop singularity in the first derivative in finite time (see, e.g., [104]). This property is conserved by the stochastic perturbation considered here.

Remark 7.19 *If we choose the functions $\xi_\beta(x)$ in equation (7.33) different from $K_\beta + H_\beta x$, not only we are no longer able to reduce equation (7.33) to a local one, but the generic solution to (7.33) is not finite dimensional. This does not mean that equation (7.33) has only infinite dimensional solutions. Indeed it is possible to verify, using the procedure proposed in [100], that equation (7.33) has infinite many families of (weak) finite dimensional solutions of the form*

$$U_t(x) = \sum_{i=1}^{k} P_t^i |x^i - Q_t^i|,$$

where $\sum_{i=1}^k P_t^i = 0$ and the process (P^i, Q^i) solves a finite dimensional SDE. The class of SPDEs possessing a large set of families of finite dimensional solutions of increasing dimension does not reduce to equations of the form (7.32) or (7.33): indeed we provide another example in Section 7.3.3. In our opinion the class of SPDEs which, despite not having all finite dimensional solutions, possess many families of finite dimensional solutions deserves more attention and a further detailed investigation.

7.3.3 A stochastic filtering model

In this section we consider an equation inspired by stochastic filtering. In particular, given two stochastic processes $X_t \in \mathcal{X}$ and $Y_t \in \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are two metric spaces (for example $\mathcal{X} = \mathbb{R}^k$ and $\mathcal{Y} = \mathbb{R}^h$), stochastic filtering theory faces the problem of describing the conditional probability $\mathbb{P}^X(\cdot|Y)$ of the process X given the process of observation Y. Although this one is in general an infinite dimensional problem, there are situations where it is possible to partially describe the probability $\mathbb{P}^X(\cdot|Y)$ using only a finite dimensional process B_t on a finite dimensional manifold B. When the filtering problem can be reduced to a finite dimensional process we speak of finite dimensional filters. Examples of such filters are the Kalman filter, the Benes filter and related ones (see [1, 17, 91]).

In many cases it is possible to reduce the problem of finding and studying finite dimensional filters to the problem of calculating finite dimensional solutions to particular SPDEs. Indeed, if $\mathcal{X} = M \subset \mathbb{R}^m$ and the process X_t, conditioned with respect to the process Y_t, solves a Markovian Brownian-motion-driven SDE, it is possible to describe the filtering problem using a second order linear SPDEs. There are different ways to obtain this description (in the following we use two of them). The most common method is to study a function $\rho_t(x)$ related to the conditional density $p_t(x)$ of the random variable X_t on M conditioned with respect to $Y_{[0,t]}$. In particular, it is possible to prove that $\rho_t(x)$ solves a second order linear SPDEs called Zakai equation. A finite dimensional filter is a filtering problem whose Zakai equation admits (some or all, depending on the definition) finite dimensional solutions.

This is the first problem where the research of finite dimensional solution to an SPDEs was studied in detail. Indeed, the theory proposed in the previous sections has been deeply influenced by the research in this field, and in particular by the works of Cohen de Lara [41, 42]. With our algorithm it is possible to calculate all the solutions to Zakai equation associated with the finite dimensional

filters appearing in the previous literature.
Instead of applying our algorithm to some already well studied finite dimensional filter, in this section we propose a new filtering problem for which we are able to calculate some finite dimensional solutions.
In particular, we consider the following SPDE

$$\begin{array}{rcl} dU_t(x) & = & \left(\frac{\sigma^2}{2}x\partial_x^2(U_t)(x) + \beta\partial_x(U_t)(x) + \alpha x\partial_x(U_t)(x) + \gamma x U_t(x) + \delta U_t(x)\right) dt \\ & & +\partial_x(U)_t(x) \circ dS_t^1 + x\partial_x(U_t)(x) \circ dS_t^2, \end{array} \tag{7.41}$$

