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Symmetries and Black Holes in $\mathcal{N} = 2$ Gauged Supergravity

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Cover image: Our galaxy's central black hole, Sagittarius A, www.nationalgeographic.com*

Abstract

The main goal of this work is to explore the role and the power of the Hamilton-Jacobi approach and solutions generating techniques to search black hole solutions in abelian gauged supergravity theories. After a brief introduction explaining how supergravity arises as effective theory of superstrings, a chapter is devoted to the detailed description of $\mathcal{N} = 2$ abelian gauged supergravity in $d = 4$ and $d = 5$.
The study of the equations of motion with general matter coupling for static black hole solutions is developed with Hamilton-Jacobi approach and new first order sets of equations are presented [1,2] extending well-known BPS sets [3,4]. In $d = 5$ the equations are completely solved for the STU model with constant hypermultiplets and a solution generalizing Maldacena-Nunez black string [4] is found [5].
Then the attention is focused on some integrability properties of minimal gauged supergravity in $d = 4$ [6] and Einstein-Maxwell-Lambda system in d dimensions [7]. Always with Hamilton-Jacobi technique, studying the symmetries of the systems the dynamics is solved algebraically for some sets of solutions.
For holographic calculations the attractor points deserve particular attention and therefore I explain their definition for the cases $AdS_2 \times \Sigma_2$ and $AdS_3 \times \Sigma_2$ and the calculation of the entropy and the central charge in general matter coupled supergravities. Moreover, Freudenthal duality is extended to the abelian gauged case and general matter coupling [8].
The last part is dedicated to the introduction of a solution generating technique based on group theory in FI $d = 4$ supergravity. Many examples with the simplest prepotentials are shown [9]. With the help of dimensional reduction this solution generating technique is used to produce a rotating black string in $d = 5$ [5].

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Articles

This thesis is based on the publications

- D. Klemm, M. Nozawa and M. Rabbiosi, “On the integrability of Einstein–Maxwell–(A)dS gravity in the presence of Killing vectors,” Class. Quant. Grav. **32**, no. 20, 205008 (2015) doi:10.1088/0264-9381/32/20/205008 [arXiv:1506.09017 [hep-th]].
- D. Klemm, N. Petri and M. Rabbiosi, “Symplectically invariant flow equations for $N = 2$, $D = 4$ gauged supergravity with hypermultiplets,” JHEP **1604** (2016) 008 doi:10.1007/JHEP04(2016)008 [arXiv:1602.01334 [hep-th]].
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- D. Klemm and M. Rabbiosi, “First order flow equations for nonextremal black holes in AdS (super)gravity,” JHEP **1710** (2017) 149 doi:10.1007/JHEP10(2017)149 [arXiv: 1706.05862 [hep-th]].
- M. Azzola, D. Klemm and M. Rabbiosi, “AdS_5 black strings in $\mathcal{N} = 2$ STU model,” to appear.

Contents

Introduction

Probably the biggest unsolved problem in theoretical physics is to reconcile the description of the four fundamental forces. Gravity is described by curved spacetime, while the strong, weak and electromagnetic interactions with consistent quantum field theories. On one hand, we have General Relativity and on the other hand the Standard Model.
They provide an accurate description of two different sides of nature under suitable limits. The gravitational interaction and the behaviour of spacetime in presence of classical matter are correctly reproduced by Einstein's theory. On the other hand, the Standard Model predictions fit in an extremely precise way the world of particle physics, when one can neglect the gravitational force. Both are field theories but their applicability is set on two different scales. The metric, the dynamical variable of General Relativity, paints the picture of the spacetime as a classical field theory, while in the Standard Model quantum fields control the behaviour of the single quanta.

However certain physical objects as black holes and events like the big bang display a singular behaviour in General Relativity. As always in theoretical physics the presence of infinity means something that requires a deeper understanding, the theory has reached its limit of applicability. In this case, we need a description that takes into account quantum gravitational effects.
The lack of a microscopic description of gravity is due to the high nonlinearity of General Relativity. The kinetic term of the gravitational force, the curvature scalar, contains second order derivatives and the inverse of the metric field. On the other hand, General Relativity is a field theory and one could expect that, expanding the theory on a fixed background, it can be promoted to a quantum field theory with the canonical quantization procedure. In fact, this is possible, but the theory results to be nonrenormalizable, i.e., one needs an infinite number of couterterms and thus of free parameters. The theory loses its predictive power. It can be thought as an effective field theory in a reduced scale of validity, but it cannot be considered as a quantum theory of gravity at the fundamental level.

This suggests that one could think that in some sense gravity must be more profound, not only the fourth force but the key for a theory of everything. M-theory and superstring theory attempt to answer in this sense to the request of a theory of quantum gravity, with the unification of all the forces [10–12]. A picture at the Planck length,

l_P, substitutes particles for strings living in a ten-dimensional spacetime self-generated from their interaction. The oscillation modes of the string produce the features of what we call particles at a lower energy scale.

By the late 1970s, a big and rich apparatus was built following the idea that the fundamental objects underlying particles are strings. Even if the predictive power of these theories is very reduced by the lack of experiments that can test them, their study is itself interesting from the mathematical and theoretical physics point of view. In algebraic geometry the discovery of mirror symmetry [13] has come from a physical conjecture born for understanding certain string models. The AdS/CFT correspondence [14] can perhaps be considered as one of the most important developments of theoretical physics of the last two decades, since it provides for the first time an explicit realization of the holographic principle [15].
The five superstring theories are supersymmetric and ten-dimensional, as required by the consistency of the theories themselves. They are linked to each other and with an underling eleven-dimensional theory, called M-theory, by a web of dualities. Each of these six theories describes a different corner of the same picture 1. Moreover, string theories contain not only open and closed strings, but also branes, which are extended objects in more than one dimension. However, we are able to describe the interactions of these object only when particular expansions in the couplings, α', the tension of the string, and g_s, the quantum coupling constant, are considered; the complete description is not known.

As an ordinary field theory, superstrings can be quantized in a pertubative regime. For each mass level the spectrum contains a supersymmetric multiplet of fields, called supermultiplet. Starting from the ground state an infinite tower of supermultiplets is created. Particular attention deserves the massless supermultiplet from which one obtains a low energy effective action called supergravity. It is a supersymmetric classical field theory and it contains General Relativity as its fundamental subsector. For each superstring theory, one can write down the corresponding supergravity.

To make contact with the lower dimensional physics and to simplify the study of the solutions one can dimensionally reduce the ten- or eleven-dimensional theories by compactification mechanisms, such as the Kaluza-Klein mechanism. Some dimensions describe a compact manifold with a resulting lower dimensional effective supergravity, in which the fields parametrize the moduli of deformations of the compactification manifold M. The compactification leads to consistent truncations only when M has peculiar properties, many families are known but an exhaustive classification is still missing. The number of the dimensions d and the supersymmetries $\mathcal{N}$ of the reduced theory depend on M.
The different types of compactifications are distinguished by the presence of fluxes or not. When some fluxes are switched on, the resulting supergravity is deformed by a gauging of some isometries of the scalar manifold and certain scalar fields may be

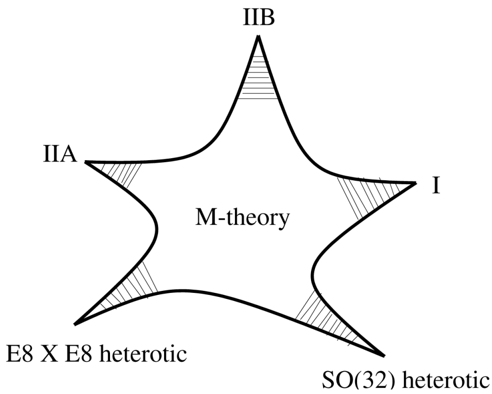

Figure 1: *M-theory contains all know superstrings in different corners.*

charged under these symmetries. Moreover, a scalar potential appears [16]. In this case, the equations are much more complicated and only a few solutions to them are known. In four- and five-dimensional $\mathcal{N} = 2$ gauged supergravity the geometric structures of the vector multiplets and hypermultiplets are very rich and not uniquely fixed as it happens for a higher amount of supersymmetry. They are obtained by compactifyng eleven-dimensional supergravity respectively on Calabi–Yau threefolds [17] and on $SU(3)$- structures [18] or type II supergravity on Calabi-Yau threefolds [19].

The bosonic field content of supergravity is General Relativity coupled to gauge fields and scalars. A solution of its equations of motion represents, not only a supergravity background, but from a higher-dimensional point of view can be considered as the effective description of a system of interacting strings and branes in particular limits. Constructing solutions in supergravity is an active technical sector in theoretical physics. The equations to solve are a complicated system of partial differential equations and therefore it is impossible to write down the general solution. Nevertheless, some interesting and useful solutions can be found in situations with a high degree of symmetry.

Typically at least stationarity and axial symmetry are taken as a starting point for defining the ansatz. For static configurations even spherical and hyperbolic symmetry are considered and constrain very much the analytical form of the configuration. However, the presence of many fields leads to complicated second order equations of motion. To simplify the problem, a first path is to search first order equations. For example, this is possible when some degree of supersymmetry is taken into account. In this case one can pose to zero the supersymmetric variations of the fermionic fields and try to solve the Killing spinor equations for a particular configuration and a certain Killing spinor. A solution to them often, considering on-shell supersymmetry, is guaranteed to be a solution also of the equations of motion. To go beyond the BPS class a more general approach is needed. Often, starting from the original higher dimensional theory specified for a

particular ansatz fields, one can discover an underlying finite dimensional dynamical system. Applying the Hamilton-Jacobi technique to this theory, one can find more general first order equations, that includes also the non-BPS class of configurations, or even solve the system algebraically.
Another powerful way to construct new solutions, when some are known, is to exploit the group of symmetries of the theory and to take advantage of some solution generating techniques. Starting from an already known solution, called seed, and applying on it a symmetry transformation, that keeps unchanged the equations of motion, one finds a final configurations will be possibly more complicated.

Moreover, symmetry is a driving tool also in the interpretation of some of these complicated configurations arising in supergravity, characterized by an AdS factor. The AdS/CFT dictionary says that for an asymptotically AdS solution one can find a CFT living on its boundary. The first way to identify the holographic dual of a certain solution is the group of symmetry defining them. Internal symmetries on the boundary must conicide with gauge symmetries in the bulk [20]. As always in physics, the presence of symmetry increases the possibility of classification.

The thesis is organized as follows. In chapter 1 a brief introduction on superstrings, highest dimensional supergravities and AdS/CFT correspondence is given. In 2 we explain in detail the structure of the $\mathcal{N} = 2$ supergravity Lagrangian in four- and five-dimensions and in 3 the effective one-dimensional field theory for a certain static ansatz is studied, first order flow equations are obtained from a Hamilton-Jacobi formalism, and some solutions are constructed. Having in mind the AdS/CFT correspondence as a main motivation, we analyze in 4 the attractor point configurations, and extract the black hole entropy and the values of the central charges. Chapter 5 is dedicated to explaining a solution generating technique of $d = 4$ supergravity.

CHAPTER 1

Superstrings and Supergravity

Inspired by [10,12] and also by the review [21], we give some basics on superstrings and ten-dimensional supergravities. After this, we introduce the idea of compactifications, dimensional reductions, and consistent truncations to explain the origin of gauged supergravities in $d = 4$ and $d = 5$ [16]. At the end of the section, to motivate the search for solutions to the equations of motion of these theories, the fundamental principles of AdS/CFT correspondence are tackled [22].

1.1 Polyakov Action and Bosonic String

In 1968, G. Veneziano [23], introduced the concept of extended objects moving in spacetime to study the strong interaction amplitudes. This idea was not fruitful for the description of the strong force, but it inspired theoretical physicists that tried to solve the unification problem of the forces. The dynamical action for a string is

$$S = \frac{T}{2} \int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{-h} h^{\alpha\beta} g_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \,, \tag{1.1}$$

used by Polyakov to create the theory of the quantum bosonic string [24]. The metric of the spacetime is $g_{\mu\nu}$ and (1.1) describes the dynamical behaviour of a string X^μ. It is parametrized by (τ, σ) generating the worldsheet whose metric is h_{ab}. This is an example of non-linear sigma model, namely a set of $d + 1$ scalar fields self-interacting via the metric dependence $g_{\mu\nu} = g_{\mu\nu}(X)$. Moreover, it is invariant under reparametrizations of the worldsheet and target space and under global Weyl rescalings of the worldsheet metric.

The first case to study is the classical free field theory $g_{\mu\nu} = \eta_{\mu\nu}$. Exploiting the invariance under worldsheet diffeomorphisms, without further restrictions we can gauge-fix

$$h_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta = e^{\rho(\tau,\sigma)} \eta_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta = e^{\rho(\tau,\sigma)} (-\mathrm{d}\tau^2 + \mathrm{d}\sigma^2) \,. \tag{1.2}$$

The action (1.1) boils down to

$$S = \frac{T}{2} \int \mathrm{d}\tau \mathrm{d}\sigma \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \tag{1.3}$$

for which the Euler-Lagrange equation reads

$$\partial^\alpha \partial_\alpha X^\mu = 0\,, \tag{1.4}$$

where $0 \leq \sigma \leq \pi$. The equation (1.4) automatically implies the stationarity of the action for closed strings, $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\pi)$, while for open strings Dirichlet or Neumann boundary conditions must be imposed

$$\delta X^\mu|_{(0,\pi)} = 0\,, \quad \text{or} \quad \partial_\sigma X^\mu|_{(0,\pi)} = 0\,. \tag{1.5}$$

Moreover the constraint

$$\frac{\delta I}{\delta h^{\alpha\beta}} = T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\eta^{\sigma\delta}\partial_\delta X^\mu \partial_\sigma X_\mu = 0\,, \tag{1.6}$$

must hold. It guarantees the local Weyl rescaling invariance. The solution to the wave equation (1.4) is easly written in terms of the light-cone coordinates $\sigma^\pm = \tau \pm \sigma$, $X^\mu = X^\mu_R + X^\mu_L$. For example, for closed string it reads

$$\begin{aligned} X^\mu_R &= \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau-\sigma) + l_s\frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-2i(\tau-\sigma)}\,, \\ X^\mu_L &= \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau+\sigma) + l_s\frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-2i(\tau+\sigma)}\,, \end{aligned} \tag{1.7}$$

where l_s is the fundamental string length and the α^μ_n and $\tilde{\alpha}^\mu_n$ are the Fourier modes expansion of the right and left movers. The reality of the solution is guarateed by the conditions

$$\alpha^\mu_n = (\alpha^\mu_{-n})^*\,, \quad \tilde{\alpha}^\mu_n = (\tilde{\alpha}^\mu_{-n})^*\,, \tag{1.8}$$

and taking x^μ and p^μ as the position and the momentum of the center of the mass.

As usual for a classical scalar field one can impose the canonical Poisson Brackets beetween the X^μ and the conjugate variables ending with the poisson bracket for the Fourier modes

$$\begin{aligned} [\alpha^\mu_m, \alpha^\nu_n] &= im\eta^{\mu\nu}\delta_{m+n,0}\,, \\ [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] &= -im\eta^{\mu\nu}\delta_{m+n,0}\,, \\ [\alpha^\mu_m, \tilde{\alpha}^\nu_n] &= 0\,. \end{aligned} \tag{1.9}$$

Moreover, the conformal invariance becomes evident once one studies the Fourier components of the stress energy tensor $T_{\alpha\beta}$. Taking $\alpha^\mu_0 = \tilde{\alpha}^\mu_0 = \frac{1}{2}p^\mu$

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}}\alpha^\mu_{m-n}\alpha_{\mu n}\,, \quad \tilde{L}_m = \frac{1}{2}\sum_{n\in\mathbb{Z}}\tilde{\alpha}^\mu_{m-n}\tilde{\alpha}_{\mu n}\,, \tag{1.10}$$

that span the direct sum of two Virasoro algebras

$$[L_m, L_n] = i(m-n)L_{m+n}\,, \quad [L_m, \tilde{L}_n] = 0\,, \quad [\tilde{L}_m, \tilde{L}_n] = i(m-n)L_{m+n}\,. \tag{1.11}$$

Following different paths now one can quantize the theory [10]. Promoting the Poisson Brackets to commutators one discovers that the Fourier modes become creator and annihilator operators from which to construct the space of physical states. The quantization procedure imposes a central extension of the Virasoro algebra because of normal ordered operators and requires $d = 26$ to have a spectrum of states without ghosts.
The first excited state is the massless spectrum and is composed by a metric, a 2-form and a scalar, the dilaton, a sector always present in a string theory. However the main problem is the fundamental state, that is a tachyon, a scalar particle with negative M^2. This instability in the spectrum and the impossibility of the description of fermions were the main motivations for the failure of bosonic string and the introduction of supersymmetry.

1.2 Superstring theories

The fundamental bricks of a field theory are fields that are the representations of the Poincaré group. However often they transform not only under spacetime symmetries, but also under a group of transformations that characterizes the specific theory, that is called internal symmetry group. The two algebras are in direct sum.
In the 1967, the Coleman-Mandula no-go theorem [25] stated, under certain physically relevant hypotesis, that it is not possibile to extend nontrivially the Lorentz algebra including the internal symmetries in a unique Lie algebra that is not a direct sum of the two factor. The idea of extending the symmetries of a theory often brings to the discovery of new physics. To overcome the obstacle underlined by the Coleman-Mandula no-go theorem was introduced the concept of Lie superalgebra as extension of a classical Lie algebra. In the 1975, the Haag–Łopuszański–Sohnius theorem [26] stated that nontrivial extensions of four-dimensional Poincaré algebra containing internal symmetries in the bosonic part can be found including fermionic generators. The introduction of supersymmetry opened a new branch of study of field theories.
The presence of fermionic generators links bosonic and fermionic degrees of freedom putting them on the same footing. A good reason to introduce supersymmetric string theory. Modifying the original Polyakov action (1.1), adding to the worldsheet certain ferionic degrees of freedom ψ_μ, the theory is described by

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{-h} \left(h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu + ... \right) , \tag{1.12}$$

where ρ^α are a two-dimensional Clifford algebra representation. The action (1.12) is supersymmetric adding two particular terms which however are zero if the gauge symmetries are fixed properly. Therefore the Euler Lagrange equation are

$$\partial^\alpha \partial_\alpha X^\mu = 0 \,, \qquad \rho^\alpha \partial_\alpha \psi = 0 \,. \tag{1.13}$$

They are easly solved in the coordinate $\sigma^\pm$. For the boson fields the expantion is (1.7) while, for closed string, the fermionic degrees of freedom

$$\psi^{\mu}(\sigma) = \psi^{\mu}_{R}(\sigma^{+}) + \psi^{\mu}_{L}(\sigma^{-}) \tag{1.14}$$

two different expansion holds, one with periodic boundary conditions, the Ramond sector,

$$\begin{aligned} \psi^{\mu}_{R} &= \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d^{\mu}_{n} e^{-in(\tau-\sigma)} \,, \\ \psi^{\mu}_{L} &= \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} \tilde{d}^{\mu}_{n} e^{-in(\tau+\sigma)} \,, \end{aligned} \tag{1.15}$$

and one with antiperiodic boundary conditions, the Neveu-Schwartz sector,

$$\begin{aligned} \psi^{\mu}_{R} &= \frac{1}{\sqrt{2}} \sum_{n\in\left(\mathbb{Z}+\frac{1}{2}\right)} b^{\mu}_{n} e^{-in(\tau-\sigma)} \,, \\ \psi^{\mu}_{L} &= \frac{1}{\sqrt{2}} \sum_{n\in\left(\mathbb{Z}+\frac{1}{2}\right)} \tilde{b}^{\mu}_{n} e^{-in(\tau+\sigma)} \,. \end{aligned} \tag{1.16}$$

With same method of the bosonic string, imposing canonical Poisson Bracket and quantizing the theory one obtains that the critical dimension must be ten and the SuperVirasoro algebras need central extensions [10]. Moreover armed of the mass formula one can find that the vacuum for the NS sector is in the vector representation of the little group in ten dimension, $SO(8)$, while the vacuum in the R sector has a spinor representation.
To preserve supersymmetry and maintain the same number of degrees of freedom between bosons and fermions for each mass level the Gliozzi-Scherk-Olive (GSO) projection is performed [10]. Two different theories of closed string are generated modifying the boundary conditions for two inequivalent choice of the zero mass state in the spectrum of the superstring.
The Lie group $SO(8)$ has two spinorial representation $\mathbf{8}$ and $\mathbf{8}'$, other than the vector one $\mathbf{8_v}$. The choice of the chirality generates two inequivalent theories. The non chiral theory is called IIA and the vacuum is

$$(\mathbf{8_v} \oplus \mathbf{8}) \otimes (\mathbf{8_v} \oplus \mathbf{8}') = (\mathbf{35} \oplus \mathbf{28} \oplus \mathbf{1}) \oplus (\mathbf{56}' \oplus \mathbf{8}) \oplus (\mathbf{56} \oplus \mathbf{8}') \oplus (\mathbf{56} \oplus \mathbf{8_v}) \,, \tag{1.17}$$

that in terms of fields are described as

$$(g_{\mu\nu} + B_{\mu\nu} + \phi) + (\psi^{\mu\prime}_{\alpha} + \lambda_{\beta}) + (\psi^{\mu}_{\alpha} + \lambda'_{\beta}) + (C^{(3)}_{\mu\nu\rho} + C^{(1)}_{\mu}) \,, \tag{1.18}$$

The chiral one is called IIB and the vacuum is

$$(\mathbf{8_v} \oplus \mathbf{8}) \otimes (\mathbf{8_v} \oplus \mathbf{8}) = (\mathbf{35} \oplus \mathbf{28} \oplus \mathbf{1}) \oplus (\mathbf{56} \oplus \mathbf{8}') \oplus (\mathbf{56} \oplus \mathbf{8}') \oplus (\mathbf{35} \oplus \mathbf{28} \oplus \mathbf{1}) \,, \tag{1.19}$$

that in terms of fields are described as

$$(g_{\mu\nu} + B_{\mu\nu} + \phi) + (\psi^{\mu}_{\alpha} + \lambda'_{\beta}) + (\psi^{\mu}_{\alpha} + \lambda'_{\beta}) + (C^{(4)}_{\mu\nu\rho\sigma} + C^{(2)}_{\mu\nu} + C^{(0)}) \,. \tag{1.20}$$

The presence of two gravitinos implies there is maximal supersymmetry.

However, three others type of superstring can be built. One is called type I and the other two are Heterotic strings, $SO(32)$ and $E_8 \times E_8$ [10]. The first can be obtained by a particular projection of type IIB spectrum and contains unoriented strings that may be open or closed. The vacuum is composed of IIA NS-NS and R-NS sector plus a Yang-Mills supermultiplet SO(32).
The last possibility is obtained by exploiting the complete decoupling of left movers and right movers. Choosing left movers of the bosonic string theory and right movers of the supersymmetric one, it is again possible to preserve supersymmetry obtaining the Heterotic superstrings. There are two of them distinguished by the internal symmetries gauge group $SO(32)$ or $E_8 \times E_8$. They share the same amount of supersymmetry of type I.

All the five superstring theories are themselves consistent, however a deeper knowledge of their relations underlines a unifing picture. They describe different expansions of the same general theory. In this analysis another corner of this picture arises, called M-theory. This is an eleven-dimensional theory supposed to not have a coupling constant and with eleven-dimensional supergravity as low energy effective action.

What makes all this machinery possible are S, T and U dualities, that go beyond the ordinary symmetries because they link different theories.
S-duality in type IIB is $\mathbb{Z}_2$ transformation that exchanges the 2-forms of the theory up to a sign and takes $g_s \to \frac{1}{g_s}$. It is a subgroup of a bigger $SL(2, \mathbb{Z})$ duality interchanging the role of perturbative objects and solitons. As we will see, a link can be easily found between zero mass level for the case of the two $SO(32)$ theories sharing the same effective low energy description.
T-duality in type II interchanges the role of momentum and winding modes in the spectrum and is intrinsically perturbative. Reducing type II on T^d one obtains $SO(d, d, \mathbb{Z})$ as the simmetry group. A similar connection exists between the two Heterotic superstrings.
U-duality in type II arises from the reduction on T^d, as the internal symmetry group of the nonlinear sigma model of scalars generated upon the dimensional reduction. These groups are called $E_{d+1(d+1)}(\mathbb{Z})$ in literature.

In type II superstrings the spectrum of some solitonic objects goes beyond the excitation modes of the fundamental string. Other objects are present and the most important are called Dp-branes and can be detected by the presence of the p-forms in the spectrum. A Dp-brane is an extended object like the string but with a worldvolume that spans $p+1$ dimension. The action is a generalization of the (1.1) to which one can add the coupling with potential p-form. In the NS-NS sector, the only p-form is the Kalb-Ramond 2-form $B_{\mu\nu}$ that can be coupled electrically to strings NS1 or magnetically NS5-branes and this can be understood by Hodge duality. Similarly, in the R-R sector, the possible Dp-brane sources depend on the p-forms allowed. A Dp-brane is an electrical source for the $C^{(p+1)}$ potential and a magnetical source for the $C^{(7-p)}$ potential. Summarizing the idea is that

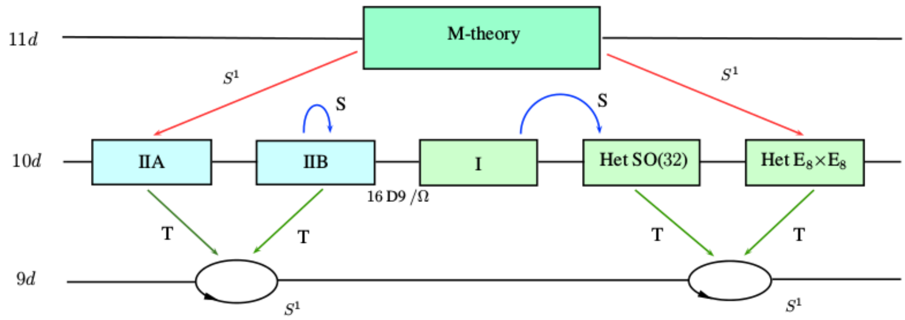

Figure 1.1: *S,T and U dualities links together the five superstrings and M-theory.*

in type IIA there are D0, D2, D4 and D6 branes[1] and in type IIB we have D(-1), D1, D3, D5 and D7 branes. In a democratic formulation of the p-forms, considering on the same foot a potential and its Hodge dual, each of these objects can be dyonically charged.

The dynamics of a Dp-brane is given by open strings ending on them. The letter D stands for Dirichlet, the boundary condition imposed on some of the worldsheet scalar excitations of this open strings. Their quantization introduces additional degrees of freedom. At low energy, they can be described by a p+1-dimensional field theory living on the subspace wrapped by Dp-branes.

The presence of these solitonic objects linked by dualities and the infinite towers of massive states are not the only reasons for which superstrings are very hard to be treated in a nonperturbative regime. The presence of two different coupling constants g_s, regulating the quantum behaviour, and even more α', inverse proportionally to the mass of the states, makes theory something different from an ordinary quantum field theory. Even if g_s is similar to the usual coupling of a quantum field theory, on the other hand, α' switches on and off massive states of superstring and their inclusion change drastically the theory. What is much more understood and studied is the low energy description. The field theory that governs the dynamical behaviour of the massless particles in the spectrum that is called Supergravity. It is natural to consider it as a classical field theory, in fact it contains General Relativity and thus it is not renormalizable.

1.3 Supergravity as Effective Superstring

Supergravity was found before the advent of the superstrings because it is itself consistent as classical field theory. It can be formulated in different d dimensions starting from the representation of the supersymmetry algebra with certain numbers $\mathcal{N}$ of supercharges. Its peculiarity is to contain always gravity in terms of a spin 2 field and have a local supersymmetry invariance thanks to the presence of the fermionic partners of the

[1] At level of supergravity, in massive IIA there is also the D8-brane and in type IIB the D9-brane can be added with a coupling with ten dimensional not propagating RR form.

metric, the gravitinos. A systematic study of these theories shows that these features constraint very much the Lagrangian, such that going up with the dimensions the possibilities become much more restricted until $d = 11$ where exists only one supergravity theory [2].

The eleven-dimensional supergravity is composed only by the gravity multiplet $(g_{\mu\nu}, \psi^{\mu}_{\alpha}, A^{(3)}_{\mu\nu\rho})$ and the bosonic Lagrangian reads

$$I_{11} = \int \left(R * 1 - \frac{1}{2} F^{(4)} \wedge * F^{(4)} + \frac{1}{6} F^{(4)} \wedge F^{(4)} \wedge A^{(3)} \right) . \tag{1.21}$$

The antisymmetric 3-form is a gauge potential for the field strenght that is $F^{(4)} = \mathrm{d}A^{(3)}$. The gauge transformation $\delta A^{(3)} = \mathrm{d}\Lambda^{(2)}$ leveas (1.21) invariant up to total derivatives. The Chern-Simons term is required to have local supersymmetry invariance of the full action, with also the gravitino terms.
The Lagrangian (1.21) can be regarded as the low energy description of the massless modes of M-theory. From the presence of a 3-form potential, following the previous discussion on Dp-branes and sources, we understand that in M-theory must be present M2 and M5 brane sources.

Even more, as shown in figure (1.1), this theory is linked to type IIA supergravity with a Kaluza-Klein (KK) reduction on S^1. This means that compactifing on one of the eleven coordinates,

$$\begin{array}{cccc} \text{11d SUGRA} & g^{(11)}_{\mu\nu} & + & A^{(3)}_{\mu\nu\rho} \\ \downarrow & \downarrow & & \downarrow \\ \text{Type IIA} & (g^{(10)}_{\mu\nu} + C^{(1)}_{\mu} + \phi) & + & (C^{(3)}_{\mu\nu\rho} + B_{\mu\nu}) \end{array}$$

we can obtain the Lagrangian of type IIA supergravity from (1.21). The explict reduction of the action (1.21) in terms of $d = 10$ fields is

$$\mathrm{d}s^2_{11} = e^{-\frac{1}{6}\phi} \mathrm{d}s^2_{10} + e^{\frac{4}{3}\phi} (\mathrm{d}y - C^{(1)})^2 , \quad A^{(3)} = C^{(3)} + B \wedge \mathrm{d}y , \tag{1.22}$$

A straightforward calculation shows the final result

$$\begin{aligned} I_{IIA} = \int \Big(& R * 1 - \frac{1}{2} \mathrm{d}\phi \wedge * \mathrm{d}\phi - \frac{1}{2} e^{-\phi} H^{(3)} \wedge * H^{(3)} - \frac{1}{2} e^{\frac{3}{2}\phi} F^{(2)} \wedge * F^{(2)} \\ & - \frac{1}{2} e^{\frac{1}{2}\phi} \tilde{F}^{(4)} \wedge * \tilde{F}^{(4)} - \frac{1}{2} F^{(4)} \wedge F^{(4)} \wedge B \Big) , \end{aligned} \tag{1.23}$$

where we have defined

$$F^{(2)} = \mathrm{d}C^{(1)} , \quad F^{(4)} = \mathrm{d}C^{(3)} , \quad H^{(3)} = \mathrm{d}B , \quad \tilde{F}^{(4)} = F^{(4)} + C^{(1)} \wedge H^{(3)} . \tag{1.24}$$

This theory admits a not trivial deformation called the Romans mass [27]. This is the dual of a ten-dimensional field strenght associated to the D8-branes presence. The theory is

[2]Over $d = 11$ representation of supersymmetry show that is necessary to add a field with irreducible spin bigger that 2.

called massive IIA and its higher dimensional origin is not known. The Lagrangian reads

$$
\begin{aligned}
I_{IIA_m} = \int \Big(& R * 1 - \frac{1}{2}\mathrm{d}\phi \wedge *\mathrm{d}\phi - \frac{1}{2}e^{-\phi} H^{(3)} \wedge *H^{(3)} - \frac{1}{2}e^{\frac{3}{2}\phi} F^{(2)} \wedge *F^{(2)} \\
& - \frac{1}{2}e^{\frac{1}{2}\phi}\tilde{F}^{(4)} \wedge *\tilde{F}^{(4)} - \frac{1}{2}F^{(4)} \wedge F^{(4)} \wedge B - \frac{1}{6}mF^{(4)} \wedge B \wedge B \wedge B \\
& + \frac{1}{40}m^2 B \wedge B \wedge B \wedge B \wedge B - \frac{1}{2}m^2 e^{\frac{5}{2}\phi} * 1 \Big) .
\end{aligned}
\tag{1.25}
$$

where we have defined

$$
\begin{aligned}
F^{(2)} =& \mathrm{d}C^{(1)} + mB\,, \quad F^{(4)} = \mathrm{d}C^{(3)}\,, \quad H^{(3)} = \mathrm{d}B\,, \\
\tilde{F}^{(4)} =& F^{(4)} + C^{(1)} \wedge H^{(3)} + \frac{1}{2}B \wedge B\,.
\end{aligned}
\tag{1.26}
$$

This theory shares many features with gauged supergravity in less dimensions. Its equation of motion are complicated by the runaway potential of the dilaton and different new gauge terms.

For the chiral theory, type IIB, exists only a pseudo Lagrangian

$$
\begin{aligned}
I_{IIB} = \int \Big(& R * 1 - \frac{1}{2}\mathrm{d}\phi \wedge *\mathrm{d}\phi - \frac{1}{2}e^{-\phi} H^{(3)} \wedge *H^{(3)} - \frac{1}{2}e^{2\phi} F^{(1)} \wedge *F^{(1)} \\
& - \frac{1}{2}e^{\phi}\tilde{F}^{(3)} \wedge *\tilde{F}^{(3)} - \frac{1}{2}\tilde{F}^{(5)} \wedge *\tilde{F}^{(5)} - \frac{1}{2}C^{(4)} \wedge F^{(3)} \wedge B \Big) ,
\end{aligned}
\tag{1.27}
$$

that must be supplemented by the self duality condition $F^{(5)} = *F^{(5)}$, imposed after the variation of (1.27). Here we have defined

$$
\begin{aligned}
F^{(1)} =& \mathrm{d}C^{(0)}\,, \quad F^{(3)} = \mathrm{d}C^{(2)}\,, \quad \tilde{F}^{(3)} = \mathrm{d}C^{(2)} + C^{(0)} \wedge H^{(3)}\,, \\
H^{(3)} =& \mathrm{d}B\,, \quad \tilde{F}^{(5)} = \mathrm{d}C^{(4)} + C^{(2)} \wedge H^{(3)}\,.
\end{aligned}
\tag{1.28}
$$

This theory enjoys an $SL(2,\mathbb{R})$ that is the continuos version of $SL(2,\mathbb{Z})$ containing S-duality in string theory. This symmetry acts on the bosonic sector as

$$
\tau \to \frac{a\tau + b}{c\tau + d}\,, \quad B^\alpha = (\Lambda^{-1})_\beta{}^\alpha B^\beta\,, \quad F^{(5)} \to F^{(5)}\,,
\tag{1.29}
$$

where the parametrization of the $SL(2,\mathbb{R})$ group is

$$
\Lambda^\alpha{}_\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{with} \quad ad - cb = 1\,,
\tag{1.30}
$$

and the complex scalar field is defined as

$$
\tau = C^{(0)} + ie^{-\phi}\,, \quad B^\alpha = (-B, C^{(2)})^t\,.
\tag{1.31}
$$

The element $b = c = 1$ and $a = d = 0$ is exactly the $\mathbb{Z}_2 \in SL(2,\mathbb{R})$ representing S-duality in type IIB.

The Heterotic supergravity are half maximal supergravities and the Lagrangian reads

$$I_H = \int \Big(R * 1 - \frac{1}{2}\mathrm{d}\phi \wedge *\mathrm{d}\phi - \frac{1}{2}e^{-\phi}\tilde{H}^{(3)} \wedge *\tilde{H}^{(3)} - \frac{1}{2}e^{-\frac{1}{2}\phi}\alpha' \mathrm{tr}(F^{(2)} \wedge *F^{(2)}\Big)\,, \quad (1.32)$$

where

$$\tilde{H}^{(3)} = \mathrm{d}B + \alpha'\omega\,, \quad \omega = \frac{1}{4}(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A)\,, \quad (1.33)$$

where $A = A^a T_a$ is a Yang Mills field in the adjoint representation. This Lagrangian is free of anomalies for the choice of the gauge group $SO(32)$, $E_8 \times E_8$ and $U(1)^{496}$. The last one does not have an origin from superstrings.
Chosing T_a as the generators of the algebra of $SO(32)$, the action (1.32) describes SO(32) Heterotic supergravity. However, from this action is possible to recover also the type I supergravity action using the S-duality transformation $\phi \to -\phi$.

This concludes the exposition on the highest dimensional supergravities. In lower dimension more possibilities are presents, a larger class of theories exhibit local supersymmetry. They are linked each other in many cases. The dimensional reduction through a compactification manifold results to be a great instrument to study this network.

1.4 Compactifications

The need of a compactification arises from the necessity of linking the ten-dimensional world of superstrings with the low energy four-dimensional physics that we experiment everyday. The idea is to suppose that six dimensions are curled up into a compact space and that the remaining four span the extended spacetime. The idea is to study what happens to the theory living in four dimensions, what is the effect of the six compact ones.

We expand the higher dimensional fields in terms of fields living on the compact background. The result is an infinite tower of states with increasing masses. However, the analysis of higher dimensional equations of motion for these states can show the possibility to truncate this tower consistently to a finite set of massless fields that describes correctly the higher dimensional theory. This means that solutions of the lower dimensional theory can always be uplifted to solutions of the higher dimensional one. In this case, the compactification is promoted to a consistent truncation.

The basic example is to consider the theory of a $(d+1)$-dimensional massless scalar ϕ on $\mathbb{R}^d \times S^1$ with radius l [28]

$$\phi(x_1, \ldots, x_d, y) = \sum_n \phi_n(x_1, \ldots, x_d) e^{\frac{2\pi i n}{l} y}\,. \quad (1.34)$$

The wave equation reads

$$\Box_{d+1}\phi = \sum_n \left(\Box_d - m_n^2\right) \phi_n e^{i m_n y} = 0\,, \quad (1.35)$$

d	G	H
9	$\mathbb{R}^+ \times SL(2)$	$SO(2)$
8	$SL(2) \times SL(3)$	$SO(2) \times SO(3)$
7	$SL(5)$	$SO(5)$
6	$SO(5,5)$	$SO(5) \times SO(5)$
5	$E_{6(6)}$	$Usp(8)$
4	$E_{7(7)}$	$SU(8)$
3	$E_{8(8)}$	$SO(16)$

Table 1.1: *G is the group of the hidden symmetries in d dimensions and H is its maximal compact subgroup arising upon the compactification of T^{10-d} and dualization of $(d-2)$-forms. The scalar fields set in the coset space G/H.*

therefore one finds n separate Klein Gordon equations with masses $m_n = \frac{2\pi n}{l}$. Setting to zero the massive modes one ends with the wave equation of a massless scalar in d dimension, which clearly statisfied also the wave equation in $d+1$ dimensions. This means that the d dimensional theory is a consistent truncation of the original one. Moreover, this truncation can be physically understood supposing the radius l to be very small and therefore very high masses of the other modes and thus a natural decoupling.
What must be underlined in the example is that all works well because the massive modes do not mix in the equations of motion and they can be set to zero separately without affecting the others. In more complicated reductions this does not happen.

The compactification on S^1 of eleven-dimensional supergravity to ten-dimensional type IIA supergravity, shown in the previous section, is a consistent truncation. There we have supposed tacitly ∂_y to be Killing vector to set to zero all massive modes of all fields. Consistent truncations construct a web between supergravities and their deformations in different dimensions. It results very powerful to study solutions. A first example is the reduction of ten-dimensional type IIA on $T^{10-d} = S^1 \times \cdots \times S^1$. Starting from IIA supergravity and rolling up one dimension on a circle and dualizing $(d-2)$-forms one can see the rising of a always bigger coset space G/H of scalars coupled to gravity and gauge fields.

The reduction on S^1 does not break any supersymmetries, therefore this results in a chain between the maximal supersymmetric ungauged supergravities in different dimensions. The group G is the continuous version of the U-duality group of superstrings and it is obtained dualizing $d-2$-forms of the reduced theory. Because of this fact, they are called hidden symmetries. Moreover, with a further reduction to two dimensions and a particular gauge choice for the two dimensional metric one can show the complete integrability of the theory [29].

More complicated cases are those leading to deformed supergravities. The deforma-

d	G	Origin
8	$SO(3)$	IIA on S^2
7	$SU(2)$	IIA$_m$ on S^3
7	$SO(5)$	11d on S^4
6	$SU(2)$	IIA$_m$ on S^4
6	$SO(5)$	IIA on S^4
5	$SO(6)$	IIB on S^5
4	$ISO(7)$	IIA$_m$ on S^6
4	$SO(8)$	11d on S^7

Table 1.2: *All the known consistent truncations of type II or eleven-dimensional supergravity on the spheres. G is the gauge group that appears in the reduced d-dimensional theory.*

tion consist in a gauging of a subgroup of the internal symmetries of the undeformed theory via the lower dimensional 2-forms field strengths. This produces covariant derivatives, in the kinetic terms of the scalars, and a scalar potential. The latter stabilizes the scalars fields in the low energy description.
All these possible self-consistent deformations are been studied in different dimensions independently. Often they are linked to each other by compactifications provided that some fluxes are switched on and these are called compactifications with geometrical fluxes. They are distinguished in metric fluxes when we consider a curved compactification manifold, and gauge fluxes, when certain charges of the higher dimensional p-forms are non zero. However sometimes a higher dimensional origin of some gaugings is not known and these are considered as non-geometrical fluxes [3].

Already in presence of metric and gauge fluxes, the consistency holds in a few cases. The nonlinear reduction with the highest amount of symmetry are coset space and interesting examples are the truncations on the spheres of eleven and ten-dimensional supergravity [30–39].

Apart from the case of massive IIA, these are maximal supersymmetric supergravities with the bigger semisimple compact gauge group. This can be understood from the rich symmetry structure of the reduction manifold. The connection between the group of the isometries of the compactification manifold and the gaugings of the lower dimensional supergravity is a general feature.

