ONLINE SUPPLEMENTAL APPENDIX: NONPARAMETRIC COVARIATE-ADJUSTED RESPONSE-ADAPTIVE DESIGN BASED ON A FUNCTIONAL URN MODEL

A. Analytic expressions. We now derive some useful analytic expressions for $\mathbf{X}_n$ and D_n. Using (2.3), we can express $\mathbf{X}_n$ as follows: on the set $\{\bar{X}_n^k = 1\}$, $k \in \{1,..,d\}$, we have, for any $j \in \{1,..,d\}$,

(A.1)

$$X_n^j(t) = \mathbb{P}\left(\sum_{i=1}^{j-1} Z_{n-1}^i(t) < U_n \leq \sum_{i=1}^{j} Z_{n-1}^i(t) \mid \mathcal{F}_{n-1}, \bar{X}_n^k = 1\right)$$
$$= \frac{\left(\min\left\{\sum_{i=1}^{j} Z_{n-1}^i(t); \sum_{i=1}^{k} Z_{n-1}^i(T_n)\right\} - \max\left\{\sum_{i=1}^{j-1} Z_{n-1}^i(t); \sum_{i=1}^{k-1} Z_{n-1}^i(T_n)\right\}\right)^+}{Z_{n-1}^k(T_n)},$$

where by convention $\sum_{i=1}^{0}(\cdot) = 0$. Note that $w(\mathbf{X}_n(t)) = 1$ for all $t \in \tau$ and $\mathbf{X}_n(T_n) = \bar{\mathbf{X}}_n$.

The definition of D_n in (2.8) may be simplified when $\hat{\pi}_s^k$, for some $s \in \tau$ and $k \in \{1,..,d\}$, is absolutely continuous or discrete. In fact, in the first case the QF is bijective, i.e. $(\hat{Q}_s^k)^{-1}(y) \equiv \hat{F}_s^k(y)$ for any $y \in S^k$, and hence from (2.8), on the sets $\{T_n = s\}$, $\{\bar{X}_n^k = 1\}$ and $\{\xi_n^k = y\}$, D_n reduces to

$$D_n^{ij}(t) \;=\; \hat{u}^{ij}(\;\hat{Q}_t^j(\;\hat{F}_s^k(y)\;)\;). \tag{A.2}$$

When $\hat{\pi}_s^k$ is discrete, for some $y \in S^k$ we have $\hat{\pi}_s^k(y) > 0$, and hence

$$(\hat{Q}_s^k)^{-1}(y) \;=\; \Big(\;\hat{F}_s^k(y^-)\;,\;\hat{F}_s^k(y)\;\Big),$$

where $y^- := (y - \epsilon)$ with $\epsilon > 0$ arbitrary small. Thus, from (2.7), on the sets $\{T_n = s\}$, $\{\bar{X}_n^k = 1\}$ and $\{\xi_n^k = y\}$, D_n reduces to

$$\begin{aligned} D_n^{ij} &= \Big(\hat{F}_s^k(y) - \hat{F}_s^k(y^-)\Big)^{-1} \int_{\hat{F}_s^k(y^-)}^{\hat{F}_s^k(y)} \hat{u}_t^{ik}(\hat{Q}_t^j(v))dv \\ &= \Big(\hat{\pi}_s^k(y)\Big)^{-1} \int_{\hat{F}_s^k(y^-)}^{\hat{F}_s^k(y)} \hat{u}_t^{ik}(\hat{Q}_t^j(v))dv, \end{aligned} \tag{A.3}$$

where we recall that $\hat{\pi}_s^k$ is the estimator of $\pi_s^k(y) = \mathbb{P}(\xi_n^k = y | T_n = s)$.

B. Proofs. This section is concerned with the proofs of the results presented in Section 3.

B.1. *Proof of the first-order asymptotic results.* We now prove Theorem 3.1. We first need to introduce some notation concerning the eigen-structure of $H(t)$.

For any $t \in \tau$, $H(t)$ is diagonalizable by (A2). Then there exists a nonsingular matrix $\widetilde{U}(t)$ such that $\widetilde{U}^\top(t)H(t)(\widetilde{U}^\top(t))^{-1}$ is diagonal with elements $\lambda_j(t) \in Sp(H(t))$. Notice that each column $\mathbf{u}_j(t)$ of $\widetilde{U}(t)$ is a left eigenvector of $H(t)$ associated with $\lambda_j(t)$. WLOG, we set $\|\mathbf{u}_j\|(t) = 1$. Moreover, when the multiplicity of some $\lambda_j(t)$ exceeds one, we assume the corresponding eigenvectors to be orthogonal. Then if we define $\widetilde{V}(t) = (\widetilde{U}^\top(t))^{-1}$, each column $\mathbf{v}_j(t)$ of $\widetilde{V}(t)$ is a right eigenvector of $H(t)$ associated with $\lambda_j(t)$ such that

$$\mathbf{u}_j^\top \mathbf{v}_j = 1 \quad \text{and} \quad \mathbf{u}_h^\top \mathbf{v}_j = 0,\ \forall h \neq j. \tag{B.1}$$

These constraints, combined with the assumptions in (A2) on H (precisely, nonnegativity, constant balance and irreducibility) imply, by the Frobenius-Perron Theorem, that, for any $t \in \tau$, $\lambda_1(t) = 1$ is an eigenvalue of $H(t)$ with multiplicity one, $\max_{j>1} \Re e(\lambda_j) < 1$ and

$$\mathbf{u}_1 = N^{-1/2}\mathbf{1}, \qquad N^{-1/2}\mathbf{1}^\top \mathbf{v}_1 = 1, \qquad v_1^j > 0\ \forall j = 1,..,d.$$

Because $\mathbf{v}(t) \in \mathcal{S}$, or equivalently $w(\mathbf{v}(t)) = 1$, in the statement of Theorem 3.1, then $\mathbf{v} = N^{-1/2}\mathbf{v}_1$.

In the sequel, we will use U and V to indicate the sub-matrices of $\widetilde{U}$ and $\widetilde{V}$, respectively, whose columns for any $t \in \tau$ are the left and the right eigenvectors of $H(t)$ associated with $Sp(H(t)) \setminus \{1\}$, given by $\{\mathbf{u}_2(t), .., \mathbf{u}_N(t)\}$ and $\{\mathbf{v}_2(t), .., \mathbf{v}_N(t)\}$, respectively.

Now, given the eigen-structure of H presented here, the matrix $\mathbf{v}\mathbf{1}^\top$ has real entries and the following relations hold:

$$V^\top \mathbf{1} = U^\top \mathbf{v} = \mathbf{0}, \quad V^\top U = U^\top V = I \quad \text{and} \quad I = \mathbf{v}\mathbf{1}^\top + VU^\top, \tag{B.2}$$

where the identity matrices above have dimensions $(d-1)$ and d, respectively. As a consequence of (B.2), the matrix $U(t)V^\top(t)$ has real entries for any $t \in \tau$. Moreover, denoting by $\Lambda(t)$ the diagonal matrix whose elements are $\lambda_j(t) \in Sp(H(t)) \setminus \{1\}$, we can decompose the functional matrix H as follows:

$$H \;=\; \mathbf{v}\mathbf{1}^\top \;+\; V\Lambda U^\top. \tag{B.3}$$

With this notation in mind, we are now ready to present the proof of the first-order results.

Proof of Theorem 3.1. The structure of the proof of part (a) is analogous to that in [1, Theorem 4.1]. Consider the urn dynamics expressed in (2.9) as follows: let $\mathbf{Y}_0 = \mathbf{1}$ and for any $n \geq 1$

$$\mathbf{Y}_n \;=\; \mathbf{Y}_{n-1} \;+\; D_n\mathbf{X}_n. \tag{B.4}$$

From (B.4), we can derive the following decomposition:

$$\mathbf{Y}_n \;=\; \mathbf{Y}_{n-1} \;+\; H\mathbf{Z}_{n-1} \;+\; \Delta\mathbf{M}_{Z,n} \;+\; \mathbf{R}_{Z,n}, \tag{B.5}$$

where

(1) $\Delta\mathbf{M}_{Z,n} := (D_n - H_n)\mathbf{X}_n + H(\mathbf{X}_n - \mathbf{Z}_{n-1})$ is a martingale increment, since

$$\mathbb{E}[(D_n - H_n)|\mathcal{F}_{n-1}, T_n, \bar{\mathbf{X}}_n] \;=\; \mathbb{E}[(\mathbf{X}_n - \mathbf{Z}_{n-1})|\mathcal{F}_{n-1}, T_n] \;=\; 0.$$

(2) $\mathbf{R}_{Z,n} := (H_n - H)\mathbf{X}_n$ is a remainder term that converges to zero a.s. due to the fact that, since $\mathbf{X}_n \in \mathcal{S}$, $\mathbb{P}(\|\mathbf{X}_n\| \leq 1) = 1$ a.s. and by Assumption (A2).

