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Comparison results about dg-categories, A_∞-categories, stable ∞-categories and noncommutative motives

Mat/03

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Ai miei genitori,
per il loro incodizionato supporto

Introduction

This thesis is divided into two parts. The first part concerns the papers [49] and [9], the second part regards the paper [50]. The first part, from chapter 1 to chapter 4, deals with the category of A_∞-categories, dg-categories and stable ∞-categories, in particular it is devoted to give a comparison between them.

We remark that the notion of triangulated category, developed by Jean-Louis Verdier and Alexander Grothendieck, plays an important role in algebraic geometry. The main example of triangulated category is the derived categories of sheaves. Their applications concern the study of the geometry of moduli spaces or some problems in birational geometry.

However triangulated categories have some serious drawbacks, for example the non-functoriality of the mapping cone or the non-existence of homotopy colimits and homotopy limits. These technical problems suggest the definitions of pretriangulated differential graded category, of pretriangulated A_∞-categories and, more recently, stable ∞-categories. Roughly speaking, such categories are dg-categories (resp. A_∞-categories, ∞-categories) whose homotopy category is a triangulated category, so they can be viewed as enhanced triangulated categories.

To summarize briefly, we prove that the category of A_∞-categories, localized over quasi-equivalences (resp. Morita equivalences), is equivalent to the category of dg-categories localized over quasi-equivalences (resp. Morita equivalences).
The idea behind the construction of these equivalences is to take a quotient of the composition of the functor bar with the functor cobar, from the category of non-unital A_∞-categories to the category of non-unital dg-categories.

Moreover, regarding the ∞-stable category, we prove that the categorical nerve of the category of dg-categories and A_∞-categories (localized over Morita equivalences) is equivalent to an idempotent complete ∞-stable category, using a result of Lee Cohn. Furthermore we prove that the A_∞-nerve sends quasi-equivalences of unital A_∞-categories to weak-equivalences of ∞-categories. We find also some particular assumptions where the converse is true.

As we said before, A_∞-categories, dg-categories and stable ∞-categories are very important for the study of the derived categories of schemes. For example, we recall that the enhancements via A_∞-categories is fundamental to prove the existence of non-Fourier-Mukai functors between derived categories, or to prove the existence of non-unique enhancements of a triangulated category [52].

Another example of why we are interested in enhancements of triangulated categories, concerns the Homological Mirror Symmetry Conjecture. The HMSC, formulated by Kontsevich, states that there exists an A_∞-equivalence between a dg enhancement of $D^b(X)$, for X a smooth projective Calabi-Yau threefold, and the Fukaya category $\mathcal{F}(Y)$ of the mirror Y of X. We recall that the Fukaya category $\mathcal{F}(Y)$, whose objects are Lagrangian submanifolds of Y, is a pretriangulated A_∞-category. It means that his homotopy category is a triangulated category. Roughly speaking, the objects of the A_∞-category $\mathcal{F}(Y)$ are Lagrangian submanifolds of Y. So, the importance of the uniqueness of dg-enhancement of $D^b(X)$ is strictly connected to the A_∞-nature of $\mathcal{F}(Y)$.

Moreover there is another important application of the study of such categories concerning noncommutative geometry. As we will explain accurately in chapter 5, there are several ways to approach, to noncommutative geometry. We cite, among the others, Connes's approach via C^*-algebras, the approach of Rosenberg, Gabriel and Kontsevich which identifies a scheme with the category of its coherent sheaves. Other variants are the approach of Kontsevich and Soibelman, related to A_∞-algebras and homological mirror symmetry, and the approach of Artin and Zhang.

In the second part of my thesis, we focus on the theory of noncommutative motives. The central point of this theory is the study of algebraic varieties by using the dg-enhancements of the derived category of their category of perfect sheaves. We remind that this approach began, in the 80's, thanks to the work of Beilinson, Bondal, Kapranov, Manin, etc. and was recently developed by Tabuada in the book [59].

Roughly speaking, taking a smooth projective scheme X, there exists a relation between the Chow ring $\mathfrak{h}(X)$, associated to X, and the noncommutative Chow ring $\mathrm{U}(\mathrm{perf}_{\mathrm{dg}}X)$, associated to any dg-enhancement of the category of perfect sheaves on X.

In particular, Tabuada et al., proved that some classical conjectures about Chow ring of X, e.g. Voevodsky nilpotence conjecture, Kimura conjecture and Schur conjecture hold, if and only if, the corresponding conjectures in the noncommutative case hold for $\mathrm{perf}_{\mathrm{dg}}X$.

We sketch briefly the results we obtain. First we prove Voevodsky nilpotence conjecture, Kimura conjecture and Schur conjecture, for smooth cubic fourfolds and ordinary generic Gushel-Mukai fourfolds. The proof of such conjecture is based on the observation that there exists a quadric fibrations from a blow-up of these fourfolds to $\mathbb{P}^3$. Then, we prove the noncommutative version of the aforementioned conjectures for the dg-enhancement (induced by the enhancement of X) of the Kuznetsov category $\mathcal{A}_X$, which is a noncommutative K3 surface in sense of Kontsevich. Finally, we prove the Voevodsky nilpotence conjecture for generic Gushel-Mukai fourfolds, containing a plane P of type $Gr(2,3)$. This provides a geometrical application of the previous result.

We believe that this approach yields a new tool for the proof of Voevodsky's conjecture for smooth projective k-schemes whose derived category of perfect complexes contains the noncommutative K3 surface $\mathcal{A}_X$.

Acknowledgement

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Contents

Ouverture

We recall that pretriangulated dg-categories, pretriangulated A_∞-category and stable ∞-category are three ways to enhance a triangulated category. From a geometric point of view, we remark that a large number of examples of triangulated categories are the derived categories of (coherent sheaves) of a scheme. The study of such category was fundamental to prove some results in algebraic geometry. Among the others, we point out the works of Stellari, Huybrechts, Kuznetsov, Orlov etc.

There is another important application of the study of the derived categories (and their enhancements) concerning the noncommutative geometry. There are several ways to approach, to non commutative geometry. We cite, among the others, Connes's approach via C^*-algebras, the approach of Rosenberg, Gabriel and Kontsevich, which identifies a scheme by the category of its coherent sheaves, the approach of Kontsevich and Soibelman, related to A_∞-algebras and homological mirror symmetry, and the approach of Artin and Zhang. Roughly speaking, the objects of study of noncommutative geometry are dg-algebras, A_∞-algebras or dg-coalgebras instead of varieties.

Enhancements of a triangulated category

Let $\mathbb{K}$ be a field (or a commutative ring).

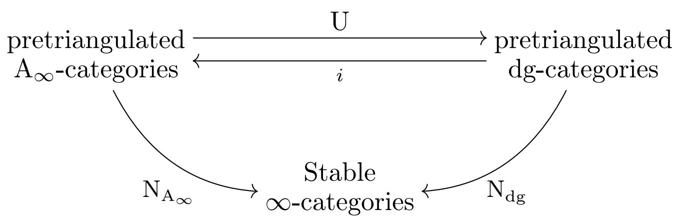
Given a $\mathbb{K}$-linear triangulated category $\mathscr{T}$. We say that $\mathscr{T}$ is *enhanced* by $\mathbf{C}$ if

$$\mathrm{Ho}(\mathbf{C}) \simeq \mathscr{T},$$

where Ho denotes the homotopy category and $\mathbf{C}$ is one of the followings:

- a pretriangulated dg-category,
- a pretriangulated A_∞-category,
- a stable ∞-category.

The aim of the first part of the thesis is to relate the, above mentioned, enhancements. Roughly speaking, we can summarize these results with the following diagram, which is commutative up to weak-equivalences:



where the functor U is widely described in Chapter 2 and the functors N_{A_∞} and N_{dg} are defined in Chapter 4. Moreover, we say that a triangulated category has a *unique* enhancement if, give two enhancements **C** and **D**, there exist a dg (or A_∞, or ∞)-functor, from **C** to **D** which is a quasi (or weak)-equivalence.
We know that, given a smooth projective $\mathbb{K}$-scheme X, the category of perfect complexes perf(X) has a unique dg enhancement $\text{perf}_{dg}(X)$ (cf. [41] or [10]). Actually, one of the most important questions, about this topic, is related to the uniqueness of enhancements. In [52], Rizzardo and Van den Bergh showed that the graded field $F = K[t, t^{-1}]$, with $K = \mathbb{K}(x_1, .., x_{n+1})$ and n even, has two, not quasi-equivalent, A_∞-structures. The homotopy category of $\text{pretr}_{A_\infty}(F)$ provide an example of a triangulated category whose enhancements are not unique[4].

Noncommutative geometry

Under mild assumption on a $\mathbb{K}$-scheme X (e.g. if $\mathbb{K}$ is a perfect field and X is quasi-projective cf. [53]) we can take a generator E of the dg-category $\text{perf}_{dg}(X)$. Moreover we denote by $\mathscr{E}$ the dg-algebra of endomorphisms $\text{Hom}(E, E)$. The dg-algebra $\mathscr{E}$ has only finitely many non zero cohomology groups (since E is perfect cf. [48]). Then, by [28], we have a quasi-equivalence between $\text{perf}_{dg}(X)$ and $\text{perf}_{dg}(\mathscr{E})$, where $\mathscr{E}$ is a cohomologically bounded dg-algebra.

This fact, allow us to suggest the following definition of a noncommutative scheme:

Definition (Noncommutative scheme)**.** We define a *noncommutative scheme* to be a $\mathbb{K}$-linear dg-category quasi-equivalent to a category of the form perf(E) where E is a cohomologically bounded differential graded $\mathbb{K}$-algebra.

Now we give two examples of a noncommutative scheme, which are not equivalent to the enhancement of the derived category of a scheme.

[4]Other, more general, examples are studied by Kajiura in [27].

Example 1: Noncommutative projective spaces

We recall that the derived category of a quiver $\mathcal{Q}$ is defined as the derived category of the abelian category of (finitely generated) modules over the $\mathbb{K}$-algebra $\mathbb{K}\mathcal{Q}$:

$$\mathcal{D}^b(\mathcal{Q}) := \mathcal{D}^b(\text{mod } \mathbb{K}\mathcal{Q})$$

Now we consider the following quiver:

$$\cdot \longrightarrow \cdot$$

The associated $\mathbb{K}$-algebra $\mathbb{K}\mathcal{Q}$ is given by

$$\mathbb{K}\mathcal{Q} = \left\{ \begin{pmatrix} k_1 & k_2 \\ 0 & k_3 \end{pmatrix} \mid k_1,\, k_2,\, k_3 \in \mathbb{K} \right\}$$

where $k_i \in \mathbb{K}$. The (primitive orthogonal) idempotent elements of $\mathbb{K}\mathcal{Q}$ are given by:

$$e_1 = \mathbb{K}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ e_2 = \mathbb{K}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Moreover the modules P_1 and P_2, defined as $\mathbb{K}\mathcal{Q}e_1$ and $\mathbb{K}\mathcal{Q}e_2$, are indecomposable and, by Krull-Schmidt theorem, they generate the abelian category mod $\mathbb{K}\mathcal{Q}$.
We recall that a *semiorthogonal decomposition* of a triangulated category is a collection of subcategories $\mathcal{A}_i$ such that $\text{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0$, for $i > j$, and, for every object T, there exists a chain of maps $0 = T_m \to T_{m-1} \to \ldots \to T_1 \to T$, with $T_i \in \mathcal{A}_i$, such that $\text{cone}(T_i \to T_{i-1}) \in \mathcal{A}_i$. Moreover we say that an object T is *exceptional* if $\text{Hom}(E, E) = \mathbb{K}$ and $\text{Ext}^i(T, T) = 0$, with $i \neq 0$. We can note that P_1 and P_2 are exceptional objects, moreover by the following exact sequence in mod $k\mathcal{Q}$

$$0 \to P_1 \to P_1 \oplus P_2 \to P_2 \to 0$$

we have the distinguished triangle

$$P_1 \to P_1 \oplus P_2 \to P_2 \to P_1[1].$$

So the derived category of the quiver $\mathcal{Q}$ is generated by P_1 and P_2, in formula

$$\mathcal{D}^b(\text{mod } \mathbb{K}\mathcal{Q}) = \langle P_1, P_2 \rangle.$$

Now by Kuznetsov [32, Theorem 7.3] we have that the Hochschild Homology of $\mathcal{D}^b(\mathcal{Q})$ is

$$\text{HH}_*(\mathcal{D}^b(\mathcal{Q})) = \text{HH}_*(P_1) \oplus \text{HH}_*(P_2).$$

Given an appropriate[5] category $\mathcal{C}$, we recall that $\text{HH}_*(\mathcal{C}) = \text{Tor}^{\text{EndFun}(\mathcal{C})}(\text{Id}_{\mathcal{C}}, \text{Id}_{\mathcal{C}})$ where $\text{EndFun}(\mathcal{C})$ denotes the category of $\mathcal{C}$-endofunctors.

[5] with "appropriate" I mean a category $\mathcal{C}$ such that $\text{EndFun}(\mathcal{C})$ can be replaced with a category having a structure to define Tor. For example if $\mathcal{C}$ is the category of modules over an algebra A we can replace $\text{EndFun}(\mathcal{C})$ by $\text{Mod}(A \otimes A^{\text{op}})$.

In order to calculate $\mathrm{HH}_0(\mathcal{D}^b(\mathcal{Q}))$ we have to calculate the 0^{th}-Hochschild Homology of P_1 and P_2. We recall ([40]) that $\mathrm{HH}_0(A) = A/[A, A]$, for a $\mathbb{K}$-algebra A.
HH_0 of P_1 is $\mathbb{K}$ (because $P_1 \simeq \mathbb{K}$), and the commutator of P_2 is given by

$$C = [P_2, P_2] = \begin{pmatrix} 0 & k_2 \\ 0 & k_3 \end{pmatrix} \begin{pmatrix} 0 & \tilde{k_2} \\ 0 & \tilde{k_3} \end{pmatrix} - \begin{pmatrix} 0 & \tilde{k_2} \\ 0 & \tilde{k_3} \end{pmatrix} \begin{pmatrix} 0 & k_2 \\ 0 & k_3 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{K} \\ 0 & 0 \end{pmatrix}$$

Thus $\mathrm{HH}_0(P_2) = P_2/C \simeq \mathbb{K}$, and

$$\mathrm{HH}_0(\mathcal{D}^b(\mathcal{Q})) := \mathbb{K} \oplus \mathbb{K}.$$

Remark 1
The calculation above is still true for all the quivers with n-arrows of the form:

(1) $$\cdot \underset{\longrightarrow}{\overset{\longrightarrow}{\ldots n\text{-arrows} \ldots}} \cdot$$

Because the indecomposable elements of mod $\mathbb{K}\mathcal{Q}$ are the same.

In general, given a smooth proper connected scheme X over an algebraically closed base field $\mathbb{K}$ of characteristic zero, using Hochschild-Konstant-Rosemberg-theorem we have that the 0^{th}-Hochschild homology is given by

$$\mathrm{HH}_0(X) = \bigoplus_{p-q=0} H^{p,q}(X) = H^{0,0}(X) \oplus ... \oplus H^{n,n}(X) \neq \mathbb{C} \oplus \mathbb{C}.$$

Because $H^{0,0}(X) \simeq H^{1,1}(X) \simeq H^{n,n}(X) \simeq \mathbb{C}$.
Moreover thanks to the work of [29, §5.2] we have that the Hochschild homology of the derived category of X is equivalent to the Hochschild homology of X in formula $\mathrm{HH}_0(\mathcal{D}^b(X)) \simeq \mathrm{HH}_0(X)$.
To conclude, if $\mathcal{Q}$ is a quiver with n arrows of the form (1), where n is not equal to 0 or 2 and $\mathbb{K} = \mathbb{C}$, then $\mathcal{D}^b(\mathcal{Q})$ is not the derived category of a scheme.

Definition (Noncommutative projective space)**.** A *n-noncommutative projective space* $N\mathbb{P}^n_{\mathbb{K}}$ is the dg-enhancement of the derived category of the quiver of the form (1).

Example 2: Cubic 3-Folds

Let X be a cubic threefold i.e. a smooth hypersurface $X \subset \mathbb{P}^4$ with deg $X = 3$, by [36], we have a semiorthogonal decomposition:

$$\mathcal{D}^b(X) = \langle \mathscr{A}_X, \mathcal{O}_X, \mathcal{O}_X(H) \rangle$$

where H is a hyperplane section and

$$\mathscr{A}_X = \{C \in \mathcal{D}^b(X) \text{ s.t. } \mathrm{Hom}_{\mathcal{D}^b(X)}(\mathcal{O}_X, C) = \mathrm{Hom}_{\mathcal{D}^b(X)}(\mathcal{O}_X(H), C) = 0\} \subset \mathcal{D}^b(X).$$

$\mathscr{A}_X$ is a (smooth and proper) noncommutative scheme and, by [32, Corollary 4.4], we know that the Serre functor of $\mathscr{A}_X$ is given by

$$S^3_{\mathscr{A}_X} \simeq [5]$$

But there does not exists a smooth projective scheme Y such that $\mathcal{D}^b(Y) \simeq \mathscr{A}_X$ because, for a smooth projective scheme, the Serre functor is given by

$$S_Y := (- \otimes \omega_Y)[\dim Y]$$

then

$$S^3_Y = (((- \otimes \omega_Y) \otimes \omega_Y) \otimes \omega_Y)[3 \dim Y]$$

and it can not be the shift functor [5].

A natural question arise:

- What kind of informations, about a scheme, we can recover from his noncommutative counterpart?

In the second part of the thesis, we give an example of the use noncommutative geometry to solve a geometrical problem.
In particular, we use the theory of noncommutative motives, developed by Tabuada in [59], to prove Voevodsky conjecture, for some kind of varieties.

Part I

A_∞-categories, dg-categories and stable ∞-categories

Chapter 1

A_∞-categories and dg-categories

In the first chapter we introduce some basic definitions about A_∞-categories and dg-categories. We stress that we distinguish unital categories and cohomological unital categories. Then, fixed two A_∞-categories, we introduce the category of A_∞-functors, and the homotopy relation between prenatural transformations which play the role of morphisms in such a category. Moreover we recall the construction of pretriangulated A_∞-categories originally introduced, for dg-categories, by Bondal and Kapranov in [8]. Roughly speaking, as in the case of dg-categories, pretriangulated A_∞-categories are categories whose homotopy category is triangulated. We refer to [56] and [30] for a complete reference about this topic. We conclude this chapter with some definitions about dg-cocategories.

Let $\mathbb{K}$ be a commutative ring.

1.1 A_∞-categories and dg-categories

Definition 1.1.1 (A_∞-category)**.** We define an *A_∞-category* to be a $\mathbb{K}$-linear category equipped by $\mathbb{K}$-linear maps

$$m^d_{\mathscr{A}} : \mathscr{A}(x_{d-1}, x_d) \otimes ... \otimes \mathscr{A}(x_0, x_1) \to \mathscr{A}(x_0, x_d)[2-d],$$

for every $d > 0$, verifying the followings:

$$(1.1) \qquad \sum_{m=1}^{d} \sum_{n=0}^{d-m} (-1)^{\ddagger_n} m^{d-m+1}_{\mathscr{A}}(a_d, ..., a_{n+m+1}, m^m_{\mathscr{A}}(a_{n+m}, ..., a_{n+1}), a_n, ..., a_1) = 0.$$

where $\ddagger_n = \deg(a_1) + ... + \deg(a_n) - n$.

Example 1.1.1. The category of differential graded chains $\mathcal{C}(\mathbb{K})$, whose morphisms are given by $\operatorname{Hom}^k_{\mathcal{C}(\mathbb{K})}(X, Y) := \sum_{l \in \mathbb{Z}} \operatorname{Hom}(X^l, Y^{l+k})$, equipped by the maps:

- $m^1_{\mathcal{C}(\mathbb{K})}(f^\cdot) := df^\cdot + (-1)^{\deg f^\cdot + 1} f^\cdot d$;
- $m^2_{\mathcal{C}(\mathbb{K})}(f^\cdot, g^\cdot) := (-1)^{\deg f^\cdot (\deg g^\cdot + 1)} g^\cdot f^\cdot$;
- $m^n_{\mathcal{C}(\mathbb{K})} = 0$, for all $n > 2$;

is an A_∞-category.

Definition 1.1.2 (A_∞-opposite category)**.** We define the *opposite category* of $\mathscr{A}$ (denoted by $\mathscr{A}^{\mathrm{op}}$) to be the category defined by:

(op1) $\mathrm{Obj}(\mathscr{A}^{\mathrm{op}}) = \mathrm{Obj}(\mathscr{A})$;

(op2) $\forall x, y \in \mathscr{A}^{\mathrm{op}}$ we have $\mathrm{Hom}_{\mathscr{A}^{\mathrm{op}}}(x, y) = \mathrm{Hom}_{\mathscr{A}}(y, x)$;

(op3) $\forall n > 1$ we have $m^n_{\mathscr{A}^{\mathrm{op}}}(f_1, ..., f_n) = (-1)^{\epsilon(f_n,...,f_1)} m^n_{\mathscr{A}}(f_n, ..., f_1)$, where

$$\epsilon(f_n, ..., f_1) = \sum_{1 \leq i < j \leq k} (\deg f_i + 1)(\deg f_j + 1) + 1.$$

Definition 1.1.3 (Homotopy category)**.** We define the *homotopy category* of $\mathscr{A}$, denoted by $\mathrm{Ho}(\mathscr{A})$, to be the category[1] whose:

- objects are objects of $\mathscr{A}$,
- for every pair of objects x and y, the morphisms are given by the quotient

$$\mathrm{Ho}(\mathscr{A})(x, y) := \frac{Z^0(\mathscr{A}(x, y))}{B^0(\mathscr{A}(x, y))} = H^0(\mathscr{A}(x, y)),$$

  where $Z^0(\mathscr{A}(x, y)) := \mathrm{Ker}(m^1_{\mathscr{A}} : \mathscr{A}^0(x, y) \to \mathscr{A}^1(x, y))$ and
  $B^0(\mathscr{A}(x, y)) := \mathrm{Im}(m^1_{\mathscr{A}} : \mathscr{A}^{-1}(x, y) \to \mathscr{A}^0(x, y))$.

The composition of such a category is given by

$$fg = (-1)^{\deg(g)} m^2(f, g).$$

Let $\mathscr{A}$ be a small A_∞-category and let x be an object of $\mathscr{A}$.

Definition 1.1.4 (Unit)**.** We define the *unit* of x, denoted by e_x, to be a morphism of degree 0 such that:

- $m^2_{\mathscr{A}}(f, e_x) = f$,

  $m^2_{\mathscr{A}}(e_x, g) = (-1)^{\deg(g)} g$,
- $m^n_{\mathscr{A}}(..., e_x, ...) = 0$, for all $n \neq 2$.

We say that an A_∞-category $\mathscr{A}$ is *unital* if $\mathscr{A}$ is equipped with a unit e_x for every object x of $\mathscr{A}$.

Given a non-unital A_∞-category $\mathscr{C}$, we can associate to $\mathscr{C}$ a unital A_∞-category $\mathscr{C}_+$, defined as:

- $\mathrm{Obj}(\mathscr{C}_+) = \mathrm{Obj}(\mathscr{C})$,

[1] such a category has no, a priori, identity arrow between objects.

- $\mathscr{C}_+(x,y) := \begin{cases} \mathbb{K}1_{\mathbb{K}} \oplus \mathscr{C}(x,y) \text{ if } x=y, \\ \mathscr{C}(x,y), \text{otherwise.} \end{cases}$
- $m^1_{\mathscr{C}_+}(1_x) = 0$,

  $m^2_{\mathscr{C}_+}(1_x, f) = (-1)^{\deg(f)} f$,

  $m^2_{\mathscr{C}_+}(f, 1_x) = f$,

  $m^n(..., 1_x, ...) = 0$, for every $n > 2$.

Viceversa, given a unital A_∞-category $\mathscr{D}$, we can associate to $\mathscr{D}$ a non-unital A_∞-category $\mathscr{D}_-$, defined as:

- $\mathrm{Obj}(\mathscr{D}_-) = \mathrm{Obj}(\mathscr{D})$,
- $\mathscr{D}_-(x,y) := \begin{cases} \mathscr{D}(x,y)/\{\mathbb{K}e_x\}, \text{ if } x=y, \\ \mathscr{D}(x,y), \text{otherwise.} \end{cases}$
- $m^1_{\mathscr{D}_-} = m^1_{\mathscr{D}}$,

  $m^n_{\mathscr{D}_-} = 0$, for every $n > 1$.

Definition 1.1.5 (Cohomological unital category)**.** We say that $\mathscr{A}$ is *cohomological unital* if $\mathrm{Ho}(\mathscr{A})$ is unital.

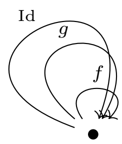
Definition 1.1.6 (Dg-category)**.** We define a *dg-category* to be a unital A_∞-category such that $m^n = 0$, for every $n > 2$.

Remark 1.1
We note that, taking a dg-category $\mathscr{A}$ and setting

$$fg = (-1)^{\deg(g)} m^2(f,g),$$

we can see $\mathscr{A}$ as a category enriched in the category of chain complexes.

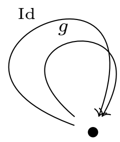
Example 1.1.2. We consider the following unital category, denoted by $\mathscr{A}$, with one object and three morphisms of degree zero:



We have $\mathscr{A}(\bullet,\bullet) = \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}$. The maps $m^i_{\mathscr{A}} = 0$ if $i \neq 2$, and:

$$\begin{aligned} m^2_{\mathscr{A}} : \mathscr{A}(\bullet,\bullet) \otimes \mathscr{A}(\bullet,\bullet) &\to \mathscr{A}(\bullet,\bullet) \\ (a, b_1, b_2), (a', b'_1, b'_2) &\mapsto (aa', ab'_1 + a'b_1, ab'_2 + a'b_2). \end{aligned}$$

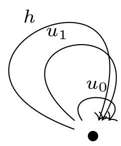
Example 1.1.3. We consider the following category, denoted by $\mathscr{B}$, with one object and two morphisms one of degree zero and one of degree 1:



We have $\mathscr{B}(\bullet,\bullet) = \mathbb{K} \oplus \mathbb{K}[1]$. The maps $m^i_{\mathscr{B}} = 0$ if $i \neq 2$, otherwise:

$$\begin{aligned} m^2_{\mathscr{B}} : \mathscr{B}(\bullet,\bullet) \otimes \mathscr{B}(\bullet,\bullet) &\to \mathscr{B}(\bullet,\bullet) \\ (a,b),(a',b') &\mapsto (aa', ab' + a'b). \end{aligned}$$

Example 1.1.4. We consider the following unital dg-category, denoted by $\mathscr{I}$, with one object and two morphisms u_0, u_1 of degree zero, such that $d(u_0) = d(u_1) = 0$, and a morphism h of degree 1, such that $h = d(u_0) = -d(u_1)$:



We have $\mathscr{I}(\bullet,\bullet) = \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}$. The product is given by:

$$\begin{aligned} \cdot : \mathscr{I}(\bullet,\bullet) \otimes \mathscr{I}(\bullet,\bullet) &\to \mathscr{I}(\bullet,\bullet) \\ (k_1u_0 + k_2u_1 + k_3h),(k_4u_0 + k_5u_1 + k_6h) &\mapsto (k_1k_4)u_0 + (k_2k_5)u_1 + (k_3k_4 + k_2k_6)h. \end{aligned}$$

The unit of $\mathscr{I}$ is the morphism $u_0 + u_1$, and the homotopy category of $\mathscr{I}$ is $\mathbb{K}$.
Moreover, given an A$_\infty$-category $\mathscr{B}$, we define the A$_\infty$-category $\mathscr{I} \otimes \mathscr{B}$ to be the category with the same objects of $\mathscr{B}$ and whose morphisms, between two objects x and y, are given by the tensor product $\mathscr{I} \otimes \mathscr{B}(x,y)$. The A$_\infty$-structure of $\mathscr{I} \otimes \mathscr{B}$ is given by the maps:

$$m^1(i \otimes b) = i \otimes m^1(b) + (-1)^{\deg(i)+\deg(b)} m^1(i) \otimes b,$$

and

$$m^d(i_d \otimes b_d, ..., i_1 \otimes b_1) = (-1)^{\sum_{j>k}(\deg(b_j)\text{-}1)\deg(i_k)} i_d ... i_1 \otimes m^d(b_d, ..., b_1),$$

if $i > 1$.

1.2 A$_\infty$-functors and dg-functors

Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be three A$_\infty$-categories.

Definition 1.2.1 (A_∞-functor)**.** We define an *A_∞-functor* $\mathscr{F} : \mathscr{A} \to \mathscr{B}$ to be: a map between the objects of $\mathscr{A}$ and $\mathscr{B}$, and a collection of $\mathbb{K}$-linear maps (for every integer $d \geq 1$):

$$\mathscr{F}^d : \mathscr{A}(x_{d-1}, x_d) \otimes ... \otimes \mathscr{A}(x_0, x_1) \to \mathscr{B}(\mathscr{F}(x_0), \mathscr{F}(x_d))[1-d]$$

such that the followings are satisfied:

$$\sum_{r\geq 1} \sum_{s_1,...,s_r} m^r_{\mathscr{B}}(\mathscr{F}^{s_r}(a_d, ..., a_{d-s_r+1}), ..., \mathscr{F}^{s_1}(a_{s_1}, ..., a_1)) =$$

$$= \sum_{m=1}^{d} \sum_{n=0}^{d-m} (-1)^{\ddagger_n} \mathscr{F}^{d-m+1}(a_d, ..., a_{n+m+1}, m^m_{\mathscr{A}}(a_{n+m}, ..., a_{n+1}), a_n, ..., a_1)$$

where $s_1 + ... + s_r = d$.
Moreover, if $\mathscr{A}$ and $\mathscr{B}$ are unital A_∞-categories. We say that $\mathscr{F}$ is a *unital* A_∞-functor, if the unit is preserved by $\mathscr{F}_1$, and $\mathscr{F}_d(..., e_x, ...) = 0$ for every $d \geq 2$.

Let us note that $\mathscr{F}$ induces a functor $\mathrm{Ho}(\mathscr{F}) : \mathrm{Ho}(\mathscr{A}) \to \mathrm{Ho}(\mathscr{B})$, whose action on the morphisms is $[f] \to [\mathscr{F}^1(f)]$.

Definition 1.2.2 (Cohomological unital functor)**.** Let $\mathscr{A}$ and $\mathscr{B}$ be two cohomological unital A_∞-category. An A_∞-functor $\mathscr{F} : \mathscr{A} \to \mathscr{B}$ is *cohomological unital* if the functor $\mathrm{Ho}(\mathscr{F})$ is unital.

Definition 1.2.3 (Dg-functor)**.** We define a *dg-functor* between two dg-categories $\mathscr{C}$ and $\mathscr{D}$ to be a (unital) A_∞-functor $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ such that $\mathscr{F}^n = 0$ for every $n \geq 1$.

Example 1.2.1. Given $\mathscr{A}$ and $\mathscr{B}$ the categories defined in Example 1.1.2 and Example 1.1.3. The functors $\mathscr{F}, \mathscr{G} : \mathscr{A} \to \mathscr{B}$ defined in the following way:

$$\mathscr{F}^n := \begin{cases} \mathscr{F}^1(a, b_1, b_2) := (a, 0), & \text{if } n = 1, \\ 0, & \text{if } n \neq 1. \end{cases}$$

$$\mathscr{G}^n := \begin{cases} \mathscr{G}^1(a, b_1, b_2) := (a, 0), & \text{if } n = 1, \\ \mathscr{G}^2((a, b_1, b_2) \otimes (a', b'_1, b'_2)) := (0, b_1 b'_2), & \text{if } n = 2, \\ 0, & \text{if } n \neq 1, 2. \end{cases}$$

are two unital A_∞-functors.

Example 1.2.2. Given $\mathscr{I}$ the category defined in Example 1.1.4 and $\mathbb{K}$ the category with one object and $\mathbb{K}(\bullet, \bullet) = \mathbb{K}$. The functors $\mathscr{F}_1$ and $\mathscr{F}_0$ from $\mathscr{I}$ to $\mathbb{K}$ defined as

$$\mathscr{F}^n_j := \begin{cases} \mathscr{F}^1_j(u_0, u_1, h) := (u_j), & \text{if } n = 1, \\ 0, & \text{if } n \neq 1, \end{cases}$$

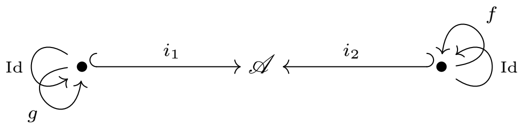
with $j = 0, 1$, are two unital A_∞-functors.

Definition 1.2.4 (Composition of A_∞-functors)**.** Let $\mathscr{F} : \mathscr{A} \to \mathscr{B}$ and $\mathscr{G} : \mathscr{B} \to \mathscr{C}$ two A_∞-functors. The composition $\mathscr{G}\mathscr{F} : \mathscr{A} \to \mathscr{C}$ is the A_∞-functor defined, for every positive integers d, as:

$$(\mathscr{G}\mathscr{F})^d(a_d, ..., a_1) = \sum_{r \geq 1} \sum_{s_1,...,s_r} \mathscr{G}^r(\mathscr{F}^{s_r}(a_d, ..., a_{d-s_r+1}), ..., \mathscr{F}^{s_1}(a_{s_1}, ..., a_1))$$

where $s_1 + ... + s_r = d$.

Remark 1.2
In the situation of Example 1.2.1. We consider the followings subcategories of $\mathscr{A}$:



We note that $\mathscr{F} i_k = \mathscr{G} i_k$ and that, the minimal A_∞-category containing $\mathrm{Im}(i_1)$ and $\mathrm{Im}(i_2)$, is $\mathscr{A}$.

Definition 1.2.5 (Quasi-equivalence)**.** Let $\mathscr{A}$ and $\mathscr{B}$ be a unital A_∞-categories, we say that an A_∞-functor $\{\mathscr{F}^n\} : \mathscr{A} \to \mathscr{B}$ is a *quasi-equivalence* if:

- $\mathrm{Ho}(\mathscr{F}) : \mathrm{Ho}(\mathscr{A}) \to \mathrm{Ho}(\mathscr{B})$ is an equivalence of categories.
- $\mathscr{F}^1 : \mathscr{A}(x, y) \to \mathscr{B}(\mathscr{F}^0(x), \mathscr{F}^0(y))$ is a quasi-isomorphism[2].

Example 1.2.3. Two dg-categories which are quasi-equivalent are quasi-equivalent as A_∞-categories.

Example 1.2.4. The functors $\mathscr{F}_0$ and $\mathscr{F}_1$, defined in Example 1.2.2, are quasi-equivalences. Moreover the functor $\mathscr{S}$ from $\mathbb{K}$ to $\mathscr{I}$ defined by $\mathscr{S}(1) \to u_0 + u_1$ is a quasi-equivalence, because $\mathscr{F}_0\mathscr{S} = \mathscr{F}_1\mathscr{S} = \mathrm{Id}_{\mathbb{K}}$ (and $\mathscr{S}\mathscr{F}_0 = \mathscr{S}\mathscr{F}_1 = \mathrm{Id}_{\mathscr{I}}$).