To obtain less supersymmetric effective supergravity one needs more general classes of manifolds. Calabi-Yau threefolds CY_3 are the classical example. They are manifold for which we have a solution of the equation

$$\nabla_\mu \epsilon = 0\,, \tag{1.36}$$

[3] The highest dimensional example can be seen in massive type IIA.

where ϵ is a Majorana spinor with a defined chirality and ∇_μ is the covariant derivative with the CY_3 metric. It easily proved that the supersymmetry equations in ten-dimensional type II with zero fluxes admit spontaneous compactifications of the type

$$\text{Mink}_4 \times CY_3\,, \tag{1.37}$$

a vacuum configuration that preserves eight supersymmetries, exactly thanks to (1.36). A more careful analysis shows that $\mathcal{N} = 2$, $d = 4$ supergravity can be constructed as effective field theory of type II superstring compactified on CY_3 and it admits Mink_4 as a maximally supersymmetric vacuum. The property (1.36) is linked to many others nice features of CY_3 as to have $SU(3)$ holonomy group, first Chern class $c_1(CY_3) = 0$ and to be Ricci flat. They are complex manifolds characterized by a complex structure (3,0)-form Ω and a complex Kähler (1,1)-form J. Their Hodge diamond reads

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The deformations of the complex structure are $\delta\Omega \in H^{1,1}(CY_3)$ and the ones for the Kähler structure $\delta J \in H^{2,1}(CY_3)$, therefore the precise content of the $d = 4$ theory, that means the number of hypermultiplets and vectormultiplets, is linked to specific numbers $h^{1,1}$ and $h^{2,1}$. In particular for type IIA one has $h^{1,1}$ vectormultiplets and $h^{2,1} + 1$ hypermultiplets, while in type IIB there are $h^{2,1}$ vectormultiplets and $h^{1,1} + 1$ hypermultiplets. This mutation of roles is what has generated mirror symmetry. A couple of mirror CY_3 have $h^{1,1}$ and $h^{2,1}$ exchanged, therefore the compactifications of the two type IIB on this couple enjoys the same moduli space. Infact the moduli space of CY_3 is a product manifold that in supergravity arises as

$$\mathcal{M} = \mathcal{M}_{\text{SK}} \times \mathcal{Q}\,, \tag{1.38}$$

the product of a spiecial Kähler manifold for the vector multiplets and a quaternionic one for the hypermultiplets.
In presence of fluxes the equations (1.36) is more refined, but one can generate supersymmetric effective theory on CY_3 obtaining the reduced gauged supergravity [40–42]. In this sense, a similar story holds for Sasaki-Einstein manifolds SE_d. They are Kähler cone over odd dimensional Einstein manifold [43].
In this thesis we are mainly interested on $\mathcal{N} = 2$ in $d = 4$ and $d = 5$ gauged supergravities. The first comes from the reductions not only of type II on CY_3 [17,19] but also

from M-theory on SE_7 [18], the other one from type II on SE_5 [44] and from M-theory on CY_3 [45].

1.5 AdS/CFT Correspondence

The AdS/CFT correspondence [20, 22] is surely one of the most intriguing result arisen from superstrings. This duality relates $d+1$ dimensional gravitational theories in anti de Sitter (AdS) [46] spacetime to conformal field theories (CFT) [47] in d dimensions. It is a successful realization of the holographic principle [15], asserting that the description of the bulk AdS spacetime is encoded on its boundary on which the CFT lives.
The key symmetry of gauge/gravity duality is the d dimensional conformal group $SO(2,d)$, the maximal semisimple extension of the Lorentz group in d dimensions $SO(1,d-1)$, and characterizing the spacetime symmetry group a CFT. On the other hand AdS in $d+1$ dimensions is a maximally symmetric space and it has $SO(2,d)$ as isometry group. This correspondence is a weak/strong coupling duality, the behaviour of strongly coupled CFT is described by classical gravity theory.

The way in which the correspondence arises in string theory is a decoupling limit of certain branes configurations in type II and in M-theory. The couples of theories linked by the duality are descriptions of different aspects of the same physical system of branes. The father example is the stack of N coincident D3-branes as sources for the 5-form in type IIB string. The presence of branes curves the spacetime and the effective picture in type IIB supergravity is the configuration [14]

$$\begin{aligned} \mathrm{d}s^2 &= H^{-\frac{1}{2}}(-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2) + H^{\frac{1}{2}}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2)\,, \\ F^{(5)} &= (1+*)\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}H^{-1}\,, \\ H &= 1 + \left(\frac{R}{r}\right)^4\,, \quad R = 4\pi g_s \alpha'^2 N\,. \end{aligned} \tag{1.39}$$

In the limit $g_s \to 0$ and $\alpha' \to 0$ for large N , the interaction between branes and the bulk gravity is small and can be negleted. The bulk system boils down to $AdS_5 \times S^5$ background with 5-form fluxes in a supergravity description and on the brane world-worlume lives $\mathcal{N}=4$, $d=4$ $SU(N)$ Super Yang-Mills (SYM), both theories preserve 32 supercharges. The R symmetry of SYM is $SU(4) \sim SO(6)$ the isometry group of the S^5, in total on both sides there is the superconformal group $SU(2,2|4)$.
Another example, nowadays well-known, is the configuartion of N M2 branes in M-theory that gives rise to the $AdS_4 \times S^7$ supergravity description, dual of a $\mathcal{N}=6$, $d=3$ Chern-Simons theory $SU(N) \times SU(N)$ called ABJM.
The basic elements in CFT are operators $\mathcal{O}(x)$ and in supergravity fields $\phi = \phi(x,z)$. The dictionary of the duality is made by the usual pairing between sources and operators in QFT

$$< e^{\int \mathrm{d}^4 x \phi_0(x)\mathcal{O}(x)} >\,, \tag{1.40}$$

where $\phi_0(x)$ is the boundary value of the supergravity field ϕ.
The main statement of the correspondence is the identification

$$Z_{CFT}[\phi_0] = e^{-S_{AdS}(\phi(x,z))|_{\phi_0}} , \tag{1.41}$$

the generating functional of the connected graphs equals the classical supergravity action evaluted on the boundary value of the fields.

The duality works well when a high amount of symmetry is present. With less symmetry, some calculations become harder and the correspondence is not again completely understood. The idea is to consider more complicated compactification than the sphere reductions, that, at the level of strings or M-theory, resides to consider more elaborate branes configurations and study supergravity solutions that are not only vacuums. With symmetry as the guide one can identify the corresponding CFT and proceed to the calculation of certain physical quantities on both side that are belived to be dual, checking the correspondence.
Supergravity solutions that represent black objects are fruitful arenas for this analysis. For example asymptotically AdS_4 black holes typically have an horizon geometry $AdS_2 \times \Sigma_2$, where Σ_2 is two dimensional Riemann surface. Solutions as static black holes with running scalars have the interpretation of a renormalization group flow across dimensions between quantum field theories. In the specific example, some recent precision tests have proved that the entropy of a certain class of AdS_4 black holes in $\mathcal{N} = 2$ supergravity have an entropy coinciding with the value of the partition function of ABJM [48] posed on $S^1 \times \Sigma_2$ [49]. Both arise from a well-defined minimization process.
A similar story holds for AdS_5 static black strings that flow in $AdS_3 \times \Sigma_2$ IR geometry. In this case, the calculation concerns the anomaly coefficients of the 2d and 4d SCFT [4,50]. These are surely the main motivation to start a deeper understanding of supergravity solutions. We will focus on these two case for the rest of the thesis.

CHAPTER 2

Abelian Gauged $\mathcal{N} = 2$ Supergravity

The supergravity theory with 8 supercharges can be formulated starting from $d = 6$, but the richest geometrical structure emerges in $d = 4$ and $d = 5$, where the moduli target space is a product manifold. Many symmetries are encoded in this theory and determine the various couplings between the fields. Only theories with an abelian gauge group are considered. We will describe their structure and write down the equations of motion for the truncation to the bosonic sector, our subject of study. Furthermore, we specialize the equations respectively for a static black hole and static black string studying the one-dimensional effective actions arising from the higher ones.
The new result of this chapter is the one-dimensional symplectically covariant effective Lagrangian for a four-dimensional static black hole with running hypermultiplets [1].

2.1 General matter coupled $d = 4$ supergravity

The supergravity multiplet of $\mathcal{N} = 2$, $d = 4$ supergravity can be coupled to a number n_V of vector multiplets and to n_H hypermultiplets. The bosonic sector then includes the vierbein $e^a{}_\mu$, $n_V + 1$ vector fields A^Λ_μ with $\Lambda = 0, \ldots n_V$ (the graviphoton plus n_V other fields from the vector multiplets), n_V complex scalar fields z^i ($i = 1, \ldots, n_V$), and $4n_H$ real hyperscalars q^u ($u = 1, \ldots, 4n_H$).

The complex scalars z^i of the vector multiplets parametrize an n_V-dimensional special Kähler manifold, i.e., a Kähler-Hodge manifold, with Kähler metric $g_{i\bar{\jmath}}(z, \bar{z})$, which is the base of a symplectic bundle with the covariantly holomorphic sections[1]

$$\mathcal{V} = \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix}, \qquad D_{\bar{\imath}}\mathcal{V} \equiv \partial_{\bar{\imath}}\mathcal{V} - \frac{1}{2}\left(\partial_{\bar{\imath}}\mathcal{K}\right)\mathcal{V} = 0\,, \tag{2.1}$$

obeying the constraint

$$\langle \mathcal{V}|\bar{\mathcal{V}}\rangle \equiv \bar{L}^\Lambda M_\Lambda - L^\Lambda \bar{M}_\Lambda = -i\,, \tag{2.2}$$

where $\mathcal{K}$ is the Kähler potential. Alternatively one can introduce the explicitly holomorphic sections of a different symplectic bundle,

[1]We use the conventions of [51].

$$v \equiv e^{-\mathcal{K}/2}\mathcal{V} \equiv \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} . \tag{2.3}$$

In appropriate symplectic frames it is possible to choose a homogeneous function of second degree $F(X)$, called prepotential, such that $F_\Lambda = \partial_\Lambda F$. In terms of the sections v the constraint (2.2) becomes

$$\langle v|\bar{v}\rangle \equiv \bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda = -ie^{-\mathcal{K}}. \tag{2.4}$$

The couplings of the vector fields to the scalars are determined by the $(n_V+1)\times(n_V+1)$ period matrix $\mathcal{N}$, defined by the relations

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma}\, L^\Sigma \,, \qquad D_{\bar{\imath}}\bar{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma}\, D_{\bar{\imath}}\bar{L}^\Sigma \,. \tag{2.5}$$

If the theory is defined in a frame in which a prepotential exists, $\mathcal{N}$ can be obtained from

$$\mathcal{N}_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2i\frac{(N_{\Lambda\Gamma}X^\Gamma)(N_{\Sigma\Delta}X^\Delta)}{X^\Omega N_{\Omega\Psi}X^\Psi} \,, \tag{2.6}$$

where $F_{\Lambda\Sigma} = \partial_\Lambda\partial_\Sigma F$ and $N_{\Lambda\Sigma} \equiv \mathrm{Im}(F_{\Lambda\Sigma})$. Introducing the matrix[2]

$$\mathcal{M} = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix}, \tag{2.7}$$

we have the important relation between the symplectic sections and their derivatives,

$$\frac{1}{2}(\mathcal{M} - i\Omega) = \Omega\bar{\mathcal{V}}\mathcal{V}\Omega + \Omega D_i\mathcal{V}g^{i\bar{\jmath}}D_{\bar{\jmath}}\bar{\mathcal{V}}\Omega \,, \tag{2.8}$$

where

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{2.9}$$

The equation (2.8) underlines the symplectic structure of the special Kähler manifold essential for its definition. The nonlinear sigma model of the scalars $g_{i\bar{\jmath}}$ has a nontrivial isometry group, called U-duality. Moreover, (2.7) is the scalar dependent matrix of the coupling with the gauge fields. The last ones with their duals build a symplectic section (2.23) and U-duality is a symmetry in the ungauged supergravity, so one needs that also the gauge symplectic section transforms under this symmetry. The complete characterization of the theory is reached only when a particular embedding of U-duality in $Sp(2n_{\rm v}+2,\mathbb{R})$ is chosen, typically fixed by supersymmetry [51–53]. The action of this on-shell symmetry is often called electromagnetic duality [54].

The $4n_H$ real hyperscalars q^u parametrize a quaternionic Kähler manifold with metric $h_{uv}(q)$. A quaternionic Kähler manifold is a $4n$-dimensional Riemannian manifold admitting a locally defined triplet $\vec{K}_u{}^v$ of almost complex structures satisfying the quaternion relation

[2] We use the notation $R = \mathrm{Re}\,\mathcal{N}$ and $I = \mathrm{Im}\,\mathcal{N}$.

$$h^{st}K^x{}_{us}K^y{}_{tw} = -\delta^{xy}h_{uw} + \varepsilon^{xyz}K^z{}_{uw}\,, \tag{2.10}$$

and whose Levi-Civita connection preserves $\vec{K}$ up to a rotation,

$$\nabla_w \vec{K}_u{}^v + \vec{\omega}_w \times \vec{K}_u{}^v = 0\,, \tag{2.11}$$

where $\vec{\omega} \equiv \vec{\omega}_u(q)\,dq^u$ is the connection of the $\mathrm{SU}(2)$-bundle for which the quaternionic manifold is the base. An important property is that the $\mathrm{SU}(2)$ curvature is proportional to the complex structures,

$$\Omega^x \equiv d\omega^x + \frac{1}{2}\varepsilon^{xyz}\omega^y \wedge \omega^z = -\,K^x\,. \tag{2.12}$$

As far as the gaugings are concerned, we shall consider only abelian symmetries of the action. Under abelian symmetries, the complex scalars z^i transform trivially, so that we will be effectively gauging abelian isometries of the quaternionic-Kähler metric h_{uv}. These are generated by commuting Killing vectors $k^u_\Lambda(q)$, i.e., $[k_\Lambda, k_\Sigma] = 0$.
This way of writing the Killing vectors $k^u_\Lambda = -\Theta^a_\Lambda k^u_a$, implies a coupling between the two manifolds of the scalars, where a is an index that runs along the isometries of the quaternionic one. Encoded in the embedding tensor Θ^a_Λ there is the physical connection between gauge fields and gauged isometries. Before having chosen a particular embedding tensor the symplectic covariance is preserved including magnetic gaugings, as we will see.
The requirement that the quaternionic Kähler structure be preserved implies the existence, for each Killing vector, of a triplet of Killing potentials, or moment maps, P^x_Λ, such that

$$\mathsf{D}_u P^x_\Lambda \equiv \partial_u P^x_\Lambda + \varepsilon^{xyz}\omega^y{}_u P^z_\Lambda = -2\Omega^x{}_{uv}k^v_\Lambda\,. \tag{2.13}$$

One of the most important relations satisfied by the moment maps is the so-called equivariance relation. For abelian gaugings it has the form

$$\frac{1}{2}\epsilon^{xyz}P^x_\Lambda P^y_\Sigma - \Omega^x_{uv}k^u_\Lambda k^v_\Sigma = 0\,. \tag{2.14}$$

The bosonic Lagrangian reads

$$\begin{aligned}\sqrt{-g}^{\,-1}\mathscr{L} = {}& \frac{R}{2} - g_{i\bar{\jmath}}\,\partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}} - h_{uv}\hat{\partial}_\mu q^u \hat{\partial}^\mu q^v \\ &+ \frac{1}{4}I_{\Lambda\Sigma}F^{\Lambda\mu\nu}F^\Sigma{}_{\mu\nu} + \frac{1}{4}R_{\Lambda\Sigma}F^{\Lambda\mu\nu}\star F^\Sigma{}_{\mu\nu} - \mathrm{g}^2 V_g(z,\bar{z},q)\,,\end{aligned} \tag{2.15}$$

where the scalar potential has the form

$$V_g = 4h_{uv}k^u_\Lambda k^v_\Sigma L^\Lambda \bar{L}^\Sigma + (g^{i\bar{\jmath}}D_i L^\Lambda D_{\bar{\jmath}}\bar{L}^\Sigma - 3L^\Lambda \bar{L}^\Sigma)P^x_\Lambda P^x_\Sigma\,, \tag{2.16}$$

the covariant derivatives acting on the hyperscalars are

$$\hat{\partial}_\mu q^u = \partial_\mu q^u + \mathrm{g}A^\Lambda_\mu k^u_\Lambda\,, \tag{2.17}$$

and

$$I_{\Lambda\Sigma} \equiv \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\,, \qquad R_{\Lambda\Sigma} \equiv \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\,, \qquad I^{\Lambda\Sigma}I_{\Sigma\Gamma} = \delta^{\Lambda}{}_{\Gamma}\,. \tag{2.18}$$

If we introduce the quantity

$$\mathcal{W}^x = \langle \mathcal{P}^x, \mathcal{V}\rangle = L^{\Lambda}P^x_{\Lambda}\,, \tag{2.19}$$

with

$$\mathcal{P}^x = \begin{pmatrix} 0 \\ P^x_{\Lambda} \end{pmatrix}, \tag{2.20}$$

and use the quaternionic relations (2.37), (2.39), (2.40), the scalar potential (2.16) can be rewritten in the form

$$V_g = \tilde{\mathbb{G}}^{AB}\mathbb{D}_A\mathcal{W}^x\mathbb{D}_B\bar{\mathcal{W}}^x - 3|\mathcal{W}^x|^2\,, \tag{2.21}$$

where we defined

$$\tilde{\mathbb{G}}^{AB} = \begin{pmatrix} g^{i\bar{\jmath}} & 0 \\ 0 & \frac{1}{3}h^{uv} \end{pmatrix}, \qquad \mathbb{D}_A = \begin{pmatrix} D_i \\ \mathsf{D}_u \end{pmatrix}. \tag{2.22}$$

As underlined by the formalism yet introduced, a natural generalization of the theory, due to the electric-magnetic duality of the $d = 4$, is to consider also the dual gauge fields $A_{\Lambda\mu}$, called magnetic gauge fields. This implies the inclusion of magnetic Killing vectors $k^{\Lambda u}$ and magnetic moment maps $P^{x\Lambda}$, referred to the gauged isometries. This formulation of gauged supergravity is typically expressed in terms of the embedding tensor $\Theta^a_M = (\Theta^a_\Lambda, \Theta^{a\Lambda})^T$[3], and the main consequence is the restoration of symplectic covariance of the theory [55,56].

In this context, one introduces the symplectic vectors

$$\mathcal{A}_\mu = \begin{pmatrix} A^{\Lambda}_{\mu} \\ A_{\Lambda\mu} \end{pmatrix}, \qquad \mathcal{K}^u = \begin{pmatrix} k^{\Lambda u} \\ k^u_{\Lambda} \end{pmatrix}, \qquad \mathcal{P}^x = \begin{pmatrix} P^{x\Lambda} \\ P^x_{\Lambda} \end{pmatrix}, \tag{2.23}$$

where the magnetic quantities $k^{\Lambda u}$ and $P^{x\Lambda}$ obey to analogous relations (2.41) and (2.40), satisfied by the electric part. As was shown in [57], the locality constraint $\langle\Theta^a, \Theta^b\rangle = 0$, namely the possibility to rotate any gauging to a frame with a purely electric one, implies also

$$\langle \mathcal{K}^u, \mathcal{P}^x\rangle = 0\,. \tag{2.24}$$

In presence of magnetic gaugings, the general action (2.15) is modified in a nontrivial way by some topological terms [55]. The consistency of the theory requires the introduction of the auxiliary 2-forms $B_a = \frac{1}{2}B_{a\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ that do not change the number of degrees of freedom. The action has the form [55,56]

[3] In this section we explicitly introduce the indices $(M, N, \ldots)$ in the fundamental representation of $\mathrm{Sp}(2n_V + 2, \mathbb{R})$ for clarity [55,56].

$$
\begin{aligned}
\sqrt{-g}^{-1}\mathscr{L} = & \frac{R}{2} - g_{i\bar{\jmath}}\,\partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}} - h_{uv}\hat{\partial}_\mu q^u \hat{\partial}^\mu q^v + \frac{1}{4} I_{\Lambda\Sigma} H^{\Lambda\mu\nu} H^{\Sigma}{}_{\mu\nu} + \\
& \frac{1}{4} R_{\Lambda\Sigma} H^{\Lambda\mu\nu} \star H^{\Sigma}{}_{\mu\nu} - \frac{\epsilon^{\mu\nu\rho\sigma}}{4\sqrt{-g}} \Theta^{a\Lambda} B_{a\mu\nu} \partial_\rho A_{\Lambda\sigma} + \\
& \frac{1}{32\sqrt{-g}} \Theta^{\Lambda a} \Theta^{b}_{\Lambda} \epsilon^{\mu\nu\rho\sigma} B_{a\mu\nu} B_{b\rho\sigma} - \mathrm{g}^2 V_g \, ,
\end{aligned}
\tag{2.25}
$$

where the modified field strength $H^{\Lambda}{}_{\mu\nu} = F^{\Lambda}{}_{\mu\nu} + \frac{1}{2}\Theta^{\Lambda a} B_{a\mu\nu}$ was introduced. The covariant derivatives of the hyperscalars and the scalar potential read respectively [40, 55, 56]

$$
\hat{\partial}_\mu q^u = \partial_\mu q^u - \mathrm{g} A^{\Lambda}_{\mu} \Theta^{a}_{\Lambda} k^u_a - \mathrm{g} A_{\Lambda\mu} \Theta^{\Lambda a} k^u_a \equiv \partial_\mu q^u - \mathrm{g} \langle \mathcal{A}_\mu, \mathcal{K}^u \rangle \, , \tag{2.26}
$$

$$
V_g = 4 h_{uv} \langle \mathcal{K}^u, \mathcal{V} \rangle \langle \mathcal{K}^v, \bar{\mathcal{V}} \rangle + g^{i\bar{\jmath}} \langle \mathcal{P}^x, D_i \mathcal{V} \rangle \langle \mathcal{P}^x, \bar{D}_{\bar{\jmath}} \bar{\mathcal{V}} \rangle - 3 \langle \mathcal{P}^x, \mathcal{V} \rangle \langle \mathcal{P}^x, \bar{\mathcal{V}} \rangle \, . \tag{2.27}
$$

Note that it is also possible to generate (2.27) from (2.21) by a symplectic rotation.

A really important truncation of this theory is when the hypermultiplets are set to be constant. In this case, even if none abelian isometries of the scalar manifold is gauged, one finds a non trivial potential arising from the gauging of a $U(1) \subset SU(2)$ of the R symmetry group and is called FI gauged supergravity

$$
\begin{aligned}
\sqrt{-g}^{-1}\mathscr{L}_{FI} = & \frac{R}{2} - g_{i\bar{\jmath}}\,\partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}} + \frac{1}{4} I_{\Lambda\Sigma} F^{\Lambda\mu\nu} F^{\Sigma}{}_{\mu\nu} + \frac{1}{4} R_{\Lambda\Sigma} F^{\Lambda\mu\nu} \star F^{\Sigma}{}_{\mu\nu} \\
& - \left(g^{i\bar{\jmath}} \langle \mathcal{G}, D_i \mathcal{V} \rangle \langle \mathcal{G}, \bar{D}_{\bar{\jmath}} \bar{\mathcal{V}} \rangle - 3 \langle \mathcal{G}, \mathcal{V} \rangle \langle \mathcal{G}, \bar{\mathcal{V}} \rangle \right) \, .
\end{aligned}
\tag{2.28}
$$

The symplectic vector

$$
\mathcal{G} = (g^{\Lambda}, g_{\Lambda})^t \, , \tag{2.29}
$$

collects the coupling constants of the theory. Restoring the symplectic covariance at level of the gauged theory is important to treat models that have stringly origin [58].

2.2 General matter coupled $d = 5$ supergravity

The supergravity multiplet of $\mathcal{N} = 2$, $d = 5$ supergravity can be coupled to a number n_V of vector multiplets and to n_H hypermultiplets. It is possible also to introduce tensor multiplets, but they can always be dualized to vector multiplets [59] in FI case.

Firstly considering only the coupling to n_{v} vector multiplets [60, 61], we have the bosonic Lagrangian of $\mathcal{N} = 2$, $d = 5$ FI-gauged supergravity is[4]

$$
e^{-1}\mathscr{L} = \frac{1}{2} R - \frac{1}{2} \mathcal{G}_{ij} \partial_\mu \phi^i \partial^\mu \phi^j - \frac{1}{4} G_{IJ} F^{I}_{\mu\nu} F^{J\mu\nu} + \frac{e^{-1}}{48} C_{IJK} \epsilon^{\mu\nu\rho\sigma\lambda} F^{I}_{\mu\nu} F^{J}_{\rho\sigma} A^{K}_{\lambda} - \mathrm{g}^2 V_5 \, , \tag{2.30}
$$

where the scalar potential reads

$$
V = V_I V_J \left(\frac{9}{2} \mathcal{G}^{ij} \partial_i h^I \partial_j h^J - 6 h^I h^J \right) \, . \tag{2.31}
$$

[4]The indices $I, J, \ldots$ range from 1 to $n_{\mathrm{v}} + 1$, while $i, j, \ldots = 1, \ldots, n_{\mathrm{v}}$.

The potential comes from a gauging of a $U(1) \subset SU(2)$ R symmetry group, as in the four dimensional case. Here, V_I are FI constants, ∂_i denotes a partial derivative with respect to the real scalar field ϕ^i, and $h^I = h^I(\phi^i)$ satisfy the condition

$$\mathcal{V} \equiv \frac{1}{6} C_{IJK} h^I h^J h^K = 1 \,. \tag{2.32}$$

Moreover, G_{IJ} and $\mathcal{G}_{ij}$ can be expressed in terms of the homogeneous cubic polynomial $\mathcal{V}$ which defines a 'very special geometry' [62],

$$G_{IJ} = -\frac{1}{2} \frac{\partial}{\partial h^I} \frac{\partial}{\partial h^J} \log \mathcal{V}|_{\mathcal{V}=1} \,, \qquad \mathcal{G}_{ij} = \partial_i h^I \partial_j h^J \, G_{IJ}|_{\mathcal{V}=1} \,. \tag{2.33}$$

Further useful relations are

$$\begin{gathered} \partial_i h_I = -\frac{2}{3} G_{IJ} \partial_i h^J \,, \qquad h_I = \frac{2}{3} G_{IJ} h^J \,, \qquad G_{IJ} = \frac{9}{2} h_I h_J - \frac{1}{2} C_{IJK} h^K \,, \\ \mathcal{G}^{ij} \partial_i h^I \partial_j h^J = G^{IJ} - \frac{2}{3} h^I h^J \,, \qquad \mathcal{G}^{ij} \partial_i h_I \partial_j h_J = \frac{4}{9} G_{IJ} - \frac{2}{3} h_I h_J \,, \\ \mathcal{G}^{ij} \partial_i h^I \partial_j h_J = -\frac{2}{3} \delta^I_J + \frac{2}{3} h^I h_J \,. \end{gathered} \tag{2.34}$$

In the special case where the tensor T_{ijk} that determines the Riemann tensor of the vector multiplet scalar manifold $\mathcal{M}$ (cf. [60] for details) is covariantly constant[5], one has also

$$C_{IJK} C_{J'(LM} C_{PQ)K'} \delta^{JJ'} \delta^{KK'} = \frac{4}{3} \delta_{I(L} C_{MPQ)} \,, \tag{2.35}$$

which is the adjoint identity of the associated Jordan algebra [60]. Using (2.35) and defining $C^{IJK} \equiv \delta^{II'} \delta^{JJ'} \delta^{KK'} C_{I'J'K'}$, one can show that

$$G^{IJ} = -6 C^{IJK} h_K + 2 h^I h^J \,. \tag{2.36}$$

Moreover we note that if the five-dimensional theory is obtained by gauging a supergravity theory coming from a Calabi-Yau compactification of M-theory, then $\mathcal{V}$ is the intersection form, h^I and $h_I \equiv \frac{1}{6} C_{IJK} h^J h^K$ correspond to the size of the two- and four-cycles and the constants C_{IJK} are the intersection numbers of the Calabi-Yau threefold [17].

We now generalize our analysis to include also the coupling to $n_{\rm H}$ hypermultiplets. The charged hyperscalars q^u ($u = 1, \cdots, 4n_{\rm H}$) parametrize, as above in the $d = 4$ case, a quaternionic Kähler manifold with metric $h_{uv}(q)$, i.e., a $4n_{\rm H}$-dimensional Riemannian manifold admitting a locally defined triplet $\vec{K}_u{}^v$ of almost complex structures satisfying the quaternion relation

$$h^{st} K^x{}_{us} K^y{}_{tw} = -\delta^{xy} h_{uw} + \varepsilon^{xyz} K^z{}_{uw} \,, \tag{2.37}$$

and whose Levi-Civita connection preserves $\vec{K}$ up to a rotation,

[5]This implies that $\mathcal{M}$ is a locally symmetric space.

$$\nabla_w \vec{K}_u{}^v + \vec{\omega}_w \times \vec{K}_u{}^v = 0\,, \tag{2.38}$$

where $\vec{\omega} \equiv \vec{\omega}_u(q)\, dq^u$ is the connection of the $\mathrm{SU}(2)$-bundle for which the quaternionic manifold is the base. The $\mathrm{SU}(2)$ curvature is proportional to the complex structures,

$$\Omega^x \equiv \mathrm{d}\omega^x + \frac{1}{2}\varepsilon^{xyz}\omega^y \wedge \omega^z = -K^x\,. \tag{2.39}$$

Here we shall consider only gaugings of abelian isometries of the quaternionic Kähler metric h_{uv}. These are generated by commuting Killing vectors $k_I^u(q)$. In $d = 5$ the dual of a 2-form is a 3-form and this breaks the $Sp(2n_\mathrm{v} + 2, \mathbb{R})$ covariance that naturally arises in $d = 4$ to an $SL(n_\mathrm{v} + 1, \mathbb{R})$ covariance. This 3-form is linked to the possibility of add also tensor multiplets [59]. However also in this case k_I^u hides the choice of the gauge fields that effectively gauged the isometries, creating a coupling between special Kähler and quaternionic manifold. For each Killing vector one can introduce a triplet of moment maps, P_I^x, such that

$$\mathsf{D}_u P_I^x \equiv \partial_u P_I^x + \varepsilon^{xyz}\omega^y{}_u P_I^z = -2\Omega^x{}_{uv} k_I^v\,. \tag{2.40}$$

One of the most important relations satisfied by the moment maps is the so-called equivariance relation. For abelian gaugings it has the form

$$\frac{1}{2}\epsilon^{xyz} P_I^y P_J^z - \Omega_{uv}^x k_I^u k_J^v = 0\,. \tag{2.41}$$

The bosonic Lagrangian is now given by[6]

$$\begin{aligned} e^{-1}\mathscr{L} \;\; = \;\; & \frac{1}{2}R - \frac{1}{2}\mathcal{G}_{ij}\partial_\mu\phi^i\partial^\mu\phi^j - h_{uv}\hat{\partial}_\mu q^u \hat{\partial}^\mu q^v - \frac{1}{4}G_{IJ}F_{\mu\nu}^I F^{J\mu\nu} \\ & + \frac{e^{-1}}{48} C_{IJK}\epsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu}^I F_{\rho\sigma}^J A_\lambda^K - \mathrm{g}^2 V_5\,, \end{aligned} \tag{2.42}$$

with the covariant derivative

$$\hat{\partial}_\mu q^u = \partial_\mu q^u + 3\mathrm{g} A_\mu^I k_I^u\,, \tag{2.43}$$

and the scalar potential

$$V = P_I^x P_J^x \left(\frac{9}{2}\mathcal{G}^{ij}\partial_i h^I \partial_j h^J - 6 h^I h^J \right) + 9 h_{uv} k_I^u k_J^v h^I h^J\,. \tag{2.44}$$

The FI case is the truncation obtained posing constant the hypermultiplets.

2.3 Construction of the r-map

The theories constructed in (2.1) and (2.2) can be related by a Kaluza-Klein compactification on S^1. The link is a standard tool in the ungauged theories [64] but is much less

[6] (2.42) can be obtained from the Lagrangian in [63] by rescaling $a_{IJ} \to \frac{2}{3}G_{IJ}$, $C_{IJK} \to \frac{1}{6}C_{IJK}$, $k_I \to 2k_I$, $A^I \to \sqrt{\frac{3}{2}}A^I$, $g \to \sqrt{\frac{3}{2}}g$.

explored [65,66] in presence of gaugings. Reviewing the original formulation we extend it to the general gauged case.

The first step is a Kaluza-Klein reduction along the z-direction (i.e., along the string), by using the ansatz[7]

$$\mathrm{d}s_5^2 = e^{\frac{\phi}{\sqrt{3}}}\mathrm{d}s_4^2 + e^{-\frac{2}{\sqrt{3}}\phi}(\mathrm{d}z + K_\mu \mathrm{d}x^\mu)^2\,, \qquad A^I = B^I\mathrm{d}z + C^I_\mu \mathrm{d}x^\mu + B^I K_\mu \mathrm{d}x^\mu\,. \tag{2.45}$$

Defining $K_{\mu\nu} = \partial_\mu K_\nu - \partial_\nu K_\mu$ and $C^I_{\mu\nu} = \partial_\mu C^I_\nu - \partial_\nu C^I_\mu$, the five-dimensional Lagrangian (2.42) reduces to[8]

$$\begin{aligned} e_4^{-1}\mathscr{L}^{(4)} = &\frac{R^{(4)}}{2} - \frac{1}{8}e^{-\sqrt{3}\phi}K^{\mu\nu}K_{\mu\nu} - \frac{1}{4}G_{IJ}e^{-\frac{\phi}{\sqrt{3}}}(C^{I\mu\nu} + B^I K^{\mu\nu})(C^J_{\mu\nu} + B^J K_{\mu\nu}) \\ &- \frac{1}{2}e^{\frac{2\phi}{\sqrt{3}}}G_{IJ}\partial_\mu B^I \partial^\mu B^J - \frac{1}{2}G_{IJ}\partial_\mu h^I \partial^\mu h^J - \frac{1}{4}\partial_\mu\phi\partial^\mu\phi - h_{uv}\hat{\partial}_\mu q^u \hat{\partial}^\mu q^v \\ &- \frac{e_4^{-1}}{16}\epsilon^{\mu\nu\rho\sigma}C_{IJK}\big(C^I_{\mu\nu}C^J_{\rho\sigma}B^K + \frac{1}{3}K_{\mu\nu}K_{\rho\sigma}B^I B^J B^K + C^I_{\mu\nu}K_{\rho\sigma}B^J B^K\big) \\ &- e^{\sqrt{3}\phi}\mathrm{g}^2 B^I k^u_I B^J k^v_J h_{uv} - \mathrm{g}^2 e^{\frac{\phi}{\sqrt{3}}}V_5\,. \end{aligned} \tag{2.46}$$

Now we want to rewrite $\mathscr{L}^{(4)}$ in the language of $\mathcal{N}=2$, $d=4$ supergravity, by using the identifications of the ungauged case [64]. The coordinates of the special Kähler manifold, Kähler potential, Kähler metric and electromagnetic field strengths are given in terms of five-dimensional data respectively by

$$\begin{aligned} z^I &= B^I + ie^{-\frac{\phi}{\sqrt{3}}}h^I\,, \qquad e^{\mathcal{K}} = \frac{1}{8}e^{\sqrt{3}\phi}\,, \\ g_{I\bar{J}} &= \frac{1}{2}e^{\frac{2\phi}{\sqrt{3}}}G_{IJ}\,, \qquad F^\Lambda_{\mu\nu} = \frac{1}{\sqrt{2}}(K_{\mu\nu}, C^I_{\mu\nu})\,, \end{aligned} \tag{2.47}$$

where capital greek indices $\Lambda, \Sigma, \ldots$ range from 0 to $n_\mathrm{v}+1$. If we introduce the matrices

$$R_{\Lambda\Sigma} = -\begin{pmatrix} \frac{1}{3}B & \frac{1}{2}B_J \\ \frac{1}{2}B_I & B_{IJ} \end{pmatrix}\,, \qquad I_{\Lambda\Sigma} = -e^{-\sqrt{3}\phi}\begin{pmatrix} 1+4g & 4g_{\bar{J}} \\ 4g_I & 4g_{I\bar{J}} \end{pmatrix}\,, \tag{2.48}$$

where we defined

$$\begin{aligned} B_{IJ} =& C_{IJK}B^K\,, \qquad B_I = C_{IJK}B^J B^K\,, \qquad B = C_{IJK}B^I B^J B^K\,, \\ g &= g_{I\bar{J}}B^I B^J\,, \qquad g_{I\bar{J}}B^J = g_I = g_{\bar{I}} = g_{\bar{I}J}B^J\,, \end{aligned} \tag{2.49}$$

the Lagrangian (2.46) can be cast into the form

$$\begin{aligned} e_4^{-1}\mathscr{L}^{(4)} =& \frac{R}{2} - g_{I\bar{J}}\partial_\mu z^I \partial^\mu \bar{z}^{\bar{J}} - h_{uv}\hat{\partial}_\mu q^u \hat{\partial}^\mu q^v \\ &+ \frac{1}{4}I_{\Lambda\Sigma}F^{\Lambda\mu\nu}F^\Sigma_{\mu\nu} + \frac{1}{8}e_4^{-1}\epsilon^{\mu\nu\rho\sigma}R_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma} - \tilde{V}\,, \end{aligned} \tag{2.50}$$

[7]In this subsection $\mu, \nu, \ldots$ are curved indices for the four-dimensional theory. Further details on the notation and the theory in $d=4$ can be found in (2.1) and (2.2).

[8]We choose $\epsilon_5^{\mu\nu\rho\sigma z} = -\epsilon_4^{\mu\nu\rho\sigma}$.

with the four-dimensional potential given by

$$\tilde{V} = \mathrm{g}^2 e^{\frac{\phi}{\sqrt{3}}} V_5 + e^{\sqrt{3}\phi} \mathrm{g}^2 h_{uv} k_I^u k_J^v B^I B^J \,. \tag{2.51}$$

The underlying prepotential of the special Kähler manifold turns out to be

$$F = \frac{1}{6} \frac{C_{IJK} X^I X^J X^K}{X^0} \,, \tag{2.52}$$

chosen the parametrization $X^I/X^0 = z^I = B^I + ie^{-\phi/\sqrt{3}} h^I$ [64].

The actual novelties with respect to the ungauged case are the potential and the covariant derivative acting on the hyperscalars. The former reads

$$\begin{aligned}
\frac{\tilde{V}}{\mathrm{g}^2} &= -9e^{\frac{\phi}{\sqrt{3}}} P_I^x P_J^x \left(h^I h^J - \frac{1}{2} G^{IJ} \right) + 9e^{\frac{\phi}{\sqrt{3}}} h_{uv} k_I^u k_J^v h^I h^J + 9e^{\sqrt{3}\phi} h_{uv} k_I^u k_J^v B^I B^J \\
&= 18 P_I^x P_J^x \left(\frac{1}{4} e^{\frac{\phi}{\sqrt{3}}} G^{IJ} + \frac{1}{2} e^{\sqrt{3}\phi} B^I B^J - 4 \frac{e^{\sqrt{3}\phi}}{8} (e^{-\frac{\phi}{\sqrt{3}}} h^I)(e^{-\frac{\phi}{\sqrt{3}}} h^J) - \frac{1}{2} e^{\sqrt{3}\phi} B^I B^J \right) \\
&\quad + 72 \frac{e^{\sqrt{3}\phi}}{8} h_{uv} k_I^u k_J^v (e^{-\frac{2\phi}{\sqrt{3}}} h^I h^J + B^I B^J) \,.
\end{aligned} \tag{2.53}$$

Now the first two terms in the second line of (2.53) combine to give $-\frac{1}{2} I^{\Lambda\Sigma}$ (the inverse of $I_{\Lambda\Sigma}$ defined above), while the last two terms yield $-4X^I \bar{X}^J$. Fixing furthermore $\mathrm{g}_4 = 3\sqrt{2}\mathrm{g}$, one has thus

$$\begin{aligned}
\tilde{V} &= \mathrm{g}_4^2 \left[P_\Lambda^x P_\Sigma^x \left(-\frac{1}{2} I^{\Lambda\Sigma} - 4X^\Lambda \bar{X}^\Sigma \right) + 4 h_{uv} k_\Lambda^u k_\Sigma^v X^\Lambda \bar{X}^\Sigma \right] \Big|_{P_0^x = 0, k_0^u = 0} \\
&= \mathrm{g}_4^2 V_4 \Big|_{P_0^x = 0, k_0^u = 0} \,,
\end{aligned} \tag{2.54}$$

which is precisely the truncated potential of the four-dimensional theory.
The final point to take care of is the covariant derivative of the hyperscalars,

$$\hat{\partial}_\mu q^u = \partial_\mu q^u + 3\mathrm{g} C_\mu^I k_I^u = \partial_\mu q^u + \mathrm{g}_4 A_{4\mu}^I k_I^u \,. \tag{2.55}$$

We have therefore shown that the r-map can be extended to the case of gauged supergravity, where the scalar fields have a potential.

2.4 Equations of motion and effective theories

The main aim of our research is to find classical solutions to the equations of motion of the theories previously described. The most interesting configurations are respectively black holes or black strings with running scalars. We will focus on static solutions and the technique that we shall adopt is to plug an ansatz on the fields into the equations of motion of the supergravity theory to obtain a set of second order differential equations plus certain algebraic constraints that can be read as equations of motions of a one-dimensional effective theory. The problem is so reduced to find solutions to a finite dimensional dynamical system.