Let $r_n := (d+n)^{-1}$. By Assumption (A1), $w(\mathbf{Y}_n) = (d+n)$ with probability one for any $n \geq 0$, and hence $\mathbf{Z}_n = r_n\mathbf{Y}_n$. Then, multiplying the dynamics (B.5) by r_n and using $r_n r_{n-1}^{-1} = (1 - r_n)$, we obtain

$$\mathbf{Z}_n \;=\; [I - r_n(I - H)]\mathbf{Z}_{n-1} \;+\; r_n\Delta\mathbf{M}_{Z,n} \;+\; r_n\mathbf{R}_{Z,n}.$$

Moreover, since $(I - H)\mathbf{v} = \mathbf{0}$ and defining $\mathbf{W}_n := (\mathbf{Z}_n - \mathbf{v})$, we obtain the following expression:

$$\mathbf{W}_n \;=\; [I - r_n(I - H)]\mathbf{W}_{n-1} \;+\; r_n\Delta\mathbf{M}_{Z,n} \;+\; r_n\mathbf{R}_{Z,n}. \tag{B.6}$$

Let us consider the $(d-1)$-dimensional complex process $\{\mathbf{W}_{U,n}; n \geq 1\}$ defined as $\mathbf{W}_{U,n} = U^\top\mathbf{W}_n$. The relation $\mathbf{W}_n = V\mathbf{W}_{U,n}$ is a consequence of (B.2) and $\mathbf{1}^\top\mathbf{W}_n = (\mathbf{1}^\top\mathbf{Z_n}) - (\mathbf{1}^\top\mathbf{v}) = 0$. Hence, to prove that $\int_\tau \|\mathbf{W}_n(t)\|\nu(dt) \stackrel{a.s.}{\to} 0$, it is enough to show that

$$\int_\tau \|\mathbf{W}_{U,n}(t)\|\nu(dt) \stackrel{a.s.}{\to} 0.$$

To this purpose, we observe that the dynamics of $\mathbf{W}_{U,n} = U^\top\mathbf{W}_n$ can be derived from (B.6), so obtaining

$$\mathbf{W}_{U,n} \;=\; [I - r_n(I - \Lambda)]\mathbf{W}_{U,n-1} \;+\; r_n U^\top\Delta\mathbf{M}_{Z,n} \;+\; r_n U^\top\mathbf{R}_{Z,n},$$

where I here indicates an identity matrix of dimension $(d-1)$. Hence, using Assumption (A2) and $\mathbb{E}[\Delta\mathbf{M}_{Z,n}\,|\,\mathcal{F}_{n-1}] = 0$, we have

$$\begin{aligned}\mathbb{E}\left[\|\mathbf{W}_{U,n}\|^2|\mathcal{F}_{n-1}\right] \;&=\; \mathbb{E}\left[\overline{\mathbf{W}}_{U,n}^\top\,\mathbf{W}_{U,n}\,|\,\mathcal{F}_{n-1}\right] \;=\\ \overline{\mathbf{W}}_{U,n-1}^\top\,\mathbf{W}_{U,n-1} &- r_n\overline{\mathbf{W}}_{U,n-1}^\top\left(2I - \overline{\Lambda} - \Lambda\right)\mathbf{W}_{U,n-1} + r_n n^{-\alpha}\psi_n,\end{aligned}$$

where $\{\psi_n; n \geq 1\}$ is a suitable bounded sequence of $\mathcal{F}_{n-1}$-measurable random variables. Now, since $\mathcal{R}e(\lambda_j(t)) < 1$ for any $\lambda_j(t) \in Sp(H(t)) \setminus \{1\}$ and $t \in \tau$, the matrix $2I - (\overline{\Lambda}(t) + \Lambda(t))$ is positive definite and hence we can write

$$\mathbb{E}\left[\int_\tau \|\mathbf{W}_{U,n}(t)\|^2\nu(dt)\,|\,\mathcal{F}_{n-1}\right] \leq \int_\tau \|\mathbf{W}_{U,n-1}(t)\|^2\nu(dt) \;+\; O(n^{-(1+\alpha)}).$$

Since $\sum_n n^{-(1+\alpha)} < +\infty$, we can conclude that the real stochastic process $\int_\tau \|\mathbf{W}_{U,n}(t)\|^2\nu(dt)$ is a positive almost supermartingale and so it converges almost surely, and in mean since it is also bounded (see [10]).

In order to prove that the limit is zero, we show the sufficient condition that

$$\mathbb{E}[\int_{\tau} \|\mathbf{W}_{U,n}(t)\|^2 \nu(dt)]$$

converges to zero. To this end, we observe that, from the above computations, we obtain

$$E[\|\mathbf{W}_{U,n}\|^2] \ \leq \ E[\overline{\mathbf{W}}_{U,n-1}^{\top}(I - r_n(I - \overline{\Lambda}))(I - r_n(I - \Lambda))\mathbf{W}_{U,n-1}] \ + \ n^{-(1+\alpha)}C_1$$

for a suitable constant $C_1 \geq 0$. Then, we note that the elements of the diagonal matrix above can be written as follows:

$$[(I - r_n(I - \overline{\Lambda}))(I - r_n(I - \Lambda))]^{jj} \ = \ 1 - 2r_n(1 - \mathcal{R}e(\lambda_j)) + r_n^2 \,|1 - \lambda_j|^2.$$

Setting $a_j(t) := 1 - \mathcal{R}e(\lambda_j(t))$ and $a^*(t) := \min_{j>1} a_j(t)$, we have that

$$\begin{aligned}
&\mathbb{E}[\overline{\mathbf{W}}_{U,n-1}^{\top}(I - r_n(I - \overline{\Lambda}))(I - r_n(I - \Lambda))\mathbf{W}_{U,n-1}] \ \leq \\
&\sum_{j=2}^{N}(1 - 2a_j r_n)\mathbb{E}[\overline{W}_{U,n-1}^{j} W_{U,n-1}^{j}] \ + C_2 n^{-(1+\alpha)} \ \leq \\
&(1 - 2a^* r_n)\mathbb{E}[\|\mathbf{W}_{U,n}\|^2] \ + C_2 n^{-(1+\alpha)},
\end{aligned}$$

for a suitable constant $C_2 \geq 0$. Since for any $t \in \tau$ $\max_{j>1} \mathcal{R}e(\lambda_j(t)) < 1$, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\nu(A_\delta) > 1 - \epsilon$, where $A_\delta := \{t \in \tau, a^*(t) > \delta\}$. Denoting $q_{\delta,n} := \mathbb{E}[\int_{A_\delta} \|\mathbf{W}_{U,n}(t)\|^2 \nu(dt)]$, we have

$$q_{\delta,n} \ \leq \ (1 - 2\delta r_n) q_{\delta,n-1} \ + \ (C_1 + C_2) n^{-(1+\alpha)}, \tag{B.7}$$

which implies $\lim_n q_{\delta,n} = 0$ (see [5]). Hence, for any $\epsilon > 0$ we have proved

$$\mathbb{E}\left[\int_{\tau} \|\mathbf{W}_{U,n}(t)\|^2 \nu(dt)\right] \ \leq \ \epsilon \ + \ q_{\delta,n} \ \to \ \epsilon.$$

This concludes the proof of part (a).

Concerning part (b), consider the decomposition $(\mathbf{N}_{t,n}/w(\mathbf{N}_{t,n}) - \mathbf{v}(t)) = (\mathbf{A}_{1,n}(t) + \mathbf{A}_{2,n}(t))$, where

$$\begin{aligned}
\mathbf{A}_{1,n}(t) \ &:= \ \frac{\sum_{i=1}^{n} \mathbb{1}_{\{T_i = t\}}(\bar{\mathbf{X}}_i - \mathbf{Z}_{i-1}(T_i))}{\sum_{j=1}^{n} \mathbb{1}_{\{T_j = t\}}}, \\
\mathbf{A}_{2,n}(t) \ &:= \ \frac{\sum_{i=1}^{n} \mathbb{1}_{\{T_i = t\}}(\mathbf{Z}_{i-1}(t) - \mathbf{v}(t))}{\sum_{j=1}^{n} \mathbb{1}_{\{T_j = t\}}}.
\end{aligned}$$

First, using [4, Theorem 1] and the assumption $\sum_{j=1}^n \mu_{j-1}(\{t\}) \overset{a.s.}{\to} \infty$, it follows that $\sum_{j=1}^n \mathbb{1}_{\{T_j=t\}} \overset{a.s.}{\to} \infty$. Hence, we can write that, for any $n_0 \geq 1$,

$$\limsup_{n\to\infty} \|\mathbf{A}_{2,n}(t)\| \leq \sup_{i\geq n_0} \|\mathbf{Z}_{i-1}(t) - \mathbf{v}(t)\|.$$

Then, $\|\mathbf{A}_{2,n}(t)\| \overset{a.s.}{\to} 0$ as a consequence of part (a). To deal with the term $\mathbf{A}_{1,n}(t)$, consider the martingale process $\{\tilde{\mathbf{A}}_{1,n}(t); n \geq 1\}$ defined as follows:

$$\tilde{\mathbf{A}}_{1,n}(t) \ := \ \sum_{i=1}^n \frac{\mathbb{1}_{\{T_i=t\}}(\bar{\mathbf{X}}_i - \mathbf{Z}_{i-1}(T_i))}{\sum_{j=1}^i \mathbb{1}_{\{T_j=t\}}},$$

and notice that $\tilde{\mathbf{A}}_{1,n}(t)$ converges a.s. since with probability one its bracket process is bounded: $\sum_{i=1}^\infty \mathbb{E}[\|\Delta\tilde{\mathbf{A}}_{1,i}(t)\|^2|\mathcal{F}_{i-1}] \leq d\sum_{i=1}^\infty i^{-2} < \infty$. Then, applying the Cesàro Lemma it follows that $\|\mathbf{A}_{1,n}(t)\| \overset{a.s.}{\to} 0$.

Concerning part (c), consider the decomposition $(\mathbf{N}_n/n - \int_\tau \mu(dt)\mathbf{v}(t)) = (\mathbf{B}_{1,n} + \mathbf{B}_{2,n} + \mathbf{B}_{3,n})$, where

$$\begin{aligned}
\mathbf{B}_{1,n} &:= n^{-1}\sum_{i=1}^n (\bar{\mathbf{X}}_i - \int_\tau \mu_{i-1}(dt)\mathbf{Z}_{i-1}(t)),\\
\mathbf{B}_{2,n} &:= n^{-1}\sum_{i=1}^n \int_\tau (\mu_{i-1}(dt) - \mu(dt))\mathbf{Z}_{i-1}(t),\\
\mathbf{B}_{3,n} &:= n^{-1}\sum_{i=1}^n \int_\tau (\mathbf{Z}_{i-1}(t) - \mathbf{v}(t))\mu(dt).
\end{aligned}$$

To deal with the term $\mathbf{B}_{1,n}$, consider the martingale process $\{\tilde{\mathbf{B}}_{1,n}(t); n \geq 1\}$ defined as follows:

$$\tilde{\mathbf{B}}_{1,n} \ := \ \sum_{i=1}^n i^{-1}(\bar{\mathbf{X}}_i - \int_\tau \mu_{i-1}(dt)\mathbf{Z}_{i-1}(t)),$$

and notice that $\tilde{\mathbf{B}}_{1,n}$ converges a.s. since with probability one its bracket process is bounded: $\sum_{i=1}^\infty \mathbb{E}[\|\Delta\tilde{\mathbf{B}}_{1,i}\|^2|\mathcal{F}_{i-1}] \leq d\sum_{i=1}^\infty i^{-2} < \infty$. Then, applying the Cesàro Lemma it follows that $\|\mathbf{B}_{1,n}\| \overset{a.s.}{\to} 0$. Notice that, for any $n_0 \geq 1$,

$$\limsup_{n\to\infty} \|\mathbf{B}_{2,n}\| \leq \sup_{i\geq n_0} \int_\tau |\mu_{i-1}(dt) - \mu(dt)|,$$

and hence $\|\mathbf{B}_{2,n}\| \overset{a.s.}{\to} 0$ using assumption $\int_\tau \|\mu_{i-1}(dt) - \mu(dt)\| \overset{a.s.}{\to} 0$. Finally, from part (a) the third term $\|\mathbf{B}_{3,n}\|$ converges to zero a.s. by the Bounded Convergence Theorem. This concludes the proof. $\square$

B.2. *Proof of the second-order asymptotic results.* This section contains the proofs of the central limit theorems (CLTs) presented in Section 3, namely Theorem 3.2 and Theorem 3.3. The key idea of these proofs consists in revisiting the functional urn dynamics in the stochastic approximation (SA) framework, in the same spirit of the recent works [2, 9, 11]. For this reason, we now show some basic tools of SA. The general theory can be found in [3, 6, 8] (cf. [9, Theorem A.2] and [11, Appendix A]) with different group of conditions.