Definition 1.2.6 (Category of dg-categories)**.** We define the category of small dg-categories, denoted by dgcat , to be the category whose:

- objects are the small dg-categories,
- the morphisms are dg-functors.

Definition 1.2.7 (Homotopy category of dg-categories)**.** We define the *homotopy category of dg-categories*, denoted by Ho(dgcat), to be the localisation of the category of dg-categories over the class of quasi-equivalences.

Definition 1.2.8 (Category of unital A_∞-categories)**.** We define the category of unital A_∞-categories, denoted by A_∞-cat, to be the category whose:

- objects are small unital A_∞-categories,

[2] $\mathscr{F}^1$ induces an equivalence $H^n(\mathscr{A}(x, y)) \simeq H^n(\mathscr{B}(\mathscr{F}^0(x), \mathscr{F}^0(y)))$, for every integer n.

- the morphisms are unital A_∞-functors.

Definition 1.2.9 (Homotopy category of unital A_∞-categories)**.** We define the *homotopy category of unital A_∞-categories*, denoted by $\mathrm{Ho}(A_\infty\text{-cat})$, to be the localisation of the category of unital A_∞-categories over the class of quasi-equivalences.

Definition 1.2.10 (Category of cohomological unital A_∞-categories)**.** We define the category of cohomological unital A_∞-categories, denoted by $A_\infty\text{-cat}^{\mathrm{cu}}$, to be the category whose:

- objects are small cohomological unital A_∞-categories,
- the morphisms are cohomological unital A_∞-functors.

Definition 1.2.11 (Homotopy category of cohomological unital A_∞-categories)**.** We define the *homotopy category of cohomological unital A_∞-categories*, denoted by $\mathrm{Ho}(A_\infty\text{-cat}^{\mathrm{cu}})$, to be the localisation of the category of cohomological unital A_∞-categories over the class of quasi-equivalences.

1.3 Prenatural transformations and homotopy between functors

Let $\mathscr{F}$ and $\mathscr{G}$ be two A_∞-functors between two A_∞-categories $\mathscr{A}$ and $\mathscr{B}$.

Definition 1.3.1 (Prenatural transformation)**.** We define a *prenatural transformation* T of degree g to be a sequence $(T^0, T^1, ..., T^d, ...)$ of $\mathbb{K}$-multilinear maps, such that, for every $d > 0$:

$$T^d : \mathscr{A}(x_{d-1}, x_d) \otimes ... \otimes \mathscr{A}(x_0, x_1) \to \mathscr{B}(\mathscr{F}^0(x_0), \mathscr{G}^0(x_d))[g-d].$$

Moreover T^0 associates to every x in $\mathscr{A}$ a morphisms in $\mathscr{B}(\mathscr{F}^0(x), \mathscr{G}^0(x))$.

We denote by $\mathrm{Hom}_{A_\infty\text{-Fun}}(\mathscr{F}, \mathscr{G})^g$ the prenatural transformations between $\mathscr{F}$ and $\mathscr{G}$ of degree g. On the other hand, $\mathrm{Hom}_{A_\infty\text{-Fun}}(\mathscr{F}, \mathscr{G})$ denotes all prenatural transformations (in any degree) between $\mathscr{F}$ and $\mathscr{G}$.

Definition 1.3.2 (Differential of prenatural transformation)**.** Given a prenatural transformation $T \in \mathrm{Hom}_{A_\infty\text{-Fun}}(\mathscr{F}, \mathscr{G})$, we define the *differential* μ^1 of T to be the prenatural transformation $\mu^1(T)$ such that, for every $d > 0$:

$$\begin{aligned}\mu^1(T)^d(a_d, ..., a_1) = &\sum_{r=1}^{d} \sum_{s_1,...,s_r} \sum_{i=1}^{r} (-1)^{\dagger_1} m^r(\mathscr{G}^{s_r}(a_d, ..., a_{d-s_r+1}), ...\\ &..., \mathscr{G}^{s_{i+1}}(..., a_{s_{1+...+s_i+1}}), T^{s_i}(a_{s_1+...+s_i}, ..., a_{s_1+...+s_{i-1}+1}),\\ &\mathscr{F}^{s_{i-1}}(a_{s_1+...+s_{i-1}}, ...), ..., \mathscr{F}^{s_1}(a_{s_1}, ..., a_1)) +\\ &+ \sum_{i=1}^{d} \sum_{j=1}^{d} (-1)^{\deg(T)+\ddagger_i} T^{d-j}(a_d, ..., m^j(a_{i+j+1}, ..., a_{i+1}), ..., a_1).\end{aligned}$$

Where $s_1 + ... + s_r = d$ and $\dagger_1 = (\deg(T)-1)(\deg(a_1) + ... + \deg(a_{s_1+...+s_{i-1}}) - s_1 - ... - s_{i-1})$ and $\ddagger_i = \deg(a_1) + ... + \deg(a_i) - i$.

We call *natural transformation*, a prenatural transformation T whose differential μ^1 is zero. The name "natural transformation" follows from the fact that, if $\mathscr{F}$ and $\mathscr{G}$ are cohomological unital, T induces a natural transformation between $\mathrm{Ho}(\mathscr{F})$ and $\mathrm{Ho}(\mathscr{G})$ in the classical sense.

Definition 1.3.3 (Product of prenatural transformations)**.** Given two prenatural transformations $T_1 \in \mathrm{Hom}_{\mathrm{A}_\infty\text{-Fun}}(\mathscr{F}_0, \mathscr{F}_1)$ and $T_2 \in \mathrm{Hom}_{\mathrm{A}_\infty\text{-Fun}}(\mathscr{F}_1, \mathscr{F}_2)$, we define the *product* $\mu^2(T_2, T_1)$ to be the prenatural transformation in $\mathrm{Hom}_{\mathrm{A}_\infty\text{-Fun}}(\mathscr{F}_0, \mathscr{F}_2)$ given by:

$$\begin{aligned}\mu^2(T_2,T_1)^d(a_d,...,a_1) = \sum_{r,i,j}\sum_{s_1,...,s_r} (-1)^{\circ} m^r_{\mathscr{B}}(&\mathscr{F}_2^{s_r}(a_d,...,a_{d-s_r+1}),...,\mathscr{F}_2^{s_{j+1}}(...)),\\ &T_2^{s_j}(a_{s_1+...+s_j},...,a_{s_1+...+s_{j-1}+1}),\mathscr{F}_1^{s_{j-1}}(...),...,\mathscr{F}_1^{s_{i+1}}(...),\\ &T_1^{s_i}(a_{s_1+...+s_{s_1+...+s_{i-1}+1}}),\mathscr{F}_0^{s_{i-1}}(...),...,\mathscr{F}_0^{s_1}(a_{s_1},...,a_1)),\end{aligned}$$

where $\circ = \displaystyle\sum_{k=1}^{s_1+...+s_{j-1}} (\deg(T_2)-1)(\deg(a_k)-1) + \sum_{k=1}^{s_1+...+s_{i-1}} (\deg(T_1)-1)(\deg(a_k)-1).$

More generally μ^d is defined, for all $d \geq 1$, and the following result holds.

Theorem 1.3. *The category of A_∞-functors between two A_∞-categories $\mathscr{A}$ and $\mathscr{B}$, together with prenatural transformations, is an A_∞-category denoted by* $\mathrm{A}_\infty\text{-Fun}(\mathscr{A},\mathscr{B})$ *(or shortly* A_∞-Fun*). Moreover if $\mathscr{B}$ is a dg-category, then* $\mathrm{A}_\infty\text{-Fun}(\mathscr{A},\mathscr{B})$ *is a dg-category.*

Proof. See §**7** of [19] or [56, pp. 19-20]. □

Example 1.3.1. Given an A_∞-category $\mathscr{A}$, we call $\mathrm{A}_\infty\text{-Fun}(\mathscr{A}^{\mathrm{op}}, \mathcal{C}(\mathbb{K}))$ the A_∞-category of *right non-unital modules.*

We say that two A_∞-functors $\mathscr{F}$ and $\mathscr{G}$ are *homotopic* if, there exists a prenatural transformation H, of degree 0, such that $H^0 = 0$ and $\mathscr{F}^d - \mathscr{G}^d = \mu^1(H)^d$, for every $n \geq 1$. We point out that homotopy is an equivalence relation.

Definition 1.3.4 (Strictly unital homotopy)**.** We define a *strictly unital homotopy* to be an homotopy H, between two A_∞-functors, such that $H^n(a_n, ..., e, ..., a_1) = 0$, for every $n \geq 1$.

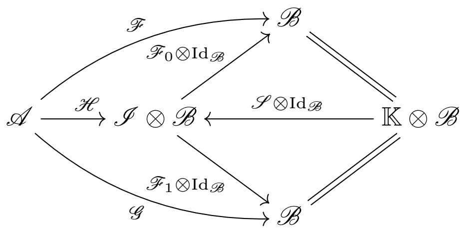
Remark 1.4

We note that, T is a homotopy between $\mathscr{F}$ and $\mathscr{G}$ if and only if, the maps

$$\begin{aligned}\mathscr{H}^d : \mathscr{A}(x_{d-1}, y) \otimes ... \otimes \mathscr{A}(x, x_1) \to &(\mathscr{I} \otimes \mathscr{B})(x, y)\\ a_d, ..., a_1 \mapsto{} & u_0 \otimes \mathscr{F}^d(a_d, ..., a_1) + u_0 \otimes \mathscr{G}^d(a_d, ..., a_1)\\ &+ (-1)^{\dagger_d} h \otimes T^d(a_d, ..., a_1).\end{aligned}$$

define an A_∞-functor, where $\mathscr{I}$, $\mathscr{I} \otimes \mathscr{B}$, $\mathscr{F}_0$ and $\mathscr{F}_1$ are defined in Example 1.1.4 and 1.2.2.

Moreover if $\mathscr{F}$, $\mathscr{G}$ and T are strictly unital, then $\mathscr{H}$ is strictly unital. We have the following commutative diagram:



where $\mathscr{S}$ is the quasi-equivalence defined in Example 1.2.4 cf. [56, Remark 1.11].

Remark 1.5
We recall, by [56, (2c)] and [56, Lemma 2.5.], that if $\mathscr{A}$ is a (strictly unital) cohomological unital A_∞-category, and $\mathscr{F}$ is homotopic to $\mathscr{G}$, then they are isomorphic as objects of $\mathrm{Ho}(A_\infty\text{-Fun})$.

1.4 Pretriangulated A_∞-categories

In this section we recall the construction of the pretriangulated envelope of an A_∞-category and some fundamental properties of such a construction. We assume a little familiarity with triangulated categories. For the non-expert reader we suggest the book by Neeman [47, Chapter 1].

The next definition is probably due to Kontsevich, we refer to [7] for the proofs. Let $\mathscr{A}$ be a $\mathbb{K}$-linear A_∞-category.

Definition 1.4.1 (Shifted category and shift functor)**.** We define the category $\Sigma(\mathscr{A})$ to be the A_∞-category such that $\mathrm{Obj}(\Sigma\mathscr{A}) = (\mathrm{Obj}(\mathscr{A})) \times \mathbb{Z}$, and morphisms are defined as follow

$$\Sigma(\mathscr{A})(x[n], y[m]) := \mathscr{A}(x, y)[m-n],$$

where x, $y \in \mathscr{A}$. We note immediately that $\mathscr{A}$ induces an A_∞-structure on $\Sigma\mathscr{A}$. Moreover the endofunctor sending $x[n]$ to $x[n+1]$ is called *shift functor*.

Definition 1.4.2 (Closure under shift)**.** We say that $\mathscr{A}$ is *closed under shift* if the inclusion functor $\mathscr{A} \hookrightarrow \Sigma(\mathscr{A})$ is a quasi-equivalence.

Definition 1.4.3 (A_∞-twisted complexes)**.** A *twisted complex* in $\mathscr{A}$ is a finite set of objects $(E_i[n_i])_{i\in\mathbb{Z}}$ of $\Sigma(\mathscr{A})$ together with maps $\alpha_{ij} \in \mathscr{A}(E_i, E_j)^{n_j - n_i + 1}$, if $i < j$, such that:

$$\sum_{k=1}^{+\infty} m_k(\alpha, ..., \alpha) = 0$$

Remark 1.6
The set of A_∞-twisted complexes, defined above, has the structure of an A_∞-category

[56], [7], we denote such an A_∞-category by $\mathrm{pretr}_{A_\infty}(\mathscr{A})$. Moreover we have an A_∞-functor $i_{A_\infty} : \mathscr{A} \hookrightarrow \mathrm{pretr}_{A_\infty}(\mathscr{A})$ cf. §3 of [56] for the complete construction. Given an A_∞-morphism $\mathscr{F}$ we denote by $\mathrm{pretr}_{A_\infty}\mathscr{F}$ the induced functor.

Definition 1.4.4 (Pretriangulated A_∞-categories)**.** We say that an A_∞-category $\mathscr{A}$ is *pretriangulated* if $\mathscr{A}$ is closed under shifts and the functor $i_{A_\infty} : \mathscr{A} \hookrightarrow \mathrm{pretr}_{A_\infty}(\mathscr{A})$ is a quasi-equivalence.

Remark 1.7
If $\mathscr{C}$ is a dg-category $\mathrm{pretr}(\mathscr{C}) = \mathrm{pretr}_{A_\infty}(\mathscr{C})$. Where $\mathrm{pretr}(\mathscr{C})$ denotes the pretriangulated envelope of the dg-category $\mathscr{C}$ according to the notation of [30].

We recall the fundamental proposition which motivates also the name "pretriangulated":

Proposition 1.8. *Let $\mathscr{C}$ be a pretriangulated dg-category, or a pretriangulated A_∞-category, then the homotopy category* $\mathrm{Ho}(\mathscr{C})$ *is a triangulated category.*

We have the following [56, Lemma 3.25.]:

Theorem 1.9. *Let $\mathscr{F} : \mathscr{A} \to \mathscr{B}$ be a quasi-equivalence between two A_∞-categories. Then* $\mathrm{pretr}_{A_\infty}\mathscr{F} : \mathrm{pretr}_{A_\infty}(\mathscr{A}) \to \mathrm{pretr}_{A_\infty}(\mathscr{B})$ *is a quasi-equivalence.*

Definition 1.4.5 (Idempotent completion)**.** We say that an additive category $\mathscr{K}$ is *idempotent complete* if any endomorphism $E : k \to k$ such that $E^2 = E$ (idempotent) is such that $k = \mathrm{Im}(E) \oplus \ker(E)$.

According to [3], in general, we can always embed an additive category in a idempotent complete category (we denote by $(-)^{ic}$ such an embedding) moreover if $\mathscr{K}$ is a triangulated category we have the following [3, Thm.1.5.]:

Proposition 1.10. *If $\mathscr{K}$ is a triangulated category, its idempotent completion $(\mathscr{K})^{ic}$ admits a unique triangulated structure such that the canonical functor $(-)^{ic}$ is exact.*

Definition 1.4.6 (Idempotent complete)**.** We say that a pretriangulated dg-category $\mathscr{T}$ (or A_∞-category) is *idempotent complete* if the homotopy category $\mathrm{Ho}(\mathscr{T})$ is idempotent complete.

Definition 1.4.7 (Morita equivalence)**.** Let $\mathscr{A}$, $\mathscr{B}$ be A_∞-categories, we say that an A_∞-functor $\{\mathscr{F}^n\} : \mathscr{A} \to \mathscr{B}$ is a *Morita-equivalence* if:

- $\mathrm{pretr}_{A_\infty}(\mathscr{F})^{ic} : \mathrm{pretr}_{A_\infty}(\mathscr{A})^{ic} \to \mathrm{pretr}_{A_\infty}(\mathscr{B})^{ic}$ is a quasi-equivalence.

Definition 1.4.8 (Homotopy (Morita) category of A_∞-categories)**.** We define the *homotopy Morita category of A_∞-category*, denoted by $\mathrm{Hmo}(A_\infty\text{-cat})$, to be the localisation of the category of unital A_∞-category over the class of Morita-equivalences.

Remark 1.11
We say that a dg-functor is a Morita equivalence if it is a Morita equivalence as A_∞-functor.

1.5 Dg cocategories

In this last section, we introduce the category of dg-cocategories that will be fundamental in the next chapter to define the bar and cobar constructions.

Definition 1.5.1 (Dg cocategory)**.** We define a *cocategory* $\mathscr{C}$ to consist of the data:

- a set of objects, denoted by $\mathrm{Obj}(\mathscr{C})$,
- a differential graded $\mathbb{K}$-module, with differential d, for every pair of objects x and y in $\mathscr{C}$,
- a comultiplication

  $$\Delta_{x,y} : \mathscr{C}(x,y) \to \sum_{z \in \mathrm{Obj}(\mathscr{C})} \mathscr{C}(x,z) \otimes \mathscr{C}(z,y),$$

  for every pair x, $y \in \mathrm{Obj}(\mathscr{C})$, such that, for every z, $w \in \mathrm{Obj}(\mathscr{C})$, the following diagram

  $$\begin{array}{ccc} \mathscr{C}(x,z) & \xrightarrow{\quad \Delta_{xyz} \quad} & \mathscr{C}(x,y) \otimes \mathscr{C}(y,z) \\ {\scriptstyle \Delta_{xwz}}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_{xwy} \times \mathrm{Id}_{\mathrm{Hom}(y,\,z)}} \\ \mathscr{C}(x,w) \otimes \mathscr{C}(w,z) & \xrightarrow[\mathrm{Id}_{\mathrm{Hom}(y,\,z)} \times \Delta_{wyz}]{} & \mathscr{C}(x,w) \otimes \mathscr{C}(w,y) \otimes \mathscr{C}(y,z) \end{array}$$

  commutes. Here Δ_{xyz} denotes the composition $\pi_y \Delta_{xz}$, where π_y is the projection on the object y.
  Moreover, we require also that Δ is of degree zero, and that the following diagram

  $$\begin{array}{ccc} \mathscr{C}(x,y) & \xrightarrow{\quad \Delta \quad} & \sum_{z \in \mathrm{Obj}(\mathscr{C})} \mathscr{C}(x,z) \otimes \mathscr{C}(z,y) \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle d \otimes 1 + 1 \otimes d} \\ \mathscr{C}(x,y) & \xrightarrow{\quad \Delta \quad} & \sum_{z \in \mathrm{Obj}(\mathscr{C})} \mathscr{C}(x,z) \otimes \mathscr{C}(z,y) \end{array}$$

  commutes, for every x, y and z.

Let $\mathscr{C}$ and $\mathscr{C}'$ be two cocategories.

Definition 1.5.2 (Counit)**.** Given an object x in $\mathscr{C}$. We define the *counit* of x to be a map $\eta : \mathscr{C}(x,x) \to \mathbb{K}$ such that the following diagram:

$$\begin{array}{ccc} \mathscr{C}(x,x) \otimes \mathscr{C}(x,y) & \xleftarrow{\quad \Delta \quad} & \mathscr{C}(x,x) \\ {\scriptstyle \eta \otimes \mathrm{Id}}\big\downarrow & \nearrow & \\ \mathbb{K} \otimes \mathscr{C}(x,y) & & \end{array}$$

commutes.

Definition 1.5.3 (Dg cofunctors)**.** A *dg cofunctor* F between $\mathscr{C}$ and $\mathscr{C}'$ is given by the following data:

- a map of sets $F_0 : \mathrm{Obj}(\mathscr{C}) \to \mathrm{Obj}(\mathscr{C}')$,
- a $\mathbb{K}$-linear map $F_1 : \mathscr{C}(x,y) \to \mathscr{C}'(F_0(x), F_0(y))$, for x and y objects in $\mathscr{C}$, such that the following diagram

$$\begin{array}{ccc} \mathscr{C}(x,y) & \xrightarrow{\Delta_{xyz}} & \mathscr{C}(x,y) \otimes \mathscr{C}(y,z) \\ \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_1 \otimes F_1} \\ \mathscr{C}'(F_0(x), F_0(y)) & \xrightarrow{\Delta'_{F_0(x)F_0(y)F_0(z)}} & \mathscr{C}'(F_0(x), F_0(y)) \otimes \mathscr{C}'(F_0(y), F_0(z)) \end{array}$$

  commutes. Here Δ denotes the comultiplication of $\mathscr{C}$, and Δ' denotes the comultiplication of $\mathscr{C}'$.
  Moreover we require that:
  $$dF_1(f) = F_1 d(f)$$
  for every pair of objects x and y and for every morphism $f \in \mathscr{C}(x,y)$.

Definition 1.5.4 (Category of dg-cocategories)**.** We define the category of dg-cocategories, denoted by dgcocat, to be the category whose:

- objects are dg-cocategories,
- morphisms are dg-cofunctors.

Definition 1.5.5 (Coderivation)**.** Let F and G be two cofuntors between $\mathscr{C}$ and $\mathscr{C}'$. We define a (F,G)-coderivation to be a map $D : \mathscr{C} \to \mathscr{C}'$ such that:

- $\Delta D = (D \otimes F_1 + G_1 \otimes D)\Delta$.

Chapter 2

Bar and cobar constructions

In this chapter we recall the bar and cobar constructions. To the best of our knowledge, the bar construction is originally due to Eilenberg and MacLane, to pass from algebras to Hopf algebras. On the other hand the cobar construction was originally develop by J. F. Adams, to define a functor from cocommutative differential graded coalgebras, to cocommutative differential graded Hopf algebras. Of course, there are several ways to use bar and cobar constructions. We suggest [62] for a detailed discussion on this topic. Roughly speaking, we use these construction, to pass from an A_∞-category (or a dg-category) to a dg-cocategory, and viceversa. Even if such constructions are functorial, they have some problems concerning the unit. For this reason, in the last section, we define a new functor which works well with respect to the unit. We point out that, most of the results of this section, were already proven in the case of algebras, in [39] or just announced, in the case of categories, by Fukaya. Hence the aim of this chapter, is to write a complete, and self contained reference about these constructions.

Notations

Given a positive integer n.

$\sum_{\star_k^n}$ denotes $\sum_{k=1}^{n}\sum_{i_1,\ldots,i_k}$ with $i_1+\ldots+i_k=n$.

$\sum_{\diamond_n}$ denotes $\sum_{k=1}^{n}\sum_{i_1,\ldots,i_k}$ with $1\leq i_1<\ldots<i_k\leq n$.

2.1 A_∞-categories vs dg cocategories

Let $\mathscr{C}$ be a small unital A_∞-category.

We define the dg-cocategory $\mathrm{Bar}(\mathscr{C})$ to be a category such that:

- $\mathrm{Obj}(\mathrm{Bar}(\mathscr{C}))=\mathrm{Obj}(\mathscr{C})$,

- $\mathrm{Bar}(\mathscr{C})(x,y) := \sum_{n\geq 0}\sum_{\dagger_n}\mathscr{C}(x_n,y)[1]\otimes ...\otimes \mathscr{C}(x,x_1)[1]$.

  Where $\dagger_0 = \mathscr{C}(x,y)[1]$ and $\dagger_n = \{\{x_1,...,x_n\}$ such that $x_i \in \mathrm{Obj}(\mathscr{C})\}$.

  We say that the element $f_n[1]\otimes ...\otimes f_1[1]$ of $\mathrm{Bar}(\mathscr{A})(x,y)$ has *length* n.
  Fixed a positive integer j, we denote by p_j the *projection* from $\mathrm{Bar}(\mathscr{A})(x,y)$ to the elements of length j.

- Fixed two objects x and y of $\mathrm{Bar}(\mathscr{C})$, we define the comultiplication:

$$\Delta : \mathrm{Bar}(\mathscr{C})(x,y) \to \sum_{z\in\mathscr{C}} \mathrm{Bar}(\mathscr{C})(z,y)\otimes \mathrm{Bar}(\mathscr{C})(x,z)$$

  as

$$\Delta(f_n[1]\otimes ...\otimes f_1[1]) = \sum_{i=1}^{n-1} f_n[1]\otimes ...\otimes f_{i+1}[1]\bigotimes f_i[1]\otimes ...\otimes f_1[1],$$

  And $\Delta(f[1]) = 0$.

- Given a morphism $f_n[1]\otimes ...\otimes f_1[1] \in \mathrm{Bar}(\mathscr{C})$, the differential $\hat{d} : \mathrm{Bar}(\mathscr{C})(x,y) \to \mathrm{Bar}(\mathscr{C})(x,y)$ is given by

$$\hat{d} = \sum_{k=1}^{n} \hat{d}_k$$

  Where

$$\hat{d}_k(f_n[1]\otimes ...\otimes f_1[1]) = \sum_{l=1}^{n-k+1}(-1)^{\ddagger_{l-1}} f_n[1]\otimes ...\otimes m_k(f_{l+k-1},...,f_l)[1]\otimes ...\otimes f_1[1],$$

  and $\ddagger_{l-1} = \deg(f_1)+...+\deg(f_{l-1})-(l-1)$.

Clearly, we have

$$\hat{d}(\hat{d}(f_n[1]\otimes ...\otimes f_1[1])) = 0,$$

because $\mathscr{A}$ is an A_∞-category.
Conversely, given a graded $\mathbb{K}$-linear category $\mathscr{C}$, if $\mathrm{Bar}(\mathscr{C})$ is a dg-cocategory then $\mathscr{C}$ is an A_∞-category, by setting:

$$m^n(f_n,...,f_1) := p^n(\hat{d}(f_n[1]\otimes ...\otimes f_1[1]))[-1].$$

It means that we have:

Theorem 2.1. *The A_∞-structures on $\mathscr{C}$ are in bijection with the dg structure on the cocategory* $\mathrm{Bar}(\mathscr{C})$.

Example 2.1.1. Let A be a dg-algebra. The construction bar induces the dg-algebra $\mathrm{Bar}(A) := \sum_{i\geq 1} A[1]^{\otimes i}$.

2.2 A$_\infty$-functors vs dg cofunctors

Let $\mathscr{F}$ from $\mathscr{A}$ to $\mathscr{B}$ be an A$_\infty$-functor.

We define a functor of dg-cocategories, denoted by Bar($\mathscr{F}$), from Bar($\mathscr{A}$) to Bar($\mathscr{B}$) in the following way:

- $\mathrm{Bar}(\mathscr{F})_0(x) = \mathscr{F}^0(x)$, for every $x \in \mathrm{Obj}(\mathscr{A})$,
- Given $f_n[1] \otimes ... \otimes f_1[1] \in \mathrm{Bar}(\mathscr{A})(x, y)$, we define

$$\mathrm{Bar}(\mathscr{F})(f_n[1] \otimes ... \otimes f_1[1]) : = \sum_{\star_k^n} \mathscr{F}^{i_1}(f_n, ..., f_{i_{n-i_1+1}})[1] \otimes ... \\ ... \otimes \mathscr{F}^{i_k}(f_{i_k}, ..., f_1)[1].$$

Example 2.2.1. Let $\mathscr{F} : \mathscr{A} \to \mathscr{B}$ be a dg-functor then

$$\mathrm{Bar}(\mathscr{F})(f_n[1] \otimes ... \otimes f_1[1]) := \mathscr{F}^1(f_n)[1] \otimes ... \otimes \mathscr{F}^1(f_1)[1].$$

Lemma 2.2. *The construction* Bar *is functorial.*

Proof. By Example 2.2.1, we have immediately that

$$\mathrm{Bar}(\mathrm{Id}_-) = \mathrm{Id}_{\mathrm{Bar}(-)},$$

because the identity is a dg functor.
Given two A$_\infty$-functors $\mathscr{F}$, $\mathscr{G}$ and a morphism $f_1[1] \otimes ... \otimes f_j[1]$, we have:

$$\begin{aligned}
&\mathrm{Bar}(\mathscr{G})(\mathrm{Bar}(\mathscr{F})(f_1[1] \otimes ... \otimes f_j[1])) = \\
&= \mathrm{Bar}(\mathscr{G})(\sum_{\star_k^j} \mathscr{F}^{i_1}(f_1, ..., f_{i_1})[1] \otimes ... \otimes \mathscr{F}^{i_k}(f_{j-i_k+1}, ..., f_j)[1]) \\
&= \sum_{\star_k^j} \mathrm{Bar}(\mathscr{G})(\mathscr{F}^{i_1}(f_1, ..., f_{i_1})[1] \otimes ... \otimes \mathscr{F}^{i_k}(f_{j-i_k+1}, ..., f_j)[1]) \\
&= \sum_{\star_s^j}(\sum_{\substack{1 \le q \le s \\ i_1^q + ... + i_{j_q}^q = t_q}} \mathscr{G}^{j_1}(\mathscr{F}^{i_1^1}(f_1, ..., f_{i_1}), ..., \mathscr{F}^{i_{j_1}^1}(f_{t_1 - i_{j_1}^1 + 1}, ..., f_{t_1}))[1] \otimes ... \\
&... \otimes \mathscr{G}^{j_s}(\mathscr{F}^{i_1^s}(f_{j-t_s+1}, ..., f_{i_1^s - j + t_s}), ..., \mathscr{F}^{i_{j_s}^s}(f_{j-i_{j_s}^s+1}, ..., f_j))[1]) \\
&= \sum_{\star_s^j} (\mathscr{G}\mathscr{F})^{t_1}(f_1, ..., f_{i_1})[1] \otimes ... \otimes (\mathscr{G}\mathscr{F})^{t_k}(f_{j-i_k+1}, ..., f_j)[1]) \\
&= \mathrm{Bar}(\mathscr{G}\mathscr{F})(f_1[1] \otimes ... \otimes f_j[1]).
\end{aligned}$$

□

Lemma 2.3. *The construction* Bar *commutes with the differential.*

Proof. We calculate:

$$d(\mathrm{Bar}(\mathscr{F})(f_j[1]\otimes ...\otimes f_1[1])) =$$
$$= d(\sum_{\star_k^j} \mathscr{F}^{i_1}(f_j, ..., f_{j-i_1+1})[1]\otimes ...\otimes \mathscr{F}^{i_k}(f_{i_k}, ..., f_1)[1])$$
$$= \sum_{\star_k^j} d(\mathscr{F}^{i_1}(f_j, ..., f_{j-i_1+1})[1]\otimes ...\otimes \mathscr{F}^{i_k}(f_{i_k}, ..., f_1)[1])$$
$$= \sum_{\star_k^j}(\sum_{1\leq r\leq k}\sum_{s}(-1)^{\star}\mathscr{F}^{i_1}(f_j, ..., f_{j-i_1+1})[1]\otimes ...\otimes \mathscr{F}^{i_{s-1}}(f_{j-i_{s-1}-1}, ...)[1]\otimes$$
$$\otimes\, m^r_{\mathscr{A}}(\mathscr{F}^{i_s}(...), ..., \mathscr{F}^{i_s+r}(...))[1]\otimes \mathscr{F}^{i_s+r+1}(f_{j-i_k+1}, ..., f_j)[1]\otimes ...\otimes \mathscr{F}^{i_k}(f_{i_k}, ..., f_1)[1])$$
$$= \mathrm{Bar}(\mathscr{F})(d(f_j[1]\otimes ...\otimes f_1[1])).$$

Where $\star = \deg(\mathscr{F}^{i_k}(f_{i_k}, ..., f_1)) + ... + \deg(\mathscr{F}^{i_s+r+1}(f_{j-i_k+1}, ..., f_j)) - r$. □

Lemma 2.4. $(\mathrm{Bar}(\mathscr{F})\otimes \mathrm{Bar}(\mathscr{F}))\Delta = \Delta\mathrm{Bar}(\mathscr{F})$.

Proof. We calculate:

$$(\mathrm{Bar}(\mathscr{F})\otimes \mathrm{Bar}(\mathscr{F}))\Delta(f_n[1]\otimes ...\otimes f_1[1]) =$$
$$= \sum_{j=1}^{n}(\mathrm{Bar}(\mathscr{F})\otimes \mathrm{Bar}(\mathscr{F}))(f_n[1]\otimes ...\otimes f_{j+1}[1])\bigotimes(f_j[1]\otimes ...\otimes f_1[1])$$
$$= \sum_{j=1}^{n}\mathrm{Bar}(\mathscr{F})(f_n[1]\otimes ...\otimes f_{j+1}[1])\bigotimes \mathrm{Bar}(\mathscr{F})(f_j[1]\otimes ...\otimes f_1[1])$$
$$= \sum_{j=1}^{n}(\sum_{\star_s^{n-j}}\mathscr{F}^{t_s}(f_n, ..., f_{n-t_s+1})[1]\otimes ...\otimes \mathscr{F}^{t_1}(f_{j+t_1}, ..., f_{j+1})[1])\bigotimes$$
$$\bigotimes(\sum_{\star_k^j}\mathscr{F}^{i_1}(f_j, ..., f_{j-i_1+1})[1]\otimes ...\otimes \mathscr{F}^{i_k}(f_{i_k}, ..., f_1)[1])$$
$$= \Delta(\sum_{\star_k^n}\mathscr{F}^{i_1}(f_n, ..., f_{n-i_1+1})[1]\otimes ...\otimes \mathscr{F}^{i_k}(f_{i_k}, ..., f_1)[1])$$
$$= \Delta(\mathrm{Bar}(\mathscr{F})^n(f_n[1]\otimes ...\otimes f_1[1])).$$

Where $\star_s^{n-j}$ means $j+1\leq s\leq n$ and $t_1+...+t_s = n-j$. □

Theorem 2.5. $\mathrm{Bar}(\mathscr{F})$ *is a dg-cofunctor.*

Proof. The proof is a consequence of Lemma 2.3 and Lemma 2.4. □

2.3 Prenatural transformations vs coderivations

Let T be a prenatural transformation between the A_∞-functors $\mathscr{F}$ and $\mathscr{G}$.

