Modularity Results for Interpolation, Amalgamation and Superamalgamation

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Abstract

Wolter in [38] proved that the Craig interpolation property transfers to fusion of normal modal logics. It is well-known [21] that for such logics Craig interpolation corresponds to an algebraic property called *superamalgamability*. In this paper, we develop model-theoretic techniques at the level of first-order theories in order to obtain general combination results transferring *quantifier-free interpolation* to unions of theories over non-disjoint signatures. Such results, once applied to equational theories sharing a common Boolean reduct, can be used to prove that superamalgamability is modular also in the non-normal case. We also state that, in such non-normal context, superamalgamability corresponds to a strong form of interpolation that we call "comprehensive interpolation property" (which consequently transfers to fusions).

Keywords: Interpolation, Fusion, Modal Logic, Superamalgamability

1. Introduction

Craig's interpolation theorem [7] is a model-theoretic result which applies to first order formulae and states that whenever a formula ϕ entails a formula ψ, then it is possible to find a third formula θ which can be interpolated beetwen ϕ and ψ, and which is defined over their common symbols. Interpolation theory has been recently introduced in verification, after the work of McMillan (see, e.g., [23]), and it has also a long tradition in non-classical logics (see for instance the seminal papers by L.L. Maksimova [20],[21]). In

particular, the specific form of interpolation for modal logic is the following: a modal logic L is said to enjoy the (local) interpolation property iff, whenever we consider two modal formulae t_1 and t_2 such that $\vdash_L t_1 \to t_2$ holds, it is possible to find a modal formula u such that (i) $\vdash_L t_1 \to u$, (ii) $\vdash_L u \to t_2$, and (iii) the variables of u are in common with both t_1 and t_2. In this context, of great importance is the study of combination of logics, focusing on the transfer of significant properties like interpolation. The simplest way of combining modal logics is given by the well-known notion of *fusion*: considering two modal logics L_1 and L_2 over the modal signatures Σ_M^1 and Σ_M^2 such that $\Sigma_M^1 \cap \Sigma_M^2 = \emptyset$, the *fusion* $L_1 \oplus L_2$ is the least modal logic, over the modal signature $\Sigma_M^1 \cup \Sigma_M^2$, that contains $L_1 \cup L_2$.

In [38] Wolter proved that the fusion of two interpolating *normal* modal logics is also interpolating. However, the non-normal case remained open and in this paper we try to attack it: *we show that superamalgamability transfers to fusions in the general non-normal context*. It is well-known that superamalgamability (which is an algebraic condition) is equivalent to interpolation in the normal case [21]. Thus, our result implies Wolter's result and we prove that in the general non-normal case our result gives a fusion transfer theorem for a new strong form of interpolation (covering both local and global interpolation) which we call "comprehensive interpolation property".

The above result is obtained as a *corollary of combination techniques for first-order theories*: in fact, we specialize to the modal context modular conditions of combinability that generalize various previous works. The study of the modularity property of *quantifier-free* interpolation in first-order theories was first started in [40], where the disjoint signatures convex case was solved; in [5] - the journal version of [4] - the non-convex (still disjoint) case was also thoroughly investigated. In attacking combination problems for non-disjoint signatures, we follow the model-theoretic approach successfully employed in [11], [2], [14], [29], [26], [27], [28] for combined satisfiability; this approach relies on model-theoretic notions like T_0-compatibility.

The paper is organized in five sections. In Section 2, we introduce notations and basic ingredients concerning first order logic. In Section 3 we obtain a first general result (Theorem 3.1) which gives sufficient conditions for the transfer of quantifier-free interpolation in the non-disjoint signatures case; Theorem 3.1 has all known results for disjoint signatures case [5], [40] as an immediate consequence. In Section 4 we focus our attention on universal Horn theories, in order to obtain a modular condition (Theorem 4.1) referring to "minimal" amalgams. In Section 5, we apply Theorem 4.1 to modal logic:

2

we prove that superamalgamability is a modular condition (Corollary 5.2), since it is equivalent to the combination condition of Theorem 4.1 in case the background theory is the theory of Boolean Algebras. Then, we syntactically characterize superamalgamability, by defining the notion of "comprehensive interpolation". Comprehensive interpolation, in the normal case, is nothing but standard interpolation, but in the non-normal case it looks like a stronger property, which transfers to fusions as a consequence of our results (Theorem 5.1).

2. Formal Preliminaries in First Order Logic

We adopt the usual first-order syntactic notions of signature, term, atom, (ground) formula, sentence, and so on. Let Σ be a first-order signature; we assume the binary equality predicate symbol '=' to be added to any signature (so, if $\Sigma = \emptyset$, then Σ just contains equality). The signature obtained from Σ by adding to it a set $\underline{a}$ of new constants (i.e., 0-ary function symbols) is denoted by $\Sigma^{\underline{a}}$. A *literal* is an atomic formula or the negation of an atomic formula; a *clause* is a disjunction of literals and a *positive clause* is a disjunction of atoms. A formula is *quantifier-free* (or open) iff it does not contain quantifiers. A Σ-*theory* T is a set of sentences (called the axioms of T) in the signature Σ and it is *universal* iff it has universal closures of open formulae as axioms.

We also assume the usual first-order notion of interpretation and truth of a formula, with the proviso that the equality predicate = is always interpreted as the identity relation. A formula φ is *satisfiable* in $\mathcal{M}$ iff its *existential* closure is true in $\mathcal{M}$. A Σ-structure $\mathcal{M}$ is a *model* of a Σ-theory T (in symbols $\mathcal{M} \models T$) iff all the sentences of T are true in $\mathcal{M}$. If φ is a formula, $T \models \varphi$ ('φ *is a logical consequence of T*') means that the universal closure of φ is true in all the models of T; T is *consistent* iff it has a model. A sentence φ is T-consistent iff $T \cup \{\varphi\}$ is consistent. A Σ-theory T is *complete* iff for every Σ-sentence φ, either φ or $\neg\varphi$ is a logical consequence of T. T admits *quantifier elimination* iff for every formula $\varphi(\underline{x})$ there is a quantifier-free formula $\varphi'(\underline{x})$ such that $T \models \varphi(\underline{x}) \leftrightarrow \varphi'(\underline{x})$ (notations like $\varphi(\underline{x})$ mean that φ has free variables only among the tuple $\underline{x}$).

If $\Sigma_0 \subseteq \Sigma$ is a subsignature of Σ and if $\mathcal{M}$ is a Σ-structure, the Σ_0-*reduct* of $\mathcal{M}$ is the Σ_0-structure $\mathcal{M}_{|\Sigma_0}$ obtained from $\mathcal{M}$ by forgetting the interpretation of function and predicate symbols from $\Sigma \setminus \Sigma_0$. A Σ-*homomorphism* (or, simply, a homomorphism) between two Σ-structures $\mathcal{M}$ and $\mathcal{N}$ is any

3

mapping $\mu : |\mathcal{M}| \longrightarrow |\mathcal{N}|$ among the support sets $|\mathcal{M}|$ of $\mathcal{M}$ and $|\mathcal{N}|$ of $\mathcal{N}$ satisfying the condition

$$\mathcal{M} \models \varphi \quad \Rightarrow \quad \mathcal{N} \models \varphi \tag{1}$$

for all $\Sigma^{|\mathcal{M}|}$-atoms φ (here $\mathcal{M}$ is regarded as a $\Sigma^{|\mathcal{M}|}$-structure, by interpreting each additional constant $a \in |\mathcal{M}|$ into itself and $\mathcal{N}$ is regarded as a $\Sigma^{|\mathcal{M}|}$-structure by interpreting each additional constant $a \in |\mathcal{M}|$ into $\mu(a)$). In case condition (1) holds for all $\Sigma^{|\mathcal{M}|}$-literals, the homomorphism μ is said to be an *embedding* and if it holds for all first order formulae, the embedding μ is said to be *elementary*. If $\mu : \mathcal{M} \longrightarrow \mathcal{N}$ is an embedding which is just the identity inclusion $|\mathcal{M}| \subseteq |\mathcal{N}|$, we say that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ or that $\mathcal{N}$ is an *extension* of $\mathcal{M}$. A Σ-structure $\mathcal{M}$ is said to be *generated by* a set X included in its support $|\mathcal{M}|$ iff there are no proper substructures of $\mathcal{M}$ including X.

Given a signature Σ and a Σ-structure $\mathcal{A}$, we indicate with $\Delta_\Sigma(\mathcal{A})$ the *diagram* of $\mathcal{A}$: this is the set of sentences obtained by first expanding Σ with a fresh constant $\bar{a}$ for every element a from $|\mathcal{A}|$ and then taking the set of ground $\Sigma^{|\mathcal{A}|}$-literals which are true in $\mathcal{A}$ (under the natural expanded interpretation mapping $\bar{a}$ to a).

2.1. Model completion and T_0-compatibility

We recall a standard notion in Model Theory, namely the notion of a *model completion* of a first order theory [6] (we limit the definition to universal theories, because we shall use only this case):

Definition 2.1. Let T_0 be a universal Σ_0-theory and let $T_0^\star \supseteq T_0$ be a further Σ_0-theory; we say that $T_0^\star$ is a model completion of T_0 iff: (i) every model of T_0 can be embedded into a model of $T_0^\star$; (ii) for every model $\mathcal{M}$ of T_0, we have that $T_0^\star \cup \Delta_{\Sigma_0}(\mathcal{M})$ is a complete theory in the signature $\Sigma_0^{|\mathcal{M}|}$.

Since T_0 is universal, condition (ii) is equivalent to the fact that $T_0^\star$ *has quantifier elimination*; we recall also that the model completion $T_0^\star$ of a theory T_0 is unique, if it exists (see [6] for these results and for examples).

We also recall the concept of T_0-compatibility [11, 14], which is crucial for our combination technique.

Definition 2.2. Let T be a theory in the signature Σ and let T_0 be a universal theory in a subsignature $\Sigma_0 \subseteq \Sigma$. We say that T is T_0-*compatible* iff $T_0 \subseteq T$ and there is a Σ_0-theory $T_0^\star$ such that:

4

(i) $T_0^\star$ is a model completion of T_0;

(ii) every model of T can be embedded, as a Σ-structure, into a model of $T \cup T_0^\star$.

Notice that if T_0 is the empty theory over the empty signature, then $T_0^\star$ is the theory axiomatizing an infinite domain, and the requirement of T_0-compatibility is equivalent to the stably infinite requirement of the Nelson-Oppen schema [25, 36] (in the sense that T is T_0-compatible iff it is stably infinite). We remind that a theory T is stably infinite iff every T-satisfiable quantifier-free formula (from the signature of T) is satisfiable in an infinite model of T. By compactness, it is possible to show that T is stably infinite iff every model of T embeds into an infinite one.

We underline that T_0-compatibility is a modular condition. The following result is proved in [11] (as Proposition 4.4):

Proposition 2.1. *Let T_1 be a Σ_1-theory and let T_2 be a Σ_2-theory; suppose they are both compatible with respect to a Σ_0-theory T_0 (where $\Sigma_0 := \Sigma_1 \cap \Sigma_2$). Then $T_1 \cup T_2$ is T_0-compatible too.*

2.2. Interpolation and Amalgamation

We say that a theory T *has quantifier-free interpolation* iff the following hold, for every pair of quantifier free formulae $\varphi(\underline{x}, \underline{y}), \psi(\underline{y}, \underline{z})$: if $T \models \varphi(\underline{x}, \underline{y}) \to \psi(\underline{y}, \underline{z})$, then there exists a quantifier-free formula $\theta(\underline{y})$ such that $T \models \varphi(\underline{x}, \underline{y}) \to \theta(\underline{y})$ and $T \models \theta(\underline{y}) \to \psi(\underline{y}, \underline{z})$. We underline that the requirement that θ is quantifier-free is essential: in general such a $\theta(\underline{y})$ exists by the Craig interpolation theorem, but it is not quantifier-free even if φ, ψ are such.