Consider an $\mathcal{F}_n$-measurable multivariate process $\{\mathbf{W}_n; n \geq 1\}$ which evolves as follows:

$$\forall\, n \geq 1, \quad \Delta\mathbf{W}_n = -\frac{1}{n} f(\mathbf{W}_{n-1}) + \frac{1}{n}(\Delta\mathbf{M}_n + \mathbf{R}_n), \tag{B.8}$$

where f is a differentiable function, $\Delta\mathbf{M}_n$ is an $\mathcal{F}_{n-1}$-martingale increment and $\mathbf{R}_n$ is a remainder term. Then, assuming that

$$\mathbf{R}_n \xrightarrow{a.s.} \mathbf{0} \quad \text{and} \quad \sup_{n\geq 1} \mathbb{E}\left[\|\Delta\mathbf{M}_n\|^2 \,|\, \mathcal{F}_{n-1}\right] < \infty \quad \text{a.s.},$$

we have that the set $\mathcal{W}$ of the limiting values of $\mathbf{W}_n$ as $n \to \infty$ is *a.s.* a compact connected set, stable by the flow of $ODE_f \equiv \dot{\mathbf{W}} = -f(\mathbf{W})$.

Moreover, suppose that there exist a constant $\delta > 0$ and a deterministic symmetric positive semidefinite matrix Γ such that

$$\sup_{n\geq 1} \mathbb{E}[\|\Delta\mathbf{M}_n\|^{2+\delta}|\mathcal{F}_{n-1}] < \infty \quad \text{a.s.}, \qquad \mathbb{E}\left[\Delta\mathbf{M}_n\Delta\mathbf{M}_n^\top|\mathcal{F}_{n-1}\right] \xrightarrow{a.s.} \Gamma, \tag{B.9}$$

and, for any $\epsilon > 0$, $n\mathbb{E}[\|\mathbf{R}_n\|^2 \mathbb{1}_{\{\|\mathbf{W}_n - \mathbf{W}\| \leq \epsilon\}}] \longrightarrow 0$. Then, considering an equilibrium point $\mathbf{W}$ of $\{\mathbf{w} : f(\mathbf{w}) = 0\}$ such that all the eigenvalues of $\mathcal{D}f(\mathbf{W})$ have real parts bigger than $1/2$, we have that $\sqrt{n}(\mathbf{W}_n - \mathbf{W}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma)$, where $\Sigma = \int_0^\infty e^{u(I/2 - \mathcal{D}f(\mathbf{W}))} \Gamma e^{u(I/2 - \mathcal{D}f(\mathbf{W}))^\top} du$.

Proof of Theorem 3.2. Initially, we need to express in the SA form (B.8) the joint dynamics of the following processes:

(1) the urn proportion in correspondence of all the covariate profiles,

$$\mathbf{Z}_n := (\mathbf{Z}_n(t), t \in \tau)^\top;$$

(2) the proportion of subjects of all covariate profiles assigned to the treatments,

$$\tilde{\mathbf{N}}_n := (\tilde{\mathbf{N}}_{t,n}, t \in \tau)^\top, \qquad \text{where} \qquad \tilde{\mathbf{N}}_{t,n} := \frac{\mathbf{N}_{t,n}}{w(\mathbf{N}_{t,n})};$$

(3) the adaptive estimators of features of interest related with the response distributions conditioned on each covariate profile,

$$\hat{\theta}_n := (\hat{\theta}_{t,n}, t \in \tau)^\top, \qquad \text{where} \qquad \hat{\theta}_{t,n} := (\hat{\theta}^j_{t,n}, j \in \{1, .., d\})^\top;$$

(4) the proportion of subjects with all covariate profiles observed in the trial,

$$\mathbf{Q}_n := (Q_{t,n}, t \in \tau)^\top, \qquad \text{where} \qquad Q_{t,n} := \frac{w(\mathbf{N}_{t,n})}{n} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{T_i=t\}}.$$

Then, the CLT follows by applying to this joint dynamics the standard theory of the SA.

First using (2.9), we express the joint dynamics of $(\mathbf{Z}_n(t), \mathbf{N}_{t,n})$ as follows: let $\mathbf{N}_0 = \mathbf{0}$ and $\mathbf{Y}_0 = \mathbf{1}$, and for any $n \geq 0$

$$\text{(B.10)} \qquad \begin{cases} \mathbf{Y}_n(t) \;=\; \mathbf{Y}_{n-1}(t) \;+\; D_n(t)\mathbf{X}_n(t), \\ \mathbf{N}_{t,n} \;=\; \mathbf{N}_{t,n-1} \;+\; \bar{\mathbf{X}}_n \mathbb{1}_{\{T_n=t\}}. \end{cases}$$

Notice that, by defining $r_n := (d+n)^{-1}$ and using Assumption (A1), $r_n\mathbf{Y}_n = \mathbf{Y}_n/w(\mathbf{Y}_n) = \mathbf{Z}_n$. Then in (B.10), if we multiply the dynamics of $\mathbf{Y}_n(t)$ by r_n and the dynamics of $\mathbf{N}_{t,n}$ by $w(\mathbf{N}_{t,n})^{-1}$, we obtain

$$\text{(B.11)} \qquad \begin{cases} \mathbf{Z}_n(t) - \mathbf{Z}_{n-1}(t) \;=\; -r_n(\mathbf{Z}_{n-1}(t) - D_n(t)\mathbf{X}_n(t)), \\ \dfrac{\mathbf{N}_{t,n}}{w(\mathbf{N}_{t,n})} - \dfrac{\mathbf{N}_{t,n-1}}{w(\mathbf{N}_{t,n-1})} \;=\; -\dfrac{\mathbb{1}_{\{T_n=t\}}}{w(\mathbf{N}_{t,n})}\left(\dfrac{\mathbf{N}_{t,n-1}}{w(\mathbf{N}_{t,n-1})} - \bar{\mathbf{X}}_n\right), \end{cases}$$

where in (B.11) we have used the relations $r_n r_{n-1}^{-1} = (1 - r_n)$ and

$$\frac{w(\mathbf{N}_{t,n-1})}{w(\mathbf{N}_{t,n})} = (1 - \mathbb{1}_{\{T_n=t\}} w(\mathbf{N}_{t,n})^{-1}).$$

Then, recalling $\tilde{\mathbf{N}}_{t,n} = \mathbf{N}_{t,n}/w(\mathbf{N}_{t,n})$ and adding to (B.11) the dynamics of $\{\hat{\theta}^j_{t,n}; n \geq n_0\}$ expressed in (3.3), we obtain

$$\text{(B.12)} \qquad \begin{cases} \Delta\mathbf{Z}_n(t) \;=\; -r_n(\mathbf{Z}_{n-1}(t) - D_n(t)\mathbf{X}_n(t)), \\ \Delta\tilde{\mathbf{N}}_{t,n} \;=\; -\dfrac{\mathbb{1}_{\{T_n=t\}}}{w(\mathbf{N}_{t,n})}\left(\tilde{\mathbf{N}}_{t,n-1} - \bar{\mathbf{X}}_n\right), \\ \Delta\hat{\theta}^j_{t,n} \;=\; -\dfrac{\bar{X}^j_n \mathbb{1}_{\{T_n=t\}}}{N^j_{t,n}}(f_{t,j}(\hat{\theta}^j_{t,n-1}) - \Delta\mathbf{M}_{t,j,n} - \mathbf{R}_{t,j,n}). \end{cases}$$

Now, let $Q_{t,n} := w(\mathbf{N}_{t,n})/n$, where by assumption $f_{\mu,t}(\cdot) \geq \epsilon > 0$ we have $\liminf_n Q_{t,n} \geq \liminf_n \mu_n(t) \geq \epsilon > 0$ with probability one. Notice that

$$r_n^{-1}\frac{\mathbb{1}_{\{T_n=t\}}}{w(\mathbf{N}_{t,n})} \;=\; r_n^{-1}\frac{\mathbb{1}_{\{T_n=t\}}}{w(\mathbf{N}_{t,n-1})+1} \;=\; \frac{\mathbb{1}_{\{T_n=t\}}}{Q_{t,n-1}} + \frac{\psi_{\theta^j_t,n}}{n},$$

and analogously,

$$r_n^{-1}\frac{\bar{X}^j_n \mathbb{1}_{\{T_n=t\}}}{N^j_{t,n}} \;=\; r_n^{-1}\frac{\bar{X}^j_n \mathbb{1}_{\{T_n=t\}}}{N^j_{t,n-1}+1} \;=\; \frac{\bar{X}^j_n \mathbb{1}_{\{T_n=t\}}}{\tilde{N}^j_{t,n-1}Q_{t,n-1}} + \frac{\psi_{N_t,n}}{n},$$

where $\{\psi_{N_t,n}; n \geq 1\}$ and $\{\psi_{\theta_t^j,n}; n \geq 1\}$ are suitable bounded sequence of $\mathcal{F}_n$-measurable random variables. Then, using the above relations in (B.12) we obtain

