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An educational path for the magnetic vector potential and its physical implications

S Barbieri[1], **M Cavinato**[2] **and M Giliberti**[2]

[1] Università degli Studi di Palermo, Palermo, Italy
[2] Dipartimento di Fisica, Università degli Studi di Milano, Milano, Italy

E-mail: marco.giliberti@unimi.it

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Abstract
We present an educational path for the magnetic vector potential **A** aimed at undergraduate students and pre-service physics teachers. Starting from the generalized Ampère–Laplace law, in the framework of a slowly varying time-dependent field approximation, the magnetic vector potential is written in terms of its empirical references, i.e. the conduction currents. Therefore, once the currents are known, our approach allows for a clear and univocal physical determination of **A**, overcoming the mathematical indeterminacy due to the gauge transformations. We have no need to fix a gauge, since for slowly varying time-dependent electric and magnetic fields, the 'natural' gauge for **A** is the Coulomb one. We stress the difference between our approach and those usually presented in the literature. Finally, a physical interpretation of the magnetic vector potential is discussed and some examples of the calculation of **A** are analysed.

1. Introduction

The magnetic vector potential **A** is very useful in many physical situations. Besides its obvious relevance in the standard quantization of the electromagnetic field and in electromagnetic gauge theories, **A** is fundamental in understanding both some classical phenomena (i.e. the Maxwell–Lodge effect [1, 2]) and some quantum physics phenomena (i.e. the Aharonov–Bohm effect [3] and the Mercereau effect [4]). Moreover it allows the possibility of introducing superconductivity in a simple and meaningful way, at least within a phenomenological approach [5–7]. Nevertheless, from a teaching point of view, in many introductory textbooks on electromagnetism, and also in undergraduate up to graduate lectures, the magnetic vector potential is generally presented only as a useful mathematical tool, disregarding its physical meaning. Even though many papers can be found in the literature that clarify that vector potential does indeed have a precise physical meaning [1, 8–10], nonetheless a clear educational path on vector potential is, to the best of our knowledge, still missing. Therefore, as part of a

PhD study on physics education, we have developed an approach to magnetic vector potential that is now being tested in three different experimentations [11]. The first one is addressed to third year college students in mathematics and is being carried out in a basic course on electromagnetism. The second one is addressed to pre-service physics teachers and is being carried out in a course on electromagnetic induction education. The last one is a laboratory course addressed to graduate students in mathematics.

In what follows we present the general framework of our educational path on the magnetic vector potential. Our intention is to show a particularly meaningful way to introduce the vector potential for slowly varying time-dependent fields in terms of an empirical reference, i.e. the conduction current density. The 'natural' gauge condition is then discussed and some thoughts on the physical meaning of **A**, with examples, are finally offered. The results obtained in the previously mentioned experimentations will be discussed in a forthcoming paper.

2. Our educational path for the magnetic vector potential

Maxwell, in his book 'A treatise on electricity and magnetism', introduced the notion of magnetic vector potential through an integral relation. Following Maxwell, but with modern symbology, we can say that the magnetic vector potential **A** is a vector such that the flux of the magnetic field **B** through any surface Σ is equal to the circulation of **A** around the boundary $\partial\Sigma$ of Σ [12], that is:

$$\int_{\Sigma} \mathbf{B}\cdot\mathbf{n}\,\mathrm{d}\Sigma = \oint_{\partial\Sigma} \mathbf{A}\cdot\mathrm{d}\mathbf{s}. \qquad (1)$$

By contrast, textbooks usually introduce the magnetic vector potential via a differential equation that is the local form of equation (1); **A** is defined as the vector such that:

$$\mathbf{B} = \nabla\times\mathbf{A}. \qquad (2)$$

As is well known, neither equation (1) nor equation (2) give an explicit definition of **A**. In fact, for a specified magnetic field **B**, there are many possible solutions for **A** of both equation (1) and (2); that is, **A** is not univocally defined by **B**. This fact is at the basis of the so-called gauge invariance that will be discussed later, and is one of the main reasons for many difficulties encountered in understanding the physical meaning of the magnetic vector potential [13]. In the following, we develop a path for the magnetic vector potential that we believe is much more meaningful than the traditional one; it is addressed to both undergraduate students and secondary school teachers.

