Elsevier Editorial System(tm) for Journal of Approximation Theory
Manuscript Draft

Manuscript Number:

Title: Quasi uniform convexity - revisited

Article Type: Regular Article

Keywords: Quasi uniformly convex Banach space; Chebyshev center; Approximate center.

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We show that (QUC) is equivalent to existence and continuous dependence (in the Hausdorff metric) of Chebyshev centers of bounded sets.
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Together with Chebyshev centers, we consider also asymptotic centers
for bounded sequences or nets (of points or sets).

***Cover Letter**

To: Journal of Approximation Theory

Dear Sirs,

I am sending you my manuscript
"Quasi uniform convexity - revisited"
as a contribution to the Journal of Approximation Theory.

Yours sincerely,

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# Quasi uniform convexity – revisited

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**Abstract**

Quasi uniform convexity (QUC) is a geometric property of Banach spaces, introduced in 1973 by J.R. Calder et al., which implies existence of Chebyshev centers for bounded sets. We extend and strengthen some known results about this property. We show that (QUC) is equivalent to existence and continuous dependence (in the Hausdorff metric) of Chebyshev centers of bounded sets. If $X$ is (QUC) then the space $C(K;X)$ of continuous $X$-valued functions on a compact $K$ is (QUC) as well. We also show that a sufficient condition introduced by L. Pevac already implies (QUC), and we provide a couple of new sufficient conditions for (QUR). Together with Chebyshev centers, we consider also asymptotic centers for bounded sequences or nets (of points or sets).

*Keywords:* Quasi uniformly convex Banach space, Chebyshev center, Approximate center.

## 1. Introduction

Let $X$ be a Banach space. A point $x \in X$ is a *Chebyshev center* of a nonempty bounded set $A \subset X$ if, roughly speaking, $x$ is the center of a closed ball $B$ of smallest radius such that $A \subset B$ (see Definition 3.3). An important question in Approximation Theory is the question about existence of Chebyshev centers. An easy $w^*$-compactness argument shows that if $X$ is a dual space then every bounded set in $X$ admits at least one Chebyshev center. And the same conclusion now follows for every Banach space that is norm-one

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complemented in its bidual, like $L_1[0,1]$ (indeed, take a Chebyshev center of $A$ in $X^{**}$ and project it to $X$ by a contractive projection to get a Chebyshev center in $X$).

However, some spaces (like $c_0$, $c$, $C[0,1]$), in which all bounded sets admit Chebyshev centers, are not complemented in their biduals. In 1973, Calder, Coleman and Harris [5] introduced a new geometric condition, subsequently called "quasi uniform convexity" (here denoted by (QUC), see Definition 2.1), which is sufficient for existence of Chebyshev centers, and is satisfied by the spaces $c_0$, $c$, $C[0,1]$, and similar. Roughly speaking, existence of Chebyshev centers is based on completeness instead of compactness, in this case.

The following three theorems collect main known results concerning (QUC) spaces. For the definition of Chebyshev radius see Definition 3.1.

**Theorem 1.1** ([2]). *If $X$ is (QUC) then every bounded set $A \subset X$ has a Chebyshev center, and the Chebyshev-center map (assigning to any bounded closed subset of $X$ the set of its Chebyshev centers) is uniformly Hausdorff continuous on families of sets with equi-bounded Chebyshev radii.*

Let us remark that existence of Chebyshev centers in (QUC) spaces was proved already in [5]. The following result somehow motivates the terminology. It was essentially proved in [5], for a better presentation see [2].

**Theorem 1.2** ([5], [2]). *$X$ is uniformly convex if and only if $X$ is (QUC) and strictly convex.*

**Theorem 1.3.** *The following spaces are (QUC):*

*(a)* $\ell_\infty$, $c_0$, $c$, $C[0,1]$ (see [5]);

*(b) the space $C(K;X)$ of $X$-valued continuous functions on a compact Hausdorff space $K$, provided $X$ is uniformly convex* (see [2]).

*On the other hand, infinite-dimensional $L_1(\mu)$ spaces are not (QUC)* (see [2]).

Let us remark that a (formally) weaker version of (b), saying that if $X$ is uniformly convex then every bounded set $A \subset C(K;X)$ has a Chebyshev center and the corresponding Chebyshev-center map is Hausdorff continuous, was proved already in [1].

Another important relative notion is the notion of *asymptotic center* of a bounded sequence or net (of points or sets); see Definitions 3.2 and 3.3, and

Remark 3.4. Also here a compactness argument works to prove existence of asymptotic centers, but only for reflexive spaces (indeed, the function to minimize is convex and continuous, and hence also weakly lower semicontinuous). In dual spaces, however, asymptotic centers may exist or not (see [3], pp. 421–422). The paper [3], considering a related geometric property in complete metric spaces, more or less contains a proof that in a (QUC) Banach space every bounded net (or sequence) admits an asymptotic center (see [3, Theorem 3]); however, the idea of proof is practically the same as for existence of Chebyshev centers in (QUC) spaces (cfr. Theorem 3.7 below).

The present paper contains some new results on quasi uniform convexity that extend or strengthen some known ones. Let us briefly describe them. In Section 2, we characterize (QUC) by two formally weaker properties, and then we show that a (bit complicated) sufficient condition for existence of Chebyshev centers, introduced by Pevac in [9] (see also [10]), gives nothing more general since it already implies (QUC) (see Corollary 2.3).

In Section 3, we characterize (QUC) by, roughly speaking, existence of Chebyshev centers or asymptotic centers plus some kind of uniform continuity property in the Hausdorff metric. In particular, it follows that also the reverse implication holds in the above Theorem 1.1. For the case of relative Chebyshev centers, see Remark 3.12.

In Section 4, we consider some spaces of $X$-valued functions (or sequences). In particular, we generalize Theorem 1.3(b) by showing that $C(K;X)$ is (QUC) whenever $X$ is (QUC). As an easy corollary, we obtain that if $X$ is (QUC) then, for instance, the space $C_0(L;X)$ of continuous $X$-valued functions that vanish at infinity (where $L$ is a locally compact topological space) is (QUC) as well.

In the last section we show that if $X$ either has dimension two or is finite-dimensional and polyhedral then it is (QUC). In particular, for such $X$ the spaces $C(K;X)$ and $C_0(L;X)$ are (QUC).