We define

$$\mathrm{Bar}(T)(f_n[1]\otimes ...\otimes f_1[1]) = \sum_{s_1,s_2,s_3}(-1)^{\deg(T)\dagger_1}\mathrm{Bar}(\mathscr{G})^{s_3}(f_n[1]\otimes ...\otimes f_{n-s_3+1}[1])\otimes$$

$$\otimes T^{s_2}(f_{n-s_3},...,f_{s_1+1})[1] \otimes \mathrm{Bar}(\mathscr{F})^{s_1}(f_{s_1}[1] \otimes ... \otimes f_1[1]).$$

Where s_1, s_2, s_3 are positive integers with $s_1 + s_2 + s_3 = n$ and $\dagger_1 = \deg(f_{s_1}) + ... + \deg(f_1) - s_1$.
The following lemma follows immediately from the definition:

Lemma 2.6. *Given a prenatural transformation T. We have:*

$$p_1(\mathrm{Bar}(T)^n(f_n[1] \otimes ... \otimes f_1[1]))[-1] = T^n(f_n,...,f_1).$$

Theorem 2.7. $\mathrm{Bar}(T)$ *is a* $(\mathrm{Bar}(\mathscr{G}),\mathrm{Bar}(\mathscr{F}))$*-coderivation.*

Proof. We calculate:

$$(\mathrm{Bar}(\mathscr{G}) \otimes \mathrm{Bar}(T) + \mathrm{Bar}(T) \otimes \mathrm{Bar}(\mathscr{F}))\Delta(f_n[1] \otimes ... \otimes f_1[1]) =$$
$$= (\mathrm{Bar}(\mathscr{G}) \otimes \mathrm{Bar}(T) + \mathrm{Bar}(T) \otimes \mathrm{Bar}(\mathscr{F}))\sum_{i=0}^{n-1}(f_n[1] \otimes ... \otimes f_{i+1}[1]) \bigotimes (f_i[1] \otimes ... \otimes f_1[1])$$
$$= \sum_{i=0}^{n-1}(\mathrm{Bar}(\mathscr{G})^{n-1}(f_n[1] \otimes ... \otimes f_{i+1}[1]) \otimes \mathrm{Bar}(T)^i(f_i[1] \otimes ... \otimes f_1[1])$$
$$+ \mathrm{Bar}(T)^{n-i}(f_n[1] \otimes ... \otimes f_{i+1}[1]) \otimes \mathrm{Bar}(\mathscr{F})^i(f_i[1] \otimes ... \otimes f_1[1]))$$
$$= (-1)^{\deg(T)}\sum_{i=0}^{n-1}(\mathrm{Bar}(\mathscr{G})^{n-1}(f_n[1] \otimes ... \otimes f_{i+1}[1]) \otimes \sum_{s_1,s_2,s_3} \mathrm{Bar}(\mathscr{G})^{s_3}(f_i[1] \otimes ... \otimes f_{i-s_3+1}[1]) \otimes$$
$$\otimes T^{s_2}(f_{i-s_3},...,f_{s_1+1})[1] \otimes \mathrm{Bar}(\mathscr{F})^{s_1}(f_{s_1}[1] \otimes ... \otimes f_1[1])$$
$$+ \sum_{s_1,s_2,s_3} \mathrm{Bar}(\mathscr{G})^{s_3}(f_n[1] \otimes ... \otimes f_{n-s_3+1}[1])$$
$$\otimes T^{s_2}(f_{i-s_3},...,f_{s_1+i+1})[1] \otimes \mathrm{Bar}(\mathscr{F})^{s_1}(f_{s_1+i}[1] \otimes ... \otimes f_{i+1}[1])$$
$$\otimes \mathrm{Bar}(\mathscr{F})^i(f_i[1] \otimes ... \otimes f_1[1]))$$
$$= \Delta\mathrm{Bar}(T)^n(f_n[1] \otimes ... \otimes f_1[1]).$$

□

2.4 Properties of bar construction

Let $\mathscr{C}$ and $\mathscr{D}$ be two A_∞-categories.

The following lemma characterizes every functor between the image of two A_∞-categories via the bar construction.

Lemma 2.8. *Let F be a dg-cofunctor from* $\mathrm{Bar}(\mathscr{C})$ *to* $\mathrm{Bar}(\mathscr{D})$.
For every positive integer n, we have:

$$F_1(f_1[1] \otimes ... \otimes f_n[1]) = \sum_{\diamond_n} p_1F_1(f_1[1] \otimes ... \otimes f_{i_1}[1]) \otimes ... \otimes p_1F_1(f_{i_k+1}[1] \otimes ... \otimes f_n[1]).$$

Proof. We immediately note that $F_1(f[1]) = p_1(F_1(f[1]))$ and, fixed a positive integer n, we can write $F_1(f_1[1] \otimes ... \otimes f_n[1])$ as

$$p_1(F_1(f_1[1] \otimes ... \otimes f_n[1])) + p_2(F_1(f_1[1] \otimes ... \otimes f_n[1])) + ... + p_n(F_1(f_1[1] \otimes ... \otimes f_n[1])).$$

We prove the lemma by induction on the length of $f_1[1] \otimes ... \otimes f_n[1]$.

Let $n = 2$, then $F_1(f_1[1] \otimes f_2[1]) = p_1 F_1(f_1[1] \otimes f_2[1]) + p_2 F_1(f_1[1] \otimes f_2[1])$.
We have:

$$\Delta F_1(f_1[1] \otimes f_2[1]) = p_2 F_1(f_1[1] \otimes f_2[1]),$$

and

$$(F_1 \otimes F_1)\Delta(f_1[1] \otimes f_2[1]) = F_1(f_1[1]) \otimes F_1(f_2[1]).$$

Comparing the two equation above we have:

$$p_2 F_1(f_1[1] \otimes f_2[1]) = F_1(f_1[1]) \otimes F_1(f_2[1])$$

i.e.

$$F_1(f_1[1] \otimes f_2[1]) = p_1 F(f_1[1] \otimes f_2[1]) + p_1 F_1(f_1[1]) \otimes p_1 F_1(f_2[1]).$$

Now, we assume that

$$F_1(f_1[1] \otimes ... \otimes f_{n-1}[1]) = \sum_{\diamond_{n-1}} p_1 F_1(f_1[1] \otimes ... \otimes f_{i_1}[1]) \otimes ... \otimes p_1 F_1(f_{i_k}[1] \otimes ... \otimes f_{n-1}[1]).$$

We want to prove

$$F_1(f_1[1] \otimes ... \otimes f_n[1]) = \sum_{\diamond_n} p_1 F_1(f_1[1] \otimes ... \otimes f_{i_1}[1]) \otimes ... \otimes p_1 F_1(f_{i_k}[1] \otimes ... \otimes f_n[1]).$$

We calculate

$$\Delta F_1(f_1[1] \otimes ... \otimes f_n[1]) = \sum_{j=2}^{n} (j-1) p_j F_1(f_1[1] \otimes ... \otimes f_n[1]) \tag{2.1}$$

and

$$F_1 \otimes F_1 \Delta(f_1[1] \otimes ... \otimes f_n[1]) = \sum_{i=0}^{n-1} \sum_{\alpha=1}^{i} \sum_{\beta=1}^{n-i} p_\alpha F_1(f_1[1] \otimes ... \otimes f_i[1]) \otimes p_\beta F_1(f_{i+1}[1] \otimes ... \otimes f_n[1]). \tag{2.2}$$

Comparing (2.1) and (2.2), for every fixed $j \leq n-1$, we have

$$(j-1) p_j F_1(f_1[1] \otimes ... \otimes f_n[1]) =$$
$$= \sum_{\alpha=1}^{j-1} \sum_{i=1}^{\alpha} p_\alpha F_1(f_1[1] \otimes ... \otimes f_i[1]) \otimes p_{j-\alpha} F_1(f_{i+1}[1] \otimes ... \otimes f_n[1]).$$

Using the induction hypothesis on $p_\alpha F_1(f_1[1]\otimes ... \otimes f_i[1])$ and $p_{j-\alpha}F_1(f_{i+1}[1]\otimes ... \otimes f_n[1])$ we have

$$\begin{aligned}(j-1)&p_jF_1(f_1[1]\otimes ... \otimes f_n[1]) = \\ &=(j-1)\sum_{\triangle_{j-1}} p_1F_1(f_1[1]\otimes ... \otimes f_{i_1}[1])\otimes ... \otimes p_1F_1(f_{i_{j-1}+1}[1]\otimes ... \otimes f_n[1]),\end{aligned}$$

where $\triangle_{j-1} = \{i_1, ..., i_{j-1}\}$ with $1 \leq i_1 < ... < i_{j-1} \leq n$, and we are done. □

Remark 2.9
In the proof of Lemma 2.8 we only used that $\Delta F = (F \otimes F)\Delta$.

Lemma 2.10. *Setting*

$$\mathscr{F}^n(f_n, ..., f_1) := p_1F_1(f_n[1]\otimes ... \otimes f_1[1])[-1],$$

$\mathscr{F}$ *defines an* A_∞*-functor.*

Proof. By definition of dg-cofunctor we have $\hat{d}F_1(f_n[1]\otimes ... \otimes f_1[1]) = F_1(\hat{d}(f_n[1]\otimes ... \otimes f_1[1]))$. Applying the projection p_1, we have

$$p_1\hat{d}F_1(f_n[1]\otimes ... \otimes f_1[1]) = p_1F_1(\hat{d}(f_n[1]\otimes ... \otimes f_1[1]))$$

The left hand side is given by

$$\begin{aligned}p_1\hat{d}F_1(f_n[1]\otimes ... \otimes f_1[1]) =& m^1(p_1F_1(f_n[1]\otimes ... \otimes f_1[1])) \\ &+m^2(p_1F_1(f_n[1]\otimes ... \otimes f_2[1]), p_1F_1(f_1[1])) \\ &+... \\ &+m^n(p_1F_1(f_n[1]), ..., p_1F_1(f_1[1])) \\ &=\sum_{r=1}^{n}\sum_{s_1,...,s_r} m^r(\mathscr{F}^{s_r}(f_n, ..., f_{n-s_r+1}), ..., \mathscr{F}^{s_1}(f_{s_1}, ..., f_1)).\end{aligned}$$

Where $s_1 + ... + s_r = n$.
The right hand side is given by

$$\begin{aligned}p_1F_1(\hat{d}(f_n[1]\otimes ... \otimes f_1[1])) =& p_1F_1(\sum_j(-1)^{\ddagger_i}f_n[1]\otimes ... \otimes m^j(...)\otimes f_i[1]\otimes ... \otimes f_1[1]) \\ &=\sum_{j=1}^{n}(-1)^{\ddagger_j}\mathscr{F}^{n-j+1}(f_n, ..., f_{n-j+2}, m^j(f_{n-j+1}, ... \\ &..., f_{j+1}), f_j, ..., f_1).\end{aligned}$$

Thus we are done. □

Combining Lemma 2.8 and Lemma 2.10 we have the following:

Theorem 2.11. *The functor* Bar *induces the following bijection:*

- A_∞*-functors between* $\mathscr{A}$ *and* $\mathscr{B}$ $\longleftrightarrow$ *dg cofunctors between* $\mathrm{Bar}(\mathscr{A})$ *and* $\mathrm{Bar}(\mathscr{B})$.

2.5 Homotopy between A$_\infty$-functors

Lemma 2.12. *Let T be a prenatural transformation and $a_d[1] \otimes ... \otimes a_1[1] \in \mathrm{Bar}(\mathscr{A})$, we have:*

$$\begin{aligned}\mu^1(T)^d(a_d, ..., a_1)[1] =& p_1((-1)^{\deg(T)}\mathrm{Bar}(T)(\hat{d}(a_d[1] \otimes ... \otimes a_1[1]))\\ &+\hat{d}(\mathrm{Bar}(T)^d(a_d[1] \otimes ... \otimes a_1[1]))).\end{aligned}$$

Proof. We calculate the right hand side of the equality:

$$\begin{aligned}&p_1(\mathrm{Bar}(T)(\hat{d}(a_d[1] \otimes ... \otimes a_1[1])) =\\ &= p_1(\mathrm{Bar}(T)(\sum_{k=1}^{n}\sum_{j}(-1)^{\ddagger_i}(a_d[1] \otimes ... \otimes m^j(a_{i+j+1}, ..., a_{i+1})[1] \otimes ... \otimes a_1[1]))\\ &= \sum_{k=1}^{n}\sum_{j}(-1)^{\ddagger_i}p_1(\mathrm{Bar}(T)^{d-j}(a_d[1] \otimes ... \otimes m^j(a_{i+j+1}, ..., a_{i+1})[1] \otimes ... \otimes a_1[1]))\\ &= \sum_{k=1}^{n}\sum_{j}(-1)^{\ddagger_i}T^{d-j}(a_d, ..., m^j(a_{i+j+1}, ..., a_{i+1}), ..., a_1)[1],\end{aligned}$$

and

$$\begin{aligned}&p_1(\hat{d}(\mathrm{Bar}(T)^d(a_d[1] \otimes ... \otimes a_1[1])))[-1] =\\ &= p^1(\hat{d}(\sum_{\substack{s_1,s_2,s_3\\ s_1+s_2+s_3=d}}(-1)^{\deg(T)\dagger_1}\mathrm{Bar}(\mathscr{G})^{s_3}(a_d[1] \otimes ... \otimes a_{d-s_3+1}[1])\otimes\\ &\otimes T^{s_2}(a_{d-s_3}, ..., a_{s_1+1})[1] \otimes \mathrm{Bar}(\mathscr{F})^{s_1}(a_{s_1}[1] \otimes ... \otimes a_1[1])))\\ &= p^1(\hat{d}(\sum_{\substack{s_1,s_2,s_3\\ s_1+s_2+s_3=d}}(-1)^{\deg(T)\dagger_1}\\ &\sum_{k=1}^{s_3}\sum_{\substack{i_1,...,i_k\\ i_1+...+i_k=s_3}}\mathscr{G}^{i_1}(a_d, ..., a_{d-i_1})[1] \otimes ... \otimes \mathscr{G}^{i_k}(..., a_{d-s_3+1})[1]\otimes\\ &\otimes T^{s_2}(a_{d-s_3}, ..., a_{s_1+1})[1]\otimes\\ &\otimes\sum_{t=1}^{s_1}\sum_{\substack{r_1,...,r_t\\ r_1+...+r_t=s_1}}\mathscr{F}^{r_1}(a_{s_1}, ...)[1] \otimes ... \otimes \mathscr{F}^{r_t}(a_{r_t}, ..., a_1)[1]))\\ &= \sum_{t=1}^{d}\sum_{k=0}^{t-1}\sum_{i_1,...,i_t}(-1)^{\deg(T)\dagger_2}m^t(\mathscr{G}^{i_t}(a_d, ..., a_{d-i_t+1}), ...\\ &..., \mathscr{G}^{i_{k+2}}(a_{i_k+2}, ...), T^{i_{k+1}}(a_{i_{k+1}}, ..., a_{i_k+1}), \mathscr{F}^{i_k}(a_{i_k}, ...), ..., \mathscr{F}^{i_1}(a_{i_1}, ..., a_1))[1].\end{aligned}$$

Where $\dagger_1 = \deg(a_{s_1}) + ... + \deg(a_1) - s_1$ and $\dagger_2 = \deg(a_{i_k}) + ... + \deg(a_1) - i_k$. This is enough to conclude. □

Remark 2.13
Using the lemma above we have:

$$\mathrm{Bar}(\mu^1(T)^d(a_d, ..., a_1)) =(-1)^{\deg(T)}\mathrm{Bar}(T)(\hat{d}(a_d[1] \otimes ... \otimes a_1[1]))$$

$$+\hat{d}(\mathrm{Bar}(T)^d(a_d[1] \otimes ... \otimes a_1[1])).$$

So

$$\hat{d}(\mathrm{Bar}(\mu^1(T))^d(a_d[1] \otimes ... \otimes a_1[1])) = (-1)^{\deg(T)}\mathrm{Bar}(\mu^1(T))\hat{d}(a_d[1] \otimes ... \otimes a_1[1]).$$

Theorem 2.14. *Given $\mathscr{F}$ an A_∞-functor. If $\mathscr{F}' = \mathscr{F} + \mu^1(\Xi)$ where Ξ is a prenatural transformation then $\mathscr{F}'$ is an A_∞-functor.*

Proof. By Theorem 2.11 we show that $G = \mathrm{Bar}(\mathscr{G})$ is a dg cofunctor.
We take $\mathrm{Bar}(\mathscr{F}) = F$, and $T = \mathrm{Bar}(\mu^1(\Xi))$.

$$\begin{aligned}
\Delta G_1 &= \Delta(F_1 + T_1) \\
&= (F_1 \otimes F_1)\Delta + (F_1 \otimes T_1 + T_1 \otimes G_1)\Delta \\
&= (F_1 \otimes F_1 + F_1 \otimes T_1 + T_1 \otimes G_1)\Delta \\
&= (F_1 \otimes (F_1 + T_1) + T_1 \otimes (F_1 + T_1))\Delta \\
&= ((F_1 + T_1) \otimes (F_1 + T_1))\Delta \\
&= (G_1 \otimes G_1)\Delta
\end{aligned}$$

By Remark 2.13, G is also a dg-functor. □

2.6 Cobar construction

Using a construction which is very similar to the bar one, we can associate to a dg-cocategory a non-unital dg-category. We explain below such a construction, called *cobar construction*.
Let D be a small dg-cocategory.

We define the (not unital) dg-category $\Omega(D)$ to be the category such that:

- $\mathrm{Obj}(\Omega(D)) = \mathrm{Obj}(D)$,
- $\Omega(D)(x, y) := \sum_{n\geq 0} \sum_{\dagger_n} D(x_n, y)[-1] \otimes ... \otimes D(x, x_1)[-1]$.
  Here $\dagger_0 = D(x, y)[-1]$ and $\dagger_n = \{\{x_1, ..., x_n\}$ such that $x_i \in \mathrm{Obj}(D)\}$.

  We say that the element $f_n[-1] \otimes ... \otimes f_1[-1] \in \Omega(D)(x, y)$ has *length* n. Fixed a positive integer j, we denote by p_j the *projection* from $\Omega(D)(x, y)$ to the elements of length j.

- Fixed three objects x, y and z in $\Omega(D)$, we define the multiplication:

$$m^2_{\Omega(D)} : \Omega(D)(y,z) \otimes \Omega(D)(x,y) \to \Omega(D)(x,z)$$

as

$$\begin{aligned} m^2_{\Omega(D)}(f_n[-1] \otimes ... \otimes f_1[-1], g_m[-1] \otimes ... \otimes g_1[-1]) = \\ = f_n[-1] \otimes ... \otimes f_1[-1] \bigotimes g_m[-1] \otimes ... \otimes g_1[-1]. \end{aligned}$$

Here $f_n[-1] \otimes ... \otimes f_1[-1] \in \Omega(D)(x,y)$ and $g_m[-1] \otimes ... \otimes g_1[-1] \in \Omega(D)(y,z)$.

- The differential is given by

$$\begin{aligned} d(f_n[-1] \otimes ... \otimes f_1[-1]) = & \sum_{i=1}^{n} (-1)^{\deg(f_1)+...+\deg(f_{i-1})} f_n[-1] \otimes ... \otimes f_{i+1}[-1] \otimes \\ & \otimes \Delta(f_i)[-1] \otimes f_{i-1}[-1] \otimes ... \otimes f_1[-1] + \\ & + \sum_{i=1}^{n} (-1)^{\deg(f_1)+...+\deg(f_n)} f_n[-1] \otimes ... \otimes f_{i+1}[-1] \otimes \\ & \otimes d(f_i)[-1] \otimes f_{i-1}[-1] \otimes ... \otimes f_1[-1]. \end{aligned}$$

Let F be a cofunctor then

$$\Omega(F)(f_n[-1] \otimes ... \otimes f_1[-1]) = F_1(f_n)[-1] \otimes ... \otimes F_1(f_1)[-1].$$

It means $\Omega(F)(f_n[-1] \otimes ... \otimes f_1[-1]) = \Omega(F)(f_n)[-1] \otimes ... \otimes \Omega(F)(f_1)[-1]$.

Let T be a (F,G)-coderivation, where F and G are dg-cofunctors from C to D, then $\Omega(T)$ is the prenatural transformation defined as follow:

$$\begin{aligned} \Omega(T)^d : \Omega(C)(x_{d-1}, x_d) \otimes ... \otimes \Omega(C)(x_0, x_1) \to \Omega(D)(\mathscr{F}_0 x_0, \mathscr{F}_0 x_d) \\ (f^d_{n_d}[-1] \otimes ... \otimes f^d_1[-1]) \otimes ... \otimes (f^1_{n_1}[-1] \otimes ... \otimes f^1_1[-1]) \mapsto T(f^d_{n_d})[-1] \otimes ... \otimes T(f^d_1)[-1] \otimes ... \\ ... \otimes T(f^1_{n_1})[-1] \otimes ... \otimes T(f^1_1)[-1]. \end{aligned}$$

As before, we prove that such a construction is functorial and sends dg-cofunctors to dg-functors.

Lemma 2.15. *The construction cobar commutes with differentials.*

Proof. We calculate:

$$\begin{aligned} d(\Omega(F)(f_n[-1] \otimes ... \otimes f_1[-1])) &= d(F_1(f_n)[-1] \otimes ... \otimes F_1(f_1)[-1]) \\ &= \sum_{i=1}^{n} (-1)^{\deg(f_1)+...+\deg(f_{i-1})} F_1(f_n)[-1] \otimes ... \otimes F_1(f_{i+1})[-1] \otimes \end{aligned}$$

$$\begin{aligned}
&\otimes \Delta F_1(f_i)[-1] \otimes F_1(f_{i-1})[-1] \otimes ... \otimes F_1(f_1)[-1]+\\
&+\sum_{i=1}^{n}(-1)^{\deg(f_1)+...+\deg(f_n)} F_1(f_n)[-1] \otimes ... \otimes F_1(f_{i+1})[-1]\otimes\\
&\otimes d(F_1(f_i))[-1] \otimes F_1(f_{i-1})[-1] \otimes ... \otimes F_1(f_1)[-1]\\
=&\sum_{i=1}^{n}(-1)^{\deg(f_1)+...+\deg(f_{i-1})} F_1(f_n)[-1] \otimes ... \otimes F_1(f_{i+1})[-1]\otimes\\
&\otimes F_1(\Delta^1 f_i)[-1] \otimes F_1(\Delta^2 f_i)[-1] \otimes F_1(f_{i-1})[-1] \otimes ... \otimes F_1(f_1)[-1]+\\
&+\sum_{i=1}^{n}(-1)^{\deg(f_1)+...+\deg(f_n)} F_1(f_n)[-1] \otimes ... \otimes F_1(f_{i+1})[-1]\otimes\\
&\otimes F_1(df_i)[-1] \otimes F_1(f_{i-1})[-1] \otimes ... \otimes F_1(f_1)[-1]\\
=&\Omega(F)(d(f_n[-1] \otimes ... \otimes f_1[-1])).
\end{aligned}$$

□

Lemma 2.16. *The construction* cobar *is functorial.*

Proof. We have immediately that

$$\Omega(\mathrm{Id}_-) = \mathrm{Id}_{\Omega(-)}.$$

Moreover, given two dg-cofunctors F and G and a morphism $f_n[-1]\otimes...\otimes f_1[-1]$, we have:

$$\begin{aligned}
\Omega(F)(\Omega(G)(f_n[-1] \otimes ... \otimes f_1[-1])) =&\Omega(F)(G_1(f_n)[-1] \otimes ... \otimes G_1(f_1)[-1])\\
=&(F_1G_1)(f_n)[-1] \otimes ... \otimes (F_1G_1)(f_1)[-1]\\
=&\Omega(FG)(f_n[-1] \otimes ... \otimes f_1[-1])).
\end{aligned}$$

□

2.7 The functor U

Using the constructions bar and cobar, we get a functor from the category of A_∞-categories to the category of dg-categories, given by the composition $\Omega(\mathrm{Bar}(-))_+$.
In this section, using the constructions bar and cobar, we propose a functor U, strongly inspired by [39, Dèfinition 2.3.4.2] and [52, Appendix C], from A_∞-categories to dg-categories. As the reader can see, given a unital A_∞-category $\mathcal{A}$, the composition of the bar and cobar constructions does not preserve the unit. The great advantage of the functor U is that it does not "forget" the existence of the unit in the category $\mathcal{A}$.

Let $\mathscr{A}$ be a unital A_∞-category.

- $\mathrm{Obj}(\mathrm{U}(\mathscr{A})) = \mathrm{Obj}(\Omega(\mathrm{Bar}(\mathscr{A}))_+)$.
- The morphisms $\mathrm{U}(\mathscr{A})(x,y) := \Omega(\mathrm{Bar}(\mathscr{A}))_+(x,y)/_\sim$ where:

  i $(f_n[1] \otimes ... \otimes e[1] \otimes ... \otimes f_1[1])[-1] \sim 0$, ii $(e_x[1])[-1] \sim 1_x$, where 1_x is the unit in $\Omega(\mathrm{Bar}(\mathscr{A}))_+(x,x)$.

- The composition and the differential are the same of $\Omega(\mathrm{Bar}(\mathscr{A}))_+$.

We give an explicit description of the morphisms of $\mathrm{U}(\mathscr{A})$:

- The morphisms $\mathrm{U}(\mathscr{A})(x,y)$ are generated by
  - the morphisms of the form
  $$(f^m_{n_m}[1] \otimes ... \otimes f^m_1[1])[-1] \otimes ... \otimes (f^1_{n_1}[1] \otimes ... \otimes f^1_1[1])[-1],$$
  if $x \neq y$,here $f^i_j \in \mathscr{A}(x^i_{j+1}, x^i_j)$, $f \neq e$, $x^1_1 = x$ and $x^m_{n_m+1} = y$.
  - the morphisms of the form
  $$(f^m_{n_m}[1] \otimes ... \otimes f^m_1[1])[-1] \otimes ... \otimes (f^1_{n_1}[1] \otimes ... \otimes f^1_1[1])[-1] \text{ or } \mathbb{K} \cdot 1_x,$$
  if $x = y$ here $f^i_j \in \mathscr{A}(x^i_{j+1}, x^i_j)$, $f \neq e$ and $x^m_{n_m+1} = x^1_1 = x$.
- The composition $m^2(\mathscr{A})(-,-)$ is given by:
$$m^2(g_m[-1] \otimes ... \otimes g_1[-1], h_l[-1] \otimes ... \otimes h_1[-1]) = \\ = h_l[-1] \otimes ... \otimes h_1[-1] \otimes g_m[-1] \otimes ... \otimes g_1[-1],$$
where $g_m[-1] \otimes ... \otimes g_1[-1] \in \mathrm{U}(\mathscr{A})(y,z)$ and $h_l[-1] \otimes ... \otimes h_1[-1] \in \mathrm{U}(\mathscr{A})(x,y)$.
- The differential is given by the following formula:
$$\begin{aligned} d((f^m_{n_m}[1] \otimes ... \otimes f^m_1[1])[-1] \otimes ... \otimes (f^1_{n_1}[1] \otimes ... \otimes f^1_1[1])[-1]) = \\ = \sum_{i=1}^{m} (-1)^{\deg(g_1)+...+\deg(g_{i-1})} g_m[-1] \otimes ... \otimes g_{i+1}[-1] \otimes \\ \otimes \sum_{j=2}^{n_i} (f^i_{n_i}[1] \otimes ... \otimes f^i_j[1])[-1] \bigotimes (f^i_{j-1}[1] \otimes ... \otimes f^i_1[1])[-1] \otimes \\ \otimes g_{i-1}[-1] \otimes ... \otimes g_1[-1] + \\ + \sum_{i=1}^{m} (-1)^{\deg(g_1)+...+\deg(g_m)} g_m[-1] \otimes ... \otimes g_{i+1}[-1] \otimes \\ \otimes \sum_{k=1}^{n_i} \sum_{j=1}^{n_i-k+1} (-1)^{\deg(f^i_1)+...+\deg(f^i_{k-1})} (f^i_{n_i}[1] \otimes ... \otimes f^i_{k+j}[1] \otimes \\ \otimes m^j_{\mathscr{A}}(f^i_{k+j-1}, ..., f^i_k)[1] \otimes f^i_{k-1}[1] \otimes ... \otimes f^i_1[1])[-1] \otimes \\ \otimes g_{i-1}[-1] \otimes ... \otimes g_1[-1]. \end{aligned}$$

Where $(f^m_{n_m}[1] \otimes ... \otimes f^m_1[1])[-1] \otimes ... \otimes (f^1_{n_1}[1] \otimes ... \otimes f^1_1[1])[-1] = g_m[-1] \otimes ... \otimes g_1[-1]$.

Given an A_∞-functor $\mathscr{F}$, we note immediately that the construction U induce a dg-functor $\mathrm{U}(\mathscr{F})$ between $\mathrm{U}(\mathscr{A})$ and $\mathrm{U}(\mathscr{B})$. Moreover, we have that the construction U is functorial, i.e. U defines a functor between unital A_∞-categories and dg-categories.

Chapter 3

Categories of A_∞-categories and dg-categories

The aim of this chapter is to provide a natural equivalence between the localisations of the categories of A_∞-categories and dg-categories, over some classes of morphisms. In detail, we prove that: the homotopy categories of unital A_∞-categories (resp. cohomological unital), dg-categories and the category of unital A_∞-categories (resp. cohomological unital) localized over the homotopy relation, are equivalent. We prove also that the localisations of the categories of A_∞-categories and dg-categories, over Morita equivalences, are equivalent.

A similar result (for A_∞-algebras) was already obtained in [39]. In 2002 indeed, Lefèvre provided a canonical model structure (without limits) on the category of A_∞-algebras. Making use of such a model structure, he proved an equivalence between the homotopy category of dg-algebras and the homotopy category of A_∞-algebras. For categories, finding an appropriate model structure is much more complicated. We avoid this problem, with direct computation which uses the constructions of previous chapter.

The results of this section are part of the paper [9], which I wrote during the last year of PhD with Prof. Canonaco and Prof. Stellari.

We assume that all the dg-categories and A_∞-categories are small. From now on $\mathbb{K}$ stands for a field. This hypothesis is necessary for the proof of Lemma 3.18, some other results works even $\mathbb{K}$ is a commutative ring.

3.1 Completeness and cocompleteness

First we discuss some properties about completeness and cocompleteness, of the categories of dg-categories and A_∞-categories.

Proposition 3.1. *The category of dg-categories is complete and cocomplete.*

Proof. The final object is the already mentioned category $\mathbb{K}$, and the initial object is the empty set $\emptyset$. The pullback is defined in the canonical way, and the existence of pushouts is guaranteed by Proposition 7.2, 7.4 and Lemma 6.6. of [16]. □

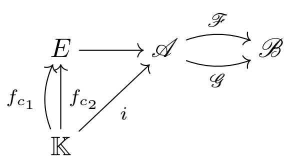
In the case of the category of A$_\infty$-categories, initial and final objects are the same as for the category of dg-categories. However the situation for pullback and pushout is different. The fact that the category of A$_\infty$-algebras (or A$_\infty$-categories) is not complete is well know, but we could not find an example in the existing literature. The next two propositions and lemmas are devoted to prove that the category of unital A$_\infty$-categories has no equalizers.

Proposition 3.2. *In the category of A_∞-categories the diagram*

$$D := \mathscr{A} \underset{\mathscr{G}}{\overset{\mathscr{F}}{\rightrightarrows}} \mathscr{B}$$

(*where the categories and the functors are defined in Example 1.2.1*) *does not admit limit.*

Proof. We suppose that D admits the equalizer $\mathrm{Eq}(\mathscr{F},\mathscr{G}) = (E, e)$ where E is a unital A$_\infty$-category and e is a monomorphism. We suppose that c_1 and c_2 are two objects in E. Let we consider the following diagram



where f_{c_i} denotes the morphism sending the (only) object of $\mathbb{K}$ to c_i. Using the universal property of the equalizer E, by the previous diagram we deduce that $f_{c_1} = f_{c_2}$ so $c_1 = c_2$. It means that the equalizer has just one object, i.e. E is an A$_\infty$-algebra. Now if E is an equalizer, then by Remark 1.2, it has to contain $\mathscr{A}$. But it cannot be possible because $\mathscr{F} \neq \mathscr{G}$. □

Now we prove that the category of cohomological unital A$_\infty$-categories is not complete.

Proposition 3.3. *The diagram of Proposition 3.2 does not admit limit in the category* A$_\infty$-cat$^{\mathrm{cu}}$.

Proof. The proof is the same of Proposition 3.2 using Lemma 3.5 and Lemma 3.4. □

Lemma 3.4. *Given an element $d \in E$ (as in Proposition 3.2) there exists an A_∞-functor $F_d : \mathbb{K} \to E$ which is cohomologically unital and such that* $\mathrm{Im}(F_d^0) = d$.

Proof. By Lemma 3.6 we have a quasi-equivalence ϕ from the category E to a unital A$_\infty$-category D' whose objects are exactly the objects of E and, thanks to [56, Theorem 2.9.], we have a quasi-equivalence ψ from D' to E. Then, using Proposition 3.2, we get an A$_\infty$-functor:

$$\mathbb{K} \xrightarrow{f_d} D' \xrightarrow{\psi} E$$

The composition of the functors above yields $F_{d'}$. □

Lemma 3.5. *The functor $d : E \to \mathscr{A}$ is injective on objects.*

Proof. Let d_1 and d_2 two objects in E such that $d(d_1) = d(d_2)$. Let us consider the functors F_{d_1} and F_{d_2} as in Lemma 3.5. We take the composition of A$_\infty$-cohomological unital functors $dF_{d_j} : \mathbb{K} \to \mathscr{A}$ (where $j = 1, 2$). Since $\mathbb{K}$ and $\mathscr{A}$ are unital in degree zero, the only non trivial component of such an A$_\infty$-functor is

$$(dF_{d_j})_1 = d_1(F_{d_j})_1.$$

But $H(d_1(F_{d_j})_1)(1) = 1_{d(d_j)}$, since $A_1 = H^0(A_1)$, we have $d_1(F_{d_j})_1(1) = 1_{d(d_j)}$.
Thus, for $j = 1, 2$, we have $(dF_{d_1})_1 = (dF_{d_2})_1$. Since they are the only non trivial component of dF_{d_j}, we get $dF_{d_1} = dF_{d_2}$. Then d is a monomorphism so $F_{d_1} = F_{d_2}$ and $d_1 = d_2$. □

3.2 Equivalence between Ho(A$_\infty$-cat$^{\mathrm{cu}}$) and Ho(A$_\infty$-cat)

In this section we prove that the homotopy category of the unital A$_\infty$-categories and the homotopy category of the cohomological unital A$_\infty$-categories are equivalent.
We begin with three lemmas:

Lemma 3.6. *Let $\mathscr{A}$ be a cohomological unital A_∞-category. There exists a unital A_∞-category $\mathscr{A}'$ quasi equivalent to $\mathscr{A}$.*

Lemma 3.7. *Let $\mathscr{A}$, $\mathscr{A}'$ be two A_∞-categories. Given a cohomological unital functor $\mathscr{F} : \mathscr{A} \to \mathscr{A}'$, we can find a strictly unital A_∞-functor $\mathscr{F}'$ homotopic to $\mathscr{F}$. Moreover the homotopy Ξ' from $\mathscr{F}$ to $\mathscr{F}'$ is unital.*

Lemma 3.8. *Let $\mathscr{A}$, $\mathscr{A}'$ be two strictly unital A_∞-categories and let $\mathscr{F}$, $\mathscr{F}'$ be two unital A_∞-functors. Given a cohomological unital homotopy $H : \mathscr{F} \to \mathscr{F}'$, we can find a unital homotopy H' from $\mathscr{F}$ to $\mathscr{F}'$.*

We are ready to prove the main theorem of the section.