Given a general distribution of conduction current density **J** and taking also into account the displacement current density, the magnetic field **B** at position **r** and time t, in vacuum, is given by the generalized Ampère–Laplace law:

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\left[\mathbf{J}(\mathbf{r}',t') + \varepsilon_0 \frac{\partial\mathbf{E}(\mathbf{r}',t')}{\partial t}\right]\times\Delta\mathbf{r}}{(\Delta r)^3}\,\mathrm{d}V', \qquad (3)$$

where V' is the region containing the currents and

$$\Delta\mathbf{r} \equiv \mathbf{r}-\mathbf{r}', \quad \Delta r \equiv |\Delta\mathbf{r}|, \quad t' \equiv t - \frac{\Delta r}{c}, \qquad (4)$$

where t' is the retarded time. If we now adopt the quasi-static approximation, that is if we consider only fields that are slowly varying in time, we can neglect all the time derivative multiplied by $1/c$ (but not time-dependent terms alone). Therefore the contribution of the displacement currents in equation (3) can be disregarded, thanks to the presence of the constant $\varepsilon_0\mu_0 = 1/c^2$ that multiplies the time derivative of $\mathbf{E}$. Moreover, the retarded time t' of equation (4) also can be considered equal to t. So we are left with

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}(\mathbf{r}', t) \times \Delta\mathbf{r}}{(\Delta r)^3}\,\mathrm{d}V'. \tag{5}$$

Observing that:

$$\nabla\left(\frac{1}{\Delta r}\right) = -\frac{\Delta\mathbf{r}}{(\Delta r)^3}, \tag{6}$$

and commutating the factors of the vector product in the integrand of equation (5), we can write:

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int_{V'} \nabla\left(\frac{1}{\Delta r}\right) \times \mathbf{J}(\mathbf{r}', t)\,\mathrm{d}V'. \tag{7}$$

Keeping in mind that if f is a scalar field while $\mathbf{v}$ is a vector field one has the identity:

$$\nabla \times (f\mathbf{v}) = \nabla f \times \mathbf{v} + f\nabla \times \mathbf{v} \tag{8}$$

and using the fact that $\nabla \times \mathbf{J}(\mathbf{r}', t) = 0$ because $\mathbf{J}$ depends on primed variables while the curl is done with respect to unprimed ones, we obtain:

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \left(\frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}(\mathbf{r}', t)}{\Delta r}\,\mathrm{d}V'\right). \tag{9}$$

Equation (9) clearly shows that we can introduce a vector:

$$\mathbf{A}(\mathbf{r}, t) \equiv \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}(\mathbf{r}', t)}{\Delta r}\,\mathrm{d}V' \tag{10}$$

such that

$$\nabla \times \mathbf{A}(\mathbf{r}, t) = \mathbf{B}(\mathbf{r}, t). \tag{11}$$

Equation (11) shows that the vector $\mathbf{A}$ of equation (10) is a magnetic vector potential and that, moreover, in the framework of our slowly varying time-dependent approximation, it is the magnetic vector potential to which one is naturally led by physical considerations.

In addition, equation (10) proves a clear analogy between magnetic vector potential and electric scalar potential

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho(\mathbf{r}', t)}{\Delta r}\,\mathrm{d}V', \tag{12}$$

where $\rho(\mathbf{r}', t)$ is the charge density at point $\mathbf{r}'$ and time t. The vector potential given by equation (10) is a precise function of the current density. Therefore (in our slowly varying field approximation) once the currents are known, $\mathbf{A}$ is univocally determined. Instead, if we start from equation (2), $\mathbf{A}$ is not unique. In fact, the Helmholtz theorem (and its generalized version [14]) states that a quasi-static vector field vanishing at infinity more quickly than $1/r$, is completely determined once both its curl *and* its divergence are known. Therefore an additional condition (the so-called gauge condition) is clearly needed. This is generally done by arbitrary fixing the divergence of $\mathbf{A}$. On the contrary, in our approach, we have no need to fix a gauge. Nevertheless it is interesting to directly calculate $\nabla \cdot \mathbf{A}$.