## 2. The properties

In what follows, $X$ denotes a (real or complex) Banach space of dimension at least two, whose closed unit ball and unit sphere will be denoted by $B_X$ and $S_X$, respectively. The closed ball with center $x \in X$ and radius $r \geq 0$ is $B(x,r) := x + rB_X$.

**Definition 2.1.**

- We say that $X$ is *(QUC)* (*quasi uniformly convex*, [5]) if:

for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for each $x \in X$ there is $y \in B(0, \varepsilon)$ with $B(0, 1 + \delta) \cap B(x, 1) \subset B(y, 1)$.

- We say that $X$ is *(Pquc)* (*"Pevac quasi uniformly convex"*) if:

  there exists $\alpha \in [0, 1)$ such that:

  (i) there exists a function $\varepsilon\colon (0, 1) \to (0, 1)$ such that $\varepsilon(\delta) \to 0$ as $\delta \to 0^+$, and for each $x \in X$ there is $y \in B(0, \varepsilon(\delta))$ such that $B(0, 1 + \delta) \cap B(x, 1) \subset B(y, 1 + \alpha\delta)$;

  (ii) either $\alpha = 0$ or $\sum_{n=1}^{+\infty} \varepsilon(\alpha^n) < \infty$.

Our definition of (QUC) is a (clearly equivalent) slight reformulation of the original definition in [5, Definition 2.3] (under different terminology) and [2] (for $Y = X$). Let us underline that we only consider here "absolute" (QUC) and not "relative" (QUC) w.r.t. a closed subspace – see Remark 3.12.

The property (Pquc) was introduced by L. Pevac [9, 10] under a different terminology: he calls $X$ to be "$\alpha$-approximative" if it satisfies (i), and then proves that if $X$ is $\alpha$-approximative for some $\alpha \in [0, 1)$ and satisfies (ii) then every bounded set in $X$ admits a Chebyshev center. We are going to show that (Pquc) in fact implies (QUC), and hence the theorem by Pevac already follows from previously known results on (QUC) spaces.

Let us start with characterizing (QUC) by two formally weaker properties.

**Theorem 2.2.** *The following properties are equivalent.*

*(i) $X$ is* (QUC)*.*

*(ii) For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for each $x \in X$ and $\beta > 0$ there is $y \in B(0, \varepsilon)$ with $B(0, 1 + \delta) \cap B(x, 1) \subset B(y, 1 + \beta)$.*

*(iii) There exist sequences $(\varepsilon_n)$ and $(\delta_n)$ of positive reals such that $\delta_n \to 0$, $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, and for each $n \in \mathbb{N}$ and $x \in X$ there is $y \in B(0, \varepsilon_n)$ with $B(0, 1 + \delta_n) \cap B(x, 1) \subset B(y, 1 + \delta_{n+1})$.*

*Proof.* The implication $(i) \Rightarrow (ii)$ is obvious.

To show $(ii) \Rightarrow (iii)$, fix an arbitrary sequence $(\varepsilon_n)$ of positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, and put $\delta_n := \delta(\varepsilon_n)$ for each $n$. The inclusion in (iii) follows by considering $\beta = \delta_{n+1}$. It remains to show that $\delta_n \to 0$. For each $n$, take some $x$ of norm $1 + \delta_n$. By (ii), for each $\beta > 0$ there is $y \in B(0, \varepsilon_n)$

such that $x \in B(y, 1+\beta)$. Now, $1+\delta_n = \|x\| \le \|x-y\| + \|y\| \le 1+\beta+\varepsilon_n$. Arbitrariety of $\beta > 0$ implies $\delta_n \le \varepsilon_n$, and hence $(\delta_n)$ is null.

Finally, let us show that *(iii)* $\Rightarrow$ *(i)*. Fix an arbitrary $\varepsilon > 0$. By omitting first finitely many terms of $(\varepsilon_n)$ and $(\delta_n)$, we can (and do) suppose that $\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon$. Let $x \in X$. By inductively applying (ii), we easily get a sequence $(y_n)$ in $X$ such that

$$\begin{aligned} B(0,1+\delta_1) \cap B(x,1) &\subset B(y_1, 1+\delta_2) \cap B(x,1) \\ &\subset B(y_2, 1+\delta_3) \cap B(x,1) \\ &\subset \dots \end{aligned}$$

with $\|y_1\| \le \varepsilon_1$ and $\|y_n - y_{n-1}\| \le \varepsilon_n$ ($n \ge 2$). Since $(y_n)$ is a Cauchy sequence, it converges to some $\bar{y} \in X$. Moreover, $\|\bar{y}\| \le \|y_1\| + \sum_{n=2}^{\infty} \|y_n - y_{n-1}\| < \varepsilon$, and

$$B(0,1+\delta_1) \cap B(x,1) \subset \bigcap_{n=1}^{\infty} B(y_n, 1+\delta_{n+1}) \subset B(\bar{y}, 1).$$

Thus (i) is satisfied with $\delta(\varepsilon) = \delta_1$. The proof is complete. □

**Corollary 2.3.** *If $X$ is* (Pquc) *then it is also* (QUC).

*Proof.* Let $X$ be (Pquc). If $\alpha = 0$, it is easy to see that $X$ is (QUC). If $\alpha \in (0,1)$, the condition (iii) in Theorem 2.2 is satisfied with $\delta_n = \alpha^n$ and $\varepsilon_n = \varepsilon(\alpha^n)$. □

## 3. Asymptotic centers and Chebyshev centers

By $\mathcal{BC}(X)$ we denote the complete metric space of all nonempty, bounded, closed subsets of $X$, equipped with the Hausdorff metric

$$\begin{aligned} h(A,B) &= \max\big\{\sup_{a\in A} d(a,B)\,,\, \sup_{b \in B} d(b,A)\big\} \\ &= \inf\{\varepsilon > 0 :\ A \subset B + \varepsilon B_X \text{ and } B \subset A + \varepsilon B_X\}. \end{aligned}$$

*In what follows, continuity from or into $\mathcal{BC}(X)$ is always intended in the Hausdorff metric.*

**Definition 3.1.** Given $A \in \mathcal{BC}(X)$, we denote

$$\begin{aligned} r(A,x) &:= \sup_{a \in A} \|x - a\| \qquad (x \in X), \\ r(A) &:= \inf_{x \in X} r(A,x), \\ Z_r(A) &:= \{x \in X : r(A,x) \le r\} \qquad (r \ge r(A)), \\ Z(A) &:= Z_{r(A)}(A). \end{aligned}$$

The nonnegative real number $r(A)$ is the *Chebyshev radius* of $A$, and the (possibly empty) set $Z(A)$ is the set of *Chebyshev centers* of $A$.