$$
\text{(B.13)}\qquad \begin{cases}
\Delta \mathbf{Z}_n(t) \;=\; -r_n(\mathbf{Z}_{n-1}(t) - D_n(t)\mathbf{X}_n(t)),\\
\Delta \tilde{\mathbf{N}}_{t,n} \;=\; -r_n \dfrac{\mathbb{1}_{\{T_n=t\}}}{Q_{t,n-1}} \left(\tilde{\mathbf{N}}_{t,n-1} - \bar{\mathbf{X}}_n\right) + r_n \mathbf{R}_{N_t,n},\\
\Delta \hat{\theta}_{t,n}^j \;=\; -r_n \dfrac{\bar{X}_n^j \mathbb{1}_{\{T_n=t\}}}{\tilde{N}_{t,n-1}^j Q_{t,n-1}} (f_{t,j}(\hat{\theta}_{t,n-1}^j) - \Delta \mathbf{M}_{t,j,n}) + r_n \mathbf{R}_{\theta_t^j,n},
\end{cases}
$$

where $\mathbf{R}_{N_t,n}, \mathbf{R}_{\theta_t^j,n} \in \mathcal{F}_n$ are suitable random variables that converges to zero a.s. and $(\mathbb{E}[\|\mathbf{R}_{N_t,n}\|^2] + \mathbb{E}[\|\mathbf{R}_{\theta_t^j,n}\|^2]) = o(n)$. Now, in order to express the dynamics in (B.13) in the SA form (B.8), we need also to consider the process $\{Q_{t,n}; n \geq 1\}$ and to rewrite (B.13) as follows:

$$
\begin{cases}
\Delta \mathbf{Z}_n(t) \;=\; -r_n f_{Z,t}(\mathbf{Z}_{n-1}(t)) \;+\; r_n \Delta \mathbf{M}_{Z(t),n},\\
\Delta \tilde{\mathbf{N}}_{t,n} \;=\; -r_n f_{N,t}(\mathbf{Z}_{n-1}(t), \tilde{\mathbf{N}}_{t,n-1}, \hat{\theta}_{t,n-1}, Q_{t,n-1}) \;+\; r_n(\Delta \mathbf{M}_{N_t,n} + r_n \mathbf{R}_{N_t,n}),\\
\Delta \hat{\theta}_{t,n}^j \;=\; -r_n f_{\theta,t}(\mathbf{Z}_{n-1}(t), \tilde{\mathbf{N}}_{t,n-1}, \hat{\theta}_{t,n-1}, Q_{t,n-1}) + r_n(\Delta \mathbf{M}_{\theta_t^j,n} + \mathbf{R}_{\theta_t^j,n}),\\
\Delta Q_{t,n} \;=\; -r_n f_{Q,t}(\tilde{\mathbf{N}}_{t,n-1}, \hat{\theta}_{t,n-1}, Q_{t,n-1}) \;+\; r_n \Delta M_{Q_t,n},
\end{cases}
$$

where

$$
\begin{aligned}
&f_{Z,t}(\mathbf{Z}_{n-1}(t)) := (I - H(t))\mathbf{Z}_{n-1}(t) + \mathbf{v}(t)(\mathbf{1}^\top \mathbf{Z}_{n-1}(t) - 1),\\
&f_{N,t}(\mathbf{Z}_{n-1}(t), \tilde{\mathbf{N}}_{t,n-1}, \hat{\theta}_{t,n-1}, Q_{t,n-1}) := \frac{\mu_{n-1}(t)}{Q_{t,n-1}} \left(\tilde{\mathbf{N}}_{t,n-1} - \mathbf{Z}_{n-1}(t)\right),\\
&f_{\theta,t}(\mathbf{Z}_{n-1}(t), \tilde{\mathbf{N}}_{t,n-1}, \hat{\theta}_{t,n-1}, Q_{t,n-1}) := \frac{\mu_{n-1}(t) Z_{n-1}^j(t)}{\tilde{N}_{t,n-1}^j Q_{t,n-1}} f_{t,j}(\hat{\theta}_{t,n-1}^j),\\
&f_{M,t}(\tilde{\mathbf{N}}_{t,n-1}, \hat{\theta}_{t,n-1}, Q_{t,n-1}) := (Q_{t,n-1} - \mu_{n-1}(t)),
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta \mathbf{M}_{Z(t),n} \;&:= (D_n \mathbf{X}_n - H\mathbf{Z}_{n-1})(t),\\
\Delta \mathbf{M}_{N_t,n} \;&:= (\mathbb{1}_{\{T_n=t\}} - \mu_{n-1}(t)) \frac{(\tilde{\mathbf{N}}_{t,n-1} - \mathbf{Z}_{n-1}(t))}{Q_{t,n-1}} - \frac{\mathbb{1}_{\{T_n=t\}}}{Q_{t,n-1}} (\bar{\mathbf{X}}_n - \mathbf{Z}_{n-1}(t)),\\
\Delta \mathbf{M}_{\theta_t^j,n} \;&:= (\bar{X}_n^j \mathbb{1}_{\{T_n=t\}} - \mu_{n-1}(t) Z_{n-1}^j(t)) \frac{f_{t,j}(\hat{\theta}_{t,n-1}^j)}{\tilde{N}_{t,n-1}^j Q_{t,n-1}} + \frac{\bar{X}_n^j \mathbb{1}_{\{T_n=t\}}}{\tilde{N}_{t,n-1}^j Q_{t,n-1}} \Delta \mathbf{M}_{t,j,n},\\
\Delta M_{Q,n}(t) \;&:= (\mathbb{1}_{\{T_n=t\}} - \mu_{n-1}(t)),
\end{aligned}
$$

are martingale increments since $\mathbb{E}[D_n(t)|T_n, \bar{\mathbf{X}}_n] = H(t)$, $\mathbb{E}[\bar{\mathbf{X}}_n|\mathcal{F}_{n-1}, T_n] = \mathbf{Z}_{n-1}(T_n)$, $\mathbb{E}[\Delta \mathbf{M}_{t,j,n}|\mathcal{F}_{n-1}, T_n, \bar{\mathbf{X}}_n] = 0$, $\mathbb{E}[\mathbb{1}_{\{T_n=t\}}|\mathcal{F}_{n-1}] = \mu_{n-1}(t)$.

Let us now introduce the joint processes $\{\mathbf{W}_n, n \geq 1\}$ defined as $\mathbf{W}_n := (\mathbf{Z}_n, \tilde{\mathbf{N}}_n, \hat{\theta}_n, \mathbf{Q}_n)^\top$, and note that its dynamics can be expressed in the SA form (B.8) as follows:

$$\Delta \mathbf{W}_n \;=\; -r_n f_W(\mathbf{W}_{n-1}) \;+\; r_n(\Delta \mathbf{M}_{W,n} + \mathbf{R}_{W,n}), \tag{B.14}$$

where

(i) $f_W := (f_Z, f_N, f_\theta, f_Q)^\top$, where $f_Z := (f_{Z(t)}, t \in \tau)^\top$, $f_N := (f_{N_t}, t \in \tau)^\top$, $f_\theta := (f_{\theta_t^j}, t \in \tau, j \in \{1,..,d\})^\top$, $f_Q := (f_{Q_t}, t \in \tau)^\top$;

(ii) $\Delta\mathbf{M}_{W,n} := (\Delta\mathbf{M}_{Z,n}, \Delta\mathbf{M}_{N,n}, \Delta\mathbf{M}_{\theta,n}, \Delta\mathbf{M}_{Q,n})^\top$, where $\Delta\mathbf{M}_{Z,n} := (\Delta\mathbf{M}_{Z(t),n}, t \in \tau)^\top$, $\Delta\mathbf{M}_{N,n} := (\Delta\mathbf{M}_{N_t,n}, t \in \tau)^\top$, $\Delta\mathbf{M}_{\theta,n} := (\Delta\mathbf{M}_{\theta_t^j,n}, t \in \tau, j \in \{1,..,d\})^\top$, $\Delta\mathbf{M}_{Q,n} := (\Delta\mathbf{M}_{Q_t,n}, t \in \tau)^\top$;

(iii) $\mathbf{R}_{W,n} := (\mathbf{0}, \mathbf{R}_{N,n}, \mathbf{R}_{\theta,n}, \mathbf{0})^\top$, where $\mathbf{R}_{N,n} := (\mathbf{R}_{N_t,n}, t \in \tau)^\top$, $\mathbf{R}_{\theta,n} := (\mathbf{R}_{\theta_t^j,n}, t \in \tau, j \in \{1,..,d\})^\top$.

Since $\mathbf{R}_{W,n} \xrightarrow{a.s.} 0$ and, using (3.4) in (A6a), $\sup_n \mathbb{E}[\|\Delta\mathbf{M}_{W,n}\|^2] < \infty$, we have that the set $\mathcal{W}$ of the limiting values of $\mathbf{W}_n$ is a stable set by the flow of $\dot{\mathbf{W}} = -f_W(\mathbf{W})$. Notice that the set $\{\mathbf{w} : f_W(\mathbf{w}) = \mathbf{0}\}$ is composed only of the element $\mathbf{W} := (\mathbf{v}, \mathbf{v}, \theta, \mu)^\top$, where $\mathbf{v} := (\mathbf{v}(t), t \in \tau)^\top$ and $\mu := (\mu(t), t \in \tau)^\top$. Moreover, we recall from Theorem 3.1 that we have $\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{v}$ and $\tilde{\mathbf{N}}_n \xrightarrow{a.s.} \mathbf{v}$, and by (A6a), we have $\hat{\theta}_n \xrightarrow{a.s.} \theta$. Since $\mu_n(t) = f_{\mu,t}(\tilde{\mathbf{N}}_{t,n}, \hat{\theta}_{t,n}) \xrightarrow{a.s.} f_{\mu,t}(\mathbf{v}(t), \theta_t) = \mu(t)$ and $\mathbf{Q}_n - \sum_{i=1}^n \mu_{i-1}/n = \sum_{i=1}^n \Delta\mathbf{M}_{Q,n}/n \xrightarrow{a.s.} \mathbf{0}$, we also have $\mathbf{Q}_n \xrightarrow{a.s.} \mu$, which implies $\mathbf{W}_n \xrightarrow{a.s.} \mathbf{W}$.