With the same symbology of equation (8), we have the following vector identity:

$$\nabla \cdot (f\mathbf{v}) = \nabla f \cdot \mathbf{v} + f\nabla \cdot \mathbf{v}, \tag{13}$$

therefore, from equation (10) we obtain:

$$\begin{aligned}\nabla \cdot \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi}\int_{V'} \nabla\left(\frac{1}{\Delta r}\right) \cdot \mathbf{J}(\mathbf{r}', t)\;\mathrm{d}V' + \frac{\mu_0}{4\pi}\int_{V'} \frac{1}{\Delta r}\nabla \cdot \mathbf{J}(\mathbf{r}', t)\,\mathrm{d}V' \\ &= -\frac{\mu_0}{4\pi}\int_{V'} \nabla'\left(\frac{1}{\Delta r}\right) \cdot \mathbf{J}(\mathbf{r}', t)\,\mathrm{d}V', \end{aligned} \tag{14}$$

where the operator ∇' acts on the primed variables. We note that $\nabla \cdot \mathbf{J}(\mathbf{r}', t) = 0$, because $\mathbf{J}$ depends only on the primed variables while the divergence is done with respect to the unprimed ones and $\nabla(\frac{1}{\Delta r}) = -\nabla'(\frac{1}{\Delta r})$. Moreover (again keeping in mind equation (13)) the integrand of the last term in equation (14) can be written as follows:

$$\nabla'\left(\frac{1}{\Delta r}\right)\cdot \mathbf{J}(\mathbf{r}', t) = \nabla' \cdot \left[\frac{\mathbf{J}(\mathbf{r}', t)}{\Delta r}\right] - \frac{1}{\Delta r}\nabla' \cdot \mathbf{J}(\mathbf{r}', t)$$
$$= \nabla' \cdot \left[\frac{\mathbf{J}(\mathbf{r}', t)}{\Delta r}\right] + \frac{1}{\Delta r}\frac{\partial \rho(\mathbf{r}', t)}{\partial t}, \tag{15}$$

where, in the last equality, we have used the continuity equation:

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0. \tag{16}$$

Therefore we have:

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi}\int_{V'} \nabla' \cdot \left[\frac{\mathbf{J}(\mathbf{r}', t)}{\Delta r}\right] \mathrm{d}V' - \frac{\mu_0}{4\pi}\int_{V'} \frac{1}{\Delta r}\frac{\partial \rho(\mathbf{r}', t)}{\partial t} \mathrm{d}V'. \tag{17}$$

The first integral in equation (17) is zero thanks to the divergence theorem. In fact

$$-\frac{\mu_0}{4\pi}\int_{V'} \nabla' \cdot \left[\frac{\mathbf{J}(\mathbf{r}', t)}{\Delta r}\right] \mathrm{d}V' = -\frac{\mu_0}{4\pi}\oint_{\Sigma'} \frac{\mathbf{J}(\mathbf{r}', t)}{\Delta r} \cdot \mathbf{n}\, \mathrm{d}\Sigma', \tag{18}$$

where $\Sigma' \equiv \partial V'$ is the boundary of the region V' and $\mathbf{n}$ is the outer normal to Σ'. Since V' must contain all the currents that generate $\mathbf{A}$ at each time, it can be taken so large that $\mathbf{J}$ can be considered zero upon Σ' and therefore the right-hand side integral in equation (18) vanishes. For the second integral in the right-hand side of equation (17), since V' is time-independent, we get:

$$\frac{\mu_0}{4\pi}\int_{V'} \frac{1}{\Delta r}\frac{\partial \rho(\mathbf{r}', t)}{\partial t} \mathrm{d}V' = \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int_{V'} \frac{\rho(\mathbf{r}', t)}{\Delta r} \mathrm{d}V' = \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}[4\pi\varepsilon_0\varphi(\mathbf{r}, t)], \tag{19}$$

where $\varphi(\mathbf{r}, t)$ is the electric scalar potential given by equation (12). From equations (17)–(19) we finally obtain:

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = -\varepsilon_0\mu_0\frac{\partial}{\partial t}\varphi(\mathbf{r}, t) = -\frac{1}{c^2}\frac{\partial}{\partial t}\varphi(\mathbf{r}, t). \tag{20}$$

Equation (20) is the well-known Lorenz gauge. In the quasi-static approximation we adopt in this paper, the right-hand term of equation (20) can be considered to be zero, and therefore we are left in the so-called Coulomb gauge:

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = 0. \tag{21}$$

The most common attitude is to arbitrary fix $\nabla \cdot \mathbf{A}(\mathbf{r}, t) = 0$ from the beginning, so that equation (10) is obtained as a final result. On the contrary, we have followed an inverse path in which we have been naturally led to select the magnetic vector potential expressed in terms of the current density (that can therefore be seen as the source of the potential), as we have done in equation (10). As a consequence of this choice, we found that the magnetic vector potential of equation (10) is given in the Coulomb gauge, that therefore can be seen as the 'natural' gauge for slowly varying fields.