where $\sigma, \beta, \alpha, \gamma, \delta$ are some constants and S^1, S^2 are the semimartingales driving the equation (below some restrictions on these constants and semimartingales will be discussed), which is related to several problems of stochastic filtering.
For example, if $S_t^1 = 0$ and $\beta = -\tilde{\beta} < 0, \alpha = -\tilde{\alpha} \leq 0, \gamma = 0, \delta = -\tilde{\alpha}$ and $\sigma^2 > \tilde{\beta}$ equation (7.41) is the Zakai equation giving the density of the conditioned probability of the following filtering problem

$$dX_t = (\beta + \alpha X_t)dt + \sigma\sqrt{X_t}dW_t + X_t d\tilde{S}_t^2, \tag{7.42}$$

with observation given by

$$d\tilde{Y}_t = d\tilde{S}_t^2, \tag{7.43}$$

where $\tilde{S}_t^2$ is any semimartingale independent from W_t and $S^2 = \tilde{S}_t^2 - \frac{1}{2}[\tilde{S}, \tilde{S}]_t$. Equation (7.42) can be considered as a general affine continuous process perturbed by a noise linearly dependent on the process itself. It is well known that one dimensional continuous markovian affine processes admit closed form for their probability densities. Unfortunately the perturbation (7.42) does not admit closed form solution even in the simplest case where $\tilde{S}_t^2$ is a Brownian motion.
In this case the interesting solutions to the SPDE (7.41) should satisfy $U_t(0) = 0$, $U_t(x) \geq 0$. These two conditions guarantee that, if $\int_0^{+\infty} U_0(x)dx = 1$, then $\int_0^{+\infty} U_t(x)dx = 1$. Therefore, using the techniques of [135], we can prove that any solution (smooth in space) to equation (7.41) is also a solution to the filtering problem (7.42) and (7.43). In this case, it is simple to prove that

$$\tilde{F} = \frac{\sigma^2}{2}xu_{xx} + \beta u_x, \quad G_1 = xu_x, \quad G_2 = u$$

form a three dimensional Lie algebra. For this reason, whenever we know a solution to the equation

$$\partial_a(f(x,a)) = \tilde{F}(f(x,a))$$

with $f(0,a) = 0$, $f(x,0) \geq 0$ and $\int_0^{+\infty} f(x,0)dx = 1$, we can apply our technique to equation (7.41).
In particular we consider the two dimensional manifold with boundary $\mathcal{H}$

$$u - f(x,a) = 0,$$

for $a \geq 0$. Since the characteristic vector fields

$$\begin{array}{rcl} \bar{V}_{G_1} & = & -x\partial_x + u_x\partial_{u_x} + ... + nu_{(n)}\partial_{u_{(n)}} + ... \\ \bar{V}_{G_2} & = & u\partial_u + u_x\partial_{u_x} + ... + u_{(n)}\partial_{u_{(n)}} + ... \end{array}$$

have characteristic flows

$$\begin{array}{rcl} \Phi_b^{1,*}(x) & = & e^{-b}x \\ \Phi_b^{1,*}(u) & = & u \\ \Phi_c^{2,*}(x) & = & x \\ \Phi_c^{2,*}(u) & = & e^c u, \end{array}$$

the manifold $\mathcal{K}$ is defined by the union on $(a,b,c) \in \mathbb{R}_+ \times \mathbb{R}^2$ solution to

$$e^c u - f(e^{-b}x, a) = 0,$$

and all its differential consequences. Using the coordinate system (x,a,b,c) on $\mathcal{K}$, by Theorem 7.18, we have

$$\begin{aligned} V_{\tilde{F}} &= e^{-b}\partial_a \\ V_{G_1} &= -\partial_b \\ V_{G_2} &= -\partial_c. \end{aligned}$$