Quantifier-free interpolation property can be semantically characterized using the following notions, introduced in [3, 5] (see [18] for several examples):

Definition 2.3. A theory T has the *sub-amalgamation property* iff, for given models $\mathcal{M}_1$ and $\mathcal{M}_2$ of T sharing a common substructure $\mathcal{A}$, there exists a further model $\mathcal{M}$ of T endowed with embeddings $\mu_1 : \mathcal{M}_1 \longrightarrow \mathcal{M}$ and $\mu_2 : \mathcal{M}_2 \longrightarrow \mathcal{M}$ whose restrictions to the support of $\mathcal{A}$ coincide. The triple $(\mathcal{M}, \mu_1, \mu_2)$ (or, by abuse, $\mathcal{M}$ itself) is said to be a *T-sub-amalgam* of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{A}$.

Definition 2.4. A theory T has the *strong sub-amalgamation property* if the T-sub-amalgam $(\mathcal{M}, \mu_1, \mu_2)$ of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{A}$ can be chosen so as to satisfy the

following additional condition: if for some m_1, m_2 we have $\mu_1(m_1) = \mu_2(m_2)$, then there exists an element a in $|\mathcal{A}|$ such that $m_1 = a = m_2$.

If T is universal, then every substructure of a model of T is itself a model of T: in these cases, we shall drop the prefix sub- and directly speak of 'amalgamability', 'strong amalgamability' and 'T-amalgam'. The following fact is proved in [5], as Theorem 3.3:

Theorem 2.1. *A theory T has the sub-amalgamation property iff it admits quantifier-free interpolants.*

3. Conditions for Combination

The main result from [5] says that if T_1, T_2 have disjoint signatures, are both stably infinite and both enjoy the strong sub-amalgamation property, then the combined theory $T_1 \cup T_2$ also has the strong sub-amalgamation property[1] (and so it has quantifier-free interpolation).

In this paper, we try to extend the above results to the non-disjoint signatures case. The idea, already shown to be fruitful for combined satisfiability problems in [11], is to use T_0-compatibility as the proper generalization of stable infiniteness.

We shall first obtain a rather abstract sufficient condition for the transfer of quantifier-free interpolation property to combined theories; nevertheless, we show that such a sufficient condition generalizes the disjoint signatures result from [5]. Then we move to the case in which the shared theory T_0 is Horn and obtain as a corollary a specialized result which is quite effective in modal logic applications.

3.1. Sub-amalgamation schemata

Let T_0, T be theories in their respective signatures Σ_0, Σ such that $\Sigma_0 \subseteq \Sigma$, T_0 is universal and $T_0 \subseteq T$. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are Σ-models of T with a common substructure $\mathcal{A}$, we call the triple $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ a T-*fork* (or, simply, a *fork*).

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[1]It is possible to characterize syntactically strong sub-amalgamability in terms of a suitable 'equality interpolating' condition [5]. That sub-amalgamability needs to be strenghtened to strong sub-amalgamability in order to get positive combination results is demonstrated by converse facts also proved in [5].

6

The *sub-amalgamation schema* $\sigma_{T_0}^T$ (of T over T_0) is the following function, associating sets of T_0-amalgams with T-forks: [2]

$$\sigma_{T_0}^T[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})] := \left\{ \begin{array}{l} \text{the set of all } (\mathcal{B}, \nu_1, \nu_2) \text{ s.t.} \\ (i) \ (\mathcal{B}, \nu_1, \nu_2) \text{ is a } T_0\text{-amalgam of the } \Sigma_0\text{-reducts of} \\ \qquad \mathcal{M}_1 \text{ and } \mathcal{M}_2 \text{ over the } \Sigma_0\text{-reduct of } \mathcal{A}; \\ (ii) \ \mathcal{B} \text{ is generated, as a } \Sigma_0\text{-structure, by the union} \\ \qquad \text{of the images of } \nu_1 \text{ and } \nu_2; \\ (iii) \ (\mathcal{B}, \nu_1, \nu_2) \text{ is embeddable in the } \Sigma_0\text{-reduct of a} \\ \qquad T\text{-sub-amalgam of the fork } (\mathcal{M}_1, \mathcal{M}_2, \mathcal{A}). \end{array} \right\}$$

Condition (iii) means that there is a T-sub-amalgam $(\mathcal{M}, \mu_1, \mu_2)$ such that $\mathcal{B}$ is a Σ_0-substructure of $\mathcal{M}$ and that μ_1, μ_2 coincide with ν_1, ν_2 on their domains.

Condition (ii) ensures that, disregarding isomorphic copies, $\sigma_{T_0}^T[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$ is a set and not a proper class. Recall that T_0 is universal, so that substructures of models of T_0 are also models of T_0. This ensures that the following Proposition trivially holds:

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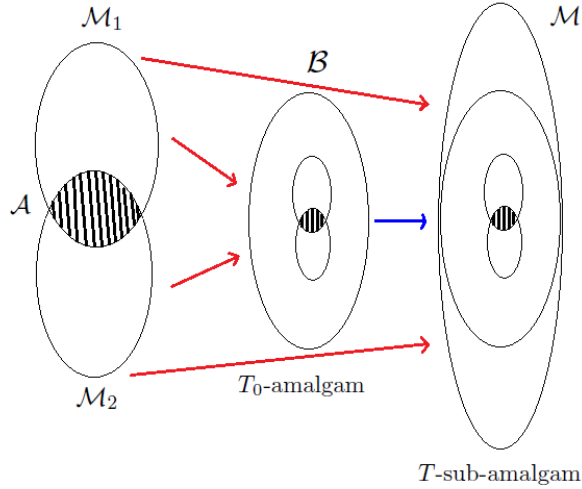
[2]It is not difficult to realize (using well-known Löwenheim-Skolem theorems [6]) that one can get all the results in the paper by limiting this definition to forks among structures whose cardinality is bounded by the cardinality of the set of the formulae in our signatures (signatures are finite or countable in all practical cases).

Proposition 3.1. *T is sub-amalgamable iff $\sigma_{T_0}^T$ is not empty (i.e. iff we have that $\sigma_{T_0}^T[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})] \neq \emptyset$, for all forks $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$).*

We are ready to formulate the main combination result, which is the following Theorem: one side of the inclusion of the statement is immediate, whereas, for the other one, T_0-compatibility is needed.

Theorem 3.1 (Combination Theorem). *Let T_1 and T_2 be two theories in their respective signatures Σ_1, Σ_2; assume that they are both T_0-compatible where T_0 is a further universal theory in the signature $\Sigma_0 := \Sigma_1 \cap \Sigma_2$. The following holds for the sub-amalgamation schema of $T_1 \cup T_2$ over T_0:*

$$\sigma_{T_0}^{T_1 \cup T_2}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})] \;=\; \sigma_{T_0}^{T_1}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_1}] \cap \sigma_{T_0}^{T_2}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_2}]$$

for every $(T_1 \cup T_2)$-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ (here, with $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_i}$ we indicate the T_i-fork obtained by taking reducts to the signature Σ_i).

Proof of Combination Theorem. The proof of Theorem 3.1 splits into a few steps. We begin by recalling some standard results from model theory and by introducing some preliminary lemmata. The following easy fact is proved in [5], as Lemma 3.7:

Lemma 3.1. *Let Σ_1, Σ_2 be two signatures and $\mathcal{A}$ be a $\Sigma_1 \cup \Sigma_2$-structure; then $\Delta_{\Sigma_1 \cup \Sigma_2}(\mathcal{A})$ is logically equivalent to $\Delta_{\Sigma_1}(\mathcal{A}) \cup \Delta_{\Sigma_2}(\mathcal{A})$.*

Proof. As a general fact, let us first observe that every diagram is equivalent to its flat subdiagram, in the following sense. Let $\mathcal{B}$ be a Σ-structure; the flat Σ-subdiagram of $\mathcal{B}$ is the set $\Delta_\Sigma^f(\mathcal{B})$ of literals of the kind

$$f(a_1, \ldots, a_n) = b, \quad a_1 \neq a_2, \quad P(a_1, \ldots, a_n), \quad \neg P(a_1, \ldots, a_n)$$

that are true in $\mathcal{B}$ (here $a_1, \ldots, a_n, b$ are free constants naming elements from $|\mathcal{B}|$). That every diagram is equivalent to its flat subdiagram can be easily proved from the fact (to be shown by induction) that for every $\Sigma \cup |\mathcal{B}|$-ground term t there is $a \in |\mathcal{B}|$ such that $\Delta_\Sigma^f(\mathcal{B}) \vdash t = a$.

Now we have that $\Delta_{\Sigma_1 \cup \Sigma_2}(\mathcal{A})$ is logically equivalent to $\Delta_{\Sigma_1 \cup \Sigma_2}^f(\mathcal{A})$ and the latter is $\Delta_{\Sigma_1}^f(\mathcal{A}) \cup \Delta_{\Sigma_2}^f(\mathcal{A})$ which in turn is equivalent to $\Delta_{\Sigma_1}(\mathcal{A}) \cup \Delta_{\Sigma_2}(\mathcal{A})$. $\square$

An easy but nevertheless important basic result, called the *Robinson Diagram Lemma* [6], says that, given any Σ-structure $\mathcal{B}$, the embeddings

$\mu : \mathcal{A} \longrightarrow \mathcal{B}$ are in bijective correspondence with expansions of $\mathcal{B}$ to $\Sigma^{|\mathcal{A}|}$-structures which are models of $\Delta_{\Sigma}(\mathcal{A})$. The expansions and the embeddings are related in the obvious way: $\bar{a}$ is interpreted as $\mu(a)$.

The following Lemma is proved using this property of diagrams:

Lemma 3.2. *Let T_0, T be theories in their respective signatures Σ_0, Σ such that $\Sigma_0 \subseteq \Sigma$ and $T_0 \subseteq T$; let $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ be a T-fork. For a T_0-amalgam $(\mathcal{B}, \nu_1, \nu_2)$ the following conditions are equivalent (we suppose that the support of $\mathcal{B}$ is disjoint from the supports of $\mathcal{M}_1, \mathcal{M}_2$):*

(i) *$(\mathcal{B}, \nu_1, \nu_2) \in \sigma_{T_0}^{T}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$;*

(ii) *the following theory $(*)$ is consistent*

$$T \cup \Delta_{\Sigma}(\mathcal{M}_1) \cup \Delta_{\Sigma}(\mathcal{M}_2) \cup \Delta_{\Sigma_0}(\mathcal{B}) \cup$$
$$\cup \{\bar{a}_1 = \bar{b} \parallel b \in |\mathcal{B}|, a_1 \in |\mathcal{M}_1|, \nu_1(a_1) = b\} \cup$$
$$\cup \{\bar{a}_2 = \bar{b} \parallel b \in |\mathcal{B}|, a_2 \in |\mathcal{M}_2|, \nu_2(a_2) = b\}.$$

Furthermore, in case T is T_0-compatible, we can equivalently replace T by $T \cup T_0^{\star}$ in the theory $(*)$ mentioned in (ii) *above.*

Proof. By the above mentioned property of diagrams, the consistency of $(*)$ means that there is a model $\mathcal{N} \models T$ and there are three embeddings

$$\mu_1 : \mathcal{M}_1 \longrightarrow \mathcal{N}, \quad \mu_2 : \mathcal{M}_2 \longrightarrow \mathcal{N}, \quad \nu : \mathcal{B} \longrightarrow \mathcal{N}$$

(the last one is a Σ_0-embedding, the first two are Σ-embeddings) such that $\nu \circ \nu_1 = \mu_1$ and $\nu \circ \nu_2 = \mu_2$. Since μ_1, μ_2 agree on the support of $\mathcal{A}$, the triple $(\mathcal{N}, \mu_1, \mu_2)$ is a T-sub-amalgam of the fork. To make $\mathcal{B}$ a substructure of $\mathcal{N}$, it is sufficient to make a renaming of the elements in the image of ν (so that ν becomes an inclusion). Thus consistency of $(*)$ means precisely that $(\mathcal{B}, \nu_1, \nu_2) \in \sigma_{T_0}^{T}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$.