As is well known, the relations which give a link among the fields $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$ and the potentials are:

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t) - \nabla\varphi(\mathbf{r}, t) \tag{22}$$

and equation (11). In the general case, when the following transformations (called gauge transformations) are performed:

$$\varphi \to \varphi' = \varphi - \frac{\partial \Lambda}{\partial t}, \tag{23}$$

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\Lambda, \tag{24}$$

where Λ is a scalar function, the electric and the magnetic fields remain unchanged. Here we want to stress that our choice of a vector potential given in terms of the conduction current (equation (10)) is not equivalent to the choice of the Coulomb gauge, where $\mathbf{A}$ is determined only up to the gradient of a harmonic function, as can be seen from equation (24).

3. The physical meaning of the magnetic vector potential

We cannot directly assign a physical meaning to the electric scalar potential itself. The physical meaning can only be assigned to its spatial derivative or to the difference between the potentials at two separate points. However, in many physical situations, we can choose one of the two different points at the spatial infinity, where we can fix the value of the scalar potential equal to zero. A particularly interesting case is when the magnetic vector potential $\mathbf{A}$ is time-independent. In fact, in this condition, we can define the potential energy $U(\mathbf{r}, t)$ of a point-like charge q set at position $\mathbf{r}$ at time t, as the work (independent of the chosen path) necessary to move the charge q from infinity, where the electric field is zero, to the point $\mathbf{r}$, against the forces of the electric field; that is:

$$U(\mathbf{r}, t) = -\int_{\infty}^{\mathbf{r}} q\mathbf{E}(\mathbf{r}', t) \cdot \mathrm{d}\mathbf{r}'. \tag{25}$$

The electric scalar potential can therefore be written as:

$$\varphi(\mathbf{r}, t) = -\int_{\infty}^{\mathbf{r}} \mathbf{E}(\mathbf{r}', t) \cdot \mathrm{d}\mathbf{r}'. \tag{26}$$

It has the clear physical meaning of potential energy per unit charge and can be identified with the function φ of equation (22). We note that the integrals of equations (25) and (26) are performed only over the spatial coordinates while the time coordinate is a fixed parameter.

Likewise, we cannot assign a physical meaning even to the magnetic vector potential itself, while it can be assigned to its time derivative or to the difference between the vector potentials at two different times. Similarly to what is done for the scalar potential, in many physical situations we can choose one of the two different times at $t = -\infty$ and fix $\mathbf{A}(t = -\infty) = 0$. An interesting situation arises when $\nabla\varphi(\mathbf{r}, t) = 0$ and the magnetic field is slowly varying in time. In this case we can give a physical meaning to the vector potential $\mathbf{A}$ of equation (22), which is univocally determined and naturally given by equation (10). To do this, we have to exchange the roles of the variables $\mathbf{r}$ and t; that is, we have to perform an integral over the time coordinate while the point $\mathbf{r}$ remains fixed. Let us consider a point-like charge q in the position $\mathbf{r}$ at a time, which we will indicate as $-\infty$, when the currents and consequently the magnetic field are zero. Let us now slowly switch the currents on. They will generate a magnetic field $\mathbf{B}$, a vector potential $\mathbf{A}$ and therefore an electric field $\mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t)$ that will act on q. In order to keep q fixed in $\mathbf{r}$, an impulse must be applied against the field forces, and this is given by:

$$\boldsymbol{\Upsilon}(\mathbf{r}, t) = -\int_{-\infty}^{t} q\mathbf{E}(\mathbf{r}, t')\,\mathrm{d}t' = \int_{-\infty}^{t} q\frac{\partial}{\partial t'}\mathbf{A}(\mathbf{r}, t')\,\mathrm{d}t' = q\mathbf{A}(\mathbf{r}, t), \tag{27}$$

where we have put $\mathbf{A}(t = -\infty) = 0$, since the currents are zero at that time. The impulse against the magnetic force does not depend on the time dependence of $\mathbf{A}$, provided the field is slowly varying. The magnetic vector potential can thus be interpreted as the total momentum per unit charge that must be transferred to a charge, during the time interval $(-\infty, t)$, in order to keep this charge at rest at the point **r** while the field varies slowly from zero to the value **B** in that time interval. Magnetic vector potential can be therefore considered as a 'momentum vector' per unit charge, while the electric scalar potential can be seen as an energy component. As $q\varphi$ is called potential energy, $q\mathbf{A}$ can be called potential momentum (of the charge q, at point **r** and time t) [10].