In order to show the existence of a stable attracting area which contains a neighborhood of $\mathbf{W}$, it is sufficient (see [7, p. 1077]) to show that $\{\Re e(Sp(\mathcal{D}f_W(\mathbf{W}))) > 0\}$, where

$$\mathcal{D}f_W(\mathbf{W}) = \begin{pmatrix} \mathcal{D}_Z f_Z(\mathbf{W}) & 0 & 0 & 0 \\ -I & I & 0 & 0 \\ 0 & 0 & \mathcal{D}_\theta f_\theta(\mathbf{W}) & 0 \\ 0 & \mathcal{D}_N f_Q(\mathbf{W}) & \mathcal{D}_\theta f_Q(\mathbf{W}) & I \end{pmatrix}, \tag{B.15}$$

and all the terms in (B.15) are block-diagonal matrices, whose t^{th} block is: $[\mathcal{D}_Z f_Z(\mathbf{W})]^{tt} = (I - H(t) + \mathbf{v}(t)\mathbf{1}^\top)$, $[\mathcal{D}_\theta f_\theta(\mathbf{W})]^{tt} = diag(\mathcal{D}f_{t,j}(\theta_t^j), j \in \{1,..,d\})$, $[\mathcal{D}_N f_Q(\mathbf{W})]^{tt} = \mathcal{D}_N f_{\mu,t}(\mathbf{W})$ and $[\mathcal{D}_N f_Q(\mathbf{W})]^{tt} = \mathcal{D}_\theta f_{\mu,t}(\mathbf{W})$. Note from the structure of $\mathcal{D}f_W(\mathbf{W})$ in (B.15) that $\{\Re e(Sp(\mathcal{D}f_W(\mathbf{W}))) > 0\}$ follows by establishing that for any $t \in \tau$ and $j \in \{1,..,d\}$

$$\{\Re e(Sp(I - H(t) + \mathbf{v}(t)\mathbf{1}^\top)) > 0\} \;\text{ and }\{\Re e(Sp(\mathcal{D}f_{t,j}(\theta_t^j))) > 0\}.$$

Since $(I - H(t)) = V(t)(I - \Lambda(t))U(t)$ from (B.2) and (B.3), we have that

$$Sp(I - H(t) + \mathbf{v}(t)\mathbf{1}^\top) = \{1\} \cup \{1 - \lambda(t), \lambda(t) \in Sp(H(t)) \setminus \{1\}\}.$$

Then $\{\Re e(Sp(\mathcal{D}f_W(\mathbf{W}))) > 0\}$ follows by $\{\max_{t\in\tau} \mathcal{R}e(\lambda_H^*(t)) < 1/2\}$ from (A5) and $\{\min_{t\in\tau} \mathcal{R}e(\lambda_{\theta_t^j}^*) > 1/2\}$ from (A6a).

We now show that the assumptions of the CLT for processes in the SA form are satisfied by the dynamics in (B.14) of the joint process $\{\mathbf{W}_n, n \geq 1\}$. First, note that using the above arguments we obtain $\{\Re e(Sp(\mathcal{D}f_W(\mathbf{W})) > 1/2\}$. Then, it is immediate to see that $\mathbb{E}[\|\mathbf{R}_{W,n}\|^2] = o(n)$ and the first condition in (B.9) is satisfied using (3.4) in Assumption (A6a). Concerning the second condition in (B.9), we need to show that there exists a deterministic symmetric positive semidefinite matrix Γ such that

$$\mathbb{E}[\Delta\mathbf{M}_{W,n}(\Delta\mathbf{M}_{W,n})^\top|\mathcal{F}_{n-1}] \xrightarrow{a.s.} \Gamma = \begin{pmatrix} \Gamma_{ZZ} & \Gamma_{ZN} & \Gamma_{Z\theta} & \Gamma_{ZQ} \\ \Gamma_{ZN}^\top & \Gamma_{NN} & \Gamma_{N\theta} & \Gamma_{NQ} \\ \Gamma_{Z\theta}^\top & \Gamma_{N\theta}^\top & \Gamma_{\theta\theta} & \Gamma_{\theta Q} \\ \Gamma_{ZQ}^\top & \Gamma_{NQ}^\top & \Gamma_{\theta Q}^\top & \Gamma_{QQ} \end{pmatrix}.$$

First, note that since $(\tilde{\mathbf{N}}_{t,n-1} - \mathbf{Z}_{n-1}(t)) \xrightarrow{a.s.} 0$ and $f_{t,j}(\hat{\theta}^j_{t,n-1}) \xrightarrow{a.s.} 0$, these terms do not contribute to Γ; hence in the following calculations they will be omitted by $\Delta\mathbf{M}_{N_t,n}$ and $\Delta\mathbf{M}_{\theta^j_t,n}$, respectively. Moreover, let us introduce for any $t, s \in \tau$ and $j \in \{1,..,d\}$, a vector $\mathbf{g}(t,s,\mathbf{e}_j) \in \mathcal{S}$ such that $g^k(t,s,\mathbf{e}_j)$, $k \in \{1,..,d\}$, is defined as follows:

$$\text{(B.16)} \qquad \frac{\left(\min\left\{\sum_{i=1}^k v^i(t); \sum_{i=1}^j v^i(s)\right\} \; - \; \max\left\{\sum_{i=1}^{k-1} v^i(t); \sum_{i=1}^{j-1} v^i(s)\right\} \right)^+}{v^j(s)}.$$

Then, before computing the terms in Γ we show that for any $t \in \tau$

$$\text{(B.17)} \qquad \mathbb{E}[\|\mathbf{X}_n(t) - \mathbf{g}(t_1, T_n, \mathbf{X}_n)\||\mathcal{F}_{n-1}, T_n, \bar{\mathbf{X}}_n] \xrightarrow{a.s.} 0,$$

To this end, first note from (A.1) that $\mathbf{X}_n(t)$ is a continuous function of $\{\mathbf{Z}_{n-1}(s), s \in \tau\}$ conditioned on $\mathcal{F}_{n-1}$, T_n and $\bar{\mathbf{X}}_n$; then (B.17) follows by Theorem 3.1 which states $\mathbf{Z}_{n-1}(t) \xrightarrow{a.s.} \mathbf{v}(t)$ for any $t \in \tau$, since τ has a finite number of elements. We now compute the terms in Γ.

Computation of $\Gamma_{ZZ} := a.s. - \lim_n \mathbb{E}[\Delta\mathbf{M}_{Z,n}(\Delta\mathbf{M}_{Z,n})^\top|\mathcal{F}_{n-1}]$. For any $t_1, t_2 \in \tau$, we have

$$\begin{aligned} &\mathbb{E}[\Delta\mathbf{M}_{Z(t_1),n}(\Delta\mathbf{M}_{Z(t_2),n})^\top|\mathcal{F}_{n-1}] = \mathbb{E}[D_n(t_1)\mathbf{X}_n(t_1)(\Delta\mathbf{M}_{Z(t_2),n})^\top|\mathcal{F}_{n-1}] \\ &= \mathbb{E}[D_n(t_1)\mathbf{X}_n(t_1)\mathbf{X}_n^\top(t_2)D_n^\top(t_2)|\mathcal{F}_{n-1}] - H(t_1)\mathbf{Z}_{n-1}(t_1)\mathbf{Z}_{n-1}^\top(t_2)H^\top(t_2) \end{aligned}$$

Consider the decomposition $D_n(t_1)\mathbf{X}_n(t_1)\mathbf{X}_n^\top(t_2)D_n^\top(t_2) = (B_{1n} + B_{2n})$, where

$$\begin{aligned} B_{1n} &:= D_n(t_1)(\mathbf{X}_n(t_1)\mathbf{X}_n^\top(t_2) - \mathbf{g}(t_1, T_n, \bar{\mathbf{X}}_n)\mathbf{g}^\top(t_2, T_n, \bar{\mathbf{X}}_n))D_n^\top(t_2) \\ B_{2n} &:= D_n(t_1)\mathbf{g}(t_1, T_n, \bar{\mathbf{X}}_n)\mathbf{g}^\top(t_2, T_n, \bar{\mathbf{X}}_n)D_n^\top(t_2). \end{aligned}$$

Using (B.17) and since D_n is a.s. bounded, it follows by the the Dominated Convergence Theorem that $\mathbb{E}[B_{1n}|\mathcal{F}_{n-1}] \xrightarrow{a.s.} 0$. In addition, since the probability distribution of the random variables in B_{2n}, i.e. $(T_n, \bar{\mathbf{X}}_n, \bar{\xi}_n)$, conditioned on $\mathcal{F}_{n-1}$, converges a.s. as n increases to infinity, we obtain $\mathbb{E}[B_{2n}|\mathcal{F}_{n-1}] \xrightarrow{a.s.} \mathbb{E}[D(t_1)\mathbf{g}(t_1,T,\bar{\mathbf{X}})\mathbf{g}^\top(t_2,T,\bar{\mathbf{X}})D^\top(t_2)]$. Hence, we have proved the following:

$$\Gamma_{ZZ}^{t_1t_2} := \mathbb{E}[D(t_1)\mathbf{g}(t_1,T,\bar{\mathbf{X}})\mathbf{g}^\top(t_2,T,\bar{\mathbf{X}})D^\top(t_2)] - \mathbf{v}(t_1)\mathbf{v}^\top(t_2).$$

Computation of $\Gamma_{NN} := a.s.{-}\lim_n \mathbb{E}[\Delta\mathbf{M}_{N,n}(\Delta\mathbf{M}_{N,n})^\top|\mathcal{F}_{n-1}]$. Note that $\mathbb{E}[\Delta\mathbf{M}_{N_{t_1},n}(\Delta\mathbf{M}_{N_{t_2},n})^\top|\mathcal{F}_{n-1}] = 0 = \Gamma_{NN}^{t_1t_2}$ for any $t_1 \neq t_2$, while for $t_1 = t_2 = t$ we have

$$\begin{aligned}
&\mathbb{E}[\Delta\mathbf{M}_{N_t,n}(\Delta\mathbf{M}_{N_t,n})^\top|\mathcal{F}_{n-1}] \\
&= Q_{t,n-1}^{-2}\mu_{n-1}(t)\mathbb{E}[(\bar{\mathbf{X}}_n - \mathbf{Z}_{n-1}(t))(\bar{\mathbf{X}}_n - \mathbf{Z}_{n-1}(t))^\top|\mathcal{F}_{n-1}, T_n = t] \\
&= Q_{t,n-1}^{-2}\mu_{n-1}(t)(diag(\mathbf{Z}_{n-1}(t)) - \mathbf{Z}_{n-1}(t)\mathbf{Z}_{n-1}^\top(t)) \\
&\xrightarrow{a.s.} \Gamma_{NN}^{tt} := \mu^{-1}(t)(diag(\mathbf{v}(t)) - \mathbf{v}(t)\mathbf{v}^\top(t)).
\end{aligned}$$