Therefore, the solutions to SPDE (7.41) can be found solving the following triangular system

$$\begin{aligned} dA_t &= e^{-B_t}dt \\ dB_t &= -(\alpha dt + dS_t^2) \\ dC_t &= -\delta dt, \end{aligned}$$

and the finite dimensional solution to the Zakai equation is given by

$$U_t(x) = e^{-C_t} f(e^{-B_t}x, A_t).$$

Another interesting problem described by equation (7.41) for S_t^1 not equal to zero is the filtering problem

$$dX_t = (\beta + \alpha X_t)dt + \sigma\sqrt{X_t}dW_t + d\tilde{S}_t^1 + X_t d\tilde{S}_t^2 \tag{7.44}$$

with observations

$$\begin{aligned} dY_t^1 = d\tilde{S}_t^1 \\ dY_t^2 = d\tilde{S}_t^2, \end{aligned} \tag{7.45}$$

where we suppose that the $\mathbb{R}^2$ semimartingale $(\tilde{S}^1, \tilde{S}^2)$ is independent from W_t (instead we do not request that $\tilde{S}^1$ and $\tilde{S}^2$ are independent). Although for a general noise $\tilde{S}^1$ the solution X_t to equation (7.44) does not remain positive for all the times t, it is possible to provide sufficient conditions in order to ensure that this is the case. Suppose that $\tilde{S}^1$ is almost surely of bounded variation. This means that there are an increasing predictable process $\hat{S}^{1,i}$ and a decreasing predictable process $\hat{S}^{1,d}$ such that $\tilde{S}^1 = \hat{S}^{1,i} + \hat{S}^{1,d}$. If $\hat{S}^{1,d}$ is absolutely continuous and

$$\frac{d\hat{S}_t^{1,d}}{dt} + \beta > \frac{\sigma^2}{2},$$

for $t > 0$ and for any solution X_t to equation (7.44) such that $X_0 > 0$ almost surely, we have that $X_t > 0$ almost surely for any $t > 0$. The Zakai equation of filtering problem (7.44) and (7.45) has exactly the form (7.41). Unfortunately, for a deep reason that will be clarified below, we cannot deal directly with the Zakai equation of the filtering problem (7.44) and (7.45) and we have to consider another SPDE related with this filtering problem.
Given a bounded function $g \in C^2((0,+\infty))$, let us consider the process dependent on the space parameter $x \in (0,+\infty)$

$$V_t(x) = \mathbb{E}\left[g(X_T)e^{\int_t^T(\gamma X_s + \delta)ds} | \{X_t = x\} \vee \mathcal{G}_{t,T}\right], \tag{7.46}$$

where

$$\mathcal{G}_{t,T} = \sigma\{\tilde{S}_T^1 - \tilde{S}_s^1, \tilde{S}_T^2 - \tilde{S}_s^2 | s \in [t,T]\}.$$

The process V_t is adapted with respect to the inverse filtration $\mathcal{G}_{t,T}$ with $t \in (0,T]$. If $\tilde{S}^2_T - \tilde{S}^2_{T-t}$ is a semimartingale with respect to the filtration $\mathcal{G}_T, t$ (an example of such processes is given by Brownian motions or solutions to Markovian Brownian motion driven SDEs), we can generalize Theorem 2.1 of [148] (see also [18, 149]) proving that $U_t(x) = V_{T-t}(x)$ solves equation (7.41) with $S^1_t = \tilde{S}^1_T - \tilde{S}^1_{T-t}$ and $S^2_t = \tilde{S}^2_T - \tilde{S}^2_{T-t} - \frac{1}{2}[\tilde{S}^2_T - \tilde{S}^2_{T-t}, \tilde{S}^2_T - \tilde{S}^2_{T-t}]_t$. If we can explicitly find solutions to equation (7.41), we have a closed formula for the conditional expected value (7.46), extending in this way the closed formula of some expected values of Markovian continuous affine one dimensional processes.
It is important to note that any bounded smooth solution to equation (7.41) is a solution to the problem (7.46) since such kinds of solutions are unique (this fact can be proven using the coordinate change $\tilde{x} = e^{-S^2_t}x$ and then using some standard reasoning based on the maximum principle for parabolic PDEs see, e.g. Theorem 4.1 and Theorem 4.3 of [70]). For all these reasons we are interested in finding solutions to equation (7.41) when $\beta > 0$ and $\gamma < 0$.
We remark that, if S^2 is not identically zero, we do not have a finite dimensional Lie algebra. Indeed, in this case, if we put