Static black hole in d=4

A straightforward application of the Euler-Lagrange operator to 2.25 extracts the equations of motion of the theory.
The Einstein's equations can be divided in tracless and trace part

$$
\begin{aligned}
&R_{\mu\nu} - 2g_{i\bar{\jmath}}\partial_\mu z^i \partial_\nu \bar{z}^{\bar{\jmath}} - 2h_{uv}\hat{\partial}_\mu q^u \hat{\partial}_\nu q^v - I_{\Lambda\Sigma}\left(H^\Lambda{}_{\mu\rho}H^\Sigma{}_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}H^\Lambda{}_{\sigma\rho}H^{\Sigma\sigma\rho}\right) - g_{\mu\nu}V_g = 0\,,\\
&R - 2h_{uv}\hat{\partial}_\mu q^u \hat{\partial}^\mu q^v - 2g_{i\bar{\jmath}}\partial_\mu z^i \partial^\mu z^{\bar{\jmath}} - 4V_g = 0\,.
\end{aligned}
\tag{2.56}
$$

The equations of motion for $A_{\Lambda\mu}$, A^Λ_μ and $B_{a\mu\nu}$ following from (2.25) are

$$
\begin{aligned}
&\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\partial_\mu B_{a\nu\rho}\Theta^{\Lambda a} = -2\sqrt{-g}h_{uv}\Theta^{\Lambda a}k^u_a\hat{\partial}^\sigma q^v\,,\\
&G_{\Lambda\mu\nu}\Theta^{\Lambda a} = \Theta^{\Lambda a}(F_{\Lambda\mu\nu} - \frac{1}{2}\Theta^b_\Lambda B_{b\mu\nu})\,,\\
&\partial_\mu\left(\sqrt{-g}I_{\Lambda\Sigma}H^{\Sigma\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}R_{\Lambda\Sigma}H^\Sigma{}_{\rho\sigma}\right) = 2\sqrt{-g}h_{uv}\Theta^a_\Lambda k^u_a\hat{\partial}^\nu q^v\,,
\end{aligned}
\tag{2.57}
$$

where $G_{\Lambda\mu\nu}$ is defined by

$$
G_\Lambda = -\frac{2}{\sqrt{-g}}\star\frac{\delta\mathscr{L}}{\delta F^\Lambda}\,. \tag{2.58}
$$

The scalars must satisfy

$$
\begin{aligned}
&\nabla_\mu(g_{i\bar{\jmath}}\partial^\mu\bar{z}^{\bar{\jmath}}) - \frac{\partial g_{k\bar{\jmath}}}{\partial z^i}\partial_\mu z^k\partial^\mu\bar{z}^{\bar{\jmath}} + \frac{1}{4}\frac{\partial I_{\Lambda\Sigma}}{\partial z^i}H^{\Lambda\mu\nu}H^\Sigma{}_{\mu\nu} + \frac{1}{4}\frac{\partial R_{\Lambda\Sigma}}{\partial z^i}H^{\Lambda\mu\nu}\star H^\Sigma{}_{\mu\nu} - \frac{\partial V_g}{\partial z^i} = 0\,,\\
&2\nabla_\mu(h_{sv}\hat{\partial}^\mu q^v) - 2h_{uv}\langle\frac{\partial\mathcal{K}^u}{\partial q^s},\mathcal{A}_\mu\rangle\hat{\partial}^\mu q^v - \frac{\partial h_{uv}}{\partial q^s}\hat{\partial}_\mu q^u\hat{\partial}^\mu q^v - \frac{\partial V_g}{\partial q^s} = 0\,.
\end{aligned}
\tag{2.59}
$$

Now the point is to choose a good ansatz from the point of view of the physics and a well-defined truncation about the equations of motion. At this aim symmetry can be the guide for this selection and the most general static metric with spherical/hyperbolic symmetry has the form

$$
\mathrm{d}s^2 = -e^{2U(r)}\mathrm{d}t^2 + e^{-2U(r)}\mathrm{d}r^2 + e^{2(\psi(r)-U(r))}\mathrm{d}\Omega^2_\kappa\,, \tag{2.60}
$$

where $\mathrm{d}\Omega^2_\kappa = \mathrm{d}\theta^2 + f^2_\kappa(\theta)\mathrm{d}\varphi^2$ is the metric on the two-dimensional surfaces $\Sigma = \{\mathrm{S}^2, \mathrm{H}^2\}$ of constant scalar curvature $R = 2\kappa$, with $\kappa \in \{1, -1\}$, and

$$
f_\kappa(\theta) = \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}\theta) = \begin{cases} \sin\theta & \kappa = 1\,,\\ \sinh\theta & \kappa = -1\,.\end{cases} \tag{2.61}
$$

The scalar fields are assumed to depend only on the radial coordinate,

$$
z^i = z^i(r)\,, \qquad q^u = q^u(r)\,, \tag{2.62}
$$

while the abelian gauge fields and the 2-forms are,

$$\begin{aligned} A^\Lambda &= A_t^\Lambda \mathrm{d}t - \kappa p^\Lambda f_\kappa'(\theta)\mathrm{d}\phi\,, \qquad & A_\Lambda &= A_{\Lambda t}\mathrm{d}t - \kappa e_\Lambda f_\kappa'(\theta)\mathrm{d}\phi\,, \\ B^\Lambda &= 2\kappa p'^\Lambda f_\kappa'(\theta)\mathrm{d}r\wedge \mathrm{d}\phi\,, \qquad & B_\Lambda &= -2\kappa e'_\Lambda f_\kappa'(\theta)\mathrm{d}r\wedge\mathrm{d}\phi\,, \end{aligned} \tag{2.63}$$

where the magnetic and electric charges (p^Λ, e_Λ) are defined as

$$p^\Lambda(r) = \frac{1}{\mathrm{vol}(\Sigma_\kappa)}\int_{\Sigma_\kappa} H^\Lambda\,, \quad e_\Lambda(r) = \frac{1}{\mathrm{vol}(\Sigma_\kappa)}\int_{\Sigma_\kappa} G_\Lambda\,, \quad \mathrm{vol}(\Sigma_\kappa) = \int f_\kappa(\theta)\mathrm{d}\theta\wedge\mathrm{d}\phi\,. \tag{2.64}$$

These imply a structure of the field strengths

$$\begin{aligned} {H^\Lambda}_{tr} &= e^{2(U-\psi)} I^{\Lambda\Sigma}(R_{\Sigma\Gamma}p^\Gamma - e_\Sigma)\,, \qquad {H^\Lambda}_{\theta\phi} = p^\Lambda f_\kappa(\theta)\,, \\ G_{\Lambda tr} &= e^{2(U-\psi)}\left(I_{\Lambda\Sigma}p^\Sigma + R_{\Lambda\Gamma}I^{\Gamma\Omega}R_{\Omega\Sigma}p^\Sigma - R_{\Lambda\Gamma}I^{\Gamma\Omega}e_\Omega\right)\,, \qquad G_{\Lambda\theta\phi} = e_\Lambda f_\kappa(\theta)\,. \end{aligned} \tag{2.65}$$

Introducing the symplectic matrix

$$\mathcal{H} = (\mathcal{K}^u)^T h_{uv}\mathcal{K}^v\,, \tag{2.66}$$

and plugging the above ansatz into (2.57), one obtains

$$\mathcal{A}_t' = -e^{2(U-\psi)}\Omega\mathcal{M}\mathcal{Q}\,, \qquad \mathcal{Q}' = -2e^{2\psi-4U}\mathcal{H}\Omega\mathcal{A}_t\,, \tag{2.67}$$

where the constraints

$$\mathcal{H}\Omega\mathcal{Q} = 0\,, \qquad \mathcal{K}_u q'^u = 0 \tag{2.68}$$

have to be imposed to have spherical/hyperbolic.

Putting the ansatz just described in to the equations of motion plus the constraints (2.68) one finds certain second order ODEs. All the content of these equations can be derived as equations of motion of the effective action

$$\begin{aligned} S &= \int \mathrm{d}r\left[e^{2\psi}(U'^2 - \psi'^2 + h_{uv}q'^u q'^v + g_{i\bar{\jmath}}z'^i\bar{z}'^{\bar{\jmath}} + \frac{1}{4}e^{4(U-\psi)}\mathcal{Q}'^T\mathcal{H}^{-1}\mathcal{Q}') - \tilde{V}\right]\,, \\ \tilde{V} &= -e^{2(U-\psi)}V_{\mathrm{BH}} + \kappa - e^{2(\psi-U)}V_g\,, \end{aligned} \tag{2.69}$$

plus a constraint that in terms of this dynamical system reads zero value of the Hamiltonian associated, $H=0$. It's been introduced the black hole potential

$$V_{\mathrm{BH}} = -\frac{1}{2}\mathcal{Q}^T\mathcal{M}\mathcal{Q}\,, \qquad \mathcal{Q} \equiv \begin{pmatrix} p^\Lambda \\ e_\Lambda \end{pmatrix}\,, \tag{2.70}$$

with the symplectic matrix defined in (2.7). Moreover note that $\mathcal{H}$ is not invetible, but it is symmetric

$$\mathcal{H} = O^t D O\,, \tag{2.71}$$

where O is orthogonal and D is diagonal. Therefore defining $\mathcal{H}^{-1} = O^t D^{-1} O$ [9], the following property holds

[9] Here D^{-1} denotes the inverse of the diagonal matrix in the non degenere subspace, leaving zero for the zero eigeinvalues.

$$\mathcal{H}\mathcal{H}^{-1}\mathcal{H} = \mathcal{H}\,. \tag{2.72}$$

In a slight abuse of notation, $\mathcal{H}^{-1}$ denotes a weaker notion of the inverse matrix. This property will be enough for the next manipulations.

Static black string in d=5

Analogously to the previuos subsection, one can find the equations of motion of the theory 2.42.

The Einstein's equations can be divided in tracless and trace part

$$\begin{aligned}
&R_{\mu\nu} - \mathcal{G}_{ij}\partial_\mu\phi^i\partial_\nu\phi^j - 2h_{uv}\hat{\partial}_\mu q^u\hat{\partial}_\nu q^v - G_{IJ}\left(F^I_{\mu\sigma}F^{J\sigma}_{\nu} - \frac{1}{6}g_{\mu\nu}F^I_{\sigma\rho}F^{J\sigma\rho}\right) + \frac{2}{3}g^2 g_{\mu\nu}V_5 = 0\,,\\
&R - \mathcal{G}_{ij}\partial_\mu\phi^i\partial^\mu\phi^j - 2h_{uv}\hat{\partial}_\mu q^u\hat{\partial}^\mu q^v - \frac{1}{6}G_{IJ}F^I_{\mu\nu}F^{J\mu\nu} + \frac{10}{3}g^2V_5 = 0\,,
\end{aligned} \tag{2.73}$$

The Maxwell equations are

$$\nabla_\mu(G_{IK}F^{I\mu\nu}) + \frac{1}{4}C_{IJK}F^I_{\mu\sigma}F^J_{\rho\lambda}\epsilon^{\mu\sigma\rho\lambda\nu} = 6gk^u_K h_{uv}\hat{\partial}^\nu q^v\,, \tag{2.74}$$

and the scalars must satisfy

$$\begin{aligned}
&\nabla_\mu(\mathcal{G}_{ij}\partial^\mu\phi^j) - \frac{1}{2}\frac{\partial\mathcal{G}_{kj}}{\partial\phi^i}\partial_\mu\phi^k\partial^\mu\phi^j - \frac{1}{4}\frac{\partial G_{IJ}}{\partial\phi^i}F^{I\mu\nu}F^J{}_{\mu\nu} - g^2\frac{\partial V_5}{\partial\phi^i} = 0\,,\\
&2\nabla_\mu(h_{sv}\hat{\partial}^\mu q^v) - 2h_{uv}\frac{\partial k^u_I}{\partial q^s}A^I_\mu\hat{\partial}^\mu q^v - \frac{\partial h_{uv}}{\partial q^s}\hat{\partial}_\mu q^u\hat{\partial}^\mu q^v - g^2\frac{\partial V_5}{\partial q^s} = 0\,.
\end{aligned} \tag{2.75}$$

Very special real Kähler manifolds can be viewed as the pre-image of the supergravity r-map [65, 67]. This suggests to consider the five-dimensional spacetime as a Kaluza-Klein uplift of the usual static black holes in four dimensions 2.3[10]. Moreover, a pure string solution in $d = 5$ supports only magnetic charges, thus the field configuration reads

$$\begin{aligned}
&\mathrm{d}s^2 = e^{2T(r)}\mathrm{d}z^2 + e^{-T(r)}\left(-e^{2U(r)}\mathrm{d}t^2 + e^{-2U(r)}\mathrm{d}r^2 + e^{2\psi(r)-2U(r)}\mathrm{d}\sigma^2_\kappa\right)\,,\\
&F^I = p^I f_\kappa(\theta)\mathrm{d}\theta\wedge\mathrm{d}\phi\,, \qquad \phi^i = \phi^i(r)\,, \qquad q^u = q^u(r)\,,
\end{aligned} \tag{2.76}$$

where $\mathrm{d}\sigma^2_\kappa = \mathrm{d}\theta^2 + f^2_\kappa(\theta)\mathrm{d}\varphi^2$ is the metric on the two-dimensional surfaces $\Sigma = \{\mathrm{S}^2, \mathrm{H}^2\}$ of constant scalar curvature $R = 2\kappa$, with $\kappa \in \{1, -1\}$, and

$$f_\kappa(\theta) = \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}\theta) = \begin{cases} \sin\theta & \kappa = 1\,,\\ \sinh\theta & \kappa = -1\,.\end{cases} \tag{2.77}$$

Plugging the ansatz (2.76) into the equations of motion following from (2.30), imposing the constraints

[10]The identification is $T = -\phi/\sqrt{3}$.

$$h_{uv}k_I^u q'^{\,v} = 0\,, \qquad k_I^u p^I = 0\,. \tag{2.78}$$

yields a set of ordinary differential equations that can be derived from the one-dimensional effective action

$$\begin{aligned} S_{\text{eff}} &= \int \mathrm{d}r \left[e^{2\psi}\left(U'^2 + \frac{3}{4}T'^2 - \psi'^2 + \frac{1}{2}\mathcal{G}_{ij}\phi'^{\,i}\phi'^{\,j} + h_{uv}q'^{\,u}q'^{\,v} \right) - V_{\text{eff}} \right]\,, \\ V_{\text{eff}} &= \kappa - e^{2\psi-2U-T}\mathrm{g}^2 V_5 - \frac{1}{2}e^{2U+T-2\psi}G_{IJ}p^I p^J\,, \end{aligned} \tag{2.79}$$

supplemented by the Hamiltonian constraint $H_{\text{eff}} = 0$.

CHAPTER 3

First Order Flow and Solutions

In the previous section, the problem of finding a supergravity solution is reduced to that of solving a dynamical system of a finite number of degrees of freedom. However, this involves the resolution of a complicated system of second order ordinary differential equations, but some techniques to simplify the problem again are known. They consist in finding first order ordinary differential equations systems for which the solutions are also configurations that satisfy the equations of motion. A well-known example in analytic mechanics is the Hamilton-Jacobi technique that allows to find the system of first-order equations equivalent, in the better case in which one can solve the first order Hamilton-Jacobi partial differential equation, to that of the equations of motion. Another more technical but formally clear way to obtain a similar result in a supersymmetric theory is posing to zero the supersymmetric variations of the fermionic fields, obtaining the Killing spinor equations. Handling these expressions leads to certain first order equations that define the supersymmetric solutions of the theory. However, helped by the symmetries, we will focus on the HJ approach that in certain cases is exactly equivalent. The new results of this chapter are the integration of the Hamilton-Jacobi equation for electric Reissner-Nordström-Einstein-(A)dS black hole in d-dimensions [7], the complete integrability of the effective theory for dyonic Reissner-Nordström Taub-NUT (A)dS black hole in $d = 4$ [6], a symplectically covariant first order system for $d = 4$ black hole with running hypermultiplets [1], a first order system for $d = 5$ black string with running hypermultiplets [2] and, with the integration of the latter, an explicit generalization of the Maldacena-Nunez black string [5].

3.1 Hamilton-Jacobi to square the action

In complete generality, we start from a system of n degrees of freedom like

$$I = \int \mathrm{d}r L(\dot{q}^\Lambda, q^\Lambda) = \int \mathrm{d}r \left[\frac{1}{2}\mathscr{G}_{\Lambda\Sigma}\dot{q}^\Lambda\dot{q}^\Sigma - U(q^\Lambda)\right] , \tag{3.1}$$

where r is a radial variable (the 'flow' direction), the $q^\Lambda(r)$ denote collectively the dynamical variables, $U(q^\Lambda)$ is the potential and $\mathscr{G}_{\Lambda\Sigma}(q^\Gamma)$ the metric on the target space

parametrized by the q^Λ, with inverse $\mathscr{G}^{\Lambda\Sigma}$.
A possibile way to solve a system like (3.1) is to use Hamilton-Jacobi technique. It consists in finding a canonical transformation for which the conjugate momenta p^Λ are constant. This can be achieved solving the non-linear partial differential equation for the Hamilton-Jacobi's principal function $S(q^\Lambda, r)$ [68]

$$\frac{\partial S}{\partial r} + H\left(\frac{\partial S}{\partial q^\Lambda}, q^\Lambda\right) = 0\,, \tag{3.2}$$

where $H(p_\Lambda, q^\Lambda)$ is the Hamiltonian relative to Lagrangian $L(\dot{q}^\Lambda, q^\Lambda)$. Moreover for (3.1) the Hamiltionian is a first integral of motion and without loss of generality the energy can be posed constant $H = E$. In this case the principal Hamilton-Jacobi function is $S = W(q^\Lambda) - Er$ and (3.2) boils down to

$$U(q^\Lambda) = E - \frac{1}{2}\mathscr{G}^{\Lambda\Sigma}\frac{\partial W}{\partial q^\Lambda}\frac{\partial W}{\partial q^\Sigma}\,. \tag{3.3}$$

This is a first order partial differential equation whose complete solution $W(q^\Lambda, a_\Lambda, E)$ is a function of the n degrees of freedom q^Λ and of the $n+1$ arbitrary constants a_Λ and E. Having found this complete integral, one can proceed to the algebraic resolution of the dynamics in terms of the equations

$$\frac{\partial S}{\partial E} = B\,, \quad \frac{\partial S}{\partial a_\Lambda} = c^\Lambda\,, \tag{3.4}$$

where B and c_Λ are $n+1$ arbitrary constants. Often it is impossible to write down the complete integral. However, even only knowing a particular solution to (3.3), the action (3.1) can be rewritten as

$$I = \int \mathrm{d}r \left[\frac{1}{2}\mathscr{G}_{\Lambda\Sigma}\left(\dot{q}^\Lambda - \mathscr{G}^{\Lambda\Omega}\frac{\partial W}{\partial q^\Omega}\right)\left(\dot{q}^\Sigma - \mathscr{G}^{\Sigma\Delta}\frac{\partial W}{\partial q^\Delta}\right) + \frac{\mathrm{d}}{\mathrm{d}r}(W - Er)\right]\,, \tag{3.5}$$

which is up to a total derivative equal to

$$I = \int \mathrm{d}r\, \frac{1}{2}\mathscr{G}_{\Lambda\Sigma}\left(\dot{q}^\Lambda - \mathscr{G}^{\Lambda\Omega}\frac{\partial W}{\partial q^\Omega}\right)\left(\dot{q}^\Sigma - \mathscr{G}^{\Sigma\Delta}\frac{\partial W}{\partial q^\Delta}\right)\,. \tag{3.6}$$

A sufficient condition for the stationarity of the latter is that

$$\dot{q}^\Lambda = \mathscr{G}^{\Lambda\Omega}\frac{\partial W}{\partial q^\Omega} \tag{3.7}$$

hold. These equations (3.7) represent the expression for the conjugate momenta $p_\Lambda = \partial\mathscr{L}/\partial\dot{q}^\Lambda = \mathscr{G}_{\Lambda\Sigma}\dot{q}^\Sigma$ in the Hamilton-Jacobi theory[1].
Hamilton-Jacobi approach is not an essential tool to solve the dynamics of (3.1). One could try to integrate the second order equations of motion derived from the Lagrangian

[1]For further discussions of the relationship between the Hamilton-Jacobi formalism and the first-order equations derived from a (fake) superpotential cf. [69,70].

as usual. However we find that Hamilton-Jacobi approach is particularly interesting because recollect all the knowledge about the dynamics in a sigle equation (3.2), that even if it is a partial differential equation, in general, it is more tractable than the system of the equations of motion.

3.2 General first order flow

An example in which the integration of the Hamilton-Jacobi equation can be carried out is the d-dimensional Einstein-Maxwell-Lambda system. In this section we show the integration procedure for this system, commenting on more complicated cases, at the end.

Static black holes in Einstein-Maxwell-Lambda gravity

We consider d-dimensional Einstein-Maxwell-Lambda gravity, whose action is given by

$$S = \frac{1}{16\pi G_d} \int \mathrm{d}^d x \sqrt{-g} \left(R - F_{\mu\nu}F^{\mu\nu} - 2\Lambda\right) , \tag{3.8}$$

with $d > 3$. This is the simplest model that can be embedded (at least for some d) in $\mathcal{N} = 2$ gauged supergravity. The equations of motion following from (3.73) are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 2\left(F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{1}{4}g_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}\right) , \qquad \nabla_\mu F^{\mu\nu} = 0 , \tag{3.9}$$

where $F = \mathrm{d}A$. For future convenience we report the trace and the traceless part of the Einstein equations that respectively read

$$\begin{aligned} &R - \frac{2d}{d-2}\Lambda - \frac{d-4}{d-2}F^{\mu\nu}F_{\mu\nu} = 0 , \\ &R_{\mu\nu} - 2F_{\mu}{}^{\sigma}F_{\nu\sigma} - \frac{2}{d-2}\Lambda g_{\mu\nu} + \frac{1}{d-2}g_{\mu\nu}F^{\sigma\rho}F_{\sigma\rho} = 0 . \end{aligned} \tag{3.10}$$

We shall consider electrically charged static black holes whose horizon is a $(d-2)$-dimensional Einstein space[2]. The metric and the gauge field have the form

$$\mathrm{d}s_d^2 = -e^{-2(d-3)U}\mathrm{d}t^2 + e^{2U-2(d-4)\psi}\mathrm{d}r^2 + e^{2(U+\psi)}\mathrm{d}\Omega^2_{\kappa,d-2} , \qquad A = A_t \mathrm{d}t , \tag{3.11}$$

where the functions U, ψ and A_t depend only on the coordinate r. The metric in (3.11) has the warped product structure

$$\mathrm{d}s_d^2 = \tilde{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b + f^2(x)\hat{g}_{ij}\mathrm{d}y^i\mathrm{d}y^j , \tag{3.12}$$

where the $(d-2)$-dimensional fiber with metric $\hat{g}_{ij}\mathrm{d}y^i\mathrm{d}y^j = \mathrm{d}\Omega^2_{\kappa,d-2}$ is a generic Einstein space, i.e., $\hat{R}_{ij} = (d-3)\kappa\hat{g}_{ij}$. The nonvanishing components of the Ricci tensor in d dimensions are thus given by [71]

[2]For $d > 5$ this does not necessarily imply that the horizon has constant curvature.

$$
\begin{aligned}
R_{ab} &= \tilde{R}_{ab} - \frac{d_F}{f}\tilde{\nabla}_a\tilde{\nabla}_b f\,, \\
R_{ij} &= \hat{R}_{ij} - \hat{g}_{ij}\left(f\tilde{\nabla}_a\tilde{\nabla}^a f + (d_F-1)\tilde{g}^{ab}\partial_a f\partial_b f\right)\,,
\end{aligned}
\tag{3.13}
$$

where $d_F > 1$ is the dimension of the fiber and $\tilde{\nabla}_a$ denotes the covariant derivative constructed with the Levi-Civita connection for $\tilde{g}_{ab}$.

The Maxwell equations for the ansatz (3.11) are solved by

$$
F = -Qe^{-2(d-3)(U+\psi)}\mathrm{d}t \wedge \mathrm{d}r\,, \tag{3.14}
$$

where Q is an integration constant corresponding to the electric charge. Using (3.13) it is straightforward to show that the Einstein equations (3.10) boil down to three ordinary differential equations that can be derived form the one-dimensional effective action

$$
S_{\text{eff}} = \int \mathrm{d}r L = \int \mathrm{d}r\left(e^{2(d-3)\psi}(U'^2 - \psi'^2) - V_{\text{eff}}\right)\,, \tag{3.15}
$$

with the potential

$$
V_{\text{eff}} = \kappa - \frac{2Q^2}{(d-3)(d-2)}e^{-2(d-3)(U+\psi)} - \frac{2\Lambda}{(d-3)(d-2)}e^{2(U+\psi)}\,, \tag{3.16}
$$

if we impose in addition the zero energy condition

$$
e^{2(d-3)\psi}(U'^2 - \psi'^2) + V_{\text{eff}} = 0\,. \tag{3.17}
$$

To be concrete, the equation of motion for U is proportional to the tt-component of (3.10), while the one for ψ is a linear combination of the tt- and rr-components. Moreover, from the first of (3.10) and the tt-component one gets (3.17). The Einstein equations along the fiber are automatically satisfied.

The conjugate momenta and Hamiltonian of the dynamical system (3.15) are respectively given by

$$
\begin{aligned}
p_U &= \frac{\partial L}{\partial U'} = 2e^{2(d-3)\psi}U'\,, \qquad p_\psi = \frac{\partial L}{\partial \psi'} = -2e^{2(d-3)\psi}\psi'\,, \\
H_{\text{eff}}(p_U, p_\psi, U, \psi) &= \frac{1}{4}e^{-2(d-3)\psi}(p_U^2 - p_\psi^2) + V_{\text{eff}}\,.
\end{aligned}
\tag{3.18}
$$

Integration of the Hamilton-Jacobi equation

The Hamilton-Jacobi equation associated to (3.18) reads

$$
H_{\text{eff}}(\partial_U S, \partial_\psi S, U, \psi) + \frac{\partial S}{\partial r} = 0\,. \tag{3.19}
$$

Since H_{eff} does not depend explicitely on r we set

$$
S = 2W(U,\psi) - Er\,, \tag{3.20}
$$

such that (3.19) reduces to

$$e^{-2(d-3)\psi}(W_U^2 - W_\psi^2) + V_{\text{eff}} = E\,, \tag{3.21}$$

where W_U and W_ψ are respectively the partial derivatives of W w.r.t. U and ψ. Inspired by [6,72], we define a new set of coordinates

$$X = e^{(d-3)(U+\psi)}\,, \qquad Y = e^{-2(d-3)U}\,, \tag{3.22}$$

for which (3.21) becomes

$$\frac{4(d-3)^2}{X^2}\left(YW_Y^2 - XW_XW_Y\right) - \frac{2Q^2}{(d-2)(d-3)X^2} - \frac{2\Lambda X^{\frac{2}{d-3}}}{(d-2)(d-3)} = \hat{E}\,, \tag{3.23}$$

where $\hat{E} = E - \kappa$. To avoid loss of information E will be set to zero, as required by (3.17), only at the end of the integration procedure. This because to solve the dynamics algebraically one needs (3.27) and (3.37), therefore we set $E = 0$ only after these equations have been obtained. In the ungauged case, with a suitable change of coordinate the value of E can be associated to the parameter of the non-extremality [69].

First solution Applying the method of characteristics yields

$$\frac{\mathrm{d}W_Y}{W_Y} = \frac{\mathrm{d}X}{X}\,, \tag{3.24}$$

and thus $W_Y = aX$, where a is an integration constant. The solution of this equation boils down to $W(X,Y) = aYX + \omega(X)$ that inserted into (3.23) leads to an ODE

$$-4a(d-3)^2\omega_X - \frac{2Q^2}{(d-2)(d-3)X^2} - \frac{2\Lambda X^{\frac{2}{(d-3)}}}{(d-2)(d-3)} = \hat{E}\,, \tag{3.25}$$

that can be easily integrated to give

$$S_1 = 2aYX + \frac{1}{2a(d-3)^2}\left(\frac{2Q^2}{(d-2)(d-3)X} - \frac{2\Lambda X^{\frac{d-1}{d-3}}}{(d-1)(d-2)} - \hat{E}X\right) - Er + C\,. \tag{3.26}$$

This contains three integration constants C, E and a, where the latter must be different from zero. Using

$$\left.\frac{\partial S_1}{\partial E}\right|_{E=0} = c_1\,, \qquad \left.\frac{\partial S_1}{\partial a}\right|_{E=0} = c_2\,, \tag{3.27}$$

where c_1 and c_2 denote arbitrary constants, the dynamics can be solved algebraically, with the result

$$\begin{aligned} X &= -2a(d-3)^2(r+c_1)\,, \\ Y &= \frac{c_2}{2X} + \frac{Q^2}{2a^2(d-2)(d-3)^3X^2} + \frac{\kappa}{4a^2(d-3)^2} - \frac{\Lambda X^{\frac{2}{d-3}}}{2a^2(d-1)(d-2)(d-3)^2}\,. \end{aligned} \tag{3.28}$$

In terms of Y and the new radial coordinate $R = X^{\frac{1}{d-3}}$, the solution (3.11) becomes

$$
\begin{aligned}
&\mathrm{d}s_d^2 = -Y\mathrm{d}t^2 + \frac{\mathrm{d}R^2}{Y} + R^2\mathrm{d}\Omega^2_{\kappa,d-2}\,, \qquad F = \frac{Q}{R^{d-2}}\mathrm{d}t\wedge\mathrm{d}r\,,\\
&Y = \kappa - \frac{2M}{R} + \frac{2Q^2}{(d-2)(d-3)R^{2(d-3)}} - \frac{2\Lambda R^2}{(d-1)(d-2)}\,.
\end{aligned}
\tag{3.29}
$$

Here we fixed $a^2 = \frac{1}{4(d-3)^2}$ (which can always be achieved by rescaling the coordinates appropriately) and defined $c_2 = -4M$. (3.29) is the most general solution to the equations of motion following from (3.15), and represents a generalization of the d-dimensional Reissner-Nordström-(A)dS black hole to the case where the horizon is an arbitrary Einstein space.

In the original coordinates, Hamilton's characteristic function reads

$$
W_1(U,\psi) = ae^{(d-3)(\psi-U)} + \frac{Q^2e^{-(d-3)(U+\psi)}}{2a(d-2)(d-3)^3} - \frac{\Lambda e^{(d-1)(U+\psi)}}{2a(d-1)(d-2)(d-3)} + \frac{\kappa e^{(d-3)(U+\psi)}}{4a(d-3)^2}\,.
$$

The expressions for the conjugate momenta

$$
p_U = 2\frac{\partial W_1}{\partial U}\,, \qquad p_\psi = 2\frac{\partial W_1}{\partial \psi}\,, \tag{3.30}
$$

together with (3.18), lead to the first order flow equations

$$
U' = e^{-2(d-3)\psi}\partial_U W_1(U,\psi)\,, \qquad \psi' = -e^{-2(d-3)\psi}\partial_\psi W_1(U,\psi)\,, \tag{3.31}
$$

that are satisfied by the nonextremal black holes (3.29). Notice also that, using (3.31), the action (3.15) can be written as a sum of squares. This clarifies also the reason for the very existence of first order equations for nonextremal black holes, namely they are just the expressions for the conjugate momenta in terms of derivatives of the principal function in a Hamilton-Jacobi formalism.

In the BPS case for $d = 4$, one would expect to recover the supergravity BPS flow [73], in absence of vector multiplets, that is driven by[3]

$$
W_{\text{BPS}}(U,\psi) = e^{-U}Q + e^{2\psi+U}g\,, \tag{3.32}
$$

where g is related to the cosmological constant by $\Lambda = -3g^2$. However, it is easy to see that there is no limit in which (3.32) can arise from W_1. We shall come back to this issue in the next subsection.

Second solution Similar to what was done in [70] for $\mathcal{N} = 2$, $d = 4$ ungauged supergravity, we introduce the quantity

$$
\mathbb{Q} \equiv e^{2(d-3)\psi}\frac{U'+\psi'}{d-3} + W\,. \tag{3.33}
$$

[3] To derive (3.32) from the results of [73], take the prepotential $F = -i(X^0)^2$ and a purely magnetic gauging with FI-parameter proportional to g.

Using (3.31) and the equations of motion following from the action (3.15), one easily shows that $\mathbb{Q}' = 0$, and thus $\mathbb{Q}$ is a constant of motion that can be used to simplify (3.23). In phase space we have

$$\mathbb{Q} = \frac{W_U - W_\psi}{d-3} + W = -2YW_Y + W \tag{3.34}$$

that implies $W(X,Y) = \mathbb{Q} + \sqrt{Y\omega(X)}$. Plugging this into (3.23) one gets the ODE

$$-4(d-3)^2 \frac{\partial}{\partial X}\left(\frac{\omega}{X}\right) - \frac{2Q^2}{(d-2)(d-3)X^2} - \frac{2\Lambda X^{\frac{2}{d-3}}}{(d-2)(d-3)} = \hat{E}\,. \tag{3.35}$$

A final integration leads to the solution of the original differential equation (3.23) [4]

$$S_2 = 2\mathbb{Q} - Er + 2\sqrt{-4AXY + \frac{2Q^2Y}{(d-2)(d-3)^3} - \frac{\hat{E}X^2Y}{(d-3)^2} - \frac{2\Lambda X^{\frac{2d-4}{d-3}}Y}{(d-1)(d-2)(d-3)^2}}\,, \tag{3.36}$$

which has three arbitrary integration constants $\mathbb{Q}, E, A$, but in this case the parameter domain is the whole $\mathbb{R}^3$. Using

$$\left.\frac{\partial S_2}{\partial E}\right|_{E=0} = c_3\,, \qquad \left.\frac{\partial S_2}{\partial A}\right|_{E=0} = c_4\,, \tag{3.37}$$

gives back (3.28), where

$$a = -\frac{2}{c_4}\,, \qquad c_1 = c_3\,, \qquad c_2 = -\frac{Ac_4^2}{2}\,. \tag{3.38}$$

To complete the comparison we evaluate

$$\mathbb{Q}|_{W_1} = -aXY + \frac{\kappa X}{4a(d-3)^2} + \frac{Q^2}{2a(d-2)(d-3)^2X} - \frac{\Lambda X^{\frac{d-1}{d-3}}}{2a(d-1)(d-2)(d-3)^2}\,.$$

Plugging the solution (3.28) into the rhs yields $2\mathbb{Q} = C - ac_2$. In terms of U and ψ, W_2 reads (setting $E = 0$)

$$\begin{aligned} W_2(U,\psi) &= \mathbb{Q} \\ &+ \sqrt{Ae^{(d-3)(\psi-U)} + \frac{2Q^2e^{-2(d-3)U}}{(d-2)(d-3)^3} + \frac{\kappa e^{2(d-3)\psi}}{(d-3)^2} - \frac{2\Lambda e^{2(U+(d-2)\psi)}}{(d-1)(d-2)(d-3)^2}} \end{aligned} \tag{3.39}$$

which leads to the first order flow equations

$$U' = e^{-2(d-3)\psi}\partial_U W_2(U,\psi)\,, \qquad \psi' = -e^{-2(d-3)\psi}\partial_\psi W_2(U,\psi)\,. \tag{3.40}$$

(3.40) and (3.31) have different analytic forms, but share the same general class of physical solutions. Notice also that, contrary to W_1, there is a well-defined limit in which (3.39) reduces to the BPS superpotential (3.32) for $d = 4$, by setting $A = 0$, $\Lambda = -3g^2$ and

[4] This solution was already found in [74] and for $\kappa = 0$ but with magnetic fluxes swithed on in [75].

imposing the Dirac-type quantization condition $2gQ = \kappa$.
The authors of [74] found that the potential (3.16) can be expressed in terms of a superpotential. One easily verifies that their superpotential (2.5) coincides with (3.39) and that eq. (2.4) of [74] is just the Hamilton-Jacobi equation for zero energy. The fact that a nonextremal black hole solution arises from a first-order system via a superpotential construction is thus not surprising at all [76], also in the gauged case.

Matter-coupled $\mathcal{N} = 2$, $d = 4$ gauged supergravity

In this section, we shall discuss possible generalizations of our formalism to $\mathcal{N} = 2$ supergravity in four dimensions coupled to vector multiplets and with Fayet-Iliopoulos gauging. The analogue of the one-dimensional effective action (3.15) is then given by

$$S_{\text{eff}} = \int \mathrm{d}r \left(e^{2\psi}(U'^2 - \psi'^2 + g_{i\bar{\jmath}} z^{i\prime} \bar{z}^{\bar{\jmath}\prime}) - V_{\text{eff}} \right) , \tag{3.41}$$

with the potential

$$V_{\text{eff}} = \kappa - e^{-2(U+\psi)} V_{\text{BH}} - e^{2(U+\psi)} V_g(z, \bar{z}) , \tag{3.42}$$

where [73]

$$V_{\text{BH}} = g^{i\bar{\jmath}} D_i Z \bar{D}_{\bar{\jmath}} \bar{Z} + |Z|^2 = -\frac{1}{2} Q^T \mathcal{M} Q , \qquad V_g = g^{i\bar{\jmath}} D_i \mathcal{L} \bar{D}_{\bar{\jmath}} \bar{\mathcal{L}} - 3|\mathcal{L}|^2 \tag{3.43}$$

denote respectively the black hole- and scalar potential. In (3.43), D_i is the Kähler-covariant derivative, $Z = \langle Q, \mathcal{V} \rangle$, $\mathcal{L} = \langle \mathcal{G}, \mathcal{V} \rangle$, with the symplectic section $\mathcal{V}$ and the symplectic vectors of charges Q and gauge couplings $\mathcal{G}$. $\mathcal{M}$ is the matrix defined in (2.7). Moreover

$$\langle A, B \rangle \equiv A^T \Omega B = A_\Lambda B^\Lambda - A^\Lambda B_\Lambda . \tag{3.44}$$

Note that the target space of the one-dimensional sigma model (3.41) is equipped with the metric

$$d\sigma^2 = e^{2\psi}(-d\psi^2 + dU^2 + g_{i\bar{\jmath}} dz^i d\bar{z}^{\bar{\jmath}}) , \tag{3.45}$$

and is thus a Lorentzian cone over a special Kähler manifold times a line, as can be seen by setting $\tau = e^{\psi}$. The conjugate momenta and Hamiltonian read

$$\begin{aligned} p_U &= 2e^{2\psi} U' , \qquad p_\psi = -2e^{2\psi} \psi' , \qquad p_i = e^{2\psi} g_{i\bar{\jmath}} \bar{z}^{\bar{\jmath}\prime} , \qquad \bar{p}_{\bar{\jmath}} = e^{2\psi} g_{i\bar{\jmath}} z^{i\prime} , \\ H_{\text{eff}} &= e^{-2\psi} \left(\frac{1}{4} p_U^2 - \frac{1}{4} p_\psi^2 + g^{i\bar{\jmath}} p_i p_{\bar{\jmath}} \right) + V_{\text{eff}} . \end{aligned} \tag{3.46}$$

If we set $S = 2W - Er$, the reduced Hamilton-Jacobi equation becomes

$$e^{-2\psi} \left(W_U^2 - W_\psi^2 + 4 g^{i\bar{\jmath}} \frac{\partial W}{\partial z^i} \frac{\partial W}{\partial \bar{z}^{\bar{\jmath}}} \right) + V_{\text{eff}} = E . \tag{3.47}$$

As was shown for ungauged [70] and gauged supergravity [1], the quantity

$$\mathbb{Q} \equiv e^{2\psi}(U' + \psi') + W , \tag{3.48}$$

is a first integral also in presence of the scalar fields z^i. $\mathbb{Q}$ is the Noether charge related to the symmetry

$$\delta U = U_\epsilon - U = \epsilon\,, \qquad \delta\psi = \psi_\epsilon - \psi = -\epsilon\,, \tag{3.49}$$

that leaves the potential (3.42) and the action (3.41) invariant (the latter up to boundary terms). In fact, a function W, satisfying (3.47) with $E = 0$, drives a first order flow

$$U' = e^{-2\psi} W_U\,, \qquad \psi' = -e^{-2\psi} W_\psi\,, \qquad z^{i\prime} = 2e^{-2\psi} g^{i\bar{\jmath}} \frac{\partial W}{\partial z^{\bar{\jmath}}}\,, \tag{3.50}$$

and therefore the variation of (3.41) for infinitesimal ϵ can be written as

$$\begin{aligned} \delta S = S(U_\epsilon, \psi_\epsilon) - S(U, \psi) =& -2\epsilon \int \mathrm{d}r \left(e^{2\psi}(U'^2 - \psi'^2 + 4 g_{i\bar{\jmath}} z^{i\prime} \bar{z}^{\bar{\jmath}\prime}) \right) \\ =& -2\epsilon \int \mathrm{d}r \left(U' W_U + \psi' W_\psi + z^{i\prime} \frac{\partial W}{\partial z^i} + \bar{z}^{\bar{\jmath}\prime} \frac{\partial W}{\partial \bar{z}^{\bar{\jmath}}} \right) \\ =& -2\epsilon \int \mathrm{d}r \frac{dW}{dr}\,, \end{aligned} \tag{3.51}$$

which vanishes if we choose appropriate boundary conditions. Note that the transformation (3.49) is generated by the vector field $\partial_U - \partial_\psi = \partial_U - \tau\partial_\tau$, which is a conformal Killing vector of the Lorentzian cone (3.45). The fact that $\mathbb{Q}$ is the Noether charge related to (3.49) follows also from the inverse Noether theorem[5]: If $\mathbb{Q}$ is a conserved charge, then the transformation

$$\delta q^I = [q^I, \epsilon\mathbb{Q}] = \epsilon \frac{\partial \mathbb{Q}}{\partial p_I}\,, \qquad \delta p_I = [p_I, \epsilon\mathbb{Q}] = -\epsilon \frac{\partial \mathbb{Q}}{\partial q^I}\,, \tag{3.52}$$

where [,] denotes the Poisson bracket, is a symmetry of the action.
As before, we introduce the coordinates

$$X = e^{U+\psi}\,, \qquad Y = e^{-2U}\,. \tag{3.53}$$

Then the first integral (3.48) becomes

$$\mathbb{Q} = -2Y W_Y + W\,, \tag{3.54}$$

which can be easily integrated to give

$$W(X, Y, z, \bar{z}) = \mathbb{Q} + \sqrt{Y \omega(X, z, \bar{z})}\,, \tag{3.55}$$

where ω is an integration 'constant'. Using (3.55), the Hamilton-Jacobi equation (3.47) boils down to

$$-\partial_X \frac{\omega}{X} + \frac{1}{\omega X^2} g^{i\bar{\jmath}} \frac{\partial \omega}{\partial z^i} \frac{\partial \omega}{\partial \bar{z}^{\bar{\jmath}}} - X^2 V_g - \frac{1}{X^2} V_{\text{BH}} + \kappa = E\,. \tag{3.56}$$

A particular solution to (3.56) is the one found in [73] by squaring the action for the BPS case,

[5] See [77] for a nice review.

$$\omega_{\text{BPS}} = (Z - iX^2\mathcal{L})(\bar{Z} + iX^2\bar{\mathcal{L}}) = |Z|^2 + X^4|\mathcal{L}|^2 - iX^2(\mathcal{L}\bar{Z} - \bar{\mathcal{L}}Z)\,. \tag{3.57}$$

Imposing $E = 0$, as required by Einstein's equations, and using

$$\frac{\partial \omega}{\partial z^i} = (\bar{Z} + iX^2\bar{\mathcal{L}})(D_i Z - iX^2 D_i\mathcal{L})\,, \tag{3.58}$$

as well as the special Kähler geometry identity

$$\frac{1}{2}(\mathcal{M} - i\Omega) = \Omega\bar{\mathcal{V}}\mathcal{V}\Omega + \Omega D_i\mathcal{V}g^{i\bar{\jmath}}D_{\bar{\jmath}}\bar{\mathcal{V}}\Omega\,, \tag{3.59}$$

it is only matter of some algebra to show that (3.57) solves (3.56) if one imposes the Dirac charge quantization condition

$$\langle \mathcal{G}, \mathcal{Q}\rangle = -\kappa\,. \tag{3.60}$$

In the following subsection we shall consider a particular prepotential, for which the effective action (3.41) has additional symmetries, that allow a further reduction of the Hamilton-Jacobi equation (3.56).