Since, by T_0-compatibility, every model of T can be embedded into a model of $T \cup T_0^{\star}$, the consistency of $(*)$ is the same of the consistency of $T_0^{\star} \cup (*)$. □

We need a further result from model theory to be found in textbooks like [6]; it can be seen as a combination result 'ante litteram':

Lemma 3.3 (Joint Consistency). *Let Θ_1, Θ_2 be two signatures and let $\Theta_0 := \Theta_1 \cap \Theta_2$; suppose that the Θ_1-theory U_1 and the Θ_2-theory U_2 are both consistent and that there is a Θ_0-theory U_0 which is complete and included both in U_1 and in U_2. Then, $U_1 \cup U_2$ is also consistent.*

Proof. There are basically two proofs of this result, one by Craig's interpolation Theorem and another one by a double chain argument. The interested reader is referred to [6]. $\qquad\square$

We can now *prove Theorem 3.1;* the Theorem concerns theories T_1, T_2 (in their respective signatures Σ_1, Σ_2) which are both T_0-compatible with respect to a universal theory T_0 in the shared signature $\Sigma_0 := \Sigma_1 \cap \Sigma_2$.

Fix a $T_1 \cup T_2$-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$. On one side, it is evident that if $(\mathcal{B}, \nu_1, \nu_2)$ belongs to $\sigma_{T_0}^{T_1 \cup T_2}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$, then it also belongs to $\sigma_{T_0}^{T_1}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_1}] \cap \sigma_{T_0}^{T_2}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_2}]$.

Vice versa, suppose that $(\mathcal{B}, \nu_1, \nu_2)$ belongs to $\sigma_{T_0}^{T_1}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_1}]$ and to $\sigma_{T_0}^{T_2}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_2}]$; in order to show that it belongs to $\sigma_{T_0}^{T_1 \cup T_2}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$, in view of Lemmas 3.1 and 3.2 (recall also Proposition 2.1), we need to show that the following theory (let us call it U) is consistent:

$$
\begin{aligned}
&T_1 \cup T_2 \cup T_0^\star \cup \Delta_{\Sigma_1}(\mathcal{M}_1) \cup \Delta_{\Sigma_1}(\mathcal{M}_2) \cup \Delta_{\Sigma_0}(\mathcal{B})\ \cup \\
&\cup \Delta_{\Sigma_2}(\mathcal{M}_1) \cup \Delta_{\Sigma_2}(\mathcal{M}_2)\ \cup \\
&\cup \{\bar{a}_1 = \bar{b} \parallel b \in |\mathcal{B}|, a_1 \in |\mathcal{M}_1|, \nu_1(a_1) = b\}\ \cup \\
&\cup \{\bar{a}_2 = \bar{b} \parallel b \in |\mathcal{B}|, a_2 \in |\mathcal{M}_2|, \nu_2(a_2) = b\}.
\end{aligned}
$$

The idea is to use Robinson Joint Consistency Lemma 3.3 and split U as $U_1 \cup U_2$. Now U is a theory in the signature $\Sigma_1 \cup \Sigma_2 \cup |\mathcal{M}_1| \cup |\mathcal{M}_2| \cup |\mathcal{B}|$; we let (for $i = 1, 2$) U_i be the following theory in the signature $\Sigma_i \cup |\mathcal{M}_1| \cup |\mathcal{M}_2| \cup |\mathcal{B}|$:

$$
\begin{aligned}
&T_i \cup T_0^\star \cup \Delta_{\Sigma_i}(\mathcal{M}_1) \cup \Delta_{\Sigma_i}(\mathcal{M}_2) \cup \Delta_{\Sigma_0}(\mathcal{B})\ \cup \\
&\cup \{\bar{a}_1 = \bar{b} \parallel b \in |\mathcal{B}|, a_1 \in |\mathcal{M}_1|, \nu_1(a_1) = b\}\ \cup \\
&\cup \{\bar{a}_2 = \bar{b} \parallel b \in |\mathcal{B}|, a_2 \in |\mathcal{M}_2|, \nu_2(a_2) = b\}.
\end{aligned}
$$

Notice that U_i is consistent by Lemma 3.2 because our assumption is that $(\mathcal{B}, \nu_1, \nu_2)$ belongs to $\sigma_{T_0}^{T_i}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_i}]$. We now only have to identify a complete theory U_0 included in $U_1 \cap U_2$. The shared signature of U_1 and U_2

is $\Sigma_0 \cup |\mathcal{M}_1| \cup |\mathcal{M}_2| \cup |\mathcal{B}|$ and we take as U_0 the theory

$$T_0^\star \cup \Delta_{\Sigma_0}(\mathcal{M}_1) \cup \Delta_{\Sigma_0}(\mathcal{M}_2) \cup \Delta_{\Sigma_0}(\mathcal{B}) \cup$$
$$\cup \{\bar{a}_1 = \bar{b} \parallel b \in |\mathcal{B}|, a_1 \in |\mathcal{M}_1|, \nu_1(a_1) = b\} \cup$$
$$\cup \{\bar{a}_2 = \bar{b} \parallel b \in |\mathcal{B}|, a_2 \in |\mathcal{M}_2|, \nu_2(a_2) = b\}.$$

By the definition of a model-completion ($T_0^\star$ is a model-completion of T_0), we know that $T_0^\star \cup \Delta_{\Sigma_0}(\mathcal{B})$ is a complete theory in the signature $\Sigma_0 \cup |\mathcal{B}|$. Now it is sufficient to observe that every $\Sigma_0 \cup |\mathcal{M}_1| \cup |\mathcal{M}_2| \cup |\mathcal{B}|$-sentence is equivalent, modulo $U_0 \supseteq T_0^\star \cup \Delta_{\Sigma_0}(\mathcal{B})$, to a $\Sigma_0 \cup |\mathcal{B}|$-sentence: this is clear because U_0 contains the sentences

$$\{\bar{a}_1 = \bar{b} \parallel b \in |\mathcal{B}|, a_1 \in |\mathcal{M}_1|, \nu_1(a_1) = b\} \cup$$
$$\cup \{\bar{a}_2 = \bar{b} \parallel b \in |\mathcal{B}|, a_2 \in |\mathcal{M}_2|, \nu_2(a_2) = b\}.$$

which can be used to eliminate the constants from $|\mathcal{M}_1| \cup |\mathcal{M}_2|$. $\qquad\square$

Despite its abstract formulation, Theorem 3.1 is powerful enough to imply the main disjoint signatures result of [5] and also to work out interesting examples arising from software verification: in [12] it is shown how to use Theorem 3.1 to prove that suitable unions of theories describing data structures (like lists and binary trees) are quantifier-free interpolating. We explain in more detail how to derive the disjoint signatures result of [5] from Theorem 3.1:

Example 3.1 (The disjoint signatures case)**.** Let S_0, S_1, S_2 be sets such that $S_0 \subseteq S_1, S_0 \subseteq S_2$; the amalgamated sum $S_1 +_{S_0} S_2$ of S_1, S_2 over S_0 is just the set-theoretic union $S_1 \cup S_2$ in which elements from $S_1 \setminus S_0$ are renamed away so as to be different from the elements of $S_2 \setminus S_0$. With this terminology, a theory T is strongly sub-amalgamable iff its sub-amalgamation schema over the empty theory T_0 is such that $\sigma_{T_0}^T[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$ always contains the amalgamated sum of the supports of $\mathcal{M}_1, \mathcal{M}_2$ over the support of $\mathcal{A}$. Thus, Theorem 3.1 says in particular that if T_1, T_2 are both stably infinite and strongly sub-amalgamable, then so is $T_1 \cup T_2$ (and the last theory is in particular quantifier-free interpolating). $\qquad\square$

4. When the shared theory is Horn

Theorem 3.1 gives modular information to determine the combined sub-amalgamation schema, but it is not a modular result itself. In fact, a modular

result should identify a condition C on a single (standing alone) theory such that whenever T_1, T_2 satisfy C, then $T_1 \cup T_2$ also satisfies C. To get a modular sufficient condition for quantifier-free interpolation, we need to specialize our framework. In doing that, we are still guided by what happens in the disjoint signatures case. Although satisfactory results could be obtained without Horn hypotheses (see for a reference [15]), we prefer to assume here that the shared theory is universal Horn, on one side in order to simplify the statement of the results below, on the other side because universal Horn theories are sufficient for our applications to Modal Logic. Recall that a Σ-theory T is *universal Horn* iff it can be axiomatized via Horn clauses (i.e. via formulae of the form $A_1 \wedge \cdots \wedge A_n \rightarrow B$, where the A_i are atoms and B is either an atom or $\bot$). For simplicity (and because our applications to modal logic do not require more), in this section we shall consider only universal Horn theories axiomatized by clauses containing *exactly one* positive literal. We fix such a T for the remaining part of this section (we let also Σ be the signature of T).

4.1. Minimal amalgam and presentations

In universal Horn theories, it is possible to show that if amalgamation holds, then there is always a (unique up to isomorphisms) minimal amalgam this fact is basically due to the universal property of pushouts. We introduce the relevant notions:

Definition 4.1. Let $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ be a T-fork; a *pushout of the fork* is a triple $(\mathcal{M}, \mu_1, \mu_2)$, where $\mathcal{M}$ is a T-model and $\mu_1 : \mathcal{M}_1 \longrightarrow \mathcal{M}$, $\mu_2 : \mathcal{M}_2 \longrightarrow \mathcal{M}$ are Σ-homomorphisms whose restrictions to the support of $\mathcal{A}$ coincide, such that for every other triple $(\mathcal{M}', \mu'_1, \mu'_2)$ with the same properties, there is a unique homomorphism (called the comparison homomorphism) $\theta : \mathcal{M} \longrightarrow \mathcal{M}'$ such that $\theta \circ \mu_i = \mu'_i$ $(i = 1, 2)$.
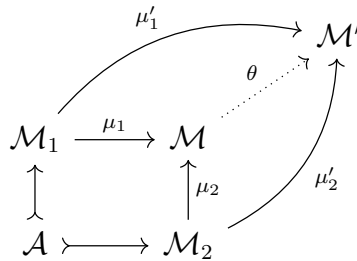
Definition 4.2. If the pushout $(\mathcal{M}, \mu_1, \mu_2)$ of the T-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ is an amalgam (i.e. if μ_1, μ_2 are monomorphisms), it is called the *minimal T-amalgam* of the T-fork.