$$\begin{aligned} F &= xu_{xx} \\ G_3 &= u_x \\ G_4 &= xu, \end{aligned}$$

we have

$$[F,[F,\underbrace{[...[F}_{n \text{ times}},u_x]...]]] = n!u_{(n+1)} \tag{7.47}$$

and so F, G_1, G_2, G_3, G_4 cannot form a finite dimensional Lie algebra on all the space $J^\infty(\mathbb{R}_+,\mathbb{R})$. This means that the solution $U_t(x)$ to equation (7.41) with a general initial condition $U_t(x) = f(x)$ is not finite dimensional. This is why we choose to consider the problem (7.46) instead of the Zakai equation related to the filtering problem (7.44) and (7.45). Indeed, a smooth solution to the Zakai equation on $(0,+\infty)$ should satisfy $U_t(0) = 0$ in order to be the conditional probability density of the filtering problem (7.44) and (7.45). Unfortunately we are not able to construct solutions to equations of the form (7.41) satisfying this property if $S^1 \neq 0$. This is due to the fact that $F, G_1, ..., G_4$ do not form a finite dimensional Lie algebra. Conversely a sufficient condition ensuring that a smooth solution $U_t(x)$ to equation (7.41) represents the integral (7.46) is that $U_t(x)$ is bounded in $(0,+\infty)$. We are able to construct bounded finite dimensional solutions to the equation (7.41), so giving (for suitable functions g) the explicit expression of the conditional expectation (7.46).
In order to construct families of finite dimensional solutions to equation (7.41) we exploit the particular form of the commutators (7.47). Indeed let $\mathcal{K}$ be the finite dimensional submanifold of $J^\infty(\mathbb{R}_+,\mathbb{R})$ defined by

$$h = u_{(n)} + \sum_{k=0}^{n-1} \mu^k u_{(k)} = 0, \tag{7.48}$$

and all its differential consequences with respect to x considering μ^k as constants. Using that

$$\begin{aligned} [F, u_{(k)}] &= ku_{(k+1)} \\ [G_1, u_{(k)}] &= ku_{(k)} \\ [G_2, u_{(k)}] &= 0 \\ [G_3, u_{(k)}] &= 0 \\ [G_4, u_{(k)}] &= ku_{(k-1)}, \end{aligned} \tag{7.49}$$

we are able to prove that $V_F, V_{G_1}, ..., V_{G_4} \in T\mathcal{K}$ on $\mathcal{K}$. It is important to note that the previous relation does not hold if we consider the submanifold $\tilde{\mathcal{K}}$ defined by equation (7.48) with all its

differential consequences where μ^k are fixed constant and not variable constants (with respect to x). This situation is similar to the previous section where $\mathcal{K}$ is defined by $\mathbf{\Phi}^*_\alpha(h^i)$ and all its differential consequences, where $h^i=0$ defines the submanifold $\mathcal{H}$.
If we choose on the manifold $\mathcal{K}$ the coordinate system given by $(x,\mu^0,...,\mu^{n-1},u,....,u_{(n-1)})$, it is possible to prove that $V_F(\mu^k),V_{G_i}(\mu^k)$ depend only on $\mu^0,...,\mu^k$. Furthermore, using relation (7.49), it is possible to compute $V_F(\mu^k),V_{G_i}(\mu^k)$. In order to illustrate the explicit calculations, we consider the submanifold $\mathcal{K}$ defined by

$$h=u_{xx}+\lambda u_x+\mu u=0$$

and we calculate V_F. We have that

$$\begin{aligned}
V_F(h)|_{\mathcal{K}} &= (V_F(u_{xx})+\lambda V_F(u_x)+\mu V_F(u)+V_F(\lambda)u_x+V_F(\mu)u)\,|_{\mathcal{K}}\\
&= ([F,u_{xx}]+\lambda[F,u_x]+\mu[F,u]+V_F(\lambda)u_x+V_F(\mu)u)\,|_{\mathcal{K}}\\
&= (2u_{xxx}+\lambda u_{xx}+V_F(\lambda)u_x+V_F(\mu)u)\,|_{\mathcal{K}}\\
&= (2(-\lambda u_{xx}-\mu u_x)+\lambda(-\lambda u_x-\mu u)+V_F(\lambda)u_x+V_F(\mu)u)\,|_{\mathcal{K}}\\
&= \big((V_F(\lambda)-2\mu+\lambda^2)u_x+(V_F(\mu)+\lambda\mu)u\big)\,|_{\mathcal{K}}.
\end{aligned}$$