The above notations and notions have the following analogues for bounded nets in $\mathcal{BC}(X)$.

**Definition 3.2.** Given a bounded nonincreasing (w.r.t. the set inclusion) net $\mathcal{A} = (A_i)_{i\in I}$ in $\mathcal{BC}(X)$, we denote

$$\begin{aligned}
\varphi(\mathcal{A}, x) &:= \lim_{i\in I} r(A_i, x) = \inf_{i\in I} r(A_i, x) \qquad (x \in X),\\
r(\mathcal{A}) &:= \inf_{x\in X} \varphi(\mathcal{A}, x) = \inf_{x\in X} \inf_{i\in I} r(A_i, x),\\
Z_r(\mathcal{A}) &:= \{x \in X : \varphi(\mathcal{A}, x) \le r\} \qquad (r \ge r(\mathcal{A})),\\
Z(\mathcal{A}) &:= Z_{r(\mathcal{A})}(\mathcal{A}).
\end{aligned}$$

The nonnegative real number $r(\mathcal{A})$ is the *asymptotic radius* of the net $\mathcal{A}$, and the (possibly empty) set $Z(\mathcal{A})$ is the set of *asymptotic centers* of $\mathcal{A}$.

Let us also recall the classical notion of asymptotic radius and center of a bounded sequence.

**Definition 3.3.** Given a bounded sequence $(a_n)$ in $X$, we denote

$$\begin{aligned}
\rho((a_n), x) &:= \limsup_{n\to\infty} \|x - a_n\| \qquad (x \in X),\\
r(a_n) &:= \inf_{x\in X} \rho((a_n), x),\\
Z_r(a_n) &:= \{x \in X : \rho((a_n), x) \le r\} \qquad (r \ge r(a_n)),\\
Z(a_n) &:= Z_{r(a_n)}(a_n).
\end{aligned}$$

The nonnegative real number $r(a_n)$ is the *asymptotic radius* of the sequence $(a_n)$, and the (possibly empty) set $Z(a_n)$ is the set of *asymptotic centers* of $(a_n)$.

**Remark 3.4.** Let us see that Definition 3.2 is, in a sense, the most general one since it "includes" the other two and their formal extensions.

(a) Definition 3.1 could be formulated for any nonempty bounded (not necessarily closed) set $A \subset X$, but this would give nothing new: the notations and notions therein would be the same for $A$ as for $\overline{A}$.

(b) Definition 3.2 can be reformulated for any equi-bounded net $\mathcal{B} = (B_i)_{i\in I}$ of nonempty (not necessarily closed) sets in $X$ in a similar way as in Definition 3.3, by defining $\varphi(\mathcal{B}, x) := \inf_{i\in I} \sup_{j\geq i} r(B_j, x)$; but then the nonincreasing net $\mathcal{A} = (A_i)_{i\in I}$ in $\mathcal{BC}(X)$, given by $A_i := \overline{\bigcup_{j\geq i} B_j}$, satisfies $\varphi(\mathcal{B}, \cdot) = \varphi(\mathcal{A}, \cdot)$, and the right-hand side function is the same as in Definition 3.2.

(c) Definitons 3.3 and 3.1 can be seen as particular cases of Definition 3.2. Indeed, a bounded sequence $(a_n) \subset X$ can be viewed as the net $\mathcal{B} = (\{a_n\})_{n\in\mathbb{N}}$ of singletons which generates a nonincreasing net $\mathcal{A} = (A_n)_{n\in\mathbb{N}} \subset \mathcal{BC}(X)$ via (b); and a set $A \in \mathcal{BC}(X)$ can be assigned a constant net $\mathcal{A} = (A)_{i\in I}$ (with any upper-directed index set $I$).

We are going to show that (QUC) implies existence and a kind of uniform continuity for asymptotic centers. We start with two auxiliary facts.

**Observation 3.5.** *Let $X$ be* (QUC)*, and $\delta = \delta(\varepsilon)$ as in the corresponding definition. Assume that $x, u \in X$, $r_1 > r_2 > 0$, and $\frac{r_1}{r_2} - 1 \leq \delta$. Then there exists $y \in B(x, r_2\varepsilon)$ with $B(x, r_1) \cap B(u, r_2) \subset B(y, r_2)$.*

*Proof.* This is a trivial consequence of the inclusion $B(x, r_1) \cap B(u, r_2) \subset B(x, r_2(1 + \delta)) \cap B(u, r_2)$. □

**Lemma 3.6.** *Let $\mathcal{A} = (A_i)_{i\in I}$ be a nonincreasing net in $\mathcal{BC}(X)$, and $\varepsilon > 0$. If $r(\mathcal{A}) \leq r < r'$, $r < \frac{\varepsilon}{3}$ and $r' - r < \frac{\varepsilon}{3}$, then $h(Z_r(\mathcal{A}), Z_{r'}(\mathcal{A})) \leq \varepsilon$.*

*Proof.* We have $r' < \frac{2\varepsilon}{3}$. Moreover, for each $x \in Z_{r'}(\mathcal{A})$ and $y \in Z_r(\mathcal{A})$ there exists $i \in I$ such that $(\emptyset \neq)\, A_i \subset B(x, r') \cap B(y, r)$; but this leads to $\|x - y\| \leq \varepsilon$. Now, the assertion clearly follows. □

As already remarked in Introduction, the following theorem is only formally new since the idea of its proof is practically the same as that for Chebyshev centers (proof of Theorem 1.1; see also [3]). For the sake of completeness, we present a complete proof of this version which will be useful for us.