Computation of $\Gamma_{ZN} := a.s. - \lim_n \mathbb{E}[\Delta\mathbf{M}_{Z,n}(\Delta\mathbf{M}_{N,n})^\top|\mathcal{F}_{n-1}]$. For any $t_1, t_2 \in \tau$ we have that

$$\begin{aligned}
&\mathbb{E}[\Delta\mathbf{M}_{Z(t_1),n}(\Delta\mathbf{M}_{N_{t_2},n})^\top|\mathcal{F}_{n-1}] = \mathbb{E}[D_n(t_1)\mathbf{X}_n(t_1)(\Delta\mathbf{M}_{N_{t_2},n})^\top(t_2)|\mathcal{F}_{n-1}] \\
&= Q_{t_2,n-1}^{-1}\mu_{n-1}(t_2)\left(\mathbb{E}[D_n(t_1)\mathbf{X}_n(t_1)\bar{\mathbf{X}}_n^\top|\mathcal{F}_{n-1}, T_n = t_2] - H(t_1)\mathbf{Z}_{n-1}(t_1)\mathbf{Z}_{n-1}^\top(t_2)\right).
\end{aligned}$$

Note that the above term $\mathbb{E}[D_n(t_1)\mathbf{X}_n(t_1)\bar{\mathbf{X}}_n^\top|\mathcal{F}_{n-1}, T_n = t_2]$ can be expressed as follows:

$$\begin{aligned}
&\mathbb{E}[\mathbb{E}[D_n(t_1)|\mathcal{F}_{n-1}, T_n = t_2, \bar{\mathbf{X}}_n]\mathbf{X}_n(t_1)\bar{\mathbf{X}}_n^\top|\mathcal{F}_{n-1}, T_n = t_2] = \\
&H(t_1)\mathbb{E}[\mathbf{X}_n(t_1)\bar{\mathbf{X}}_n^\top|\mathcal{F}_{n-1}, T_n = t_2].
\end{aligned}$$

Then, since using (B.17) it follows by the the Dominated Convergence Theorem that

$$\mathbb{E}[\mathbf{X}_n(t_1) - \mathbf{g}(t_1,t_2,\bar{\mathbf{X}}_n)|\mathcal{F}_{n-1}, T_n = t_2] \xrightarrow{a.s.} 0,$$

we can directly consider $H(t_1)\mathbb{E}[\mathbf{g}(t_1,t_2,\bar{\mathbf{X}}_n)\bar{\mathbf{X}}_n^\top|\mathcal{F}_{n-1}, T_n = t_2]$; then, since the probability distribution of $\bar{\mathbf{X}}_n$ conditioned on $\mathcal{F}_{n-1}$ and T_n converges a.s. as n increases to infinity, we obtain

$$\mathbb{E}[\mathbf{g}(t_1,t_2,\bar{\mathbf{X}}_n)\bar{\mathbf{X}}_n^\top|\mathcal{F}_{n-1}, T_n = t_2] \xrightarrow{a.s.} \mathbb{E}[\mathbf{g}(t_1,t_2,\bar{\mathbf{X}}_n)\bar{\mathbf{X}}^\top|T = t_2].$$

Hence, we have proved the following:

$$\Gamma_{ZN}^{t_1t_2} := H(t_1)G(t_1,t_2)diag(\mathbf{v}(t_2)) - \mathbf{v}(t_1)\mathbf{v}^\top(t_2).$$

Computation of $\Gamma_{\theta\theta} := a.s. - \lim_n \mathbb{E}[\Delta\mathbf{M}_{\theta,n}(\Delta\mathbf{M}_{\theta,n})^\top|\mathcal{F}_{n-1}]$. Since for any $j_1 \neq j_2$ or $t_1 \neq t_2$ we have $\mathbb{E}[\Delta\mathbf{M}_{\theta_{t_1}^{j_1},n}(\Delta\mathbf{M}_{\theta_{t_2}^{j_2},n})^\top|\mathcal{F}_{n-1}] = 0$, we have that $\Gamma_{\theta\theta}$ is a block-diagonal matrix. In particular, for any $t \in \tau$ we have that $\Gamma_{\theta\theta}^{tt} = diag([\Gamma_{\theta\theta}^{tt}]^{jj}, j \in \{1,..,d\})^\top$, where

$$\begin{aligned}
&\mathbb{E}[\Delta\mathbf{M}_{\theta_t^j,n}(\Delta\mathbf{M}_{\theta_t^j,n})^\top|\mathcal{F}_{n-1}] \\
&= (\tilde{N}_{t,n-1}^j Q_{t,n-1})^{-2}\mu_{n-1}(t)Z_{n-1}^j(t) \times \mathbb{E}[\Delta\mathbf{M}_{t,j,n}(\Delta\mathbf{M}_{t,j,n})^\top|\mathcal{F}_{n-1}, T_n = t, \bar{X}_n^j = 1] \\
&\xrightarrow{a.s.} [\Gamma_{\theta\theta}^{tt}]^{jj} := (v^j(t)\mu(t))^{-1}\mathbb{E}[\Delta\mathbf{M}_{t,j}(\Delta\mathbf{M}_{t,j})^\top|T = t, \bar{X}^j = 1].
\end{aligned}$$

Computation of $\Gamma_{Z\theta}$. For any $t_1, t_2 \in \tau$ and $j \in \{1,..,d\}$ we have that

$$\mathbb{E}[\Delta\mathbf{M}_{Z(t_1),n}(\Delta\mathbf{M}_{\theta^j_{t_2},n})^\top|\mathcal{F}_{n-1}] = \mathbb{E}[D_n(t_1)\mathbf{X}_n(t_1)(\Delta\mathbf{M}_{\theta^j_{t_2},n})^\top|\mathcal{F}_{n-1}]$$
$$= (\tilde{N}^j_{t_2,n-1}Q_{t_2,n-1})^{-1}\mu_{n-1}(t_2)Z^j_{n-1}(t_2)\mathbb{E}[D_n(t_1)\mathbf{X}_n(t_1)(\Delta\mathbf{M}_{t_2,j,n})^\top|\mathcal{F}_{n-1}, T_n = t_2, \bar{X}^j_n = 1].$$

Then, since using (B.17) it follows by the the Dominated Convergence Theorem that $\mathbb{E}[\mathbf{X}_n(t_1) - \mathbf{g}(t_1,t_2,\mathbf{e}_j)|\mathcal{F}_{n-1}, T_n = t_2, \bar{X}^j_n = 1] \xrightarrow{a.s.} 0$, we can directly consider

$$\mathbb{E}[D_n(t_1)\mathbf{g}(t_1,t_2,\mathbf{e}_j)(\Delta\mathbf{M}_{t_2,j,n})^\top|\mathcal{F}_{n-1}, T_n = t_2, \bar{X}^j_n = 1];$$

then, since the probability distribution of $\bar{\xi}_n$ conditioned on $\mathcal{F}_{n-1}$, T_n and $\bar{\mathbf{X}}_n$ does not change, we have proved that

$$[\Gamma^{t_1t_2}_{Z\theta}]^{jj} := \mathbb{E}[D(t_1)\mathbf{g}(t_1,t_2,\mathbf{e}_j)(\Delta\mathbf{M}_{t_2,j})^\top|T = t_2, \bar{X}^j = 1].$$

Computation of $\Gamma_{N\theta} := a.s. - \lim_n \mathbb{E}[\Delta\mathbf{M}_{N,n}(\Delta\mathbf{M}_{\theta,n})^\top|\mathcal{F}_{n-1}]$. For any $t_1, t_2 \in \tau$ we have that

$$\mathbb{E}[\Delta\mathbf{M}_{N_{t_1},n}(\Delta\mathbf{M}_{\theta^j_{t_2},n})^\top|\mathcal{F}_{n-1}] = \mathbb{E}[\Delta\mathbf{M}_{N_{t_1},n}\mathbb{E}[(\Delta\mathbf{M}_{\theta^j_{t_2},n})^\top|\mathcal{F}_{n-1}, T_n, \bar{\mathbf{X}}_n]|\mathcal{F}_{n-1}] = 0 = \Gamma^{t_1t_2}_{N\theta}.$$

Computation of $\Gamma_{QQ} := a.s. - \lim_n \mathbb{E}[\Delta\mathbf{M}_{Q,n}(\Delta\mathbf{M}_{Q,n})^\top|\mathcal{F}_{n-1}]$. It is immediate to see that for any $t_1 \neq t_2$

$$\mathbb{E}[\Delta Q_{t_1,n}\Delta Q_{t_2,n}|\mathcal{F}_{n-1}] = -\mu_{n-1}(t_1)\mu_{n-1}(t_2) \xrightarrow{a.s.} \Gamma^{t_1t_2}_{QQ} := -\mu(t_1)\mu(t_2),$$

while for $t_1 = t_2 = t$ we have

$$\mathbb{E}[\Delta Q^2_{t,n}|\mathcal{F}_{n-1}] = \mu_{n-1}(t)(1-\mu_{n-1}(t)) \xrightarrow{a.s.} \Gamma^{tt}_{QQ} := \mu(t)(1-\mu(t)).$$

Remaining terms in Γ. Finally, we have that for any $t_1 \neq t_2$,

$$\mathbb{E}[\Delta\mathbf{M}_{Z(t_1),n}\Delta M_{Q_{t_2},n}|\mathcal{F}_{n-1}] = \mathbb{E}[\mathbb{E}[\Delta\mathbf{M}_{Z(t_1),n}|\mathcal{F}_{n-1}, T_n]\Delta M_{Q_{t_2},n}|\mathcal{F}_{n-1}] = 0 = \Gamma^{t_1t_2}_{ZQ},$$
$$\mathbb{E}[\Delta\mathbf{M}_{N_{t_1},n}\Delta M_{Q_{t_2},n}|\mathcal{F}_{n-1}] = \mathbb{E}[\mathbb{E}[\Delta\mathbf{M}_{N_{t_1},n}|\mathcal{F}_{n-1}, T_n]\Delta M_{Q_{t_2},n}|\mathcal{F}_{n-1}] = 0 = \Gamma^{t_1t_2}_{NQ},$$
$$\mathbb{E}[\Delta\mathbf{M}_{\theta_{t_1},n}\Delta M_{Q_{t_2},n}|\mathcal{F}_{n-1}] = \mathbb{E}[\mathbb{E}[\Delta\mathbf{M}_{\theta_{t_1},n}|\mathcal{F}_{n-1}, T_n]\Delta M_{Q_{t_2},n}|\mathcal{F}_{n-1}] = 0 = \Gamma^{t_1t_2}_{\theta Q}.$$

Since the assumptions are all satisfied, we can apply the CLT of the SA to the dynamics (B.14), so obtaining a Gaussian asymptotic distribution for the process $\{\mathbf{W}_n; n \geq 1\}$, with asymptotic variance

$$\Sigma := \int_0^\infty e^{u(\frac{\mathbf{I}}{2} - \mathcal{D}f_W(\mathbf{W}))}\Gamma e^{u(\frac{\mathbf{I}}{2} - \mathcal{D}f_W(\mathbf{W}))^\top} du.$$

This concludes the proof. □

Proof of Theorem 3.3. The structure of this proof is analogous to the proof of Theorem 3.2. In particular, we initially need to express in the SA form (B.8) the joint dynamics of the following processes:

(1) the urn proportion in correspondence of all the covariate profiles,
$$\mathbf{Z}_n := (\mathbf{Z}_n(t), t \in \tau)^\top;$$
(2) the proportion of subjects assigned to the treatments in the study, $\tilde{\mathbf{N}}_n := \mathbf{N}_n/n$;
(3) the adaptive estimators of features of interest related with the family of response distributions conditioned on the covariates, $\hat{\beta}_n := (\hat{\beta}_n^j, j \in \{1, .., d\})^\top$.