Theorem 3.9. *The categories* Ho(A$_\infty$-cat$^{\mathrm{cu}}$) *and* Ho(A$_\infty$-cat) *are equivalents.*

Proof. By Lemma 3.6 we have that, for every cohomological unital A$_\infty$-category $\mathscr{A}$, there exists a strictly unital A$_\infty$-category $\mathscr{A}'$ such that $[\mathscr{A}] = [\mathscr{A}']$ in Ho(A$_\infty$-cat$^{\mathrm{cu}}$).
Then, by Remark 1.4 and by Lemma 3.7 for every cohomological unital A$_\infty$-functor $\mathscr{F}$, there exists a strictly unital A$_\infty$-functor $\mathscr{F}'$ such that $[\mathscr{F}] = [\mathscr{F}']$ in Ho(A$_\infty$-cat$^{\mathrm{cu}}$).
We define the functor

$$R : \mathrm{Ho}(\mathrm{A}_\infty\text{-cat}^{\mathrm{cu}}) \to \mathrm{Ho}(\mathrm{A}_\infty\text{-cat})$$

such that:

$$[\mathscr{A}] \mapsto [\mathscr{A}']$$
$$[\mathscr{F}] \mapsto [\mathscr{F}'].$$

By Lemma 3.8, we have that the functor R is well defined. Moreover the inclusion is the inverse of R. □

Proofs of Lemmas 3.6, 3.7 and 3.8

Proof of Lemma 3.6. This lemma is well known in the literature see [39], [37], [19] or [56]. What follows is just a sketch of a construction, due to Seidel, turning a cohomological unital A_∞-category $\mathscr{A}$ in a unital A_∞-category $\mathscr{A}'$. For a complete proof of the Lemma cf. [56, Lemma 2.1.]. Roughly speaking the strategy of the proof is to find a formal diffeomorphism ϕ i.e. an arbitrary sequence of maps such that ϕ^1 is a linear automorphism of $\mathrm{Hom}_{\mathscr{A}}(x, y)$. Then, solving recursively the conditions of Definition 1.2.1, we get a new A_∞-structure $m_{\mathscr{A}'}$ such that $\phi : \mathscr{A} \to \mathscr{A}'$ is an A_∞-functor ([56, §(1c)]). We write $\mathscr{A}' = \phi_* \mathscr{A}$.
At first, for each object x, we choose a cocycle $e_x \in \mathrm{Hom}^0_{\mathscr{A}}(x, x)$ representing the identity in the homotopy category of $\mathscr{A}$. We can find a chain map

$$m^2_{\mathscr{A}'} : \mathrm{Hom}_{\mathscr{A}}(y, z) \otimes \mathrm{Hom}_{\mathscr{A}}(x, y) \to \mathrm{Hom}_{\mathscr{A}}(x, z)$$

representing the composition in $\mathrm{Ho}(\mathscr{A})$, and which satisfies the unital condition of Definition 1.2.1.
We can take the formal diffeomorphism with $\phi^1_1 = \mathrm{Id}$, such that each ϕ^2_1 is a chain homotopy between $m^2_{\mathscr{A}}$ to $m^2_{\mathscr{A}'}$. Then $m^1_{\mathscr{A}'} = m^1_{\mathscr{A}}$ and the composition is the chosen $m^2_{\mathscr{A}'}$.
Now the construction becomes recursive. We suppose that we have an A_∞-category $\mathscr{A}$ together with a unit e_x, moreover we consider the following family of conditions $(U_{d,n})$, indexed by $d > 2$ and $0 \leq n \leq d$:

$(U_{d,n})$ $m^i_{\mathscr{A}}(a_{i-1}, ..., a_{j+1}, e_{x_j}, a_j, ..., a_1) = 0$, for $i < d$ and arbitrary j, as well as for $i = d$ and $j < n$.

We suppose that $\mathscr{A}$ already satisfies $(U_{d,n})$ for some $n < d$. We can take a formal diffeomorphism ϕ_1 such that

$$\phi^1_1 = \mathrm{Id},$$

$$\phi^k_1 = 0 \text{ for } 2 \leq n \leq d-1,$$

$$\phi^{d-1}_1(a_{d-1}, ..., a_1) = (-1)^{\ddagger_n} m^d_{\mathscr{A}}(a_{d-1}, ..., a_{n+1}, e_{x_n}, a_n, ..., a_1),$$

$$\phi^d_1(a_d, ..., a_1) = (-1)^{\ddagger_n} m^{d+1}_{\mathscr{A}}(a_d, ..., a_{n+1}, e_{x_n}, a_n, ..., a_1).$$

From the definition the composition maps in $\mathscr{A}'$ agree with those of $\mathscr{A}$ for orders $\leq d-1$. Moreover, more involved computation shows that $\mathscr{A}'$ has property $(U_{d,n+1})$.
Now we can repeat the process getting a formal diffeomorphism ϕ_2 such that the A_∞-structure given by $(\phi_2\phi_1)_*\mathscr{A}$ satisfies more of the strict unitality condition. By repeating this process we get a sequence of formal diffeomorphisms ϕ_1, ϕ_2,...
Now we take the transfinite composition $\phi = \phi_k \cdot ... \cdot \phi_1$ with $k \to \infty$, then $\phi_*\mathscr{A}$ is the strictly unital A_∞-category $\mathscr{A}'$. □

Proof of Lemma 3.7. By definition we have $\mathscr{F}(e) = e - \mu^1(f)$, we define $\mathscr{F}_{(1)} := \mathscr{F} - \mu^1(\Xi)$, where Ξ is a prenatural transformation such that $\Xi^1(e) := -f$ and zero otherwise. We have

$$\mathscr{F}^1_{(1)}(e) = \mathscr{F}^1(e) - \mu^1(\Xi)^1(e) = e.$$

Now we suppose that, given two positive integers n and k such that $k < n$,

$$\mathscr{F}^s(a_s, ..., e, ..., a_1) = 0$$

for every $s < n$, and $\mathscr{F}^n(a_n, ..., a_j, ..., a_r, e, a_{r-1}, ..., a_1) = 0$ for every r such that $2 \leq r \leq j - 1$.
Moreover we suppose that $\mathscr{F}^n(a_n, ..., a_j, e, a_{j-1}, ..., a_1) \neq 0$.
Let $\Xi_{(1)}$ be a prenatural transformation of degree 1:

$$\Xi^n_{(1)} : \mathrm{Hom}_{\mathscr{A}}(x_{n-1}, x_n) \otimes ... \otimes \mathrm{Hom}_{\mathscr{A}}(x_0, x_1) \to \mathrm{Hom}_{\mathscr{A}}(\mathscr{F}_0(x_0), \mathscr{F}_0(x_n))[1-n].$$

defined as:

$$\Xi^m_{(1)} = 0, \text{ if } m \neq n-1 \text{ or } n,$$
$$\Xi^{n-1}_{(1)}(a_{n-1}, ..., a_1) = (-1)^{\ddagger_{j-1}} \mathscr{F}^n(a_{n-1}, ..., e, a_{j-1}, ..., a_1),$$
$$\Xi^n_{(1)}(a_n, ..., a_1) = (-1)^{\ddagger_j} \mathscr{F}^{n+1}(a_n, ..., e, a_{j-1}, ..., a_1).$$

We define $\mathscr{F}_{(1)} : \mathscr{A} \to \mathscr{A}'$ as

$$\mathscr{F}^n_{(1)}(a_n, ..., a_1) := \mathscr{F}^n(a_n, ..., a_1) - \mu^1(\Xi_{(1)})^n(a_n, ..., a_1)$$

By definition, we have that $\mathscr{F}^t_{(1)} = \mathscr{F}^t$ for every $t < n$.
Now we prove that $\mathscr{F}^n_{(1)}(a_n, ..., e, a_{j-1}, ..., a_1) = 0$.

$$\begin{aligned}
\mathscr{F}^n_{(1)}(a_n, ..., e, a_{j-1}, ..., a_1) := \; & \mathscr{F}^n(a_n, ..., e, a_{j-1}, ..., a_1) \\
& -\mu^1(\Xi_{(1)})^n(a_n, ..., e, a_{j-1}, ..., a_1) \\
& = \mathscr{F}^n(a_n, ..., e, a_{j-1}, ..., a_1) \\
& -[m^1(\Xi^n(a_n, ..., e, a_{j-1}, ..., a_1)) + \\
& + m^2(\mathscr{F}^1(a_n), \Xi^{n-1}_{(1)}(a_{n-1}, ..., e, a_{j-1}, ..., a_1)) \\
& + m^2(\Xi^{n-1}_{(1)}(a_n, ..., e, a_{j-1}, ..., a_2), \mathscr{F}^1(a_1)) \\
& - \sum_{l=0}^{n-2} (-1)^{\ddagger_l} \Xi^{n-1}_{(1)}(a_n, ..., m^2(a_{l+2}, a_{l+1}), a_l, ..., e, a_{j-1}, ..., a_1) \\
& - \sum_{l=0}^{n-1} (-1)^{\ddagger_l} \Xi^n_{(1)}(a_n, ..., m^1(a_{l+1}), a_l, ..., e, a_{j-1}, ..., a_1)] \\
& = \mathscr{F}^n(a_n, ..., e, a_{j-1}, ..., a_1) \\
& -[(-1)^{\ddagger_j} m^1(\mathscr{F}^{n+1}(a_n, a_{n-1}, ..., e, e, a_{j-1}, ..., a_1)) \\
& +(-1)^{\ddagger_j} m^2(\mathscr{F}^1(a_n), \mathscr{F}^n(a_{n-1}, ..., e, e, a_{j-1}, ..., a_1)) \\
& +(-1)^{\ddagger_j} m^2(\mathscr{F}^n(a_n, ..., e, e, ..., a_2), \mathscr{F}^1(a_1))
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{l=0}^{n-2}(-1)^{\ddagger_l+\ddagger_j}\mathscr{F}^n(a_n,...,m^2(a_{l+2},a_{l+1}),a_l,...,e,e,a_{j-1},...,a_1)\\
&-\sum_{l=0}^{n-1}(-1)^{\ddagger_l+\ddagger_j}\mathscr{F}^{n+1}(a_n,...,m^1(a_{l+1}),a_l,...,e,e,a_{j-1},...,a_1)]\\
=&\mathscr{F}^n(a_n,...,e,a_{j-1},...,a_1)\\
&+(-1)^{\ddagger_j-1}[m^1(\mathscr{F}^{n+1}(a_n,a_{n-1},...,e,e,a_{j-1},...,a_1)\\
&+m^2(\mathscr{F}^1(a_n),\mathscr{F}^n(a_{n-1},...,e,e,a_{j-1},...,a_1))\\
&+m^2(\mathscr{F}^n(a_n,...,e,e,...,a_2),\mathscr{F}^1(a_1))\\
&-\sum_{l=0}^{n-2}(-1)^{\ddagger_l}\mathscr{F}^n(a_n,...,m^2(a_{l+2},a_{l+1}),a_l,...,e,e,a_{j-1},...,a_1)\\
&-\sum_{l=0}^{n-1}(-1)^{\ddagger_l}\mathscr{F}^{n+1}(a_n,...,m^1(a_{l+1}),a_l,...,e,e,a_{j-1},...,a_1)]\\
=&\mathscr{F}^n(a_n,...,e,a_{j-1},...,a_1).
\end{aligned}
$$

Moreover we prove that $\mathscr{F}^n_{(1)}(a_n,...,a_j,...,a_r,e,...,a_1)=0$, if $2\leq r\leq j$.
Firstly we suppose that $2<r$, we have:

$$
\begin{aligned}
\mathscr{F}^n_{(1)}(a_n,...,a_j,...,e,...,a_1):=&\mathscr{F}^n(a_n,...,a_j,...,e...,a_1)\\
&-\mu^1(\Xi_{(1)})^n(a_n,...,a_j,...,e...,a_1)\\
=&\mathscr{F}^n(a_n,...,m^2(e,a_{j-1}),...,e...,a_1)\\
&+-[(-1)^{\ddagger_j}m^1(\mathscr{F}^{n+1}(a_n,...,e,a_{j-1},...,a_r,e,...,a_1))\\
&+(-1)^{\ddagger_{j-1}}m^2(\mathscr{F}^1(a_n),\mathscr{F}^n(a_{n-1},...,e,a_{j-1},...,e,...,a_1))\\
&+(-1)^{\ddagger_{j-1}}m^2(\mathscr{F}^n(a_n,...,e,a_j,...,e,...,a_2),\mathscr{F}^1(a_1))\\
&-\sum_{l=0}^{n-2}(-1)^{\ddagger_l}\Xi^{n-1}_{(1)}(a_n,...,m^2(a_{l+2},a_{l+1}),a_l,...,a_j,...,e,...,a_1)\\
&-\sum_{l=0}^{n-1}(-1)^{\ddagger_l}\mathscr{F}^{n+1}(a_n,...,m^1(a_{l+1}),a_l,...\\
&\quad ...,e,a_{j-1},...,e,...,a_1)]\\
=&\mathscr{F}^n(a_n,...,m^2(e,a_{j-1}),...,e...,a_1)\\
&+(-1)^{\ddagger_{j-1}}[m^1(\mathscr{F}^{n+1}(a_n,...,e,a_{j-1},...,a_r,e,...,a_1))\\
&+m^2(\mathscr{F}^1(a_n),\mathscr{F}^n(a_{n-1},...,e,a_{j-1},...,e,...,a_1))\\
&+m^2(\mathscr{F}^n(a_n,...,e,a_j,...,e,...,a_2),\mathscr{F}^1(a_1))\\
&-\sum_{l=j}^{n-2}(-1)^{\ddagger_l}\mathscr{F}^n(a_n,...,m^2(a_{l+2},a_{l+1}),a_l,...\\
&\quad ...,e,a_{j-1},...,a_r,e,...,a_1)\\
&-\sum_{l=0}^{j-2}(-1)^{\ddagger_l}\mathscr{F}^n(a_n,...,e,a_j,...
\end{aligned}
$$

$$
\begin{aligned}
&..., m^2(a_{l+2}, a_{l+1}), a_l, ..., a_r, e, ..., a_1) \\
&-\sum_{l=0}^{n-1}(-1)^{\ddagger_l}\mathscr{F}^{n+1}(a_n, ..., m^1(a_{l+1}), a_l, ... \\
&..., e, a_{j-1}, ..., e, ..., a_1)] \\
=&(-1)^{\ddagger_{j-1}}[m^1(\mathscr{F}^{n+1}(a_n, ..., e, a_{j-1}, ..., a_r, e, ..., a_1)) \\
&-(-1)^{\ddagger_{r-2}}\mathscr{F}^n(a_n, ..., e, a_j, ..., m^2(a_r, e), a_{r-2}, ..., a_1) \\
&-(-1)^{\ddagger_{r-3}}\mathscr{F}^n(a_n, ..., e, a_j, ..., a_r, m^2(e, a_{r-2}), ..., a_1) \\
&-\sum_{l=0}^{n-1}(-1)^{\ddagger_l}\mathscr{F}^{n+1}(a_n, ..., m^1(a_{l+1}), a_l, ... \\
&..., e, a_{j-1}, ..., e, ..., a_1)] \\
=&(-1)^{\ddagger_{j-1}+1}[(-1)^{\ddagger_{r-2}}\mathscr{F}^n(a_n, ..., e, a_j, ..., a_r, a_{r-2}, ..., a_1) \\
&+(-1)^{\ddagger_{r-2}+1}\mathscr{F}^n(a_n, ..., e, a_j, ..., a_r, a_{r-2}, ..., a_1) \\
=&0.
\end{aligned}
$$

Now we suppose that $r = 2$, we have:

$$
\begin{aligned}
\mathscr{F}^n_{(1)}(a_n, ..., a_j, ..., e) :=&\mathscr{F}^n(a_n, ..., a_j, ..., e) \\
&-\mu^1(\Xi_{(1)})^n(a_n, ..., a_j, ..., e) \\
=&\mathscr{F}^n(a_n, ..., m^2(e, a_{j-1}), ..., e) \\
&+-[(-1)^{\ddagger_j}m^1(\mathscr{F}^{n+1}(a_n, ..., e, a_{j-1}, ..., a_2, e)) \\
&+(-1)^{\ddagger_{j-1}}m^2(\mathscr{F}^1(a_n), \mathscr{F}^n(a_{n-1}, ..., e, a_{j-1}, ..., a_2, e)) \\
&+(-1)^{\ddagger_{j-1}}m^2(\mathscr{F}^n(a_n, ..., e, a_j, ..., e, ..., a_2), \mathscr{F}^1(e)) \\
&-\sum_{l=0}^{n-2}(-1)^{\ddagger_l}\Xi^{n-1}_{(1)}(a_n, ..., m^2(a_{l+2}, a_{l+1}), a_l, ..., a_j, ..., e) \\
&-\sum_{l=0}^{n-1}(-1)^{\ddagger_l}\mathscr{F}^{n+1}(a_n, ..., m^1(a_{l+1}), a_l, ..., e, a_{j-1}, ..., e)] \\
=&(-1)^{\ddagger_j-1}[m^2(\mathscr{F}^n(a_n, ..., e, a_j, ..., a_2), \mathscr{F}^1(e)) \\
&-\sum_{l=j}^{n-2}(-1)^{\ddagger_l}\mathscr{F}^n(a_n, ..., m^2(a_{l+2}, a_{l+1}), a_l, ..., e, a_{j-1}, ..., e) \\
&-\sum_{l=0}^{j-2}(-1)^{\ddagger_l}\mathscr{F}^n(a_n, ..., e, a_j, ..., m^2(a_{l+2}, a_{l+1}), a_l, ..., e) \\
=&(-1)^{\ddagger_{j-1}}[m^2(\mathscr{F}^n(a_n, ..., e, a_j, ..., a_2), \mathscr{F}^1(e)) \\
&-\sum_{l=0}^{j-2}(-1)^{\ddagger_l}\mathscr{F}^n(a_n, ..., e, a_j, ..., m^2(a_{l+2}, a_{l+1}), a_l, ..., e) \\
=&(-1)^{\ddagger_{j-1}}[\mathscr{F}^n(a_n, ..., e, a_j, ..., a_2))
\end{aligned}
$$

$$-\sum_{l=0}^{j-2}(-1)^{\ddagger_l}\mathscr{F}^n(a_n,...,e,a_j,...,m^2(a_{l+2},a_{l+1}),a_l,...,e)$$
$$=(-1)^{\ddagger_{j-1}}[\mathscr{F}^n(a_n,...,e,a_j,...,a_2))-\mathscr{F}^n(a_n,...,e,a_j,...,a_2)]$$
$$=0.$$

Theorem 2.14 proves that $\mathscr{F}_{(1)}$ is an A$_\infty$-functor.
By repeating this process, we get a sequence of A$_\infty$-functors $\mathscr{F}_{(1)}$,...,$\mathscr{F}_{(n)}$,... homotopically equivalent to $\mathscr{F}$. By definition we have that $\mathscr{F}_{(k)}$ are increasingly close to a strictly unital functor. As in the previous proof, taking the transfinite composition $\mathscr{F}_{(k)}\cdot ... \cdot \mathscr{F}_{(1)}$ with $k\to\infty$, we get the desired strictly unital A$_\infty$-functor $\mathscr{F}'$. Moreover, if we define Ξ' as $\Xi_{(k)}$ with $k\to\infty$, we obtain a strictly unital homotopy from $\mathscr{F}$ to $\mathscr{F}'$. □

Proof of Lemma 3.8. Let n and k be two positive integers such that $k<n$. We suppose that $H^s(a_s,...,e,...,a_1)=0$ for every $s<n$, and

$$H^n(a_n,...,a_j,...,a_r,e,a_{r-1},...,a_1)=0,$$

for every r such that $1\leq r\leq j$. Moreover we suppose that

$$H^n(a_n,...,a_j,e,a_{j-1},...,a_1)\neq 0.$$

Let $\Upsilon_{(1)}$ be a prenatural transformation of degree $\deg(H)+1$ defined as:

$\Upsilon^m_{(1)}=0$, if $m\neq n-1$,

$\Upsilon^{n-1}_{(1)}(a_{n-1},...,a_1)=(-1)^{\deg(H)+\ddagger_{j-1}}\mathscr{F}^n(a_{n-1},...,e,a_{j-1},...,a_1)$,

$\Upsilon^n_{(1)}(a_n,...,a_1)=(-1)^{\deg(H)+\ddagger_j}\mathscr{F}^n(a_n,...,e,a_{j-1},...,a_1)$.

We define $H_{(1)}:\mathscr{F}\to\mathscr{F}'$ as

$$H^n_{(1)}(a_n,...,a_1):=H^n(a_n,...,a_1)-\mu^1(\Upsilon_{(1)})^n(a_n,...,a_1)$$

By definition, we have that $H^t_{(1)}=H^t$ for every $t<n$.
Now we prove that $H^n_{(1)}(a_n,...,e,a_{j-1},...,a_1)=0$.

$$\begin{aligned}H^n_{(1)}(a_n,...,e,a_{j-1},...,a_1):=&\ H^n(a_n,...,e,a_{j-1},...,a_1)\\&-\mu^1(\Upsilon_{(1)})^n(a_n,...,e,a_{j-1},...,a_1)\\&=\mu^1(H)^n(a_n,...,e,a_{j-1},...,a_1)\\&=(\mathscr{F}'^n-\mathscr{F}^n)(a_n,...,e,a_{j-1},...,a_1)\\&=0.\end{aligned}$$

Moreover we prove that $H^n_{(1)}(a_n,...,a_j,...,a_r,e,...,a_1)=0$, if $2\leq r\leq j$:

$$H^n_{(1)}(a_n,...,a_j,...,e,...,a_1):=H^n(a_n,...,a_j,...,e...,a_1)$$

$$
\begin{aligned}
&-\mu^1(\Upsilon_{(1)})^n(a_n, ..., a_j, ..., e..., a_1)\\
&=\mu^1(H)^n(a_n, ..., a_j, ..., a_r, e, ..., a_1)\\
&=(\mathscr{F}'^n - \mathscr{F}^n)(a_n, ..., a_j, ..., a_r, e, ..., a_1)\\
&=0.
\end{aligned}
$$

By repeating this process we get a sequence of prenatural transformations, once again, taking the transfinite composition $H_{(k)} \cdot ... \cdot H_{(1)}$ with $k \to \infty$, we get the desired strictly unital homotopy H'. □

3.3 Equivalences between Ho(A$_\infty$-cat) and Ho(dgcat) and between Hmo(A$_\infty$-cat) and Hmo(dgcat)

In this section we prove that the homotopy category of unital A$_\infty$-categories and the homotopy category of dg-categories are equivalent. Moreover we prove that the localisation of the category of unital A$_\infty$-categories and the one of dg-categories, over Morita equivalences, are equivalent. In order to construct these equivalences we begin with some theorems and propositions that we will prove in the next two subsections.

Through this section i denotes the inclusion dgcat $\hookrightarrow$ A$_\infty$-cat.

Proposition 3.10. *Given a quasi-equivalence $\mathscr{F} : \mathscr{A} \to \mathscr{B}$. Then* U($\mathscr{F}$) *is a quasi-equivalence.*

Theorem 3.11. *There exists a natural transformation α between* $\mathrm{Id}_{\mathrm{A}_\infty\text{-cat}}$ *and* iU *such that, for any an A$_\infty$-category $\mathscr{A}$, $\alpha_{\mathscr{A}}$ is a quasi-equivalence.*

Theorem 3.12. *There exists a natural transformation α between* Ui *and* $\mathrm{Id}_{\mathrm{dgcat}}$. *Moreover, for any $\mathscr{C}$ a dg-category, $\beta_{\mathscr{C}}$ is a dg-functor such that $\beta_{\mathscr{C}}\alpha_{\mathscr{C}} = \mathrm{Id}_{\mathscr{C}}$.*

Theorem 3.13. *The functor* U *is left adjoint to i.*

Now we can prove the main results of this section.

Theorem 3.14. *The natural transformations α and β give rise to an equivalence between the categories* Ho(A$_\infty$-cat) *and* Ho(dgcat).

Proof. First of all we note that, using the universal property of localisation and Proposition 3.10, we can define the functor $\overline{\mathrm{U}}$ by the following diagram

$$
\begin{array}{ccc}
\mathrm{A}_\infty\text{-cat} & \longrightarrow & \mathrm{Ho}(\mathrm{A}_\infty\text{-cat}) \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle \exists! \overline{U}} \\
\mathrm{dgcat} & \longrightarrow & \mathrm{Ho}(\mathrm{dgcat})
\end{array}
$$

In same vein, by Example 1.2.3, we can define $\bar{i}$ from Ho(dgcat) to Ho(A$_\infty$-cat). Now, by Theorem 3.11 and Theorem 3.12 we have that, for every dg-category $\mathscr{C}$, $\beta_{\mathscr{C}}$ is a quasi-equivalence. Then, as before, α and β induce two natural isomorphisms on the homotopy categories, in formula

$$\overline{\alpha} : \mathrm{Id}_{\mathrm{Ho}(A_\infty\text{-cat})} \to \overline{i\mathrm{U}},$$

and

$$\overline{\beta} : \overline{\mathrm{U}i} \to \mathrm{Id}_{\mathrm{Ho}(\mathrm{dgcat})}.$$

and we are done. □

Theorem 3.15. *The category* $\mathrm{Hmo}(A_\infty\text{-cat})$ *is equivalent to* $\mathrm{Hmo}(\mathrm{dgcat})$.

Proof. The first step is to prove that, if $\mathscr{F} : \mathscr{A} \to \mathscr{B}$ is a Morita equivalence, then $\mathrm{U}(\mathscr{F})$ is a Morita equivalence. By definition, we have that $\mathrm{pretr}(\mathscr{F})^{ic} : \mathrm{pretr}(\mathscr{A})^{ic} \to \mathrm{pretr}(\mathscr{B})^{ic}$ is a quasi-equivalence, then by the following diagram we deduce that $\mathrm{U}(\mathscr{F})$ is a Morita equivalence

$$\begin{array}{ccc} \mathrm{pretr}_{A_\infty}(\mathscr{A})^{ic} & \xrightarrow{\sim} & \mathrm{pretr}_{A_\infty}(\mathrm{U}(\mathscr{A}))^{ic} \\ \downarrow & & \downarrow \\ \mathrm{pretr}_{A_\infty}(\mathscr{B})^{ic} & \xrightarrow{\sim} & \mathrm{pretr}_{A_\infty}(\mathrm{U}(\mathscr{B}))^{ic}. \end{array}$$

Now, as in the previous theorem, we note that, using the universal property of localisation and Proposition 3.10, we can define a new functor $\overline{U}$ by the following diagram

$$\begin{array}{ccc} A_\infty\text{-cat} & \longrightarrow & \mathrm{Hmo}(A_\infty\text{-cat}) \\ {\scriptstyle \mathrm{U}}\downarrow & & \downarrow{\scriptstyle \exists ! \overline{\mathrm{U}}} \\ \mathrm{dgcat} & \longrightarrow & \mathrm{Hmo}(\mathrm{dgcat}) \end{array}$$

In same vein, by Remark 1.11, we can define $\overline{i}$ from $\mathrm{Hmo}(\mathrm{dgcat})$ to $\mathrm{Hmo}(A_\infty\text{-cat})$. Then we note that the natural transformations α and β provide two natural isomorphisms between the localisations over Morita equivalences, in formula

$$\overline{\alpha} : \mathrm{Id}_{\mathrm{Hmo}(A_\infty\text{-cat})} \to \overline{i\mathrm{U}},$$

and

$$\overline{\beta} : \overline{\mathrm{U}i} \to \mathrm{Id}_{\mathrm{Hmo}(\mathrm{dgcat})}.$$

and we are done. □

Proofs of Theorem 3.10 and 3.11

We define an A_∞-functor $\alpha_{\mathscr{A}} : \mathscr{A} \to \mathrm{U}(\mathscr{A})$ in the following way:

- $\alpha^0_{\mathscr{A}}(x) = x$, for every $x \in \mathrm{Obj}(\mathrm{U}(\mathscr{A}))$.
- For every integer n, we have:

$$\begin{aligned} \alpha^n_{\mathscr{A}} : \mathscr{A}(x_{n-1}, y) \otimes ... \otimes \mathscr{A}(x, x_1) &\to \mathrm{U}(\mathscr{A})(x, y)[1-n] \\ (f_n, ..., f_1) &\mapsto (f_n[1] \otimes ... \otimes f_1[1])[-1]. \end{aligned}$$

Lemma 3.16. *$\alpha_{\mathscr{A}}$ is a strictly unital A_∞-functor.*

Proof. It is sufficient to prove the following:

$$d_{\mathrm{U}(\mathscr{A})}((a_d[1]\otimes ...\otimes a_1[1])[-1])+\sum_{i=1}^{d-1}(a_d[1]\otimes ...\otimes a_{i+1}[1])[-1]\otimes(a_i[1]\otimes ...\otimes a_1[1])[-1]=$$

$$=\sum_{m,n}(-1)^{\dagger_n}(a_d[1]\otimes ...\otimes a_{n+m+1}[1]\otimes(m^m(a_{n+m},...,a_{n+1})[1])\otimes a_n[1]\otimes ...\otimes a_1[1])[-1].$$

By an easy calculation, we have:

$$\begin{aligned} d((a_d[1]\otimes ...\otimes a_1[1])[-1]) = &\sum_{j=1}^{n_i-1}(a_d[1]\otimes ...\otimes a_{j+1}[1])[-1]\bigotimes(a_j[1]\otimes ...\otimes a_1[1])[-1]+\\ &+\sum_{j=1}^{n}\sum_{k=1}^{d-j}(-1)^{\deg(a_1)+...+\deg(a_j)}(a_d[1]\otimes ...\otimes a_{k+j+1}[1]\otimes\\ &\otimes m^k_{\mathscr{A}}(a_{k+j},...,a_{j+1})[1]\otimes a_j[1]\otimes ...\otimes a_1[1])[-1]. \end{aligned}$$

□

Lemma 3.17. *α gives rise to a natural transformation between* $\mathrm{Id}_{\mathrm{A}_\infty\text{-cat}}$ *to* $i\mathrm{U}$.

Proof. Given an A$_\infty$-functor $\mathscr{F}$ from $\mathscr{A}$ to $\mathscr{B}$, we calculate:

$$\begin{aligned} (\alpha_{\mathscr{B}}\mathscr{F})^n(f_n,...,f_1) &:= \sum_{r=1}^{n}\sum_{i_1,...,i_r}\alpha^r_{\mathscr{B}}(\mathscr{F}^{i_r}(f_n,...,f_{n-i_r+1}),...,\mathscr{F}^{i_1}(f_{i_1},...,f_1))\\ &=\sum_{r=1}^{n}\sum_{i_1,...,i_r}(\mathscr{F}^{i_r}(f_n,...,f_{n-i_r+1})[1]\otimes ...\otimes\mathscr{F}^{i_1}(f_{i_1}...,f_1)[1])[-1]\\ &=(\mathrm{Bar}(\mathscr{F})(f_n[1]\otimes ...\otimes f_1[1])[-1])\\ &=(\mathrm{U}(\mathscr{F})((f_n[1]\otimes ...\otimes f_1[1])[-1])\\ &=(\mathrm{U}(\mathscr{F})\alpha_{\mathscr{A}})^n(f_n,...,f_1)\\ &=\mathrm{U}(\mathscr{F})(\alpha^n_{\mathscr{A}}(f_n,...,f_1)). \end{aligned}$$

Where $i_1+...+i_r=n$. □

Lemma 3.18. *We fix two objects x and y in $\mathscr{A}$. The functor:*

$$\begin{aligned} \alpha^1_{\mathscr{A}}:\mathscr{A}(x,y)&\to\mathrm{U}(\mathscr{A})(x,y)\\ f&\mapsto(f[1])[-1]. \end{aligned}$$

is a quasi-isomorphism.

Proof. First of all, we provide a filtration for $\mathscr{A}(x,y)$, setting

$$F_n\mathscr{A}(x,y):=\begin{cases}\mathbb{K}e_x, \text{ if } x=y \text{ and } n=0,\\ 0, \text{ if } x\neq y \text{ and } n=0,\\ \mathscr{A}(x,y), \text{ if } n>0.\end{cases}$$

Now we provide a filtration for $\mathrm{U}(\mathscr{A})(x,y)$. We set,

$$F_0\mathrm{U}(\mathscr{A})(x,y) := \begin{cases} \mathbb{K}1_x, & \text{if } x = y, \\ 0, & \text{if } x \neq y, \end{cases}$$

and $F_n\mathrm{U}(\mathscr{A})(x,y)$ is generated by the elements of the form:

$$f_1[1] \otimes ... \otimes f_{a_1}[1])[-1] \otimes ... \otimes (f_{a_m}[1] \otimes ... \otimes f_m[1])[-1],$$

with $m \leq n$.
We consider the case $x = y$, the proof in the case $x \neq y$ is similar.
The graded chains, associated to the previous filtrations, are given by:

$$\begin{aligned} \mathrm{gr}_F\mathscr{A}(x,x) &= \mathbb{K}e_x \oplus \frac{\mathscr{A}(x,x)}{\mathbb{K}e_x} \oplus \frac{\mathscr{A}(x,x)}{\mathscr{A}(x,x)} \oplus \frac{\mathscr{A}(x,x)}{\mathscr{A}(x,x)} \oplus ... \\ &= \mathbb{K}e_x \oplus \mathscr{A}(x,x)/\mathbb{K}e_x, \end{aligned}$$

and

$$\begin{aligned} \mathrm{gr}_F\mathrm{U}(\mathscr{A})(x,x) =& \mathrm{gr}^0_F(\mathrm{U}(\mathscr{A})(x,x)) \oplus \mathrm{gr}^1_F(\mathrm{U}(\mathscr{A})(x,x)) \oplus \mathrm{gr}^2_F(\mathrm{U}(\mathscr{A})(x,x)) \oplus ... \\ =& \mathrm{U}(\mathscr{A}_-)(x,x) = \mathbb{K}1_x \oplus \Omega B(\mathscr{A}_-)(x,x). \end{aligned}$$

We have that the A_∞-functor $\alpha^1_{\mathscr{A}}$ induces a functor $\mathrm{gr}^n_F\mathscr{A}(x,x) \to \mathrm{gr}^n_F\mathrm{U}(\mathscr{A})(x,x)$, for every positive integer n.
In order to prove that $\alpha^1_{\mathscr{A}}$ is a quasi-isomorphism, we have to prove that $\alpha^1_{\mathscr{A}}$ induces a quasi-isomorphism between $\mathrm{gr}^n_F\mathscr{A}(x,x) \to \mathrm{gr}^n_F\mathrm{U}(\mathscr{A})(x,x)$, for every positive integer n.
If $n = 0$ and $n = 1$, it is clearly a quasi-isomorphism.
So we have to prove that, if $n > 1$, every cycle is a boundary in $\mathrm{gr}^n\mathrm{U}(\mathscr{A})(x,x)$.
First of all, we note that the cycles in $\mathrm{gr}^n\mathrm{U}(\mathscr{A})(x,x)$ are of the form $f_n[1][-1] \otimes ... \otimes f_1[1][-1]$ with

$$\sum_{j=1}^{n}(-1)^{\dagger_{j-1}}(f_n[1])[-1] \otimes ... \otimes (df_j[1])[-1] \otimes ... \otimes (f_1[1])[-1] = 0.$$

It means that

$$\begin{aligned} d(\sum_{j=1}^{n}(f_n[1])[-1] \otimes ... \otimes (f_{j+1}[1] \otimes f_j[1])[-1] \otimes ... \otimes (f_1[1])[-1]) = \\ = (n-1)f_n[1][-1] \otimes ... \otimes f_1[1][-1] \end{aligned}$$

and we are done. □

Proof of Proposition 3.10. By Lemma 3.17 we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\alpha_{\mathscr{A}}} & \mathrm{U}(\mathscr{A}) \\ {\scriptstyle \mathscr{F}}\downarrow & & \downarrow{\scriptstyle \mathrm{U}(\mathscr{F})} \\ \mathscr{B} & \xrightarrow{\alpha_{\mathscr{A}}} & \mathrm{U}(\mathscr{B}). \end{array}$$

Using the previous lemma we deduce that $\mathrm{U}(\mathscr{F})$ is a quasi-equivalence. □

Proof of Theorem 3.12

Let $\mathscr{C}$ be a dg-category.