**Theorem 3.7.** *Let $X$ be a* (QUC) *Banach space. For every $\varepsilon > 0$ and $R > 0$ there exists $\eta > 0$ such that if $\mathcal{A} = (A_i)_{i\in I}$ is a bounded nonincreasing net in $\mathcal{BC}(X)$ with $r(\mathcal{A}) < R$, and $r, r' \in [r(\mathcal{A}), R)$ are such that $|r - r'| < \eta$, then $Z_r(\mathcal{A}), Z_{r'}(\mathcal{A}) \in \mathcal{BC}(X)$ and*

$$h(Z_r(\mathcal{A}), Z_{r'}(\mathcal{A})) \leq \varepsilon. \tag{1}$$

*In particular, the mapping $r \mapsto Z_r(\mathcal{A})$ has nonempty values and is (Hausdorff) continuous for $r \geq r(\mathcal{A})$.*

*Proof.*

(a) Let $r(\mathcal{A}) \leq r < r' < R < \infty$ be such that $r > 0$. Fix a sequence $(\varepsilon_k)$ in $(0, \infty)$ so that $\varepsilon_1 = \frac{\varepsilon}{2R}$ and $\sum_{k=1}^{\infty} \varepsilon_k \leq \frac{\varepsilon}{R}$. Let $r' - r$ be so small that

$$\frac{r'}{r} - 1 < \delta(\varepsilon_1)$$

(where $\delta(\cdot)$ is as in the definition of (QUC)). There exists a sequence $(r_k)$ of positive numbers such that $R > r_1 > r' > r_2 > r_3 > \ldots,\ r_k \to r$, and

$$\frac{r_k}{r} - 1 < \delta(\varepsilon_k) \quad \text{for each } k \in \mathbb{N}.$$

Now, let us describe an inductive procedure for defining two sequences $(x_k)$ and $(i_k)$ in $X$ and $I$, respectively.

Notice that $Z_r(\mathcal{A}) \subset Z_{r'}(\mathcal{A})$, and fix an arbitrary $x_1 \in Z_{r'}(\mathcal{A})$. There exist an index $i_1 \in I$ and $y_1 \in X$ such that $A_{i_1} \subset B(x_1, r_1) \cap B(y_1, r_2)$. Since $\frac{r_1}{r_2} - 1 < \frac{r_1}{r} < \delta(\varepsilon_1)$, we can use Observation 3.5 to get $x_2 \in X$ such that $A_{i_1} \subset B(x_2, r_2)$ and $\|x_2 - x_1\| \leq r_2 \varepsilon_1 < R\varepsilon_1$. An easy inductive procedure leads to sequences $(x_k)$, $(y_k)$ in $X$, and a nondecreasing sequence $(i_k)$ in $I$ such that

$$A_{i_k} \subset B(x_k, r_k) \cap B(y_k, r_{k+1}) \quad \text{and} \quad \|x_{k+1} - x_k\| \leq R\varepsilon_k$$

for each $k \in \mathbb{N}$. Since $(x_k)$ is a Cauchy sequence, it converges to some $\bar{x} \in X$. Since $\varphi(\mathcal{A}, x_k) < r_k$ for each $k$, we have that $\bar{x} \in Z_r(\mathcal{A})$. Moreover,

$$\|\bar{x} - x_1\| = \|\textstyle\sum_{k=1}^{\infty} (x_{k+1} - x_k)\| \leq R \sum_{k=1}^{\infty} \varepsilon_k \leq \varepsilon.$$

This shows that $Z_r(\mathcal{A}) \neq \emptyset$ and $h(Z_r(\mathcal{A}), Z_{r'}(\mathcal{A})) \leq \varepsilon$.

Notice that this shows that $Z(\mathcal{A}) \neq \emptyset$. To complete the proof, let us show that

$$\eta := \tfrac{\varepsilon}{3} \min\{1\,,\, \delta(\tfrac{\varepsilon}{2R})\}$$

works. We shall consider two cases.

(b) Case $r < \frac{\varepsilon}{3}$. If $r' - r < \eta$ then (1) follows from Lemma 3.6.

(c) Case $r \geq \frac{\varepsilon}{3}$. If $r' - r < \eta$ then $\frac{r'-r}{r} < \frac{3\eta}{\varepsilon} \leq \delta(\frac{\varepsilon}{2R}) = \delta(\varepsilon_1)$. Thus (1) holds by part (a) of the present proof. We are done. □

**Lemma 3.8.** *Let $0 < r \leq R < \infty$ and $a, b \in X$ be such that $\|a - b\| \leq r + R$. Then the (nonempty) sets*

$$A := B(a, R) \cap B(b, r) \quad \textit{and} \quad B := B(a, r) \cap B(b, R)$$

*satisfy $h(A, B) \leq 3(R - r)$.*

*Proof.* Define $c := b - a$, $z := \frac{r}{R+r}a + \frac{R}{R+r}b$, and $\widetilde{z} := \frac{R}{R+r}a + \frac{r}{R+r}b$. It is easy to see that $z \in A$, $\widetilde{z} \in B$, and

$$z + h \in A \quad \Leftrightarrow \quad \widetilde{z} - h \in B. \tag{2}$$

We *claim* that if $z + h \in A$ then $z - \frac{r}{R}h \in A$. To show this, assume that $z + h \in A$, that is,

$$\left\|\tfrac{R}{R+r}c + h\right\| \leq R \quad \text{and} \quad \left\|-\tfrac{r}{R+r}c + h\right\| \leq r. \tag{3}$$

The first inequality in (3) is equivalent to $\left\|-\frac{r}{R+r}c - \frac{r}{R}h\right\| \leq r$, which means that $z - \frac{r}{R}h \in B(b, r)$. The second inequality in (3) implies that

$$\left\|\tfrac{R}{R+r}c - \tfrac{r}{R}h\right\| \leq \tfrac{r}{R}\left\|\tfrac{r}{R+r}c - h\right\| + \tfrac{1}{R+r}(R - \tfrac{r^2}{R})\|c\| \leq \tfrac{r^2}{R} + (R - \tfrac{r^2}{R}) = R,$$

which gives that $z - \frac{r}{R}h \in B(a, R)$, and our claim is proved.

Now, if $z+h \in A$ then $z - \frac{r}{R}h \in A$, and hence $\widetilde{z} + \frac{r}{R}h \in B$ by (2). Moreover, $\|h\| \leq \operatorname{diam} B(b, r) = 2r$, and

$$\left\|(z + h) - (\widetilde{z} + \tfrac{r}{R}h)\right\| = \left\|-\tfrac{R-r}{R+r}c + (1 - \tfrac{r}{R})h\right\| \leq (R - r) + \tfrac{(R-r)2r}{R} \leq 3(R - r).$$

This shows that $A \subset B + 3(R - r)B_X$, and the assertion follows by symmetry. ☐

The following easy proposition, or some variant of it, is probably known, but we have not found any reference. Recall that a function $f\colon X \to \mathbb{R}$ is called *coercive* if $\lim_{\|x\| \to \infty} f(x) = \infty$.