Prepotential $F = -iX^0X^1$ This simple model has only one complex scalar field z parametrizing the Poincaré half-plane, with Kähler metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(z+\bar{z})^2}\,, \tag{3.61}$$

which has the three Killing vectors

$$v_1 = i(\partial_z - \partial_{\bar{z}})\,, \qquad v_2 = z\partial_z + \bar{z}\partial_{\bar{z}}\,, \qquad v_3 = \frac{i}{2}(\bar{z}^2\partial_{\bar{z}} - z^2\partial_z)\,. \tag{3.62}$$

These are all symmetries of the ungauged theory, but in presence of a potential for the scalars only a linear combination of them survives, as is shown in 5 using the symplectic representation.

If we consider a configuration with only magnetic charges and purely electric gaugings, the HJ equation (3.56) becomes for this prepotential

$$-\partial_X\frac{\omega}{X} + \frac{1}{\omega X^2}g^{z\bar{z}}\frac{\partial\omega}{\partial z}\frac{\partial\omega}{\partial\bar{z}} + X^2\frac{g_0^2 + g_1^2 z\bar{z} + 2g_0g_1(z+\bar{z})}{z+\bar{z}} - \frac{1}{X^2}\frac{{p^1}^2 + {p^0}^2 z\bar{z}}{z+\bar{z}} + \kappa = E\,. \tag{3.63}$$

The linear combination

$$v = \frac{g_0^2}{2g_1^2}v_1 + v_3 = \frac{i}{2}\left(\frac{g_0^2}{g_1^2} - z^2\right)\partial_z - \frac{i}{2}\left(\frac{g_0^2}{g_1^2} - \bar{z}^2\right)\partial_{\bar{z}} \tag{3.64}$$

generates a symmetry of (3.41) if one imposes the BPS condition [78] $p^0 g_0 = p^1 g_1$. It is straightforward to verify that this implies the existence of a further conserved charge

$$\mathbb{C} = \frac{i}{2}\left(\frac{g_0^2}{g_1^2} - z^2\right)\frac{\partial\omega}{\partial z} - \frac{i}{2}\left(\frac{g_0^2}{g_1^2} - \bar{z}^2\right)\frac{\partial\omega}{\partial\bar{z}}\,. \tag{3.65}$$

By introducing the new variables

$$z = \frac{g_0}{g_1} \tanh\left(\frac{g_0}{g_1}(u+iv)\right) , \qquad \bar{z} = \frac{g_0}{g_1} \tanh\left(\frac{g_0}{g_1}(u-iv)\right) , \tag{3.66}$$

(3.65) can easily be integrated, with the result $\omega = 2\mathbb{C}v + \alpha(u, X)$. Plugging this into (3.63), the HJ equation assumes the form

$$-\partial_X \frac{\alpha}{X} + \left(\frac{g_1}{g_0}\right)^2 \frac{\sinh^2(2g_0 u/g_1)}{4X^2} \frac{\alpha_u^2 + 4\mathbb{C}^2}{\alpha + 2\mathbb{C}v} - X^2 V_g(u) - \frac{1}{X^2} V_{\text{BH}}(u) + \kappa = E\,, \tag{3.67}$$

where

$$V_{\text{BH}}(u) = \frac{(p^0)^2 g_0}{g_1 \tanh(2g_0 u/g_1)}\,, \qquad V_g(u) = -\frac{g_0 g_1}{\tanh(2g_0 u/g_1)} - 2g_0 g_1\,. \tag{3.68}$$

It is easy to see that (3.67) can be satisfied for all v only if $\mathbb{C} = 0$[6], so that we have

$$-\partial_X \frac{\alpha}{X} + \left(\frac{g_1}{g_0}\right)^2 \frac{\sinh^2(2g_0 u/g_1)}{4X^2} \frac{\alpha_u^2}{\alpha} - X^2 V_g(u) - \frac{1}{X^2} V_{\text{BH}}(u) + \kappa = E\,. \tag{3.69}$$

For the prepotential under consideration, the BPS solution (3.57) reads

$$\omega_{\text{BPS}}(X, z, \bar{z}) = \frac{(p^1 + p^0 z - X^2(g_0 + g_1 z))(p^1 + p^0 \bar{z} - X^2(g_0 + g_1 \bar{z}))}{2(z+\bar{z})}\,. \tag{3.70}$$

Imposing $g_0 p^0 = g_1 p^1$ and using the coordinates (3.66), this leads to

$$\alpha_{\text{BPS}}(X, u) = \frac{p^1 (p^0 - g_1 X^2)^2 e^{4p^1 u/p^0}}{p^0 (e^{4p^1 u/p^0} - 1)}\,. \tag{3.71}$$

It is interesting to note that the variables X and u separate in (3.71). This suggests to use a product ansatz $\alpha(X, u) = \xi(X)\mu(u)$ in order to get something more general than (3.71). Unfortunately, plugging this into (3.69) gives back precisely (3.71). Another possibility is inspired by the comparison with (3.36) (for $d = 4$), which contains, in addition to quartic, quadratic and X-independent terms that appear also in (3.71), a linear piece in X proportional to the constant A that is essentially a nonextremality parameter (or black hole mass). One may thus try

$$\alpha(X, u) = \sum_{n=0}^{4} \alpha_n(u) X^n\,, \tag{3.72}$$

where (to be still more general) we added a cubic term as well. However, one can check that, using this ansatz in (3.69) leads to an overdetermined system that admits a solution only for $\alpha_1 = \alpha_3 = 0$, namely (3.71).

It remains to be seen if there exist additional conserved charges associated to hidden

[6] This sort of 'axion-free' condition is probably related to the special choice of purely electric gaugings and only magnetic charges, so we don't expect that $\mathbb{C}$ vanishes in a more general setting.

symmetries of the action (3.41), that would allow to completely separate the Hamilton-Jacobi equation (3.47). Note in this context that the transformation (3.49) acts only on U and ψ but not on the scalars z^i, whereas (3.64) touches only the z^i but not the metric components U and ψ. There might thus exist (at least for some specific models) more complicated symmetry transformations involving all the dynamical variables.

In this subsection we considered electrically charged static nonextremal black holes in d-dimensional Einstein-Maxwell-(A)dS gravity, whose horizon is a generic Einstein space in $d-2$ dimensions. We have shown that for this system the Hamilton-Jacobi equation is exactly integrable and admits two branches of solutions. One of them exhibits a non-simply connected domain of integration constants and does not reduce to the well-known solution for the $d=4$ BPS case. The principal functions generate two first order flows that are analytically different but support the same general solution. One of the two sets of flow equations corresponds to those found in [74] and (for $d=4$ and $\Lambda=0$) in [79]. We clarified thus also the reason for the very existence of first-order equations for nonextremal black holes, namely, they are just the expressions for the conjugate momenta in terms of derivatives of the principal function in a Hamilton-Jacobi formalism.

In the last part of our paper, we also analyzed if these integrability properties continue to hold for matter-coupled $N=2$, $d=4$ gauged supergravity. Unfortunately, it turned out that the principal function W for nonextremal black holes is not straightforwardly generalizable to this case. Still, we showed (for the example of a particular model) that there exist several conserved charges that allow a partial separation of variables in the HJ equation. We pointed out the possible existence of additional hidden symmetries of the one-dimensional effective action (3.41) that involve simultaneous transformations of the dynamical variables of both the metric and the scalar sector.

One might ask if there exist covariantly constant spinors related to the first order equations. The authors of [79] have shown that the nonextremal Reissner-Nordström solution cannot admit (generalized) Killing spinors in 3+1 dimensions, but it is supersymmetric in a lower-dimensional effective theory. It might be, however, that the nonextremal black holes considered in this paper possess so-called conformal Killing spinors (CKS, cf. e.g. [80] for a review of this topic). Note in this context that both the (nonextremal) Kerr metric and all other type II-II vacuum spacetimes do admit a CKS [81]. We hope to come back to this point in a future publication.

3.3 Hidden symmetries and 4d RN-TN-Λ

In this section, we specialize the theory studied in the previous subsection to $d=4$ and with a dimensional reduction to $d=3$ and a dualization of the vectors into scalars we show the appearing of hidden symmetries. With them, we proceed then to the integration of the HJ PDE similarly to the previous case, but with the inclusion of NUT charge.

Four dimensional EMΛ system

In this section, we focus on $3+1$-dimensional Einstein-Maxwell-Λ gravity, with action[7]

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left(R - F_{\mu\nu}F^{\mu\nu} - 2\Lambda\right)\,, \tag{3.73}$$

and equations of motion

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 2\left(F_{\mu\sigma}F_\nu^{\ \sigma} - \frac{1}{4}g_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}\right)\,, \qquad \nabla_\mu F^{\mu\nu} = 0\,. \tag{3.74}$$

The Faraday tensor can be locally expressed in terms of a gauge potential as $F = \mathrm{d}A$.

We shall investigate the integrability properties of the stationary Einstein-Maxwell-Λ system, which is an extension of the work in [72].

Dimensional reduction

Let us consider stationary spacetimes admitting a Killing field which is timelike at infinity. Applying the algorithm of Kaluza-Klein reduction along the timelike direction, the metic and the gauge field can be decomposed as

$$\mathrm{d}s^2 = -e^{-\phi}(\mathrm{d}t + K_\alpha \mathrm{d}x^\alpha)^2 + e^\phi h_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta\,, \qquad A = B(\mathrm{d}t + K_\alpha\mathrm{d}x^\alpha) + B_\alpha \mathrm{d}x^\alpha\,, \tag{3.75}$$

where early greek indices refer to three dimensions, and the fields $h_{\alpha\beta}$, K_α, B_α, ϕ and B are t-independent. Here and in what follows, the indices $\alpha, \beta, ..$ are raised and lowered by $h_{\alpha\beta}$ and its inverse. Then the effective three-dimensional Lagrangian derived from (3.73) becomes

$$\begin{aligned}\mathcal{L}^{(3)} \;=\; \sqrt{h}\Big[&R^{(3)} - \frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + \frac{1}{4}e^{-2\phi}K_{\alpha\beta}K^{\alpha\beta} + 2e^\phi\partial_\alpha B\partial^\alpha B \\ &-e^{-\phi}(G_{\alpha\beta} + K_{\alpha\beta}B)(G^{\alpha\beta} + K^{\alpha\beta}B) - 2\Lambda e^\phi\Big]\,,\end{aligned} \tag{3.76}$$

where $G_{\alpha\beta} \equiv \partial_\alpha B_\beta - \partial_\beta B_\alpha$ and $K_{\alpha\beta} \equiv \partial_\alpha K_\beta - \partial_\beta K_\alpha$. It is convenient to dualize the two vector fields to scalars, which can be implemented by adding to (3.76) a piece containing two Lagrange multipliers C and $\tilde{\psi}$ that ensure the Bianchi identities,

$$\tilde{\mathcal{L}}^{(3)} = \mathcal{L}^{(3)} + 2C\epsilon^{\alpha\beta\gamma}\partial_\alpha G_{\beta\gamma} + (\tilde{\psi} + CB)\epsilon^{\alpha\beta\gamma}\partial_\alpha K_{\beta\gamma}\,. \tag{3.77}$$

Variation of (3.77) w.r.t. $K_{\alpha\beta}$ and $G_{\alpha\beta}$ yields

$$K^{\alpha\beta} = \frac{2}{\sqrt{h}}e^{2\phi}\epsilon^{\alpha\beta\gamma}\omega_\gamma\,, \qquad \omega_\gamma \equiv \partial_\gamma\tilde{\psi} + C\partial_\gamma B - B\partial_\gamma C\,, \tag{3.78}$$

and

[7]We use the signature $(-,+,+,+)$. The Ricci tensor is defined as $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu} = \partial_\sigma\Gamma^\sigma{}_{\mu\nu} - \partial_\nu\Gamma^\sigma{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\nu}\Gamma^\sigma{}_{\sigma\rho} - \Gamma^\rho{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\rho}$.

$$G^{\alpha\beta} + K^{\alpha\beta} B = -\frac{1}{\sqrt{h}} e^{\phi} \epsilon^{\alpha\beta\gamma} \partial_\gamma C\,. \tag{3.79}$$

These equations express the field strengths in terms of the twist potential $\tilde{\psi}$ and the magnetic potential C. Plugging (3.78) and (3.79) back into (3.77) leads (after dropping a tilde on $\tilde{\mathcal{L}}^{(3)}$) to

$$\mathcal{L}^{(3)} = \sqrt{h}\left[R^{(3)} - \langle J_\alpha, J^\alpha\rangle - 2\Lambda e^{\phi}\right], \tag{3.80}$$

where we have introduced the notation

$$\langle J_\alpha, J_\beta\rangle \equiv \frac{1}{2}\left[\partial_\alpha\phi\partial_\beta\phi + 4e^{2\phi}\omega_\alpha\omega_\beta - 4e^{\phi}\left(\partial_\alpha B\partial_\beta B + \partial_\alpha C\partial_\beta C\right)\right]\,. \tag{3.81}$$

The equations of motion following from the Lagrangian (3.80) are the three-dimensional Einstein equations

$$G^{(3)}_{\alpha\beta} + \Lambda e^{\phi} h_{\alpha\beta} = \langle J_\alpha, J_\beta\rangle - \frac{1}{2} h_{\alpha\beta}\langle J_\gamma, J^\gamma\rangle\,, \tag{3.82}$$

supplemented by the divergence-type equations of motion

$$\nabla_\alpha[\partial^\alpha\phi + 2e^{\phi}(B\partial^\alpha B + C\partial^\alpha C) - 4e^{2\phi}\tilde{\psi}\omega^\alpha] = 2\Lambda e^{\phi}\,, \qquad \nabla_\alpha(e^{2\phi}\omega^\alpha) = 0\,, \tag{3.83}$$

$$\nabla_\alpha(e^{\phi}\partial^\alpha B - 2e^{2\phi}C\omega^\alpha) = 0\,, \qquad \nabla_\alpha(e^{\phi}\partial^\alpha C + 2e^{2\phi}B\omega^\alpha) = 0\,. \tag{3.84}$$

(3.80) describes a nonlinear σ-model with pseudo-Riemannian target space coupled to Euclidean gravity in $d = 3$, with a potential. The latter breaks part of the target space isometries.

Nonlinear σ-model and broken symmetries

The target space Φ of the scalars in (3.80) is a Bergmann space corresponding to a non-compact version of $\mathbb{C}\mathrm{P}^2$ [29,82,83], namely it describes a coset space $\mathrm{SU}(2,1)/\mathrm{S}(\mathrm{U}(1,1)\times\mathrm{U}(1))$, endowed with the metric

$$\mathrm{d}s^2_\Phi = \mathcal{G}_{IJ}(\varphi)\mathrm{d}\varphi^I\mathrm{d}\varphi^J = \mathrm{d}\phi^2 + 4e^{2\phi}(\mathrm{d}\tilde{\psi} + C\mathrm{d}B - B\mathrm{d}C)^2 - 4e^{\phi}(\mathrm{d}B^2 + \mathrm{d}C^2)\,, \tag{3.85}$$

where $\varphi^I = (\phi, \tilde{\psi}, B, C)$. One can easily verify that

$$R_{IJ} = -\frac{3}{2}\mathcal{G}_{IJ}\,, \qquad C_{IJKL} = -\frac{1}{2}\epsilon_{IJMN}C^{MN}{}_{KL}\,, \qquad D_I R_{JKLM} = 0\,. \tag{3.86}$$

Here R_{IJKL} and C_{IJKL} are the Riemann and Weyl tensors constructed from the target space metric $\mathcal{G}_{IJ}$ and the covariant derivative D_I. The Bergmann space is a special Kähler-Einstein manifold with negative curvature. The last equation of (3.86) is a differential characterization of a symmetric space, the second equation implies a quaternionic structure [84].

The eight Killing vectors of Φ generating the isometry algebra $\mathfrak{su}(2,1)$ are given by

$$
\begin{aligned}
&\xi_1 = \partial_{\tilde{\psi}}\,, \qquad \xi_2 = C\partial_{\tilde{\psi}} + \partial_B\,, \qquad \xi_3 = -B\partial_{\tilde{\psi}} + \partial_C\,,\\
&\xi_4 = -C\partial_B + B\partial_C\,, \qquad \xi_5 = -2\partial_\phi + 2\tilde{\psi}\partial_{\tilde{\psi}} + B\partial_B + C\partial_C\,,\\
&\xi_6 = 4\tilde{\psi}\partial_\phi + \left[\frac{1}{2}(e^{-\phi} - (B^2+C^2))^2 - 2\tilde{\psi}^2\right]\partial_{\tilde{\psi}}\\
&\qquad + \left[C(e^{-\phi} - (B^2+C^2)) - 2\tilde{\psi}B\right]\partial_B - \left[B(e^{-\phi} - (B^2+C^2)) + 2\tilde{\psi}C\right]\partial_C\,,\\
&\xi_7 = -4B\partial_\phi + \left[2\tilde{\psi}B - C(e^{-\phi} - (B^2+C^2))\right]\partial_{\tilde{\psi}}\\
&\qquad +(e^{-\phi} + B^2 - 3C^2)\partial_B + (4BC - 2\tilde{\psi})\partial_C\,,\\
&\xi_8 = -4C\partial_\phi + \left[2\tilde{\psi}C + B(e^{-\phi} - (B^2+C^2))\right]\partial_{\tilde{\psi}}\\
&\qquad +(4BC + 2\tilde{\psi})\partial_B + (e^{-\phi} + C^2 - 3B^2)\partial_C\,.
\end{aligned}
\tag{3.87}
$$

The first five Killing vectors represent infinitesimal transformations that are linear in the scalars and comprehend a twist transformation, two electromagnetic gauge transformations, an internal $\mathrm{U}(1)$ transformation and a scaling one. The remaining three are the most interesting, due to the nonlinearity in the fields, and they are usually called generalized Ehlers transformation (ξ_6) [85] and two Harrison transformations (ξ_7, ξ_8) [86].

In order to see that these Killing vectors indeed generate the $\mathrm{SU}(2,1)$ symmetry, let us define

$$
\begin{aligned}
&{E_2}^1 = -\frac{1}{4}[\xi_7 + i\xi_8 + i(\xi_3 - i\xi_2)]\,, \qquad {E_2}^3 = -\frac{1}{4}[-(\xi_7 + i\xi_8) + i(\xi_3 - i\xi_2)]\,,\\
&{E_1}^3 = \frac{1}{4}(2\xi_5 + i\xi_1 + 2i\xi_6)\,, \quad {E_1}^1 = H_1 + {E_3}^3\,, \qquad {E_2}^2 = H_2 + {E_3}^3\,,\\
&{E_3}^3 = -\frac{1}{3}(H_1 + H_2)\,, \qquad {E_1}^2 = -({E_2}^1)^*\,, \qquad {E_3}^1 = ({E_1}^3)^*\,, \qquad {E_3}^2 = ({E_2}^3)^*\,,
\end{aligned}
\tag{3.88}
$$

where H_1, H_2 are Cartan generators defined by

$$
H_1 = \frac{i}{2}\xi_1 - i\xi_6\,, \qquad H_2 = \frac{i}{4}(\xi_1 - 6\xi_4 - 2\xi_6)\,, \qquad [H_1, H_2] = 0\,.
$$

One can easily verify that these vectors ${E_i}^j$ $(i, j = 1, 2, 3)$ satisfy the $\mathfrak{su}(2,1)$ algebra

$$
[{E_i}^j, {E_k}^l] = {\delta_k}^j {E_i}^l - {\delta_i}^l {E_k}^j\,, \qquad {E_i}^i = 0\,. \tag{3.89}
$$

Note that the dependence of the scalar potential

$$
V(\phi) = -2\Lambda e^{\phi} \tag{3.90}
$$

on the dilaton ϕ breaks the invariance under nonlinear isometries and scalings. It is easy to see that the latter is recovered if we admit a rescaling of Λ.

The five unbroken generators close themselves to form one-dimensional Heisenberg subalgebra in semidirect sum with $\mathbb{R}^2$,

$$
[\xi_2, \xi_3] = -2\xi_1\,, \qquad [\xi_2, \xi_1] = [\xi_3, \xi_1] = 0\,,
$$

$$[\xi_i, \xi_4] = (\sigma_4)_i^{\ j} \xi_j \,, \qquad [\xi_i, \xi_5] = (\sigma_5)_i^{\ j} \xi_j \,, \tag{3.91}$$

where $i, j = 1, 2, 3$ and

$$\sigma_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} , \qquad \sigma_5 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

We call the group generated by this algebra G_{res}. The Heisenberg algebra (3.91) realizes the fact that the constant ϕ space constitutes a Nil manifold, viz, one can view the four-dimensional metric (3.85) as a Wick-rotated Bianchi-II universe. The theory described by (3.80) is thus invariant only under G_{res}.

The well-known solution-generating techniques [87–89] based on group theory can thus not be applied in presence of a cosmological constant. Moreover, the broken symmetries are also a first sign of the loss of complete integrability, valid for $\Lambda = 0$ after another dimensional reduction [90, 91]. This implies also the inapplicability of the inverse scattering method [89, 92]. In what follows, we shall perform an analysis of some remaining integrability properties, extending the results of [72].

Hamiltonian formalism and first integrals

In the spacetime admitting a single Killing field, the sigma model still couples to the base space $h_{\alpha\beta}$ represented by three-dimensional Einstein gravity according to (3.82). Because of the intricacy of this system, we usually simplify the problem by assuming further symmetries. In the absence of Λ, the base space is decoupled from the sigma model by assuming an axial Killing field. More precisely, the metric without Λ can be cast into the Weyl-Papapetrou form [93], and the base space part can be obtained by quadrature once the sigma model on $\mathbb{R}^2$ is solved. Unfortunately, this decoupling does not occur in the presence of Λ.

In this section, we follow a different path to arrive at an integrable system. Along the lines of the argument in [72], we consider the case in which the base space admits only a single degree of freedom. Now we suppose that $h_{\alpha\beta}$ describes a warped product space $\mathbb{R} \times \Sigma$, with Σ a two-dimensional manifold. Moreover we assume that all the scalar fields depend only on the coordinate representing $\mathbb{R}$. To capture this more conveniently, let us introduce another scalar field k that describes a rescaling of the three-dimensional metric $h_{\alpha\beta}$,

$$h_{\alpha\beta} = k \hat{h}_{\alpha\beta} \,. \tag{3.92}$$

Absorbing the warp factor into k, $\hat{h}_{\alpha\beta}$ can be taken to be an unwarped product,

$$\hat{h}_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta = \mathrm{d}\sigma^2 + \mathrm{d}\Omega^2 \,, \tag{3.93}$$

where $\mathrm{d}\Omega^2$ is the line element on Σ. Under these settings, every quantity depends only on a single valuable σ. In this case the trace and the $\sigma\sigma$-component of the Einstein equations (3.82) become respectively

$$\hat{R}^{(3)} = \frac{1}{2k^2}\left(\frac{\mathrm{d}k}{\mathrm{d}\sigma}\right)^2 - \langle J_\sigma, J_\sigma\rangle + 2\Lambda k e^{\phi}\,, \tag{3.94}$$

$$\frac{1}{k}\left(\frac{\mathrm{d}^2 k}{\mathrm{d}\sigma^2}\right) = \frac{1}{k^2}\left(\frac{\mathrm{d}k}{\mathrm{d}\sigma}\right)^2 - \langle J_\sigma, J_\sigma\rangle - 2\Lambda k e^{\phi}\,. \tag{3.95}$$

It is clear that the scalar curvature $\hat{R}^{(3)}$ must be constant as a consequence of the fact that the r.h.s. of (3.94) depends only on σ and the l.h.s. is independent of σ. Without further resrictions we can thus take $\hat{R}^{(3)} = 2l$ with $l = 0, \pm 1$, so that Σ must be a maximally symmetric space, $\mathrm{d}\Omega_l^2 = \mathrm{d}\theta^2 + f_l^2(\theta)\mathrm{d}\varphi^2$, where

$$f_l(\theta) = \frac{1}{\sqrt{l}}\sin(\sqrt{l}\theta) = \begin{cases} \sin\theta\,, & l = 1\,,\\ \theta\,, & l = 0\,,\\ \sinh\theta\,, & l = -1\,. \end{cases} \tag{3.96}$$

One obtains then a classical dynamical system with five degrees of freedom, with action

$$S = \int \mathrm{d}\sigma k^{\frac{1}{2}}\left[\frac{1}{2k^2}\left(\frac{\mathrm{d}k}{\mathrm{d}\sigma}\right)^2 - \langle J_\sigma, J_\sigma\rangle + 2l - 2\Lambda k e^{\phi}\right]\,. \tag{3.97}$$

For future convenience we introduce a new evolution parameter τ defined by

$$k^{\frac{3}{2}} e^{\phi}\mathrm{d}\sigma = \mathrm{d}\tau\,.$$

With the new potential $\hat{V} = 2\Lambda - \frac{2l}{k}e^{-\phi}$ and $\omega \equiv \omega_\tau$, the action (3.97) can be expressed as $S = \int L\mathrm{d}\tau$ with a Lagrangian

$$L = \frac{1}{2}\left[e^{\phi}k'^2 - k^2 e^{\phi}\phi'^2 - 4k^2 e^{3\phi}\omega^2 + 4e^{2\phi}k^2(B'^2 + C'^2)\right] - \hat{V}\,, \tag{3.98}$$

where a prime denotes a derivative w.r.t. τ. It is easy to see that (3.94) is the constraint $H \equiv L + 2\hat{V} = 0$. It then turns out more convenient to pass to a Hamiltonian formulation rather than working in a Lagrangian description. After a Legendre transformation one gets

$$\begin{aligned} H &= \frac{1}{2}\left[e^{-\phi}p_k^2 - \frac{e^{-\phi}}{k^2}p_\phi^2 - \frac{e^{-3\phi}}{4k^2}p_{\tilde{\psi}}^2\right. \\ &+ \left.\frac{e^{-2\phi}}{4k^2}\left(p_B^2 + p_C^2 - 2Cp_Bp_{\tilde{\psi}} + 2Bp_Cp_{\tilde{\psi}} + (B^2 + C^2)p_{\tilde{\psi}}^2\right)\right] + \hat{V}\,. \end{aligned} \tag{3.99}$$

The solution of this dynamical system is highly linked to the existence of commuting constants of motion. The Killing vector fields of $\mathfrak{su}(2,1)$ can be promoted to functions in phase space, realizing a Lie algebra isomorphism, by means of the substitutions[8]

$$\partial_{\varphi^I} \mapsto p_{\varphi^I}\,, \qquad [\cdot,\cdot] \mapsto \{\cdot,\cdot\}_{\mathrm{PB}}\,, \qquad \xi_i \mapsto -C_i\,, \tag{3.100}$$

[8]Our convention for the Poisson bracket is $\{A, B\} \equiv \Omega^{MN}\partial_M A \partial_N B = \sum_I\left(\frac{\partial A}{\partial q^I}\frac{\partial B}{\partial p_I} - \frac{\partial A}{\partial p_I}\frac{\partial B}{\partial q^I}\right)$, where $\Omega = i\sigma_2$ is the symplectic form.

where $\{\varphi^I\} = \{\phi, \tilde{\psi}, B, C\}$ and $i = 1, \ldots, 8$. The minus sign in front of C_i reflects the fact that the infinitesimal generators and the corresponding charges obey the same algebra up to the sign of the structure constants[9]. The only nonvanishing Poisson brackets between the C_i and the Hamiltonian are given by

$$\{H, C_5\} = -2H + 4\Lambda\,, \qquad \{H, C_6\} = 4H\tilde{\psi} - 8\Lambda\tilde{\psi}\,,$$
$$\{H, C_7\} = -4BH + 8\Lambda B\,, \qquad \{H, C_8\} = -4HC + 8\Lambda C\,. \tag{3.101}$$

Since the C_i do not depend explicitely on τ, we find immediately that C_1, C_2, C_3, C_4 are four constants of motion besides H. Moreover if we define the modified function $\tilde{C}_5 \equiv C_5 - 4\Lambda\tau$ and use the constraint $H = 0$, we recover the constant of motion linked to a scale trasformation ξ_5,

$$\frac{\mathrm{d}\tilde{C}_5}{\mathrm{d}\tau} = -2H = 0\,. \tag{3.102}$$

The modification of C_5 to $\tilde{C}_5$ is a consequence of the necessity to rescale also Λ in order to maintain invariance under scale transformations.

The only nonvanishing Poisson brackets between the constants of motion read

$$\{C_2, C_3\} = -2C_1\,, \qquad \{C_2, C_4\} = C_3\,, \qquad \{C_3, C_4\} = -C_2\,, \tag{3.103}$$
$$\{\tilde{C}_5, C_1\} = -2C_1\,, \qquad \{\tilde{C}_5, C_2\} = -C_2\,, \qquad \{\tilde{C}_5, C_3\} = -C_3\,. \tag{3.104}$$

Among $C_1, C_2, C_3, C_4, \tilde{C}_5$ and the operators composed of them, the maximal set of commuting first integrals is given by $H, C_1, C_4, C_2^2 + C_3^2$, and we fix the values of these first integrals with four constants E, v, K_1, K_2,

$$H = E\,, \quad p_{\tilde{\psi}} = 4v\,, \quad Bp_C - Cp_B = K_1\,, \quad (p_B + Cp_{\tilde{\psi}})^2 + (p_C - Bp_{\tilde{\psi}})^2 = K_2\,. \tag{3.105}$$

We want to use these equations to solve the system, so we shall set $E = 0$ only at the end of the integration procedure.

Integrability: RN-TN-Λ solution

Using (3.105), the Hamiltonian can be rewritten as

$$H = \frac{e^{-\phi}}{2}p_k^2 - \frac{e^{-\phi}}{2k^2}p_\phi^2 - \frac{e^{-3\phi}}{8k^2}(4v)^2 + \frac{e^{-2\phi}}{8k^2}(K_2 + 16vK_1) + \hat{V}\,, \tag{3.106}$$

and thus the electromagnetic and twist part has decoupled from the other fields. In order to solve the Hamilton-Jacobi equation

[9] This can be shown as follows. Let $Q_i = Q_i(q^I, p_I)$ be first integrals obeying the Lie algebra $\{Q_i, Q_j\} = f^k{}_{ij}Q_k$ and let us denote the corresponding Hamiltonian vector fields by $V_i^M = \Omega^{MN}\partial_N Q_i$. For any function $F = F(q^I, p_I)$ in phase space, we have a formula $V_i^M\partial_M F = -\{Q_i, F\}$. It follows that for the vector field $V_{ij}^M = \Omega^{MN}\partial_N\{Q_i, Q_j\} = f^k{}_{ij}V_k^M$, we obtain $V_{ij}^M\partial_M F = -\{\{Q_i, Q_j\}, F\} = -\{Q_i, \{Q_j, F\}\} + \{Q_j, \{Q_i, F\}\} = -[V_i, V_j]^M\partial_M F$, where at the second equality we used the Jacobi identity. This establishes $[V_i, V_j] = -f^k{}_{ij}V_k$, as desired.

$$H\left(k, \phi, \frac{\partial S}{\partial k}, \frac{\partial S}{\partial \phi}\right) + \frac{\partial S}{\partial \tau} = 0\,, \tag{3.107}$$

we use the separation ansatz

$$S = W(k, \phi) - E\tau\,, \tag{3.108}$$

which leads to

$$\frac{e^{-\phi}}{2}\left(\frac{\partial W}{\partial k}\right)^2 - \frac{e^{-\phi}}{2k^2}\left(\frac{\partial W}{\partial \phi}\right)^2 - \frac{e^{-3\phi}}{8k^2}(4v)^2 + \frac{e^{-2\phi}}{8k^2}\left(K_2 + 16vK_1\right) + \hat{V} = E\,. \tag{3.109}$$

(3.109) can be solved by defining the new variables $x = ke^{\phi}$, $y = e^{-\phi}$ and applying the Charpit-Lagrange method. The result is

$$\begin{aligned} W(x, y) \;&=\; \frac{1}{6a^2}\sqrt{2ax - v^2}\left(\tilde{E}(v^2 + ax) - 6al - 12a^2 y\right) \\ &\quad + \frac{1}{8v}(K_2 + 16vK_1)\text{arccot}\left(\frac{v}{\sqrt{2ax - v^2}}\right)\,, \end{aligned} \tag{3.110}$$

where $\tilde{E} \equiv 2\Lambda - E$ and a is an integration constant. Following the Hamilton-Jacobi technique we can introduce two other constants β_1, β_2 according to

$$\beta_1 = \frac{\partial S}{\partial \tilde{E}}\,, \qquad \beta_2 = \frac{\partial S}{\partial a}\,. \tag{3.111}$$

Using the dynamical constraint $H = 0$, they are given by

$$\beta_1 = \frac{1}{6a^2}\sqrt{2ax - v^2}(v^2 + ax) + \tau\,, \tag{3.112}$$

$$\beta_2 = \frac{\Lambda(2v^4 - 2av^2x - a^2x^2)}{3a^3\sqrt{2ax - v^2}} + \frac{K_2 + 16vK_1}{16a\sqrt{2ax - v^2}} - \frac{lv^2 - lax + 2a^2xy}{a^2\sqrt{2ax - v^2}}\,. \tag{3.113}$$

To simplify the solution, it is convenient to define a new evolution parameter r by

$$\tau = \frac{1}{\sqrt{2a}}\left(\frac{r^3}{3} + r\frac{v^2}{2a}\right)\,. \tag{3.114}$$

To solve the two algebraic equations (3.112) and (3.113), we note that it is possible to set $\beta_1 = 0$ without loss of generality by shifting τ. Then (3.112) gives

$$x = r^2 + \frac{v^2}{2a}\,. \tag{3.115}$$

Plugging this into (3.113) yields

$$y = \frac{1}{2a\left(r^2 + \frac{v^2}{2a}\right)}\left[\frac{K_2 + 16vK_1}{16} - \sqrt{2}\beta_2 a^{3/2} r + lr^2 - \frac{lv^2}{2a} - \frac{\Lambda}{3}\left(r^4 + \frac{3r^2v^2}{a} - \frac{3v^4}{4a^2}\right)\right]\,.$$

Using the original expression for H (3.99), the Hamilton equations for the electromagnetic part become

$$\frac{\mathrm{d}p_B}{\mathrm{d}r} = -\frac{v}{\sqrt{2a}\left(r^2+\frac{v^2}{2a}\right)}(p_C+4vB)\,, \qquad \frac{\mathrm{d}p_C}{\mathrm{d}r} = -\frac{v}{\sqrt{2a}\left(r^2+\frac{v^2}{2a}\right)}(-p_B+4vC)\,, \tag{3.116}$$

$$\frac{\mathrm{d}B}{\mathrm{d}r} = \frac{1}{4\sqrt{2a}\left(r^2+\frac{v^2}{2a}\right)}(p_B-4vC)\,, \qquad \frac{\mathrm{d}C}{\mathrm{d}r} = \frac{1}{4\sqrt{2a}\left(r^2+\frac{v^2}{2a}\right)}(p_C+4vB)\,. \tag{3.117}$$

Using the gauge freedom generated by ξ_1 and ξ_2, we can implement a boundary condition in such a way that B and C vanish at infinity [29]. This eliminates two integration constants and the solutions are given by

$$B = \frac{\beta_3 + r\beta_4}{r^2+\frac{v^2}{2a}}\,, \qquad C = \frac{\sqrt{2a}}{v}\frac{r\beta_3 - \frac{v^2\beta_4}{2a^2}}{r^2+\frac{v^2}{2a}}\,. \tag{3.118}$$

Finally, the twist potential $\tilde{\psi}$ can be found by inverting the equation $p_{\tilde{\psi}} = 4v$, which leads to

$$\tilde{\psi} = \int \mathrm{d}r\left(-\frac{v}{\sqrt{2a}}\frac{e^{-\phi}}{\left(r^2+\frac{v^2}{2a}\right)} - C\frac{\mathrm{d}B}{\mathrm{d}r} + B\frac{\mathrm{d}C}{\mathrm{d}r}\right)\,. \tag{3.119}$$

The integration procedure is now complete. Since the constants defining the solution are not very illuminating, we define the new constants

$$m = \frac{\beta_2 a^{3/2}}{\sqrt{2}}\,, \quad n = \frac{v}{\sqrt{2a}}\,, \quad Q = \sqrt{2a}\beta_4\,, \quad P = -\frac{\sqrt{2a}\beta_3}{n}\,, \quad 2a = m^2 + l^2 n^2\,, \tag{3.120}$$

which give $K_1 = 0$ and $K_2 = 16(P^2+Q^2)$. It turns out that the four-dimensional metric and $\mathrm{U}(1)$ gauge field take the form of the RN-TN-(A)dS solution [94],

$$\mathrm{d}s^2 = -e^{-\phi}(\mathrm{d}t + K_\varphi \mathrm{d}\varphi)^2 + ke^{\phi}\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2 + f_l^2(\theta)\mathrm{d}\varphi^2\right)\,, \tag{3.121}$$

$$A_\mu \mathrm{d}x^\mu = B\mathrm{d}t + A_\varphi \mathrm{d}\varphi\,, \tag{3.122}$$

where

$$\begin{aligned} &\Delta = l(r^2-n^2) - 2mr - \frac{\Lambda}{3}(r^4 + 6r^2n^2 - 3n^4) + P^2 + Q^2\,, \\ &k = \frac{\Delta}{m^2+l^2n^2}\,, \qquad e^{-\phi} = \frac{k}{r^2+n^2}\,, \qquad K_\varphi = -4n\sqrt{m^2+l^2n^2}f_l^2(\theta/2)\,, \\ &B = \frac{Qr - nP}{\sqrt{m^2+l^2n^2}(r^2+n^2)}\,, \qquad A_\varphi = \frac{2f_l^2(\theta/2)\left(P(n^2-r^2) - 2nQr\right)}{n^2+r^2}\,. \end{aligned} \tag{3.123}$$

Note that the fields K_φ and A_φ are obtained from the dualization (3.78) and (3.79), that involves

$$\tilde{\psi} = \frac{n}{3(m^2+l^2n^2)}\left(\Lambda r + \frac{3lr - 3m - 4\Lambda n^2 r}{r^2+n^2}\right)\,, \qquad C = -\frac{nQ + rP}{\sqrt{m^2+l^2n^2}(r^2+n^2)}\,.$$

For $P = Q = 0$ we recover the results of [72], and thus the integrability properties described in [72] are still valid in the case of nonvanishing electromagnetic charges. We saw that, even if the cosmological constant reduces the internal symmetry group from $\mathrm{SU}(2,1)$ to G_{res}, it hasn't spoiled integrability once we restrict to the subspace (3.93). This condition reduces the infinite number of degrees of freedom to effectively five. Only the three nonlinear generators of $\mathfrak{su}(2,1)$ are broken and the remaining commuting first integrals are enough to decouple the electromagnetic and twist potentials and to integrate the system in three steps. The general case remains unsolved and is highly linked to the broken affine Kac-Moody algebra arising after another dimensional reduction [91]. The action of G_{res} on the fields generates a transformation on the parameter space, and in particular, a scale transformation requires a rescaling also of Λ. Unfortunately, these surviving symmetries alone are useless to produce new interesting solutions.

3.4 Supersymmetry equation and BPS first order flow

In presence of many fields, as we have shown at the end of the subsection 3.2, the resolution of the Hamilton-Jacobi equation is an insuperable technical obstacle. On the other hand in supersymmetric field theory, the infinitesimal action of supersymmetry on the fermionic fields typically can be used to find a supersymmetric first order flow. This system selects a subclass of the general solution that preserves some of the supersymmetry of the theory and it is called BPS system [95,96]. The same set of equations can be found using the HJ technique, without solving directly the partial differential equation, but through an ansatz on the principal function [97]. This implies that we lose the dependence from the arbitrary integration constants and therefore the algebraic resolution of the dynamics, however, a first order flow is achieved. In this subsection, we will show this way of proceeding in matter coupled gauged supergravity for a black hole in $d=4$ and for a black string in $d=5$.

BPS first order flow for d=4 black hole

The Hamilton-Jacobi equation for the action (2.69) reads

$$e^{-2\psi}\Big((\partial_U W)^2-(\partial_\psi W)^2+4g^{i\bar{\jmath}}\partial_i W\partial_{\bar{\jmath}}W+h^{uv}\partial_u W\partial_v W+4e^{4(\psi-U)}(\partial_{\mathcal{Q}}W)^T\mathcal{H}\partial_{\mathcal{Q}}W\Big)$$
$$-e^{2(\psi-U)}V_g-e^{2(U-\psi)}V_{\mathrm{BH}}+\kappa=0\,. \tag{3.124}$$

Inspired by [97] one can introduce the function

$$W=e^U|\mathcal{Z}+i\kappa e^{2\psi-2U}\mathcal{L}|=e^U\mathrm{Re}(e^{-i\alpha}\mathcal{Z})-\kappa e^{2\psi-U}\mathrm{Im}(e^{-i\alpha}\mathcal{L})\,, \tag{3.125}$$

where the phase α is

$$e^{2i\alpha}=\frac{\mathcal{Z}+i\kappa e^{2(\psi-U)}\mathcal{L}}{\bar{\mathcal{Z}}-i\kappa e^{2(\psi-U)}\bar{\mathcal{L}}}\,,\qquad \text{or}\qquad \mathrm{Im}(e^{-i\alpha}\mathcal{Z})=-\kappa e^{2(\psi-U)}\mathrm{Re}(e^{-i\alpha}\mathcal{L})\,, \tag{3.126}$$

where $\mathcal{Z}=\langle\mathcal{Q},\mathcal{V}\rangle$ is the central charge of the supersymmetry algebra and $\mathcal{L}=\mathcal{Q}^x\langle\mathcal{P}^x,\mathcal{V}\rangle$, with $\mathcal{Q}^x=\langle\mathcal{P}^x,\mathcal{Q}\rangle$.