Besides its physical meaning, the magnetic vector potential gives us also the possibility to write some physical relations in a clearer and more understandable way. For instance, a mechanical harmonic plane wave of amplitude S_0 and angular frequency ω, propagating in a medium of density ϱ with velocity v, carries an intensity given by:

$$I = \tfrac{1}{2}\rho\omega^2 S_0^2 v. \tag{28}$$

If we consider an electromagnetic linearly polarized, harmonic, plane wave of amplitude E_0, angular frequency ω, propagating in a medium of absolute dielectric permittivity ε with velocity v, its intensity is usually written without explicitly showing the angular frequency, that is as:

$$I = \tfrac{1}{2}\varepsilon E_0^2 v. \tag{29}$$

The magnetic vector potential gives the possibility to write equation (29) in a form completely similar to equation (28). In fact, from the first term of equation (22) and denoting the vector potential amplitude with A_0, we immediately have:

$$I = \tfrac{1}{2}\varepsilon\omega^2 A_0^2 v. \tag{30}$$

Equation (30) shows that vector potential plays for the electromagnetic field the same role played by the displacement from the equilibrium position for a mechanical wave propagating in a medium (see equation (28)). Moreover, since the intensity, the frequency and the velocity of propagation can be all measured, equation (30) immediately yields A_0 [8].

Equations (27) and (30) and their interpretations clearly show that the magnetic vector potential is not a simple mathematical tool, but it has a deep physical meaning and can greatly help visualization.

4. Some examples

In the following, we give some examples of the calculation of vector potential with different and simple strategies, just to show how easy it can be to visualize vector potential in space (see also [15]) and how problems can be approached from different points of view. Furthermore, we make some physical considerations that can help highlight the link between the magnetic vector potential and the electric and magnetic fields.

4.1. Magnetic vector potential of a solenoid

The expression of the magnetic vector potential generated by an infinite solenoid carrying a current density **J** is well known and can be found in text books and many papers. It seems to us that a very intuitive way of presenting to students this and similar calculations can be based on the fact that the mathematical relation between **A** and **B** is the same as that between **B** and $\mu_0\mathbf{J}$. In fact, for slowly varying fields, the magnetic field **B** is linked to the current density vector **J** by the Maxwell equation:

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J}, \tag{31}$$

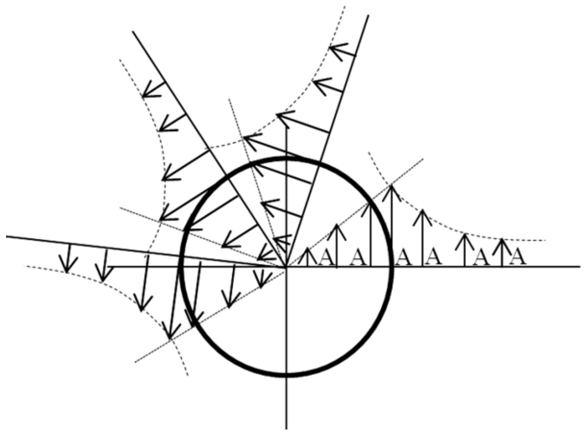

Figure 1. Field lines of the magnetic vector potential generated by an infinite solenoid carrying a current.

while the relation between the magnetic vector potential **A** and **B** is given by:

$$\nabla \times \mathbf{A} = \mathbf{B}. \tag{32}$$

Moreover, both **A** and **B** are solenoidal fields. Therefore, once we know the dependence of **B** from **J**, we also immediately know the dependence of **A** from **B** in situations where they have similar symmetry [10]. The structure of equations (31) and (32) could induce interpreting **B** as the source of **A** (in analogy with the fact that **J** is the source of **B**). From a didactical point of view we want to stress that this is only a formal analogy, since the sources of **A** are the currents, while the fields can be obtained by deriving **A**.