**Proposition 3.9.** *Let $f\colon X \to \mathbb{R}$ be a coercive continuous convex function. Then $\alpha_0 := \inf f(X) > -\infty$, the sets $C_\alpha := \{x \in X : f(x) \leq \alpha\}$, $\alpha > \alpha_0$, belong to $\mathcal{BC}(X)$, and*

$$h(C_b, C_a) \leq \frac{\operatorname{diam} C_b}{b - \alpha_0}(b - a) \quad \textit{whenever } \alpha_0 < a < b.$$

*Proof.* The (closed) sets $C_\alpha$ are bounded by coercivity of $f$. Moreover, $f$ is bounded below since it is minorized by a continuous affine function and such affine functions are bounded on the bounded set $C_{\alpha_0+1}$.

To show the last part, fix $x \in C_b$, $\alpha_1 \in (\alpha_0, a)$, and $y \in C_{\alpha_1}$. For $t \in (0,1)$, put $z_t := ty + (1-t)x$ and notice that $f(z_t) \le t\alpha_1 + (1-t)b$ by convexity. An easy calculation shows that for $s := \frac{b-a}{b-\alpha_1}$ we have $z_s \in C_a$. Moreover, $\|z_s - x\| = s\|y - x\| \le \frac{b-a}{b-\alpha_1} \operatorname{diam} C_b$. We have proved that $h(C_b, C_a) \le \frac{\operatorname{diam} C_b}{b-\alpha_1}(b-a)$ whenever $\alpha_1 \in (\alpha_0, a)$. Passing to the limit for $\alpha_1 \to \alpha_0$ concludes the proof. □

Now we are ready for the main result of the present section.

**Theorem 3.10.** *For a Banach space $X$, the following assertions are equivalent:*

*(i)* $X$ *is* (QUC)*;*

*(ii) for every bounded nonincreasing net $\mathcal{A}$ in $\mathcal{BC}(X)$, $Z(\mathcal{A})$ is nonempty, the mapping $r \mapsto Z_r(\mathcal{A})$ is continuous on $[r(\mathcal{A}), \infty)$, and this continuity is uniform for nets with equi-bounded approximate radii;*

*(iii) every $A \in \mathcal{BC}(X)$ has a Chebyshev center, the mapping $r \mapsto Z_r(A)$ is continuous on $[r(A), \infty)$, and this continuity is uniform for sets with equi-bounded Chebyshev radii;*

*(iv) for every (some) $r_0 > 0$, the mapping $(A, r) \mapsto Z_r(A)$ has values in $\mathcal{BC}(X)$ and is uniformly continuous on the set*

$$\{(A, r) \in \mathcal{BC}(X) \times [0, \infty) : r(A) \le r \le r_0\};$$

*(v) for every (some) $r_0 > 0$, the mapping $Z_{r_0}(\cdot)$ has values in $\mathcal{BC}(X)$ and is uniformly continuous on the set $\{A \in \mathcal{BC}(X) : r(A) \le r_0\}$;*

*(vi) for every (some) $r_0 > 0$, the Chebyshev-center map $Z(\cdot)$ is nonempty-valued and uniformly continuous on the set $\{A \in \mathcal{BC}(X) : r(A) = r_0\}$.*

*Moreover, any of (i)–(vi) implies the property*

*(vii) every bounded sequence $(a_n)$ in $X$ has an asymptotic center, the mapping $r \mapsto Z_r(a_n)$ is continuous on $[r(a_n), \infty)$, and this continuity is uniform for sequences with equi-bounded approximate radius.*

*Finally, if $X$ is separable then (vii) is equivalent to any of (i)–(vi).*

*Proof.* The implication $(i) \Rightarrow (ii)$ is Theorem 3.7, and $(ii) \Rightarrow (iii)$ is clear.

$(iii) \Rightarrow (iv)$. Given $r_0 > 0$ and $\varepsilon > 0$, (iii) implies existence of $\delta \in (0,1)$ such that $h(Z_r(A), Z_{r'}(A)) < \varepsilon$ whenever $r(A) < R$ and $r, r' \in [r(A), R+1]$ are such that $|r - r'| < 2\delta$. Now, let $A, B \in \mathcal{BC}(X)$ and $t, s \in \mathbb{R}$ be such that $r(A) \le t \le r_0$, $r(B) \le s \le r_0$, $h(A,B) < \delta$ and $|t-s| < \delta$. Fix an arbitrary $x \in Z_t(A)$. Since $A \subset B(x,t)$, we have $B \subset B(x, t+\delta)$, and hence $x \in Z_{t+\delta}(B)$. Since $|(t+\delta) - s| < 2\delta$, there exists $y \in Z_s(B)$ with $\|x-y\| < \varepsilon$. By interchanging the role of $A, B$, we conclude that $h(Z_t(A), Z_s(B)) \le \varepsilon$.

The implications $(iv) \Rightarrow (v) \Rightarrow (vi)$ are obvious. The proof of the first part will be complete once we show that $(vi) \Rightarrow (iii) \Rightarrow (i)$.

$(vi) \Rightarrow (iii)$. Assume that $r(A) \le r_0$. By Proposition 3.9, if $r(A) < r < r' \le r_0 + 1$ then

$$h\big(Z_{r'}(A), Z_r(A)\big) \le \tfrac{\operatorname{diam} Z_{r'}(A)}{r' - r(A)}(r' - r) \le \tfrac{2(r_0+1)}{r - r(A)}(r' - r).$$

This shows that $r \mapsto Z_r(A)$ is always continuous on $(r(A), \infty)$, uniformly for $r(A) \le r_0$. Let us prove continuity at $r = r(A)$. Given $\varepsilon > 0$, we are looking for $\delta > 0$ such that

$$r(A) \le r_0,\ r(A) < r < r(A) + \delta \quad \Rightarrow \quad h\big(Z_r(A), Z(A)\big) \le \varepsilon. \tag{4}$$

(a) Case $r(A) < \frac{\varepsilon}{3}$. In this case $\delta = \frac{\varepsilon}{3}$ works by Lemma 3.6.
(b) Case $r(A) \ge \frac{\varepsilon}{3}$. By (vi), there exists $\Delta > 0$ such that

$$r(P) = r(Q) = 1,\ h(P,Q) < \Delta \quad \Rightarrow \quad h\big(Z(P), Z(Q)\big) < \tfrac{\varepsilon}{r_0}.$$

To show that $\delta = \frac{\varepsilon\Delta}{9}$ works, assume $r(A) < r < r(A) + \frac{\varepsilon\Delta}{9}$ and fix an arbitrary $x \in Z_r(A)$. Choose any $y \in Z(A)$. Consider the sets

$$C := B(x,r) \cap B(y, r(A)) \quad \text{and} \quad D := B(x, r(A)) \cap B(y, r).$$