Notice that, even when the pushout is an amalgam, comparison morphisms need not be injective.

Pushouts of T-forks (more generally, of T-spans) always exist: this is essentially a consequence of well-known co-completeness results for our categories of models [1], nevertheless we shall use an explicit construction in the sequel (it will be useful in our application to modal logic).

The key feature of T is that it *admits presentations*, in the following sense. A T-presentation is a pair (X, Γ), where X is a set of fresh constants and Γ is a set of Σ^X-atoms. To a T-presentation we associate the T-model $F_T(X, \Gamma)$ built as follows:

(i) the support $|F_T(X, \Gamma)|$ of $F_T(X, \Gamma)$ is formed by the equivalence classes of Σ^X-ground terms with respect to the equivalence relation $\sim_\Gamma$ defined by: $t \sim_\Gamma u$ iff $T \cup \Gamma \models t = u$;

(ii) function and relation symbols are interpreted so as to have that $F_T(X, \Gamma) \models A$ holds iff $T \cup \Gamma \models A$, for every Σ^X-atom A (clearly $F_T(X, \Gamma)$ is a Σ^X-structure, with constants from X interpreted as their own equivalence classes).[3]

Thanks to the current assumption on T (i.e. that the axioms of T are all Horn clauses with exactly one positive literal), we can easily check that $F_T(X, \Gamma) \models T$.

The fundamental property of $F_T(X, \Gamma) \models T$ is the following (this is very similar to the diagrams property): *for every T-model $\mathcal{M}$, there is a bijective correspondence between Σ-homomorphisms $F_T(X, \Gamma) \longrightarrow \mathcal{M}$ and expansions of $\mathcal{M}$ to Σ^X-structures which are models of Γ.*

Every model $\mathcal{M}$ of T is isomorphic to a model of the kind $F_T(X, \Gamma)$: this is because it is easily seen that $\mathcal{M} \simeq F_T(|\mathcal{M}|, \Delta_\Sigma^+(\mathcal{M}))$, where $\Delta_\Sigma^+(\mathcal{M})$ (the *positive diagram* of $\mathcal{M}$) is given by $\{A \mid A$ is an atomic formula of $\Sigma^{|\mathcal{M}|}$ s.t. $\mathcal{M} \models A\}$. We call $F_T(|\mathcal{M}|, \Delta_\Sigma^+(\mathcal{M}))$ the *canonical presentation of $\mathcal{M}$.*

Existence of minimal amalgam for any universal Horn theory T having the amalgamation property is an immediate consequence of the following:

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[3]In more detail, in $F_T(X, \Gamma)$ an n-ary function symbol f is interpreted as the function mapping the tuple of equivalence classes $[t_1], \dots, [t_n]$ to the equivalence class $[f(t_1, \dots, t_n)]$; an n-ary relation symbol R is interpreted as the set of tuples of equivalence classes $[t_1], \dots, [t_n]$ such that $T \cup \Gamma \models R(t_1, \dots, t_n)$.

Proposition 4.1. *Every T-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ has a pushout $(\mathcal{M}, \mu_1, \mu_2)$; if T has the amalgamation property, the pushout is the minimal T-amalgam of the T-fork.*

Proof. Up to renamings, we can freely suppose that $|\mathcal{M}_1| \cap |\mathcal{M}_2| = |\mathcal{A}|$. Let us take

$$\mathcal{M} \;\; := \;\; F_T(|\mathcal{M}_1| \cup |\mathcal{M}_2|, \Delta_\Sigma^+(\mathcal{M}_1) \cup \Delta_\Sigma^+(\mathcal{M}_2)) \tag{2}$$

and let μ_1 (resp. μ_2) be the map associating an element from $|\mathcal{M}_1|$ (resp. $|\mathcal{M}_2|$) to its own equivalence class in $\mathcal{M}$. The fact that this is a pushout is guaranteed by the fundamental property of the presentations: indeed, given a T-model $\mathcal{M}'$, the Σ-homomorphisms $\mathcal{M} \longrightarrow \mathcal{M}'$ are in bijective correspondence with expansions of $\mathcal{M}'$ to a $\Sigma^{|\mathcal{M}_1| \cup |\mathcal{M}_2|}$-structure modeling $\Delta_\Sigma^+(\mathcal{M}_1) \cup \Delta_\Sigma^+(\mathcal{M}_2)$ and the latter are in bijective correspondence with pairs of Σ-homomorphisms $\mathcal{M}_1 \longrightarrow \mathcal{M}'$ and $\mathcal{M}_2 \longrightarrow \mathcal{M}'$, agreeing on $|\mathcal{M}_1| \cap |\mathcal{M}_2| = |\mathcal{A}|$.[4]

If T has the amalgamation property, then there exists a T-amalgam $\mathcal{N}$, with embeddings $\nu_i : \mathcal{M}_i \longrightarrow \mathcal{N}$ ($i = 1, 2$); by the above property of the pushout, there is $\theta : \mathcal{M} \longrightarrow \mathcal{N}$ with $\nu_i = \theta \circ \mu_i$ ($i = 1, 2$). Since the ν_i are injective and $\nu_i = \theta \circ \mu_i$, then also the μ_i are injective; the same argument shows that the μ_i reflect relations and hence they are embeddings: if a relation R holds in $\mathcal{M}$ (when applied to some parameters from $\mathcal{M}_i$), then it holds in $\mathcal{N}$ because θ is a homomorphism, hence it holds also in $\mathcal{M}_i$, since the ν_i are embeddings. Thus, if T has the amalgamation property, the pushout (2) is an amalgam (minimal by the definition of pushout) of the fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$. $\qquad\square$

It is useful to have a formula like (2) operating in the case where we are given T-presentations of $\mathcal{M}_1, \mathcal{M}_2$ which might not be the canonical ones. We first introduce T-presentations of embeddings and of T-forks.

Suppose that we are given an embedding among T-models; up to an isomorphism, we can assume that it is a substructure inclusion $\mathcal{M}_1 \subseteq \mathcal{M}_2$. Passing to canonical presentations, again up to isomorphisms, we have an embedding

$$F_T(|\mathcal{M}_1|, \Delta_\Sigma^+(\mathcal{M}_1)) \longrightarrow F_T(|\mathcal{M}_2|, \Delta_\Sigma^+(\mathcal{M}_2)) \tag{3}$$

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[4]A slight modification of this construction shows the existence of pushouts also when the maps of $\mathcal{A}$ into $\mathcal{M}_i$ are Σ-homomorphisms (not just substructure inclusions).

which is also 'canonical', in the sense that for every $a \in |\mathcal{M}_1|$, the embedding (3) maps the equivalence class of a in $F(|\mathcal{M}_1|, \Delta_\Sigma^+(\mathcal{M}_1))$ into the equivalence class of a in $F(|\mathcal{M}_2|, \Delta_\Sigma^+(\mathcal{M}_2))$. Notice that the reason why we have an embedding here is that $\Delta_\Sigma^+(\mathcal{M}_2)$ is *conservative* over $\Delta_\Sigma^+(\mathcal{M}_1)$, meaning that for every $\Sigma^{|\mathcal{M}_1|}$-atom A we have $T \cup \Delta_\Sigma^+(\mathcal{M}_2) \models A$ iff $T \cup \Delta_\Sigma^+(\mathcal{M}_1) \models A$ (it is so, because (3) is obtained from the substructure inclusion $\mathcal{M}_1 \subseteq \mathcal{M}_2$).

Vice versa, a T-*presentation of an embedding* is a pair of T-presentations $(X_1, \Gamma_1), (X_2, \Gamma_2)$ with $X_1 \subseteq X_2$ and with Γ_2 conservative over Γ_1; to this presentation corresponds the embedding (which we also call *canonical*) $F_T(X_1, \Gamma_1) \longrightarrow F_T(X_2, \Gamma_2)$ given by the map associating, for every $a \in X_1$, the equivalence class of a in $F_T(X_1, \Gamma_1)$ with the equivalence class of a in $F_T(X_2, \Gamma_2)$.

Similar considerations apply to T-forks: given a T-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$, we can associate with it a pair of canonical embeddings among canonical presentations

$$F_T(|\mathcal{M}_1|, \Delta_\Sigma^+(\mathcal{M}_1)) \longleftarrow F_T(|\mathcal{A}|, \Delta_\Sigma^+(\mathcal{A})) \longrightarrow F_T(|\mathcal{M}_2|, \Delta_\Sigma^+(\mathcal{M}_2))$$

A T-*presentation of a fork* is a triple of T-presentations $(X_0, \Gamma_0), (X_1, \Gamma_1), (X_2, \Gamma_2)$ with $X_1 \cap X_2 = X_0$, $X_0 \subseteq X_i$, and Γ_i conservative over Γ_0 ($i = 1, 2$). To this T-presentation it corresponds a pair of canonical embeddings

$$F_T(X_1, \Gamma_1) \longleftarrow F_T(X_0, \Gamma_0) \longrightarrow F_T(X_2, \Gamma_2) \tag{4}$$

(which is a T-fork, up to renamings). An argument analogous to that used in the proof of Proposition 4.1 shows that the pushout of (4) is given by

$$F_T(X_1, \Gamma_1) \xrightarrow{\mu_1} F_T(X_1 \cup X_2, \Gamma_1 \cup \Gamma_2) \xleftarrow{\mu_2} F_T(X_2, \Gamma_2) \tag{5}$$

Notice that (for $i = 1, 2$) the map μ_i still associates, for every $a \in X_i$, the equivalence class of a in $F_T(X_i, \Gamma_i)$ with the equivalence class of a in $F_T(X_1 \cup X_2, \Gamma_1 \cup \Gamma_2)$, but this map *does not need to be* an embedding, because one cannot in general infer that $\Gamma_1 \cup \Gamma_2$ is conservative over Γ_i from the fact that Γ_i is conservative over Γ_0: this is indeed the key property leading to amalgamability (in the case of our Horn theories, if the property fails, amalgamability fails and vice versa).

4.2. Modularity of Combination Theorem

In this subsection we will show that Theorem 3.1 allows us to obtain a modular condition of combination.

Let now T be a Σ-theory and let $T_0 \subseteq T$ be a universal Horn Σ_0-theory having the amalgamation property (with $\Sigma_0 \subseteq \Sigma$).

We say that T is T_0-*strongly sub-amalgamable* if the sub-amalgamation schema $\sigma_{T_0}^T$ always contains the minimal T_0-amalgam (meaning that for every T-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$, we have that the minimal T_0-amalgam of $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ belongs to $\sigma_{T_0}^T[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$). Notice that, whenever T_0 is the empty theory in the empty signature, being T_0-strongly sub-amalgamable is the same as being strongly sub-amalgamable.

Theorem 3.1 immediately implies the following:

Theorem 4.1. *If T_1, T_2 are both T_0-compatible and T_0-strongly sub-amalgamable (over an amalgamable universal Horn theory T_0 in their common subsignature Σ_0), then so is $T_1 \cup T_2$.*

Proof. Since T_1 and T_2 are T_0-strongly sub-amalgamable, their sub-amalgamation schemata $\sigma_{T_0}^{T_i}$ $(i = 1, 2)$ always contain minimal T_0-amalgams. By Theorem 3.1 (T_1 and T_2 are also T_0-compatible), this implies that for every $T_1 \cup T_2$-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$, the minimal amalgam $\mathcal{B}$ of $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_0}$ belongs to the set $\sigma_{T_0}^{T_1 \cup T_2}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$. Using Proposition 2.1, we conclude that also $T_1 \cup T_2$ is T_0-compatible and T_0-strongly sub-amalgamable. $\square$

Theorem 4.1 has surprising applications to modal logic. As a first step in this direction, we show that it applies to equational theories with a Boolean reduct.