A straightforward calculation, allowing for the rules of special Kähler and quaternionic geometry, imposing the quantization condition

$$\mathcal{Q}^x\mathcal{Q}^x = 1, \tag{3.127}$$

the HJ equation (3.124) is satisfied. Through (3.165) and discarding total derivatives, the action (2.69) can be cast into the form

$$\begin{aligned}S = \int \mathrm{d}r \Big[& e^{2\psi}\big(U' + e^{-2\psi}\partial_U W\big)^2 - e^{2\psi}\big(\psi' - e^{-2\psi}\partial_\psi W\big)^2 + \\ & e^{2\psi} g_{i\bar{\jmath}}\big(z'^{\,i} + 2e^{-2\psi} g^{i\bar{k}}\partial_{\bar{k}} W\big)\big(\bar{z}'^{\,\bar{\jmath}} + 2e^{-2\psi} g^{\bar{\jmath}l}\partial_l W\big) + \\ & e^{2\psi} h_{uv}\big(q'^{\,u} + e^{-2\psi} h^{us}\partial_s W\big)\big(q'^{\,v} + e^{-2\psi} h^{vt}\partial_t W\big) + \\ & \frac{1}{4} e^{4U-2\psi}\big(\mathcal{Q}' + 4e^{2\psi-4U}\mathcal{H}\partial_{\mathcal{Q}} W\big)^T \mathcal{H}^{-1}\big(\mathcal{Q}' + 4e^{2\psi-4U}\mathcal{H}\partial_{\mathcal{Q}} W\big)\Big] .\end{aligned} \tag{3.128}$$

All first-order equations following from (3.128) except the one for z^i are symplectically covariant. Computing explicitely $\partial_{\bar{k}} W$, the latter reads

$$z'^{\,i} = -e^{i\alpha} g^{i\bar{\jmath}}\left(e^{U-2\psi} D_{\bar{\jmath}}\bar{\mathcal{Z}} - i\kappa e^{-U} D_{\bar{\jmath}}\bar{\mathcal{L}}\right) . \tag{3.129}$$

Contracting this with $D_i\mathcal{V}$ and using (2.8), one obtains a symplectically covariant equation for the section $\mathcal{V}$,

$$\begin{aligned}\mathcal{V}' + iA_r\mathcal{V} \;&=\; e^{i\alpha} e^{U-2\psi}\left(-\frac{1}{2}\Omega\mathcal{M}\mathcal{Q} - \frac{i}{2}\mathcal{Q} + \bar{\mathcal{V}}\mathcal{Z}\right) \\ &\quad -i\kappa e^{i\alpha} e^{-U}\left(-\frac{1}{2}\Omega\mathcal{M}\mathcal{P}^x\mathcal{Q}^x - \frac{i}{2}\mathcal{P}^x\mathcal{Q}^x + \bar{\mathcal{V}}\mathcal{L}\right) ,\end{aligned} \tag{3.130}$$

where $A_r = \mathrm{Im}(z'^{\,i}\partial_i\mathcal{K})$ is the U(1) Kähler connection. Calculating the remaining derivatives of W, the first-order flow equations become

$$\begin{aligned}& U' = -e^{U-2\psi}\mathrm{Re}\tilde{\mathcal{Z}} - \kappa e^{-U}\mathrm{Im}\tilde{\mathcal{L}} , \\ & \psi' = -2\kappa e^{-U}\mathrm{Im}\tilde{\mathcal{L}} , \\ & q'^{\,u} = \kappa e^{-U} h^{uv}\mathrm{Im}(e^{-i\alpha}\partial_v\mathcal{L}) , \\ & \mathcal{Q}' = -4e^{2\psi-3U}\mathcal{H}\Omega\mathrm{Re}\tilde{\mathcal{V}} , \\ & \mathcal{V}' = e^{i\alpha} e^{U-2\psi}\left(-\frac{1}{2}\Omega\mathcal{M}\mathcal{Q} - \frac{i}{2}\mathcal{Q} + \bar{\mathcal{V}}\mathcal{Z}\right) \\ & - i\kappa e^{i\alpha} e^{-U}\left(-\frac{1}{2}\Omega\mathcal{M}\mathcal{P}^x\mathcal{Q}^x - \frac{i}{2}\mathcal{P}^x\mathcal{Q}^x + \bar{\mathcal{V}}\mathcal{L}\right) - iA_r\mathcal{V} .\end{aligned} \tag{3.131}$$

These equations have a more useful form if one consider the phase α as a dynamical variable. Introducing the quantity $\mathcal{S} = \mathcal{Z} + i\kappa e^{2(\psi-U)}\mathcal{L}$, the relations (3.126) and (3.125) can be rewritten as

$$e^{2i\alpha} = \frac{\mathcal{S}}{\bar{\mathcal{S}}} , \qquad \mathrm{Im}(e^{-i\alpha}\mathcal{S}) = 0 , \qquad W = e^U\mathrm{Re}(e^{-i\alpha}\mathcal{S}) , \qquad W^2 = e^{2U}\mathcal{S}\bar{\mathcal{S}} . \tag{3.132}$$

One has thus

$$\alpha' = \frac{\mathrm{Im}(e^{-i\alpha}\mathcal{S}')}{e^{-U}W}\,, \qquad \mathcal{S}' = U'\partial_U\mathcal{S} + \psi'\partial_\psi\mathcal{S} + \mathcal{V}'\partial_\mathcal{V}\mathcal{S} + q'^{\,u}\partial_u\mathcal{S} + \mathcal{Q}'^{\,T}\partial_\mathcal{Q}\mathcal{S}\,. \tag{3.133}$$

Inserting (3.131) and the derivatives of $\mathcal{S}$ in this last expression, one gets

$$\alpha' + A_r = 2\kappa e^{-U}\mathrm{Re}(e^{-i\alpha}\mathcal{L})\,. \tag{3.134}$$

Finally, plugging the equation for U into the expression of $\mathrm{Im}\tilde{\mathcal{V}}'$, one can write the first-order flow equations in the form

$$\begin{aligned}
&2e^{2\psi}\left(e^{-U}\mathrm{Im}(e^{-i\alpha}\mathcal{V})\right)' - \kappa e^{2(\psi-U)}\Omega\mathcal{M}\mathcal{Q}^x\mathcal{P}^x + 4e^{2\psi-U}(\alpha'+A_r)\mathrm{Re}(e^{-i\alpha}\mathcal{V}) + \mathcal{Q} = 0\,,\\
&\psi' = -2\kappa e^{-U}\mathrm{Im}(e^{-i\alpha}\mathcal{L})\,,\\
&\alpha' + A_r = 2\kappa e^{-U}\mathrm{Re}(e^{-i\alpha}\mathcal{L})\,,\\
&q'^{\,u} = \kappa e^{-U}h^{uv}\mathrm{Im}(e^{-i\alpha}\partial_v\mathcal{L})\,,\\
&\mathcal{Q}' = -4e^{2\psi-3U}\mathcal{H}\Omega\mathrm{Re}\tilde{\mathcal{V}}\,,
\end{aligned} \tag{3.135}$$

where also (2.68) and (3.127) must hold together with

$$2e^U\mathcal{H}\Omega\mathrm{Re}\tilde{\mathcal{V}} = \mathcal{H}\Omega\mathcal{A}_t\,, \tag{3.136}$$

since the last equ. of (3.135) has to coincide with (2.67).

With the Hamilton-Jacobi formalism we obtain a system of first order flow that is symplectic covariant that it was not known using Killing spinor equations. Moreover one can show that the scalar potential (2.21), can be expressed in terms of the superpotential $\mathcal{L}$ as

$$V_g = \mathbb{G}^{AB}\mathbb{D}_A\mathcal{L}\,\mathbb{D}_B\bar{\mathcal{L}} - 3|\mathcal{L}|^2\,, \tag{3.137}$$

where

$$\mathbb{G}^{AB} = \begin{pmatrix} g^{i\bar{\jmath}} & 0 \\ 0 & h^{uv}\end{pmatrix}\,, \qquad \mathbb{D}_A = \begin{pmatrix} D_i \\ \mathsf{D}_u \end{pmatrix}\,, \tag{3.138}$$

provided that the quantization condiction (3.127) holds.

BPS first order flow for d=5 black string

The Hamilton-Jacobi equation for the effective action (2.79) reads

$$e^{-2\psi}\left((\partial_U W)^2 - (\partial_\psi W)^2 + \frac{4}{3}(\partial_T W)^2 + 2\mathcal{G}^{ij}\partial_i W\partial_j W + h^{uv}\partial_u W\partial_v W\right) + V_{\text{eff}} = 0\,. \tag{3.139}$$

Guided by the previous four dimensional case, we use the ansatz for the principal HJ function

$$W = ce^{U+\frac{T}{2}}\mathcal{Z} + de^{2\psi-U-\frac{T}{2}}\mathcal{L}\,, \tag{3.140}$$

where

$$\mathcal{Z} = p^I h_I\,, \qquad \mathcal{L} = \mathcal{Q}^x \mathcal{W}^x\,, \qquad \mathcal{Q}^x = p^I P^x_I\,, \qquad \mathcal{W}^x = h^J P^x_J\,. \tag{3.141}$$

Throughout some relations of very special geometry (2.34) as well as (2.37), (2.39) and (2.41), one can show that (3.140) solves indeed (3.139) provided that

$$c = -\frac{3}{4}\,, \qquad d = -\frac{9}{2}\kappa \mathrm{g}^2\,, \qquad \mathcal{Q}^x\mathcal{Q}^x = \frac{1}{9\mathrm{g}^2}\,. \tag{3.142}$$

The solution (3.140) leads then to the first-order flow equations

$$\begin{aligned} U' &= -\frac{3}{4}e^{U+\frac{T}{2}-2\psi}\mathcal{Z} + \frac{9}{2}\kappa\mathrm{g}^2 e^{-U-\frac{T}{2}}\mathcal{L}\,, \qquad T' = \frac{2}{3}U'\,,\\ \psi' &= 9\kappa\mathrm{g}^2 e^{-U-\frac{T}{2}}\mathcal{L}\,,\\ \phi'^{\,i} &= \mathcal{G}^{ij}\left(-\frac{3}{2}e^{U+\frac{T}{2}-2\psi}\partial_j\mathcal{Z} - 9\kappa\mathrm{g}^2 e^{-U-\frac{T}{2}}\partial_j\mathcal{L}\right)\,,\\ q'^{\,u} &= -\frac{9}{2}\kappa\mathrm{g}^2 e^{-U-\frac{T}{2}}h^{uv}\partial_v\mathcal{L}\,. \end{aligned} \tag{3.143}$$

One can recast (3.143) into a form very similar to that of the first-order flow in four dimensions (3.135). Integrating $T' = \frac{2}{3}U'$ and plugging this into the remaining equations of (3.143), one gets

$$\begin{aligned} T' &= -\frac{1}{2}e^{2T-2\psi}\mathcal{Z} + 3\kappa\mathrm{g}^2 e^{-2T}\mathcal{L}\,,\\ \psi' &= 9\kappa\mathrm{g}^2 e^{-2T}\mathcal{L}\,,\\ \phi'^{\,i} &= \mathcal{G}^{ij}\left(-\frac{3}{2}e^{2T-2\psi}\partial_j\mathcal{Z} - 9\kappa\mathrm{g}^2 e^{-2T}\partial_j\mathcal{L}\right)\,,\\ q'^{\,u} &= -\frac{9}{2}\kappa\mathrm{g}^2 e^{-2T}h^{uv}\partial_v\mathcal{L}\,. \end{aligned} \tag{3.144}$$

Using the equation for $\phi'^{\,i}$ together with $(h^I)' = \phi'^{\,i}\partial_i h^I$ and (2.34), the equations for T and ϕ^i can be rewritten as

$$e^{2\psi}\left(e^{-2T}h^I\right)' + 9\mathrm{g}^2\kappa e^{2\psi-4T}\mathcal{Q}^x P^x_J G^{IJ} - p^I = 0\,. \tag{3.145}$$

Note that the FI case can be recovered imposing $P^1_I = P^2_I = 0$ and $P^3_I = V_I$. Then the charge quantization condition $\mathcal{Q}^x\mathcal{Q}^x = 1/(9\mathrm{g}^2)$ boils down to $\mathcal{Q}^3 = p^I V_I = \pm\kappa/(3\mathrm{g})$ (use $\kappa^2 = 1$), while $\mathcal{L}$ in (3.141) becomes $\mathcal{L} = \pm\frac{\kappa}{3\mathrm{g}}h^J V_J$. The two signs correspond to the two equivalent BPS branches.

3.5 Example of solution: generalizing Maldacena-Nunez

The of BPS equations (3.144) can be simplified to a system of the number of equations equal to the number of the scalar fields in the model. The idea is substantially the same of [96]. Introducing a new radial coordinate and rescaled scalars

$$\mathrm{d}R = e^{-\psi}\mathrm{d}u\,, \quad y^I = e^{\psi-2T}h^I\,, \tag{3.146}$$

the system boils down to

$$\begin{aligned} &\psi = \int 9\kappa \mathrm{g}^2 \mathcal{L}_y \mathrm{d}R\,, \quad e^{3\psi-6T} = \frac{1}{6} C_{IJK} y^I y^J y^K\,, \\ &(y^I)' - 9\mathrm{g}^2\kappa(\mathcal{L}_y y^I - \mathcal{Q}^x P^x_J G^{IJ}_y) - p^I = 0\,, \\ &q'^{\,u} = -\frac{9}{2}\kappa \mathrm{g}^2 h^{uv}\partial_v \mathcal{L}_y\,, \end{aligned} \tag{3.147}$$

where are been defined

$$\mathcal{L}_y = \mathcal{Q}^x P^x_I y^I\,, \quad G^{IJ}_y = -C^{IJK} C_{KLM} y^L y^M + 2y^I y^J\,. \tag{3.148}$$

Even if the complete integration of these equations in a particular model remains an hard task, this partial integration can be considered as a step towards the resolution. Numerical and asymptotic analysis of this type of models can be found in [98], where a particular truncation of $\mathcal{N} = 8$, $d = 5$ gauged supergravity is studied.

However, truncating the hypermultiplet sector a set of a simple ODEs is found. For example, for the STU model, described by the symmetric tensor with only nontrivial component $C_{123} = 1$, up to permutations, if we pose

$$\begin{aligned} \mathcal{Q}^x = &\frac{\kappa}{3\mathrm{g}}\,, \quad P^x_I = \frac{g}{3\mathrm{g}}\,, \quad p^I = \kappa q^I\,, \\ &V = T\,, \quad W = -2T - \psi\,, \end{aligned} \tag{3.149}$$

following the appendix of [96] the problem of finding $\frac{1}{4}$-BPS solution with running scalars for this model is reduced to solve a system of three first order differential equation. The configuration is defined by the metric

$$\mathrm{d}s^2 = e^{2V}(-\mathrm{d}t^2 + \mathrm{d}z^2) + e^{2W}(\mathrm{d}u^2 + \mathrm{d}\Omega^2_\kappa)\,, \tag{3.150}$$

and the fluxes are purely magnetic

$$F^I_{\theta\phi} = \kappa q^I F_\kappa(\theta)\,, \quad F_\kappa(\theta) = \begin{cases} \sin\theta\,, & \kappa = 1 \\ \sinh\theta\,, & \kappa = -1 \end{cases} \tag{3.151}$$

The warp factors are defined as

$$e^{2V} = (x^1x^2x^3)^{-\frac{1}{3}} e^{-g\int(x^1+x^2+x^3)\mathrm{d}u}\,, \quad e^{2W} = (x^1x^2x^3)^{\frac{2}{3}}\,. \tag{3.152}$$

and the scalar fields are $h^I = x^I/(x^1x^2x^3)^{\frac{1}{3}}$ and the linear combinations

$$y^1 = x^1 + x^2 - x^3\,, \quad y^2 = x^1 - x^2 - x^3\,, \quad y^3 = x^1 - x^2 + x^3\,, \tag{3.153}$$

satisfy a non-homogeneous version of the Nahm system

$$\begin{aligned} y^{1\prime} &= gy^2y^3 + Q^1\,, \\ y^{2\prime} &= gy^1y^3 + Q^2\,, \\ y^{3\prime} &= gy^1y^2 + Q^3\,. \end{aligned} \tag{3.154}$$

where the parameters Q^I are related to the physical fluxes with

$$
\begin{aligned}
Q^1 &= -\kappa(q^1 + q^2 - q^3)\,, \\
Q^2 &= -\kappa(q^1 - q^2 - q^3)\,, \\
Q^3 &= -\kappa(q^1 - q^2 + q^3)\,,
\end{aligned}
\tag{3.155}
$$

and the quantization condiction $g(q^1 + q^2 + q^3) = 1$ must hold. The functions $h^I = h^I(\phi^1, \phi^2)$ are

$$
h^1 = e^{-\frac{\phi^1}{\sqrt{6}} - \frac{\phi^2}{\sqrt{2}}}\,, \quad h^2 = e^{\frac{2\phi^1}{\sqrt{6}}} \quad h^3 = e^{-\frac{\phi^1}{\sqrt{6}} + \frac{\phi^2}{\sqrt{2}}}\,,
\tag{3.156}
$$

and therefore the constraint $h^1 h^2 h^3 = 1$ is satisfied and moreover, the expressions of the physical fields are

$$
\frac{2\phi_1}{\sqrt{6}} = \log\left(\frac{x^2}{(x^1 x^2 x^3)^{\frac{1}{3}}}\right)\,, \quad \sqrt{2}\phi_2 = \log\left(\frac{x^3}{x^1}\right)\,.
\tag{3.157}
$$

Imposing $Q^1 = Q^2 = 0$, that means $q^2 = 0$ and $q^1 = q^3$, one of the two configurations studied in [4]. Defining $y_+ = y^1 + y^2$ and $y_- = y^1 - y^2$, one finds that

$$
\begin{aligned}
&y_+ = k_+ e^{g\int y^3 \mathrm{d}u}\,, \quad y_- = k_- e^{g\int y^3 \mathrm{d}u}\,, \\
&y^{3\prime} = \frac{g}{4}\left(k_+^2 e^{2g\int y^3 \mathrm{d}u} - k_-^2 e^{-2g\int y^3 \mathrm{d}u}\right) + Q\,,
\end{aligned}
\tag{3.158}
$$

where $Q = Q^3 = -2\kappa q^1$. Now we can introduce a new coordinate $y = \int y^3 \mathrm{d}u$ so that the last equation becomes

$$
y'' = \frac{g}{4}\left(k_+^2 e^{2gy} - k_-^2 e^{-2gy}\right) + Q\,,
\tag{3.159}
$$

and can be integrated to

$$
\frac{1}{2}y'^2 = \frac{1}{8}\left(k_+^2 e^{2gy} + k_-^2 e^{-2gy}\right) + Qy + A\,,
\tag{3.160}
$$

therefore we obtain [10]

$$
y' = \frac{\mathrm{d}y}{\mathrm{d}u} = y^3 = \sqrt{\frac{1}{4}\left(k_+^2 e^{2gy} + k_-^2 e^{-2gy}\right) + 2Qy + 2A}\,.
\tag{3.161}
$$

The integration constant A can be set to zero by a shift of the y coordinate and the three functions x^I read

$$
\begin{aligned}
x^1 &= \frac{1}{4}\left(k_+ e^{gy} + k_- e^{-gy} + \sqrt{k_+^2 e^{2gy} + k_-^2 e^{-2gy} + 8Qy}\right)\,, \\
x^2 &= \frac{1}{2}k_- e^{-gy}\,, \\
x^3 &= \frac{1}{4}\left(-k_+ e^{gy} + k_- e^{-gy} + \sqrt{k_+^2 e^{2gy} + k_-^2 e^{-2gy} + 8Qy}\right)\,,
\end{aligned}
\tag{3.162}
$$

[10] The minus sign corresponds to an unphysical solution with negative defined scalrs h^I.

where the algebraic restriction $2gq^1 = 1$ must hold. Applying the inversion $y \to -y$ the configuration becomes

$$
\begin{aligned}
\mathrm{d}s^2 &= (x^1x^2x^3)^{-\frac{1}{3}} e^{g\int \frac{x^1+x^2+x^3}{y^3}\mathrm{d}y}(-\mathrm{d}t^2+\mathrm{d}z^2) + (x^1x^2x^3)^{\frac{2}{3}}\left(\frac{1}{(y^3)^2}\mathrm{d}y^2 + \mathrm{d}\Omega_\kappa^2\right) ,\\
x^1 &= \frac{1}{4}\left(k_+e^{-gy} + k_-e^{gy} + \sqrt{k_+^2e^{-2gy} + k_-^2e^{2gy} + \frac{8\kappa y}{g}}\right) ,\\
x^2 &= \frac{1}{2}k_-e^{gy} ,\\
x^3 &= \frac{1}{4}\left(-k_+e^{-gy} + k_-e^{gy} + \sqrt{k_+^2e^{-2gy} + k_-^2e^{2gy} + \frac{8\kappa y}{g}}\right) .
\end{aligned}
\tag{3.163}
$$

and supposing the necessary regularity condition $k_- > 0$ one can show that the asymptotic behaviour $y \to \infty$ of the metric is AdS_5

$$
\mathrm{d}s^2 = \frac{2e^{2gy}}{k_-}(-\mathrm{d}t^2+\mathrm{d}z^2) + \mathrm{d}y^2 + \frac{k_-^2e^{2gy}}{4}\mathrm{d}\Omega_\kappa^2 . \tag{3.164}
$$

For arbitrary integration constants the metric blows up at a certain point and the solution does not have an horizon, but choosing properly the values of k_+ and k_- for $\kappa = -1$[11]

$$
k_- = e^{-ag}\sqrt{\frac{2}{g^2}(2ag+1)} , \quad k_+ = -e^{ag}\sqrt{\frac{2}{g^2}(2ag-1)} , \tag{3.165}
$$

the metric has an horizon for $y = a$ and the geometry is $AdS_3 \times H^2$ [12].

If we take $g > 0$, (3.165) are well defined for $a \geq \frac{1}{2g}$. We note that for these values of k_+ and k_-

$$
y^3(y) = \frac{1}{g}\sqrt{2ag\cosh(2g(a-y)) - \sinh(2g(a-y)) - 2gy} , \tag{3.166}
$$

and therefore

$$
\frac{\mathrm{d}(y^3)^2}{\mathrm{d}y} = \frac{2}{g}\cosh(2g(a-y)) - 4a\sinh(2g(a-y)) - \frac{2}{g} \geq \frac{2}{g}(e^{2g(y-a)}-1) \geq 0 . \tag{3.167}
$$

Joint to extremal value $y^3(a) = 0$, one has that y^3 is always well defined for $y \geq a$ and the scalar fields (3.163) are positive. The parameter a is linked to the scaling transformation

$$
g \to \frac{g}{a} , \quad y \to ay , \quad \kappa \to \frac{\kappa}{a^2} , \tag{3.168}
$$

but the value $a = \frac{1}{2g}$ is special because it corresponds to the limit in which $x^1 = x^3$ that means $\phi_2 = 0$. This truncation corresponds to the solution of [4] with two nonzero and equal fluxes, infact the equation

$$
e^{2W+\frac{\phi_1}{\sqrt{6}}} = e^{2W-\frac{2\phi_1}{\sqrt{6}}} + \frac{\sqrt{6}W + 2\phi_1}{2\sqrt{6}g^2} + \frac{1}{4} , \tag{3.169}
$$

[11] The case in which $k_- = e^{-ag}\sqrt{\frac{2}{g^2}(2ag+1)}$ and $k_+ = e^{ag}\sqrt{\frac{2}{g^2}(2ag-1)}$ is linked to the $\mathbb{Z}_2$ symmetry $x^1 \to x^3$ and corresponds to the case of negative h^I.

[12] For $\kappa = 1, 0$ this cannot be done.

holds and it becomes exactly eq. (17) of pg. 13 of [4] for $g = 1$. The metric in the limit $y \to a$ can be studied from the asymptotic behaviours of

$$\begin{aligned} \frac{x^1 + x^2 + x^3}{y^3} &= \sqrt{\frac{1+2ag}{2ag^3}}\frac{1}{y-a} + O((y-a)^0)\,, \\ \frac{(y^3)^3}{x^1x^2x^3} &= 32\sqrt{\frac{2a^3g^9}{1+2ag}}(y-a)^3 + O((y-a)^4)\,, \\ x^1x^2x^3 &= \sqrt{\frac{1+2ag}{32g^6}} + O((y-a)^1)\,. \end{aligned} \tag{3.170}$$

Introducing the new radial coordinate

$$(y-a)^{\sqrt{\frac{1+2ag}{2ag^3}}} = \frac{1}{u^2}\,, \tag{3.171}$$

for $y \to a$ (5.110) boils down to

$$\mathrm{d}s^2 = \frac{1}{u^2}(-\mathrm{d}t^2 + \mathrm{d}z^2) + \frac{1}{2^{\frac{2}{3}}(1+2ag)^{\frac{2}{3}}}\frac{\mathrm{d}u^2}{u^2} + \left(\frac{1+2ag}{32g^6}\right)^{\frac{1}{3}}\mathrm{d}\Omega^2_{-1}\,, \tag{3.172}$$

that is $AdS_3 \times H^2$.

The central charge of the two dimensional SCFT dual to the horizon cofiguration $AdS_3 \times H^2$ is (4.20)

$$c = \frac{3R_{\mathrm{AdS}_3}}{2G_3} = \frac{6\pi(\mathfrak{g}-1)R_{\mathrm{AdS}_3}R^2_{H^2}}{G_5}\,, \tag{3.173}$$

with $\mathfrak{g} = 2, 3, \ldots$, the genus of the Riemann surface H^2. In particular the values of the radii are

$$R_{AdS_3} = \frac{1}{2^{\frac{1}{3}}(1+2ag)^{\frac{1}{3}}}\,, \quad R^2_{H^2} = \frac{Q(1+2ag)^{\frac{1}{3}}}{2^{\frac{5}{3}}g}\,, \tag{3.174}$$

where the dependence from the scalars is expressed by

$$2ag = \sqrt{1 + \left(\frac{g^2k_+k_-}{2}\right)^2}\,. \tag{3.175}$$

For the truncation $\phi_2 = 0$, that means $k_+ = 0$, $R_{AdS_3} = 2^{-\frac{2}{3}}$ that is exactly the value found in [4]. However the central charge is independent from ϕ_2

$$c = \frac{6\pi(\mathfrak{g}-1)}{G_5}\frac{Q}{4g}\,, \tag{3.176}$$

Moreover near the conformal boundary the scalar fields (3.157) behave like

$$\frac{2\phi_1}{\sqrt{6}} \sim 2Qye^{-2gy}\,, \quad \sqrt{2}\phi_2 \sim -\frac{k_+}{k_-}e^{-2gy}\,, \tag{3.177}$$

therefore, in the dual SCFT, are red respectively as an insertion and an expectation value of an operator of scaling dimension $\Delta = 2$.

The relevant deformation of the dual superpotential relative to ϕ_1 is described in [4], while ϕ_2 is a marginal deformation of the two-dimensional $\mathcal{N} = (4,4)$ SYM and does not describe the gravity dual of two-dimensional $\mathcal{N} = (2,2)^*$ SYM theory [99]. The constant a represent the physical scale of the energy in the renormalization group flow at which the IR fixed point appears, but being a CFT, the details of the theory are independent of the energy scale.

CHAPTER 4

Attractor Points, Entropy and Central Charge

Particular solutions to supergravity theories are the attractor points configurations. These are solutions for which the metric assumes the simple form of a product space between a two dimensional Riemann surface and an AdS factor. They are fixed points for the scalar flow. Here we show their form and behaviuor for the solutions treated in this thesis, a static black string in $d = 5$ and a static black hole in $d = 4$. They are especially important for testing AdS/CFT correspondence. We will briefly comment about the index calculation and the value of the central charge in the dual CFT. Moreover, we elucidate on the role of a nonlinear symmetry of four-dimensional static black hole Freudenthal duality to generate new attractor points with the same value of the entropy.
The new results of this chapter are the formula for the central charge in AdS_3 in presence of a generic coupling to hypermultiplets [2] and the exstension of Freudenthal duality in gauged supergravity [8].

4.1 $AdS_3 \times \Sigma_2$ attractor points

In this section, we want to investigate the near-horizon configurations of a static black string 3.4. To keep things simple, we shall first concentrate on the hyperless FI-gauged case and set $\mathrm{g} = 1$. The geometry is of the type $\mathrm{AdS}_3 \times \Sigma_2$ with $\Sigma_2 = \{\mathrm{S}^2, \mathrm{H}^2\}$, and we assume that the scalars stabilize regularly at the horizon, i.e., $\phi'^{\,i} = 0$. Note that a similar problem was solved in four dimensions in [100] for the case of symmetric special Kähler manifolds with cubic prepotential. Supersymmetric Bianchi attractors in $\mathcal{N} = 2$, $d = 5$ gauged supergravity coupled to vector- and hypermultiplets were analyzed recently in [101].

Starting from (2.76) and introducing the coordinates (t, R, z, θ, ϕ), where the new radial coordinate R and the warp factors f and ρ are such that

$$U = \frac{3}{2}f\,, \qquad \psi = 2f + \rho\,, \qquad T = f\,, \qquad \frac{\mathrm{d}R}{\mathrm{d}r} = e^{-3f}\,, \tag{4.1}$$

now the metric (2.76) takes the form of

$$\mathrm{d}s^2 = e^{2f}(-\mathrm{d}t^2 + \mathrm{d}R^2 + \mathrm{d}z^2) + e^{2\rho}\mathrm{d}\sigma_\kappa^2\,. \tag{4.2}$$

Taking the limit of constant hypermultiplets that means

$$P_I^3 = \frac{V_I}{\mathrm{g}}\,, \quad \mathcal{Q}^3 = \frac{\kappa}{3\mathrm{g}}\,, \tag{4.3}$$

the first-order flow equations (3.143) becomes

$$\begin{aligned} f' &= -e^f(h^I V_I + \frac{1}{2}e^{-2\rho}\mathcal{Z})\,, \\ \rho' &= -e^f(h^I V_I - e^{-2\rho}\mathcal{Z})\,, \\ \phi'^{\,i} &= 3\mathcal{G}^{ij}e^f(\partial_j h^I V_I - \frac{1}{2}e^{-2\rho}\partial_j\mathcal{Z})\,, \end{aligned} \tag{4.4}$$

where the primes now denote derivatives w.r.t. R. For a product space $\mathrm{AdS}_3 \times \Sigma_2$ we have

$$e^{2f} = \frac{R^2_{\mathrm{AdS}_3}}{R^2}\,, \qquad e^{2\rho} = R_{\mathrm{H}}^2\,. \tag{4.5}$$

Plugging this together with $\phi'^{\,i} = 0$ into (4.4), one obtains a system of algebraic equations whose solution fixes the near-horizon values of the scalars in terms of the charges and the FI parameters,

$$h^I V_I = \frac{2}{3R_{\mathrm{AdS}_3}}\,, \qquad \mathcal{Z} = R_{\mathrm{H}}^2 h^I V_I\,, \qquad \partial_i\mathcal{Z} = 2R_{\mathrm{H}}^2\partial_i h^I V_I\,. \tag{4.6}$$

For the ansatz (4.5), the FI-version of (3.145) (obtained by taking $\mathcal{Q}^x P_J^x = \mathcal{Q}^3 P_J^3 = -\kappa V_J/(3\mathrm{g})$) reduces to

$$e^{f+2\rho}(e^{-2f}h^I)' - 3e^{2\rho}G^{IJ}V_J - p^I = 0\,. \tag{4.7}$$

Using (4.5) and (4.6), this can be rewritten as

$$p^I + 3R_{\mathrm{H}}^2 G^{IJ}V_J = 3\mathcal{Z}h^I\,. \tag{4.8}$$

We want to solve the attractor equations (4.6) (or equivalently (4.8)) in order to express R_{AdS_3}, R_{H} and h^I in terms of p^I and V_I. To this end, contract the third relation of (2.34) with V_I to get

$$\mathcal{G}^{ij}\partial_i h^I V_I \partial_j h_J = -\frac{2}{3}V_J + \frac{2}{3}h^I V_I h_J\,. \tag{4.9}$$

With (4.6), this becomes

$$R_{\mathrm{H}}^2 V_J = -\frac{3}{4}\mathcal{G}^{ij}\partial_i\mathcal{Z}\partial_j h_J + \mathcal{Z}h_J\,. \tag{4.10}$$

Using $h_I = \frac{1}{6}C_{IJK}h^J h^K$ and (2.34), one obtains

$$R_{\mathrm{H}}^2 V_J = \frac{1}{6}C_{JKL}p^K h^L\,. \tag{4.11}$$

Let us introduce the charge-dependent matrix

$$C_{p\,IJ} \equiv C_{IJK} p^K \,. \tag{4.12}$$

Using the adjoint identity (2.35), one easily shows that $C_{p\,IJ}$ is invertible, with inverse

$$C_p^{IJ} = 3\frac{C^{IJK} C_{KMN} p^M p^N - p^I p^J}{C_p}\,, \tag{4.13}$$

where $C_p = C_{IJK} p^I p^J p^K$. (4.11) implies then

$$h^I = 6 R_{\rm H}^2 C_p^{IJ} V_J\,. \tag{4.14}$$

Plugging (4.14) into (5.108), one can derive a general expression for $R_{\rm H}$ in terms of the intersection numbers, the charges and the FI parameters,

$$R_{\rm H}^2 = (36 C_{IJK} C_p^{IM} C_p^{JN} C_p^{KP} V_M V_N V_P)^{-\frac{1}{3}}\,. \tag{4.15}$$

Using this in (4.14) gives the values of the scalars at the horizon,

$$h^I = \frac{6 C_p^{IJ} V_J}{(36 C_{KLM} C_p^{KN} C_p^{LP} C_p^{MR} V_N V_P V_R)^{\frac{1}{3}}}\,. \tag{4.16}$$

Contracting (4.14) with V_I and using the first equation of (4.6) as well as (4.15), we obtain an expression for the AdS$_3$ curvature radius $R_{{\rm AdS}_3}$,

$$R_{{\rm AdS}_3} = \frac{(36 C_{IJK} C_p^{IM} C_p^{JN} C_p^{KP} V_M V_N V_P)^{\frac{1}{3}}}{9 C_p^{RS} V_R V_S}\,. \tag{4.17}$$

Finally, one can plug (4.13) into (4.15), (4.16) and (4.17), and use (2.35) to write the solutions of (4.6) and (4.8) as

$$\begin{aligned} R_{\rm H}^2 &= (\mathscr{C}^{IJK}(p) V_I V_J V_K)^{-\frac{1}{3}}\,, \\ h^I &= \frac{6\kappa}{C_p} \frac{p^I + 3\kappa C^{IJK} C_{KLM} p^L p^M V_J}{(\mathscr{C}^{NPR}(p) V_N V_P V_R)^{\frac{1}{3}}}\,, \\ R_{{\rm AdS}_3} &= \frac{C_p}{27} \frac{(\mathscr{C}^{IJK}(p) V_I V_J V_K)^{\frac{1}{3}}}{C^{LMN} C_{NRS}\, p^R p^S V_L V_M - \frac{1}{9}}\,, \end{aligned} \tag{4.18}$$

where

$$\mathscr{C}^{IJK}(p) = -\frac{108}{C_p}\left[2 C^{IJK} - \frac{9}{C_p} p^{(I} C^{JK)M} C_{MNP} p^N p^P + \frac{9}{C_p} p^I p^J p^K\right]\,. \tag{4.19}$$

The central charge of the two-dimensional conformal field theory that describes the black strings in the infrared [4,50,66], is given by [102]

$$c = \frac{3 R_{{\rm AdS}_3}}{2 G_3}\,, \tag{4.20}$$

where G_3 denotes the effective Newton constant in three dimensions, related to G_5 by

$$\frac{1}{G_3} = \frac{R_{\rm H}^2 \text{vol}(\Sigma)}{G_5} \,. \tag{4.21}$$

In what follows, we assume Σ_2 to be compactified to a Riemann surface of genus $\mathfrak{g}$, with $\mathfrak{g} = 0, 2, 3, \ldots$. The unit Σ_2 has Gaussian curvature $K = \kappa$, and thus the Gauss-Bonnet theorem gives

$$\text{vol}(\Sigma) = \frac{4\pi(1-\mathfrak{g})}{\kappa} \,. \tag{4.22}$$

Using (4.21) and (4.22) in (4.20) yields for the central charge

$$c = \frac{6\pi(1-\mathfrak{g}) R_{\rm AdS_3} R_{\rm H}^2}{\kappa G_5} \,. \tag{4.23}$$

The curvature radii $R_{\rm AdS_3}$ and $R_{\rm H}$ can be expressed in terms of the constants C_{IJK}, the magnetic charges p^I and the FI parameters V_I by means of (4.18). This leads to

$$c = \frac{2\pi(1-\mathfrak{g}) C_p}{\kappa G_5 (9 C^{IJK} C_{KMN} p^M p^N V_I V_J - 1)} \,. \tag{4.24}$$

If the hyperscalars are running, one has to consider also the near-horizon limit of the last equation of (3.144). Assuming $q'^{\,u} = 0$ at the horizon and using (2.40), one easily derives the algebraic condition

$$k_I^u h^I = 0 \,. \tag{4.25}$$

As far as the remaining equations of (3.144) are concerned, one can follow the same steps as in this section, with the only difference that V_I has to be replaced everywhere by $-3\kappa \mathcal{Q}^x P_I^x$.

However, the result in presence of hypermultiplets is much more model dependent. A specific and interesting case is the theory describing the compactification of ten-dimensional IIB supergravity on the conifold $T^{1,1}$. This model is tackled in [103] finding perfect agreement between the gravity side and the dual CFT not only in the IR, the horizon configuration but also in UV, the asymptotics [104].

4.2 $AdS_2 \times \Sigma_2$ attractor points

In ungauged supergravity, the attractor mechanism [105–109] essentially states that, at the horizon of an extremal black hole, the scalar fields ϕ of the theory are always attracted to the same values $\phi_{\rm h}$ (fixed by the black hole charges $\mathcal{Q}$), independently of their values ϕ_∞ at infinity. When the so-called black hole potential has flat directions, it may happen that some moduli are not stabilized, i.e., their values at the horizon are not fixed in terms of the black hole charges. Yet, the Bekenstein-Hawking entropy turns out to be independent of these unstabilized moduli. Notice that this does not hold anymore for nonextremal black holes, for which the horizon is not necessarily an attractor point. The

$\phi_{\rm h}$ are critical points of the black hole potential $V_{\rm BH}(\mathcal{Q}, z^i)$, where in $\mathcal{N} = 2$, $d = 4$ supergravity the z^i denote only the scalars in the vector multiplets, since hypermultiplets can be consistently decoupled. The horizon values $z^i_{\rm h}(\mathcal{Q})$ are thus determined by the criticality conditions

$$\partial_i V_{\rm BH}(\mathcal{Q}, z^i)|_{z^i_{\rm h}(\mathcal{Q})} = 0\,, \tag{4.26}$$

and the Bekenstein-Hawking entropy is given by

$$S_{\rm BH} = \pi V_{\rm BH}(\mathcal{Q}, z^i)|_{z^i_{\rm h}(\mathcal{Q})}\,. \tag{4.27}$$

In gauged supergravity, the scalar fields generically have a potential V, which contributes to the $\phi_{\rm h}(\mathcal{Q})$ as well. Both for U(1) Fayet-Iliopoulos gauging [110] and for abelian gauged hypermultiplets [111], the black hole potential in (4.26) has to be replaced by the effective potential

$$V_{\rm eff} = \frac{\kappa - \sqrt{\kappa^2 - 4V_{\rm BH}V}}{2V}\,, \tag{4.28}$$

where $\kappa = 0, 1, -1$ corresponds to flat, spherical and hyperbolic horizons respectively. The limit for $V \to 0$ of $V_{\rm eff}$ only exists for $\kappa = 1$, in which case $V_{\rm eff} \to V_{\rm BH}$, so one recovers correctly the black hole potential that governs the attractor mechanism in ungauged supergravity. The fact that this limit does not exist for $\kappa = 0, -1$ is not surprising since flat or hyperbolic horizon geometries are incompatible with vanishing scalar potential. As before, the critical points of the effective potential determine the horizon values of the moduli,

$$\partial_i V_{\rm eff}(\mathcal{Q}, q^u, z^i)|_{z^i_{\rm h},\, q^u_{\rm h}} = 0\,, \qquad \partial_u V_{\rm eff}(\mathcal{Q}, q^u, z^i)|_{z^i_{\rm h},\, q^u_{\rm h}} = 0\,, \tag{4.29}$$

(q^u are the hyperscalars), and the entropy density reads

$$s_{\rm BH} \equiv \frac{S_{\rm BH}}{\mathrm{vol}(\Sigma_2)} = \frac{V_{\rm eff}(\mathcal{Q}, q^u, z^i)|_{z^i_{\rm h},\, q^u_{\rm h}}}{4}\,, \tag{4.30}$$

where Σ_2 denotes the unit E^2, S^2 or H^2.

A particular class of attractors is the BPS one that can be studied starting from the BPS equations (3.135). We make some assumptions on the behaviour of the fields in the near-horizon limit, where we require all the fields and their derivatives to be regular. To get the near-horizon geometry $\mathrm{AdS}_2 \times \Sigma_2$ with $\Sigma_2 = \{\mathrm{S}^2, \mathrm{H}^2\}$ [1], the warp factors must have the form

$$U = \log\left(\frac{r}{r_A}\right)\,, \qquad \psi = \log\left(\frac{r_S}{r_A} r\right)\,, \tag{4.31}$$

where r_A and r_S denote the curvature radii of AdS_2 and Σ_2 respectively. It is easy to show that $W = 0$ at the horizon $r = 0$; in fact the flow equations for U and ψ can be rewritten as

$$U' = -e^{-2(A+U)}(W - \partial_A W)\,, \qquad A' = e^{-2(A+U)} W\,, \tag{4.32}$$

[1]For semplicity we does not treat the case $\kappa = 0$.

where $A = \psi - U$ and $A \to \log(r_S)$ for $r \to 0$. $W = 0$ implies

$$\mathcal{Z} = -i\kappa r_S^2 \mathcal{L}\,. \tag{4.33}$$

Assuming $z'^{\,i} = 0$ and $q'^{\,u} = 0$ at the horizon, it follows that

$$D_i\mathcal{Z} = -i\kappa r_S^2 D_i\mathcal{L}\,, \qquad \mathrm{D}_u\mathcal{L} = 0\,, \tag{4.34}$$

and $\alpha' = 0$. From $\mathrm{D}_u\mathcal{L} = 0$ we get

$$\langle \mathcal{K}^v, \mathcal{V}\rangle = 0\,, \tag{4.35}$$

if we use also the algebraic relation (2.68) together with (2.37), (2.39) and (2.40). As in [111], we can choose the gauge $\mathcal{A}_t = 0$ at the horizon. Then, from (3.136) and the last equation of (3.135), one obtains $\mathcal{Q}' = 0$.