We define the following:

$$\beta_{\mathscr{C}} : \mathrm{U}(\mathscr{C})(x,y) \to \mathscr{C}(x,y)$$

$$\beta_{\mathscr{C}}((f^m_{n_m}[1]\otimes ... \otimes f^m_1[1])[-1]\otimes ... \otimes (f^1_{n_1}[1]\otimes ... \otimes f^1_1[1])[-1]) := \begin{cases} f^m_1 ... f^1_1, \text{ if } n_j = 1 \\ 0, \text{ otherwise,} \end{cases}$$

Lemma 3.19. *$\beta_{\mathscr{C}}$ is a dg-functor.*

Proof. We show that $\beta_{\mathscr{C}}$ is compatible with composition:

$$\begin{aligned} &\beta_{\mathscr{C}}(((f_n[1])[-1]\otimes ... \otimes (f_{j+1}[1])[-1])\otimes (f_j[1])[-1]\otimes ... \otimes (f_1[1])[-1]) = \\ &=(f_n...f_{j+1})(f_j...f_1) \\ &=\beta_{\mathscr{C}}((f_n[1])[-1]\otimes ... \otimes (f_{j+1}[1])[-1])\beta_{\mathscr{C}}((f_j[1])[-1]\otimes ... \otimes (f_1[1])[-1]). \end{aligned}$$

Now we prove that $\beta_{\mathscr{C}}$ commutes with differential:

$$\begin{aligned} &d(\beta_{\mathscr{C}}((f_n[1])[-1]\otimes ... \otimes (f_1[1])[-1])) = d(f_n...f_1) = \sum_{j=1}^{n-1}(-1)^{\dagger_j} f_n...(df_{j+1})f_j...f_1 \\ &= \beta_{\mathscr{C}}(\sum_{j=1}^{n-1}(-1)^{\dagger_j}(f_n[1])[-1]\otimes ... \otimes (df_{j+1}[1])[-1]\otimes (f_j[1])[-1]\otimes ... \otimes (f_1[1])[-1]) \\ &= \beta_{\mathscr{C}}(d(f_n[1])[-1]\otimes ... \otimes (f_1[1])[-1]))) \end{aligned}$$

□

Lemma 3.20. *$\beta_{\mathscr{C}}$ gives rise to a natural transformation between* Ui *and* $\mathrm{Id}_{\mathrm{dgcat}}$*, where i denotes the inclusion* dgcat $\hookrightarrow$ A$_\infty$-cat*.*

Proof. Trivial by construction. □

Lemma 3.21. *Given a dg-category $\mathscr{C}$, we have $(\beta_{\mathscr{C}}\alpha_{\mathscr{C}}) = \mathrm{Id}_{\mathscr{C}}$. It implies that $\beta_{\mathscr{C}}$ is a quasi-equivalence.*

Proof. If $n = 1$, then $(\beta_{\mathscr{C}}\alpha_{\mathscr{C}})^1(f) = \beta_{\mathscr{C}}((f[1])[-1]) = f$.
If $n > 1$, then $(\beta_{\mathscr{C}}\alpha_{\mathscr{C}})^n(f_n,...,f_1) = \beta^1_{\mathscr{C}}(\alpha^n_{\mathscr{C}}(f_n,...,f_1)) = \beta^1_{\mathscr{C}}((f_n[1]\otimes...\otimes f_1)[-1]) = 0$. □

Proof of Proposition 3.13. Let $\mathscr{A}$ be an A$_\infty$-category, and let $\mathscr{B}$ be a dg-category. We define the isomorphism ψ

$$\begin{aligned} \psi : \mathrm{Hom}_{\mathrm{dgcat}}(\mathrm{U}(\mathscr{A}),\mathscr{B}) \to &\mathrm{Hom}_{\mathrm{A}_\infty\text{-cat}}(\mathscr{A},\mathscr{B}) \\ F \mapsto &F\alpha_{\mathscr{A}}, \end{aligned}$$

with the inverse

$$\begin{aligned}\psi^{-1} : \mathrm{Hom}_{A_\infty\text{-cat}}(\mathscr{A},\mathscr{B}) \to & \mathrm{Hom}_{\mathrm{dgcat}}(\mathrm{U}(\mathscr{A}),\mathscr{B}) \\ \mathscr{F} \mapsto & \beta_{\mathscr{A}} \mathrm{U}(\mathscr{F}).\end{aligned}$$

□

3.4 Equivalence between Ho(A_∞-cat) and A_∞-cat$/_\sim$

First of all, given a dg-category $\mathscr{C}$, we define the dg-category $\underline{\mathrm{Mor}}(\mathscr{C})$ in the following way:

- the objects are the triples (x, y, f), where x, y are objects in $\mathscr{C}$ and f is an arrow from x to y.
- The morphisms $\mathrm{Hom}_{\underline{\mathrm{Mor}}(\mathscr{C})}((x,y,f),(x',y',f'))$ are given by the lower triangular matrices of the form:
$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix},$$
where $a : x \to x'$, $b : x \to y'$ and $c : y \to y'$,
- the differentials are given by
$$\begin{pmatrix} d(a) & 0 \\ d(b) + (-1)^n(f'a - cf) & d(c) \end{pmatrix},$$

Moreover we have two dg-functors, S and T from $\underline{\mathrm{Mor}}(\mathscr{C})$ to $\mathscr{C}$, defined respectively, on the objects, as $S : (x, y, f) \mapsto x$ and $T : (x, y, f) \mapsto y$, and on morphisms as

$$S : \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mapsto a,$$

and

$$T : \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mapsto c,$$

We recall also that, given two cohomologically unital A_∞-functors $\mathscr{F}$, $\mathscr{G}$ between two dg-categories, $\mathscr{C}'$ and $\mathscr{C}'$, and a prenatural transformation H, between $\mathscr{F}$ and $\mathscr{G}$ such that $\mu^1(H) = 0$, we can define an A_∞-functor:

$$\psi_H : \mathscr{C}' \to \underline{\mathrm{Mor}}(\mathscr{C}),$$

setting, for every integer d:

$$\psi_H^d(f_d, ..., f_1) := (\mathscr{F}^d(f_d, ..., f_1), \mathscr{G}^d(f_d, ..., f_1), H^d(f_d, ..., f_1)).$$

The fact that ψ_H is an A_∞-functor follows by Lemma 4.8. of [20]. We note that if $\mathscr{F}$ and $\mathscr{G}$ are cohomological unital (resp. unital) then, also ψ_H is cohomological unital (resp.

unital).
Moreover one can see immediately that $S\psi = \mathscr{F}$ and $T\psi = \mathscr{G}$.

Given two cohomological unital (resp. unital) A$_\infty$-categories $\mathscr{A}$ and $\mathscr{B}$.
Let $\mathscr{F}_0$ and $\mathscr{F}_1$ be two cohomological unital (resp. unital) A$_\infty$-functors from $\mathscr{A}$ and $\mathscr{B}$.

Definition 3.4.1. We say that $\mathscr{F}_0 \sim \mathscr{F}_1$ (resp. $\mathscr{F}_0 \sim_u \mathscr{F}_1$) if and only if $\mathscr{F}_0$ and $\mathscr{F}_1$ are isomorphic, as objects, in Ho(A$_\infty$-Fun$^{cu}(\mathscr{A},\mathscr{B})$) (resp. Ho(A$_\infty$-Fun$(\mathscr{A},\mathscr{B})$)).

Here A$_\infty$-Fun$^{cu}(\mathscr{A},\mathscr{B})$ denotes the category whose objects are cohomological unital A$_\infty$-functors from $\mathscr{A}$ to $\mathscr{B}$, and whose morphisms are prenatural transformations. Similarly, from now on A$_\infty$-Fun$(\mathscr{A},\mathscr{B})$ will denote the category whose objects are unital A$_\infty$-functors from $\mathscr{A}$ to $\mathscr{B}$ (cf. Chapter 1, §1.3).

We point out that $\sim$ and $\sim_u$ are equivalence relations. Thus we define two categories A$_\infty$-cat$^{cu}/_\sim$ and A$_\infty$-cat$/_{\sim_u}$.

Theorem 3.22. *The category* A$_\infty$-cat$^{cu}/_\sim$ *is equivalent to* Ho(A$_\infty$-cat^{cu}).

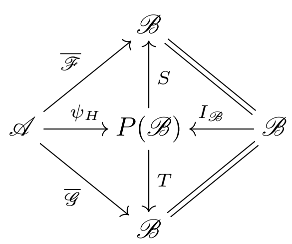
Proof. Let $\mathscr{F}_0$ and $\mathscr{F}_1$ be two functors from two cohomological unital A$_\infty$-categories $\mathscr{A}$ and $\mathscr{B}$, such that $\mathscr{F}_0 \sim \mathscr{F}_1$.
By [56, Theorem 2.9.], we can replace $\mathscr{A}$ and $\mathscr{B}$ by two dg-categories $\overline{\mathscr{A}}$ and $\overline{\mathscr{B}}$. Moreover we have the A$_\infty$-functors

$$\mathscr{Q} : \overline{\mathscr{A}} \to \mathscr{A} \qquad \text{and} \qquad \mathscr{P} : \overline{\mathscr{B}} \to \mathscr{B}$$

with

$$\mathscr{Q}^{-1} : \mathscr{A} \to \overline{\mathscr{A}} \qquad \text{and} \qquad \mathscr{P}^{-1} : \mathscr{B} \to \overline{\mathscr{B}}$$

such that $\mathscr{Q}^{-1}\mathscr{Q} \sim \mathrm{Id}$, $\mathscr{Q}\mathscr{Q}^{-1} \sim \mathrm{Id}$, and $\mathscr{P}^{-1}\mathscr{P} \sim \mathrm{Id}$, $\mathscr{P}\mathscr{P}^{-1} \sim \mathrm{Id}$.
So we get a functor $\overline{(\ )}$ from A$_\infty$-cat$^{cu}/_\sim$ to dgcat$/_\sim$, by setting $\mathscr{F} \mapsto \mathscr{P}^{-1}\mathscr{F}\mathscr{Q}$.
Then, by the fact that $\overline{\mathscr{F}} \sim \overline{\mathscr{G}}$, we have that there exist $H : \mathscr{F} \to \mathscr{G}$ and $T : \mathscr{G} \to \mathscr{F}$ such that $\mu^1(H) = \mu^1(T) = 0$ and $HT = \mathrm{Id}$ and $TH = \mathrm{Id}$ in Ho(A$_\infty$-Fun$^{cu}(\mathscr{A},\mathscr{B})$).
It means that the functor ψ_H factors through the full dg-subcategory $P(\mathscr{B})$ of $\underline{\mathrm{Mor}}(\mathscr{B})$, whose objects are the triples (x, y, f), such that f is invertible in Ho($\mathscr{B}$).
So we have the following commutative diagram



where $I_{\mathscr{B}}$ denotes the quasi-equivalence defined on objects by $x \to (x, x, e_x)$, and on morphisms by $f \to (f, f, 0)$. It is sufficient to prove that $[\overline{\mathscr{F}}]$=$[\overline{\mathscr{G}}]$ in Ho(A$_\infty$-cat^{cu}). We

get $\phi_1 : A_\infty\text{-cat}^{cu}/_\sim \to \mathrm{Ho}(A_\infty\text{-cat}^{cu})$.
Conversely, given a quasi-equivalence $\mathscr{F}$ we have, by again by Theorem 2.9. of [56], a quasi-equivalence $\overline{\mathscr{I}}$ such that $\overline{\mathscr{I}}\mathscr{I} \sim \mathrm{id}$ and $\mathscr{I}\overline{\mathscr{I}} \sim \mathrm{Id}$. It means that $\mathscr{I}$ is invertible in $A_\infty\text{-cat}^{cu}/_\sim$ and the functor $A_\infty\text{-cat}^{cu} \to A_\infty\text{-cat}^{cu}/_\sim$, given by the localisation, factor through $\phi_2 : \mathrm{Ho}(A_\infty\text{-cat}^{cu}) \to A_\infty\text{-cat}^{cu}/_\sim$.
A direct computation shows that $\phi_1\phi_2 \simeq \mathrm{Id}$ and $\phi_2\phi_1 \simeq \mathrm{Id}$. □

Theorem 3.23. *The category* $A_\infty\text{-cat}/_\sim$ *is equivalent to* $\mathrm{Ho}(A_\infty\text{-cat})$.

Proof. Using the same procedure of the previous Theorem, we get a functor from $A_\infty\text{-cat}/_\sim$ to $\mathrm{Ho}(A_\infty\text{-cat})$.
Conversely, by Theorem 9.2.0.4. of [39] we have an equivalence between $\mathrm{Ho}(A_\infty\text{-cat})$ and $A_\infty\text{-cat}/_\approx$, where $\mathscr{F} \approx \mathscr{G}$ if and only if they are A_∞-equivalent (in sense of Definition 9.2.0.1. of [39]). Moreover, by Lemma 9.1.0.4. of [39] we have that $\mathscr{F}$ and $\mathscr{G}$ are A_∞-equivalent if and only if they are isomorphic in $\mathrm{Ho}(A_\infty\text{-Fun}(\mathscr{A}, \mathscr{B}))$. □

Chapter 4

∞-categories

As we said in the introduction, we can enhance a triangulated category as a dg-category, an A_∞-category or a stable ∞-category. We point out that, due to Theorem 3.14, we have an equivalence, up to quasi-equivaleces, between the category of dg-categories and A_∞-categories. In this chapter we complete the comparison between the above mentioned categories. In particular we prove that the ∞-stable categories N(Hmo(dgcat)) and N(Hmo(A_∞-cat)) are equivalent. Moreover, given a pretriangulated A_∞-category $\mathscr{A}$ we prove that $N_{A_\infty}(\mathscr{A})$ is a stable ∞-category.

At the beginning, we introduce the language of ∞-categories, then using a result in [11] we prove that the categorical nerves of the category of A_∞-categories and of dg-categories, localised over Morita equivalences, are both ∞-equivalent to the category of stable idempotent complete ∞-categories. Moreover, we prove that the A_∞-nerve of two quasi-equivalent A_∞-categories are weak equivalent in the Joyal model structure, a consequence of this fact is that the A_∞-nerve of a pretriangulated A_∞-category is a stable ∞-category.

4.1 Brief background on ∞-categories

Definition 4.1.1 (Minimal $\mathbb{K}$-linear category)**.** Let n be a nonnegative integer. We define the *minimal* $\mathbb{K}$*-linear category* $[n]_\mathbb{K}$ to be the category such that the objects are the positive integers $\{0, 1, 2, ..., n\}$ and the morphisms are defined by

$$\mathrm{Hom}_{[n]_\mathbb{K}}(i,k) = \begin{cases} 0_\mathbb{K}, & \text{if } i > k \\ \langle j_{ik} \rangle_\mathbb{K}, & \text{if } i < k \\ \langle 1_\mathbb{K} \rangle_\mathbb{K}, & \text{if } i = k. \end{cases}$$

Here $0_\mathbb{K}$ is the zero vector space and $\langle j_{ik} \rangle_\mathbb{K}$ is the $\mathbb{K}$-vector space generated by the element j_{ik}. The composition is defined as follow, let $i_1 < i_2 < i_3$ be positive integers. Then:

$$\cdot := \mathrm{Hom}_{[n]_\mathbb{K}}(i_2, i_3) \otimes_\mathbb{K} \mathrm{Hom}_{[n]_\mathbb{K}}(i_1, i_2) \to \mathrm{Hom}_{[n]_\mathbb{K}}(i_1, i_3)$$

is such that

$$j_{i_2 i_3} \cdot j_{i_1 i_2} = j_{i_1 i_3},$$

where $j_{i_1 i_3}$ is the unique morphism in $\mathrm{Hom}_{[n]_{\mathbb{K}}}(i_1, i_3)$.

Remark 4.1
The definition above works even without the $\mathbb{K}$-linear enrichment.

Definition 4.1.2 (Simplex category)**.** We define the *simplex category* to be the category whose objects are the minimal $\mathbb{K}$-linear categories $[n]$ and whose morphisms are the functors f such that $f(i) \leq i$ and $f(i_1) \leq f(i_2)$ if $i_1 \leq i_2$. We denote the resulting category by Δ.

Definition 4.1.3 (Simplicial set)**.** We define a *simplicial set* to be a contravariant functor from the simplex category Δ to the category of sets.

We will denote by sSet the category of simplicial sets whose morphisms are the natural transformations.

Remark 4.2
We can consider, instead of Δ, the category Δ_+ which is the category Δ with the empty ordinal (formally denoted by $[-1]$). We define the *augmented simplicial set* to be a controvariant functor from Δ_+ to Set. We denote the category of augmented simplicial set by sSet_+. Moreover the inclusion functor $\Delta \hookrightarrow \Delta_+$ induces a pair of adjoint functors:

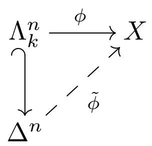
$$t^* : \mathrm{sSet}_+ \rightleftarrows \mathrm{sSet} : t_*$$

Example 4.1.1. Given a positive integer n, the functor Δ^n defined as $\mathrm{Hom}_\Delta(-, [n]) : \Delta^{\mathrm{op}} \to \mathrm{Sets}$ is a simplicial set. Moreover for each $0 \leq i \leq n$ the functor generated by all the maps $d^j : [n-1] \to [n]$ (which are the injective maps not having j in the image), with $i \neq j$, is a subsimplicial set of Δ^n. We call such a simplicial set (n, i)-*horn* and we denote it by Λ^n_i.

Example 4.1.2. Given a positive integer n, we call *the i^{th}-face* $\partial_i \Delta^n$ of Δ^n the simplicial subset generated by $d^i \in \Delta^n_{n-1}$.

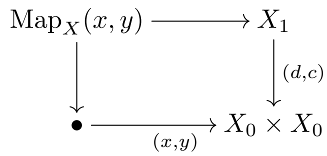
Example 4.1.3. Given a positive integer n, we define the *simplicial n-sphere* $\partial\Delta^n$ to be the simplicial subset of Δ^n given by the union of the faces $\partial_0\Delta^n, ..., \partial_n\Delta^n$.

Definition 4.1.4 (∞-category)**.** We define an ∞-*category* to be a simplicial set X such that, for every positive integer n and every natural transformation $\phi : \Lambda^n_k \to X$, with $0 < k < n$, there exists (at least) one map $\tilde{\phi}$ such that the following diagram:



commutes.

Let X be an ∞-category, the *objects* of X are given by the elements of the set X_0 and the *set of morphisms* from x to y, denoted by $\mathrm{Map}_X(x, y)$, is given by the pullback of the following diagram:

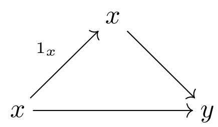


where $d = X(d_1) : X_1 \to X_0$, and $c = X(d_0) : X_1 \to X_0$.
We say that an object x in X is *terminal* if every morphism

$$\phi : \partial\Delta^n \to X$$

with target x, can be extended to a map $\tilde{\phi} : \Delta^n \to X$.

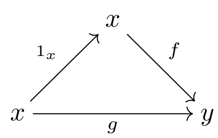
Example 4.1.4. Let X be an ∞-category. Fixing two elements x and $y \in X_0$, we get a simplicial set, denoted by $\mathrm{Hom}^R_X(x, y)$, whose 0-simplices are 1-simplices in X from x to y, whose 1-simplices are 2-simplices of the form:



and whose n-simplices are $(n + 1)$-simplices whose target is y and whose $(n + 1)^{\text{th}}$-face degenerates at x.

Example 4.1.5. Let $\mathscr{C}$ be a category, the simplicial set defined as the set of the compositions of n-arrows of $\mathscr{C}$, for every $n > 0$, and as the set of objects of $\mathscr{C}$, if $n = 0$, is an ∞-category. We call such a simplicial set the *nerve* of $\mathscr{C}$ and we denote it by $\mathrm{N}_{\mathrm{Cat}}(\mathscr{C})$ (or shortly $\mathrm{N}(\mathscr{C})$).

Given an ∞-category X and two morphisms $f, g \in \mathrm{Map}_X(x, y)$ we say that f is *homotopic* to g if there exists a natural transformation $\sigma : \Delta^2 \to X$ of the form:



The homotopy relation is an equivalence relation.

Definition 4.1.5 (Homotopy category)**.** We define the *homotopy category* of an ∞-category X to be the category whose objects are the elements of X_0 and whose morphisms, fixed two objects x and y, are given by the quotient of $\mathrm{Map}_X(x, y)$ by the homotopy relation defined above. We denote such a category by $\mathrm{Ho}(X)$.

Limits (and colimits) in an ∞-category

In this subsection we want to give just a sketch of the definition of limit (resp. colimit) in an ∞-category. We refer to [24, §16] for the details.

We start defining an operation in the category sSet_+ and sSet:

Definition 4.1.6 (Join of augmented simplicial sets)**.** The *join* $X \star Y$ of two augmented simplicial sets X and Y is defined to be the augmented simplicial set given by

$$(X \star Y)_n := \bigsqcup_{i+j=n} X_i \times Y_j$$

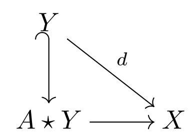
for every positive integer n.

We have that the operation $\star$ on sSet_+ induces a monoidal structure on sSet. Using the previous formula we can define the join $X \star Y$ of two simplicial sets X and Y by the following:

$$(X \star Y)_n := X_n \sqcup Y_n \sqcup \bigsqcup_{i+1+j=n} X_i \times Y_j$$

for every positive integer n.

Now we fix a simplicial set Y, avoiding technical details (you can find a complete exposition in [24, §6]), we can define a functor I_Y sending a simplicial set X in the inclusion $X \hookrightarrow X \star Y$. The functor I_Y has a right adjoint R with target in sSet. We denote by X/d the image of $d : Y \to X$ via the right adjoint R. Moreover, if X is an ∞-category, we have that the simplicial set X/d is an ∞-category. We point out that, for any simplicial set A, there is a bijection between the maps $A \to X/d$ and the maps $A \star Y \to X$ which extends d along the inclusion $Y \hookrightarrow A \star Y$:



Now we are ready to define the limits (and colimits) in an ∞-category X. First of all we define a *diagram* in X to be a morphism of simplicial sets $D \to X$. For example we call $\Delta^1 \times \Delta^1 \to X$ a commutative square in X. We call *projective cone* a map c from $1 \star D$ to X, where 1 denotes the category with one object. The base of the cone is given by the composition ci where i denotes the inclusion $i : D \hookrightarrow 1 \star D$. In particular the set of objects X/d_0 is given by the cones $1 \star D \to X$ whose base is d.

Definition 4.1.7 (Exact cone)**.** We say that a cone $c : 1 \star D \to X$ with base $d : D \to X$ is *exact* if it is a terminal object of the ∞-category X/d.

Definition 4.1.8 (Limit)**.** We define the *limit* l of the diagram $d : D \to X$ to be the image of the unique object of 1 via c.

If the limit of a diagram exists then it is unique up to homotopy. Dually we can define the *colimit* of a diagram the notions of *initial* object and of *coexact* inductive cone.

4.2 Stable ∞-categories

Definition 4.2.1 (Zero object in ∞-category)**.** Let X be an ∞-category, we define the *zero object* 0 to be an object of X that is both initial and final, i.e.

$$\mathrm{Map}_X(c, 0) \simeq \mathrm{Map}_X(0, c) \simeq *$$

for all $c \in X_0$.

Remark 4.3
The zero object (if it exists) is unique up to equivalence.

Definition 4.2.2 (Pointed ∞-category)**.** We define a *pointed ∞-category* to be an ∞-category equipped with a zero object.

Definition 4.2.3 (Fiber (cofiber) sequence)**.** Let X be a pointed ∞-category, we consider the functor of simplicial sets $T : \Delta^1 \times \Delta^1 \to X$ of the form:

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & z \end{array}$$

We call T a triangle in X. If T is a pullback square we call it *fiber sequence* (fiber of g), if T is a pushout square we call it *cofiber sequence* (cofiber of f).

Remark 4.4
It easy to check that a triangle T is the datum of:

- Two morphisms f, $g \in X_1$.
- Two 2-simplices in X_2 of the form:

$$\begin{array}{ccc} x & & \\ \downarrow & \overset{h}{\searrow} & \\ 0 & \longrightarrow & z \end{array} \qquad\qquad \begin{array}{ccc} x & \xrightarrow{f} & y \\ & \underset{h}{\searrow} & \downarrow{\scriptstyle g} \\ & & z \end{array}$$

We will denote the *triangle* T by

$$x \xrightarrow{f} y \xrightarrow{g} z.$$

Definition 4.2.4 (Stable ∞-category)**.** We say that X is a *stable ∞-category* if

(S1) X is an ∞-category equipped with zero object (pointed ∞-category).

(S2) Every morphism has fibers and cofibers.

(S3) Every triangle in X is a fiber sequence if and only if it is a cofiber sequence.

Given a stable ∞-category X, we have an auto-equivalence

$$\Sigma : X \to X$$

called *suspension* functor, with inverse Ω called *loop* functor, obtained via the category of subfunctors of $\mathrm{Fun}(\Delta^1 \times \Delta^1, X)$ generated by the following pullbacks and pushouts in $\Delta^1 \times \Delta^1 \to X$:

$$\begin{array}{ccc} x & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & x_\Sigma \end{array} \qquad\qquad \begin{array}{ccc} x_\Omega & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & x \end{array}$$

where 0 and $0'$ are zero objects in X (cf. Chapter 1 of [43] for a precise definition). If $n > 0$ we will denote by $x[n]$ the Σ functor applied n-times to $x \in X$, if $n < 0$ we will denote by $x[n]$ the Ω functor applied n-times to x.
We have the following fundamental theorem:

Theorem 4.5. *Let X be a stable ∞-category. Then the homotopy category* $\mathrm{Ho}(X)$ *is a triangulated category with Σ the suspension functor as shift functor. Distinguished triangles are given by the following $\Delta^2 \times \Delta^1 \to X$ diagram:*

$$\begin{array}{ccc} x & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ y & \longrightarrow & z \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & w. \end{array}$$

We denote by Cat_∞^{St} the category of *stable ∞-categories* whose objects are the stable ∞-categories and whose morphisms are the functors of ∞-categories.

A functor between ∞-categories "a priori" does not give information about the zero object and the fiber sequences, so in the case of stable ∞-categories we prefer use the following definition of functors.

Definition 4.2.5 (Exact functor)**.** Let $F : X \to X'$ be a functor between stable ∞-categories. We say that F is *exact* if the followings are satisfied:

(E1) $F(0_X) = 0_{X'}$.

(E2) F carries fiber sequences to fiber sequences.

Remark 4.6
If (E1) holds, than F carries triangles to triangles. Moreover F satisfies (E2) if and only if F carries cofiber sequences to cofiber sequences.

Example 4.2.1. The identity functor of a stable ∞-category and the composition of two exact functors are exact functors.

We denote by Cat_∞^{Ex} the *exact stable ∞-category* whose objects are the stable ∞-categories and whose morphisms are the exact functors.

4.3 Nerves

The nerves are useful tools to pass from a category to an ∞-category. In this section we will define the A_∞-nerve, originally defined in [18], which is a generalization of the dg-nerve of Lurie.

The nerve of a category $\mathscr{A}$, is the ∞-category defined, for every positive integer n, as:

$$\mathrm{N}(\mathscr{A})_n \ = \mathrm{Hom}_{\mathrm{Cat}}([n]_{\mathbb{K}}, \mathscr{A}),$$

where Cat denotes the category of (small) categories.
We recall that the nerve of Hmo(dg-cat) is equivalent to the ∞-category of $\mathbb{K}$-linear stable ∞-categories. Then, using Theorem 3.15 and [11, Corollary 5.4.], we have the following:

Theorem 4.7. $\mathrm{N}(\mathrm{Hmo}(\mathrm{A}_\infty\text{-cat}))$ *is equivalent to* $\mathrm{N}(\mathrm{Hmo}(\mathrm{dgcat}))$*, which is equivalent to the ∞-category of $\mathbb{K}$-linear stable ∞-categories.*

A_∞-nerve

Proposition 4.8. *Let n be a positive integer and $\mathscr{C}$ be an A_∞-category (unital). Every map $\{\mathscr{F}^n\} \in \mathrm{Hom}_{\mathrm{A}_\infty\text{-Cat}}([n]_{\mathbb{K}}, \mathscr{C})$ is uniquely determined by:*

1. *$n+1$-objects $\{X_i\}_{0\leq i\leq n}$ of $\mathscr{C}$,*
2. *A set of morphisms f_I for all set of integers $I = \{i_0 < i_1 < ... < i_m < i_{m+1}\}$ where $0 \leq i_0 < i_{m+1} \leq n$ satisfying the following:*

$$\begin{aligned} m^1_{\mathscr{C}}(f_I) = &\sum_{1\leq j\leq m} (-1)^{j-1} f_{I-i_j} + \sum_{1\leq j\leq m} (-1)^{1+(m+1)(j-1)} m^2_{\mathscr{C}}(f_{i_j\ldots i_{m+1}}, f_{i_0\ldots i_j}) \\ &+ \sum_{r>2}\sum_{\ddagger_r} (-1)^{1+\epsilon_r} m^r_{\mathscr{C}}(f_{i_{m+1-s_r}\ldots i_{m+1}}, ..., f_{i_0\ldots i_{s_1}}). \end{aligned} \tag{4.1}$$

where

$$\ddagger_r = \{s_1,..., \ s_r \in \mathbb{N} \mid \sum_{j=1}^r s_j = m+1\}$$

$$\epsilon_r(i_1, ..., i_r) = \sum_{2\leq k\leq r} (1 - i_k + i_{k-1}) i_{k-1}.$$

Proof. Given an A_∞-unital functor $\mathscr{F} = \{\mathscr{F}_m\}_{m\geq 0} : [n]_{\mathbb{K}} \to \mathscr{C}$ the image of the map $\mathscr{F}_0$ is uniquely determined by $n+1$ objects $\{X_i\}_{0\leq i\leq n}$ in $\mathscr{C}$ because $[n]_{\mathbb{K}}$ has exactly

$n+1$ objects. Moreover fixed two integers i_- and $i_+ \in [n]$ such that $i_- < i_+$, for every $0 \leq m \leq n$ we consider the map:

$$\mathscr{F}_m : \mathrm{Hom}_{[n]_{\mathbb{K}}}(i_{m-1}, i_+) \otimes ... \otimes \mathrm{Hom}_{[n]_{\mathbb{K}}}(i_-, i_1) \to \mathrm{Hom}_{\mathscr{C}}(\mathscr{F}X_-, \mathscr{F}X_+)[1-m]$$

the unique non-trivial ones are those such that $i_- < i_1 < i_2 < ... < i_{m-1} < i_m < i_+$. So the image of $\mathscr{F}_m$ is non-zero if and only if we have a set I of $m+1$-elements in $[n]$ such that $I = \{i_- < i_1 < i_2 < ... < i_{m-1} < i_m < i_+\}$. Then $\mathscr{F}_m$ is uniquely determined by the image $f_I = \mathscr{F}_m(j_{i_{m-1}i_+}, ..., j_{i_-i_1})$ where j_{kl} denotes the only one non trivial map in $\mathrm{Hom}_{[n]_{\mathbb{K}}}(i_k, i_l)$, and clearly they satisfy (4.1) because they are the image of the A_∞-functor $\mathscr{F}$. □

Proposition 4.9. *Given a map $\alpha : [m] \to [n]$ in Δ and $\mathscr{C}$ as above, we have an induced map*

$$\mathrm{Hom}_{\mathrm{A}_\infty\text{-}\mathrm{Cat}}(\alpha, \mathscr{C})$$

given by:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{A}_\infty\text{-}\mathrm{Cat}}(\alpha, \mathscr{C}) : \mathrm{Hom}_{\mathrm{A}_\infty\text{-}\mathrm{Cat}}([n]_{\mathbb{K}}, \mathscr{C}) &\to \mathrm{Hom}_{\mathrm{A}_\infty\text{-}\mathrm{Cat}}([m]_{\mathbb{K}}, \mathscr{C}) \\ (\{X_i\}_{0\leq i\leq n}, \{f_I\}) &\mapsto (\{X_{\alpha(j)}\}_{0\leq j\leq m}, \{g_J\}),\end{aligned}$$

where g_J is:

$$g_J = \begin{cases} f_{\alpha(J)}, & \text{if } \alpha_{|J} \text{ is injective} \\ 1_{X_i}, & \text{if } J = \{j, j'\} \text{ and } \alpha(j) = \alpha(j') = X_i \\ 0, & \text{otherwise,} \end{cases}$$

such that, given $\alpha : [m] \to [n]$ and $\beta : [n] \to [l]$, then

$$\mathrm{Hom}_{\mathrm{A}_\infty\text{-}\mathrm{Cat}}(\beta \cdot \alpha, \mathscr{C}) = \mathrm{Hom}_{\mathrm{A}_\infty\text{-}\mathrm{Cat}}(\alpha, \mathscr{C}) \cdot \mathrm{Hom}_{\mathrm{A}_\infty\text{-}\mathrm{Cat}}(\beta, \mathscr{C}).$$

Moreover given $\mathrm{Id} : [n] \to [n]$ *then*

$$\mathrm{Hom}_{\mathrm{A}_\infty\text{-}\mathrm{Cat}}(\mathrm{Id}, \mathscr{C}) = \mathrm{Id}_{\mathrm{Hom}_{\mathrm{A}_\infty\text{-}\mathrm{Cat}}([n], \mathscr{C})}.$$

Proof. First of all, we want to associate to α an A_∞-unital functor (denoted by $\{\alpha\}$) between the minimal categories $[m]_{\mathbb{K}} \to [n]_{\mathbb{K}}$. We define the A_∞-functor $\{\alpha_n\}_{n\geq 0} : [m]_{\mathbb{K}} \to [n]_{\mathbb{K}}$ in the following way:

- if $k = 0$, $\alpha_k = \alpha$,
- if $k = 1$,
$$\alpha_1 : \mathrm{Hom}_{[n]_{\mathbb{K}}}(l, s) \to \mathrm{Hom}_{[m]_{\mathbb{K}}}(\alpha(l), \alpha(s))$$
$$j_{ls} \mapsto \alpha_1(j_{ls}) = \begin{cases} 0, & \text{if } l > s \\ 1, & \text{if } l = s \\ j_{\alpha(l)\alpha(s)}, & \text{if } l < s \end{cases}.$$
- if $k > 1$, $\alpha_k = 0$.