Since u_x,u can take any values on $\mathcal{K}$ we can find the expression for $V_F(\lambda),V_F(\mu)$ on $\mathcal{K}$. Using similar methods we have

$$\begin{aligned}
V_F &= (-\lambda^2+2\mu)\partial_\lambda-\mu\lambda\partial_\mu-(\lambda u_x+\mu u)\partial_{u_x}\\
V_{G_1} &= \lambda\partial_\lambda+2\mu\partial_\mu+xu_x\partial_u+(u_x-x(\lambda u_x+\mu u))\partial_{u_x}\\
V_{G_2} &= u\partial_u+u_x\partial_{u_x}\\
V_{G_3} &= u_x\partial_u-(\lambda u_x+\mu u)\partial_{u_x}\\
V_{G_4} &= -2\partial_\lambda-\lambda\partial_\mu+xu\partial_u+(xu_x+u)\partial_{u_x}.
\end{aligned}$$

The SDE for $L_t,M_t,U_t(0),U_{x,t}(0)$ becomes

$$\begin{aligned}
dL_t &= \left(\frac{\sigma^2}{2}(-L_t^2+2M_t)+\alpha L_t-2\gamma\right)dt+L_t\circ dS_t^2\\
dM_t &= \left(-\frac{\sigma^2}{2}L_tM_t+2\alpha M_t-\gamma L_t\right)dt+2M_t\circ dS_t^2\\
\begin{pmatrix} dU_t(0)\\ dU_{x,t}(0)\end{pmatrix} &= \begin{pmatrix} \delta & \beta\\ -M_t\left(\frac{\sigma^2}{2}+\beta\right)+\gamma & -L_t\left(\frac{\sigma^2}{2}+\beta\right)+\alpha+\delta\end{pmatrix}\cdot\begin{pmatrix} U_t(0)\\ U_{x,t}(0)\end{pmatrix}dt+\\
&\quad+\begin{pmatrix} 0 & 1\\ -M_t & -L_t\end{pmatrix}\cdot\begin{pmatrix} U_t(0)\\ U_{x,t}(0)\end{pmatrix}\circ dS_t^1+\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix} U_t(0)\\ U_{x,t}(0)\end{pmatrix}\circ dS_t^2
\end{aligned}$$

The solution to SPDE (7.41) can be obtained solving the system

$$\begin{aligned}
\partial_x(U_t)(x) &= U_{x,t}(x)\\
\partial_x(U_{x,t}(x)) &= -L_tU_{x,t}(x)-M_tU_t(x).
\end{aligned}$$

Thus we have two possibilities. If $\lambda_0^2-4\mu_0>0$ then

$$U_t(x)=A_te^{C_tx}+B_te^{D_tx},$$

where

$$
\begin{aligned}
C_t &= \frac{-L_t + \sqrt{L_t^2 - 4M_t}}{2} \\
D_t &= \frac{-L_t - \sqrt{L_t^2 - 4M_t}}{2} \\
A_t &= \frac{D_t U_t(0) - U_{x,t}(0)}{D_t - C_t} \\
B_t &= \frac{-C_t U_t(0) + U_{x,t}(0)}{D_t - C_t}.
\end{aligned}
$$

If $\lambda_0^2 - 4\mu_0 < 0$ we have

$$U_t(x) = e^{R_t x}(A_t \cos(O_t x) + B_t \sin(O_t x)),$$

where

$$
\begin{aligned}
R_t &= \frac{-L_t}{2} \\
O_t &= \frac{\sqrt{4M_t - L_t^2}}{2} \\
A_t &= U_t(0) \\
B_t &= \frac{-R_t U_t(0) + U_{x,t}(0)}{O_t}.
\end{aligned}
$$

Future developments

This thesis can be seen as part of a wider project aiming at developing a stochastic Lie symmetry theory for the study of stochastic differential equations. In this section we sketch some possible future lines of development based on the results of this work.