With these assumptions, the BPS flow equations (3.135) become

$$\begin{aligned} &4\mathrm{Im}(\bar{\mathcal{Z}}\mathcal{V}) - \kappa r_S^2 \Omega\mathcal{M}\mathcal{Q}^x\mathcal{P}^x + \mathcal{Q} = 0\,,\\ &\mathcal{Z} = -\frac{r_S^2}{2r_A}e^{i\alpha}\,,\\ &\langle \mathcal{K}^v, \mathcal{V}\rangle = 0\,, \end{aligned} \tag{4.36}$$

that must be supplemented by the constraints $\mathcal{Q}^x\mathcal{Q}^x = 1$ and $\mathcal{H}\Omega\mathcal{Q} = 0$. If one rotates to a frame with purely electric gauging, $\mathcal{Q}^x$ boils down to $p^\Lambda P^x_\Lambda$, and the magnetic charges p^Λ become constant. One can then use a local (on the quaternionic Kähler manifold) $\mathrm{SU}(2)$ transformation to set $\mathcal{Q}^1 = \mathcal{Q}^2 = 0$, and the equations (4.36) reduce to the ones obtained in [57].

The solutions of (4.36) are the horizon values of the scalars in terms of the charges and the gaugings. Furthermore, taking in consideration homogeneous models and solving the attractor equations for r_S^2, one can derive the Bekenstein-Hawking entropy written in [57] with the substitution $\mathcal{P}^3 \to -\kappa\mathcal{Q}^x\mathcal{P}^x$. The main difference w.r.t. the FI case consists in the dependence of $\mathcal{Q}^x\mathcal{P}^x$ on the hypers, whose horizon values are fixed by (4.35) and by the first of 2.68.
In the FI case, the value of the entropy of black holes in AdS_4 finds a perfect agreement with the topological twisted index [112] of ABJM [113] theories. These represent the first microstates counting of a black hole in this dimension [49, 114, 115]. It's again not clear as the duality works in the more general case as for the quivers describe in [116].

4.3 Freudenthal duality

In this section we shall briefly review the Freudenthal duality in ungauged supergravity [117–119], and subsequently generalize it to the abelian gauged case. It results to be a nonlinear symmetry of the BH entropy highly linked to the properties of special Kähler geometry of the vector multiplets coset.

Ungauged supergravity

Following [118], we introduce the scalar field dependent Freudenthal duality operator $\mathfrak{F}_z$ by

$$\mathfrak{F}_z(\mathcal{Q}) \equiv \hat{\mathcal{Q}} = -\Omega\mathcal{M}\mathcal{Q}\,, \qquad \mathfrak{F}_z(\mathcal{V}) \equiv \mathcal{V}\,, \tag{4.37}$$

where $\mathcal{Q}$ denotes the symplectic vector of charges, while the covariantly holomorphic symplectic section $\mathcal{V}$ and the matrices $\mathcal{M}$, Ω were defined in (2.3), (2.7) and (2.9) respectively. They satisfy the relations

$$\mathcal{M}^t = \mathcal{M}\,, \qquad \mathcal{M}\Omega\mathcal{M} = \Omega\,, \qquad \mathcal{M}\mathcal{V} = i\Omega\mathcal{V}\,, \qquad \mathcal{M}D_i\mathcal{V} = -i\Omega D_i\mathcal{V}\,, \tag{4.38}$$

with D_i the Kähler-covariant derivative. Moreover, the black hole potential can be written in terms of $\mathcal{Q}$ and $\mathcal{M}$ as

$$V_{\text{BH}} = -\frac{1}{2}\mathcal{Q}^t\mathcal{M}\mathcal{Q}\,. \tag{4.39}$$

As a consequence of (4.38), it follows that the action of $\mathfrak{F}_z$ on $\mathcal{Q}$ is anti-involutive, $\mathfrak{F}_z^2(\mathcal{Q}) = -\mathcal{Q}$. Using again (4.38), one shows that

$$\mathfrak{F}_z(V_{\text{BH}}(\mathcal{Q}, z^i)) = -\frac{1}{2}\hat{\mathcal{Q}}^t\mathcal{M}\hat{\mathcal{Q}} = V_{\text{BH}}(\mathcal{Q}, z^i)\,, \tag{4.40}$$

i.e., the black hole potential is invariant under Freudenthal duality. Moreover, the second equation of (4.38) yields

$$\partial_i\mathcal{M} = \mathcal{M}\Omega(\partial_i\mathcal{M})\Omega\mathcal{M}\,. \tag{4.41}$$

The direct application of this identity implies that under $\mathfrak{F}_z$, $\partial_i V_{\text{BH}}$ flips sign[2],

$$\mathfrak{F}_z(\partial_i V_{\text{BH}}(\mathcal{Q}, z^i)) = -\frac{1}{2}\hat{\mathcal{Q}}^t(\partial_i\mathcal{M})\hat{\mathcal{Q}} = -\partial_i V_{\text{BH}}(\mathcal{Q}, z^i)\,. \tag{4.42}$$

Since the $z^i_{\text{h}}(\mathcal{Q})$ are the critical points of V_{BH}, one has

$$0 = \partial_i V_{\text{BH}}|_{z^i_{\text{h}}(\mathcal{Q})} = -\mathfrak{F}_z(\partial_i V_{\text{BH}})|_{z^i_{\text{h}}(\mathcal{Q})} = \frac{1}{2}\hat{\mathcal{Q}}^t(\partial_i\mathcal{M})\hat{\mathcal{Q}}|_{z^i_{\text{h}}(\mathcal{Q})} = \frac{1}{2}\hat{\mathcal{Q}}^t_{\text{h}}\partial_i\mathcal{M}(z^i_{\text{h}}(\mathcal{Q}))\hat{\mathcal{Q}}_{\text{h}}\,, \tag{4.43}$$

where we introduced Freudenthal duality $\mathfrak{F}$ at the horizon as

$$\mathfrak{F}(\mathcal{Q}) = \mathfrak{F}_z(\mathcal{Q})|_{z^i_{\text{h}}(\mathcal{Q})} = -\Omega\mathcal{M}_{\text{h}}\mathcal{Q} = \hat{\mathcal{Q}}_{\text{h}}\,. \tag{4.44}$$

On the other hand, applying (4.26) to the charge configuration $\hat{\mathcal{Q}}_{\text{h}}$ leads to

$$0 = -\partial_i V_{\text{BH}}(\hat{\mathcal{Q}}_{\text{h}}, z^i)|_{z^i_{\text{h}}(\hat{\mathcal{Q}}_{\text{h}})} = \frac{1}{2}\hat{\mathcal{Q}}^t_{\text{h}}\partial_i\mathcal{M}(z^i_{\text{h}}(\hat{\mathcal{Q}}_{\text{h}}))\hat{\mathcal{Q}}_{\text{h}}\,. \tag{4.45}$$

Comparing (4.43) and (4.45), one can conclude that the attractor configuration

$$z^i_{\text{h}}(\hat{\mathcal{Q}}_{\text{h}}) = z^i_{\text{h}}(\mathcal{Q})\,, \tag{4.46}$$

[2]Since the operator $\mathfrak{F}_z$ does not commute with ∂_i, it is important to specify that $\mathfrak{F}_z$ acts always after the action of ∂_i. Notice that (4.42) corrects eq. (3.11) of [118].

is a solution also for (4.45) [118]. Eq. (4.46) can be interpreted as the stabilization of the near horizon configuration under Freudenthal duality, but an explicit verification of this claim is possible only if all the charges are different from zero. In any case one can always verify that $z^i_{\rm h}$ is critical point for both $V_{\rm BH}(\mathcal{Q}, z^i)$ and $V_{\rm BH}(\hat{\mathcal{Q}}_{\rm h}, z^i)$.

This fact turns out to be crucial in order to extend (4.37) to a symmetry of the black hole entropy $S_{\rm BH}$. In fact, using (4.27), (4.40) and (4.46), one obtains

$$\begin{aligned}\frac{1}{\pi}\mathfrak{F}(S_{\rm BH}) &= \mathfrak{F}\left(-\frac{1}{2}\mathcal{Q}^t\mathcal{M}(z^i_{\rm h}(\mathcal{Q}))\mathcal{Q}\right) = -\frac{1}{2}\hat{\mathcal{Q}}^t_{\rm h}\mathcal{M}(z^i_{\rm h}(\hat{\mathcal{Q}}_{\rm h}))\hat{\mathcal{Q}}_{\rm h} \\ &= -\frac{1}{2}\mathcal{Q}^t\mathcal{M}_{\rm h}\mathcal{Q} = \frac{S_{\rm BH}}{\pi}\,. \end{aligned} \tag{4.47}$$

Thus, the entropy pertaining to the charge configuration $\mathcal{Q}$ is the same as the one pertaining to the Freudenthal dual configuration $\mathfrak{F}(\mathcal{Q})$. Since $\mathfrak{F}(\mathcal{Q})$ in (4.44) is homogeneous of degree one (but generally nonlinear) in $\mathcal{Q}$, (4.47) results in the quite remarkable fact that the Bekenstein-Hawking entropy of a black hole in ungauged supergravity is invariant under an intrinsically nonlinear map acting on charge configurations. Note that no assumption has been made on the underlying special Kähler geometry, nor did we use supersymmetry.

$\mathrm{U}(1)$ FI-gauged $\mathcal{N}=2$, $d=4$ supergravity

In U(1) FI-gauged $\mathcal{N}=2$, $d=4$ supergravity, the parameters in terms of which the scalars z^i stabilize at the horizon, are doubled by the gauge couplings $\mathcal{G}$. The entropy density and the horizon values of the scalars are now determined by the effective potential (4.28), which contains both $V_{\rm BH}$ and the scalar potential V.

As a first step, we extend the action of the field-dependent Freudenthal duality $\mathfrak{F}_z$ by acting on both $\mathcal{Q}$ and $\mathcal{G}$ according to

$$\mathfrak{F}_z(\mathcal{Q}) = \hat{\mathcal{Q}} = -\Omega\mathcal{M}\mathcal{Q}\,, \qquad \mathfrak{F}_z(\mathcal{G}) = \hat{\mathcal{G}} = -\Omega\mathcal{M}\mathcal{G}\,, \tag{4.48}$$

while, by definition, $\mathfrak{F}_z$ leaves the symplectic section $\mathcal{V}$ (and its covariant derivatives) invariant. Now use (4.38), (4.41), and the fact that the scalar potential can be written as [73]

$$V = g^{i\bar{\jmath}}D_i\mathcal{L}\bar{D}_{\bar{\jmath}}\bar{\mathcal{L}} - 3|\mathcal{L}|^2 = -\frac{1}{2}\mathcal{G}^t\mathcal{M}\mathcal{G} - 4|\mathcal{L}|^2\,, \tag{4.49}$$

where

$$\mathcal{L} \equiv \mathcal{G}^t\Omega\mathcal{V} = \langle\mathcal{G},\mathcal{V}\rangle\,, \tag{4.50}$$

to obtain

$$\begin{aligned}\mathfrak{F}_z(V(\mathcal{G},z^i)) &= -\frac{1}{2}\hat{\mathcal{G}}^t\mathcal{M}\hat{\mathcal{G}} - 4\hat{\mathcal{L}}\hat{\bar{\mathcal{L}}} = V(\mathcal{G},z^i)\,, \\ \mathfrak{F}_z(\partial_i V(\mathcal{G},z^i)) &= -\frac{1}{2}\hat{\mathcal{G}}^t(\partial_i\mathcal{M})\hat{\mathcal{G}} - 4(D_i\hat{\mathcal{L}})\hat{\bar{\mathcal{L}}} = -\partial_i V(\mathcal{G},z^i)\,. \end{aligned} \tag{4.51}$$

Since V_{eff} and $\partial_i V_{\text{eff}}$ (cf. (2.26) of [110]) can be written as functions of V_{BH}, V, $\partial_i V_{\text{BH}}$ and $\partial_i V$, (4.51), together with (4.40) and (4.42) implies

$$\mathfrak{F}_z(V_{\text{eff}}(\mathcal{Q},\mathcal{G},z^i)) = V_{\text{eff}}(\mathcal{Q},\mathcal{G},z^i)\,, \qquad \mathfrak{F}_z(\partial_i V_{\text{eff}}(\mathcal{Q},\mathcal{G},z^i)) = -\partial_i V_{\text{eff}}(\mathcal{Q},\mathcal{G},z^i)\,. \quad (4.52)$$

Using the second relation of (4.52), one has then

$$\begin{aligned} 0 &= -\partial_i V_{\text{eff}}|_{z^i_{\text{h}}(\mathcal{Q},\mathcal{G})} = \mathfrak{F}_z(\partial_i V_{\text{eff}})|_{z^i_{\text{h}}(\mathcal{Q},\mathcal{G})} \\ &= \partial_i V_{\text{eff}}(\hat{\mathcal{Q}},\hat{\mathcal{G}},z^i)|_{z^i_{\text{h}}(\mathcal{Q},\mathcal{G})} = \partial_i V_{\text{eff}}(\hat{\mathcal{Q}}_{\text{h}},\hat{\mathcal{G}}_{\text{h}},z^i_{\text{h}}(\mathcal{Q},\mathcal{G}))\,. \end{aligned} \quad (4.53)$$

Let us define Freudenthal duality at the horizon by

$$\begin{aligned} \mathfrak{F}(\mathcal{Q}) &= \mathfrak{F}_z(\mathcal{Q})|_{z^i_{\text{h}}(\mathcal{Q},\mathcal{G})} = -\Omega\mathcal{M}_{\text{h}}\mathcal{Q} = \hat{\mathcal{Q}}_{\text{h}}\,, \\ \mathfrak{F}(\mathcal{G}) &= \mathfrak{F}_z(\mathcal{G})|_{z^i_{\text{h}}(\mathcal{Q},\mathcal{G})} = -\Omega\mathcal{M}_{\text{h}}\mathcal{G} = \hat{\mathcal{G}}_{\text{h}}\,. \end{aligned} \quad (4.54)$$

From the comparison of (4.53) with the definition

$$0 = \partial_i V_{\text{eff}}(\hat{\mathcal{Q}}_{\text{h}},\hat{\mathcal{G}}_{\text{h}},z^i)|_{z^i_{\text{h}}(\hat{\mathcal{Q}}_{\text{h}},\hat{\mathcal{G}}_{\text{h}})} = \partial_i V_{\text{eff}}(\hat{\mathcal{Q}}_{\text{h}},\hat{\mathcal{G}}_{\text{h}},z^i_{\text{h}}(\hat{\mathcal{Q}}_{\text{h}},\hat{\mathcal{G}}_{\text{h}}))\,, \quad (4.55)$$

it follows that

$$z^i_{\text{h}}(\hat{\mathcal{Q}}_{\text{h}},\hat{\mathcal{G}}_{\text{h}}) = z^i_{\text{h}}(\mathcal{Q},\mathcal{G}) \quad (4.56)$$

is a solution also for (4.55), thus it is a critical point for both V_{eff} and $\mathfrak{F}(V_{\text{eff}})$.

Eqns. (4.30), (4.52) and (4.56) imply that s_{BH} is invariant under Freudenthal duality,

$$\begin{aligned} 4\mathfrak{F}(s_{\text{BH}}) &= V_{\text{eff}}(\hat{\mathcal{Q}}_{\text{h}},\hat{\mathcal{G}}_{\text{h}},z^i_{\text{h}}(\hat{\mathcal{Q}}_{\text{h}},\hat{\mathcal{G}}_{\text{h}})) = V_{\text{eff}}(\hat{\mathcal{Q}}_{\text{h}},\hat{\mathcal{G}}_{\text{h}},z^i_{\text{h}}(\mathcal{Q},\mathcal{G})) \\ &= V_{\text{eff}}(\mathcal{Q},\mathcal{G},z^i_{\text{h}}(\mathcal{Q},\mathcal{G})) = 4s_{\text{BH}}\,. \end{aligned} \quad (4.57)$$

It is immediate to see that in the limit $\mathcal{G} \to 0$, one recovers the results of the ungauged case. Notice that the origin of Freudenthal duality is firmly rooted into the properties (4.38). The action of $\mathfrak{F}$ yields a new attractor-supporting configuration $(\hat{\mathcal{Q}}_{\text{h}},\hat{\mathcal{G}}_{\text{h}})$ that, in general, belongs to a physically different theory, specified by a different choice of gauge couplings.

It is worthwhile to note that no assumption has been made on the special Kähler geometry of the scalars in the vector multiplets. The invariance (4.57) holds thus also in models with non-homogeneous special Kähler manifolds, like e.g. the quantum STU model recently treated in [120].

As an illustrative example, let us check the action of Freudenthal duality for the simple model with prepotential $F = -iX^0X^1$ and purely electric FI gauging, cf. [78] for details[3]. To keep things simple, we assume that the electric charges vanish. One has thus

[3]As discussed in sec. 10 of [121], the Freudenthal duality of $\mathcal{N} = 2$, $D = 4$ supergravity minimally coupled to a certain number of vector multiplets in the ungauged case is nothing but a particular anti-involutive symplectic transformation of the U-duality.

$$\mathcal{Q} = \begin{pmatrix} p^0 \\ p^1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathcal{G} = \begin{pmatrix} 0 \\ 0 \\ g_0 \\ g_1 \end{pmatrix}. \tag{4.58}$$

This model has just one complex scalar $z = x + iy$, and the matrix $\mathcal{M}$ is given by

$$\mathcal{M} = \begin{pmatrix} -\frac{x^2+y^2}{x} & 0 & \frac{y}{x} & 0 \\ 0 & -\frac{1}{x} & 0 & -\frac{y}{x} \\ \frac{y}{x} & 0 & -\frac{1}{x} & 0 \\ 0 & -\frac{y}{x} & 0 & -\frac{x^2+y^2}{x} \end{pmatrix}. \tag{4.59}$$

The black hole and scalar potential read respectively

$$\begin{aligned} V_{\text{BH}} &= -\frac{1}{2}\mathcal{Q}^t\mathcal{M}\mathcal{Q} = \frac{x^2+y^2}{2x}(p^0)^2 + \frac{(p^1)^2}{2x}, \\ V &= -\frac{1}{2x}(g_0^2 + 4g_0g_1x + g_1^2(x^2+y^2)). \end{aligned} \tag{4.60}$$

Plugging this into the effective potential (4.28), one shows that the latter is extremized for

$$x = x_{\text{h}} = \frac{ug_0}{g_1}, \qquad y = y_{\text{h}} = 0, \tag{4.61}$$

where u is a solution of the quartic equation

$$\left[(1-\nu^2)u + 2(u^2-\nu^2)\right]^2 = k(1-u^2)(\nu^2-u^2), \tag{4.62}$$

with

$$\nu \equiv \frac{g_1p^1}{g_0p^0}, \qquad k \equiv \frac{\kappa^2}{(g_0p^0)^2}. \tag{4.63}$$

Note that positivity of the kinetic terms in the action requires $x > 0$. Depending on the sign of g_0/g_1, this means that either only negative or only positive roots of (4.62) are allowed, and such roots may not exist for all values of ν and k. Notice also that in the special case where

$$(2g_0p^0)^2 = (2g_1p^1)^2 = \kappa^2, \tag{4.64}$$

the effective potential (4.28) becomes completely flat,

$$V_{\text{eff}} = -\frac{\kappa}{2g_0g_1}, \tag{4.65}$$

and the scalar z is thus not stabilized at the horizon, a fact first noted in [78]. (Nonetheless, the entropy is still independent of the arbitrary value z_{h}, in agreement with the attractor mechanism). (4.64) corresponds to the BPS conditions found in [78], or to a sign-flipped modification of them[4]. It would be interesting to see whether the appearance of

[4]In the BPS case, g_0p^0 and g_1p^1 must have the same sign.

flat directions is a generic feature of the BPS case, or just a consequence of the simplicity of the model under consideration. A large class of supersymmetric black holes in gauged supergravity satisfies a Dirac-type quantization condition [78] (that corresponds to a twisting of the dual superconformal field theory [4]), i.e., one has a relation between $\mathcal{Q}$ and $\mathcal{G}$, that enter into V_{BH} and V respectively. This indicates that flat directions of V_{eff} might be generic in the supersymmetric case.

Using (4.62), one can derive the near-horizon value of V_{eff}, and thus the entropy density (4.30),

$$s_{\text{BH}} = \frac{V_{\text{eff}}(\mathcal{Q},\mathcal{G},z^i)|_{z^i_{\text{h}}(\mathcal{Q},\mathcal{G})}}{4} = \frac{g_0 {p^0}^2[(1-\nu^2)u + 2(u^2-\nu^2)]}{4\kappa g_1(1-u^2)} \,. \tag{4.66}$$

We now determine the action of Freudenthal duality on the charges and the FI parameters. The definitions (4.54) yield

$$\mathfrak{F}(\mathcal{Q}) \equiv \begin{pmatrix} 0 \\ 0 \\ \hat{q}_0 \\ \hat{q}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p^0 x_{\text{h}} \\ p^1/x_{\text{h}} \end{pmatrix}, \qquad \mathfrak{F}(\mathcal{G}) \equiv \begin{pmatrix} \hat{g}^0 \\ \hat{g}^1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -g_0/x_{\text{h}} \\ -g_1 x_{\text{h}} \\ 0 \\ 0 \end{pmatrix}. \tag{4.67}$$

The dual configuration is thus electrically charged and has purely magnetic gaugings. For the transformed potentials one gets

$$\begin{aligned} \mathfrak{F}(V_{\text{BH}}) &= -\frac{1}{2}\hat{\mathcal{Q}}^t_{\text{h}}\mathcal{M}\hat{\mathcal{Q}}_{\text{h}} = \frac{x^2+y^2}{2x}\hat{q}_1^2 + \frac{\hat{q}_0^2}{2x}\,, & (4.68) \\ \mathfrak{F}(V) &= -\frac{1}{2}\hat{\mathcal{G}}^t_{\text{h}}\mathcal{M}\hat{\mathcal{G}}_{\text{h}} - 4|\langle\hat{\mathcal{G}}_{\text{h}},\mathcal{V}\rangle|^2 = -\frac{1}{2x}\left((\hat{g}^1)^2 + 4\hat{g}^0\hat{g}^1 x + (\hat{g}^0)^2(x^2+y^2)\right)\,. \end{aligned}$$

These are identical to (4.60), except for the replacements

$$(p^0)^2 \to \hat{q}_1^2\,, \qquad (p^1)^2 \to \hat{q}_0^2\,, \qquad g_0^2 \to (\hat{g}^1)^2\,, \qquad g_1^2 \to (\hat{g}^0)^2\,, \qquad g_0 g_1 \to \hat{g}^0\hat{g}^1\,.$$

The critical points of $\mathfrak{F}(V_{\text{eff}})$ are thus $\hat{x}_{\text{h}} = \hat{g}^1\hat{u}/\hat{g}^0$ and $\hat{y}_{\text{h}} = 0$, where $\hat{u}$ satisfies

$$\left[(1-\hat{\nu}^2)\hat{u} + 2(\hat{u}^2-\hat{\nu}^2)\right]^2 = \hat{k}(1-\hat{u}^2)(\hat{\nu}^2-\hat{u}^2)\,, \tag{4.69}$$

with

$$\hat{\nu} \equiv \frac{\hat{g}^0\hat{q}_0}{\hat{g}^1\hat{q}_1}\,, \qquad \hat{k} \equiv \frac{\kappa^2}{(\hat{g}^1\hat{q}_1)^2}\,. \tag{4.70}$$

Now, using (4.67), one easily shows that

$$\hat{\nu}^2 = \frac{1}{\nu^2}\,, \qquad \hat{k} = \frac{k}{\nu^2}\,.$$

Plugging this into (4.69) and multiplying with $\nu^4/\hat{u}^4$ yields

$$\left[(1-\nu^2)\hat{u}^{-1} + 2(\hat{u}^{-2}-\nu^2)\right]^2 = k(1-\hat{u}^{-2})(\nu^2-\hat{u}^{-2})\,. \tag{4.71}$$

Comparing with (4.62), we see that u and $\hat{u}^{-1}$ satisfy the same equation, and have thus the same set of solutions. Hence, up to permutations of possible multiple roots, one gets $u = \hat{u}^{-1}$, which, by means of (4.67), leads to $\hat{x}_{\text{h}} = x_{\text{h}}$, and therefore V_{eff} and $\mathfrak{F}(V_{\text{eff}})$ share the same critical points.

The transformed entropy density is given by

$$\mathfrak{F}(s_{\text{BH}}) = \frac{V_{\text{eff}}(\mathfrak{F}(\mathcal{Q}), \mathfrak{F}(\mathcal{G}), z^i)|_{\hat{z}^i_{\text{h}}(\mathfrak{F}(\mathcal{Q}),\mathfrak{F}(\mathcal{G}))}}{4} = \frac{\hat{g}^1 \hat{q}_1^2[(1-\hat{\nu}^2)\hat{u} + 2(\hat{u}^2 - \hat{\nu}^2)]}{4\kappa \hat{g}^0 (1-\hat{u}^2)} \,. \tag{4.72}$$

Using again (4.67), it is easy to see that this coincides with (4.66), so that the entropy is indeed invariant under Freudenthal duality.

Coupling to hypermultiplets

In this section, we generalize our analysis to include also hypermultiplets, and consider the case where abelian isometries of the quaternionic hyperscalar target manifold are gauged. The dynamics of the attractor mechanism is now governed by the potentials $V_{\text{BH}}(\mathcal{Q}, z^i)$ and $V(\mathcal{P}^x(q^u), \mathcal{K}^u, z^i)$, where $\mathcal{P}^x = (\mathcal{P}^{x\Lambda}, \mathcal{P}^x_\Lambda)$ denote the triholomorphic moment maps, and $\mathcal{K}^u = (k^{\Lambda u}, k^u_\Lambda)$ are the Killing vectors that define the gauging. Note the presence of magnetic moment maps $\mathcal{P}^{x\Lambda}$ and magnetic Killing vectors $k^{\Lambda u}$. In what follows, we introduce the collective index $A = (i, u)$ and represent the scalars as

$$\phi^A = (z^i, q^u) \,. \tag{4.73}$$

As was shown (3.137), the scalar potential can be written in the symplectically covariant form provided the quantization condition (3.127) holds.

The field-dependent Freudenthal duality is again defined by (4.37), supplemented with

$$\mathfrak{F}_z(\mathcal{P}^x) \equiv \hat{\mathcal{P}}^x = -\Omega\mathcal{M}\mathcal{P}^x \,, \qquad \mathfrak{F}_z(\mathcal{K}^u) \equiv \hat{\mathcal{K}}^u = -\Omega\mathcal{M}\mathcal{K}^u \,. \tag{4.74}$$

One easily shows that $\mathfrak{F}_z(\mathcal{Q}^x) = \mathcal{Q}^x$ and, with slightly more effort, that

$$\begin{aligned} &\mathfrak{F}_z(V_{\text{eff}}(\mathcal{Q}, \mathcal{P}^x(q^u), \mathcal{K}^u(q^u), z^i)) = V_{\text{eff}}(\mathcal{Q}, \mathcal{P}^x(q^u), \mathcal{K}^u(q^u), z^i) \,, \\ &\mathfrak{F}_z(\partial_A V_{\text{eff}}(\mathcal{Q}, \mathcal{P}^x(q^u), \mathcal{K}^u(q^u), z^i)) = -\partial_A V_{\text{eff}}(\mathcal{Q}, \mathcal{P}^x(q^u), \mathcal{K}^u(q^u), z^i) \,. \end{aligned} \tag{4.75}$$

Thus, in analogy to the U(1) FI case, one has to consider the criticality conditions (4.29) and apply the second relation of (4.75),

$$\begin{aligned} 0 &= -\partial_A V_{\text{eff}}(\mathcal{Q}, \mathcal{P}^x, \mathcal{K}^u, z^i)|_{\phi^A_{\text{h}}} = \mathfrak{F}_z(\partial_A V_{\text{eff}}(\mathcal{Q}, \mathcal{P}^x, \mathcal{K}^u, z^i))|_{\phi^A_{\text{h}}} = \\ &= \partial_A V_{\text{eff}}(\hat{\mathcal{Q}}, \hat{\mathcal{P}}^x, \hat{\mathcal{K}}^u, z^i)|_{\phi^A_{\text{h}}} = \partial_A V_{\text{eff}}(\hat{\mathcal{Q}}_{\text{h}}, \hat{\mathcal{P}}^x_{\text{h}}(q^u_{\text{h}}), \hat{\mathcal{K}}^u(q^u_{\text{h}}), z^i_{\text{h}}) \,, \end{aligned} \tag{4.76}$$

where

$$\hat{\mathcal{P}}^x_{\text{h}}(q^u) = -\Omega\mathcal{M}_{\text{h}}\mathcal{P}^x(q^u) \tag{4.77}$$

is the dual expression for the moment maps that depends on the scalar fields, the charges and the parameters contained in the quaternionic Killing vectors. Defining $\hat{\mathcal{Q}}_{\text{h}}$ as in

(4.54), the criticality condition of the attractor points $\hat{\phi}^A_{\rm h}$ for the dual configuration of $(\mathcal{Q}, \mathcal{P}^x(q^u))$, namely for $(\hat{\mathcal{Q}}_{\rm h}, \hat{\mathcal{P}}^x_{\rm h}(q^u))$, reads

$$0 = \partial_A V_{\rm eff}(\hat{\mathcal{Q}}_{\rm h}, \hat{\mathcal{P}}^x_{\rm h}, \hat{\mathcal{K}}^u, z^i)|_{\hat{\phi}^A_{\rm h}} = \partial_A V_{\rm eff}(\hat{\mathcal{Q}}_{\rm h}, \hat{\mathcal{P}}^x_{\rm h}(\hat{q}^u_{\rm h}), \hat{\mathcal{K}}^u(\hat{q}^u_{\rm h}), \hat{z}^i_{\rm h})\,. \tag{4.78}$$

Thus a comparison between (4.76) and (4.78) shows that the configuration

$$\phi^A_{\rm h} = \hat{\phi}^A_{\rm h} \tag{4.79}$$

is a solution for both criticality conditions. It follows that

$$\begin{aligned} 4\mathfrak{F}(s_{\rm BH}) =& V_{\rm eff}(\hat{\mathcal{Q}}_{\rm h}, \hat{\mathcal{P}}^x_{\rm h}(\hat{q}^u_{\rm h}), \hat{z}^i_{\rm h}) = V_{\rm eff}(\hat{\mathcal{Q}}_{\rm h}, \hat{\mathcal{P}}^x_{\rm h}(q^u_{\rm h}), z^i_{\rm h}) \\ &= V_{\rm eff}(\mathcal{Q}, \mathcal{P}^x_{\rm h}(q^u_{\rm h}), z^i_{\rm h}) = 4 s_{\rm BH}\,, \end{aligned} \tag{4.80}$$

namely the entropy density of the two configurations related by the Freudenthal operator is the same.

CHAPTER 5

Symmetries as a solution generating technique

The structure of $d = 4, \mathcal{N} = 2$ supergravity exhibits a very large symmetry group and the electromagnetic duality is responsible for this fact [54]. This symmetry is generically spoilt when a potential appears. We show two methods to use the residual symmetry to generate more complicated solutions starting from a given seed. The first one is systematical and always applicable, the second resides in some particular form of the potential that defines the symmetry and extends the possibility to add axions to the original solution. Moreover mixing this technique with r-map construction 2.3 one can generate dyonic stationary black string in $d = 5$.

The new results of this chapter are a characterization of the symmetries useful to generate new solutions in $d = 4$, $\mathcal{N} = 2$ FI gauged supergravity [9] and its application to generate rotating black strings after a dimensional reduction [5].

5.1 Reparametrization and U-duality algebra

A symplectic reparametrization of the section $\mathcal{V}$ (2.3) for a prepotential $F = F(X)$ is a transformation

$$\mathcal{V} = (X^\Lambda, F_\Lambda)^t \longmapsto \tilde{\mathcal{V}} = (\tilde{X}^\Lambda, \tilde{F}_\Lambda)^t \,. \tag{5.1}$$

In the new frame a prepotential does not necessarily exist. We are interested in the subgroup of $\mathrm{Sp}(2n_\mathrm{V} + 2, \mathbb{R})$ that leaves the prepotential invariant [57,122,123],

$$F(\tilde{X}) = \tilde{F}(\tilde{X}) \,. \tag{5.2}$$

Its algebra is determined by the equation

$$X^\Lambda S_{\Lambda\Sigma} X^\Sigma - F_\Lambda R^{\Lambda\Sigma} F_\Sigma - 2X^\Lambda {Q^t}_\Lambda{}^\Sigma F_\Sigma = 0 \,, \tag{5.3}$$

where Q, R and S parametrize the symplectic algebra,

$$U = \begin{pmatrix} Q & R \\ S & -Q^t \end{pmatrix} , \qquad R = R^t \,, \qquad S = S^t \,.$$

A reparametrization of this type, in special projective coordinates, leaves $\mathcal{V}$ invariant up to a Kähler transformation. It is called the U-duality algebra and is an internal symmetry of the ungauged theory.

5.2 Symplectic embedding

The choice of the symplectic embedding of the non-linear sigma model isometry group is necessary to completely specify the special Kähler structure over a manifold [51, 122–125].

Symplectically equivalent embeddings

The way in which the isometry group is embedded in the symplectic group is fixed by supersymmetry [124]. In particular for a quadratic prepotential the fundamental representaion of $\mathrm{Sp}(2n_\mathrm{V}+2,\mathbb{R})$ has the branching rule to $\mathrm{SU}(1,n_\mathrm{V})$

$$\mathbf{2n}_\mathrm{V}+\mathbf{2}\to(\mathbf{n}_\mathrm{V}+\mathbf{1})\oplus(\mathbf{n}_\mathrm{V}+\mathbf{1}) \tag{5.4}$$

and for the STU model the fundamental representaion of $\mathrm{Sp}(8,\mathbb{R})$ has the branching rule to $\mathrm{SU}(1,1)\times\mathrm{SO}(2,2)$

$$\mathbf{8}\to\mathbf{2}\otimes(\mathbf{2}\oplus\mathbf{2})\,. \tag{5.5}$$

These embeddings are not unique since one can always act by conjugation with a symplectic matrix to construct a symplectically equivalent embedding. There are choices for the section $\mathcal{V}$ such that the isometry group sits in the symplectic group in a simple way, but the existence of a prepotential in that frame is in general not guaranteed. On the other hand, many symplectically equivalent embeddings are encoded by different prepotentials. Two physically interesting examples are [126, 127]

$$\mathcal{S}_1=\begin{pmatrix}1&1&0&0\\1&-1&0&0\\0&0&\frac{1}{2}&\frac{1}{2}\\0&0&\frac{1}{2}&-\frac{1}{2}\end{pmatrix},\qquad -iX^0X^1\longmapsto\frac{i}{4}({X^1}^2-{X^0}^2)\,, \tag{5.6}$$

$$\mathcal{S}_2=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0\\0&-1&0&0&0&0&0&0\\0&0&-1&0&0&0&0&0\\0&0&0&-1&0&0&0&0\end{pmatrix},\qquad -\frac{X^1X^2X^3}{X^0}\longmapsto-2i\sqrt{X^0X^1X^2X^3}\,. \tag{5.7}$$

A physically less important transformation, which is nevertheless useful for practical purposes, is for instance

$$\mathcal{S}_a = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}, \qquad \frac{i}{4} X^\Lambda \eta_{\Lambda\Sigma} X^\Sigma \longmapsto \frac{i}{4a^2} X^\Lambda \eta_{\Lambda\Sigma} X^\Sigma \,. \tag{5.8}$$

One can also construct inequivalent embeddings over the same manifold, the simplest example being SU$(1,1)$/U(1) [124].

Special Kähler structure over SU$(1, n_\mathrm{V})/($U$(1) \times$ SU$(n_\mathrm{V}))$

For this noncompact version of $\mathbb{C}\mathrm{P}^n$ a simple way to embed SU$(1, n_\mathrm{V})$ into Sp$(2n_\mathrm{V}+2, \mathbb{R})$ is obtained from the fact that

$$\mathrm{Sp}(2n_\mathrm{V}+2,\mathbb{R}) \cong \mathrm{Usp}(1+n_\mathrm{V}, 1+n_\mathrm{V}) = \mathrm{Sp}(2n_\mathrm{V}+2,\mathbb{C}) \cap \mathrm{U}(1+n_\mathrm{V}, 1+n_\mathrm{V})\,. \tag{5.9}$$

This isomorphism is provided by conjugation with the Cayley matrix,

$$C_\alpha : \mathrm{Sp}(2n_\mathrm{V}+2,\mathbb{R}) \longrightarrow \mathrm{Usp}(1+n_\mathrm{V}, 1+n_\mathrm{V})\,, \qquad U \longmapsto \hat{\mathcal{C}}_\alpha U \hat{\mathcal{C}}_\alpha^{-1}\,, \tag{5.10}$$

where

$$\hat{\mathcal{C}}_\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{\alpha}} I_{n_\mathrm{V}+1} & i\sqrt{\alpha}\eta \\ \frac{1}{\sqrt{\alpha}} I_{n_\mathrm{V}+1} & -i\sqrt{\alpha}\eta \end{pmatrix}, \tag{5.11}$$

and η is the Minkowski metric in $n_\mathrm{V}+1$ dimensions. In fact Usp$(1+n_\mathrm{V}, 1+n_\mathrm{V})$ is defined by the conditions

$$\mathcal{U}\mathbb{H}\mathcal{U}^\dagger = \mathbb{H}\,, \qquad \mathcal{U}\tilde{\Omega}\mathcal{U}^t = \tilde{\Omega}\,. \tag{5.12}$$

If the invariant bilinear forms are chosen as

$$\mathbb{H} = \begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix}, \qquad \tilde{\Omega} = \begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix}, \tag{5.13}$$

(5.12) becomes

$$\mathcal{U} = \begin{pmatrix} A & C^* \\ C & A^* \end{pmatrix}, \qquad A\eta A^\dagger - C^*\eta C^t = \eta\,, \qquad A^*\eta C^t - C\eta A^\dagger = 0\,. \tag{5.14}$$

The first of (5.4) is obtained by restricting the action of $\iota_\alpha \equiv C_\alpha^{-1}$ to the subgroup with $C = 0$. One can also explicitly verify that in this frame the prepotential exists and is given by $F = -\frac{i}{2\alpha} X^\Lambda \eta_{\Lambda\Sigma} X^\Sigma$.

Special Kähler structure over SU$(1,1)/$U$(1) \times$ SO$(2,2)/($SO$(2) \times$ SO$(2))$

This manifold belongs to the infinite sequence SU$(1,1)$/U$(1) \times$ SO$(2,n)$/(SO$(2) \times$ SO(n)), which for $n = 2$ is isomorphic to (SL$(2,\mathbb{R})$/SO$(2))^3$. To find the symplectic embedding it is useful to choose a frame [124, 128–130] in which the symplectic section cannot be integrated to have a prepotential. In this frame the Calabi-Visentini parametrization appears in a natural way. The embedding problem is solved by

$$\mathrm{SO}(2,2) \ni L \longmapsto \begin{pmatrix} L & 0 \\ 0 & L^{-1\,t} \end{pmatrix} \in \mathrm{Sp}(8,\mathbb{R}) \,, \tag{5.15}$$

$$\mathrm{SL}(2,\mathbb{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & b\hat{\eta} \\ c\hat{\eta} & d \end{pmatrix} \in \mathrm{Sp}(8,\mathbb{R}) \,, \tag{5.16}$$

where $\hat{\eta}$ is the metric preserved by SO$(2,2)$. A symplectic transformation that leads to a frame in which a prepotential exists is highly nontrivial to find [124].

5.3 Producing axions with the stabilizier of $\mathcal{G}$ under U-duality

The kinetic part of (2.15) corresponds to the action of the ungauged theory, whose on-shell global symmetry group is called U-duality, consisting of the isometries of the non-linear sigma model that act linearly also on the field strengths via the symplectic embedding [29]. For purely electric gaugings, the scalar potential generically spoils this invariance, but, as is clear from (2.16), for dyonic gauging one recovers the whole U-duality invariance, at the price of changing the vector of gauge couplings and so the physical theory [131]. We will call this group U_{fi}, that stands for fake internal symmetry group[1]. The action of U_{fi} on a solution is the mapping to other solutions of other theories, in the same way in which some elements of the symplectic group map solutions of theories with different prepotential into each other [73], cf. e.g. (5.6), (5.7).

Given U_{fi}, we fix a choice of the coupling constants $\mathcal{G}$ and, at least at the beginning, we suppose that they are generic. We want to underline that for abelian dyonic gaugings, the Maxwell equations remain homogeneous and so the action (2.15) doesn't have topological terms [55].

The true internal symmetry group U_{i} of the gauged supergravity theory is $S_{\mathcal{G}}$, the stabilizer of $\mathcal{G}$ under the action of U_{fi}. This is obvious from the definition of the stabilizer,

$$S_{\mathcal{G}} = \{ g \in U_{\text{fi}} \,|\, g\mathcal{G} = \mathcal{G} \} \,, \tag{5.17}$$

which means that we impose to stay in the same theory, and this restricts of course the group of internal symmetries.

By acting with $S \in S_{\mathcal{G}}$ on a given seed solution $(\mathcal{V}, \mathcal{G}, \mathcal{F}_{\mu\nu})$[2] of the equations of motion, we can generate another configuration via the map

$$(\mathcal{V}, \mathcal{G}, \mathcal{F}_{\mu\nu}) \mapsto (\tilde{\mathcal{V}}, \tilde{\mathcal{G}}, \tilde{\mathcal{F}}_{\mu\nu}) := (S\mathcal{V}, S\mathcal{G}, S\mathcal{F}_{\mu\nu}) = (S\mathcal{V}, \mathcal{G}, S\mathcal{F}_{\mu\nu}) \,. \tag{5.18}$$

[1]When the special Kähler manifold is symmetric we define the Lie algebra $\mathfrak{u}_{\text{fi}}$ of U_{fi} through the equations (5.3). The corresponding definition for nonsymmetric special Kähler manifolds requires more care.

[2]Actually we should write $(\mathcal{V}, \mathcal{G}, \mathcal{F}_{\mu\nu}, g_{\mu\nu})$, but since $S_{\mathcal{G}}$ does not act on the metric, we shall suppress the dependence on $g_{\mu\nu}$.