The induced map $\mathrm{Hom}_{\mathrm{A}_\infty\text{-Cat}}(\alpha, \mathscr{C})$ is given by the composition with the A_∞-functor $\{\alpha_n\}_{n\geq 0}$. Let $\mathscr{F} \in \mathrm{Hom}_{\mathrm{A}_\infty\text{-Cat}}([n]_\mathbb{K}, \mathscr{C})$. For all $t \geq 1$ we have:

$$(\mathscr{F}\alpha)_t = \sum_{r=1}^{t} \sum_{i_1+\ldots+i_r=t} \mathscr{F}_r(\alpha_{i_r}, ..., \alpha_{i_1}).$$

Since only α_1 is non-trivial, we have $r = t$, $i_1 = i_2 = ... = i_t = 1$ and $(\mathscr{F}\alpha)_t$ becomes:

$$(\mathscr{F}\alpha)_t = \mathscr{F}_t(\alpha_1, ..., \alpha_1).$$

Therefore

$$\mathscr{F}_1(\alpha_1(j_{i_0 i_1})) = \mathscr{F}_1(j_{\alpha(i_0)\alpha(i_1)}),$$
$$\mathscr{F}_2(\alpha_1(j_{i_0 i_1}), \alpha_1(j_{i_1 i_2})) = \mathscr{F}_2(j_{\alpha(i_0)\alpha(i_1)}, j_{\alpha(i_1)\alpha(i_2)}),$$
$$\ldots$$
$$\mathscr{F}_n(\alpha_1(j_{i_0 i_1}), \alpha_1(j_{i_1 i_2}), ..., \alpha_1(j_{i_{n-1} i_n})) = \mathscr{F}_n(j_{\alpha(i_0)\alpha(i_1)}, j_{\alpha(i_1)\alpha(i_2)}, ..., j_{\alpha(i_{n-1}),\alpha(i_n)}).$$

Of course, i_k are positive integers smaller than m (because $\alpha : [m] \to [n]$). So if we take an element in $\mathrm{Hom}_{\mathrm{A}_\infty\text{-Cat}}([n]_\mathbb{K}, \mathscr{C})$ denoted by $(\{X_i\}_{0\leq i\leq n}, \{f_I\})$ this is sent to

$$(\{X_{\alpha(j)}\}_{0\leq j\leq m}, \{g_J\}),$$

where g_J is:

$$g_J = \begin{cases} f_{\alpha(J)}, & \text{if } \alpha_{|J} \text{ is injective} \\ 1_{X_i}, & \text{if } J = \{j, j'\} \text{ and } \alpha(j) = \alpha(j') = X_i \\ 0, & \text{otherwise.} \end{cases}$$

This concludes the proof. □

Definition 4.3.1 (A_∞-nerve)**.** Let $\mathscr{C}$ be a unital A_∞-category. We define the *A_∞-nerve* of $\mathscr{C}$ to be the simplicial set (denoted by $\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C})$) such that for all positive integers n

$$\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C})_n := \mathrm{Hom}_{\mathrm{A}_\infty\text{-Cat}}([n]_\mathbb{K}, \mathscr{C}).$$

And for every $\alpha : [m] \to [n] \in \Delta$ the element $(\{X_i\}_{0\leq i\leq n}, \{f_I\}\})$ in $\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C})_n$ is sent to $(\{X_{\alpha(j)}\}_{0\leq j\leq m}, \{g_J\}\})$ where g_J is:

$$g_J = \begin{cases} f_{\alpha(J)}, & \text{if } \alpha_{|J} \text{ is injective} \\ 1_{X_i}, & \text{if } J = \{j, j'\} \text{ and } \alpha(j) = \alpha(j') = X_i \\ 0, & \text{otherwise.} \end{cases}$$

Remark 4.10
Note that if $\mathscr{C}$ is a dg-category then $\mathrm{N}_{\mathrm{A}_\infty}(i(\mathscr{C})) = \mathrm{N}_{\mathrm{dg}}(\mathscr{C})$ where N_{dg} is the dg-nerve defined in [44, 1.3.1.6].

Theorem 4.11. *Let $\mathscr{C}$ be an A_∞-category, then $\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C})$ is an ∞-category.*

Proof. [18, Prop. 2.2.12.]. □

4.4 Properties of the A_∞-nerves

This section is divided into three parts: in the first one we will introduce some notions that will be useful to characterize the mapping space of the A_∞-nerve. In the second we will recall some classical results about model categories. Finally we will prove the main theorem of the chapter that will be the fundamental tool to give a comparison between A_∞-categories and stable ∞-categories.

Simplicial Objects and DK-correspondence

Let $\mathcal{A}$ be an abelian category, we denote by $\mathrm{Ch}^{\geq 0}_{\mathcal{A}}$ the category of bounded above chain complexes. In particular if $\mathcal{A}$ is the category of $\mathbb{K}$-modules, we denote by $\mathrm{Ch}^{\geq 0}_{\mathbb{K}}$ the the category of bounded above chain complexes of $\mathbb{K}$-modules.

Definition 4.4.1 (Simplicial Object)**.** A *simplicial object* A in $\mathcal{A}$ is a functor $X : \Delta^{\mathrm{op}} \to \mathcal{A}$.

We have a functor $\mathbf{N}_* : \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A}) \to \mathrm{Ch}^{\geq 0}_{\mathcal{A}}$ that associates to each simplicial object A the chain:

$$... \longrightarrow \mathbf{N}_2(A) \xrightarrow{A(d_0)} \mathbf{N}_1(A) \xrightarrow{A(d_0)} \mathbf{N}_0(A) \longrightarrow 0 \longrightarrow ...$$

where:

$$\mathbf{N}_n(A.) := \bigcap_{1\leq i\leq n} \ker(A(d_i))$$

and $d_j : [n-1] \to [n]$ is the natural injective map such that $j \notin \mathrm{Im}(d_j)$.

We have also a functor $\mathrm{DK}_\bullet : \mathrm{Ch}^{\geq 0}_{\mathcal{A}} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A})$ that associates to each chain $C^\bullet$ the simplicial object $\mathrm{DK}_*(C^\bullet) : \Delta^{\mathrm{op}} \to \mathcal{A}$ defined, for every n, to be:

$$\mathrm{DK}_n(C^\bullet) := \bigoplus_{\alpha:[n]\to[k]} C_k,$$

where α is a surjective map.
Moreover, given a map $\beta : [n'] \to [n]$, we define $\mathrm{DK}_\bullet(\beta)$ to be the matrix with (α, α') entries:

$$(f_{\alpha,\alpha'}) : \bigoplus_{\alpha} C_k \to \bigoplus_{\alpha'} C_{k'}$$

such that:

$$f_{\alpha,\alpha'} = \begin{cases} 1_{C_k}, & \text{if } \alpha \text{ and } \alpha' \text{ fit the diagram } \begin{array}{ccc} [n] & \xrightarrow{\beta} & [n'] \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha'} \\ [k'] & = & [k] \end{array} \\ d_k, & \text{if } \alpha \text{ and } \alpha' \text{ fit the diagram } \begin{array}{ccc} [n] & \xrightarrow{\beta} & [n'] \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha'} \\ [k-1] & \xrightarrow[d_0]{} & [k] \end{array} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.12. *The functors* $\mathrm{DK}_\bullet$ *and* $\mathbf{N}_*$ *are adjoints in both directions i.e.* $\mathrm{DK}_\bullet \vdash \mathbf{N}_*$ *and* $\mathbf{N}_* \vdash \mathrm{DK}_\bullet$.

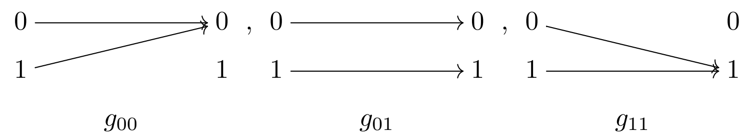
Proof. [15, Satz 3.6]. □

Let $\mathbf{Z}\Delta^n$ denote the free abelian group generated by $\Delta^n[j]$, for every j. Let us build the chain associated $\mathbf{N}_*(\mathbf{Z}\Delta^n)$.

Example 4.4.1. Let us consider $\Delta^0 = \mathrm{Hom}_\Delta(-,[0])$. If $n = 0$, then $\mathbf{N}_0(\mathbf{Z}\Delta^0) = \ker(\mathbf{Z}\Delta^0_0 \to 0) = \mathbf{Z}\Delta^0_0 = \{1 \text{ generator } g_0\}$. Moreover if $n = 1$, by definition, $\mathbf{N}_1(\mathbf{Z}\Delta^0) = \ker(d^1 : \mathbf{Z}\Delta^0_1 \to \mathbf{Z}\Delta^0_0) = 0$, because $\mathbf{Z}\Delta^0_1$ is generated by g_{00} and $d^1(g_{00}) = g_0 \neq 0$. We can procede in the same way for any other $n \geq 1$. Hence the chain associated to $\mathbf{Z}(\Delta^0)$ is given by:

$$... \longrightarrow 0 \xrightarrow{d^0} 0 \longrightarrow < g_0 > \longrightarrow 0 \longrightarrow ...$$

Example 4.4.2. Let us consider $\Delta^1 = \mathrm{Hom}_\Delta(-,[1])$. If $n = 0$ we have $\mathbf{N}_0(\mathbf{Z}\Delta^1) = \ker(\mathbf{Z}\Delta^1_0 \to 0) = \mathbf{Z}\Delta^1_0 = \{2 \text{ generators } g_0 \text{ and } g_1\}$. If $n = 1$, we have $\mathbf{N}_1(\mathbf{Z}\Delta^1) = \ker(d^1 : \mathbf{Z}\Delta^1_1 \to \mathbf{Z}\Delta^1_0)$. In $\mathbf{Z}\Delta^1_1$ we have three generators g_{00}, g_{01} and g_{11} given by the following maps:



$\mathbf{N}_1(\mathbf{Z}\Delta^1)$ is given by the elements $\mathbf{Z}\Delta^1_1$ of the form $\alpha_{00}g_{00} \oplus \alpha_{01}g_{01} \oplus \alpha_{11}g_{11}$ such that $d^1 = 0$, where $\alpha_{ij} \in \mathbb{K}$. By definition:

$$\begin{aligned} d^1(\alpha_{00}g_{00} \oplus \alpha_{01}g_{01} \oplus \alpha_{11}g_{11}) &= \alpha_{00}g_0 \oplus \alpha_{01}g_0 \oplus \alpha_{11}g_1 \\ &= (\alpha_{00} + \alpha_{01})g_0 \oplus \alpha_{11}g_1. \end{aligned} \tag{4.2}$$

and it is zero only if $\alpha_{00} + \alpha_{01} = 0$ and $\alpha_{11} = 0$.
Hence $\ker(\mathbf{Z}\Delta^1_1 \to \mathbf{Z}\Delta^1_0) =< g_{00} - g_{01} >$.
Then the associated chain $\mathbf{Z}(\Delta^1)$ is given by:

$$... \longrightarrow 0 \longrightarrow < g_{00} - g_{01} > \xrightarrow{d_0} < g_0 > \oplus < g_1 > \longrightarrow 0 \longrightarrow ...$$

such that $d_0 < g_{00} - g_{01} >= g_0 - g_1$

Let $\mathscr{C}$ be a dg-category and x, y two fixed objects in $\mathscr{C}$. By Example 4.4.1, we can identify the homomorphisms of complexes $f : \mathbf{N}_*(\mathbf{Z}\Delta^0) \to \mathrm{Hom}_\mathscr{C}(x,y)$ with the maps $f : x \to y$ of degree zero such that $df = 0$. By Example 4.4.2, we can identify the homomorphisms of complexes $f : \mathbf{N}_*(\mathbf{Z}\Delta^1) \to \mathrm{Hom}_\mathscr{C}(x,y)$ with the set of the maps $f_{02}, f_{12}, f_{012} : x \to y$ such that $\deg f_{02} = \deg f_{12} = 0$, $\deg f_{012} = -1$, $df_{012} = f_{02} - f_{12}$ and $df_{02} = df_{12} = 0$.
More generally let us discuss an important lemma (implicitly assumed by Lurie [43, pg. 66]) which characterizes the maps between $\mathbf{N}_*(\mathbf{Z}\Delta^n)$ and $\mathrm{Hom}_\mathscr{C}(x,y)$.

Lemma 4.13. *We can identify* $f : \mathbf{N}_*(\mathbf{Z}\Delta^n) \to \mathrm{Hom}_{\mathscr{C}}(x, y)$ *to the maps* $f_I : x \to y$ *of degree* $|I| - 2$ *for all subset* $I = \{0 \leq i_0 < ... < i_j < j + 1 \leq n\}$ *such that:*

$$(\dagger) \qquad df_I = \sum_{0 \leq k \leq j} (-1)^k f_{I-k}.$$

Proof. We denote by $g_{i_0 \ldots i_j}$ the free generator associated to the map $[j] \to [n]$ which sends the integer $k \in [j]$ to $i_k \in [n]$. It follows immediately that

$$\langle \bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} g_{i_0 \ldots i_j} \rangle = \mathbf{Z}\Delta^n_j.$$

By definition, an element $\bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} \alpha_{i_0 \ldots i_j} g_{i_0 \ldots i_j}$ is in $\mathbf{N}_j(\mathbf{Z}\Delta^n)$ if and only if

$$(4.3) \qquad \begin{cases} d^j(\bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} \alpha_{i_0 \ldots i_j} g_{i_0 \ldots i_j}) = 0 \\ \qquad \ldots \\ d^1(\bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} \alpha_{i_0 \ldots i_j} g_{i_0 \ldots i_j}) = 0 \end{cases}$$

Now, if we focus on the first row in (4.3), we have that

$$(4.4) \qquad d^j(\bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} \alpha_{i_0 \ldots i_j} g_{i_0 \ldots i_j}) = 0$$

if and only if

$$\sum_{i_j = i_{j-1}+1}^{n} \alpha_{i_0 \ldots i} = -\alpha_{i_0 \ldots i_{j-1} i_{j-1}}.$$

So we can rewrite (4.3) in terms of the following system of $j - 1$ equations

$$(4.5) \qquad \begin{cases} d^{j-1}(\bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} \alpha_{i_0 \ldots i_j} (g_{i_0 i_1 \ldots i_j} - g_{i_0 \ldots i_{j-1} i_{j-1}}) = 0 \\ \qquad \ldots \\ d^1(\bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} \alpha_{i_0 \ldots i_j} (g_{i_0 i_1 \ldots i_j} - g_{i_0 \ldots i_{j-1} i_{j-1}}) = 0. \end{cases}$$

Proceeding as for the first row, we obtain the following system of $j-2$ equations equivalent to (4.5)

$$(4.6) \qquad \begin{cases} d^{j-2}(\bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} \alpha_{i_0 \ldots i_j} (g_{i_0 i_1 \ldots i_j} - g_{i_0 \ldots i_{j-2} i_{j-2} i_j} + \\ \qquad -(g_{i_0 i_1 \ldots i_{j-1}} - g_{i_0 \ldots i_{j-2} i_{j-2} i_{j-1}})) = 0 \\ \qquad \ldots \\ \qquad \ldots \\ d^1(\bigoplus_{0 \leq i_0 \leq \ldots \leq i_j \leq n} \alpha_{i_0 \ldots i_j} (g_{i_0 i_1 \ldots i_j} - g_{i_0 \ldots i_{j-2} i_{j-2} i_j} + \\ \qquad -(g_{i_0 i_1 \ldots i_{j-1}} - g_{i_0 \ldots i_{j-2} i_{j-2} i_{j-1}})) = 0. \end{cases}$$

We can go on as before by removing one by one the equations from the system. In the end we have that $\bigoplus_{0\le i_0\le\ldots\le i_j\le n}\alpha_{i_0\ldots i_j}g_{i_0\ldots i_j}$ is in $\mathbf{N}_j(\mathbf{Z}\Delta^n)$ if it is of the form

$$\bigoplus_{0\le i_0\le\ldots\le i_j\le n}\alpha_{i_0\ldots i_j}\Big(\sum_{0\le k_1^0,\ldots,k_j^{j-1}\le 1}(-1)^{\triangle_{i_1^{k_1^0}\ldots i_j^{k_j^{j-1}}}}g_{i_0 i_1^{k_1^0}\ldots i_j^{k_j^{j-1}}}\Big)$$

where

$$i_l^{k_{l_2}^{l_1}}=\begin{cases} i_{l_2}, & \text{if } k_{l_2}^{l_1}=0\\ i_{l_1}, & \text{if } k_{l_2}^{l_1}=1\end{cases}$$

and

$$\triangle_{i_1^{k_1^0}\ldots i_j^{k_j^{j-1}}}=k_1^0+\ldots+k_j^{j-1}.$$

We note that, if there exists p such that $i_p=i_{p-1}$, then

$$\sum_{0\le k_1^0,\ldots,k_j^{j-1}\le 1}(-1)^{\triangle_{i_1^{k_1^0}\ldots i_j^{k_j^{j-1}}}}g_{i_0 i_1^{k_1^0}\ldots i_j^{k_j^{j-1}}}=0.$$

This means that $\mathbf{N}_j(\mathbf{Z}\Delta^n)=0$ if $j>n$. Otherwise $\mathbf{N}_j(\mathbf{Z}\Delta^n)$ is generated by

$$\bigoplus_{0\le i_0<\ldots<i_j\le n}\Big(\sum_{0\le k_1^0,\ldots,k_j^{j-1}\le 1}(-1)^{\triangle_{i_1^{k_1^0}\ldots i_j^{k_j^{j-1}}}}g_{i_0 i_1^{k_1^0}\ldots i_j^{k_j^{j-1}}}\Big). \tag{4.7}$$

Now, every map of complexes $f:\mathbf{N}_*(\mathbf{Z}\Delta^n)\to \mathrm{Hom}_{\mathscr{C}}(x,y)$ is uniquely determined, for every integer j, by the image of the generators in (4.7). We will denote such image by $f_{i_0\ldots i_j(j+1)}$. Moreover f is a chain of complexes. So

$$\begin{aligned} d^j(f_{i_0\ldots i_j(j+1)}) &= f_{j-1}\Big(\sum_{0\le k_1^0,\ldots,k_j^{j-1}\le 1}(-1)^{\triangle_{i_1^{k_1^0}\ldots i_j^{k_j^{j-1}}}}g_{i_1^{k_1^0}i_2^{k_2^1}\ldots i_j^{k_j^{j-1}}}\Big)\\ &= f_{j-1}\Big(\sum_{0\le k_2^1,\ldots,k_j^{j-1}\le 1}(-1)^{\triangle_{i_2^{k_2^1}\ldots i_j^{k_j^{j-1}}}}\big(g_{i_1 i_2^{k_2^1}\ldots i_j^{k_j^{j-1}}}-g_{i_0 i_2^{k_2^1}\ldots i_j^{k_j^{j-1}}}\big)\Big)\\ &= f_{i_1\ldots i_j(j+1)}-f_{j-1}\Big(\sum_{0\le k_2^1,\ldots,k_j^{j-1}\le 1}(-1)^{\triangle_{i_2^{k_2^1}\ldots i_j^{k_j^{j-1}}}}\big(g_{i_0 i_2^{k_2^1}\ldots i_j^{k_j^{j-1}}}\big)\Big). \end{aligned} \tag{4.8}$$

Note that, for every t, we have

$$\begin{aligned} g_{i_1^{k_1^0}i_2^{k_2^1}\ldots i_{t-2}^{k_{t-2}^{t-3}}i_t^{k_t^{t-1}}\,i_{t+1}^{k_{t+1}^t}\ldots i_j^{k_j^{j-1}}} &= g_{i_1^{k_1^0}i_2^{k_2^1}\ldots i_{t-2}^{k_{t-2}^{t-3}}i_t\ldots i_j^{k_j^{j-1}}}-g_{i_1^{k_1^0}i_2^{k_2^1}\ldots i_{t-2}^{k_{t-2}^{t-3}}i_{t-1}\ldots i_j^{k_j^{j-1}}}\\ &= g_{i_1^{k_1^0}i_2^{k_2^1}\ldots i_{t-2}^{k_{t-2}^{t-3}}i_t^{k_t^{t-2}}\,i_{t+1}^{k_{t+1}^t}\ldots i_j^{k_j^{j-1}}}+\\ &- g_{i_1^{k_1^0}i_2^{k_2^1}\ldots i_{t-2}^{k_{t-2}^{t-3}}i_{t-1}^{k_{t-1}^{t-2}}\ldots i_j^{k_j^{j-1}}}. \end{aligned}$$

This means that equation (4.8) gives precisely the condition (†). □

Remark 4.14
By Theorem 4.12 we have that

$$\mathrm{Hom}(\mathbf{Z}\Delta^n, \mathrm{DK}_\bullet(\tau_{\geq 0}\mathrm{Hom}_{\mathscr{C}}(x,y))) \simeq \mathrm{Hom}_{\mathrm{Ch}_{\mathbb{K}}}(\mathbf{N}_*(\mathbf{Z}\Delta^n), \tau_{\geq 0}\mathrm{Hom}_{\mathscr{C}}(x,y)).$$

Using the characterization in Lemma 4.13 we have that the morphisms f_I with the property (†) are in bijection with $\mathrm{DK}_n(\tau_{\geq 0}\mathrm{Hom}_{\mathscr{C}}(x,y))$.

Model structures

We briefly recall some classical notions about model structures on categories. Basics definitions and examples are treated in the Appendix, a complete reference about model structures is [23].

Example 4.4.3. The category of (small) dg-categories has two canonically model structures due to Tabuada [58] [57]: the first one has as weak equivalences the quasi-equivalences and the second one has as weak equivalences the Morita equivalences. We recall, from Chapter 1, that $F : \mathscr{C} \to \mathscr{C}'$ is a Morita equivalence if:

(Me1) F induces an equivalence on perfect-complexes

$$\mathrm{Ho}(F) : \mathrm{Ho}(\mathrm{pretr}(\mathscr{C}))^{ic} \to \mathrm{Ho}(\mathrm{pretr}(\mathscr{C}'))^{ic}$$

(Me2) $\mathrm{Hom}_{\mathscr{C}}(x,y) \to \mathrm{Hom}_{\mathscr{C}'}(F(x), F(y))$ is a quasi-isomorphism for all x, $y \in \mathscr{C}$.

Clearly every weak equivalence in the first model structure is a Morita equivalence.

Remark 4.15
A functor between pretriangulated idempotent complete dg-categories is a weak equivalence if and only if it is a Morita equivalence.

Definition 4.4.2 (Weak equivalence [25])**.** Let X, Y be ∞-categories, $F : X \to Y$ is a *weak equivalence* if:

- $\mathrm{Ho}(X) \simeq \mathrm{Ho}(Y)$ (as categories),
- $\forall x, y \in X$ the geometric realization of the morphism

  $$\mathrm{Hom}^R_X(x,y) \to \mathrm{Hom}^R_Y(F_0(x), F_0(y))$$

  is a weak homotopy equivalence of topological spaces.

Weak equivalences together with monomorphisms (i.e. $F_n : X_n \to Y_n$ monomorphisms for all $n > 0$) as cofibrations and fibrations, defined by the right left property (cf. Definition 1.1.2. [23]), forms a model structure over sSet called *Joyal model structure*.

Remark 4.16
We can see a simplicial object as a simplicial set.

Using [51, Thm. 4] we can endow the category of simplicial objects with a model structure by defining weak equivalences (resp. fibrations) as the morphisms of simplicial objects where the underling functor is a weak equivalence (resp. Kan fibrations) of simplicial sets.

Remark 4.17
Let x and y be objects of $\mathscr{C}$, where $\mathscr{C}$ is a unital A_∞-category. The simplicial set $\mathrm{Hom}^R_{N_{A_\infty}}(x,y)$ can be naturally enriched over the monoidal category of modules over the commutative ring $\mathbb{K}$. So $\mathrm{Hom}^R_{N_{A_\infty}}(x,y) \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbb{K}\text{-Mod})$ and the identification $\mathrm{Hom}^R_{N_{A_\infty}}(x,y) \simeq \mathrm{DK}_\bullet(\tau_{\geq 0}\mathrm{Hom}_{\mathscr{C}}(x,y))$ makes sense.

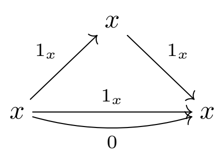
Remark 4.18
The category $\mathrm{Ch}^{\geq 0}_{\mathbb{K}}$ has a model structure where weak equivalences are quasi-isomorphisms, fibrations are degreewise epimorphisms and cofibrations are degreewise monomorphisms with degreewise projective cokernels.
Moreover the functors $\mathrm{DK}_\bullet$ and $\mathbf{N}_*$ match cofibrations, fibrations and weak equivalences of the model structures on $\mathrm{Ch}^{\geq 0}_{\mathbb{K}}$ and of the above model structure over the simplicial objects $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbb{K}\text{-Mod})$ [55, 4.1].

Characterization of mapping spaces of A_∞-nerves

Now we are ready to prove some new results about A_∞-nerves that will be useful, in the last section, to give a comparison between pretriangulated A_∞-categories and stable ∞-categories. Let X be a simplicial set and let x, y be two elements in X_0. Let $\mathscr{C}$ be an A_∞-category.

Definition 4.4.3 (Degenerate simplex)**.** We define the *degenerate n-simplex* on x to be the image of x via $X(\sigma)$, where $\sigma : [n] \to [0]$.

Example 4.4.4. A degenerate 2-simplex on x in $N_{A_\infty}(\mathscr{C})$ is represented by the following diagram:



Definition 4.4.4 (Mapping space)**.** For every couple of elements of $\mathscr{C}$, we define the *mapping space* $\mathrm{Hom}^R_X(x,y)$ to be the ∞-category whose n-simplexes are the $n+1$-simplices of X_{n+1} such that $X_{|_{\{n+1\}}} = y$ and $X_{|_{\{0,\dots,n\}}}$ is the degenerate n-simplex on x.

Lemma 4.19. *The mapping space* $\mathrm{Hom}^R_{N_{A_\infty}(\mathscr{C})}(x,y)$ *is equivalent to* $\mathrm{DK}_\bullet(\tau_{\geq 0}\mathrm{Hom}_{\mathscr{C}}(x,y))$.

Proof. First of all we calculate the degenerate n-simplex in $N_{A_\infty}(\mathscr{C})$. Let us consider the degenerate map $\sigma : [n] \to [0]$. Using Theorem 4.9, the image of x in $N_{A_\infty}(\mathscr{C})_n$ via $N_{A_\infty}(\sigma)$ is given by:

- $n+1$ copies of x, because $\alpha(i_0) = \dots = \alpha(i_n) = 0$;

- identity maps between x and itself, because $\alpha(j_{i_0 i_1}) = 1_{X_{i_0}}$;
- all the higher maps $f_{i_0 i_1 i_2,\ldots}$ are zero, because $[0]$ has only one object.

By definition we have that, for every integer n, $\mathrm{Hom}^R_{\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C})}(x,y)_n \subset \mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C})_{n+1}$. Then an element of $\mathrm{Hom}^R_{\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C})}(x,y)_n$ is a set of elements satisfying (4.1) for all sets $I = \{0 \leq i_0 < i_1 < \ldots < i_m < i_{m+1} \leq n+1\}$.
Now, using the previous calculation on degenerate n-simplices, we have that every $f_{i_p i_q}$ with $i_q \neq n+1$ is the identity and every $f_{i_p \ldots i_q}$, with $q \neq n+1$, is 0.
Then we can say that every element in $\mathrm{Hom}^R_{\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C})}(x,y)_n$ is given by the identity maps on the vertex x and, for all subsets $I = \{0 \leq i_0 < i_1 < \ldots < i_m < i_{m+1} = n+1\}$, the maps f_I (i.e. the maps with target y) satisfy:

$$m_1^{\mathscr{C}}(f_I) = \sum_{1 \leq j \leq m} (-1)^{j-1}(f_{I-i_j}) - (-1)^0 m_2^{\mathscr{C}}(f_{i_1\ldots i_{m+1}}, f_{i_0 i_1}) + \sum_{r>2}\sum_{\ddagger_r}(-1)^{1+\epsilon_r}0.$$

This means that

$$\begin{aligned} m_1^{\mathscr{C}}(f_I) &= \sum_{1 \leq j \leq m} (-1)^{j-1}(f_{I-i_j}) - (-1)^0 m_2^{\mathscr{C}}(f_{i_1\ldots i_{m+1}}, f_{i_0 i_1}) + \sum_{r>2}\sum_{\ddagger_r}(-1)^{1+\epsilon_r}0 \\ &= -f_{i_1\ldots i_{m+1}} + \sum_{1 \leq j \leq m} (-1)^{j-1}(f_{I-i_j}) \\ &= \sum_{0 \leq j \leq m} (-1)^{j+1}(f_{I-i_j}) \end{aligned}$$

Hence, after a change of sign, all the maps in $\mathrm{Hom}^R_{\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C})}(x,y)$ satisfy (†). So, using Remark 4.14 and Theorem 4.12, we have

$$\mathrm{Hom}^R_{\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C})}(x,y) \simeq \mathrm{DK}_\bullet(\tau_{\geq 0}\mathrm{Hom}_{\mathscr{C}}(x,y))$$

This is what we wanted to prove. □

Theorem 4.20. *Let $\mathscr{C}$, $\mathscr{D}$ be A_∞-categories unital and let $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ be a quasi-equivalence of A_∞-categories. Then $\mathrm{N}_{\mathrm{A}_\infty}\mathscr{F} : \mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C}) \to \mathrm{N}_{\mathrm{A}_\infty}(\mathscr{D})$ is a weak equivalence in the Joyal model structure.*

Proof. If $\{\mathscr{F}^n\}$ is a quasi-equivalence then, by definition, the functor induced between the homotopy categories $\mathrm{Ho}(\mathscr{C})$ and $\mathrm{Ho}(\mathscr{D})$ is an equivalence (we1). We observe that the homotopic category of an ∞-category X is given by the category having as objects the elements of X_0 and as morphisms the elements of X_1 that are quotient by the homotopy relation. So $\mathrm{Ho}(\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C}))$ has the same objects as $\mathscr{C}$ and as morphisms the set $Z^0(\mathrm{Hom}_{\mathscr{C}}(x,y))$ such that $f \simeq g$ if and only if there exists $h \in \mathrm{Hom}_{\mathscr{C}}(x,y)^{-1}$ such that $dh = f - g$. It follows that $\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{F})$ induces an equivalence between the homotopy categories of $\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C})$ e $\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{D})$.
Now we have to prove that, given two objects x, $y \in \mathscr{C}$, the map

$$\mathrm{Hom}^R_{\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C})}(x,y) \to \mathrm{Hom}^R_{\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C})}(\mathscr{F}_0(x), \mathscr{F}_0(y)) \tag{4.9}$$

is an homotopy equivalence between the corresponding Kan complexes. Using Lemma 4.19, we have that it is enough to prove that

$$\mathrm{DK}_\bullet(\tau_{\geq 0}\mathrm{Hom}_{\mathscr{C}}(x,y)) \to \mathrm{DK}_\bullet(\tau_{\geq 0}\mathrm{Hom}_{\mathscr{D}}(\mathscr{F}_0(x),\mathscr{F}_0(y))) \tag{4.10}$$

is a weak equivalence, and this is true because the functor $\mathrm{DK}_\bullet$ preserves weak equivalences and the map of complexes $\mathrm{Hom}_{\mathscr{C}}(x,y) \to \mathrm{Hom}_{\mathscr{D}}(\mathscr{F}_0(x),\mathscr{F}_0(y))$, induced by $\mathscr{F}$, is a quasi-isomorphism by (we2). □

Corollary 4.21. *Given a unital A_∞-category $\mathscr{C}$, we have that the following ∞-categories are weak equivalent:*

$$\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C}) \simeq \mathrm{N}_{\mathrm{A}_\infty}(\mathrm{U}(\mathscr{C})) \simeq \mathrm{N}_{\mathrm{dg}}(\mathrm{U}(\mathscr{C})).$$

Proof. The first weak equivalence is a consequence of Theorem 4.20 using the fact that α is a weak equivalence of A_∞-categories, the second weak equivalence is a straightforward consequence of Remark 4.10. □

Remark 4.22
In the case of dg-categories, Lurie proved in [43, Prop.1.3.1.20] that the dg-nerve induces a right Quillen functor from the classical model structure on the category of (small) dg-categories (the first one in Example 4.4.3) to the Joyal model structure over sSet. Unfortunatly in the case of the category of A_∞-categories there is currently no known model structure. However there exists a canonical model structure (without limits) on the category of A_∞-algebras due to Lefèvre [39] and Le Grignou proves in [38] that $\mathrm{N}_{\mathrm{A}_\infty}$ preserves weak equivalences and fibrations in such a structure. Obviously this correspondence between equivalences and fibrations do not guarantee the existence of a right Quillen functor due to the lack of limits.

Remark 4.23
Given a weak equivalence $F : \mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C}) \to \mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C}')$, we have that F induces an equivalence between the homotopy categories $\mathrm{Ho}(\mathscr{C})$ and $\mathrm{Ho}(\mathscr{D})$. Moreover given two objects x and $y \in \mathscr{C}$ we have a quasi-isomorphism $\mathscr{F}_1$ between $\tau_{\geq 0}\mathrm{Hom}_{\mathscr{C}}(x,y)$ and $\tau_{\geq 0}\mathrm{Hom}_{\mathscr{C}'}(F_0(x),F_0(y))$ given by the following diagram:

$$\begin{array}{ccc}
\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C}) & \xrightarrow{F} & \mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C}') \\
\downarrow & & \uparrow \\
\mathrm{Map}^R_X(x,y) & \xrightarrow{F} & \mathrm{Map}^R_Y(F_0(x),F_0(y)) \\
\downarrow & & \downarrow \\
\mathrm{DK}_\bullet(\tau_{\geq 0}\mathrm{Hom}_{\mathscr{C}}(x,y)) & \longrightarrow & \mathrm{DK}_\bullet(\tau_{\geq 0}\mathrm{Hom}_{\mathscr{C}'}(F_0(x),F_0(y))) \\
\downarrow\sim & & \downarrow\sim \\
\tau_{\geq 0}\mathrm{Hom}_{\mathscr{C}}(x,y) & \longrightarrow & \tau_{\geq 0}\mathrm{Hom}_{\mathscr{C}'}(F_0(x),F_0(y))
\end{array}$$

More explicitly, we set $\mathscr{F}_1 = \mathrm{DK}_\bullet \circ F \circ \mathrm{N}_*$.
Unfortunately in general it is not true that, given a weak equivalence $F : \mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C}) \to \mathrm{N}_{\mathrm{A}_\infty}(\mathscr{C}')$, then $\mathscr{C}$ and $\mathscr{C}'$ are quasi-equivalent as A_∞. For example if we take the category $\mathscr{K}$ with two objects x and y and a morphism $g : x \to y$ of degree -1 such that $dg = 0$ and the category $\mathscr{K}'$ with two objects without nontrivial morphisms then $\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{K}) = \mathrm{N}_{\mathrm{A}_\infty}(\mathscr{K}')$ but $\mathscr{K} \not\simeq \mathscr{K}'$. In the last section we will see, that under specific hypotheses Theorem 4.20 has a converse.

4.5 Stable ∞-categories vs pretriangulated A_∞-categories

In this section we will prove that the pretriangulated A_∞-categories are identified to the stable ∞-categories, via the A_∞-nerve.