Since $A \subset C \subset B(y, r(A))$ and $D = x + y - C$, we must have $r(C) = r(A) = r(D)$ and $x \in Z(D)$. By Lemma 3.8, $h(C,D) \le 3(r - r(A)) < \frac{\varepsilon\Delta}{3}$. The sets $\widetilde{C} := \frac{C}{r(A)}$ and $\widetilde{D} := \frac{D}{r(A)}$ are of Chebyshev radius 1 and satisfy $h(\widetilde{C}, \widetilde{D}) < \frac{\varepsilon\Delta}{3r(A)} \le \Delta$. Hence $h(Z(\widetilde{C}), Z(\widetilde{D})) < \frac{\varepsilon}{r_0}$, and therefore $h(Z(C), Z(D)) < \varepsilon$. Since $x \in Z(D)$, there is $z \in Z(C)$ with $\|z - x\| < \varepsilon$. Since $A \subset C \subset$

$B(z, r(A))$, we have that $z \in Z(A)$. This shows that $h(Z_r(A), Z(A)) \leq \varepsilon$, which completes the proof of (4).

*(iii)* $\Rightarrow$ *(i)*. Given $\varepsilon > 0$, fix a $\delta \in (0,1)$ such that

$$r(A) \leq 1,\ r(A) \leq r < r' \leq 2,\ r' - r < \delta \quad \Rightarrow \quad h\big(Z_{r'}(A), Z_r(A)\big) < \varepsilon.$$

Given $x \in (2+\delta)B_X$, consider the set

$$A := B(0, 1+\delta) \cap B(x, 1).$$

Clearly, $r(A) \leq 1$ and $0 \in Z_{1+\delta}(A)$. Hence there is $z \in Z_1(A)$ with $\|z\| < \varepsilon$. Since $A \subset B(z, 1)$, we have (i). The proof of equivalence of (i)–(vi) is complete.

Further, *(ii)* $\Rightarrow$ *(vii)* is clear by Remark 3.4(c). Finally, let $X$ be separable with (vii). Given $A \in \mathcal{BC}(X)$, fix a countable dense set $E \subset A$, and form a sequence $(a_n)$ in $E$ in which every element of $E$ appears infinitely many times. Then we have $r(A, \cdot) = \rho((a_n), \cdot)$, $r(A) = r(a_n)$, and $Z_r(A) = Z_r(a_n)$. Therefore (iii) holds. We are done. ☐

Concerning the proof of the last part of Theorem 3.10 about separable case, its simple idea has been used already in [7, Theorem 3.4].

**Remark 3.11** (Further extension)**.** The conditions (iv)–(vi) in Theorem 3.10 admit analogues for nets (with similar but formally more complicated proofs) if we define a distance between nets in an appropriate way. This can be done as follows. Given two nonincreasing bounded nets $\mathcal{A} = (A_i)_{i \in I}$ and $\mathcal{B} = (B_j)_{j \in J}$, define

$$d(\mathcal{A}, \mathcal{B}) := \{\varepsilon > 0 :\ \forall i,\ \exists j,\ B_j \subset A_i + \varepsilon B_X\,;\ \text{and}\ \forall j,\ \exists i,\ A_i \subset B_j + \varepsilon B_X\}.$$

This "distance" is a pseudometric which works for our purposes, providing additional three equivalent conditions to Theorem 3.10,(i)–(vi).

For bounded sequences in $X$, we can proceed in the same way by defining $d((x_n),(y_n)) := d(\mathcal{A}, \mathcal{B})$ where $\mathcal{A}, \mathcal{B}$ are the (countable) nonincreasing nets associated by Remark 3.4(c,b) to $(x_n), (y_n)$, respectively. In this way we obtain analogues of Theorem 3.10,(iv)–(vi), that are equivalent to Theorem 3.10(vii). Let us remark that this pseudometric on bounded sequences was considered in [4].

**Remark 3.12** (Relative centers w.r.t. a subspace)**.** Let $Y$ be a closed subspace of $X$. The above proofs can be easily modified to get analogous results for "relative w.r.t. $Y$" notions. Instead of (QUC) one has to consider *(QUC) w.r.t. $Y$*, as defined in [2]; and instead of Chebyshev centers one has to consider *Chebyshev centers w.r.t. $Y$*, that is, minimizers of the function $r(A, \cdot)$ restricted to $Y$ (and analogously for asymptotic centers). We have chosen to present all proofs in their "absolute" versions just for sake of simplicity.

## 4. Spaces of bounded continuous functions

In this section, we consider spaces of bounded $X$-valued functions where $X$ is a (QUC) Banach space. This needs a bit of preparation.

**Lemma 4.1.** *Let $r_1, r_2 > 0$ and $E \subset X \times X$ be such that*

$$B(x_1, r_1) \cap B(x_2, r_2) \neq \emptyset \quad \textit{for each } (x_1, x_2) \in E.$$

*Then for every $\varepsilon > 0$ the mapping*

$$\Phi(x_1, x_2) := B(x_1, r_1) \cap B(x_2, r_2 + \varepsilon)\,, \qquad (x_1, x_2) \in E,$$

*is Hausdorff continuous on $E$.*

*Proof.* Clearly, the values of $\Phi$ on $E$ have nonempty interiors. The rest follows, e.g., from [6, Proposition 2.3]. □

Let us recall the notions of Hausdorff and topological upper/lower semicontinuity for multivalued mappings. Let $T$ be a topological space, $F \colon T \to \mathcal{BC}(X)$, $t_0 \in T$. Let $\mathcal{U}(t_0)$ be the family of neighborhoods of $t_0$.
$F$ is said to be *H-u.s.c.* [*H-l.s.c.*] at $t_0$ if

$$\forall \varepsilon > 0,\ \exists U \in \mathcal{U}(t_0),\ \forall t \in U,\ F(t) \subset F(t_0) + \varepsilon B_X\ [F(t_0) \subset F(t) + \varepsilon B_X].$$

$F$ is said to be *u.s.c.* (*l.s.c.*) at $x_0$ if

$$\forall V \subset X \text{ open},\ F(t_0) \subset V\ [F(t_0) \cap V \neq \emptyset],\ \exists U \in \mathcal{U}(t_0),$$
$$\forall t \in U,\ F(t) \subset V\ [F(t) \cap V \neq \emptyset].$$

The following fact is well-known: (a),(b) are trivial, (c) is an easy exercise.