It is easy to verify that the equation of motion are invariant. The transformed fields solve the field equations by construction[3]. In general, the scalars transform nonlinearly under the corresponding isometry, the field strengths are rotated and the metric is functionally invariant. Technically, in order to determine $S_{\mathcal{G}}$, it is simpler to work with the corresponding algebra

$$\mathfrak{s}_{\mathcal{G}} = \{a \in \mathfrak{u}_{\text{fi}} \,|\, a\mathcal{G} = 0\} \,. \tag{5.19}$$

If one is interested only in the bosonic equations the group of the symmetries $U_{\text{i (bos)}}$ is enlarged to $S_{\mathcal{G}} \cup S_{\mathcal{M}}$. The group $S_{\mathcal{M}}$ is the stabilizer of $\mathcal{M}$ (2.7) under the action of $\text{Sp}(2n_v + 2)$[4]

$$S_{\mathcal{M}} = \{T \in \text{Sp}(2n_v + 2) \,|\, T^t \mathcal{M} T = \mathcal{M}\} \,, \tag{5.20}$$

By acting with $T \in S_{\mathcal{M}}$ on a given seed solution $(\mathcal{V}, \mathcal{G}, \mathcal{F}_{\mu\nu})$ of the equations of motion, we can generate another configuration via the map

$$(\mathcal{V}, \mathcal{G}, \mathcal{F}_{\mu\nu}) \mapsto (\mathcal{V}, \mathcal{G}, T\mathcal{F}_{\mu\nu}) \,. \tag{5.21}$$

In this case the supersymmetry of the solution is preserved only if T has also the property of stabilizing the FI parameters $\mathcal{G}$. However in some special cases symmetrical structures of the specific model can enlarge $U_{\text{i(bos)}}$ (see for example the discussion around (5.58)).

Stabilization and symmetries for some prepotentials

Now we want to apply these techniques to some specific prepotentials. Each of them exhibits different peculiar features related to the geometry of the underlying special Kähler manifold, namely to the symplectic embedding of the isometry group of the non-linear sigma model (cf. app. 5.2).

Prepotential $F = -iX^0X^1$ This prepotential encodes a particular special Kähler structure on the symmetric manifold $\text{SU}(1,1)/\text{U}(1)$. The symplectic section is

$$\mathcal{V} = (X^0, X^1, -iX^1, -iX^0)^t \,, \tag{5.22}$$

and we fix the couplings in a completely electric frame, $\mathcal{G} = (0, 0, g_0, g_1)^t$. The solution to (5.3) defines the algebra $\mathfrak{u}_{\text{fi}}$,

$$b_1 t_1 + b_2 t_2 + b_3 t_3 + b_4 t_4 = \begin{pmatrix} b_4 & 0 & b_1 & b_2 \\ 0 & -b_4 & b_2 & b_3 \\ -b_3 & -b_2 & -b_4 & 0 \\ -b_2 & -b_1 & 0 & b_4 \end{pmatrix} ,$$

[3]As it is clear from the formalism introduced in [73], the application of $S \in S_{\mathcal{G}}$ on a static solution of the BPS flow preserves the same amount of supersymmetry as the original configuration. In the rotating case, the same is true if one considers electric gaugings only [132].

[4]In the following we will consider only the continuos part of $S_{\mathcal{M}}$; the study of discret subgroups can be found in [97].

to be the U-duality su$(1,1)$ plus a u(1), generated by t_2, which acts trivially on the z^i, as we will see shortly. From the stability equation (5.19) one finds that $\mathfrak{s}_{\mathcal{G}}$ is generated by

$$s = t_2 - \frac{g_1}{g_0} t_1 - \frac{g_0}{g_1} t_3 \,, \tag{5.23}$$

so that $S_{\mathcal{G}} \subseteq \mathrm{U}(1,1)$ is the 1-parameter subgroup

$$S = e^{\beta s} = \begin{pmatrix} \cos^2\beta & \frac{g_1}{g_0}\sin^2\beta & -\frac{g_1}{g_0}\cos\beta\sin\beta & \cos\beta\sin\beta \\ \frac{g_0}{g_1}\sin^2\beta & \cos^2\beta & \cos\beta\sin\beta & -\frac{g_0}{g_1}\cos\beta\sin\beta \\ \frac{g_0}{g_1}\sin\beta\cos\beta & -\cos\beta\sin\beta & \cos^2\beta & \frac{g_0}{g_1}\sin^2\beta \\ -\cos\beta\sin\beta & \frac{g_1}{g_0}\cos\beta\sin\beta & \frac{g_1}{g_0}\sin^2\beta & \cos^2\beta \end{pmatrix} . \tag{5.24}$$

On the other hand, the $\mathrm{U}(1)$ generated by t_2 is given by

$$T_\alpha = e^{\alpha t_2} = \begin{pmatrix} \cos\alpha & 0 & 0 & \sin\alpha \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ -\sin\alpha & 0 & 0 & \cos\alpha \end{pmatrix} , \tag{5.25}$$

and it transforms the section $\mathcal{V}$ according to

$$T_\alpha \mathcal{V} = e^{-i\alpha}\mathcal{V} \,. \tag{5.26}$$

The projective special Kähler coordinates are thus insensible to its action. The matrix $\mathcal{M}$ defined in (2.7) transforms as

$$T_\alpha^t \mathcal{M} T_\alpha = \mathcal{M} \,. \tag{5.27}$$

One can thus act with T_α on $\mathcal{F}_{\mu\nu}$ only, leaving the equations of motion still invariant. T_α is an example for a 'field rotation matrix' that is commonly used to generate non-BPS solutions, a technique first introduced in [133, 134] and subsequently applied to gauged supergravity in [97, 135]. In conclusion, the on-shell symmetry group of this model is $U_{\text{i(bos)}} = \mathrm{U}(1) \times \mathrm{U}(1) \supset S_{\mathcal{G}}$, with the two $\mathrm{U}(1)$ factors identified respectively with S and T_α.

Prepotential $F = \frac{i}{4} X^\Lambda \eta_{\Lambda\Sigma} X^\Sigma$ The prepotential $F = \frac{i}{4} X^\Lambda \eta_{\Lambda\Sigma} X^\Sigma$, with $\eta_{\Lambda\Sigma} = \mathrm{diag}(-1, 1, ..., 1)$, describes a special Kähler structure on the symmetric manifolds $\mathrm{SU}(1, n_{\mathrm{V}})/(\mathrm{U}(1)\times\mathrm{SU}(n_{\mathrm{V}}))$. The symplectic section reads

$$\mathcal{V} = (X^\Lambda, \frac{i}{2}\eta_{\Lambda\Sigma}X^\Sigma)^t \,. \tag{5.28}$$

Due to the linearity of $\mathcal{V}$ in the coordinates X^Λ, one can easily construct the one-parameter subgroup

$$L_\alpha = \begin{pmatrix} \cos\alpha & 0 & 2\sin\alpha & 0 \\ 0 & I_{n_{\mathrm{V}}}\cos\alpha & 0 & -2I_{n_{\mathrm{V}}}\sin\alpha \\ -\frac{1}{2}\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & \frac{1}{2}I_{n_{\mathrm{V}}}\sin\alpha & 0 & I_{n_{\mathrm{V}}}\cos\alpha \end{pmatrix}$$

of Sp$(2n_{\mathrm{V}}+2,\mathbb{R})$, under which the section $\mathcal{V}$ transforms as

$$L_\alpha \mathcal{V} = e^{-i\alpha}\mathcal{V}\,. \tag{5.29}$$

Since

$$L_\alpha^t \mathcal{M} L_\alpha = \mathcal{M}\,, \tag{5.30}$$

we can add a new parameter to all the solutions of this model by acting with L_α on $\mathcal{F}_{\mu\nu}$ only.

The stability equation is slightly more involved. Notice that the case with only one vector multiplet is symplectically equivalent to $F = -iX^0X^1$, and thus the results for $n_{\mathrm{V}} = 1$ can be obtained from the previous subsection by an appropriate symplectic rotation (5.2). Let us discuss the general case of $n_{\mathrm{V}} = n$ vector multiplets. Eq. (5.3) defining the algebra $\mathfrak{u}_{\mathrm{fi}}$ is equivalent to

$$Q^t = -\eta Q \eta\,, \qquad S = -\frac{1}{4}\eta R \eta\,. \tag{5.31}$$

These equations define an embedding of U$(1,n)$ into Sp$(2n+2,\mathbb{R})$. To see this, let $z = A+iB \in \mathfrak{u}(1,n)$. Then, $z^t\eta + \eta z = 0$ implies

$$A^t = -\eta A \eta\,, \qquad B^t\eta = \eta B\,, \tag{5.32}$$

so ηB is symmetric. This suggests an embedding

$$\iota_\alpha\,:\,\mathfrak{u}(1,n) \longrightarrow \mathfrak{sp}(2n+2,\mathbb{R})\,, \qquad A+iB \longmapsto \begin{pmatrix} A & \alpha B\eta \\ -\frac{1}{\alpha}\eta B & -A^t \end{pmatrix}, \tag{5.33}$$

for any real $\alpha \neq 0$. This is indeed an injective Lie algebra morphism, and its image consists of the elements of $\mathfrak{sp}(2n+2,\mathbb{R})$ which solve (5.3) with $F_\Lambda = \frac{i}{\alpha}\eta_{\Lambda\Sigma}X^\Sigma$. In particular, (5.31) selects ι_2.

A basis for $\mathfrak{u}(1,n)$ is given by the matrices

$$\{A_a\}_{a=1}^{n(n+1)/2}\,, \qquad \{iB_k\}_{k=0}^{n(n+3)/2}\,, \tag{5.34}$$

where A_a are a basis for the space of $(n+1)\times(n+1)$ real matrices A such that ηA is antisymmetric, and B_k generate the space of $(n+1)\times(n+1)$ real matrices B such that ηB is symmetric, with $B_0 = I$, the identity matrix. The embedding extends obviously to the group level via the exponential map, and, in particular, notice that

$$\exp(\alpha\iota_2(iB_0)) = L_\alpha\,. \tag{5.35}$$

Let us now consider the symmetry group $S_{\mathcal{G}}$. If we set

$$\mathcal{G} = (\underline{0},\underline{g})^t = (0,\vec{0},g_0,\vec{g})^t\,, \tag{5.36}$$

with $\vec{g} = (g_1,\ldots,g_n)$, then we see that the invariance of $\mathcal{G}$ is defined by the equations

$$A^t\underline{g} = 0\,, \qquad B\eta\underline{g} = 0\,, \tag{5.37}$$

which define a maximal compact subgroup[5] $\mathrm{U}(n)$ of $\mathrm{U}(1,n)$. To see this, let us first put[6]

$$\hat{g} := \sqrt{-\underline{g}^2}\,, \tag{5.38}$$

and define $\Lambda_{\underline{g}} \in \mathrm{SO}(1,n)$ by

$$(g_0, \vec{g}) = (\hat{g}, \vec{0})\Lambda_{\underline{g}}\,. \tag{5.39}$$

Thus, A (or ηB^t) has $\underline{g}$ in the cokernel if and only if $\Lambda_{\underline{g}} A \Lambda_{\underline{g}}^{-1}$ (or $\Lambda_{\underline{g}} \eta B^t \Lambda_{\underline{g}}^{-1}$) has $(\hat{g}, \vec{0})$ in the cokernel. From this we immediately get that $\mathfrak{s}_{\mathcal{G}}$ is generated by the elements of $\mathfrak{u}(1,n)$ of the form

$$z_{\underline{g}} = \Lambda_{\underline{g}}^{-1} z \Lambda_{\underline{g}}\,, \tag{5.40}$$

where $z \in \mathfrak{u}(1,n)$ has vanishing first row and first column. Thus, $z_{\underline{g}} \in \mathrm{U}(n)$.
This provides also a way to realize an explicit construction of the group elements of $S_{\mathcal{G}}$. One can choose e.g. a generalized Gell-Mann basis [136] for $\mathfrak{su}(n)$, add the identity matrix I_n and then embed the basis into $\mathfrak{u}(1,n)$ by adding a first row and column of zeros. If we call $\{z_I\}_{I=0}^{n^2-1}$ such a basis for the compact subalgebra $\mathfrak{u}(n)$ of $\mathfrak{su}(1,n)$, then

$$\{\iota_2(z_I)\}_{I=0}^{n^2-1}$$

is a basis for $\mathfrak{s}_{\mathcal{G}_0}$, where $\mathcal{G}_0 \equiv (0, \vec{0}, \hat{g}, \vec{0})$. Then we can explicitly construct the group elements by means of the Euler construction of $S_{\mathcal{G}_0}$[7], as in [136, 137]. Finally we have

$$S_{\mathcal{G}} = \tilde{\Lambda}_{\underline{g}}^{-1} S_{\mathcal{G}_0} \tilde{\Lambda}_{\underline{g}}\,, \tag{5.41}$$

with

$$\tilde{\Lambda}_{\underline{g}} = \begin{pmatrix} \Lambda_{\underline{g}} & 0 \\ 0 & \Lambda_{\underline{g}}^{-1} \end{pmatrix}\,. \tag{5.42}$$

For practical purposes we can take $\Lambda_{\underline{g}}$ defined by

$$\Lambda_{\underline{g}}{}^0{}_0 = \frac{g_0}{\hat{g}}\,, \qquad \Lambda_{\underline{g}}{}^i{}_0 = \Lambda_{\underline{g}}{}^0{}_i = \frac{g_i}{\hat{g}}\,, \qquad \Lambda_{\underline{g}}{}^i{}_j = \frac{g_0 - \hat{g}}{\hat{g}\vec{g}^2} g_i g_j + \delta^i{}_j\,, \tag{5.43}$$

whose inverse is obtained by the replacement $\vec{g} \to -\vec{g}$.
Let us focus on the first nontrivial case $\mathrm{SU}(1,2)/(\mathrm{U}(1)\times\mathrm{SU}(2))$. We fix the couplings in a completely electric frame, $\mathcal{G} = (0,0,0,g_0,g_1,g_2)^t$. A basis for $\mathfrak{u}(2)$ (relative to the vector $\mathcal{G}_0 = (0,\vec{0},\hat{g},\vec{0})$) is

$$t_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}\,, \quad t_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}\,, \quad t_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\,, \quad t_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}\,, \tag{5.44}$$

[5] To be precise, this is the subgroup $\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))$.
[6] We assume $\underline{g}$ to be timelike future-directed, i.e., $\eta^{\Lambda\Sigma} g_\Lambda g_\Sigma < 0$, $g_0 > 0$.
[7] In a similar way one can use the Iwasawa construction to obtain the whole group U_{fi}, whose compact part is just $S_{\mathcal{G}}$ [125].

which, by means of ι_2, defines the basis of $\mathfrak{s}_{\mathcal{G}_0}$

$$T_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}. \tag{5.45}$$

Note that

$$T_0^2 = -\Delta\,, \qquad [T_i, T_j]_+ = -\delta_{ij}\Delta\,, \quad 1 \le i \le j \le 3\,,$$

with

$$\Delta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.46}$$

from which we immediately get the expression for a generic element of $S_{\mathcal{G}_0}$,

$$\begin{aligned} S_0(x^0, \vec{x}) &= e^{x^0 T_0} e^{\vec{x}\cdot\vec{T}} \\ &= (I_6 - 2\sin^2\frac{x^0}{2}\Delta + \sin x^0 T_0)(I_6 - 2\sin^2\frac{|\vec{x}|}{2}\Delta + \sin|\vec{x}|\,\vec{x}\cdot\vec{T})\,, \end{aligned} \tag{5.47}$$

where $\vec{x} = (x^1, x^2, x^3)$, $|\vec{x}| = \sqrt{\vec{x}\cdot\vec{x}}$, $\vec{T} = (T_1, T_2, T_3)$ and $\vec{x}\cdot\vec{T} = \sum_{i=1}^3 x^i T_i$.
Finally, after setting

$$T_\mu^g = \tilde{\Lambda}_{\underline{g}}^{-1} T_\mu \tilde{\Lambda}_{\underline{g}}\,, \quad \mu = 0,1,2,3\,, \qquad \Delta_{\underline{g}} = \tilde{\Lambda}_{\underline{g}}^{-1}\Delta\tilde{\Lambda}_{\underline{g}}\,, \tag{5.48}$$

we get for a generic element of $S_{\mathcal{G}}$

$$\begin{aligned} S_{\underline{g}}(x^0, \vec{x}) &= \tilde{\Lambda}_{\underline{g}}^{-1} S_0(x^0, \vec{x}) \tilde{\Lambda}_{\underline{g}} \\ &= (I_6 - 2\sin^2\frac{x^0}{2}\Delta_{\underline{g}} + \sin x^0 T_0^g)(I_6 - 2\sin^2\frac{|\vec{x}|}{2}\Delta_{\underline{g}} + \sin|\vec{x}|\,\vec{x}\cdot\vec{T}^g)\,. \end{aligned} \tag{5.49}$$

In order to have even more manageable expressions for the matrices, it may be convenient to change to the basis R_μ defined by

$$R_0 = T_0^g\,, \quad R_1 = \frac{g_1^2 - g_2^2}{g_1^2 + g_2^2}T_1^g - \frac{2g_1g_2}{g_1^2+g_2^2}T_3^g\,, \quad R_2 = T_2^g\,, \quad R_3 = \frac{g_1^2 - g_2^2}{g_1^2 + g_2^2}T_3^g + \frac{2g_1g_2}{g_1^2+g_2^2}T_1^g\,.$$

Prepotential $F = -X^1X^2X^3/X^0$ This prepotential describes a special Kähler structure on the symmetric manifold $(\mathrm{SU}(1,1)/\mathrm{U}(1))^3$, the well-known stu model. This is symplectically equivalent to the model with $F = -2i(X^0X^1X^2X^3)^{1/2}$, for which supersymmetric black holes with purely electric gaugings are known analytically [78]. After a symplectic transformation to $F = -X^1X^2X^3/X^0$, the electric gaugings considered in [78] become $\mathcal{G} = (0, g^1, g^2, g^3, g_0, 0, 0, 0)^t$, so we shall concentrate on this case in what follows. The symplectic section reads

$$\mathcal{V} = (X^0, X^1, X^2, X^3, X^1X^2X^3/(X^0)^2, -X^2X^3/X^0, -X^1X^3/X^0, -X^2X^1/X^0)^t\,.$$

Let us now look at the solutions of (5.3). To this end, we define

$$\boldsymbol{X} \equiv \begin{pmatrix} X^{0^3} \\ X^{0^2}X^1 \\ X^{0^2}X^2 \\ X^{0^2}X^3 \end{pmatrix}, \qquad \boldsymbol{F} \equiv \begin{pmatrix} X^1X^2X^3 \\ -X^0X^2X^3 \\ -X^0X^1X^3 \\ -X^0X^1X^2 \end{pmatrix}, \tag{5.50}$$

so that (5.3) becomes

$$\boldsymbol{X}S\boldsymbol{X} - \boldsymbol{F}R\boldsymbol{F} - 2\boldsymbol{X}Q^t\boldsymbol{F} = 0\,. \tag{5.51}$$

Since the lhs is a homogeneous polynomial of degree 6 in (X^0, X^1, X^2, X^3), the coefficients of each monomial must be zero. The simplest way to get the general solutions is then to look at the powers of X^0. The possible powers of X^0 in $p_S \equiv \boldsymbol{X}S\boldsymbol{X}$, $p_R \equiv \boldsymbol{F}R\boldsymbol{F}$ and $p_Q \equiv \boldsymbol{X}Q^t\boldsymbol{F}$ are $(6,5,4)$, $(2,1,0)$ and $(4,3,2)$ respectively. Since S and R are symmetric, p_S and p_R can vanish only if S and R are zero. Thus, we are left with the following three possibilities:

1. $R = 0$ and p_Q cancels p_S. The only common power for X^0 is 4, so we have to take matrices which generate only this power and equal degrees for the remaining variables. A quick inspection gives the solutions[8]

$$S_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ -1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ -1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \end{pmatrix},$$

[8]To avoid confusion, note that S denotes the 4×4 matrix in (5.51), while S_1, S_2 and S_3 defined below are 8×8 matrices.

$$U_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{5.52}$$

2. $S = 0$ and p_Q cancels p_R. The only common power for X^0 is 2, so we have to take matrices generating only this and equal degrees for the remaining variables. The solution is
$$S_2 = S_1^t\,, \qquad T_2 = T_1^t\,, \qquad U_2 = U_1^t\,. \tag{5.53}$$

3. $R = S = 0$ and Q satisfies $p_Q = 0$. This implies that Q must be diagonal and that the space of such solutions is 3-dimensional. The simplest way to fix a basis of this space is to choose
$$S_3 = [S_1, S_2]\,, \qquad T_3 = [T_1, T_2]\,, \qquad U_3 = [U_1, U_2]\,. \tag{5.54}$$

In this way the nine matrices $\vec{S}$, $\vec{T}$ and $\vec{U}$ generate the group $U_{\text{fi}} = (\text{SL}(2,\mathbb{R}))^3$.
In order to determine the symmetry algebra $\mathfrak{s}_\mathcal{G}$ we have to consider the equation (using the same notation as in the previous subsection)

$$(\vec{x}\cdot\vec{S} + \vec{y}\cdot\vec{T} + \vec{z}\cdot\vec{U})\mathcal{G} = 0\,, \tag{5.55}$$

whose general solution is given by

$$\mathcal{U}(x,z) = g_0 g^3 x S_1 + g^1 g^2 x S_2 - g_0 g^2 (x+z) T_1 - g^1 g^3 (x+z) T_2 + g_0 g^1 z U_1 + g^2 g^3 z U_2\,,$$

for arbitrary $x, z \in \mathbb{R}$. A convenient basis is

$$\mathcal{U}_1 = \mathcal{U}(1,-1)\,, \qquad \mathcal{U}_2 = \mathcal{U}(1,0)\,, \tag{5.56}$$

which defines a two-dimensional abelian algebra. Notice that

$$\text{tr}\,\mathcal{U}_1^2 = \text{tr}\,\mathcal{U}_2^2 = 8 g_0 g^1 g^2 g^3\,, \tag{5.57}$$

so that the algebra is compact (and thus defines the group $\text{U}(1)\times\text{U}(1)$) if and only if $g_0 g^1 g^2 g^3 < 0$. One can easily verify that, unfortunately, none of these continuous symmetries survives for the truncation to the t^3 model [124, 138] with prepotential $F = -(X^1)^3/X^0$.
It is worth noting that a particular situation arises for $g^1 = g^2 = g^3 = -g_0 \equiv g$. As was

shown in [139], there is an enhancement of the internal symmetry group of the bosonic part of the action. This happens because the scalar potential V can be written in terms of fundamental objects that define the nonlinear sigma model of the non-homogeneous projective coordinates $z^i = x^i + iy^i$ [29, 139], namely

$$V = g^2 \sum_{i=1}^{3} \mathrm{tr} M_i \,, \qquad M_i = \begin{pmatrix} y^i + \frac{x^{i\,2}}{y^i} & \frac{x^i}{y^i} \\ \frac{x^i}{y^i} & \frac{1}{y^i} \end{pmatrix} . \tag{5.58}$$

In fact, the transformation property of M_i,

$$M_i \longmapsto \mathcal{T}^t M_i \mathcal{T} \,, \tag{5.59}$$

implies the invariance of the potential only if $\mathcal{T}\mathcal{T}^t = 1$. Going back to the symplectic formalism we see that this condition is equivalent to require for the symmetry group to be orthogonal, which, in terms of the elements of $\mathfrak{u}_{\mathrm{fi}}$ amounts to consider just the subspace of antisymmetric matrices. Thus, the symmetry algebra is generated by

$$W_1 = S_1 - S_2 \,, \qquad W_2 = T_1 - T_2 \,, \qquad W_3 = U_1 - U_2 \,, \tag{5.60}$$

while the subalgebra leaving $\mathcal{G}$ fixed is generated by $W_2 - W_1$ and $W_3 - W_2$. The full symmetry group of the bosonic part of the action is therefore an extension $U_{\mathrm{i(bos)}} = \mathrm{U}(1)^3$ of $S_{\mathcal{G}} = \mathrm{U}(1)^2$. The extra $U(1)$ symmetry breaks supersymmetry, it is not a symmetry of the BPS equation but only of the bosonic equations of motion. Infact U, respresting this extra $U(1)$ symmetry, maintains invariant the equations of motion under the action

$$(\mathcal{V}, \mathcal{G}, \mathcal{F}_{\mu\nu}) \mapsto (U\mathcal{V}, \mathcal{G}, U\mathcal{F}_{\mu\nu}) \,. \tag{5.61}$$

It can be cosidered as an accidental symmetry present thanks to the special structure 5.58 of the potential.

Prepotential $F = X^1 X^2 X^3 / X^0 - \frac{A}{3}(X^3)^3 / X^0$ The base manifold for this prepotential is neither symmetric nor homogeneous and it has been studied in [120]. The symplectic section is given by $\mathcal{V} = (X^\Lambda, F_\Lambda)^t$, with

$$X^{\Lambda\,t} = \begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} , \qquad F_\Lambda^t = \begin{pmatrix} -X^1 X^2 X^3/(X^0)^2 + \frac{A}{3}(X^3)^3/(X^0)^2 \\ X^2 X^3 / X^0 \\ X^1 X^3 / X^0 \\ X^1 X^2 / X^0 - A (X^3)^2 / X^0 \end{pmatrix} . \tag{5.62}$$

The solution to (5.3) is obtained by proceeding exactly like in the previous subsection. After introducing the vectors

$$\boldsymbol{X} = \begin{pmatrix} {X^0}^3 \\ {X^0}^2 X^1 \\ {X^0}^2 X^2 \\ {X^0}^2 X^3 \end{pmatrix} , \qquad \boldsymbol{F} = \begin{pmatrix} \frac{A}{3} {X^3}^3 - X^1 X^2 X^3 \\ X^0 X^2 X^3 \\ X^0 X^1 X^3 \\ X^0 X^1 X^2 - A X^0 {X^3}^2 \end{pmatrix} , \tag{5.63}$$

we reduce the equations to a polynomial identity, and looking at the coefficients we get a five-dimensional space of solutions generated by the symplectic matrices

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2A & 0 & 0 & 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.64}$$

A direct comparison with the results of [120] shows that this algebra strictly contains the U-duality algebra. This is due to the fact that the group of symmetries of the scalar potential is larger than the symmetry group of the whole Lagrangian. Indeed the generator D_2 does not leave the metric invariant. Thus, the U-duality group is generated by the algebra

$$\langle S_1, S_2, S_3, D_1 \rangle_{\mathbb{R}}\,. \tag{5.65}$$

Notice that the S_i are nilpotent of order 4 for $i = 1$ and order 2 for $i = 2, 3$. They are indeed eigenmatrices for the adjoint action of D_1, all with eigenvalue -2. The stability equation (5.19) has a nontrivial solution only if $A = -g^1 g^2/(g^3)^2$. With this choice for A one gets a one-dimensional algebra $\mathfrak{s}_G$ generated by

$$s = S_1 - \frac{g^1}{g^3} S_3 - \frac{g^2}{g^3} S_2 \,. \tag{5.66}$$

It is nilpotent of order 4 so that $U_{\rm i} = \mathcal{S}_{\mathcal{G}}$ is a unipotent group of order 4. It is worthwhile to note that for $g^1 = g^2 = g^3$ one gets $A = -1$, which is the physically most interesting case, since the corresponding prepotential arises in the context of type IIA string theory compactified on Calabi-Yau manifolds [140].

Scalar hair and dyonic solutions

We shall now use the results of the previous section in order to generate new supergravity solutions from a given seed. The transformations in $U_{\rm i}$ add new parameters to a given solution and leave not only the equations of motion invariant, but also some potential first-order flow equations (if these are satisfied by the seed). The transformed field configuration preserves thus the same amount of supersymmetry as the one from which we started.

As was stressed in [139], the latter statement is not true in the STU model for the additional $\mathrm{U}(1)$ that arises for equal couplings, whose action generically leads to a non-BPS solution. The same story holds also in the quadratic models for T_α and L_α, due to the properties (5.27) and (5.30) [97].

In what follows we will consider several relevant examples for some well-studied prepotentials, but there is no obstacle to extending this method to other solutions and prepotentials as well. We underline that in the static case, owing to the existence of the black hole potential $V_{\rm BH}$ [105, 107], one can directly rotate the charges $\mathcal{Q}$ instead of the field strengths $\mathcal{F}_{\mu\nu}$.

Prepotential $F = -iX^0X^1$ For this prepotential, we have $U_{\rm i(bos)} = \mathrm{U}(1)^2$, whose action on the static and magnetic BPS seed solution of [78] is

$$(\mathcal{V}, \mathcal{G}, \mathcal{Q}) \quad \longmapsto \quad (\tilde{\mathcal{V}}, \tilde{\mathcal{G}}, \tilde{\mathcal{Q}}) = (S\mathcal{V}, \mathcal{G}, T_\alpha S \mathcal{Q}) \,. \tag{5.67}$$

Using the results of section 5.3 and the constraints on the seed parameters (cf. [78]), one gets

$$\begin{aligned} \tilde{\mathcal{Q}} &= (p^0 \cos\alpha, p^1 \cos\alpha, -p^1 \sin\alpha, -p^0 \sin\alpha)^t \,, \\ \tilde{z} &= \frac{\tilde{X}^1}{\tilde{X}^0} = \frac{g_0}{g_1} \cdot \frac{g_1 z \cos\beta + i g_0 \sin\beta}{g_0 \cos\beta + i g_1 z \sin\beta} \,, \qquad z \equiv \frac{X^1}{X^0} \,. \end{aligned} \tag{5.68}$$

The parameter β does not modify the supersymmetry of the solution; for $\alpha = 0$ the new configuration satisfies again the BPS flow equations of [73, 78]. For $\alpha \neq 0$ one gets a solution that still obeys a first-order flow, but this time a non-BPS one [97], driven by the fake superpotential

$$W = e^U |\langle T_{-\alpha} \tilde{\mathcal{Q}}, \tilde{\mathcal{V}} \rangle - i e^{2(\psi - U)} \tilde{\mathcal{L}}| \,, \tag{5.69}$$

where $U(r)$ and $\psi(r)$ are functions appearing in the metric

$$ds^2 = -e^{2U}dt^2 + e^{-2U}dr^2 + e^{2(\psi-U)}(d\theta^2 + \sinh^2\theta d\phi^2)\,, \tag{5.70}$$

and $\mathcal{L}$ was defined in section 3.4. The first-order equations following from (5.69) imply the equations of motion provided the Dirac-type charge quantization condition

$$\langle \mathcal{G}, \mathcal{Q} \rangle = 1 \tag{5.71}$$

holds [97]. From (5.68) we see that for $\alpha \neq 0$ one generates a dyonic solution from a purely magnetic one, while β adds scalar hair to the seed. Note that this result was first obtained in [139].

As another example for the action of $U_{\text{i(bos)}}$ we consider the Chow-Compère solution [141], that solves the equations of motion following from the Lagrangian (2.12) of [141],

$$\begin{aligned}\mathscr{L} \;&=\; R\star 1 - \frac{1}{2}\star d\varphi\wedge d\varphi - \frac{1}{2}e^{2\varphi}\star d\chi\wedge d\chi - e^{-\varphi}\star F^1\wedge F^1 + \chi F^1\wedge F^1 \\ &\quad -\frac{1}{1+\chi^2e^{2\varphi}}\left(e^{\varphi}\star F^2\wedge F^2 + \chi e^{2\varphi}F^2\wedge F^2\right) + g^2\left(4 + e^{\varphi} + e^{-\varphi} + \chi^2 e^{\varphi}\right)\star 1\,,\end{aligned} \tag{5.72}$$

which is obtained from (2.15) by setting

$$z = \frac{g_0}{g_1}\left(e^{-\varphi} - i\chi\right)\,, \qquad g_0 g_1 = g^2\,, \tag{5.73}$$

and redefining[9]

$$F^0 \longrightarrow \sqrt{\frac{g_1}{g_0}}F^1\,, \qquad F^1 \longrightarrow \sqrt{\frac{g_0}{g_1}}F^2\,. \tag{5.74}$$

The dyonic rotating black hole solution of [141] is given by

$$ds^2 = -\frac{R}{W}\left(dt - \frac{a^2 - u_1u_2}{a}d\phi\right)^2 + \frac{W}{R}dr^2 + \frac{U}{W}\left(dt - \frac{r_1r_2 + a^2}{a}d\phi\right)^2 + \frac{W}{U}du^2\,, \tag{5.75}$$

where

$$\begin{aligned} R(r) &= r^2 - 2mr + a^2 + g^2 r_1r_2(r_1r_2 + a^2)\,, \\ U(u) &= -u^2 + 2nu + a^2 + g^2u_1u_2(u_1u_2 - a^2)\,, \\ W(r,u) &= r_1r_2 + u_1u_2\,, \qquad r_{1,2} = r + \Delta r_{1,2}\,, \qquad u_{1,2} = u + \Delta u_{1,2}\,, \end{aligned} \tag{5.76}$$

and $\Delta r_{1,2}$, $\Delta u_{1,2}$ are constants defined by

$$\begin{aligned} \Delta r_1 &= m[\cosh(2\delta_1)\cosh(2\gamma_2) - 1] + n\sinh(2\delta_1)\sinh(2\gamma_1)\,, \\ \Delta r_2 &= m[\cosh(2\delta_2)\cosh(2\gamma_1) - 1] + n\sinh(2\delta_2)\sinh(2\gamma_2)\,, \\ \Delta u_1 &= n[\cosh(2\delta_1)\cosh(2\gamma_2) - 1] - m\sinh(2\delta_1)\sinh(2\gamma_1)\,, \\ \Delta u_2 &= n[\cosh(2\delta_2)\cosh(2\gamma_1) - 1] - m\sinh(2\delta_2)\sinh(2\gamma_2)\,. \end{aligned} \tag{5.77}$$

[9] We assume $g_0/g_1 > 0$.

Below we shall also use the linear combinations

$$\begin{aligned} \Sigma_{\Delta r} &= \frac{1}{2}(\Delta r_1 + \Delta r_2)\,, \qquad \Delta_{\Delta r} = \frac{1}{2}(\Delta r_2 - \Delta r_1)\,, \\ \Sigma_{\Delta u} &= \frac{1}{2}(\Delta u_1 + \Delta u_2)\,, \qquad \Delta_{\Delta u} = \frac{1}{2}(\Delta u_2 - \Delta u_1)\,. \end{aligned} \tag{5.78}$$

The complex scalar field has the very simple form

$$z = \frac{g_0}{g_1}\frac{r_1 - iu_1}{r_2 - iu_2}\,, \tag{5.79}$$

while the gauge fields and their duals read

$$\begin{aligned} A^1 &= \zeta^1(dt - ad\phi) + \frac{r_2 u_2 \tilde{\zeta}_1}{a} d\phi\,, \qquad A^2 = \zeta^2(dt - ad\phi) + \frac{r_1 u_1 \tilde{\zeta}_2}{a} d\phi\,, \\ \tilde{A}_1 &= \tilde{\zeta}_1(dt - ad\phi) - \frac{r_1 u_1 \zeta^1}{a} d\phi\,, \qquad \tilde{A}_2 = \tilde{\zeta}_2(dt - ad\phi) - \frac{r_2 u_2 \zeta^2}{a} d\phi\,, \end{aligned} \tag{5.80}$$

where the three-dimensional electromagnetic scalars are

$$\begin{aligned} \zeta^1 &= \frac{1}{2W}\frac{\partial W}{\partial \delta_1} = \frac{Q_1 r_2 - P^1 u_2}{W}\,, \qquad \tilde{\zeta}_1 = \frac{Q_1 u_1 + P^1 r_1}{W}\,, \\ \zeta^2 &= \frac{1}{2W}\frac{\partial W}{\partial \delta_2} = \frac{Q_2 r_1 - P^2 u_1}{W}\,, \qquad \tilde{\zeta}_2 = \frac{Q_2 u_2 + P^2 r_2}{W}\,. \end{aligned} \tag{5.81}$$

Here, $Q_{1,2}$ and $P^{1,2}$ denote respectively the electric and magnetic charges given by [141]

$$Q_1 = \frac{1}{2}\frac{\partial r_1}{\partial \delta_1}\,, \qquad Q_2 = \frac{1}{2}\frac{\partial r_2}{\partial \delta_2}\,, \qquad P^1 = -\frac{1}{2}\frac{\partial u_1}{\partial \delta_1}\,, \qquad P^2 = -\frac{1}{2}\frac{\partial u_2}{\partial \delta_2}\,. \tag{5.82}$$

The solution is thus specified by the 7 parameters m, n, a, $\gamma_{1,2}$ and $\delta_{1,2}$ that are related to the mass, NUT charge, angular momentum, two electric and two magnetic charges. Notice that a similar class of rotating black holes containing one parameter less was constructed in [142].

Let us now consider the action of S defined in (5.24). For the transformed scalar we get

$$\tilde{z} = \frac{\tilde{X}^1}{\tilde{X}^0} = \frac{g_0}{g_1}\frac{r + \Delta r_1' - i(u + \Delta u_1')}{r + \Delta r_2' - i(u + \Delta u_2')}\,, \tag{5.83}$$

where

$$\begin{pmatrix} \Delta r_1' \\ \Delta r_2' \\ \Delta u_1' \\ \Delta u_2' \end{pmatrix} = \begin{pmatrix} \cos^2\beta & \sin^2\beta & -\cos\beta\sin\beta & \cos\beta\sin\beta \\ \sin^2\beta & \cos^2\beta & \cos\beta\sin\beta & -\cos\beta\sin\beta \\ \cos\beta\sin\beta & -\cos\beta\sin\beta & \cos^2\beta & \sin^2\beta \\ -\cos\beta\sin\beta & \cos\beta\sin\beta & \sin^2\beta & \cos^2\beta \end{pmatrix} \begin{pmatrix} \Delta r_1 \\ \Delta r_2 \\ \Delta u_1 \\ \Delta u_2 \end{pmatrix}. \tag{5.84}$$

Note that the quantities $\Sigma_{\Delta r}$ and $\Sigma_{\Delta u}$ defined in (5.78) remain invariant under (5.84), while $\Delta_{\Delta r}$ and $\Delta_{\Delta u}$ transform as

$$\begin{pmatrix} \Delta'_{\Delta r} \\ \Delta'_{\Delta u} \end{pmatrix} = \begin{pmatrix} \cos 2\beta & -\sin 2\beta \\ \sin 2\beta & \cos 2\beta \end{pmatrix} \begin{pmatrix} \Delta_{\Delta r} \\ \Delta_{\Delta u} \end{pmatrix} . \tag{5.85}$$

The transformed gauge fields can be easily inferred from

$$\begin{pmatrix} A^1 + A^2 \\ \frac{g_1}{g_0}\tilde{A}_1 + \frac{g_0}{g_1}\tilde{A}_2 \\ A^2 - A^1 \\ \frac{g_0}{g_1}\tilde{A}_2 - \frac{g_1}{g_0}\tilde{A}_1 \end{pmatrix}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos 2\beta & -\sin 2\beta \\ 0 & 0 & \sin 2\beta & \cos 2\beta \end{pmatrix} \begin{pmatrix} A^1 + A^2 \\ \frac{g_1}{g_0}\tilde{A}_1 + \frac{g_0}{g_1}\tilde{A}_2 \\ A^2 - A^1 \\ \frac{g_0}{g_1}\tilde{A}_2 - \frac{g_1}{g_0}\tilde{A}_1 \end{pmatrix} . \tag{5.86}$$

In conclusion, S adds one more parameter β to the solution of [141].

Under the action of T_α (cf. (5.130)) the scalar z does not change. It turns out that the new gauge fields can again be written in the form (5.80), but with the three-dimensional electromagnetic scalars replaced by

$$\begin{pmatrix} \sqrt{\frac{g_1}{g_0}}\zeta^1 \\ \sqrt{\frac{g_0}{g_1}}\zeta^2 \\ \sqrt{\frac{g_1}{g_0}}\tilde{\zeta}_1 \\ \sqrt{\frac{g_0}{g_1}}\tilde{\zeta}_2 \end{pmatrix} \longmapsto \begin{pmatrix} \cos\alpha & 0 & 0 & \sin\alpha \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ -\sin\alpha & 0 & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} \sqrt{\frac{g_1}{g_0}}\zeta^1 \\ \sqrt{\frac{g_0}{g_1}}\zeta^2 \\ \sqrt{\frac{g_1}{g_0}}\tilde{\zeta}_1 \\ \sqrt{\frac{g_0}{g_1}}\tilde{\zeta}_2 \end{pmatrix} . \tag{5.87}$$

In other words, they transform (up to prefactors) with the same matrix T_α. This invariance can be used to generate additional charges by starting from a given seed. Set e.g. $\gamma_2 = \delta_2 = 0$ in (5.77), which by (5.82) implies $P^2 = Q_2 = 0$. After acting with T_α one gets a solution with all four charges nonvanishing, namely

$$Q'_1 = Q_1 \cos\alpha \,, \qquad P^{1\prime} = P^1 \cos\alpha \,, \qquad Q'_2 = \frac{g_1}{g_0} P^1 \sin\alpha \,, \qquad P^{2\prime} = -\frac{g_1}{g_0} Q_1 \sin\alpha \,.$$

On the other hand, from the point of view of the AdS/CFT the β parameter enlarges the set of the boundary conditions [143–145] and therefore deforms the dual CFT with multitrace operator [146, 147]. The simplest example shown at the begining of this section (5.68) is a fruitful arena to understand the meaning of the new parameter. If we take

$$z = e^{\sqrt{2}\phi} - i\sqrt{2}\chi \,, \tag{5.88}$$

the action is normalized as in [144] up to a global factor $1/2$. The expansion of the fields near the boundary is [10]

$$\begin{aligned} \phi &= \frac{4\beta^0 \cos 2\hat{\beta}}{a\sqrt{2}} \frac{1}{r} + \frac{8(\beta^0)^2 \sin^2(2\hat{\beta})}{a^2\sqrt{2}} \frac{1}{r^2} + O\left(\frac{1}{r^3}\right) , \\ \chi &= \frac{-4\beta^0 \sin 2\hat{\beta}}{a\sqrt{2}} \frac{1}{r} + \frac{16(\beta^0)^2 (\sin 3\hat{\beta} - \sin\hat{\beta})}{a^2\sqrt{2}} \frac{1}{r^2} + O\left(\frac{1}{r^3}\right) . \end{aligned} \tag{5.89}$$

[10] We take $g_0 = g_1 = 1$, the coordinate z of [78] is replaced by r and we recall the previous parameter β as $\hat{\beta}$ to avoid misleading errors in the following.

Expanding the potential

$$V = -\left(2 + \cosh\sqrt{2}\phi + e^{\sqrt{2}\phi}\chi^2\right) \tag{5.90}$$

near the extremum $\phi = \chi = 0$ one finds that the small fluctuations have $m_\phi^2 = m_\chi^2 = -2$ and due to the leading terms in expansion and to the Breitenlohner-Freedman bound [144]. The coefficient α and β defined in [143] fix the value of

$$k_\phi = 2\sqrt{2}\tan 2\hat{\beta}\,, \quad k_\chi = \frac{4\sqrt{2}(\sin 3\hat{\beta} - \sin\hat{\beta})}{\sin^2(2\hat{\beta})}\,. \tag{5.91}$$

These constants determine the couplings in the dual CFT of the multitrace operators [145] deforming the original action.