For example, the spatial dependence of **B** generated by an infinite straight wire of radius a is the same as that of **A** generated by an infinite solenoid of the same radius, when both the wire and the solenoid are carrying a uniform current density. More specifically, in both cases, the field lines are circular, concentric with the axis of symmetry and lie on planes perpendicular to this same axis. Therefore, indicating with r the distance from the symmetry axis, the expressions of the vector potential inside and outside the solenoid are given by:

$$\mathbf{A}(r,t) = \frac{1}{2}\mathbf{B}(t) \times \mathbf{r} \quad \text{and} \quad \mathbf{A}(r,t) = \frac{a^2}{2r^2}\mathbf{B}(t) \times \mathbf{r}, \quad \text{respectively,} \tag{33}$$

see figure 1.

If the current, and therefore the magnetic field, is time-dependent, as in equation (33), an electric field is generated both inside and outside the solenoid. The expressions of this electric field can be obtained by deriving relations (33):

$$\mathbf{E}(r,t) = -\frac{1}{2}\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} \times \mathbf{r} \quad \text{inside} \quad \text{and} \quad \mathbf{E}(r,t) = -\frac{a^2}{2r^2}\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} \times \mathbf{r} \quad \text{outside.} \tag{34}$$

Relations (34) are traditionally obtained from the integral Maxwell equation $\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathbf{B})$. When following this procedure, a strange situation arises because it is difficult to understand how is it possible that the outside electric field 'knows' that, inside the solenoid, **B** is changing, considering that the outside **B** is always zero (let us recall that for slowly varying fields, electromagnetic waves can be neglected). The question becomes more significant when it is remembered that the field concept has been introduced just to avoid actions at a distance. The problem is solved with the introduction of the vector potential, which is defined both inside and outside the solenoid and it is given by equation (33). From this equation, using

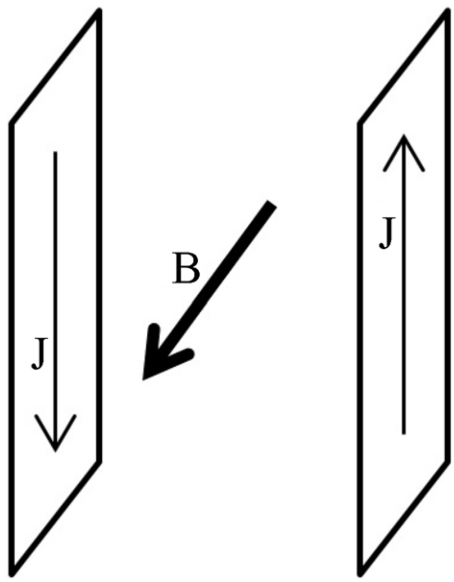

Figure 2. Magnetic field generated by two planes carrying antiparallel currents.

equation (22) and keeping in mind that no free charges are present, one can obtain the electric field. Thus, it is the local time-dependent **A** that generates **E**.

We note that the use of the local form $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ tells us that the electric field is irrotational only outside the solenoid (even if it is obviously not conservative since the region is not simply connected) and does not give the explicit expression for the electric field.

4.2. Magnetic vector potential of two parallel planes

Let us now consider two parallel planes carrying, in opposite directions, a uniform current of density **J** per unit length. It is straightforward to understand that between the planes there is a uniform magnetic field **B**, while outside the planes the magnetic field is zero (see figure 2). To determine the vector potential we could integrate equation (10) or, in a simpler way, we can still start from equation (10), but now just to understand the symmetries of **A** in terms of those of **J**, to later obtain **A**, by solving equation (1). From figure 2 it follows immediately that the magnetic vector potential is parallel to the currents in the planes. Therefore we can choose as the surface Σ of equation (1) a rectangle lying in a plane normal to the planes of the currents and with symmetry axis in the median plane, as shown in figure 3.

With the symbology of figure 3, from equation (1) we get:

$$\mathbf{A}(x) = Bx\mathbf{u}_y \quad \text{between the planes,} \tag{35}$$

while outside the planes

$$\mathbf{A}(x) = -Bx_0\mathbf{u}_y \quad \text{for } x < 0; \quad A(x) = Bx_0\mathbf{u}_y \quad \text{for } x > 0, \tag{36}$$

where $\mathbf{u}_y$ is the versor with the same direction as the density current vector **J** in the plane at positive x.

The field lines of **A** are given in figure 4. It is interesting to observe that, as in the previous case, the vector potential is different from zero both inside and outside the planes.