**Fact 4.2.** *Let $T, F, t_0$ be as above.*

*(a) If $F$ is H-l.s.c. at $t_0$ then it is l.s.c. at $t_0$.*

*(b) If $F$ is u.s.c. at $t_0$ then it is H-u.s.c. at $t_0$.*

*(c) If $F(t_0)$ is compact, the inverse implications in (a),(b) hold as well.*

**Lemma 4.3.** *Let $X$ be a (QUC) Banach space, $T$ a topological space, $t_0 \in T$, and $F\colon T \to 2^X$ a mapping with nonempty values which is Hausdorff u.s.c. at $t_0$. Let $r > 0$ be such that*

$$G(t) := \{x \in X : F(t) \subset B(x, r)\} \neq \emptyset \quad \textit{for each } t \in T.$$

*Then the mapping $G$ is Hausdorff l.s.c. at $t_0$.*

*Proof.* Fix $\varepsilon > 0$ and $x_0 \in G(t_0)$. Let $\delta = \delta(\frac{\varepsilon}{r})$ be as in the definition of (QUC). There exists a neighborhood $U$ of $t_0$ such that $F(t) \subset F(t_0) + r\delta B_X$ for each $t \in U$.

Now, let $t \in U$ and $x_t \in G(t)$. Since $F(t) \subset B(x_t, r)$ and $F(t_0) \subset B(x_0, r)$, we have $F(t) \subset B(x_0, r+r\delta) \cap B(x_t, r)$. By (QUC) (more precisely, by a variant of Observation 3.5 for (QUC)), there exists $y_t \in B(x_0, r(\varepsilon/r)) = B(x_0, \varepsilon)$ such that $F(t) \subset B(y_t, r)$. Thus $y_t \in G(t) \cap B(x_0, r)$ for any $t \in U$, and we are done. □

**Proposition 4.4.** *If $X$ is* (QUC)*, then it satisfies the condition (ii) from Theorem 2.2 in such a way that, for each $\beta > 0$, the point $y = y(x)$ therein can be taken so that it depends continuously on $x$.*

*Proof.* Given $\varepsilon > 0$, let $\delta = \delta(\varepsilon)$ be as in the definition of (QUC). Given $0 < \beta < \delta$, fix an arbitrary $\sigma \in (0, \beta)$. For $x \in (2 + \delta)B_X$, the set $F(x) := B(0, 1 + \delta) \cap B(x, 1)$ is nonempty. Since $\frac{1+\delta}{1+\sigma} - 1 < \delta$, we can use an analogue of Observation 3.5 to conclude that for each $x \in (2 + \delta)B_X$ there exists $y \in B(0, (1 + \sigma)\varepsilon)$ with $\widetilde{F}(x) := B(0, 1 + \delta) \cap B(x, 1 + \sigma) \subset B(y, 1 + \beta)$.

Notice that, by Lemma 4.1, $\widetilde{F}$ is Hausdorff continuous on $(2+\delta)B_X$. Now, Lemma 4.3 implies that the mapping

$$G(x) := \begin{cases} \{y \in X : \widetilde{F}(x) \subset B(y, 1 + \beta)\} & \text{for } x \in (2 + \delta)B_X, \\ X & \text{otherwise}, \end{cases}$$

is l.s.c. on $X$. Since $\widetilde{G}(x) := G(x) \cap B^0(0, (1 + \beta)\varepsilon) \neq \emptyset$ (since it contains $y$) for each $x \in X$, Michael's selection theorem [8] implies that each point of $X$ has a neighborhood on which the mapping $\widetilde{G}$ has a continuous selection. Then a standard partition-of-unity argument implies that $\widetilde{G}$ has a continuous selection $y = y(x)$ on $X$. □

**Definition 4.5.** Given a topological space $T$, we denote by $C_b(T;X)$ the Banach space of all bounded continuous $X$-valued functions on $T$, equipped with the norm $\|\xi\|_\infty := \sup_{t\in T} \|\xi(t)\|$.

The main result of this section is the next theorem which generalizes Theorem 1.3(b).

**Theorem 4.6.** *If $X$ is a* (QUC) *Banach space, and $T$ a topological space, then then also the space $C_b(T;X)$ is* (QUC)*.*

*Proof.* Let $\varepsilon > 0$. By Proposition 4.4, there exists $\delta > 0$ such that for every $x \in X$ and $\alpha > 0$ there is a continuous function $y\colon X \to B(0,\varepsilon)$ such that

$$B(0, 1+\delta) \cap B(x, 1) \subset B(y(x), 1+\alpha) \qquad (x \in X).$$

Let $B_\infty(\eta, r)$ denote the ball in $C_b(T;X)$, with center $\eta$ and radius $r$. Now, given $\xi \in C_b(T;X)$, we immediately obtain that

$$B_\infty(0, 1+\delta) \cap B_\infty(\xi, 1) \subset B_\infty(y \circ \xi, 1+\alpha).$$

Since $\|y \circ \xi\|_\infty \leq \varepsilon$, we conclude that $C_b(T;X)$ satisfies the condition (ii) from Theorem 2.2. The rest follows from Theorem 2.2. □

**Definition 4.7.** Let $T$ be a topological space, $T_0 \subset T$ a closed set, $L$ a locally compact topological space, and $\Gamma$ a nonempty set.

- $C_b(T, T_0; X)$ denotes the closed subspace of $C_b(T;X)$, consisiting of all functions that vanish on $T_0$.
- $C_0(L;X)$ is the closed subspace of $C_b(L;X)$, consisiting of all functions $\xi$ that vanish at infinity, i.e., the sets $\{t \in L : |\xi(t)| \geq \varepsilon\}$, $\varepsilon > 0$, are compact.
- $\ell_\infty(\Gamma;X) := C_b(\Gamma;X)$ and $c_0(\Gamma;X) := C_0(\Gamma;X)$, where $\Gamma$ is equipped with the discrete topology.

**Theorem 4.8.** *For a Banach space $X$, the following assertions are equivalent.*

*(i) $X$ is* (QUC)*.*

*(ii) $C_b(T;X)$ is* (QUC) *for every (nonempty) topological space $X$.*

*(iii)* $C_b(T, T_0; X)$ *is* (QUC) *for every (nonempty) topological space* $X$ *and every closed set* $T_0 \subset T$.