Prepotential $F = \frac{i}{4}((X^1)^2 + (X^2)^2 - (X^0)^2)$ In this case the most interesting feature of $U_{\rm i}$ is the non-abelianity of $\mathcal{S}_{\mathcal{G}}$, cf. sec. 5.3. As far as L_α is concerned, its effect is the same as the one of T_α for $F = -iX^0X^1$, namely the transformed configuration solves non-BPS first-order flow equations.

The nonabelian part acts nontrivially on the special scalars. With the 1-parameter subgroups $\exp(\alpha_\mu R_\mu)$ ($\mu = 0, \ldots, 3$, no summation over μ), where the R_μ are defined in section 5.3, one can describe the action of $\mathcal{S}_{\mathcal{G}}$ on a static seed solution with charge vector $\mathcal{Q}$ as

$$\begin{aligned}
(\mathcal{V}, \mathcal{G}, \mathcal{Q}) \quad &\longmapsto \quad (\tilde{\mathcal{V}}, \tilde{\mathcal{G}}, \tilde{\mathcal{Q}}) = (e^{\alpha_0 R_0}\mathcal{V}, \mathcal{G}, e^{\alpha_0 R_0}\mathcal{Q})\,,\\
\tilde{z}^1 &= \frac{-g_1(g_0 + g_1 z^1 + g_2 z^2) + e^{i\alpha_0}(g_0 g_1 + (g_0^2 - g_2^2)z^1 + g_1 g_2 z^2)}{g_0(g_0 + g_1 z^1 + g_2 z^2) - e^{i\alpha_0}(g_1^2 + g_2^2 + g_0 g_2 z^2 + g_0 g_1 z^1)}\,,\\
\tilde{z}^2 &= \frac{-g_2(g_0 + g_1 z^1 + g_2 z^2) + e^{i\alpha_0}(g_0 g_2 + (g_0^2 - g_1^2)z^2 + g_1 g_2 z^1)}{g_0(g_0 + g_1 z^1 + g_2 z^2) - e^{i\alpha_0}(g_1^2 + g_2^2 + g_0 g_2 z^2 + g_0 g_1 z^1)}\,,
\end{aligned}$$

$$\begin{aligned}
(\mathcal{V}, \mathcal{G}, \mathcal{Q}) \quad &\longmapsto \quad (\tilde{\mathcal{V}}, \tilde{\mathcal{G}}, \tilde{\mathcal{Q}}) = (e^{\alpha_1 R_1}\mathcal{V}, \mathcal{G}, e^{\alpha_1 R_1}\mathcal{Q})\,,\\
\tilde{z}^1 &= \frac{-g_1(g_0 + g_1 z^1 + g_2 z^2) + (g_0 g_1 + g_0^2 z^1 - g_2^2 z^1 + g_1 g_2 z^2)\cos\alpha_1 - \hat{g}(g_2 + g_0 z^2)\sin\alpha_1}{g_0(g_0 + g_1 z^1 + g_2 z^2) - (g_1^2 + g_0 g_1 z^1 + g_2^2 + g_0 g_2 z^2)\cos\alpha_1 + \hat{g}(g_1 z^2 - g_2 z^1)\sin\alpha_1}\,,\\
\tilde{z}^2 &= \frac{-g_2(g_0 + g_1 z^1 + g_2 z^2) + (g_0 g_2 + g_0^2 z^2 - g_1^2 z^2 + g_2 g_1 z^1)\cos\alpha_1 + \hat{g}(g_1 + g_0 z^1)\sin\alpha_1}{g_0(g_0 + g_1 z^1 + g_2 z^2) - (g_1^2 + g_0 g_1 z^1 + g_2^2 + g_0 g_2 z^2)\cos\alpha_1 + \hat{g}(g_1 z^2 - g_2 z^1)\sin\alpha_1}\,,
\end{aligned}$$

$$\begin{aligned}
(\mathcal{V}, \mathcal{G}, \mathcal{Q}) \quad &\longmapsto \quad (\tilde{\mathcal{V}}, \tilde{\mathcal{G}}, \tilde{\mathcal{Q}}) = (e^{\alpha_2 R_2}\mathcal{V}, \mathcal{G}, e^{\alpha_2 R_2}\mathcal{Q})\,,\\
\tilde{z}^1 &= \frac{-g_1(g_0 + g_1 z^1 + g_2 z^2) + f(g_1, g_2, z^1, z^2)\cos\alpha_2 - h(g_1, g_2, z^1, z^2)\sin\alpha_2}{g_0(g_0 + g_1 z^1 + g_2 z^2) - (g_1^2 + g_0 g_1 z^1 + g_2(g_2 + g_0 z^2))\cos\alpha_2 + i\hat{g}(g_2 z^1 - g_1 z^2)\sin\alpha_2}\,,\\
\tilde{z}^2 &= \frac{-g_2(g_0 + g_1 z^1 + g_2 z^2) + f(g_2, g_1, z^2, z^1)\cos\alpha_2 + h(g_2, g_1, z^2, z^1)\sin\alpha_2}{g_0(g_0 + g_1 z^1 + g_2 z^2) - (g_1^2 + g_0 g_1 z^1 + g_2(g_2 + g_0 z^2))\cos\alpha_2 + i\hat{g}(g_2 z^1 - g_1 z^2)\sin\alpha_2}\,,
\end{aligned}$$

$$
\begin{aligned}
(\mathcal{V},\mathcal{G},\mathcal{Q}) \quad &\longmapsto \quad (\tilde{\mathcal{V}},\tilde{\mathcal{G}},\tilde{\mathcal{Q}}) = (e^{\alpha_3 R_3}\mathcal{V},\mathcal{G},e^{\alpha_3 R_3}\mathcal{Q})\,,\\
\tilde{z}^1 &= \frac{-g_1(g_1^2+g_2^2)(g_0+g_1z^1+g_2z^2)+e^{i\alpha_3}k(g_1,g_2,z^1,z^2)+e^{-i\alpha_3}g_2\hat{g}^2(g_2z^1-g_1z^2)}{(g_1^2+g_2^2)\left(g_0(g_0+g_1z^1+g_2z^2)-e^{i\alpha_3}(g_1^2+g_0g_1z^1+g_2^2+g_0g_2z^2)\right)}\,,\\
\tilde{z}^2 &= \frac{-g_2(g_1^2+g_2^2)(g_0+g_1z^1+g_2z^2)+e^{i\alpha_3}k(g_2,g_1,z^2,z^1)+e^{-i\alpha_3}g_1\hat{g}^2(g_1z^2-g_2z^1)}{(g_1^2+g_2^2)\left(g_0(g_0+g_1z^1+g_2z^2)-e^{i\alpha_3}(g_1^2+g_0g_1z^1+g_2^2+g_0g_2z^2)\right)}\,,
\end{aligned}
$$

where we used the definitions

$$
\begin{aligned}
&\hat{g} = \sqrt{g_0^2-g_1^2-g_2^2}\,, \qquad f(g_1,g_2,z^1,z^2) = g_0g_1+g_0^2z^1+g_1g_2z^2-g_2^2z^1\,,\\
&h(g_1,g_2,z^1,z^2) = \frac{i\hat{g}}{g_1^2+g_2^2}(2g_0g_1g_2z^1+g_1^2(g_2-g_0z^2)+g_2^2(g_2+g_0z^2))\,,\\
&k(g_1,g_2,z^1,z^2) = g_0g_1(g_1^2+g_0g_1z^1+g_2^2+g_0g_2z^2)\,.
\end{aligned}
\tag{5.92}
$$

The explicit expressions for $\tilde{\mathcal{Q}}$ are not particularly enlightening, so we don't report them here. One may apply the above transformations to the static and magnetic BPS seed given by eqns. (3.100) and (3.101) of [78] to generate dyonic and axionic solutions.
Note that the form of (5.49) splits the dependence of the group coordinates from the couplings. Defining the section $\mathcal{V}_{\underline{g}} = (\boldsymbol{X}_{\underline{g}},\boldsymbol{F}_{\underline{g}})^t \equiv \tilde{\Lambda}_{\underline{g}}\mathcal{V}$, the action of $\mathcal{S}_{\mathcal{G}}$ becomes $\tilde{\mathcal{V}}_{\underline{g}} = S_0(x^0,\vec{x})\mathcal{V}_{\underline{g}}$ that more explicitly reads

$$
\tilde{\boldsymbol{X}}_{\underline{g}} = \begin{pmatrix} X^0_{\underline{g}} \\ e^{ix^0}\left(X^1_{\underline{g}}\cos|\vec{x}| + i((x^1+ix^2)X^2_{\underline{g}}+ix^3X^1_{\underline{g}})\sin|\vec{x}|\right) \\ e^{ix^0}\left(X^2_{\underline{g}}\cos|\vec{x}| + i((x^1-ix^2)X^1_{\underline{g}}-ix^3X^2_{\underline{g}})\sin|\vec{x}|\right)\end{pmatrix}\,. \tag{5.93}
$$

This split is independent of the parametrization of the group and so one can also use that of [136, 137].

Prepotential $F = -X^1X^2X^3/X^0$ This model is related to the one with $F = -2i(X^0X^1X^2X^3)^{1/2}$ by a symplectic rotation with the matrix (5.7). As a seed solution we shall thus take the static magnetic BPS black holes given by eqns. (3.31)-(3.34) of [78], transformed to $F = -X^1X^2X^3/X^0$. In this new frame, the vectors of charges and couplings are respectively given by

$$
\mathcal{Q} = (p^0,0,0,0,0,q_1,q_2,q_3)^t\,, \qquad \mathcal{G} = (0,g^1,g^2,g^3,g_0,0,0,0)^t\,. \tag{5.94}
$$

Assuming $g_0g^1g^2g^3 < 0$ and defining $A \equiv (-g_0g^1g^2g^3)^{1/2}$, the finite transformations $\exp(\alpha_1\mathcal{U}_1)$ and $\exp(\alpha_2\mathcal{U}_2)$ generated by (5.56) act as

$$
\begin{aligned}
(\mathcal{V},\mathcal{G},\mathcal{Q}) \quad &\longmapsto \quad (\tilde{\mathcal{V}},\tilde{\mathcal{G}},\tilde{\mathcal{Q}}) = (e^{\alpha_1\mathcal{U}_1}\mathcal{V},\mathcal{G},e^{\alpha_1\mathcal{U}_1}\mathcal{Q})\,,\\
\tilde{z}^1 &= \frac{Az^1\cos(A\alpha_1)+g_0g^1\sin(A\alpha_1)}{A\cos(A\alpha_1)+z^1g^2g^3\sin(A\alpha_1)}\,,\\
\tilde{z}^2 &= z^2\,,\\
\tilde{z}^3 &= \frac{Az^3\cos(A\alpha_1)-g_0g^3\sin(A\alpha_1)}{A\cos(A\alpha_1)-z^3g^1g^2\sin(A\alpha_1)}\,,
\end{aligned}
\tag{5.95}
$$

$$
\begin{aligned}
(\mathcal{V},\mathcal{G},\mathcal{Q}) \quad &\longmapsto \quad (\tilde{\mathcal{V}},\tilde{\mathcal{G}},\tilde{\mathcal{Q}}) = (e^{\alpha_2 \mathcal{U}_2}\mathcal{V},\mathcal{G},e^{\alpha_2 \mathcal{U}_2}\mathcal{Q})\,,\\
\tilde{z}^1 &= z^1\,,\\
\tilde{z}^2 &= \frac{Az^2\cos(A\alpha_2)+g_0 g^2\sin(A\alpha_2)}{A\cos(A\alpha_2)+z^2 g^1 g^3\sin(A\alpha_2)}\,,\\
\tilde{z}^3 &= \frac{Az^3\cos(A\alpha_2)-g_0 g^3\sin(A\alpha_2)}{A\cos(A\alpha_2)-z^3 g^1 g^2\sin(A\alpha_2)}\,.
\end{aligned}
\tag{5.96}
$$

Again, the expressions for $\tilde{\mathcal{Q}}$ are not particularly enlightening, so we shall not report them here. Notice that the transformations (5.95), (5.96) preserve the supersymmetry of the seed.

As we pointed out in section 5.3, in the special case $\mathcal{G} = (0,g,g,g,-g,0,0,0)^t$ there is an enhancement of the symmetry group to $\mathrm{U}(1)^3$ generated by (5.60). If we define $T = \exp[\frac{\alpha_3}{3}(W_1+W_2+W_3)]$, the action of the extra $\mathrm{U}(1)$ is

$$
\begin{aligned}
(\mathcal{V},\mathcal{G},\mathcal{Q}) \quad &\longmapsto \quad (\tilde{\mathcal{V}},\tilde{\mathcal{G}},\tilde{\mathcal{Q}}) = (T\mathcal{V},\mathcal{G},T\mathcal{Q})\,,\\
\tilde{z}^1 &= \frac{z^1\cos\alpha_3-\sin\alpha_3}{z^1\sin\alpha_3+\cos\alpha_3}\,,\\
\tilde{z}^2 &= \frac{z^2\cos\alpha_3-\sin\alpha_3}{z^2\sin\alpha_3+\cos\alpha_3}\,,\\
\tilde{z}^3 &= \frac{z^3\cos\alpha_3-\sin\alpha_3}{z^3\sin\alpha_3+\cos\alpha_3}\,,
\end{aligned}
\tag{5.97}
$$

plus an expression for the charges $\tilde{\mathcal{Q}}$. (5.95), (5.96) and (5.97) where first obtained in [139]. Note that T breaks supersymmetry, since it does not belong to the stabilizer $\mathcal{S}_{\mathcal{G}}$. In fact,

$$
T\mathcal{G} \equiv \mathcal{G}_{\alpha_3} = g(\sin\alpha_3,\cos\alpha_3,\cos\alpha_3,\cos\alpha_3,-\cos\alpha_3,\sin\alpha_3,\sin\alpha_3,\sin\alpha_3)^t\,. \tag{5.98}
$$

However, the transformed solution still satisfies first-order non-BPS flow equations driven by the fake superpotential [97][11]

$$
W = e^U|\langle\tilde{Q},\tilde{\mathcal{V}}\rangle - ie^{2(\psi-U)}\langle\mathcal{G}_{\alpha_3},\tilde{\mathcal{V}}\rangle|\,, \tag{5.99}
$$

provided the charge quantization condition $\langle\mathcal{G},\mathcal{Q}\rangle = -\kappa$ holds, where $\kappa = 0,1,-1$ for flat, spherical or hyperbolic horizons respectively.

Prepotential $F = X^1X^2X^3/X^0 + \frac{g^1g^2}{3(g^3)^2}(X^3)^3/X^0$ In this case the only known solution with running scalars is that of [120], with static metric and purely imaginary scalar fields,

$$
X^1/X^0 = z^1 = -i\lambda^1\,, \qquad X^2/X^0 = z^2 = -i\lambda^2\,, \qquad X^3/X^0 = z^3 = -i\lambda^3\,. \tag{5.100}
$$

The charges and coupling constants are given by

[11]Notice that this flow is a BPS flow for a theory with gaugings given by $\mathcal{G}_{\alpha_3}$.

$$\mathcal{Q} = (p^0, 0, 0, 0, 0, q_1, q_2, q_3)^t\,, \qquad \mathcal{G} = (0, g^1, g^2, g^3, g_0, 0, 0, 0)^t\,. \tag{5.101}$$

Applying the finite transformation generated by (5.66) yields for the scalars

$$\tilde{z}^1 = -i\lambda^1 - \frac{g^1}{g^3}c\,, \qquad \tilde{z}^2 = -i\lambda^2 - \frac{g^2}{g^3}c\,, \qquad \tilde{z}^2 = -i\lambda^3 + c\,, \tag{5.102}$$

and for the charges

$$\tilde{\mathcal{Q}} = \begin{pmatrix} p^0 \\ -(cg^1p^0)/g^3 \\ -(cg^2p^0)/g^3 \\ cp^0 \\ -(4c^3g^1g^2p^0)/(3{g^3}^2) + (g^1q_1 + g^2q_2 - g^3q_3)/g^3 \\ q_1 - c^2g^2p^0/g^3 \\ q_2 - c^2g^1p^0/g^3 \\ q_3 + 2c^2g^1g^2p^0/{g^3}^2 \end{pmatrix}, \tag{5.103}$$

where c is a group parameter. This solution is again BPS but has also nontrivial (constant) axions turned on and all charges are nonvanishing.

Extension to hypermultiplets

A possible generalization to include also hypermultiplets is described here. In this case the situation is more involved, since the coupling constants are replaced by the moment maps $\mathcal{P}^x$. However, when only abelian isometries of the quaternionic hyperscalar target space are gauged, the scalar potential can be cast into the form (3.137)

$$V = \mathbb{G}^{AB}\mathbb{D}_A\mathcal{L}\,\mathbb{D}_B\bar{\mathcal{L}} - 3|\mathcal{L}|^2\,. \tag{5.104}$$

The most general symmetry transformation of the nonlinear sigma model is a linear combination of the isometries of the quaternionic and the special Kähler manifold. Let us define the formal operator

$$\delta = k^u\mathsf{D}_u + U\mathcal{V}\frac{\delta}{\delta\mathcal{V}} + U\bar{\mathcal{V}}\frac{\delta}{\delta\bar{\mathcal{V}}} + U\mathcal{A}_\mu\frac{\delta}{\delta\mathcal{A}_\mu} + k^i\partial_i + k^{\bar{\imath}}\partial_{\bar{\imath}}\,, \tag{5.105}$$

where k^u is a Killing vector of the quaternionic manifold, U an element of the U-duality algebra, k^i the corresponding holomorphic special Kähler Killing vector, and $\mathcal{A}_\mu$ is the symplectic vector of the gauge potentials. Then it is clear from (5.104) that a sufficient condition for $\delta V = 0$ is $\delta\mathcal{L} = 0$[12], that holds if and only if

$$k^u\mathsf{D}_u\hat{\mathcal{P}}^x = U\hat{\mathcal{P}}^x\,, \tag{5.106}$$

where we added a hat to the quaternionic quantities that define the gaugings. Moreover the invariance of the kinetic term of the hyperscalars [51] leads to

[12]Note that, as in the FI case, $\delta\mathcal{L} = 0$ is in general sufficient but not necessary.

$$(\mathcal{L}_k \hat{k})^v = U \hat{k}^v \,, \tag{5.107}$$

where $\mathcal{L}$ denotes the Lie derivative. After choosing a specific model, these equations can in principle be solved for the parameters that define the linear combination of Killing vectors (5.105). In practice, (5.106) and (5.107) represent a highly constrained and very model-dependent system, and it is a priori not guaranteed that a nontrivial solution exists in general. In the FI limit, (5.106) boils down to the stabilization equation for the coupling constants $\mathcal{G}$ and (5.107) is trivially satisfied, as it must be.
An interesting class of these models are the $N = 2$ truncations of M-theory described in [3, 18]. In this case the solution of (5.106) and (5.107) could simplify the study of the attractor equations, necessary to work along the lines of [113], namely to compare the gravity side with the recent field theory results of [114–116].

5.4 Stationary dyonic black string in the STU model

In the supergravity theory with eight supercharges a particular interesting position is reserved to the STU model that comes both in $d = 4$ and $d = 5$ as an effective model of string compactification. With the r-map construction 2.3 the two cases are linked and this can be used to analyze solutions in the two frameworks. In particular, it is possible to use the rich techniques developed in $d = 4$ to generate more complicated solution physically relevant $d = 5$. We show an example of how this works in a very simple example.
In five dimensions STU model is characterized by the section constraint (5.108)

$$\mathcal{V} = h^1 h^2 h^3 = 1 \,, \tag{5.108}$$

that means a symmetric C_{IJK} tensor with only nontrivial component $C_{123} = 1$, up to permutations. The functions $h^I = h^I(\phi^1, \phi^2)$ are (5.118). By the r-map construction 2.3 it can be equivalently descibed in $d = 4$ FI gauged supergravity with a prepotential and vector-couplings

$$F = \frac{-X^1 X^2 X^3}{X^0} \,, \quad \mathcal{G} = (0, 0, 0, 0, 0, g_1, g_2, g_3)^t \,. \tag{5.109}$$

Domain wall solution in $d = 4$ and its uplift

In [78] various solutions to models in $d = 4$ $\mathcal{N} = 2$ gauged supergravity are found. In particular, for the prepotential $F = -X^1 X^2 X^3 / X^0$, the solution reads [13]

$$\mathrm{d}s^2 = -4b^2 \mathrm{d}t^2 + \frac{1}{b^2} \frac{y \mathrm{d}y^2}{cy + 2gp} + \frac{y^3}{b^2} (\mathrm{d}\theta^2 + \sinh^2 \theta \mathrm{d}\phi^2) \,, \tag{5.110}$$

where

[13] To make the comparison between 2.1 and [78] one has to take $g_{CK} \to g/2$ and $(G)\pi \to \frac{1}{8}$

$$b^4 = \frac{8g_1g_2g_3y^{\frac{9}{2}}}{H^0(cy+2gp)^{\frac{3}{2}}}\,, \quad H^0 = \frac{2q_0}{3g^2p^2y^{\frac{3}{2}}}(cy+2gp)^{\frac{1}{2}}(cy-gp)+h^0\,, \tag{5.111}$$

with field strenghts and scalars

$$F^0 = 4\mathrm{d}t\wedge\mathrm{d}(H^0)^{-1}\,, \quad F^I = \frac{p^I}{2}\sinh\theta\mathrm{d}\theta\wedge\mathrm{d}\phi\,, \quad z^i = i\tau^i = i\frac{\sqrt{g_1g_2g_3}}{\sqrt{2}g_I}\frac{\sqrt{H^0}y^{\frac{3}{4}}}{(cy+2gp)^{\frac{1}{4}}}\,. \tag{5.112}$$

This field configuration is $\frac{1}{4}$-BPS and the near horizon metric is a $\frac{1}{2}$-BPS attractor point $AdS_2\times H^2$. The range of the parameters is $q_0<0, p^I>0, c>0, gp>0, h^0>\frac{2|q_0|c^{\frac{3}{2}}}{3g^2p^2}$ and coupling constants and magnetic charges are constrained by the three equations $g_Ip^I = gp$ [14], so it has effectively only one magnetic and one electric free parameters. This implies that the configuration, imposing the physical requirement $g_I = g$, lives in the truncation, called t^3, characterized by the prepotential $F = -(X^1)^3/X^0$. Moreover this solution has an horizon for $y = 0$ and it is not asymptotically AdS_4.
A useful change of coordinates is

$$\frac{\sqrt{y}\mathrm{d}y}{(cy+2gp)^{\frac{1}{2}}} = \mathrm{d}Y\,, \quad t\to\frac{t}{2}\,, \tag{5.113}$$

for which the metric assumes the form

$$\mathrm{d}s^2 = -e^{2U}\mathrm{d}t^2 + e^{-2U}\left(\mathrm{d}Y^2 + e^{2\psi}\mathrm{d}\Omega^2_{H^2}\right)\,, \tag{5.114}$$

with $e^{2U} = b^2$ and $e^{2\psi} = y^3$, where y must be red as an implicit function of Y. Looking at sec:gen4 is immediate to find the graviphoton

$$F^0 = -\frac{1}{(H^0)^2}\frac{4q_0}{y^3}\mathrm{d}t\wedge\mathrm{d}Y = -e^{2(U-\psi)}I^{00}\frac{q_0}{2}\mathrm{d}t\wedge\mathrm{d}Y\,, \tag{5.115}$$

and all the informations on the fluxes can be described by the components of the fluxes on H^2

$$F_0 = \frac{q_0}{2}\sinh\theta\mathrm{d}\theta\wedge\mathrm{d}\phi\,, \quad F^I = \frac{p^I}{2}\sinh\theta\mathrm{d}\theta\wedge\mathrm{d}\phi\,. \tag{5.116}$$

The dictionary described in (2.3) can be used to uplift the field configuration (5.110), (5.112), in $d = 5$. With the change of radial variable $y = r^2$ the metric reads

$$\mathrm{d}s^2 = 2\left(\frac{1}{H^0b}\right)^{\frac{2}{3}}\left(\frac{1}{b^2}\frac{4r^4\mathrm{d}r^2}{cr^2+2gp}+\frac{r^6}{b^2}\mathrm{d}\Omega^2_{H^2}\right)+\frac{1}{4}(H^0b)^{\frac{4}{3}}\left(\mathrm{d}z^2-\frac{8\sqrt{2}}{H^0}\mathrm{d}t\mathrm{d}z\right)\,, \tag{5.117}$$

with the functions defined as

$$b^4 = \frac{8g_1g_2g_3r^9}{H^0(cr^2+2gp)^{\frac{3}{2}}}\,, \quad H^0 = \frac{2q_0}{3g^2p^2r^3}(cr^2+2gp)^{\frac{1}{2}}(cr^2-gp)+h^0\,. \tag{5.118}$$

The fluxes and the scalars are

[14] No summation over I.

$$F^I = \frac{p^I}{\sqrt{2}} \sinh\theta \mathrm{d}\theta \wedge \mathrm{d}\phi\,, \quad h^I = \frac{(g_1 g_2 g_3)^{\frac{1}{3}}}{g_I}\,. \tag{5.119}$$

The metric (5.117) can be simplified to

$$\mathrm{d}s^2 = \frac{2}{(g_1g_2g_3)^{\frac{2}{3}}}\frac{\mathrm{d}r^2}{r^2} + \frac{cr^2+2gp}{2(g_1g_2g_3)^{\frac{2}{3}}}\mathrm{d}\Omega^2_{H^2} + 2\sqrt{2}\frac{(g_1g_2g_3)^{\frac{1}{3}}r^3}{\sqrt{cr^2+2gp}}\mathrm{d}z\left(\frac{H^0}{4\sqrt{2}}\mathrm{d}z - \mathrm{d}t\right)\,, \tag{5.120}$$

and it appears as a particular case of the general class described in [148]. It's worth noting that this connection permits to generate static four-dimensional black hole solution that break the $SO(2,1)$ symmetry, through the explicit dependence from other coordinates.

The supersymmetry variations of the theory reads [149][15]

$$\begin{aligned} \delta\psi_\mu &= \left(D_\mu + \frac{i}{8}h_I(\Gamma_\mu{}^{\nu\rho} - 4\delta_\mu{}^\nu\Gamma^\rho)F^I_{\nu\rho} + \frac{1}{6\sqrt{2}}\Gamma_\mu h^I g_I\right)\epsilon\,, \\ \delta\lambda_i &= \left(\frac{3}{8}\Gamma^{\mu\nu}F^I_{\mu\nu}\partial_i h_I - \frac{i}{2}\mathcal{G}_{ij}\Gamma^\mu\partial_\mu\phi^i + \frac{1}{2\sqrt{2}}g_I\partial_i h^I\right)\epsilon\,, \end{aligned} \tag{5.121}$$

where

$$D_\mu\epsilon = \left(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\Gamma^{ab} - \frac{i}{2\sqrt{2}}g_I A^I_\mu\right)\epsilon\,. \tag{5.122}$$

and the configuration (5.117) and (5.119) satisfies them (5.121) with a Killing spinor

$$\epsilon = Y(r)(1+i\Gamma^{32})(1-\Gamma^1)\epsilon_0\,, \tag{5.123}$$

where ϵ_0 is a generic constant Dirac spinor and therefore it is $\frac{1}{4}$-BPS, once the quantization condition $g_I p^I = 2$ is imposed. The fuction H^0 remains undeterminated by the BPS equation, it is fixed by the zz component of the Einstein equation as required for the null class [148,150].

It has an horizon in $r=0$, where the metric approach $AdS_3 \times H^2$, while asymptotically becomes a magnetic AdS_5. In the null class of minimal gauged supergravity, this solution is the most general with these limits. This happens in a unusual set of coordinates, infact in the near horizon limit, changing the coordinates as

$$t \to \frac{3\sqrt{3}}{4\sqrt{2}}(g_1g_2g_3)^{\frac{1}{3}}t\,, \quad z \to \frac{\sqrt{3gp}}{(g_1g_2g_3)^{\frac{1}{6}}}z\,, \quad r^3 \to \rho\,, \tag{5.124}$$

the metric assumes the nice form

$$\mathrm{d}s^2 = R^2_{H^2}\mathrm{d}\Omega^2_{H^2} + (\mathrm{d}z + 2l\rho\mathrm{d}t)^2 + l^2\left(\frac{\mathrm{d}\rho^2}{4\rho^2} - 4\rho^2\mathrm{d}t^2\right)\,, \tag{5.125}$$

where the radii read

[15] We use the metric $\eta^{ab} = (-,+,+,+,+)$, the Clifford algebra $[\Gamma^a,\Gamma^b]_+ = 2\eta^{ab}$, the spin connection $\omega_{\mu ab}$ and $\Gamma^{a_1\cdots a_n} = \Gamma^{[a_1}\ldots\Gamma^{a_n]}$.

$$R_{H^2} = \frac{\sqrt{gp}}{(g_1 g_2 g_3)^{\frac{1}{6}}}\,, \quad l = \frac{2\sqrt{2}}{3(g_1 g_2 g_3)^{\frac{1}{3}}}\,. \tag{5.126}$$

The metric (5.125) is a particular set of coordinates that shows a product space of an H^2 and an Hopf-like fibration $AdS_3 \to AdS_2$. The cosmological constant is $\Lambda = -2/l^2 = -\frac{9}{4}(g_1 g_2 g_3)^{\frac{2}{3}}$.
The limit for $r \to \infty$ shows a metric

$$\mathrm{d}s^2 = \frac{cr^2}{2(g_1 g_2 g_3)^{\frac{2}{3}}}\mathrm{d}\Omega^2_{H^2} + \frac{2}{(g_1 g_2 g_3)^{\frac{2}{3}}}\frac{\mathrm{d}r^2}{r^2} + \frac{(g_1 g_2 g_3)^{\frac{1}{3}} r^2}{2\sqrt{c}}\mathrm{d}z\left(\left(\frac{2q_0 c^{\frac{3}{2}}}{3g^2p^2} + h^0\right)\mathrm{d}z - 4\sqrt{2}\mathrm{d}t\right)\,. \tag{5.127}$$

that with approprite rescaling and a linear diffeomorphism reads

$$\mathrm{d}s^2 = \frac{\alpha^2}{r^2}\mathrm{d}r^2 + \frac{r^2}{\alpha^2}\mathrm{d}z^2 - \frac{r^2}{\alpha^2}\mathrm{d}t^2 + \frac{r^2}{\beta^2}\mathrm{d}\Omega^2_{H^2}\,, \tag{5.128}$$

with $\alpha = \sqrt{2}/(g_1 g_2 g_3)^{\frac{1}{3}}$ and $\beta = \sqrt{2}(g_1 g_2 g_3)^{\frac{1}{3}}/\sqrt{c}$. Therefore asymptotically we have a conformal boundary

$$R_{\mu\nu} = \frac{2}{3}\Lambda g_{\mu\nu} + O(r^0)_{\mu\nu}\,, \quad \Lambda = -3(g_1 g_2 g_3)^{\frac{2}{3}}\,. \tag{5.129}$$

The metric (5.128) is called magnetic AdS_5 [151], because it is not diffeomorphic to exactly AdS_5, but in the limit in which r is large, the $O(r^0)_{\mu\nu}$ can be negleted.

Residual symmetries and dyonic black string

The residual symmetries of the $d = 4$ theory consists in evaluating the stabilizer of the U-duality group acting in the symplectic representation on the vector of the couplings of the theory $\mathcal{G} = (0,0,0,0,0,g,g,g)^t$. In this case the stabilizer algebra reads

$$T(a_1,a_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_1-a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_1 & -a_2 & a_1+a_2 \\ 0 & 0 & a_1+a_2 & -a_2 & 0 & 0 & 0 & 0 \\ 0 & a_1+a_2 & 0 & -a_1 & 0 & 0 & 0 & 0 \\ 0 & -a_2 & -a_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\,, \tag{5.130}$$

a two dimensional abelian nilpotent subalgebra of order three of $sp(8,\mathbb{R})$ algebra.
The informations on the fluxes is recollected in the symplectic vector $\mathcal{Q}$, defined with the dual field strenghts

$$F_{\Lambda\mu\nu} = R_{\Lambda\Sigma}F^{\Sigma}{}_{\mu\nu} - \frac{1}{2}I_{\Lambda\Sigma}\epsilon_{\mu\nu\rho\sigma}\sqrt{-g}F^{\Sigma\rho\sigma}\,. \tag{5.131}$$

For a static solution

$$\mathrm{d}s^2 = -e^{2U}\mathrm{d}t^2 + e^{-2U}\left(\mathrm{d}r^2 + e^{2\psi}\mathrm{d}\Omega^2_{H^2}\right)\,, \tag{5.132}$$

the Maxwell equation is solved for

$$F^\Lambda{}_{tr} = e^{2(U-\psi)} I^{\Lambda\Sigma}(R_{\Sigma\Gamma}p^\Gamma - q_\Sigma)\,, \qquad F^\Lambda{}_{\theta\phi} = p^\Lambda f_\kappa(\theta)\,, \tag{5.133}$$

with corresponding dual field strenghts

$$F_{\Lambda tr} = e^{2(U-\psi)}\left(I_{\Lambda\Sigma}p^\Sigma + R_{\Lambda\Gamma}I^{\Gamma\Omega}R_{\Omega\Sigma}p^\Sigma - R_{\Lambda\Gamma}I^{\Gamma\Omega}q_\Omega\right)\,, \qquad F_{\Lambda\theta\phi} = q_\Lambda f_\kappa(\theta)\,. \tag{5.134}$$

The components $\theta\phi$ fix uniquely the fluxes.

Following the recipe of 5.3 one can generate a new configuration from the seed solution (5.132) and the respective fluxes (5.116). Considering that $q_0 = -|q_0|$, starting form the fluxes [16]

$$\mathcal{Q} = (0, p, p, p, -|q_0|, 0, 0, 0)^t\,, \tag{5.135}$$

the action of $U = e^{T(a_1,a_2)}$ (5.130) generates

$$\mathcal{Q}' = \begin{pmatrix} 0 \\ p \\ p \\ p \\ -|q_0| + (a_1a_2 + a_1^2 + a_2^2)p \\ -a_1p \\ -a_2p \\ (a_1+a_2)p \end{pmatrix}\,. \tag{5.136}$$

The scalars acquire a constant axion

$$z^{1\prime} = z^1 + a_1\,, \quad z^{2\prime} = z^2 + a_2\,, \quad z^{3\prime} = z^3 - a_1 - a_2\,. \tag{5.137}$$

From (5.136) one could think that choosing properly the parameters a_1, a_2 the graviphoton could be set to zero, but the form of the field strenghts in presence of axions read

$$\begin{aligned} F^\Lambda &= \frac{e^{2U-2\psi}}{2} I^{\Lambda\Sigma}(R_{\Sigma\Gamma}p^\Gamma - q_\Sigma)\mathrm{d}t\wedge\mathrm{d}Y + \frac{p^\Lambda}{2}\sinh\theta\mathrm{d}\theta\wedge\mathrm{d}\phi = \\ &= \frac{1}{2}S^\Lambda\mathrm{d}t\wedge\mathrm{d}Y + \frac{p^\Lambda}{2}\sinh\theta\mathrm{d}\theta\wedge\mathrm{d}\phi\,, \end{aligned} \tag{5.138}$$

where the vectors are

$$S^\Lambda = \frac{|q_0|}{\tau^1\tau^2\tau^3}(1, a_1, a_2, -a_1-a_2)^t\,, \quad p^\Lambda = (0, p, p, p)^t\,. \tag{5.139}$$

The form of the last vector shows explicitly that for $q_0 = 0$ the solution cannot be dyonic. In terms of the r and t coordinates, the metric of the uplift is

[16] Note that the restriction made $g_1 = g_2 = g_3$ plus constraints $g_I p^I = gp$ impose $p^1 = p^2 = p^3 = p$.

$$\mathrm{d}s^2 = \frac{2}{g^2}\frac{\mathrm{d}r^2}{r^2} + \frac{cr^2+2gp}{2g^2}\mathrm{d}\Omega^2_{H^2} + 2\sqrt{2}\frac{gr^3}{\sqrt{cr^2+2gp}}\mathrm{d}z\left(\frac{H^0}{4\sqrt{2}}\mathrm{d}z - \mathrm{d}t\right)\,, \tag{5.140}$$

The uplifted field strenghts and scalars result

$$F^I = \frac{b^2}{r^6}S^I\frac{2\sqrt{2}r^2}{(cr^2+2gp)^{\frac{1}{2}}}\mathrm{d}t\wedge\mathrm{d}r + \frac{p^I}{\sqrt{2}}\sinh\theta\mathrm{d}\theta\wedge\mathrm{d}\phi\,,\quad h^I = 1\,, \tag{5.141}$$

where

$$S^I = \frac{|q_0|}{\tau^1\tau^2\tau^3}(a_1, a_2, -a_1-a_2)^t\,,\quad p^I = (p,p,p)^t\,. \tag{5.142}$$

and τ^I are defined in (5.112), up to the definition of the new coordinate $y = r^2$. This solution results to be a flow between magnetic AdS_5 and $AdS_3\times H^2$, in fact the metric remains untouched by the duality rotation. New features are the two charge's parameters that however maintain the same degree of supersymmetry of the configuration, as can be easily seen inserting (5.142) in the supersymmetry variations (5.121). Moreover the structure (5.142) shows that the new solution can no more be embedded in the t^3 truncation of the STU model.

Dyonic black string rotating along ∂_ϕ and ∂_z

The idea is to generate a genuine rotating black string solution starting from the previous one applying the same solution generanting technique, but this time along the other killing vector ∂_ϕ. The seed is (5.140) with respective fluxes and scalars (5.141). After the reduction the four-dimensional configuration reads

$$\begin{aligned}\mathrm{d}s^2 &= \sinh\theta\left(\frac{\sqrt{2}\left(cr^2+2gp\right)^{\frac{1}{2}}}{g^3}\frac{\mathrm{d}r^2}{r^2} + \frac{\left(cr^2+2gp\right)^{\frac{3}{2}}}{2\sqrt{2}g^3}\mathrm{d}\theta^2 + 2r^3\mathrm{d}z\left(\frac{H^0}{4\sqrt{2}}\mathrm{d}z - \mathrm{d}t\right)\right)\,,\\ A^\Lambda &= \left(0, \frac{4q_I}{H^0}\mathrm{d}t\right)\,,\quad z^i = \frac{p^I}{\sqrt{2}}\cosh\theta + i\frac{\left(cr^2+2gp\right)^{\frac{1}{2}}\sinh\theta}{\sqrt{2}g_I}\,,\end{aligned} \tag{5.143}$$

where $q_3 = -q_1 - q_2$. Now one can apply the same duality transformation of the previous section. The graviphoton is switched on

$$\mathcal{Q} = (0,0,0,0,0,q_1,q_2,-q_1-q_2) \to \mathcal{Q}' = (0,0,0,0,\omega,q_1,q_2,-q_1-q_2)^t\,, \tag{5.144}$$

where $\omega = -a_1(2q_1+q_2) - a_2(q_1+2q_2)$, and the scalar fields aquire a real constant part as in (5.137). The lifting of the solution reads

$$\begin{aligned}\mathrm{d}s^2 = &\frac{2}{g^2}\frac{\mathrm{d}r^2}{r^2} + \frac{cr^2+2gp}{2g^2}\mathrm{d}\Omega^2_{H^2} + \frac{2\sqrt{2}g}{(cr^2+2gp)^{\frac{1}{2}}}r^3\mathrm{d}z\left(\frac{H^0}{4\sqrt{2}}\mathrm{d}z - \mathrm{d}t\right)\\ &+ \frac{cr^2+2gp}{2g^2}\sinh^2\theta\left(\frac{8\sqrt{2}\omega}{H^0}\mathrm{d}\phi\mathrm{d}t + \frac{32\omega^2}{(H^0)^2}\mathrm{d}t^2\right)\,,\end{aligned} \tag{5.145}$$

with functions defined in (5.118) and the gauge fields and scalars, up to gauge transformation, read

$$A^I = \frac{p^I}{\sqrt{2}} \cosh\theta \mathrm{d}\phi + \frac{4q_I}{H^0}\mathrm{d}t + \left(s^I + \frac{p^I}{\sqrt{2}}\cosh\theta\right)\frac{4\omega}{H^0}\mathrm{d}t\,, \quad h^I = 1\,, \tag{5.146}$$

where $s^I = (a_1, a_2, -a_1 - a_2)$.
The near horizon limit of (5.145), $r \to 0$, leads to the metric

$$\mathrm{d}s^2 = \frac{2}{g^2}\frac{\mathrm{d}r^2}{r^2} + \frac{p}{g}\mathrm{d}\Omega^2_{H^2} + \frac{2g}{\sqrt{gp}}r^3\mathrm{d}z\left(\frac{\tilde{H}}{4\sqrt{2}}\mathrm{d}z - \mathrm{d}t\right) + \frac{p\sinh^2\theta}{g}\left(\frac{8\sqrt{2}\omega}{\tilde{H}}\mathrm{d}\phi\mathrm{d}t + \frac{32\omega^2}{\tilde{H}^2}\mathrm{d}t^2\right)\,, \tag{5.147}$$

where

$$\tilde{H} = h^0 - \frac{2\sqrt{2}|q_0|}{3\sqrt{gp}}\frac{1}{r^3}\,. \tag{5.148}$$

The metric (5.145) at infinity becomes

$$\begin{aligned}\mathrm{d}s^2 = \frac{2}{g^2}\frac{\mathrm{d}r^2}{r^2} + \frac{cr^2}{2g^2}\mathrm{d}\Omega^2_{H^2} + \frac{2\sqrt{2}g}{\sqrt{c}}r^2\mathrm{d}z\left(\frac{H^0}{4\sqrt{2}}\mathrm{d}z - \mathrm{d}t\right) \\ + \frac{cr^2\sinh^2\theta}{2g^2}\left(\frac{8\sqrt{2}\omega}{H^0}\mathrm{d}\phi\mathrm{d}t + \frac{32\omega^2}{(H^0)^2}\mathrm{d}t^2\right)\,,\end{aligned} \tag{5.149}$$

that with approprite rescaling and a linear diffeomorphism reads

$$\mathrm{d}s^2 = \frac{\alpha^2}{r^2}\mathrm{d}r^2 + \frac{r^2}{\alpha^2}\mathrm{d}z^2 - \frac{r^2}{\alpha^2}\mathrm{d}t^2 + \frac{r^2}{\beta^2}\mathrm{d}\Omega^2_{H^2}\,, \tag{5.150}$$

with $\alpha = \sqrt{2}/g^{\frac{1}{3}}$ and $\beta = \sqrt{2}g^{\frac{1}{3}}/\sqrt{c}$. The metric (5.149) approaches again a magnetic AdS_5 (5.129) and the ω parameter is linked to a rotation property of the solution along the ϕ direction.

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