Regarding the Brownian-motion-driven SDEs there are many possible generalizations of the results proposed in this thesis. A first possibility is to extend the group of weak stochastic transformations proposed in Chapter 1 in order to include a change of measure, exploiting the well-known Girsanov theorem. This extension is quite promising since it could explain probabilistically all the infinitesimal symmetries of the Kolmogorov equation associated with the considered SDE. This achievement will be very useful in the study of affine processes (see [57]) since, despite being considered as a prototypical class of integrable stochastic systems, they do not have the weak symmetries. Another possible extension is to enlarge our family of transformations in order to consider stochastic transformations of non-Markovian type. This new class of transformations may be studied exploiting the path-dependent stochastic calculus (see [44]) or the rough paths theory (see [73]). The latter extension could be important for understanding some strange phenomena occurring in Lie symmetry analysis of SDEs such as the difficulty of including explicitly the time in stochastic transformations (in this thesis we consider only autonomous equations and the symmetries of non-autonomous ones can only recovered using an enlargement of the set of dependent variables) or the fact that we cannot recover the integrability property of scalar linear SDEs by directly studying the symmetries of the equations but only by embedding it in a two dimensional more symmetric system (see Section 2.3.2).
Another possible development of our work concerns the concept of invariant numerical schemes introduced in Chapter 3 and further developed in Section 5.3. In particular it would be interesting to find a method for a direct construction of symmetry-preserving discretization without using coordinate changes and the standard Euler and Milstein discretization schemes as done in Chapter 3. Such a result would be essential for generalizing the theoretical estimates given by Theorem 3.5 and Theorem 3.6 for linear SDEs to more general SDEs.

Regarding SDEs driven by general semimartingales in this thesis we give the first concrete method for finding symmetries in particular by introducing the determining equations (5.9). A peculiarity of these determining equations is that they are non-linear and non-local with respect to the coefficients of the infinitesimal symmetries and so they are not so easily solvable as the determining equations of deterministic differential equations or of Brownian-motion-driven SDEs. For this reason it would be important, for the applicability of the theory, to find some methods for solving equations (5.9). Since looking for a family of symmetries for an ODE is much more simple than looking for a single symmetry, in order to simplify equation (5.9) it could be useful considering a one-parameter family of SDE instead of a single equation.
A remarkable case included in the symmetry theory proposed in Part II is the case of iterated

random maps. In this particular setting our theory can be widely extended and simplified as the example of the determination of weak symmetries of numerical schemes in Section 5.3 suggests.
Finally in Part II we introduce the new concepts of gauge and time symmetries of a semimartingale proposing an almost completely unexplored concept of invariance for semimartingales. In this framework it would be interesting looking for a characterization of all the semimartingales with a fixed gauge symmetry group or a fixed time symmetry, providing in this way a suitable generalization of the celebrated de Finetti theorem (see [109]), which characterizes the class of random variables which are invariant with respect to permutations. Another interesting development could be the extension of the concept of gauge symmetries from the case of an action induced by the deterministic action of a Lie group to more general actions.

Finally regarding the SPDEs many extensions can be proposed. First of all, in this thesis and in the previous research on the subject of finite dimensional solutions to SPDEs, a great emphasis is given to the case where all the solutions of the considered SPDE are finite dimensional with the same dimension. In Section 7.3 we provide some interesting examples of equations admitting infinite many finite dimensional solutions of different dimensions. This case seems to be more common than the previous one, seems to have some relations with the theory of infinite dimensional integrable systems and, in our opinion, deserves further investigation.
Moreover, in this thesis we consider SPDEs very regular in space but, if the noise of a SPDE is too irregular, the typical solution is very irregular. For this reason it would be very interesting to apply Lie symmetry analysis techniques to irregular equations exploiting the stochastic calculus in infinite dimension (see [48]) or the more recent theories of regularity structures (see [88]) and of paracontrolled distributions (see [84]).

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