Theorem 4.24. *Let $\mathscr{A}$ be a pretriangulated A_∞-category. Then $\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{A})$ is a stable ∞-category. The functor induced between the homotopy categories is an equivalence of triangulated categories. Moreover $\mathscr{A}$ is idempotent complete if and only if $\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{A})$ is an idempotent complete stable ∞-category.*

Proof. If $\mathscr{A}$ is pretriangulated, then $\mathrm{U}(\mathscr{A})$ is a pretriangulated dg-category. By [18, Thm. 4.3.1.] we have that the dg nerve $\mathrm{N}_{\mathrm{dg}}(\mathrm{U}(\mathscr{A}))$ is a stable ∞-category. By Corollary 4.21, we have that $\mathrm{N}_{\mathrm{dg}}(\mathrm{U}(\mathscr{A}))$ is weak equivalent to $\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{A})$. Hence it is a stable ∞-category. Moreover, by Lemma 1.2.4.6 in [43], a stable ∞-category is idempotent complete if and only if the homotopy category is idempotent complete, so $\mathscr{A}$ is idempotent complete if and only if $\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{A})$ is idempotent complete. □

Lemma 4.25. *Let $F : \mathrm{N}_{\mathrm{A}_\infty}(\mathscr{A}) \to \mathrm{N}_{\mathrm{A}_\infty}(\mathscr{A}')$ be an exact functor between A_∞-nerves then, for every object x, $F_0(\Sigma(x)) \simeq \Sigma(F_0(x))$.*

Proof. If $\mathscr{A}$ is pretriangulated A_∞-category, then $\mathrm{N}_{\mathrm{A}_\infty}(\mathscr{A})$ is a stable ∞-category. Moreover, in [18], it is proven that, given a morphism $g \in \mathscr{A}$ of degree 0 with trivial differential (i.e. $g \in \mathrm{N}_{\mathrm{A}_\infty}(\mathscr{A})_1$), the diagram

$$\begin{array}{ccc} x & \xrightarrow{g} & y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Cone}(g) \end{array}$$

is a cofiber sequence. In particular, if we take $g = 0$ and $Y = 0$, then we have that

$$\begin{array}{ccc} x & \xrightarrow{0} & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Cone}(0) \end{array}$$

is a cofiber sequence. Using the axioms **TR1** and **TR2** of triangulated categories in $\mathrm{Ho}(\mathscr{A})$ (see Definition 1.1.2. [47]), we have that $\mathrm{Cone}(0) \simeq \Sigma(x)$. Hence the diagram

$$\begin{array}{ccc} x & \xrightarrow{0} & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma(x) \end{array}$$

is a cofiber sequence. By definition, F carries cofiber sequences to cofiber sequences. In particular, the diagram above will be carried to a cofiber sequence,

$$\begin{array}{ccc} F_0(x) & \xrightarrow{0} & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & F_0(\Sigma(x)) \end{array}$$

in $\mathrm{N}_{\mathrm{A}_\infty}(\mathcal{A}')$. Therefore we have that $F_0(\Sigma(x)) \simeq \Sigma(F_0(x))$, for every object $x \in \mathscr{A}$. $\square$

Now we are ready to give a converse to Theorem 4.20.

Theorem 4.26. *Let $\mathscr{A}$, $\mathscr{A}'$ be two pretriangulated A_∞-categories. A weak equivalence $\mathrm{F} : \mathrm{N}_{\mathrm{A}_\infty}(\mathscr{A}) \to \mathrm{N}_{\mathrm{A}_\infty}(\mathscr{A}')$ in Cat_∞^{Ex} induces a quasi-equivalence between $\mathscr{A}$ and $\mathscr{A}'$.*

Proof. A weak equivalence F induces an equivalence between the categories $\mathrm{Ho}(\mathscr{A})$ and $\mathrm{Ho}(\mathscr{A}')$ (see Remark 4.23). So, for all $x, y \in \mathscr{A}$, we have the following equivalence of categories

$$H^0(\mathrm{Hom}^{\cdot}_{\mathscr{A}}(x,y)) \xrightarrow{\quad\sim\quad} H^0(\mathrm{Hom}^{\cdot}_{\mathscr{A}'}(\mathscr{F}_0(x), \mathscr{F}_0(y)))$$

Moreover, for all $n \in \mathbb{Z}$, we have

$$H^n(\mathrm{Hom}^{\cdot}_{\mathscr{A}}(x,y)) \simeq H^0(\mathrm{Hom}^{\cdot}_{\mathscr{A}}(x,y)[n]) \simeq H^0(\mathrm{Hom}^{\cdot}_{\mathscr{A}}(x[n],y)),$$

because $\mathcal{A}$ is pretriangulated. By the previous equivalence, we have

$$H^0(\mathrm{Hom}^{\cdot}_{\mathscr{A}}(x[n],y)) \simeq H^0(\mathrm{Hom}^{\cdot}_{\mathscr{A}'}(F_0(x[n]), F_0(y))).$$

Now, by Lemma 4.25, we have

$$\mathrm{Hom}^{\cdot}_{\mathscr{A}'}(F_0(x[n]), F_0(y)) \simeq \mathrm{Hom}^{\cdot}_{\mathscr{A}'}(F_0(x)[n], \mathscr{F}_0(y)).$$

Then $H^0(\mathrm{Hom}^{\cdot}_{\mathscr{A}'}(F_0(x[n]), F_0(y))) \simeq H^n(\mathrm{Hom}^{\cdot}_{\mathscr{A}'}(F_0(x), F_0(y)))$ and we are done. $\square$

Part II

Noncommutative motives

Chapter 5

Noncommutative motives

The first part of this chapter is devoted to recall the theory of pure motives. In particular, we are interested in Voevodsky's nilpotence conjecture (formulated in [64]) that states that the group of algebraic cycles of X, modulo the smash-nilpotence equivalence relation, coincides with the group of algebraic cycles of X, modulo the classical numerical equivalence relation.
Furthermore, in the second part of the chapter, we introduce the theory of noncommutative motives developed by Tabuada in [59]. In particular, we focus on the relation between this new theory and the classical theory of motives.
We recall that making use of noncommutative motives, Voevodsky's conjecture was proven for quadric fibrations, intersection of quadrics, linear sections of Grassmannians, etc. (cf. [5] and [6]).

5.1 Background in pure motives

In this first section we give some information about the theory of pure motives. In particular, we define the group of algebraic cycles and some adequate equivalence relations on it. Then, we give an idea of the construction of the category of Chow motives; cf. [2] for a complete exposition of such a construction. Finally, we recall some basic properties of rational motivic decomposition that we will use in the next part.

The letter k will stand for a field.

Definition 5.1.1 (Group of algebraic cycles)**.** Let X be a (smooth) projective k-scheme. We define the group of algebraic cycles $\mathcal{Z}^*(X)$ to be the direct sum $\bigoplus_{d\in\mathbb{N}} \mathcal{Z}^d(X)$, where $\mathcal{Z}^d(X)$ denotes the group

$$\mathcal{Z}^d(X) := \left\{ V = \sum_i n_i V_i, \quad \begin{matrix} \text{s.t. } n_i \in \mathbb{Z} \text{ and } V_i \text{ is an irreducible reduced} \\ \text{closed subscheme with } \operatorname{codim}_X(V_i) = d \end{matrix} \right\}.$$

We call *d-cycle* an element of $\mathcal{Z}^d(X)$.

Remark 5.1
Given a field of characteristic zero F, we set $\mathcal{Z}^*(X)_F = \mathcal{Z}^*(X) \otimes F$.

Now we suppose that X is smooth. For any pair α, $\beta \in \mathcal{Z}^*(X)_F$, we denote by $\alpha\cdot\beta$ the intersection product. In order to define a ring structure on the group of algebraic cycles induced by the intersection product, it is necessary to quotient the group by an adequate equivalence relation. We give some examples of adequate equivalence relation.

Example 5.1.1. (Rational equivalence) We say that two algebraic cycles α and β in $\mathcal{Z}^d(X)_F$ are rationally equivalent ($\alpha \sim_{\mathrm{rat}} \beta$) if there exists an algebraic cycle $\gamma \in \mathcal{Z}^d(X \times \mathbb{P}^1)_F$, flat over $\mathbb{P}^1$, such that $i_0^{-1}\gamma - i_\infty^{-1}\gamma = \alpha - \beta$. The maps $i_0 : X \times \{0\} \to X \times \mathbb{P}^1$ and $i_\infty : X \times \{\infty\} \to X \times \mathbb{P}^1$ are the corresponding inclusions. In the case of divisors, the condition above is equivalent to say that there exists a rational function f on X such that $\alpha - \beta = Z(f)$. We call *Chow ring* the ring $\mathcal{Z}^*(X)_F/_{\sim_{\mathrm{rat}}}$ and we denote by $\mathrm{CH}^d(X)$ the quotient of $\mathcal{Z}^d(X)$ by the subgroup of d-cycles rational equivalent to zero.

Example 5.1.2 (Smash-nilpotence equivalence)**.** We say that an algebraic cycle $\alpha \in \mathcal{Z}^*(X)_F$ is smash-nilpotent if there exists a positive integer n such that $\alpha^{\otimes n}$ is equal to 0 in $\mathcal{Z}^*(X^n)_F/_{\sim_{\mathrm{rat}}}$. Two algebraic cycles α, $\beta \in \mathcal{Z}^*(X)_F$ are smash-nilpotent equivalent ($\alpha \sim_{\otimes_{\mathrm{nil}}} \beta$) if the algebraic cycle $\alpha - \beta$ is smash-nilpotent equivalent to zero.

Example 5.1.3 (Numerical equivalence)**.** We say that an algebraic cycle $\alpha \in \mathcal{Z}^*(X)_F$ is numerically trivial if for all $\gamma \in \mathcal{Z}^{n-d}(X)_F$, $\gamma \cdot \alpha = 0 \in \mathcal{Z}^n(X)_F$. Two cycles α and β are numerically equivalent ($\alpha \sim_{\mathrm{num}} \beta$) if the algebraic cycle $\alpha - \beta$ is numerically trivial.

Roughly speaking, we can define the *category of Chow motives*, denoted by $\mathrm{Chow}(k)$, as follows. The objects are the triples (X, p, r), where X is a smooth projective k-scheme, p is an idempotent endomorphism and r is an integer, and the morphisms are:

$$\mathrm{Hom}_{\mathrm{Chow}(k)}((X, p, r), (Y, q, s)) := q\ \mathrm{CH}^{\dim(X)+r-s}(X \times Y)\ p.$$

It is well known that $\mathrm{Chow}(k)$ is an additive, idempotent complete and rigid symmetric monoidal category.
For further details about the construction of $\mathrm{Chow}(k)$ and $\mathrm{Chow}(k)_F$ consult [2].
We have a contravariant symmetric monoidal functor

$$\mathfrak{h} : \mathrm{SmProj}(k)^{\mathrm{op}} \to \mathrm{Chow}(k)$$
$$X \mapsto (X, \mathrm{Id}_X, 0),$$

where $\mathrm{SmProj}(k)$ denotes the category of smooth projective k-schemes. We list some properties of the functor $\mathfrak{h}$.

Projective space

Let us denote by $\mathbb{1}$ the $\otimes$-unit of the category $\mathrm{Chow}(k)$; we recall that $\mathfrak{h}(\mathbb{P}^1) = \mathbb{1} \oplus \mathbb{L}$, where $\mathbb{L}$ denotes the Lefschetz motive.
In more general terms, for every positive integer n, we have the decomposition

$$\mathfrak{h}(\mathbb{P}^n) = \bigoplus_{i=0}^{n} \mathbb{1}(-i),$$

where $\mathbb{1}(1)$ denotes the Tate motive (i.e. the inverse of $\mathbb{L}$, formally $\mathbb{1}(-1) = \mathbb{L}$) and $-(i)$ denotes $- \otimes \mathbb{1}(1)^{\otimes i}$. Moreover we have that $(X, p, m) = p\ \mathfrak{h}(X) \otimes \mathbb{L}^{\otimes -m}$, it means that $\mathfrak{h}(X)(r) = (X, \mathrm{Id}_X, r)$.

Blowups

The functor $\mathfrak{h}$ is "well behaved" with respect to blowups. In detail, let X be a smooth projective variety over a field k and let $j : Y \hookrightarrow X$ be a smooth closed subvariety of codimension r. Then the blowup $\pi_Y : \mathrm{Bl}_Y(X) \to X$ of X in Y induces an isomorphism of Chow motives $\mathfrak{h}(X) \oplus \bigoplus_{i=1}^{r-1} \mathfrak{h}(Y)(i) \to \mathfrak{h}(\mathrm{Bl}_Y(X))$ (see [45]). As a consequence, if $\mathrm{dim}Y \leq 2$, then $\mathrm{V}(\mathrm{Bl}_Y(X))$ holds if and only if $\mathrm{V}(X)$ holds.

Flat morphisms

Let $f : X \to B$ be a flat morphism in $\mathrm{SmProj}(k)$, with X and B of dimension d_X and d_B, respectively. We denote by X_b the fiber of f over a point b in B and let Ω be a universal domain containing k. Assume that $\mathrm{CH}^l(X_b) = \mathbb{Q}$, for all $0 \leq l < \frac{d_X - d_B}{2}$ and for all points $b \in B(\Omega)$. Then we have a direct sum decomposition of the Chow motive of X as $\mathfrak{h}(X) \simeq \bigoplus_{i=0}^{d_X - d_B} \mathfrak{h}(B)(i) \oplus (Z, r, \lfloor \frac{d_X - d_B + 1}{2} \rfloor)$, where Z is a smooth and projective variety of dimension

$$d_Z = \begin{cases} d_B - 1, & \text{if } d_X - d_B \text{ is odd,} \\ d_B, & \text{if } d_X - d_B \text{ is even.} \end{cases}$$

For a complete proof of this result, we refer to [63], Theorem 4.2 and Corollary 4.4.

Remark 5.2
We point out that the same results hold for the category $\mathrm{Chow}(k)_F$ for any commutative ring F.

5.2 Voevodsky's nilpotence conjecture, Kimura's conjecture and Schur's finiteness conjecture

First of all we state Voevodsky's nilpotence conjecture. Then, we recall two notions of finiteness for monoidal categories in order to state Schur's finiteness conjecture and Kimura's conjecture.

In [64] Voevodsky conjectured the following statement for the algebraic cycles:

Conjecture 5.2.1 (V)**.** Let X be a smooth projective k-scheme; let $\mathcal{Z}^*_{\otimes_{\mathrm{nil}}}(X)_F$ and $\mathcal{Z}^*_{\otimes_{\mathrm{num}}}(X)_F$ be the ring of algebraic cycles modulo the relation in Example 5.1.2 and in Example 5.1.3, respectively. Then $\mathcal{Z}^*_{\otimes_{\mathrm{nil}}}(X)_F$ coincides with $\mathcal{Z}^*_{\otimes_{\mathrm{num}}}(X)_F$.

Remark 5.3
Conjecture V was proven for curves, surfaces, abelian threefolds, uniruled threefolds,

quadric fibrations, intersection of quadrics, linear sections of Grassmannians, linear sections of determinantal varieties, and some homological projective duals (see [2], [26], [46], [64], [65], [5]).

Let $\mathscr{C}$ be a F-linear pseudoabelian rigid symmetric monoidal category (cf. Appendix § A.3.2).

Definition 5.2.1 (Kimura-finiteness)**.** We say that an object $x \in \mathscr{C}$ is *Kimura-finite* if x decomposes as $x = x^+ \oplus x^-$ with $\wedge^n x^+ = S^n x^- = 0$ for some $n >> 0$. We say that $\mathscr{C}$ is a *Kimura category* if $\mathrm{End}_{\mathscr{C}}(\mathbf{1}) = F$ and every object $x \in \mathscr{C}$ is Kimura-finite.

Conjecture 5.2.2 (K)**.** Let X be a smooth projective k-scheme; then the image $\mathfrak{h}(X)_F \in \mathrm{Chow}(k)_F$ of the functor $\mathfrak{h}$ on X is Kimura-finite.

Remark 5.4
Conjecture K holds for abelian varieties.

Let λ be a partition of a positive integer n. A Schur functor $S_\lambda : \mathscr{C} \to \mathscr{C}$ sends an object $x \in \mathscr{C}$ to a direct component of $x^{\otimes n}$ determined by the tuple λ. For details about the construction we refer to [14]. In particular, if $\lambda = (1, ..., 1)$ and $\lambda = (n)$, the Schur functor S_λ is respectively equal to $\wedge^n$ and S^n.

Definition 5.2.2 (Schur-finite)**.** We say that an object $x \in \mathscr{C}$ is *Schur finite* if it is annihilated by some Schur functor. Moreover, we say that $\mathscr{C}$ is a *Schur category* if $\mathrm{End}(\mathbf{1}) = F$ and all objects $x \in \mathscr{C}$ decompose as direct sum of Schur finite objects.

Remark 5.5
We recall that every Kimura-finite object is Schur-finite, but in general the converse is false. For example, in the category of super-representations of $GL(p|q)$, there exist some Schur-finite objects which are not Kimura-finite (see [31]).

Conjecture 5.2.3 (S)**.** Let X be a smooth projective k-scheme; then the image $\mathfrak{h}(X)_F \in \mathrm{Chow}(k)_F$ of the functor $\mathfrak{h}$ on X is Schur finite.

The next result explains the relation between the conjectures above.

Theorem 5.6. *If* V*(X) holds for every* $X \in \mathrm{SmProj}(k)$ *then:*

$$\mathrm{V} \Rightarrow \mathrm{K} \Rightarrow \mathrm{S}.$$

Proof. [2, Theorem 12.1.6.6.]. □

5.3 Pure motives vs noncommutative motives

The aim of this section is to give a comparison between the theory of pure motives and the theory of noncommutative motives. This section is divided into three parts. In the first we recall some basic properties about the symmetric monoidal category of dg categories.

Then, in the second part, we give the notion of noncommutative Chow motives and we state Voevodsky's nilpotence conjecture in the noncommutative case. Finally, in the last part, we relate such a conjecture with the classical Voevodsky's nilpotence conjecture for pure motives.

Noncommutative Chow motives

We briefly recall the construction of noncommutative Chow motives; for a complete explanation see [59].

Before continuing we remark that, given a dg-category $\mathscr{A}$, we can associate to $\mathscr{A}$ a derived category $\mathcal{D}(\mathscr{A})$ by taking the localisation of the category of right dg-modules over quasi-isomorphisms [30, §3]. Here the category of right dg-modules is the category of dg-functors from $\mathscr{A}^{\mathrm{op}}$ to $\mathcal{C}(k)$ (cf. Chapter 1 §1.3) and the quasi-isomorphisms are the natural transformations inducing an isomorphism in homology.

Let $\mathcal{A}$, $\mathcal{B}$ be two dg-categories, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is the dg-category whose objects are given by $\mathrm{obj}(\mathcal{A}) \times \mathrm{obj}(\mathcal{B})$ and whose morphism spaces is given by

$$(\mathcal{A} \otimes \mathcal{B})((x, y), (x', y')) = \mathcal{A}(x, y) \otimes \mathcal{B}(x', y')$$

with the natural compositions and units.
We note that the tensor product of dg-categories gives rise to a symmetric monoidal structure - $\otimes$ - on dgcat. The $\otimes$-unit is the dg category with one object k. Moreover the tensor product gives rise to a symmetric monoidal structure on $\mathrm{Hmo}(k)$, where $\mathrm{Hmo}(k)$ denotes the localisation of dgcat with respect to the class of Morita equivalences (cf. Chapter 3).
We fix two dg categories $\mathscr{A}$ and $\mathscr{B}$. We denote by $\mathrm{rep}(\mathscr{A}, \mathscr{B})$ the full triangulated subcategory of $\mathcal{D}(\mathscr{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathscr{B})$ consisting of the bimodules $M : (\mathscr{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathscr{B})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{dg}}(k)$ such that $M(\text{-}, x) : \mathscr{B}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{dg}}(k)$ is compact in $\mathcal{D}(\mathscr{B})$, for every x in $\mathscr{A}$.
We recall that $\mathscr{A} \otimes^{\mathbb{L}} \mathscr{B}$ is the tensor product $\mathscr{A}_{\mathrm{cof}} \otimes \mathscr{B}$, where $\mathscr{A}_{\mathrm{cof}}$ denotes the cofibrant replacement of $\mathscr{A}$ (cf. Example 4.4.3). We have the following bijection

$$(\dagger) \qquad \begin{aligned} \mathrm{Iso}[\mathrm{rep}(\mathscr{A}, \mathscr{B})] &\to \mathrm{Hom}_{\mathrm{Hmo}}(\mathscr{A}, \mathscr{B}) \\ M &\mapsto \text{-} \otimes^{\mathbb{L}}_{\mathscr{A}} M, \end{aligned}$$

where Iso denotes the set of isomorphism classes of objects in $\mathrm{rep}(\mathscr{A}, \mathscr{B})$. Moreover, using the bijection (†), we have an induced composition law in $\mathrm{Hmo}(k)$.
Now we define the category $\mathrm{Hmo}(k)_0$ to be the category with the same objects as $\mathrm{Hmo}(k)$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathrm{Hmo}(k)_0}(\mathscr{A}, \mathscr{B}) := K_0(\mathrm{rep}(\mathscr{A}, \mathscr{B})).$$

Here $K_0(\mathrm{rep}(\mathscr{A}, \mathscr{B}))$ denotes the Grothendieck group of the triangulated subcategory $\mathrm{rep}(\mathscr{A}, \mathscr{B})$. As before, the composition law in $\mathrm{Hmo}(k)_0$ is the one induced by the bijection (†).

We recall that the derived tensor product on $\mathrm{Hmo}(k)$ gives rise to a symmetric monoidal structure on $\mathrm{Hmo}(k)_0$.
We have a sequence of symmetric monoidal functors:

$$U : \mathrm{dgcat} \to \mathrm{Hmo}(k) \to \mathrm{Hmo}(k)_0.$$

Finally, we denote by $\mathrm{Hmo}(k)_0^{\mathrm{sp}}$ the full subcategory of the smooth and proper dg categories in $\mathrm{Hmo}(k)$. We recall that a dg category $\mathscr{A}$ is smooth if the associated bimodule

$$\begin{aligned} {}_{\mathrm{Id}}\mathcal{A} : \mathcal{A} \otimes^{\mathbb{L}} \mathcal{A}^{\mathrm{op}} &\to \mathcal{C}_{\mathrm{dg}}(k) \\ (x, y) &\mapsto \mathcal{A}(y, x). \end{aligned}$$

is compact in $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathcal{A})$. Moreover, $\mathcal{A}$ is proper if, for every couple of objects $x, y \in \mathscr{A}$, the complex of k-modules $\mathscr{A}(x, y)$ is compact in the derived category $\mathcal{D}(k)$. It is well known, that if $\mathscr{A}$ is a smooth and proper dg category, then we have an equivalence of triangulated categories

$$(\ddagger) \qquad \mathrm{rep}(\mathscr{A}, \mathscr{B}) \simeq \mathcal{D}_{\mathrm{C}}(\mathscr{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathscr{B}),$$

where $\mathcal{D}_{\mathrm{C}}(\mathscr{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathscr{B})$ denotes the subcategory of compact objects in $\mathcal{D}(\mathscr{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathscr{B})$.
Using the equivalence (‡), we can describe the morphisms of $\mathrm{Hmo}(k)_0^{\mathrm{sp}}$ as

$$\mathrm{Hom}_{\mathrm{Hmo}(k)_0^{\mathrm{sp}}}(\mathscr{A}, \mathscr{B}) \simeq K_0(\mathcal{D}_{\mathrm{C}}(\mathscr{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathscr{B})) \simeq K_0(\mathscr{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathscr{B}).$$

Now we are ready to define the rigid symmetric monoidal category of noncommutative Chow motives.

Definition 5.3.1. We define the category of *noncommutative Chow motives* to be the pseudoabelian envelope[1] of $\mathrm{Hom}_{\mathrm{Hmo}(k)_0^{\mathrm{sp}}}$. We denote such a category by $\mathrm{NChow}(k)$.

Remark 5.7
We note that the functor U extends naturally to $\mathrm{NChow}(k)$.

Remark 5.8
Let X be a smooth projective k-variety. We know that the category of perfect complexes $\mathrm{perf}(X)$ has a unique dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$ (cf. [41] or [10]), which is smooth and proper as a dg category.
Moreover, suppose that the derived category of perfect complexes on X has a semiorthogonal decomposition of the form $\mathrm{perf}(X) = \langle \mathscr{A}_1, ..., \mathscr{A}_n \rangle$ (cf. [34]). Then, by [57], we have that every dg category $\mathscr{A}_i^{\mathrm{dg}}$ is smooth and proper (where $\mathscr{A}_i^{\mathrm{dg}}$ denotes the dg enhancement of the subcategory $\mathscr{A}_i$ induced from $\mathrm{perf}_{\mathrm{dg}}(X)$) and $U(\mathrm{perf}_{\mathrm{dg}}(X)) = \mathscr{A}_1^{\mathrm{dg}} \oplus ... \oplus \mathscr{A}_l^{\mathrm{dg}}$.

Remark 5.9
Given a commutative ring F, we can define the category $\mathrm{NChow}(k)_F$ taking the F-linearization $K_0(\mathscr{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathscr{B})_F$.

[1] further details in Appendix §A.3

Voevodsky conjecture in the noncommutative case

Let $\mathscr{A}$ be a smooth and proper dg category. We denote by $K_0(\mathscr{A})$ the Grothendieck group $K_0(\mathcal{D}_{\mathrm{C}}(\mathscr{A}))$. In analogy to algebraic cycles we can define some equivalence relations on $K_0(\mathscr{A})$. We give two examples.

Example 5.3.1 ($\otimes$-nilpotence equivalence relation)**.** We say that an element $[M]$ in $K_0(\mathscr{A})$ is $\otimes$-nilpotent if there exists a positive integer n such that $[M \times n] = 0$ in the Grothendieck group $K_0(\mathscr{A}^{\otimes n})$. Given $[M]$ and $[N]$ in $K_0(\mathscr{A})$ we say that $[M]$ and $[N]$ are $\otimes$-nilpotent equivalent (shortly $[M]\sim_{\otimes_{\mathrm{nil}}}[N]$) if $[M]$-$[N]$ is $\otimes$-nilpotent.

We have a bilinear form $\chi(-,-)$ on $K_0(\mathscr{A})$ defined as

$$(M,N) \to \sum_i (-1)^i \dim \mathrm{Hom}_{\mathcal{D}_{\mathrm{C}}(\mathscr{A})}(M, N[i]).$$

The left and right kernels of $\chi(-,-)$ are the same [59, Prop. 4.24.].

Example 5.3.2 (Numerical equivalence relation)**.** We say that an element $[M]$ in $K_0(\mathscr{A})$ is numerically trivial if $\chi([M],[N]) = 0$ for all $[N] \in K_0(\mathscr{A})$. We say that $[M]$ and $[N]$ are numerically trivial equivalent (shortly $[M]\sim_{\otimes_{\mathrm{num}}}[N]$) if $[M]$-$[N]$ is numerically trivial.

Remark 5.10
If $\mathrm{char}(F) = 0$, the equivalence relations defined above give rise to well defined equivalence relations on $K_0(\mathscr{A})_F$.

In [5] Bernardara, Marcolli and Tabuada conjectured the following statement:

Conjecture 5.3.1 ($\mathrm{V_{nc}}$)**.** Let $\mathscr{A}$ be a smooth proper dg category. Then $\mathrm{K}_0(\mathscr{A})/_{\sim_{\otimes_{\mathrm{nil}}}}$ is equal to $\mathrm{K}_0(\mathscr{A})/_{\sim_{\otimes_{\mathrm{num}}}}$.

Pure motives vs noncommutative motives

Given a smooth projective k-scheme X, we know that the category of perfect complexes $\mathrm{perf}(X)$ has a unique dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$ (cf. [41] or [10]), which is smooth and proper as a dg category. Then, by [5], we have that every dg category $\mathscr{A}_i^{\mathrm{dg}}$ is smooth and proper (where $\mathscr{A}_i^{\mathrm{dg}}$ denotes the dg enhancement of the subcategory $\mathscr{A}_i$ induced from $\mathrm{perf}_{\mathrm{dg}}(X)$).

We recall that, given a smooth projective k-scheme, we have a relation between the category of Chow motives and the category of noncommutative Chow motives. In particular, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{SmProj}(k)^{\mathrm{op}} & \xrightarrow{\ \mathrm{perf}_{\mathrm{dg}}\ } & \mathrm{dgcat}(k) \\ \Big\downarrow{\scriptstyle \mathfrak{h}} & & \Big\downarrow{\scriptstyle U} \\ \mathrm{Chow}(k) & \xrightarrow[\ \Phi\cdot\eta\ ]{} & \mathrm{NChow}(k) \end{array}$$

where Φ is fully faithful and η is the functor from $\mathrm{Chow}(k)$ to the orbit category $\mathrm{Chow}(k)/_{-\otimes \mathbb{1}(1)}$ (cf. Appendix § A 3.3). Moreover, we have the following result which relates the Voevodsky's nilpotence conjecture and noncommutative motives:

Theorem (BMT)**.** *Let X be a smooth projective k-scheme. The conjecture $V(X)$ holds if and only if the conjecture* $\mathrm{V}_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$ *holds.*

Chapter 6

Voevodsky's conjecture for cubic fourfolds and Gushel-Mukai fourfolds

In this chapter we prove Voevodsky's conjecture for cubic fourfolds and ordinary generic Gushel-Mukai fourfolds. We point out that to prove this conjecture, we use the decomposition in rational Chow motives of a flat morphism computed by Vial in [63] that we recalled in §5.1.3. Then, making use of noncommutative motives, we prove the noncommutative version of Voevodsky's nilpotence conjecture for the Kuznetsov category of a cubic fourfold and an ordinary generic Gushel-Mukai fourfold. Indeed, we recall from [33] that the derived category of a cubic fourfold X has a semiorthogonal decomposition of the form $\mathcal{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}, \mathcal{O}(H), \mathcal{O}(2H)\rangle$, where $\mathcal{A}_X$ is a noncommutative K3 surface in the sense of Kontsevich. As second result we give a proof of conjecture V_{nc} for $\mathcal{A}_X^{\mathrm{dg}}$, where $\mathcal{A}_X^{\mathrm{dg}}$ denoted the dg enhancement of the category $\mathcal{A}_X$ induced from $\mathrm{perf}_{\mathrm{dg}}(X)$. As an application of this result we prove Voevodsky's conjecture for generic Gushel-Mukai fourfolds containing a τ-plane. We believe that this result provides a new tool for the proof of Voevodsky's conjecture of a smooth projective k-scheme whose derived category of perfect complexes contains the noncommutative K3 surface $\mathcal{A}_X$.

6.1 Cubic fourfolds and Gushel-Mukai varieties

In this section we briefly recall some facts about cubic fourfolds and Gushel-Mukai varieties. We are interested in finding a quadric fibration from the blow-up of a cubic fourfold (resp. a GM fourfold) over a line (resp. a surface) to the projective space $\mathbb{P}^3$. Throughout this chapter we will assume that all cubic fourfolds and Gushel-Mukai varieties are smooth. From now on the field k will be the field of complex numbers $\mathbb{C}$.

Cubic fourfolds

Definition 6.1.1 (Cubic fourfold)**.** A *cubic fourfold* is a smooth complex hypersurface of degree 3 in $\mathbb{P}^5$.

We observe that a cubic fourfold X contains (at least) a line l. Indeed, let $F(X)$ be the Fano variety of lines associated to X, which parametrizes the lines of $\mathbb{P}^5$ contained in X. By [17], [1], [4], we know that $F(X)$ is a smooth and connected projective variety of dimension 4. In particular, X contains certainly (at least) a line. We denote by l a line in X and by $\mathrm{Bl}_l(X)$ the blow-up of X in l. The aim of this paragraph is to recall the well known fact that the projection from the line l induces a flat and smooth quadric fibration from $\mathrm{Bl}_l(X)$.

Lemma 6.1. *Let X be a smooth cubic fourfold and let l be a line in X. Then the linear projection from the line l induces a smooth flat quadric fibration from the blow-up $\mathrm{Bl}_l(X)$ to $\mathbb{P}^3$.*

Proof. We can take a six-dimensional vector space V_6 such that $X \subset \mathbb{P}(V_6) \cong \mathbb{P}^5$. Moreover we can denote by V_2 a two-dimensional subvector space of V_6 such that $l = \mathbb{P}(V_2) \cong \mathbb{P}^1$ and we set $V_4 := V_6/V_2$. Let us denote by $\mathrm{Bl}_l(\mathbb{P}(V_6))$ the blow-up of $\mathbb{P}(V_6)$ over the line l. We have that the projection from the line l defines a regular map $\pi : \mathrm{Bl}_l(\mathbb{P}(V_6)) \to \mathbb{P}(V_4) \cong \mathbb{P}^3$. Let $\pi_l : \mathrm{Bl}_l(X) \to X$ be the blow-up of X along l. So the restriction of the projection π to $\mathrm{Bl}_l(X)$ induces a smooth flat conic fibration $f : \mathrm{Bl}_l(X) \to \mathbb{P}(V_4)$. To summarise, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Bl}_l(X) & \hookrightarrow & \mathrm{Bl}_l(\mathbb{P}(V_6)) \\ \downarrow{\scriptstyle \pi_l} & \searrow^{f} & \downarrow{\scriptstyle \pi} \\ X & & \mathbb{P}(V_4). \end{array}$$

□

Gushel-Mukai varieties

Let V_5 be a k-vector space of dimension 5; considering the Plücker embedding, we have that $\mathrm{Cone}(\mathrm{Gr}(2, V_5)) \subset \mathbb{P}(k \oplus \wedge^2 V_5)$. We denote by W a linear subspace of dimension $n+5$ of $\wedge^2 V_5 \oplus k$ (with $2 \leq n \leq 6$).

Definition 6.1.2 (Gushel-Mukai n-fold)**.** We define a *Gushel-Mukai n-fold* X to be a smooth and transverse intersection of the form

$$X = \mathrm{Cone}(\mathrm{Gr}(2, V_5)) \cap \mathrm{Q},$$

where Q is a quadric hypersurface in $\mathbb{P}(W)$.
We say that X is:

- *Ordinary* if X is isomorphic to a linear section of $\mathrm{Gr}(2, V_5) \subset \mathbb{P}^9$,
- *Special* if X is isomorphic to a double cover of a linear section of $\mathrm{Gr}(2, V_5)$ branched along a quadric section.

From now on, we will write GM instead of Gushel-Mukai.

Let X be a GM n-fold. We note that, if we suppose that X is smooth, then X does not contain the vertex of the cone over $\mathrm{Gr}(2, V_5)$. Thus, we have a regular map defined by the projection from the vertex:

$$\gamma_X : X \to \mathrm{Gr}(2,V_5).$$

Definition 6.1.3 (Gushel bundle)**.** Let $\mathscr{U}$ be the tautological bundle of rank 2 over $\mathrm{Gr}(2, V_5)$. We define the *Gushel bundle* to be the pullback $\mathscr{U}_X := \gamma_X^* \mathscr{U}$.