Then, the CLT follows by applying the standard theory of the SA to the joint dynamics.

Using analogous arguments to the proof of Theorem 3.2, we can obtain from (2.9) and (3.6) the following joint dynamics:

$$\text{(B.18)}\qquad \begin{cases} \Delta\mathbf{Z}_n(t) &= -r_n(\mathbf{Z}_{n-1}(t) - D_n(t)\mathbf{X}_n(t)), \\ \Delta\tilde{\mathbf{N}}_n &= -r_n\left(\tilde{\mathbf{N}}_{n-1} - \bar{\mathbf{X}}_n\right) + r_n\mathbf{R}_{N_t,n}, \\ \Delta\hat{\beta}_n^j &= -r_n\dfrac{\bar{X}_n^j}{\tilde{N}_{n-1}^j}(f_j(\hat{\beta}_{n-1}^j) - \Delta\mathbf{M}_{j,n}) + r_n\mathbf{R}_{\beta^j,n}, \end{cases}$$

where $\mathbf{R}_{N_t,n}, \mathbf{R}_{\beta^j,n} \in \mathcal{F}_n$ are suitable random variables that converges to zero a.s. and $(\mathbb{E}[\|\mathbf{R}_{N_t,n}\|^2] + \mathbb{E}[\|\mathbf{R}_{\beta^j,n}\|^2]) = o(n)$. Now, in order to express the dynamics in (B.18) in the SA form (B.8), we need to rewrite it as follows:

$$\text{(B.19)}\qquad \begin{cases} \Delta\mathbf{Z}_n(t) \;=\; -r_n f_{Z,t}(\mathbf{Z}_{n-1}(t)) \;+\; r_n\Delta\mathbf{M}_{Z(t),n}, \\ \Delta\tilde{\mathbf{N}}_n \;-r_n f_N(\mathbf{Z}_{n-1}, \tilde{\mathbf{N}}_{n-1}, \hat{\beta}_{n-1}) \;+\; r_n(\Delta\mathbf{M}_{N,n} + \mathbf{R}_{N,n}), \\ \Delta\hat{\beta}_n^j \;=\; -r_n f_{\beta^j}(\mathbf{Z}_{n-1}(t), \tilde{\mathbf{N}}_{n-1}, \hat{\beta}_{n-1}^j) + r_n(\Delta\mathbf{M}_{\beta^j,n} + \mathbf{R}_{\beta^j,n}), \end{cases}$$

where

$$f_{Z,t}(\mathbf{Z}_{n-1}(t)) := (I - H(t))\mathbf{Z}_{n-1}(t) + \mathbf{v}(t)(\mathbf{1}^\top\mathbf{Z}_{n-1}(t) - 1),$$
$$f_N(\mathbf{Z}_{n-1}, \tilde{\mathbf{N}}_{n-1}, \hat{\beta}_{n-1}) := \left(\tilde{\mathbf{N}}_{n-1} - \sum_{s=1}^{K}\mu_{n-1}(s)\mathbf{Z}_{n-1}(s)\right),$$
$$f_{\beta^j}(\mathbf{Z}_{n-1}, \tilde{\mathbf{N}}_{n-1}, \hat{\beta}_{n-1}^j) := \frac{\sum_{s=1}^{K}\mu_{n-1}(s)\mathbf{Z}_{n-1}(s)}{\tilde{N}_{n-1}^j}f_j(\hat{\beta}_{n-1}^j),$$

and

$$\Delta\mathbf{M}_{Z(t),n} := (D_n\mathbf{X}_n - H\mathbf{Z}_{n-1})(t),$$
$$\Delta\mathbf{M}_{N,n} := (\bar{\mathbf{X}}_n - \sum_{s=1}^{K}\mu_{n-1}(s)\mathbf{Z}_{n-1}(s)),$$
$$\Delta\mathbf{M}_{\beta^j,n} := (\bar{\mathbf{X}}_n - \sum_{s=1}^{K}\mu_{n-1}(s)\mathbf{Z}_{n-1}(s))\frac{f_j(\hat{\beta}_{n-1}^j)}{\tilde{N}_{n-1}^j} + \frac{\bar{X}_n^j}{\tilde{N}_{n-1}^j}\Delta\mathbf{M}_{j,n},$$

are martingale increments since $\mathbb{E}[D_n(t)|T_n, \bar{\mathbf{X}}_n] = H(t)$, $\mathbb{E}[\bar{\mathbf{X}}_n|\mathcal{F}_{n-1}] = \mathbf{Z}_{n-1}(T_n)$, $\mathbb{E}[\Delta\mathbf{M}_{j,n}|\mathcal{F}_{n-1}, T_n, \bar{\mathbf{X}}_n] = 0$, $\mathbb{E}[\mathbb{1}_{\{T_n=t\}}|\mathcal{F}_{n-1}] = \mu_{n-1}(t)$.

Let us now introduce the joint processes $\{\mathbf{W}_n, n \geq 1\}$ defined as $\mathbf{W}_n := (\mathbf{Z}_n, \tilde{\mathbf{N}}_n, \hat{\beta})^\top$, and note that its dynamics can be expressed in the SA form (B.8) as follows:

$$\Delta\mathbf{W}_n = -r_n f_W(\mathbf{W}_{n-1}) + r_n(\Delta\mathbf{M}_{W,n} + \mathbf{R}_{W,n}), \tag{B.20}$$

where

(i) $f_W := (f_Z, f_N, f_\beta)^\top$, where $f_Z := (f_{Z,t}, t \in \tau)^\top$ and $f_\beta := (f_{\beta^j}, j \in \{1, .., d\})^\top$;
(ii) $\Delta\mathbf{M}_{W,n} := (\Delta\mathbf{M}_{Z,n}, \Delta\mathbf{M}_{N,n}, \Delta\mathbf{M}_{\beta,n})^\top$, where $\Delta\mathbf{M}_{Z,n} := (\Delta\mathbf{M}_{Z(t),n}, t \in \tau)^\top$ and $\Delta\mathbf{M}_{\beta,n} := (\Delta\mathbf{M}_{\beta^j,n} j \in \{1, .., d\})^\top$;
(iii) $\mathbf{R}_{W,n} := (\mathbf{0}, \mathbf{R}_{N,n}, \mathbf{R}_{\beta,n})^\top$, where $\mathbf{R}_{\beta,n} := (\mathbf{R}_{\beta^j,n}, j \in \{1, .., d\})^\top$.

Since $\mathbf{R}_{W,n} \xrightarrow{a.s.} \mathbf{0}$ and, using (3.4) in (A6b), $\sup_n \mathbb{E}[\|\Delta\mathbf{M}_{W,n}\|^2] < \infty$, we have that the set $\mathcal{W}$ of the limiting values of $\mathbf{W}_n$ is a set stable by the flow of $\dot{\mathbf{W}} = -f_W(\mathbf{W})$. Notice that by (A7b) the set $\{\mathbf{w} : f_W(\mathbf{w}) = \mathbf{0}\}$ is composed only by the element $\mathbf{W} := (\mathbf{v}, \mathbf{x}_0, \beta)^\top$, where $\mathbf{v} := (\mathbf{v}(t), t \in \tau)^\top$. Moreover, by (A6b) we have $\hat{\beta}_n \xrightarrow{a.s.} \beta$, and from Theorem 3.1 we have $\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{v}$ and $\tilde{\mathbf{N}}_n \xrightarrow{a.s.} \mathbf{x}_0 = \sum_{s=1}^K f_{\mu,s}(\mathbf{x}_0, \beta)\mathbf{v}(s) = \sum_{s=1}^K \mu(s)\mathbf{v}(s)$, which implies $\mathbf{W}_n \xrightarrow{a.s.} \mathbf{W}$.

In order to show the existence of a stable attracting area which contains a neighborhood of $\mathbf{W}$, it is sufficient (see [7, p. 1077]) to show that $\{\Re e(Sp(\mathcal{D}f_W(\mathbf{W}))) > 0\}$, where

$$\mathcal{D}f_W(\mathbf{W}) = \begin{pmatrix} \mathcal{D}_Z f_Z(\mathbf{W}) & 0 & 0 \\ \mathcal{D}_Z f_N(\mathbf{W}) & \mathcal{D}_N f_N(\mathbf{W}) & \mathcal{D}_\beta f_N(\mathbf{W}) \\ 0 & 0 & \mathcal{D}_\beta f_\beta(\mathbf{W}) \end{pmatrix}, \tag{B.21}$$

and

(i) $\mathcal{D}_Z f_Z(\mathbf{W})$ is a block-diagonal matrix, whose t^{th} block is $[\mathcal{D}_Z f_Z(\mathbf{W})]^{tt} = (I - H(t) + \mathbf{v}(t)\mathbf{1}^\top)$;
(ii) $\mathcal{D}_Z f_N(\mathbf{W}) := -(\mu(1)I, .., \mu(K)I)$;
(iii) $\mathcal{D}_N f_N(\mathbf{W}) := I - \sum_{s=1}^K \mathbf{v}(s)\mathcal{D}_N f_{\mu,s}(\mathbf{W})^\top$;
(iv) $\mathcal{D}_\beta f_N(\mathbf{W}) := -\sum_{s=1}^K \mathbf{v}(s)\mathcal{D}_\beta f_{\mu,s}(\mathbf{W})^\top$;
(v) $\mathcal{D}_\beta f_\beta(\mathbf{W})$ is a block-diagonal matrix, whose j^{th} block is $[\mathcal{D}_\beta f_\beta(\mathbf{W})]^{jj} = \mathcal{D}f_j(\beta^j)$.