In the following, we let BA be the theory of Boolean algebras (which is well-known to be amalgamable); we use as signature for Boolean algebras the signature Σ_{BA} comprising two unary operations $\cap$ and $\cup$ (for meet and join), the 0-ary operations 0, 1 (for zero and one) and a unary operation $\sim$ for complement. We abbreviate $(\sim u) \cup t$ as $u \Rightarrow t$; $u_1 \Leftrightarrow u_2$ is defined as $(u_1 \Rightarrow u_2) \cap (u_2 \Rightarrow u_1)$. The atom $u \le t$ is defined as $u \cap t = u$ and $u < t$ stands for $u \le t \wedge u \ne t$.

A *BAO-equational theory*[5] is any theory T whose signature extends the signature of Boolean algebras and whose axioms are all equations and include the Boolean algebra axioms. In Subsection 5.2 below, we shall recall in detail how BAO-equational theories are related to modal propositional logics via Lindenbaum constructions. The *fusion* of two BAO-equational theories T_1

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[5]BAO stands for 'Boolean algebras with operations'. Notice that we do not assume any normality conditions.

and T_2 is just their combination $T_1 \cup T_2$ (when speaking of the fusion of T_1 and T_2, we assume that T_1 and T_2 share only the Boolean algebra operations and no other symbol).

The following Proposition is proved in [11]:

Proposition 4.2. *Every BAO-equational theory is BA-compatible.*

Proof. Let T be a BAO-equational theory;[6] the theory of Boolean algebras has as model completion the theory of *atomless* Boolean algebras [6],[7] so it is sufficient to show how to embed a model $\mathcal{M}$ of T into a model $\mathcal{M}'$ of T which is based on an atomless Boolean algebra. Define a sequence of models of T by: $\mathcal{M}_0 := \mathcal{M}$, $\mathcal{M}_{k+1} := \mathcal{M}_k \times \mathcal{M}_k$; define also embeddings $\delta_k : \mathcal{M}_k \longrightarrow \mathcal{M}_{k+1}$ by $\delta_k(a) := \langle a, a \rangle$. Now take as $\mathcal{M}'$ the union (more precisely, the inductive limit) of this chain: clearly $\mathcal{M}'$ is atomless as a Boolean algebra (no non-zero element is minimal in it, as any $a \in \mathcal{M}_k$ gets identified with $\langle a, a \rangle = \langle a, 0 \rangle \cup \langle 0, a \rangle$ in $\mathcal{M}_{k+1}$). $\qquad\square$

As an immediate consequence, from Theorem 4.1, we get:

Corollary 4.1. *If two BAO-equational theories T_1 and T_2 are both BA-strongly amalgamable, so is their fusion.*

The next section is devoted to the interpretation of the meaning of the above Corollary in the context of propositional modal logic.

5. Applications to modal logic

5.1. Interpolation in propositional modal logic

For simplicity, we deal only with unary modalities (and, consequently, we shall consider only BAO-theories whose non-Boolean symbols are unary function symbols), however we point out that the extension to n-ary modalities is straightforward.

A *modal signature* Σ_M is a set of unary operation symbols; from Σ_M, propositional *modal formulae* are built using countably many propositional variables, the operation symbols in Σ_M, the Boolean connectives $\cap, \cup, \sim$ and

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[6]The argument works also for universal Horn theories extending the theory of Boolean algebras.

[7]Recall that a Boolean algebra is atomless iff for every non-zero element a from its support there is a non-zero b such that $b < a$.

the constants 1 for truth and 0 for falsity. We use the letters $x, x_1, \ldots, y, y_1, \ldots$ to denote propositional variables and the letters $t, t_1, \ldots, u, u_1, \ldots$ to denote propositional formulae; $t \Rightarrow u$ and $t \Leftrightarrow u$ are abbreviations for $(\sim t) \cup u$ and for $(t \Rightarrow u) \cap (u \Rightarrow t)$, respectively. We use notations like $t(\underline{x})$ (resp. $\Gamma(\underline{x})$) to say that the modal formula t (the set of modal formulae Γ) is built up from a set of propositional variables included in the tuple $\underline{x}$.

The following definition is taken from [33], pp. 8–9:

Definition 5.1. A *classical modal logic* L based on a modal signature Σ_M is a set of modal formulae that

(i) contains all classical propositional tautologies;

(ii) is closed under uniform substitution of propositional variables by propositional formulae;

(iii) is closed under the modus ponens rule ('from t and $t \Rightarrow u$ infer u');

(iv) is closed under the replacement rules, which are specified as follows. We have one such rule for each $o \in \Sigma_M$, namely:

$$\frac{t \Leftrightarrow u}{o(t) \Leftrightarrow o(u)}$$

A classical modal logic L is said to be *normal* iff for every modal operator $o \in \Sigma_M$, L contains the modal formulae $o(1)$ and $o(y \Rightarrow z) \Rightarrow (o(y) \Rightarrow o(z))$.

Since classical modal logics (based on a given modal signature) are closed under intersections, it makes sense to speak of the least classical modal logic $[S]$ containing a certain set of propositional formulae S. If $L = [S]$, we say that S is a set of *axiom schemata* for L.

If L_1 is a classical modal logic over the modal signature Σ_M^1 and L_2 is a classical modal logic over the modal signature Σ_M^2 and $\Sigma_M^1 \cap \Sigma_M^2 = \emptyset$, the *fusion* $L_1 \oplus L_2$ is the modal logic $[L_1 \cup L_2]$ over the modal signature $\Sigma_M^1 \cup \Sigma_M^2$.

Given a modal logic L, a set of modal formulae Γ and a modal formula t, the *global consequence relation* $\Gamma \vdash_L t$ holds iff there is a finite list of modal formulae $t_0, \ldots, t_n$ such that: (i) t_n is t; (ii) each t_i is either a member of L or a member of Γ or is obtained from previous members of the list by applying one of the two inference rules from Definition 5.1 (i.e. modus ponens and replacement).

Global consequence relation should be contrasted with *local consequence relation*, to be indicated with $\vdash_L \Gamma \Rightarrow t$: this holds iff there are $g_1, \ldots, g_n \in \Gamma$ such that $\bigcap_{i=1}^{n} g_i \Rightarrow t$ belongs to L. If Γ consists of a single modal formula g, below we write $g \vdash_L t$ and $\vdash_L g \Rightarrow t$ instead of $\{g\} \vdash_L t$ and of $\vdash_L \{g\} \Rightarrow t$.

In case L is normal, one can reduce the global consequence relation to the local one: in fact, it is not difficult to see by induction that the following fact ('deduction theorem') holds:

$$\Gamma \vdash_L t \qquad \text{iff} \qquad \vdash_L o\Gamma \Rightarrow t$$

where $o\Gamma$ is some finite set of modal formulae (depending on t) obtained from Γ by prefixing a string of modal operators (i.e. elements of $o\Gamma$ are modal formulae of the kind $o_1(o_2 \cdots o_n(g) \cdots)$, for $g \in \Gamma$ and $n \geq 0$, $o_1, \ldots, o_n \in \Sigma_M$).

Due to the presence of local and global consequence relations, we can formulate two different versions of the Craig's interpolation theorem:

Definition 5.2. Let L be a classical modal logic in a modal signature Σ_M.

(i) We say that L enjoys the *local* interpolation property iff whenever we have $\vdash_L t_1(\underline{x}, \underline{y}) \Rightarrow t_2(\underline{x}, \underline{z})$ for two modal formulae t_1, t_2, then there is a modal formula $u(\underline{x})$ such that $\vdash_L t_1 \Rightarrow u$ and $\vdash_L u \Rightarrow t_2$. This is a "implication-based" interpolation property.

(ii) We say that L enjoys the *global* interpolation property iff whenever we have $t_1(\underline{x}, \underline{y}) \vdash_L t_2(\underline{x}, \underline{z})$ for two modal formulae t_1, t_2, then there is a modal formula $u(\underline{x})$ such that $t_1 \vdash_L u$ and $u \vdash_L t_2$. This is a "consequence-based" interpolation property.

Related forms of "consequence-based" interpolation (like "deductive interpolation") have been studied in detail in several works [17, 31, 8, 39, 30, 10, 9, 24, 16] (see in particular [24] for proofs and references for the connections between deductive interpolation and amalgamation through congruence extension property). For *normal* modal logics, in view of the above deduction theorem, it is easy to see that the local interpolation property implies the global one (but it is not equivalent to it, see [21]). In the non-normal case, there is no deduction theorem available, so that in order to have an interpolation property encompassing both the local and the global versions, it seems that a different notion needs to be introduced. This is what we are doing now.

Given a modal logic L and two sets of modal formulae $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z})$, let us call an $\underline{x}$-*residue chain* a tuple of modal formulae $C(\underline{x}) = g_1(\underline{x}), \dots, g_n(\underline{x})$ such that we have $\Gamma_1 \cup \{g_1, \dots, g_{2i}\} \vdash_L g_{2i+1}$ and $\Gamma_2 \cup \{g_1, \dots, g_{2j-1}\} \vdash_L g_{2j}$, for all i such that $0 \leq 2i < n$ and for all j such that $0 < 2j \leq n$.

Definition 5.3. Let L be a classical modal logic in a modal signature Σ_M.

(iii) We say that L enjoys the *comprehensive* interpolation property iff whenever we have $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z}) \vdash_L t_1(\underline{x}, \underline{y}) \Rightarrow t_2(\underline{x}, \underline{z})$ for two modal formulae t_1, t_2 and for two finite sets of modal formulae Γ_1, Γ_2, there are an $\underline{x}$-residue chain $C(\underline{x})$ and a modal formula $u(\underline{x})$ such that we have $\Gamma_1, C \vdash_L t_1 \Rightarrow u$ and $\Gamma_2, C \vdash_L u \Rightarrow t_2$.

Notice that the comprehensive interpolation property implies both the local and the global interpolation properties: the first fact is shown in the proof of Corollary 5.1; the second is stated in the following Proposition:

Proposition 5.1. *Let L be a classical modal logic which admits the comprehensive interpolation property. Then, L enjoys the global interpolation property.*

Proof. Suppose that L admits the comprehensive interpolation property. In order to show that L is globally interpolating, we consider two modal formulas $\alpha(\underline{x}, \underline{y})$ and $\beta(\underline{x}, \underline{z})$ such that $\alpha \vdash_L \beta$. By definition of comprehensive interpolation (defining $\Gamma_1 := \alpha$, $\Gamma_2 := \top$, $t_1 := \top$ and $t_2 := \beta$), there exist an $\underline{x}$-residue chain $C(\underline{x}) := g_1(\underline{x}), \dots, g_n(\underline{x})$ and a modal formula $u(\underline{x})$ such that:

- $\Gamma_1 \cup \{g_1, \dots, g_{2i}\} \vdash_L g_{2i+1}$ and $\Gamma_2 \cup \{g_1, \dots, g_{2j-1}\} \vdash_L g_{2j}$, for all i such that $0 \leq 2i < n$ and for all j such that $0 < 2j \leq n$, which implies that $\alpha \vdash_L \bigwedge C$.