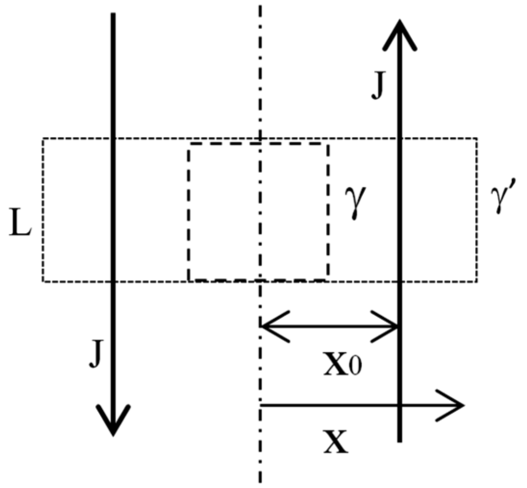

Figure 3. The vertical arrows represent the section of the planes of the currents; γ e γ' are the boundaries of the rectangles chosen to calculate the circulation of **A**.

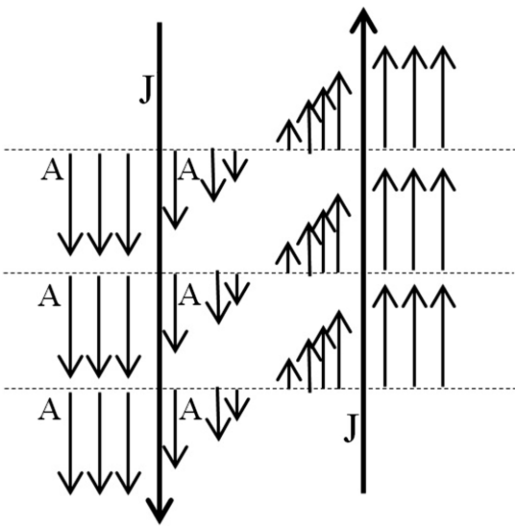

Figure 4. Field lines of the magnetic vector potential generated by two planes carrying opposite currents.

4.3. Thoughts on the link between the magnetic field and the magnetic vector potential

In the previous examples we found the expression of the vector potential starting from a known current distribution. It can be interesting now to determine **A** through equation (2), that is, starting from a known magnetic field **B**. We already know that the problem is not univocally determined. However, what was discussed in the previous section allows us to shed some light on the physical implications of this fact.

Let us imagine to calculate **A** for a uniform **B**. When we choose a particular class of close paths to calculate the circulation of the vector potential, the symmetry of the problem is broken and a particular **A** is found. To recover the lost physical symmetry, one generally considers all the vector potentials generating the same field **B** to be equivalent; and in a sense this is one of the physical meanings of the gauge invariance. Back to our example, if **B** is really uniform in the whole space we do not know whether we are inside an infinite solenoid of infinite radius or between a couple of current carrying planes, at an infinite distance apart. Therefore, even if we are in the same Coulomb gauge (in our approximation the gauge is fixed), **A** is not univocally determined by **B** because, as we have already said, the currents which could generate this field do not vanish at infinity. It is clear that the currents determine both **B** and **A**; the potential **A** determines **B**, while the vice versa is not true.

5. Conclusions

Two main facts hinder the comprehension and therefore the use of the magnetic vector potential. The first one is the non-univocity of **A** implied by its definition given by equation (2); the second one is the paucity of discussion traditionally devoted to its physical meaning.

Convinced of the educational value of the vector potential in dealing with many physical situations, we have developed a path which in our opinion can overcome the above stated difficulties. Starting from the generalized Ampère–Laplace law and driven by physical considerations, we attained a particular expression of **A** in terms of its empirical referent, i.e. the conduction current density, for slowly time-dependent electric and magnetic fields. Traditionally, this result is obtained by working with static fields or starting from the wave equation for **A**. Our approach has the advantage of being much more general than that with static fields, principally because within our quasi-static approximation we can clearly give a physical meaning to **A**. Moreover we believe that our approach can be presented in a basic course on electromagnetism before the study of the electromagnetic wave equations, thus giving students time to familiarize themselves with the concept. In our work we found a privileged gauge (the Coulomb gauge) and the physical meaning of **A** was discussed in a similar way to that of the electric scalar potential, another fact which can help comprehension. In addition, in some circumstances, the use of the vector potential allowed us a causal local description clearer than that given by the magnetic field alone and to highlight interesting parallelisms with mechanical situations. To conclude, we firmly consider the introduction of the magnetic vector potential in electromagnetism not only a good tool for making calculations, but also a useful way to better understand many physical phenomena.

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