*(iv)* $C_0(L; X)$ *is* (QUC) *for every (nonempty) locally compact space* $L$.

*(v)* $c_0(\Gamma; X)$ *is* (QUC) *for every (nonempty) set* $\Gamma$.

*(vi)* $\ell_\infty(\Gamma; X)$ *is* (QUC) *for every (nonempty) set* $\Gamma$.

*Proof.* The implication $(i) \Rightarrow (ii)$ is clear by Theorem 4.6.

$(ii) \Rightarrow (iii)$. Assume that $C_b(T; X)$ is (QUC), and $T_0 \subset T$ is a nonempty closed set. Given $\varepsilon > 0$, let $\delta = \delta(\varepsilon)$ be as in the definition of (QUC) for $C_b(T; X)$. Let us denote the balls in $C_b(T, T_0; X)$ by $B_\infty^{T_0}(\eta, r)$. Let $x \in C_b(T, T_0; X)$ be such that

$$A := B_\infty^{T_0}(0, 1 + \delta) \cap B_\infty^{T_0}(x, 1) \neq \emptyset.$$

There exists $y \in C_b(T; X)$ such that $\|y\|_\infty \leq \varepsilon$ and $A \subset B_\infty(y, 1)$. Define a continuous function $\lambda \colon [0, \infty) \to [0, 1]$ by

$$\lambda(s) = \begin{cases} 1 & \text{for } s \geq \varepsilon, \\ s/\varepsilon & \text{for } 0 \leq s \leq 1. \end{cases}$$

Put $\bar{y}(t) := \lambda(\|x(t)\|)y(t) + \big[1 - \lambda(\|x(t)\|)\big]x(t)$, $t \in T$. Then $\bar{y} \in C_b(T, T_0; X)$, and $\bar{y} \in [y, x]$ (segment in $C_b(T; X)$). For any $a \in A$, we have $y, x \in B_\infty(a, 1)$, hence $\bar{y} \in B_\infty(a, 1)$; and this proves that $A \subset B_\infty^{T_0}(\bar{y}, 1)$. Finally, it is easy to see that $\|\bar{y}\|_\infty \leq \varepsilon$.

$(iii) \Rightarrow (iv)$ is immediate since $\mathcal{C}_0(L; X) = C_b(T, T_0; X)$ where $T = L \cup \{\infty\}$ is the Alexandroff one-point compactification of $L$, and $T_0 = \{\infty\}$.

The implications $(iv) \Rightarrow (v)$ and $(ii) \Rightarrow (vi) \Rightarrow (v)$ are now obvious, while the remaining implication $(v) \Rightarrow (i)$ is a very easy observation. □

Now, results in Section 3 immediately give the following

**Corollary 4.9.** *Let* $X$ *be (QUC). Let* $W$ *be one of the spaces* $C_b(T; X)$ *and* $C_0(L; X)$ *(or other spaces appearing in Theorem 4.8). Then the Chebyshev-center map* $Z(\cdot)$ *is uniformly continuous on sets of the form* $\{A \in \mathcal{BC}(X) : r(A) \leq r_0\}$.

Let us remark that the above corollary generalizes [1, Corollary 3] (case of $X$ uniformly convex) and [11, Theorems 3.1, 3.2] (asserting existence of Chebyshev centers only for compact sets, and existence of a continuous selection for $Z(\cdot)$ on $\{A \in \mathcal{BC}(W) : A \text{ compact}\}$).

## 5. Sufficient conditions

The aim of this section is to provide some new (QUC) spaces of the form $C_b(T;X)$. The key will be the following finite-dimensional result.

**Theorem 5.1.** *Let $X$ be a Banach space of a finite dimension $d$. Let*

$$\Phi(x_1,\dots,x_d)=\bigcap_{i=1}^{d} B(x_i,1)\,,\quad x_1,\dots,x_d\in X,$$

*and $D:=\{(x_1,\dots,x^d)\in X^d:\Phi(x_1,\dots,x_d)\neq\emptyset\}$. If the mapping $\Phi\colon D\to\mathcal{BC}(X)$ is l.s.c., then $X$ is* (QUC)*.*

*Proof.* Assume that $X$ is not (QUC). There exist $\bar\varepsilon>0$, and sequences $(\delta_n)$ and $(x_n)$ in $(0,\infty)$ and $X$, respectively, such that $\delta_n\to 0$ and, for each $n$,

$$B(0,1+\delta_n)\cap B(x_n,1)\subset B(y,1)\quad\Rightarrow\quad y\notin B(0,\bar\varepsilon).$$

This implication can be rewritten as

$$\bigcap\big\{B(z,1):z\in B(0,1+\delta_n)\cap B(x_n,1)\big\}\cap B(0,\bar\varepsilon)=\emptyset\,.$$

Notice that $x_n\in B(z,1)$ for each $z\in B(0,1{+}\delta_n)\cap B(x_n,1)$. By this fact and by Helly's Intersection Theorem, for each $n$ there exists a $d$-tuple $(z_1^{(n)},\dots,z_d^{(n)})\in X^d$ such that

$$z_i^{(n)}\in B(0,1+\delta_n)\cap B(x_n,1)\quad\text{and}\quad \Phi\big(z_1^{(n)},\dots,z_d^{(n)}\big)\cap B(0,\bar\varepsilon)=\emptyset.$$

By passing to a subsequence if necessary, we can (and do) suppose that

$$\big(z_1^{(n)},\dots,z_d^{(n)}\big)\to(\bar z_1,\dots,\bar z_d)\quad\text{in } X^d.$$

Lower semicontinuity of $\Phi$ implies that $\Phi(\bar z_1,\dots,\bar z_d)\cap B^0(0,\bar\varepsilon)=\emptyset$. On the other hand, $\|\bar z_i\|\le 1$ since $\|z_i^{(n)}\|\le 1+\delta_n$. Thus we obtain $0\in\Phi(\bar z_1,\dots,\bar z_d)$, a contradiction. □

**Corollary 5.2.** *Let $X$ be a finite-dimensional Banach space satisfying* at least one *of the following conditions.*

*(a) $X$ is polyhedral (i.e., $B_X$ is a polytope).*

*(b)* $\dim X\le 2$.

*Then $X$ is* (QUC)*, and hence also every space $C_b(T;X)$ is* (QUC)*.*

*Proof.* By [12, Theorem 5.10], the mapping $\Phi$ from Theorem 5.1 is l.s.c. Apply Theorem 5.1 and Theorem 4.6. □

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