- $\Gamma_1, C \vdash_L t_1 \rightarrow u$ and $\Gamma_2, C \vdash_L u \rightarrow t_2$, which means that $\alpha, C \vdash_L u$ and $C \vdash_L u \rightarrow \beta$.

Then, it is clear that $\alpha \vdash_L (\bigwedge C) \wedge u$ and also $(\bigwedge C) \wedge u \vdash_L \beta$. If we define $\theta(\underline{x}) := (\bigwedge C)(\underline{x}) \wedge u(\underline{x})$, we get that $\alpha \vdash_L \theta$ and $\theta \vdash_L \beta$, as required. $\square$

In the normal case, via deduction theorem, it can be easily seen that the comprehensive interpolation property is equivalent to the local interpolation property (Corollary 5.1).

Remark. The formulation we used for the comprehensive interpolation property is reminiscent of intuitions coming from automated reasoning practice (we suppose we are given reasoners who cooperate by exchanging information over the common sublanguage). It is however possible to give a (trivially) equivalent formulation which is easier to compare with some definitions known from the literature. The equivalent definition is as follows: say that L has the comprehensive interpolation property iff *whenever we are given* $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z})$ - *having the property that for every* $v(\underline{x})$ *we have* $\Gamma_1 \vdash_L v$ *iff* $\Gamma_2 \vdash_L v$ - *if it happens that* $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z}) \vdash_L t_1(\underline{x}, \underline{y}) \Rightarrow t_2(\underline{x}, \underline{z})$ *for some* t_1, t_2, *then there is a modal formula* $u(\underline{x})$ *such that we have* $\Gamma_1 \vdash_L t_1 \Rightarrow u$ *and* $\Gamma_2 \vdash_L u \Rightarrow t_2$. This formulation should be compared e.g. with the (OIP) ('ordinary interpolation property') in [10], Def. 3.3: the comparison clearly shows that our 'comprehensive interpolation property' is a stronger form of (OIP) that includes both local and global aspects (i.e. in the terminology of [10], both 'conditional' and 'deductive' aspects).

Remark. As a further remark, we underline that (as it is not difficult to show), in case the congruence extension property holds, one can assume that all residue chains in the definition of the comprehensive interpolation property have length 1; in this case, again comparing with [10] Def.3.1(iii), our comprehensive interpolation property becomes just a stronger form of the (MIP) ('Maehara interpolation property') comprising both local and global aspects.

Our final result, giving an extension of Wolter's result [38] to non-normal case, is the following:

Theorem 5.1. *If the modal logics* L_1 *and* L_2 *both have the comprehensive interpolation property, so does their fusion* $L_1 \oplus L_2$.

The proof of the above Theorem is reported in the final Subsection 5.3; actually, it requires some algebraic logic background which will be introduced in the following subsection.

5.2. Algebraic logic background

We now revisit key notions from algebraic logic and recall the *bijective correspondence between modal logics and BAO-equational theories* [32]. The correspondence works as follows.

Given a logic L with modal signature Σ_M, we define the BAO-equational theory T_L as the theory having as signature $\Sigma_L := \Sigma_M \cup \Sigma_{BA}$ and as set of

axioms the set

$$BA \cup \{t = 1 \mid t \in L\}.$$

Notice that, from our notational conventions, it follows that Σ_M-*modal formulae are the same as* Σ_L-*terms.* Models of T_L will be called L-*algebras* in the following.

Vice versa, given an equational extension T of BA over the signature Σ, we define L_T as the classical modal logic over the modal signature $\Sigma \setminus \Sigma_{BA}$ axiomatized by the formulae

$$\{t \mid T \models t = 1\} \quad .$$

Notice that under the above bijection, we have $T_{L_1 \oplus L_2} = T_{L_1} \cup T_{L_2}$, i.e. *the fusion of BAO's* (as defined in Subsection 4.2) *corresponds to the fusion of modal logics* (as defined in Subsection 5.1).

Classical modal logics (in our sense) and equational extensions of BA are equivalent formalisms. For our purposes, it is important to revisit presentations, as introduced in Subsection 4.1, in terms of Lindenbaum-Tarski algebras.

Recall that a T_L-presentation (we fix a logic L) is pair (X, Γ) given by a set of fresh constants X and a set of Σ_L^X-atoms. Now, in the current situation, we can view the X as propositional variables and notice that every atom is equivalent to an atom of the kind $t = 1$, where t is a Σ_L^X-term (alias modal formula in which at most the X occur). This is so because there are no predicate symbols other than equality in Σ_L and because we can transform an atom $t_1 = t_2$ into the atom $t_1 \Leftrightarrow t_2 = 1$, modulo T_L equivalence. Thus, from now on, *a presentation will be just a pair* (X, Γ) *where the elements of* Γ *are* Σ_L^X-*terms.* We recall that the L-algebra $F_{T_L}(X, \Gamma)$ corresponding to the presentation (X, Γ) is built up from equivalence classes of Σ_L^X-terms under the equivalence relation $u_1 \sim_\Gamma u_2$ given by $T_L \cup \{t = 1 \mid t \in \Gamma\} \models u_1 = u_2$, by defining all operations on representative elements of equivalence classes.

Now, it is well-known that it is possible to build $F_{T_L}(X, \Gamma)$ in another equivalent way (the Lindenbaum-Tarski construction), directly via the global consequence relation $\vdash_L$ of the logic L. We recall how to do it and show the equivalence with the old construction of $F_{T_L}(X, \Gamma)$.

Given a presentation (X, Γ), the *Lindenbaum-Tarski algebra* $LT(X, \Gamma)$ is built as follows: we take the set of the modal formulae containing at most the propositional variables X and introduce the equivalence relation defined by $u_1 \sim_\Gamma^{LT} u_2$ iff $\Gamma \vdash_L u_1 \Leftrightarrow u_2$. The last equivalence relation is

a congruence because, for istance, if $u_1 \sim^{LT}_\Gamma u_2$ (i.e. $\Gamma \vdash_L u_1 \Leftrightarrow u_2$), then, applying the replacement rule, we also have $\Gamma \vdash_L o(u_1) \Leftrightarrow o(u_2)$, which means $o(u_1) \sim^{LT}_\Gamma o(u_2)$. The boolean cases are analogous. Defining each operation on representatives of equivalence classes, we obtain our algebra $LT(X, \Gamma)$; by construction, $LT(X, \Gamma)$ is a model of T_L and (under the natural evaluation of each variable in X as its own equivalence class) we have that all $u \in \Gamma$ evaluates to 1.

Proposition 5.2. *We have $LT(X, \Gamma) \simeq F_{T_L}(X, \Gamma)$, for every presentation (X, Γ).*

Proof. It is sufficient to observe that (by straightforward arguments) we have

$$\Gamma \vdash_L t \qquad \text{iff} \qquad T_L \cup \{u = 1 \mid u \in \Gamma\} \models t = 1$$

for every t. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Due to the above Proposition, we shall feel free to use the constructions of $LT(X, \Gamma)$ and of $F_{T_L}(X, \Gamma)$ interchangeably.

We dedicate the left subsection of this section to the proof of Theorem 5.1.

5.3. Proof of Theorem 5.1

We finally report the proof of our main result from Section 5, namely Theorem 5.1. The claim is to show that the model-theoretic condition of Theorem 4.1, i.e. T_0-strong sub-amalgamability, in case T_0 is the theory of Boolean algebras, is nothing but the superamalgamation property known from algebraic logic. Further, we will prove that superamalgamability is equivalent to comprehensive interpolation. By Corollary 4.1, we derive fusion modularity of comprehensive interpolation (and also of local interpolation in the normal case). Before doing that, we recall the definition of superamalgamation.

Definition 5.4. We say that a BAO-equational theory T has the *superamalgamation* property iff for every T-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ there exists a T-amalgam $(\mathcal{M}, \mu_1, \mu_2)$ such that for every $a_1 \in |\mathcal{M}_1|, a_2 \in |\mathcal{M}_2|$ such that $\mu_1(a_1) \leq \mu_2(a_2)$ there exists $a_0 \in |\mathcal{A}|$ such that $a_1 \leq a_0$ holds in $\mathcal{M}_1$ and $a_0 \leq a_2$ holds in $\mathcal{M}_2$.

We also state the following definition (related to superamalgamation), which involves commutative squares of algebras:

Definition 5.5. Let L be a classical modal logic. A commutative square of T_L-algebras

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\varepsilon_1} & \mathcal{A} \\ \eta_1 \uparrow & & \uparrow \varepsilon_2 \\ \mathcal{A}_0 & \xrightarrow{\eta_2} & \mathcal{A}_2 \end{array}$$

is said to have the *interpolation property* iff the following holds:

$$\forall a_1 \in |\mathcal{A}_1|, \forall a_2 \in |\mathcal{A}_2| \, (\varepsilon_1(a_1) \leq \varepsilon_2(a_2) \Rightarrow \exists b \in |\mathcal{A}_0| \, (a_1 \leq \eta_1(b) \wedge \eta_2(b) \leq a_2))$$

Lemma 5.1. *In a commutative square having the interpolation property as above, if η_1 is injective (i.e. an embedding), so is ε_2.*

Proof. Recall that a morphism μ among Boolean algebras is injective iff $1 \leq \mu(a)$ implies $1 \leq a$ for all a. Suppose that η_1 is injective and $1 \leq \varepsilon_2(a)$; then $\varepsilon_1(1) \leq \varepsilon_2(a)$, so there is $b \in |\mathcal{A}_0|$ such that $1 \leq \eta_1(b)$ and $\eta_2(b) \leq a$; this implies $b = 1$ and $1 \leq a$, as required. $\qquad\square$

Definition 5.6. Let L be a classical modal logic. Given three algebras $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$ endowed with homomorphisms $\eta_1 : \mathcal{A}_0 \to \mathcal{A}_1$ and $\eta_2 : \mathcal{A}_0 \to \mathcal{A}_2$, we say that they are *superamalgamable* if there exists another algebra $\mathcal{A}$ with homomorphisms $\varepsilon_1 : \mathcal{A}_1 \to \mathcal{A}$ and $\varepsilon_2 : \mathcal{A}_2 \to \mathcal{A}$ such that the square

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\varepsilon_1} & \mathcal{A} \\ \eta_1 \uparrow & & \uparrow \varepsilon_2 \\ \mathcal{A}_0 & \xrightarrow{\eta_2} & \mathcal{A}_2 \end{array}$$

is commutative and has the interpolation property.