Note from the structure of $\mathcal{D}f_W(\mathbf{W})$ in (B.21) that $\{\Re e(Sp(\mathcal{D}f_W(\mathbf{W}))) > 0\}$ follows by establishing that for any $t \in \tau$ and $j \in \{1, .., d\}$

$$\{\Re e(Sp(I - H(t) + \mathbf{v}(t)\mathbf{1}^\top)) > 0\}, \quad \text{and} \quad \{\Re e(Sp(\mathcal{D}f_j(\beta^j))) > 0\},$$

and $\{\Re e(Sp(\sum_{s=1}^K \mathbf{v}(s)\mathcal{D}_N f_{\mu,s}(\mathbf{W})^\top)) < 1\}$. Analogous to the proof of Theorem 3.2, the first two conditions follow from (A5) and (A6b), respectively, while the last condition follows from (A7b).

We now show that the assumptions of the CLT for processes in the SA form are satisfied by the dynamics in (B.20) of the joint process $\{\mathbf{W}_n, n \geq 1\}$. First, note that using the above arguments we obtain $\{\Re e(Sp(\mathcal{D}f_W(\mathbf{W})) > 1/2\}$. Then, it is immediate to see that $\mathbb{E}[\|\mathbf{R}_{W,n}\|^2] = o(n)$ and the first condition in (B.9) is satisfied using (3.4) in Assumption (A6b). Concerning the second condition in (B.9), we need to show that there exists a deterministic symmetric positive semidefinite matrix Γ such that

$$\mathbb{E}[\Delta\mathbf{M}_{W,n}(\Delta\mathbf{M}_{W,n})^\top|\mathcal{F}_{n-1}] \xrightarrow{a.s.} \Gamma = \begin{pmatrix} \Gamma_{ZZ} & \Gamma_{ZN} & \Gamma_{Z\beta} \\ \Gamma_{ZN}^\top & \Gamma_{NN} & \Gamma_{N\beta} \\ \Gamma_{Z\beta}^\top & \Gamma_{N\beta}^\top & \Gamma_{\beta\beta} \end{pmatrix}.$$

As in the proof of Theorem 3.2, note that since $f_j(\hat{\beta}^j_{n-1}) \xrightarrow{a.s.} 0$, this term does not contribute to Γ; hence in the following calculations they will be omitted by $\Delta\mathbf{M}_{\beta^j,n}$. We now proceed with the computation of the terms in Γ. The calculations that follow by (B.17) are here omitted since they are analogous in the proof of Theorem 3.2.

Computation of $\Gamma_{ZZ} := a.s. - \lim_n \mathbb{E}[\Delta\mathbf{M}_{Z,n}(\Delta\mathbf{M}_{Z,n})^\top|\mathcal{F}_{n-1}]$. For any $t_1, t_2 \in \tau$

$$\Gamma_{ZZ}^{t_1t_2} := \mathbb{E}[D(t_1)\mathbf{g}(t_1, T, \bar{\mathbf{X}})\mathbf{g}^\top(t_2, T, \bar{\mathbf{X}})D^\top(t_2)] - \mathbf{v}(t_1)\mathbf{v}^\top(t_2),$$

where $\mathbf{g} \in \mathcal{S}$ is a d-multivariate function defined in (B.16).

Computation of $\Gamma_{NN} := a.s. - \lim_n \mathbb{E}[\Delta\mathbf{M}_{N,n}(\Delta\mathbf{M}_{N,n})^\top|\mathcal{F}_{n-1}]$. Note that

$$\begin{aligned} &\mathbb{E}[\Delta\mathbf{M}_{N,n}(\Delta\mathbf{M}_{N,n})^\top|\mathcal{F}_{n-1}] \\ &= diag\left(\sum_{s=1}^K \mu_{n-1}(s)\mathbf{Z}_{n-1}(s)\right) - \left(\sum_{s=1}^K \mu_{n-1}(s)\mathbf{Z}_{n-1}(s)\right)\left(\sum_{s=1}^K \mu_{n-1}(s)\mathbf{Z}_{n-1}(s)\right)^\top \\ &\xrightarrow{a.s.} \Gamma_{NN} := diag\left(\sum_{s=1}^K \mu(s)\mathbf{v}(s)\right) - \left(\sum_{s=1}^K \mu(s)\mathbf{v}(s)\right)\left(\sum_{s=1}^K \mu(s)\mathbf{v}(s)\right)^\top. \end{aligned}$$

Computation of $\Gamma_{ZN} := a.s. - \lim_n \mathbb{E}[\Delta\mathbf{M}_{Z,n}(\Delta\mathbf{M}_{N,n})^\top|\mathcal{F}_{n-1}]$. For any $t \in \tau$

$$\begin{aligned} &\mathbb{E}[\Delta\mathbf{M}_{Z(t),n}(\Delta\mathbf{M}_{N,n})^\top|\mathcal{F}_{n-1}] = \mathbb{E}[D_n(t)\mathbf{X}_n(t)(\Delta\mathbf{M}_{N,n})^\top|\mathcal{F}_{n-1}] \\ &= \sum_{s=1}^K \mu_{n-1}(s)\left(\mathbb{E}[D_n(t)\mathbf{X}_n(t)\bar{\mathbf{X}}_n^\top|\mathcal{F}_{n-1}, T_n = s] - H(t)\mathbf{Z}_{n-1}(t)\mathbf{Z}_{n-1}^\top(s)\right) \\ &\xrightarrow{a.s.} \Gamma_{ZN}^t := \sum_{s=1}^K \mu(s)\left(H(t_1)G(t_1,s)diag(\mathbf{v}(s)) - \mathbf{v}(t_1)\mathbf{v}^\top(s)\right), \end{aligned}$$

where $G(t_1, s)$ is a matrix whose columns are $\{\mathbf{g}(t_1, s, \mathbf{e}_j); j \in \{1, .., d\}\}$.

Computation of $\Gamma_{\beta\beta} := a.s. - \lim_n \mathbb{E}[\Delta\mathbf{M}_{\beta,n}(\Delta\mathbf{M}_{\beta,n})^\top|\mathcal{F}_{n-1}]$. Since for any $j_1 \neq j_2$ we have $\mathbb{E}[\Delta\mathbf{M}_{\beta^{j_1},n}(\Delta\mathbf{M}_{\beta^{j_2},n})^\top|\mathcal{F}_{n-1}] = 0$, $\Gamma_{\beta\beta}$ is a block-diagonal matrix. In particular, for any $j \in \{1,..,d\}$ we have

$$\begin{aligned}
&\mathbb{E}[\Delta\mathbf{M}_{\beta^j,n}(\Delta\mathbf{M}_{\beta^j,n})^\top|\mathcal{F}_{n-1}] = \\
&(\tilde{N}^j_{n-1})^{-2}\left(\sum_{s=1}^{K}\mu_{i-1}(s)Z^j_{i-1}(s)\right)\mathbb{E}[\Delta\mathbf{M}_{j,n}(\Delta\mathbf{M}_{j,n})^\top|\mathcal{F}_{n-1},\bar{X}^j_n=1] \\
&\xrightarrow{a.s.}\Gamma^{jj}_{\beta\beta} := \left(\sum_{s=1}^{K}\mu(s)v^j(s)\right)^{-1}\mathbb{E}[\Delta\mathbf{M}_j(\Delta\mathbf{M}_j)^\top|\bar{X}^j=1].
\end{aligned}$$

Computation of $\Gamma_{Z\beta} := a.s. - \lim_n \mathbb{E}[\Delta\mathbf{M}_{Z,n}(\Delta\mathbf{M}_{\beta,n})^\top|\mathcal{F}_{n-1}]$. For any $t \in \tau$ and $j \in \{1,..,d\}$, we have that

$$\begin{aligned}
&\mathbb{E}[\Delta\mathbf{M}_{Z(t),n}(\Delta\mathbf{M}_{\beta^j,n})^\top|\mathcal{F}_{n-1}] = \mathbb{E}[D_n(t)\mathbf{X}_n(t)(\Delta\mathbf{M}_{\beta^j,n})^\top|\mathcal{F}_{n-1}] \\
&= (\tilde{N}^j_{n-1})^{-1}(\sum_{s=1}^{K}\mu_{i-1}(s)Z^j_{i-1}(s))\mathbb{E}[D_n(t)\mathbf{X}_n(t)(\Delta\mathbf{M}_{j,n})^\top|\mathcal{F}_{n-1},\bar{X}^j_n=1] \\
&\xrightarrow{a.s.}\Gamma^{tj}_{Z\beta} := \mathbb{E}[D(t)\mathbf{g}(t,T,\mathbf{e}_j)\Delta\mathbf{M}_j^\top|\bar{X}^j=1].
\end{aligned}$$

Computation of $\Gamma_{N\beta} := a.s. - \lim_n \mathbb{E}[\Delta\mathbf{M}_{N,n}(\Delta\mathbf{M}_{\beta,n})^\top|\mathcal{F}_{n-1}]$. It can immediately be seen that, for any $j \in \{1,..,d\}$,

$$\mathbb{E}[\Delta\mathbf{M}_{N,n}(\Delta\mathbf{M}_{\beta^j,n})^\top|\mathcal{F}_{n-1}] = \mathbb{E}[\Delta\mathbf{M}_{N,n}\mathbb{E}[\Delta\mathbf{M}_{\beta^j,n}|\mathcal{F}_{n-1},T_n,\bar{\mathbf{X}}_n]|\mathcal{F}_{n-1}] = 0 = \Gamma^j_{N\beta}.$$

Since the assumptions are all satisfied, we can apply the CLT of the SA to the dynamics (B.20), so obtaining a Gaussian asymptotic distribution for the process $\{\mathbf{W}_n; n \geq 1\}$, with asymptotic variance

$$\Sigma \ := \int_0^\infty e^{u(\frac{\mathbf{I}}{2}-\mathcal{D}f_W(\mathbf{W}))}\Gamma e^{u(\frac{\mathbf{I}}{2}-\mathcal{D}f_W(\mathbf{W}))^\top}du.$$

This concludes the proof. □

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