Denoting by $\pi : \mathbb{P}_X(\mathscr{U}_X) \to X$ the projectivization of the bundle $\mathscr{U}_X$, we can consider the map

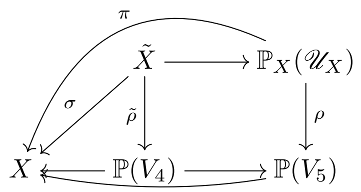
$$\rho : \mathbb{P}_X(\mathscr{U}_X) \to \mathbb{P}(V_5)$$

induced by the embedding $\mathscr{U}_X \hookrightarrow V_5 \otimes \mathcal{O}_X$. Moreover, by [13, Proposition 4.5.], we have that ρ is a quadric fibration.
Now, we suppose that X is an ordinary GM fourfold. By [13], Remark 3.15 and Remark B.4, the fibers of ρ are all conics in $\mathbb{P}^2$ except for the fiber over a point v_0 in $\mathbb{P}(V_5)$, which is a 2-dimensional quadric in $\mathbb{P}^3$. We fix a four-dimensional subvector space V_4 of V_5 such that the point v_0 is not contained in $\mathbb{P}(V_4)$. We set

$$\tilde{X} := \mathbb{P}_X(\mathscr{U}_X) \times_{\mathbb{P}(V_5)} \mathbb{P}(V_4)$$

and we denote by $\tilde{\rho}$ the restriction of ρ to $\tilde{X}$. Thus, we have the following commutative diagram

(6.1)



So, the restriction $\tilde{\rho}$ is a flat conic fibration over $\mathbb{P}(V_4) \cong \mathbb{P}^3$.

The rest of the section is devoted to prove that, when X is generic, $\tilde{\rho}$ is smooth (Lemma 6.2) and $\tilde{X}$ is the blow-up of X over a surface E (Proposition 6.3).

We begin defining E. We note that for every x in X, the fiber of σ over x is equal to $\mathbb{P}(\mathscr{U}_{X,x} \cap V_4)$. In particular, we have that $\sigma^{-1}(x)$ is a point (resp. a line) if the dimension of $\mathscr{U}_{X,x} \cap V_4$ is equal to 1 (resp. if $\mathscr{U}_{X,x} \subset V_4$). It follows that the locus of non trivial fibers of σ is the intersection

$$E := \mathrm{Gr}(2, V_4) \cap X = \mathrm{Gr}(2, V_4) \cap \mathbb{P}(W) \cap \mathrm{Q} \subset \mathbb{P}(\bigwedge^2 V_5) \cong \mathbb{P}^9. \tag{6.2}$$

Since the Grassmannian $\mathrm{Gr}(2, V_4)$ has degree 2, we have that the degree of E is at most 4. Moreover, the expected dimension of E is 2. On the other hand, by Lefschetz theorem the fourfold X cannot contain a divisor with degree less than 10, because its class has to be cohomologous to the class of a hyperplane in X. Thus, we conclude that $\dim(E) \leq 2$.

In the next lemma, we show that E is smooth under generality assumptions on $\mathbb{P}(W)$ and Q; in this case, E is a del Pezzo surface of degree 4.

Lemma 6.2. *If W is a generic vector space of dimension 9 in $\bigwedge^2 V_5$ and Q is a generic quadric hypersurface in the linear system $|\mathcal{O}_{\mathbb{P}(W)}(2)|$, then E defined in (6.2) is a smooth and irreducible surface.*

Proof. We consider the intersection $Y := \mathbb{P}(W) \cap \mathrm{Gr}(2, V_4) \subset \mathbb{P}(\bigwedge^2 V_5) \cong \mathbb{P}^9$. By Bertini's theorem on hyperplane sections (see [21]), we have that Y is smooth and irreducible, because $\mathbb{P}(W)$ is a generic hyperplane in $\mathbb{P}^9$.
Let $i : Y \hookrightarrow \mathbb{P}^8$ be the embedding of Y in $\mathbb{P}(W) \cong \mathbb{P}^8$. We note that if Y is contained in the quadric Q, then $Y = E$ would be a smooth divisor in X with degree less than 10, in contradiction with the previous observation. Hence, we have that the quadric Q does not contain Y. Again by Bertini's Theorem, the intersection $Y \cap \mathrm{Q} = E$ is smooth and irreducible. Indeed, we can consider the embedding of $\mathbb{P}^8$ in $\mathbb{P}(H^0(\mathbb{P}^8, \mathcal{O}(2))) \cong \mathbb{P}^N$ defined by $\mathcal{O}(2)$. The quadric hypersurfaces in $\mathbb{P}^8$ correspond to hyperplanes in $\mathbb{P}^N$ via this embedding. Thus, by Bertini's Theorem for hyperplane sections, we conclude that the intersection of the image of Y with the generic hyperplane in $\mathbb{P}^N$, corresponding to the generic quadric Q, is smooth and irreducible. So, we conclude that E is smooth and irreducible of dimension 2. □

As a consequence, we obtain the smoothness of the restriction to a hyperplane of the conic fibration ρ.

Proposition 6.3. *Let X be an ordinary generic GM fourfold. Then $\tilde{X}$ is the blow-up of X in E (so it is smooth) and the map $\tilde{\rho} : \tilde{X} \to \mathbb{P}(V_4)$ defined in (6.1) is a flat conic fibration.*

Proof. We observe that the quadric Q which defines X is generic in the linear system $|\mathcal{O}_{\mathbb{P}(W)}(2)|$, because X is a generic quadric section of the intersection $\mathbb{P}(W) \cap \mathrm{Gr}(2, V_5)$. On the other hand, we recall that, by [13, Lemma 2.7], there exists a functor between the groupoid of polarized GM varieties to the groupoid of GM data, which is an equivalence by [13, Theorem 2.9.]. In particular, a generic X corresponds to a generic GM data $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$. Thus, the vector spaces and the linear maps which define this GM data are generic and, then, W is a generic subvector space in $\bigwedge^2 V_5$. By Lemma 6.2, we have that the locus E defined by (6.2) is smooth and irreducible. We note that $\sigma^{-1}(E)$ is by definition the projective bundle $\mathbb{P}_E(\mathcal{U}_X) \to E$. On the other hand, the exceptional divisor of the blow-up of X in E is isomorphic to the projectivized conormal bundle $\mathbb{P}_E(\mathcal{N}^*_{E|X})$. Since E can be represented as the zero locus of a regular section of $\mathscr{U}_X^*$, the conormal bundle of E in X is isomorphic to $\mathscr{U}_X$. So, we deduce that $\tilde{X}$ is the blow-up of X in E. It follows that $\tilde{X}$ is smooth and $\tilde{\rho} : \tilde{X} \to \mathbb{P}(V_4)$ is a flat conic fibration. □

6.2 Voevodsky's nilpotence conjecture for cubic fourfolds and generic GM fourfolds

This part is devoted to prove Voevodsky's nilpotence conjecture for cubic fourfolds and ordinary generic GM fourfolds.

Theorem 6.4. *Let X be a cubic fourfold or an ordinary generic GM fourfold. Then the conjecture* $\mathrm{V}(X)$ *holds.*

Proof. Let X be a cubic fourfold and we consider the blow-up of X along a line l. By Subsection 5.1 and Lemma 6.1, the Chow motive of the blow-up decomposes as

$$\mathfrak{h}(\mathrm{Bl}_l(X)) \simeq \bigoplus_{k=0}^{1} \mathfrak{h}(\mathbb{P}^3)(k) \oplus (Z, r, 1) \simeq \bigoplus_{k=0}^{1} (\bigoplus_{i=0}^{3} \mathbb{1}(-i))(k) \oplus (Z, r, 1),$$

where $r \in \mathrm{End}(\mathfrak{h}(Z))$ and $\dim Z = \dim \mathbb{P}^3 - 1 = 2$. It means that conjecture V holds for $\mathrm{Bl}_l(X)$. By Subsection 5.1 we conclude that conjecture V holds for X.
If X is an ordinary generic GM fourfold, the same strategy applied to the conic fibration of Proposition 6.3 gives the required statement. □

6.3 Noncommutative Voevodsky's nilpotence conjecture for the Kuznetsov category

In this section we recall some facts about the decomposition of the derived category of a cubic fourfold X. In particular, we remark some properties about the Kuznetsov category $\mathcal{A}_X$ associated to X. Then we prove Voevodsky's nilpotence conjecture for the Kuznetsov category of a cubic fourfold.

Kuznetsov category

Let X be a cubic fourfold. The derived category of perfect complexes $\mathrm{perf}(X)$ admits a semiorthogonal decomposition given by

$$(\star) \qquad \mathrm{perf}(X) = \langle \mathcal{A}_X, \mathcal{O}, \mathcal{O}(H), \mathcal{O}(2H) \rangle,$$

where H is a hyperplane section and $\mathcal{A}_X$ is defined as:

$$\begin{aligned} \mathcal{A}_X &= \langle \mathcal{O}, \mathcal{O}(H), \mathcal{O}(2H) \rangle^{\perp} \\ &= \{E \in \mathrm{perf}(X) \text{ s.t. } \mathbb{R}\mathrm{Hom}_{\mathrm{perf}(X)}(\mathcal{O}_X(i), E) = 0 \text{ for } i = 0, 1, 2\} \end{aligned}$$

We call $\mathcal{A}_X$ the Kuznetsov category.
We recall that the admissible subcategory $\mathcal{A}_X$ is a Calabi-Yau category of dimension 2; indeed, the Serre functor is equal to the shift by 2, i.e. for every pair of objects F, E we have

$$\mathbb{R}\mathrm{Hom}_{\mathcal{A}_X}(E, F)^* \simeq \mathbb{R}\mathrm{Hom}_{\mathcal{A}_X}(F, E)[2].$$

Moreover, $\mathcal{A}_X$ has the same sized Hochschild (co)homology of the derived category of a K3 surface. Thus, the Kuznetsov category is a noncommutative K3 surface in the sense of Kontsevich (see [33], [36, Corollary 4.3.] and [34, Proposition 4.1.]).

Remark 6.5
We recall that if X is a cubic fourfold containing a plane, we can prove V-conjecture via noncommutative motives. In fact, if X contains a plane, we have that $\mathcal{A}_X$ is equivalent to $\mathcal{D}^b(S, \mathcal{B})$, where S is a K3 surface, $\mathcal{B}$ is a sheaf of Azumaya algebras on S and $\mathcal{D}^b(S, \mathcal{B})$ is the derived category of coherent $\mathcal{B}$-modules on S (see [33, Theorem 4.3.]). Then by [60] we have the following decomposition in NChow(k):

$$U(\mathrm{perf}_{\mathrm{dg}}(X)) \simeq U(\mathrm{perf}_{\mathrm{dg}}(S)) \oplus U(\mathbb{C}) \oplus U(\mathbb{C}) \oplus U(\mathbb{C}).$$

Since V(S) holds, we conclude that also $V(X)$ holds, as we claimed.

Similarly, let X be a GM n-fold; in [35, Proposition 4.2.], it is proved that its derived category of perfect complexes has a semiorthogonal decomposition of the form

$$(*) \quad \mathrm{perf}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathscr{U}_X^\vee, \mathcal{O}_X(H), \mathscr{U}_X^\vee(H), ..., \mathcal{O}_X((n-3)H), \mathscr{U}_X^\vee((n-3)H) \rangle,$$

where $\mathscr{U}_X^\vee$ is the dual of the Gushel bundle previously defined and $\mathcal{A}_X$ is defined as:

$$\mathcal{A}_X = \langle \mathcal{O}_X, \mathscr{U}_X^\vee, \mathcal{O}_X(H), \mathscr{U}_X^\vee(H), ..., \mathcal{O}_X((n-3)H), \mathscr{U}_X^\vee((n-3)H) \rangle^\perp.$$

Again we call $\mathcal{A}_X$ the Kuznetsov category of X.
Assume that X is a GM fourfold. Then, the Serre functor on $\mathcal{A}_X$ is the shift by two and the Hochschild cohomology of $\mathcal{A}_X$ is isomorphic to that of a K3 surface. As before the component $\mathcal{A}_X$ of a GM fourfold is a noncommutative K3 surface in the sense of Kontsevich (see [35, Proposition 5.18]).

Noncommutative Voevodsky's conjecture for the Kuznetsov category

Theorem 6.6. *Let X be a cubic fourfold or an ordinary generic GM fourfold. Then* $\mathrm{V}_{\mathrm{nc}}(U(\mathcal{A}_X^{\mathrm{dg}}))$ *holds, where $\mathcal{A}_X^{dg}$ is the dg enhancement of $\mathcal{A}_X$ induced from* $\mathrm{perf}_{\mathrm{dg}}(X)$.

Proof. First of all, we suppose that X is a cubic fourfold. Using the decomposition $\star$, we have that the dg enhancement of the triangulated category perf(X) admits the following decomposition in NChow(k):

$$U(\mathrm{perf}_{\mathrm{dg}}(X)) = U(\mathcal{A}_X^{\mathrm{dg}}) \oplus U(\mathbb{C}) \oplus U(\mathbb{C}) \oplus U(\mathbb{C}).$$

Hence, the result is a straightforward consequence of Theorem 6.4. The proof in the case of an ordinary generic GM fourfold X is analogous, applying the decomposition $*$ and Theorem 6.4. □

Remark 6.7
We point out that Theorem 6.6 holds for every cubic fourfold X even if it does not contain a plane.

6.4 Voevodsky's nilpotence conjecture for GM fourfolds containing surfaces

In this section we will prove Voevodsky's nilpotence conjecture for generic GM fourfolds containing a τ-plane and for ordinary GM fourfolds containing a quintic del Pezzo surface. The proof of the conjecture in the first case provide an application of Theorem 6.6.
Let X be a GM fourfold containing a τ-plane P, i.e. a plane P of the form $\mathrm{Gr}(2, V_3)$ for some 3-dimensional subvector space V_3 of V_5. In [35, Lemma 7.8], they proved that there exists a cubic fourfold X' containing a smooth cubic surface scroll T such that the blow-up of X in P is identified to the blow-up of X' in T. More precisely, if $p : \tilde{X} \to X$ is the blow-up of X along P and q is the regular map induced by the linear projection from P, then the diagram

$$\begin{array}{ccc} & \tilde{X} & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & & X' \end{array} \tag{6.3}$$

commutes and q is identified with the blow-up of X' along T. Moreover, they showed that, if the GM fourfold X does not contain a plane of the form $\mathbb{P}(V_1 \wedge V_4)$ for some subvectorspaces satisfying $V_1 \subset V_3 \subset V_4 \subset V_5$, then the cubic fourfold X' is smooth. We point out that this construction had already been described in [12].
They also observed that a generic GM fourfold containing a τ-plane does not contain a plane of the form $\mathbb{P}(V_1 \wedge V_4)$ as above; hence, the associated cubic fourfold X' obtained with this geometric construction is smooth. In this case, they proved that there exists an equivalence of Fourier-Mukai type

$$\phi : \mathcal{A}_X \simeq \mathcal{A}_{X'} \tag{6.4}$$

between the Kuznetsov category of X and the Kuznetsov category of X' (see [35, Theorem 1.3]).

Using this construction and Theorem 6.6, we can prove the Voevodsky's nilpotence conjecture for this class of GM fourfolds.

Theorem 6.8. *Let X be a generic GM fourfold containing a plane P of type* $\mathrm{Gr}(2, 3)$*. Then* $\mathrm{V_{nc}}(\mathrm{perf_{dg}}(X))$ *holds.*

Proof. The derived category of perfect complexes of X has the following decomposition:

$$\mathrm{perf}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{U}_X^{\vee}, \mathcal{O}_X(H), \mathcal{U}_X^{\vee}(H) \rangle.$$

Since the functor ϕ defined in (6.4) is of Fourier-Mukai type, we know that ϕ has a dg lift, thanks to the works of [42], [54] and [61]. Then the proof is a consequence of Theorem 6.6. □

Remark 6.9
Alternatively, we can prove Theorem 4.19 by observing that the isomorphism of triangulated categories $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ is induced by diagram (6.3). Then, conjecture $\mathrm{V_{nc}}(\mathrm{perf_{dg}}(X))$ follows from subsection 5.1 and Theorem 6.4.

In a similar way, we can prove conjecture V_{nc} for the category of perfect complexes of ordinary GM fourfolds containing a quintic del Pezzo surface.

Theorem 6.10. *Let X be an ordinary GM fourfold containing a quintic del Pezzo surface. Then $\mathrm{V}_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$ holds.*

Proof. By [35], Theorem 1.2 we have that there exist a K3 surface Y and an equivalence $\psi : \mathcal{A}_X \simeq \mathcal{D}^b(Y)$ of Fourier-Mukai type. Since ψ has a dg lift and conjecture V holds for Y, the proof follows from Theorem 5.3. □

Corollary 6.11. *Let X be a generic GM fourfold containing a plane P of type $\mathrm{Gr}(2,3)$ or an ordinary GM fourfold containing a quintic del Pezzo surface. Then $\mathrm{V}(X)$ holds.*

Proof. The proof is a consequence of Theorem 5.3. □

Appendix A

Tools of category theory

This appendix is devoted to recall some notions we implicitly assumed well known by the reader.

A.1 Linear algebra

Let $\mathbb{K}$ be a commutative ring.

Definition A.1.1 ($\mathbb{K}$-algebra)**.** We define a $\mathbb{K}$-*algebra* to be a $\mathbb{K}$-module endowed with: a $\mathbb{K}$-liner associative multiplication $A \otimes_{\mathbb{K}} A \to A$, and a unit 1_A such that, for all $x \in A$, $1 \otimes_{\mathbb{K}} x = x \otimes_k 1 = x$.

Definition A.1.2 ($\mathbb{K}$-linear category)**.** We define a $\mathbb{K}$-*linear category* $\mathcal{A}$ to be a category such that, for every $x, y \in \mathcal{A}$, the set of morphisms $\mathcal{A}(x, y)$ is a $\mathbb{K}$-module and, for every x, y an $z \in \mathcal{A}$, there exists a $\mathbb{K}$-linear associative map:

$$\mathcal{A}(y, z) \otimes_{\mathbb{K}} \mathcal{A}(x, y) \to \mathcal{A}(x, z)$$

that is the usual composition in category theory.

Example A.1.1. An additive category is a $\mathbb{Z}$-linear category.

Remark A.1
We recall that the tensor product of two graded maps f and g, of grade p and q respectively, between graded $\mathbb{K}$-modules, is defined using the Koszul sign rule i.e.

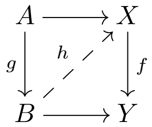
$$(f \otimes g)(x \otimes y) = (-1)^{pq} f(x) \otimes g(y).$$

Definition A.1.3 (Tensor product of modules)**.** Let f and g be two dg-modules we define the *tensor product* $f \otimes g$ to be the graded $\mathbb{K}$-module $(f \otimes g)^n = \bigoplus_{p+q=n} f^p \otimes_{\mathbb{K}} g^q$, $\forall n \in \mathbb{Z}$, with differential $d^n = d_V \otimes 1 + 1 \otimes d_W$[1].

[1] $d \otimes_{\mathbb{K}} 1$ is a tensor product of graded maps (Koszul sign rule)

A.2 Model categories

Definition A.2.1 (RLP)**.** Let $\mathscr{C}$ be a category. We say that a morphism $f : X \to Y$ has the right lifting property, respect $g : A \to B$ (RLP) if, for every commutative diagram



there exists a unique lifting h that makes the diagram commutative.

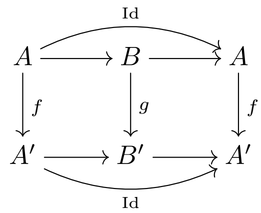
Dually we say that g has the left lifting property respect f (LLP).

Notations. $\hookrightarrow$ denote a cofibration, $\twoheadrightarrow$ denote a fibration, $\xrightarrow{\sim}$ denote a weak equivalence, $\overset{\sim}{\hookrightarrow}$ denote a weak cofibration and $\overset{\sim}{\twoheadrightarrow}$ denote a weak fibration.

Definition A.2.2 (Model category)**.** A model category $\mathscr{M}$ is a category with three classes of morphisms $\mathscr{C}$, $\mathscr{F}$, $\mathscr{W}$, called respectively cofibrations, fibrations and weak equivalences. We call weak fibrations the maps in $\mathscr{W} \cap \mathscr{F}$ and weak cofibrations the maps in $\mathscr{W} \cap \mathscr{C}$. The category $\mathscr{M}$ have to satisfy the following axioms:

(M1) $\mathscr{M}$ is complete and cocomplete.

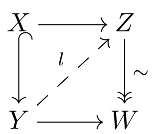
(M2) If g is a fibration (cofibration or weak equivalence) and f is *retracts* of g i.e. if we have the following commutative diagram:

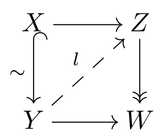


then f is in the same class of g.

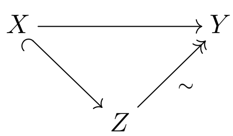
(M3) The classes, defined above, are closed by composition of maps.

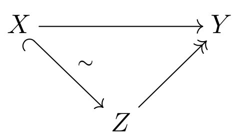
(M4) Every couple, cofibration and weak fibration (or weak cofibration and fibration), has RLP, i.e. there exist the lifting map l for the commutative diagram below:





(M5) Every map is composition of a cofibration and a weak fibrations and weak cofibration and fibration i.e.





Definition A.2.3 (Model structure)**.** We says that a, complete and cocomplete, category $\mathscr{C}$ has a model structure if there exist three distinguished classes of morphisms (fibrations, cofibrations and weak equivalences) satisfying the axioms above.

Remark A.2
If $\mathscr{C}$ is a model category then $\mathscr{C}^{\mathrm{op}}$ is a model category, cf. [23, Rem. 1.1.7].

From now on, let $\mathscr{M}$ be a model category and X an object of $\mathscr{M}$.

Definition A.2.4 (Cofibrant object)**.** We say X is *cofibrant* if the morphism $0 \to X$ is a cofibration, where 0 denoted the initial object in $\mathscr{M}$.

Definition A.2.5 (Cofibrant resolution)**.** We say that X_{cof} is a *cofibration resolution* of X, if X_{cof} is a cofibrant object weak equivalent to X, in formula:

$$0 \hookrightarrow X_{\mathrm{cof}} \xrightarrow{\sim} X$$

Definition A.2.6 (Fibrant object)**.** We say that X is *fibrant* if the morphism $X \to 1$ is a fibration, where 1 denoted the final object in $\mathscr{M}$.

Definition A.2.7 (Fibrant resolution)**.** We say that X_{fib} is a *Fibrant resolution* of X, if X_{fib} is a fibrant object weak equivalent to X, in formula:

$$X \xrightarrow{\sim} X_{\mathrm{fib}} \twoheadrightarrow 1$$

By axiom (M5) we have that the cofibrant and fibrant resolutions there always exist.

Definition A.2.8 (Right proper model category)**.** We say that $\mathscr{M}$ is right proper, if the pullback of a weak equivalence along a fibration is a weak equivalence.

Dually we have:

Definition A.2.9 (Left proper model category)**.** $\mathscr{M}$ is left proper, if the pushout of a weak equivalence along a cofibration, is a weak equivalence.

We have the following [22, Prop. 13.1.2]:

Proposition A.3. *In every model category.*

(1) Every pushout of a weak equivalence along a cofibration is a weak equivalence.

(2) Every pullback of a weak equivalence along a fibration is a weak equivalence.

Definition A.2.10 (Left Bousfield localization)**.** Let $\mathscr{M}$ be a model structure over $\mathscr{C}$, we define a *left Bousfield localization*, to be a model structure $\mathscr{M}'$ over $\mathscr{C}$, such that $\mathscr{M}'$ has the same cofibrations of $\mathscr{M}$, but weak equivalences of $\mathscr{M}$ are weak equivalences of $\mathscr{M}'$ (i.e. $\mathscr{W}_{\mathscr{M}} \subset \mathscr{W}_{\mathscr{M}'}$.)

Example A.2.1. The category $\mathcal{C}(\mathbb{K})^{\geq 0}$ of upper chain complexes over the abelian category of $\mathbb{K}$-modules has a canonical model structure, such that:

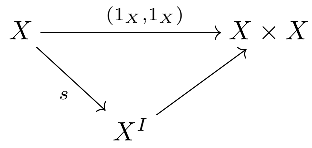
$\mathscr{W}$ The weak equivalences are quasi-isomorphisms.

$\mathscr{F}$ The fibrations are degreewise epimorphisms.

$\mathscr{C}$ Cofibrations are degreewise monomorphisms with degreewise projective cokernel.

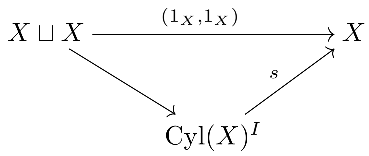
Homotopy category

Definition A.2.11 (Path object)**.** We define the *path object*, of an object X in $\mathscr{M}$, to be the an object X^I fitting in the following commutative diagram:



where s is a weak equivalence.

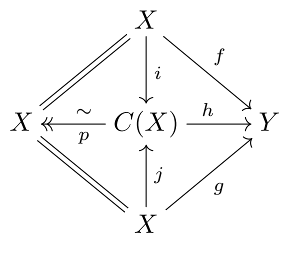
Definition A.2.12 (Cylinder object)**.** We define the *cylinder object* of an object X in $\mathscr{M}$ to be the an object $\mathrm{Cyl}(X)$ fitting in the following commutative diagram:



where s is a weak equivalence.

Definition A.2.13 (Homotopy category)**.** We define the *homotopy category* of $\mathscr{M}$ to be the localization of $\mathscr{M}$, on the weak equivalences $\mathscr{W}$, we denoted such a category by $\mathrm{Ho}(\mathscr{M})$.

Definition A.2.14 (Homotopic morphisms)**.** Two morphisms $f, g \in \mathrm{Hom}_{\mathscr{M}}(X, Y)$ are *homotopic* if there exist h, i, j, $C(X)$ such that f and g are fit in commutative diagram



and the induced morphism $i \sqcup j : X \sqcup X \to C(X)$ is cofibration.

The following theorem gives an explicit description of the homotopy category of a model category.

Theorem A.4. *Let $\mathscr{M}^{cf}/\sim$ be the category of cofibrant and fibrant objects in $\mathscr{M}$, whose morphisms are the homotopy class, then the induced functor:*

$$\mathscr{M}^{cf} \longrightarrow \mathrm{Ho}(\mathscr{M}).$$

is an equivalence of categories.

Quillen adjointness

Given two categories $\mathscr{C}$ and $\mathscr{D}$.

Let F and G functors, such that $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$.

Theorem A.5. *The followings are equivalent:*

A1 There exists a natural isomorphism $\alpha_{X,Y}$ *s.t.*

$$\mathrm{Hom}_{\mathscr{D}}(FX, Y) \simeq \mathrm{Hom}_{\mathscr{C}}(X, GY),$$

for every $X \in \mathscr{C}$ *and* $Y \in \mathscr{D}$.

A2 There exist two natural isomorphisms

$$\xi_Y : FGY \to Y,$$

$$\eta_X : X \to GFX,$$

for every $X \in \mathscr{C}$ *and* $Y \in \mathscr{D}$. *We call* ξ_- *and* η_-, *respectively, unit and counit.*

Proof.

($A1 \Rightarrow A2$) We define the counit ξ_Y as $\alpha^{-1}_{GY,Y}(1_{GY})$ and the unit η_X as $\alpha_{X,FX}(1_{FX})$.

($A1 \Leftarrow A2$) We define the natural isomorphism as $\alpha_{X,Y}(f) := Gf\eta_X$ and the inverse as $\alpha^{-1}_{X,Y}(g) := \xi_Y Fg$.

□

Definition A.2.15. We say that F and G are adjoint functors (in formula, $F \dashv G$ and $F \vdash G$) if F and G satisfies $A1$.

Example A.2.2. Every equivalence $F : \mathscr{C} \to \mathscr{D}$, give rises to a couple of adjoint functors F and F^{-1}, such that $F \dashv F^{-1}$ and $F \vdash F^{-1}$.

Definition A.2.16. Let $\mathscr{M}$ and $\mathscr{M}'$ be two model categories, we say that a couple of adjoint functors

$$F : \mathscr{M} \rightleftarrows \mathscr{M}' : G,$$

is a *Quillen adjunction* if F preserves cofibrations and acyclic cofibrations or equivalently if G preserves fibrations and acyclic fibrations.

Remark A.6
If F and G are Quillen adjoint functors, then they induce an adjunction

$$L(F) : \mathrm{Ho}(\mathscr{M}) \rightleftarrows \mathrm{Ho}(\mathscr{M}') : R(G)$$

between homotopy categories.

A.3 Monoidal categories

Definition A.3.1 (Monoidal category)**.** We define a *monoidal category* $\mathscr{M}$ to be a category equipped with a bifunctor $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ and an object I, satisfying the followings:

- Given three objects X, Y, Z, there exists a natural isomorphism $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$.
- For every $A \in \mathscr{M}$, there exist two natural isomophisms $\lambda_A : I \otimes A \to A$ and $\rho_A : A \otimes I \to A$.
- The pentagon coherence condition:

$$\begin{tikzcd} & (A \otimes (B \otimes C)) \otimes D \arrow[dr, "\alpha_{A,B\otimes C,D}"] & \\ ((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A,B,C}\otimes 1"] \arrow[d, "\alpha_{A\otimes B,C,D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[d, "1\otimes\alpha_{B,C,D}"] \\ (A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C\otimes D}"'] & & A \otimes (B \otimes (C \otimes D)) \end{tikzcd}$$

- Given a pair of objects, in $\mathscr{M}$, A and B, the following diagram

$$\begin{tikzcd} (A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A\otimes 1"'] & & A \otimes (I \otimes B) \arrow[dl, "1\otimes\lambda_B"] \\ & A \otimes B & \end{tikzcd}$$

  commutes.

Definition A.3.2 (Internal Hom)**.** Given a monoidal category $(\mathscr{M}, \otimes)$, we define an *internal Hom* in $\mathscr{M}$ to be a functor

$$[-,-] : \mathscr{M}^{op} \times \mathscr{M} \to \mathscr{M}$$

such that there exists a pair of adjoint functor

$$- \otimes X \dashv [X, -] : \mathscr{M} \to \mathscr{M}$$

for every X of $\mathscr{M}$

Definition A.3.3 (Closed monoidal category)**.** We define a *closed monoidal category* to be a monoidal category equipped with an internal Hom functor.

Definition A.3.4 (Symmetric monoidal category)**.** We define a *symmetric monoidal category* to be a monodical category equipped with a natural isomorphism $s_{A,B} : A \otimes B \to B \otimes A$, for every couple of objects A and B, such that the following diagrams commute:

- Unit coherence

$$\begin{array}{ccc} A\otimes I & \xrightarrow{s_{A,I}} & A\otimes I \\ & {\scriptstyle\rho_A}\searrow \quad \swarrow{\scriptstyle\lambda_A} & \\ & A & \end{array}$$

- The associativity coherence

$$\begin{array}{ccc} (A\otimes B)\otimes C & \xrightarrow{s_{AB}\otimes 1} & (B\otimes A)\otimes C \\ \downarrow{\scriptstyle \alpha_{A,B,C}} & & \downarrow{\scriptstyle \alpha_{B,A,C}} \\ A\otimes(B\otimes C) & & B\otimes(A\otimes C) \\ \downarrow{\scriptstyle s_{A,B\otimes C}} & & \downarrow{\scriptstyle 1\otimes s_{A,C}} \\ (B\otimes C)\otimes A & \xrightarrow[\alpha_{B,C,A}]{} & B\otimes(C\otimes A) \end{array}$$

- The inverse law:

$$\begin{array}{ccc} & B\otimes A & \\ {\scriptstyle s_{A,B}}\nearrow & & \searrow{\scriptstyle s_{B,A}} \\ A\otimes B & = = = = = & A\otimes B \end{array}$$

Example A.3.1. Let R be a ring, the category of chain complexes over $\mathscr{A}$ the abelian category of R-modules, denoted by $\mathrm{Ch}_\bullet(\mathscr{A})$ is a closed monoidal category whose internal, for every X and Y, Hom is given by:

$$[X,Y]_n := \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}(X_i, Y_{i+n}),$$

with differentials $df := d_Y f - (-1)^n f d_X$.

Rigid category

Let $(\mathscr{M}, \otimes, 1)$ be a symmetric monoidal category.

Definition A.3.5 (Dualizable object)**.** We say that a is a *dualizable object* in $\mathscr{M}$ if there exist an object $a^V \in \mathscr{M}$ and two morphisms $ev : a\otimes a^V \to 1$ and $co : 1 \to a^V \otimes a$ such that:

$$a \simeq a\otimes 1 \xrightarrow{id\otimes co} a\otimes a^V \otimes a \xrightarrow{ev\otimes id} 1\otimes a \simeq a$$

$$a^V \simeq 1\otimes a \xrightarrow{co\otimes id} a^V \otimes a \otimes a^V \xrightarrow{id\otimes ev} a^V \otimes 1 \simeq a^V$$

are the identity, a^V is called the *dual* of a.

We call a monoidal category *rigid* if every object is dualizable.

Pseudoabelian category

We recall that an idempotent morphism p is an endomorphism such that $p^2 = p$.

Definition A.3.6 (Pseudoabelian category)**.** A pseudoabelian category is a preadditive category such that every idempotent morphism has a kernel.

Given a preadditive category $\mathcal{A}$ we can construct a pseudoabelian category associated to $\mathcal{A}$, denoted by $\mathrm{kar}(\mathcal{A})$, in the following way:

- the objects of $\mathrm{kar}(\mathcal{A})$ are pairs (X, p) where X is an object of $\mathcal{A}$ and p is an idempotent of X,
- the morphisms
$$f \in \mathrm{Hom}_{\mathrm{kar}(\mathcal{A})}((X, p), (Y, q))$$
are those morphisms $f : X \to Y$ such that $f = q \circ f = f \circ p$ in $\mathcal{C}$.

The functor
$$\mathrm{kar} : \mathcal{A} \to \mathrm{kar}(\mathcal{A})$$
is given by taking $X \to (X, \mathrm{id}_X)$.

Example A.3.2. The category of associative non-unital rings (also known as rngs pronunced "rungs") together with multiplicative morphisms is pseudoabelian.

Orbit category

Let $(\mathcal{M}, \otimes, 1)$ be a F-linear, additive rigid, monoidal symmetric category.

Definition A.3.7 (Orbit category)**.** Let $\mathcal{O}$ be a $\otimes$-invertible object in $\mathcal{M}$, we define the *orbit category* associated to $\mathcal{O}$, denoted by $\mathcal{M}/_{-\otimes\mathcal{O}}$, to be the category whose objects are the objects of $\mathcal{M}$ and whose morphisms are given by
$$\mathrm{Hom}_{\mathcal{M}/_{-\otimes\mathcal{O}}}(a, b) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{M}}(a, b \otimes \mathcal{O}^{\otimes i}).$$
The composition of morphisms is given, for every k, by
$$(g \cdot f)_k := \sum_{i\in\mathbb{Z}} (g_{k-i} \otimes \mathcal{O}^{\otimes i}) f_i,$$
where $\mathrm{f} = \{f_i\} : a \to b$ and $\mathrm{g} = \{g_j\} : b \to c$

We have a canonical functor
$$\begin{aligned} \mu : \mathcal{M} &\to \mathcal{M}_{-\otimes\mathcal{O}} \\ a &\mapsto a \\ f &\mapsto \mathrm{f} := \{f_i\}_{i\in\mathbb{Z}} \end{aligned}$$
where $f_0 = f$ and $f_i = 0$. Moreover we have that the orbit category $\mathcal{M}_{-\otimes\mathcal{O}}$ is F-linear and additive, and the functor μ is symmetric monoidal.

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