Next Proposition slightly restates the superamalgamation property (basically, it shows that in order to check superamalgamation property, we only need to fill a T_L-fork into a square having the interpolation property, without caring about the fact that the square is formed by embeddings):

Proposition 5.3. *The following conditions are equivalent for a modal logic L:*

(i) *T_L has the superamalgamation property;*

24

(ii) *every T_L-fork is superamalgamable;*

(iii) *the pushout of every T_L-fork has the interpolation property.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial, whereas the implication (ii) $\Rightarrow$ (iii) is immediate by the universal property of pushouts. To show that (iii) $\Rightarrow$ (i), assume (iii) and take a T_L-fork $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_0)$; the related pushout

$$
\begin{array}{ccc}
\mathcal{A}_1 & \xrightarrow{\varepsilon_1} & \mathcal{A} \\
\big\uparrow & & \big\uparrow{\scriptstyle \varepsilon_2} \\
\mathcal{A}_0 & \rightarrowtail & \mathcal{A}_2
\end{array}
$$

has the interpolation property and by Lemma 5.1, it follows that ε_2 is injective. Exchanging $\mathcal{A}_1$ and $\mathcal{A}_2$ (i.e. considering the fork $(\mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_0)$), it follows that ε_1 is also injective (the pushout construction is symmetric). Thus, we have found a T-amalgam of the fork having the interpolation property. $\square$

Notice that, up to a renaming isomorphism, every triple $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$ of L-algebras endowed with embeddings $\eta_1 : \mathcal{A}_0 \rightarrowtail \mathcal{A}_1$ and $\eta_2 : \mathcal{A}_0 \rightarrowtail \mathcal{A}_2$ is a T_L-fork; as a consequence, T_L is superamalgamable iff *every such a triple of L-algebras connected by embeddings is superamalgamable.*

The following proposition relates the comprehensive interpolation property (which is a *linguistic* notion for propositional modal logics) to superamalgamability (which is an *algebraic* condition):

Proposition 5.4. *Let L be a modal logic. Then, T_L has the superamalgamability property iff L satisfies the comprehensive interpolation property.*

Proof. Suppose that *L has the comprehensive interpolation property.* Consider a T_L-fork; using presentations, we can suppose that the fork is given by a triple of T_L-presentations $(X_0, \Gamma_0), (X_0 \cup X_1, \Gamma_1), (X_0 \cup X_2, \Gamma_2)$ with $(X_0 \cup X_1) \cap (X_0 \cup X_2) = X_0$, and with Γ_i conservative over Γ_0 $(i = 1, 2)$. To this T_L-presentation corresponds a pair of canonical embeddings (let us write X_0, X_i instead of $X_0 \cup X_i$ for simplicity):

$$F_{T_L}(X_0, X_1, \Gamma_1) \xleftarrow{\eta_1} F_{T_L}(X_0, \Gamma_0) \xrightarrow{\eta_2} F_{T_L}(X_0, X_2, \Gamma_2) \qquad (6)$$

(we recall that we use the word 'canonical' to mean that η_i associates the equivalence class of t in $F_{T_L}(X_0, \Gamma_0)$ with the equivalence class of the same

t in $F_{T_L}(X_0, X_i, \Gamma_i)$).[8] We are using the Lindenbaum-Tarski construction for presentations, hence conservativity of Γ_i means that we have

$$\Gamma_i \vdash_L t(\underline{x}_0) \qquad \text{iff} \qquad \Gamma_0 \vdash_L t(\underline{x}_0) \tag{7}$$

for every modal formula (alias Σ_L-term) containing at most propositional variables $\underline{x}_0 \subseteq X_0$.

From (5), we know that the pushout of (6) is given by

$$F_{T_L}(X_0, X_1, \Gamma_1) \xrightarrow{\varepsilon_1} F_{T_L}(X_0, X_1, X_2, \Gamma_1 \cup \Gamma_2) \xleftarrow{\varepsilon_2} F_{T_L}(X_0, X_2, \Gamma_2) \tag{8}$$

with canonical maps $\varepsilon_1, \varepsilon_2$.

We use Proposition 5.3 and just show that the square formed by the quadruple $(\eta_1, \eta_2, \varepsilon_1, \varepsilon_2)$ has the interpolation property. To this aim, let us suppose that for modal formulae t_1, t_2 (where t_1 is built up from the variables in $X_0 \cup X_1$ and t_2 is built up from the variables in $X_0 \cup X_2$) we have

$$\varepsilon_1([t_1]) = [t_1] \leq [t_2] = \varepsilon_2([t_2])$$

in $F_{T_L}(X_0, X_1, X_2, \Gamma_1 \cup \Gamma_2)$; the last means that $\Gamma_1 \cup \Gamma_2 \vdash_L t_1 \Rightarrow t_2$ by the construction of the Lindenbaum-Tarski algebra. Since only finitely many formulae are used in a derivation, there exist two finite subsets $\Gamma_1' \subseteq \Gamma_1$ and $\Gamma_2' \subseteq \Gamma_2$ such that

$$\Gamma_1'(\underline{x}_0, \underline{x}_1), \Gamma_2'(\underline{x}_0, \underline{x}_2) \vdash_L t_1(\underline{x}_0, \underline{x}_1) \Rightarrow t_2(\underline{x}_0, \underline{x}_2)$$

(here $\underline{x}_0$ is a tuple including all the variables from X_0 occurring in $\Gamma_1', \Gamma_2', t_1, t_2$, $\underline{x}_1$ is a tuple including all the variables from X_1 occurring in Γ_1', t_1 and $\underline{x}_2$ is a tuple including all the variables from X_2 occurring in Γ_2', t_2). By the comprehensive interpolation property, there exist a formula $u(\underline{x}_0)$ and a finite set of formulae $g_1(\underline{x}_0), ..., g_n(\underline{x}_0)$ such that:

$$\Gamma_1'(\underline{x}_0, \underline{x}_1) \vdash_L g_1$$

$$\Gamma_2'(\underline{x}_0, \underline{x}_2), g_1 \vdash_L g_2$$

---

[8]Notationally, the fact that η_i is canonical allows us to write equations like $\eta_i([t]) = [t]$; there is a bit of abuse in this notation, because we do not indicate where the equivalence class $[t]$ of t is taken from, but such missed information can be easily deduced from the context.

$$\cdots$$

$$\Gamma'_1(\underline{x}_0, \underline{x}_1), g_1, ..., g_{2j-1} \vdash_L g_{2j}$$

$$\Gamma'_2(\underline{x}_0, \underline{x}_2), g_1, ..., g_{2k} \vdash_L g_{2k+1}$$

$$\cdots$$

and also such that:

$$\Gamma'_1(\underline{x}_0, \underline{x}_1), g_1, ..., g_n \vdash_L t_1 \Rightarrow u$$

$$\Gamma'_2(\underline{x}_0, \underline{x}_2), g_1, ..., g_n \vdash_L u \Rightarrow t_2$$

But, for $i = 1, 2$, by conservativity of Γ_i over Γ_0, we obtain that $\Gamma_i \vdash_L g_j$ (all $j = 1, \ldots, n$), hence $\Gamma_1 \vdash_L t_1 \Rightarrow u$ and $\Gamma_2 \vdash_L u \Rightarrow t_2$. The last two facts yield $[t_1] \leq [u] = \eta_1([u])$ in $F_{T_L}(X_0, X_1, \Gamma_1)$ and $[u] = \eta_2([u]) \leq [t_2]$ in $F_{T_L}(X_0, X_2, \Gamma_2)$, as required for the interpolation property to hold for the square $(\eta_1, \eta_2, \varepsilon_1, \varepsilon_2)$.

Conversely, suppose that *L has the superamalgamability property.* Let $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z})$ be finite sets of modal formulae and let $t_1(\underline{x}, \underline{y}), t_2(\underline{x}, \underline{z})$ be such that $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z}) \vdash_L t_1(\underline{x}, \underline{y}) \Rightarrow t_2(\underline{x}, \underline{z})$. We construct three algebras in T_L connected with (canonical) monomorphisms in order to apply the superamalgamability property. Let $\Gamma_0(\underline{x})$ be the set of the modal formulae $g(\underline{x})$ such that there is an $\underline{x}$-residue chain from $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z})$ ending up in $g(\underline{x})$. Then, we put $\Delta_1 := \Gamma_1 \cup \Gamma_0$ and $\Delta_2 := \Gamma_2 \cup \Gamma_0$; clearly Δ_1 and Δ_2 are conservative over Γ_0 by construction. Now, we are ready to take:

$$\mathcal{A}_1 := F_{T_L}(\underline{x}, \underline{y}, \Delta_1)$$

$$\mathcal{A}_2 := F_{T_L}(\underline{x}, \underline{z}, \Delta_2)$$

$$\mathcal{A}_0 := F_{T_L}(\underline{x}, \Gamma_0)$$

This is a triple of L-agebras connected by canonical embeddings. Then, by the superamalgamability property and by Proposition 5.3, the related pushout has the interpolation property. From (5), we know that such pushout is

$$F_{T_L}(\underline{x}, \underline{y}, \Delta_1) \xrightarrow{\varepsilon_1} F_{T_L}(\underline{x}, \underline{y}, \underline{z}, \Delta_1 \cup \Delta_2) \xleftarrow{\varepsilon_2} F_{T_L}(\underline{x}, \underline{z}, \Delta_2)$$

with canonical maps $\varepsilon_1, \varepsilon_2$.

From $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z}) \vdash_L t_1(\underline{x}, \underline{y}) \Rightarrow t_2(\underline{x}, \underline{z})$, it follows that we have

$$\varepsilon_1([t_1(\underline{x}, \underline{y})]) = [t_1(\underline{x}, \underline{y})] \leq [t_2(\underline{x}, \underline{z})] = \varepsilon_2([t_2(\underline{x}, \underline{z})])$$

27

in $F_{T_L}(\underline{x}, \underline{y}, \underline{z}, \Delta_1 \cup \Delta_2)$. By the interpolation property of the square, there exists a formula $u(\underline{x})$ such that we have $[t_1] \leq [u] = \eta_1[u]$ in $F_{T_L}(\underline{x}, \underline{y}, \Delta_1)$ and $\eta_2([u]) = [u] \leq [t_2]$ in $F_{T_L}(\underline{x}, \underline{z}, \Delta_2)$. This means that we get $\Gamma_1, \Gamma_0 \vdash_L t_1 \Rightarrow u$ and $\Gamma_2, \Gamma_0 \vdash_L u \Rightarrow t_2$. Since only finitely many modal formulae from Γ_0 are involved in these derivations and since all modal formulae in Γ_0 are obtained via $\underline{x}$-residue chains, the claim follows (we can obviously glue finitely many $\underline{x}$-residue chains into a single one). $\qquad\square$

The last statement of the following Corollary is well-known ([21],[13],[19]):

Corollary 5.1. *If L is normal, then L has the comprehensive interpolation property iff it has the local interpolation property. Consequently T_L has the superamalgamability property iff L satisfies the local interpolation property.*

Proof. Suppose first that L has *the local interpolation property*. Consider the deduction relation $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z}) \vdash_L t_1(\underline{x}, \underline{y}) \Rightarrow t_2(\underline{x}, \underline{z})$. Since L is normal, applying the 'deduction theorem' it is clear that the last fact is equivalent to[9] $\vdash_L (o\Gamma_1 \cap o\Gamma_2) \Rightarrow (t_1 \Rightarrow t_2)$, which implies (using propositional tautologies) that $\vdash_L (o\Gamma_1 \cap t_1) \Rightarrow (o\Gamma_2 \Rightarrow t_2)$ holds. By the local interpolation property, there exists a formula $u(\underline{x})$ such that $\vdash_L (o\Gamma_1 \cap t_1) \Rightarrow u$ and $\vdash_L u \Rightarrow (o\Gamma_2 \Rightarrow t_2)$. Applying again propositional tautologies and the 'deduction theorem', the previous statement is equivalent to $\Gamma_1 \vdash_L t_1 \Rightarrow u$ and $\Gamma_2 \vdash_L u \Rightarrow t_2$, which means that L has the comprehensive interpolation property (here, the $\underline{x}$-residue chain is empty).

For the other implication[10], suppose that L has *the comprehensive interpolation property* and that $t_2(\underline{x}, \underline{z})$ is locally deducible from $t_1(\underline{x}, \underline{y})$, i.e. that $\vdash_L t_1(\underline{x}, \underline{y}) \Rightarrow t_2(\underline{x}, \underline{z})$ holds. This fact implies, by comprehensive interpolation property, that there exist a $\underline{x}$-residue chain $C = \{g_1, \ldots, g_n\}$ and a formula $u(\underline{x})$ such that $C \vdash_L t_1 \Rightarrow u$ and $C \vdash_L u \Rightarrow t_2$. Hence, in order to achieve the aim it is sufficient to show that $\vdash_L C$ holds. In fact, since $\Gamma_i = \emptyset$, we have $\vdash_L g_1$; moreover, reasoning by induction, $\vdash_L g_1, \ldots, \vdash_L g_{n-1}$ and $g_1, ..., g_{n-1} \vdash_L g_n$ imply, replacing the g_i $(i = 1, ..., n-1)$ with their proofs, that $\vdash_L g_n$. Therefore, we conclude $\vdash_L C$, as wanted. $\qquad\square$

---

[9]Recall from Subsection 5.1, that we use the notation $o\Gamma$ to indicate the conjunction of a finite set of modal formulae of the kind $o_1(o_2 \cdots o_n(g) \cdots)$, for $g \in \Gamma$ and $n \geq 0$, $o_1, \ldots, o_n \in \Sigma_M$.

[10]Notice that the following argument does not require the normality of L. Thus, the comprehensive interpolation property implies the local interpolation property even in the non-normal case.

Classical propositional logic can be seen as the modal logic over the empty signature (as such, it is clearly also normal and Corollary 5.1 applies to it). From ordinary Craig interpolation theorem for classical propositional logic and Corollary 5.1, we can get the following very well-known fact:

Lemma 5.2. *BA has the superamalgamation property.*

Pushouts in BA can be better described:

Lemma 5.3. *Let $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_0)$ be a BA-fork and let*

$$
\begin{array}{ccc}
\mathcal{A}_1 & \xrightarrow{\ \varepsilon_1\ } & \mathcal{A} \\
\uparrow & & \uparrow{\scriptstyle \varepsilon_2} \\
\mathcal{A}_0 & \rightarrowtail & \mathcal{A}_2
\end{array}
$$

be the Boolean pushout of the fork; then all elements of $|\mathcal{A}|$ can be written as finite meets of elements of the kind $\varepsilon_1(a_1) \Rightarrow \varepsilon_2(a_2)$, for $a_1 \in |\mathcal{A}_1|$ and $a_2 \in |\mathcal{A}_2|$.

Proof. We replace our BA-fork with a canonical presentation

$$
F_{BA}(X_0, X_1, \Gamma_1) \xleftarrow{\ \eta_1\ } F_{BA}(X_0, \Gamma_0) \xrightarrow{\ \eta_2\ } F_{BA}(X_0, X_2, \Gamma_2)
$$

of it; then the pushout can be presented as

$$
F_{BA}(X_0, X_1, \Gamma_1) \xrightarrow{\ \varepsilon_1\ } F_{BA}(X_0, X_1, X_2, \Gamma_1 \cup \Gamma_2) \xleftarrow{\ \varepsilon_2\ } F_{BA}(X_0, X_2, \Gamma_2) \quad (9)
$$

Elements of $F_{BA}(X_0, X_1, X_2, \Gamma_1 \cup \Gamma_2)$ are equivalence classes of classical propositional formulae built up from the variables X_0, X_1, X_2. By conjunctive normal forms, they are conjunctions of clauses $l_1^1 \vee \cdots \vee l_n^1 \vee l_1^2 \vee \cdots \vee l_m^2$, where the literals l_j^1 are built from $X_0 \cup X_1$ and the literals l_k^2 are built up from $X_0 \cup X_2$ (this representation is of course not unique). These clauses in turn can be written as

$$
(\neg l_1^1 \wedge \cdots \wedge \neg l_n^1) \Rightarrow (l_1^2 \vee \cdots \vee l_m^2) \quad (10)
$$

and the equivalence class of (10) in $F_{BA}(X_0, X_1, X_2, \Gamma_1 \cup \Gamma_2)$ is of the desired shape. $\qquad\square$

We can now state the second main result of this paper:

Theorem 5.2. *A BAO-equational theory T has the superamalgamation property iff it is BA-strongly amalgamable.*

Proof. Let T be equal to T_L for a modal logic L.

Suppose first that T_L *has the superamalgamation property* and consider a T_L-fork $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_0)$. Since T_L is superamalgamable, we can fill the inclusions $\mathcal{A}_0 \subseteq \mathcal{A}_1$ and $\mathcal{A}_0 \subseteq \mathcal{A}_2$ into a commutative square

$$
\begin{array}{ccc}
\mathcal{A}_1 & \xrightarrow{\varepsilon_1} & \mathcal{A} \\
\Big\updownarrow & & \Big\uparrow {\scriptstyle \varepsilon_2} \\
\mathcal{A}_0 & \rightarrowtail & \mathcal{A}_2
\end{array}
$$

having the interpolation property. Considering the BA-reducts, we can build the Boolean pushout square and the Boolean comparison morphism θ as in the following diagram:


We show that θ is injective (i.e. a Boolean embedding): this is precisely what it is required by BA-strong amalgamability. Using Lemma 5.3 (and recalling that in a Boolean algebra the meet of a finite set is equal to 1 iff all elements from the set are equal to 1), it is sufficient to prove that if $\theta(\bar{\varepsilon}_1(a_1) \Rightarrow \bar{\varepsilon}_2(a_2)) = 1$, then $\bar{\varepsilon}_1(a_1) \leq \bar{\varepsilon}_2(a_2)$ (i.e. $\bar{\varepsilon}_1(a_1) \Rightarrow \bar{\varepsilon}_2(a_2) = 1$). But

$$1 = \theta(\bar{\varepsilon}_1(a_1) \Rightarrow \bar{\varepsilon}_2(a_2)) = \theta(\bar{\varepsilon}_1(a_1)) \Rightarrow \theta(\bar{\varepsilon}_2(a_2)) = \varepsilon_1(a_1) \Rightarrow \varepsilon_2(a_2)$$

and in a Boolean algebra, this is equivalent to $\varepsilon_1(a_1) \leq \varepsilon_2(a_2)$. Since the outer square has the interpolation property, we conclude that there exists an element $a \in |\mathcal{A}_0|$ such that $a_1 \leq \eta_1(a)$ and $\eta_2(a) \leq a_2$, where η_1, η_2 are the inclusions of $\mathcal{A}_0$ into $\mathcal{A}_1, \mathcal{A}_2$. Therefore, we compute $\bar{\varepsilon}_1(a_1) \leq \bar{\varepsilon}_1(\eta_1(a)) = \bar{\varepsilon}_2(\eta_2(a)) \leq \bar{\varepsilon}_2(a_2)$, as wanted.

Conversely, *suppose that T_L is BA-strongly amalgamable*. Thus, by Proposition 5.3, it is sufficient to show that every T_L-fork $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_0)$ is superamalgamable. By BA-strong amalgamability, the minimal BA-amalgam

$(\mathcal{B}, \bar{\varepsilon}_1, \bar{\varepsilon}_2)$, which is the pushout in BA, can be embedded via a θ (unique by the universal property of the pushout) into a T_L-amalgam $(\mathcal{A}, \varepsilon_1, \varepsilon_2)$.
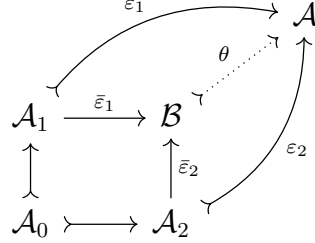
Now Lemma 5.2 states that BA has the superamalgamation property; so, by Proposition 5.3, the Boolean pushout

has the interpolation property. We show that, since θ is injective, the commutative square

has the interpolation property too. In fact, if, for $a_1 \in |\mathcal{A}_1|$ and $a_2 \in |\mathcal{A}_2|$, we have $\varepsilon_1(a_1) \leq \varepsilon_2(a_2)$ in $\mathcal{A}$, then we get $\theta(\bar{\varepsilon}_1(a_1)) \leq \theta(\bar{\varepsilon}_2(a_2))$, and also $\bar{\varepsilon}_1(a_1) \leq \bar{\varepsilon}_2(a_2)$, because θ is injective. Thus, by the fact that the Boolean pushout has the interpolation property, we conclude that:

$$\exists a_0 \in |\mathcal{A}_0| \ (a_1 \leq \eta_1(a_0) \wedge \eta_2(a_0) \leq a_2)$$

where η_1, η_2 are the inclusions of $\mathcal{A}_0$ into $\mathcal{A}_1, \mathcal{A}_2$, as wanted. $\qquad\square$

As an immediate consequence, from Corollary 4.1, we get fusion modularity of superamalgamation:

Corollary 5.2. *If two BAO-equational theories T_1 and T_2 both have the superamalgamability property, so does their fusion.*

Now, we are ready for our final main result:

Theorem 5.1 *If the modal logics L_1 and L_2 both have the comprehensive interpolation property, so does their fusion $L_1 \oplus L_2$.*

Proof. Immediate, from Proposition 5.4 and Corollary 5.2. $\square$

Finally, we state Wolter's theorem [38], which is an immediate consequence of Theorem 5.1 and of Corollary 5.1:

Theorem 5.3. *If the normal modal logics L_1 and L_2 both have the local interpolation property, so does their fusion $L_1 \oplus L_2$.*

6. Conclusions and future work

In this paper we considered the problem of transferring interpolation properties to fusions of (non normal) modal logics. We attacked the problem not via specific modal logic techniques, but via general tools applying to combined first-order theories.

Our approach is characterized by the attempt of employing concepts of general scope in several different areas of mathematical logic: in this paper, in particular, model-theoretic notions are shown to be powerful enough to solve seemingly unrelated problems coming from non-classical propositional logics. Algebraic logic, in this way, becomes the powerful tool revealing an unexpected connection to the first-order framework.

We first obtained a sufficient condition (Theorem 3.1) in terms of sub-amalgamation schemata, allowing to transfer quantifier-free interpolation property from two theories to their union, in the case where the two theories share symbols other than pure equality. We are not aware of previous papers attacking this problem (one should however mention a series of papers [34, 35, 37] analyzing the somewhat related - but different - problem of transferring, in a hierarchical way, interpolation properties to theory extensions under 'locality' assumptions). The sufficient condition of Theorem 3.1 was used to get a modular condition in case the shared theory is universal Horn (Theorem 4.1). For equational theories extending the theory of Boolean algebras, this modular condition turns out to be equivalent to the superamalgamability condition known from algebraic logic [22]. Thus, our results immediately imply the fusion transfer of local interpolation property [38] for classical normal modal logics. In the general non-normal case, the modularity of superamalgamability can be translated into a fusion transfer result

for a new kind of interpolation property (which we called 'comprehensive interpolation property').

Still, many problems need to be faced by future research. Our combinability conditions should be characterizable from a syntactic point of view and, from such syntactic characterizations, we expect to be able to design concrete combined interpolation algorithms. Concerning modal logic, besides the old question about modularity of local interpolation property in the non-normal case, new questions arise concerning the status of the new comprehensive interpolation property: is it really stronger than other forms of interpolation property (e.g. than the local one)? Are there different ways of specifying it? Is it modular also for modal logics on a non-classical basis?

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